

# Online cake cutting

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**Abstract.** We propose an online form of the cake cutting problem. This models situations where agents arrive and depart during the process of dividing a resource. We show that well known fair division procedures like cut-and-choose and the Dubins-Spanier moving knife procedure can be adapted to apply to such online problems. We propose some fairness properties that online cake cutting procedures can possess like online forms of proportionality and envy-freeness. We also consider the impact of collusion between agents. Finally, we study theoretically and empirically the competitive ratio of these online cake cutting procedures. Based on its resistance to collusion, and its good performance in practice, our results favour the online version of the cut-and-choose procedure over the online version of the moving knife procedure.

## 1 Introduction

*Congratulations. Today is your birthday so you take a cake into the office to share with your colleagues. At tea time, people slowly start to arrive. However, as some people have to leave early, you cannot wait for everyone to arrive before you start sharing the cake. How do you proceed fairly?*

This is an example of an *online* cake cutting problem. Most previous studies of fair division assume that all agents are available at the time of the division [Brams and Taylor, 1996]. Here, agents arrive and depart as the cake is being divided. Online cake cutting provides an abstract model for a range of practical problems besides birthday parties. Consider, for instance, allocating time on a large telescope. Astronomers will have different preferences for when to use the telescope depending on what objects are visible, the position of the sun, etc. How do we design a web-based reservation system so that astronomers can asynchronously choose observation times that is fair to all? As a second example, consider allocating space at an exhibition. Exhibitors will have different preferences for space depending on the size, location, cost, etc. How do allocate space when not all exhibitors arrive at the same time but those who have arrived want to start setting up immediately?

Online cake cutting poses some interesting new challenges. On the one hand, the online aspect of such problems makes fair division more difficult than in the offline case. How can we ensure that agents do not envy cake already given to other agents? On the other hand, the online aspect of such problems may make fair division easier than in the offline case. Perhaps agents do not envy cake that has already been eaten before they arrive?

## 2 Online cake cutting

We assume that agents are risk averse. That is, they declare valuations of the cake that maximizes the minimum value of the cake that they receive, regardless of what the other agents do. This is a common assumption in cake cutting. For instance, Brams, Jones and Klamler (2006) argue:

*“... As is usual in the cake-cutting literature, we postulate that the goal of each person is to maximize the value of the minimum-size piece (maximin piece) that he or she can guarantee, regardless of what the other person does. Thus, we assume that each person is risk-averse: He or she will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a maximin piece ...”*

We will formulate cake cutting as dividing the unit interval,  $[0, 1]$  between  $n$  agents.

**Definition 1 (Cutting).** A cutting of a set of intervals  $S$  is a set of intervals recursively defined as follows:  $S$  is a cutting, and if  $S$  is a cutting and  $[a, b] \in S$  then  $S \setminus \{[a, b]\} \cup \{[a, c], [c, b]\}$  is a cutting where  $a < c < b$ .

A partition of a set  $S$  is a set of subsets of  $S$  whose union equals the original set which have an empty pairwise intersection. That is,  $\{S_i \mid 1 \leq i \leq n\}$  is a partition of  $S$  iff  $S = \bigcup_{1 \leq i \leq n} S_i$  and  $S_i \cap S_j = \{\}$  for  $1 \leq i < j \leq n$ .

**Definition 2 (Division).** A division of the cake amongst  $n$  agents is a partition of some cutting of  $\{[0, 1]\}$  into  $n$  subsets.

A special type of division is where each agent receives a single continuous interval. That is, the cutting contains  $n$  intervals, and each agent receives a subset containing just one interval. Note that we suppose there is no waste and that all cake is allocated. We can either relax this assumption, or introduce an additional “dummy” agent who is allocated any remaining cake.

Agents may value parts of the cake differently. For instance, one may prefer the iced part, whilst another prefers the candied part. As a second example, as we argued before, astronomers may prefer different observation times. We capture these differences by means of valuation functions on intervals.

**Definition 3 (Valuation).** Each agent  $i$  has an additive (but possibly different) valuation function with  $v_i([0, 1]) = 1$ ,  $v_i([a, b]) = v_i([a, c]) + v_i([c, b])$  for any  $a \leq c \leq b$ , and for some set of intervals  $S$ ,  $v_i(S) = \sum_{[a, b] \in S} v_i([a, b])$ .

In an online cake cutting problem, the agents are assumed to arrive in a fixed order. We assume without loss of generality that the arrival order is agent 1 to agent  $n$ . Once agents are allocated all their cake, they depart. The order in which agents are allocated cake and depart depends on the cake cutting procedure. For example, the agent present who most values the next slice of cake could be the next to be allocated cake and to depart. We can now formally define the online cake cutting problem.

**Definition 4 (Online cake cutting).** An online cake cutting procedure is a procedure that given the total number of agents yet to arrive, a set of agents currently present, and a set of intervals  $R$ , either returns “wait” (indicating that we will wait for the next agent to arrive) or returns an agent from amongst those present and two sets of intervals  $S$  and  $T$  such that  $S \cup T$  is a cutting of  $R$ . The agent returned by the procedure is allocated  $S$ , and  $T$  is then left to be divided amongst the agents not yet allocated cake. When no agents are left to arrive and there is only one agent present, the procedure must return  $S = R, T = \{\}$ . That is, the last agent is allocated whatever is left of the cake. When no agents are left to arrive and there is more than one agent present, the procedure cannot return “wait” but must cut the cake and assign it to one agent.

Our definition of online cake cutting does not assume that all agents receive cake. Any agent can be allocated an empty set containing no intervals. However, our definition does assume the whole cake is eventually allocated, and that each agent receives all their cake at one time. We assume that at least one agent is allocated some cake before the last arrives otherwise the problem is not online. A special type of online cake cutting procedure is when the departure order is fixed in advance. For instance, if the procedure waits for the first agent to arrive, and whenever a new agent arrives, allocates cake to the longest waiting agent then the departure order is the same as the arrival order. Another special type of cake online cake cutting procedure is one in which the cake is only cut from one or other of the ends of the cake. There are many interesting possible generalisations of this problem. For example, there may only be a bound on the total number of agents to arrive (e.g. you’ve invited 20 work colleagues to share your birthday cake but not all of them might turn up). Another generalisation is when an agent is not allocated cake all at one time but at several moments during the process of division.

### 3 Fairness properties

What properties do we want from an online cake cutting procedure? The literature on cake cutting studies various notions of fairness like envy freeness, as well as various forms of strategy proofness [Brams and Taylor, 1996; Robertson and Web, 1998; Chen *et al.*, 2010]. These are all properties that we might want from an online cake cutting procedure.

**Proportionality:** A cake cutting procedure is *proportional* iff each of the  $n$  agents assigns at least  $\frac{1}{n}$  of the total value to their piece(s). We call such an allocation *proportional*.

**Envy freeness:** This is a stronger notion of fairness. A cake cutting procedure is *envy free* iff no agent values another agent’s pieces more than their own. Note that envy freeness implies proportionality but not vice versa.

**Equitability:** A cake cutting procedure is *equitable* iff agents assign the same value to the cake which they are allocated (and so no agent envies the valuation that another agent gives to their cake). For 3 or more agent, equitability and envy freeness can be incompatible [Brams and Taylor, 1996].

**Efficiency:** This is also called Pareto optimality. A cake cutting procedure is *Pareto optimal* iff there is no other allocation to the one returned that is more valuable

for one agent and at least as valuable for the others. Note that Pareto optimality does not in itself ensure fairness since allocating all the cake to one agent is Pareto optimal. A cake cutting procedure is *weakly Pareto optimal* iff there is no other allocation to the one returned that is more valuable for all agents. A cake cutting procedure that is Pareto optimal is weakly Pareto optimal but not vice versa.

**Truthfulness:** Another consideration is whether agents can profit by being untruthful about their valuations. As in [Chen *et al.*, 2010], we say that a cake cutting procedure is *weakly truthful* iff there exists some valuations of the other agents such that an agent will do at least as well by telling the truth. A stronger notion (often called strategy proofness in social choice) is that agents must not be able to profit even when they know how others value the cake. As in [Chen *et al.*, 2010], we say that a cake cutting procedure is *truthful* iff there are no valuations where an agent will do better by lying.

The fact that some agents may depart before others arrive places some fundamental limitations on the fairness of online cake cutting procedures. In particular, unlike the offline case, we can prove a strong impossibility result.

**Proposition 1** *No online cake cutting procedure is proportional, envy free or equitable.*

**Proof:** Consider any cake cutting procedure. As the procedure is online, at least one agent  $i$  departs before the final agent  $n$  arrives. Since the valuation function of agent  $n$ ,  $v_n$  is not revealed before agent  $i$  departs, the set of intervals  $S_i$  allocated to agent  $i$  cannot depend on  $v_n$ . Similarly,  $v_n$  cannot change who is first to depart. Suppose agent  $n$  has a valuation function with  $v_n(S_i) = 1$ . As  $v_n$  is additive and  $v_n([0, 1]) = 1$ , agent  $n$  only assigns value to the intervals assigned to agent  $i$ . Hence, any interval outside  $S_i$  that is allocated to agent  $n$  is of no value to agent  $n$ . Hence the procedure is not proportional. Since envy-freeness implies proportionality, by modus tollens, the procedure is also not envy-free.

To demonstrate that no cake cutting procedure is equitable, we restrict ourselves to problems in which all agents assign non-zero value to any non-empty interval. Suppose that the procedure is equitable. As all the cake is allocated, at least one agent must receive cake. Since the procedure is equitable, it follows that all agents must receive some cake. Now, the first agent  $i$  to depart and the set of intervals  $S_i$  allocated to agent  $i$  cannot depend on  $v_n$ , the valuation function of the last agent to arrive. Suppose  $v_i(S_i) = a$ . Now we have argued that  $S_i$  is non-empty. Hence, by assumption,  $a > 0$ . We now modify the valuation function of agent  $n$  so that  $v_n(S_i) = 1 - \frac{a}{2}$ . Then  $v_n(S_n) \leq \frac{a}{2} < a = v_i(S_i)$ . Hence the procedure is not equitable.  $\square$

By comparison, the other properties of Pareto optimality and truthfulness are achievable in the online setting.

**Proposition 2** *There exist online cake cutting procedures that are Pareto optimal and truthful.*

**Proof:** Consider the online cake cutting procedure which allocates all cake to the first agent to arrive. This is Pareto optimal as any other allocation will be less desirable for this agent. It is also truthful as no agent can profit by lying about their valuations.  $\square$

Of course, allocating all cake to the first agent to arrive is not a very “fair” procedure. Therefore we need to consider other weaker properties of fairness that online procedures can possess. We introduce such properties in the next section

## 4 Online properties

We define some fairness properties that are specific to online procedures.

**Proportionality:** We weaken the definition of proportionality to test whether agents receive a fair proportion of the cake that *remains* when they arrive. A cake cutting procedure is *weakly proportional* iff each agent assigns at least  $\frac{r}{k}$  of the total value of the cake to their pieces where  $r$  is the fraction of the total value assigned by the agent to the (remaining) cake when they arrive and  $k$  is the number of agents yet to be allocated cake at this point.

**Envy freeness:** We can weaken the definition of envy freeness to consider just agents allocated cake after the arrival of a given agent. A cake cutting procedure is *weakly envy free* iff agents do not value cake allocated to agents after their arrival more than their own. Note that weak envy freeness implies weak proportionality but not vice versa. Similarly, envy freeness implies weak envy freeness but not vice versa. An even weaker form of envy freeness is when an agent only envies cake allocated to other agents whilst they are present. A cake cutting procedure is *immediately envy free* iff agents do not value cake allocated to any agent after their arrival and before their departure more than their own. Weak envy freeness implies immediate envy freeness but not vice versa.

**Order monotonicity:** An agent’s allocation of cake typically depends on when they arrive. We say that a cake cutting procedure is *order monotonic* iff an agent’s valuation of their cake does not decrease when they are moved earlier in the arrival ordering and all other agents are left in the same relative positions. Note that as the moved agent can receive cake of greater value, other agents may receive cake of less value. A positive interpretation of order monotonicity is that agents are encouraged to participate as early as possible. On the other hand, order monotonicity also means that agents who have to arrive late due to reasons beyond their control may receive less value.

The online versions of the proportional and envy free properties are weaker than their corresponding offline properties. We consider next two well known offline procedures that naturally adapt to the online setting and demonstrate that they have many of the online properties introduced here.

## 5 Online Cut-and-Choose

The cut-and-choose procedure for two agents dates back to antiquity. It appears nearly three thousand years ago in Hesiod’s poem *Theogony* where Prometheus divides a cow and Zeus selects the part he prefers. Cut-and-choose is enshrined in the UN’s 1982 Convention of the Law of the Sea where it is used to divide the seabed for mining. In

cut-and-choose, one agent cuts the cake and the other takes the “half” that they most prefer. We can extend cut-and-choose to more than two agents by having one agent cut a “proportional” slice and giving this slice to the agent who values it most. We then repeat with one fewer agent. The two person cut-and-choose procedure is proportional, envy free, Pareto optimal and weakly truthful. However, it is not equitable nor truthful.

We can use cut-and-choose as the basis of an online cake cutting procedure. The first agent to arrive cuts off a slice of cake and waits for the next agent to arrive. Either the next agent to arrive chooses this slice and departs, or the next agent to arrive declines this slice and the waiting agent takes this slice and departs. If more agents are to arrive, the remaining agent cuts the cake and we repeat the process. Otherwise, the remaining agent is the last agent to be allocated cake and departs with whatever is left. We assume that all agents know how many agents will arrive. A natural extension (which we do not consider further) is when multiple agents arrive and can choose or reject the cut cake. By insisting that an agent cuts the cake before the next agent is allowed to arrive, we will make the procedure more resistant to collusion. We discuss this in more detail later.

*Example 1.* Suppose there are three agents, the first values only  $[\frac{1}{2}, 1]$ , the second values only  $[\frac{1}{3}, 1]$ , and the third values only  $[0, \frac{3}{4}]$ . We suppose that they uniformly value slices within these intervals. If we operate the online cut-and-choose procedure, the first agent arrives and cuts off the slice  $[0, \frac{2}{3}]$  as they assign this slice  $\frac{1}{3}$  the total value of the cake. The second agent then arrives. As they assign this slice with  $\frac{1}{2}$  the total value of the cake and they are only expecting  $\frac{1}{3}$  of the total, the second agent is happy to take this slice and depart. The first agent then cuts off the slice  $[\frac{2}{3}, \frac{5}{6}]$  as they assign this  $\frac{1}{3}$  of the total value of the cake (and  $\frac{1}{2}$  of the value remaining after the second agent departed with their slice). The third agent then arrives. As they assign the slice  $[\frac{2}{3}, \frac{5}{6}]$  with all of the total value of the remaining cake and they are only expecting  $\frac{1}{2}$  of whatever remains, the third agent is happy to take this slice and depart. The first agent now takes what remains, the slice  $[\frac{5}{6}, 1]$ . We can argue that everyone is happy as the first agent received a “fair” proportion of the cake, whilst the other two agents received slices that were of even greater proportional value to them.

The online cut-and-choose procedure has almost all of the online fairness properties just introduced.

**Proposition 3** *The online cut-and-choose procedure is weakly proportional, immediately envy free, and weakly truthful. However, it is not proportional, (weakly) envy free, equitable, (weakly) Pareto optimal, truthful or order monotonic.*

**Proof:** Suppose agent  $i$  cuts the slice  $c_i$ . As agent  $i$  is risk averse, and as there is a chance that agent  $i$  is allocated  $c_i$ , agent  $i$  will cut  $c_i$  to ensure that  $v_i(c_i) \geq \frac{r}{k}$  where  $k$  is the number of agents still to be allocated cake and  $r$  is the fraction of cake remaining when agent  $i$  arrived. Similarly as there is a chance that agent  $i$  is not allocated  $c_i$ , but will have to take a share of what remains, they will cut  $c_i$  so that  $v_i(c_i) \leq \frac{r}{k}$ . Hence,  $v_i(c_i) = \frac{r}{k}$ , and the procedure is both weakly proportional and weakly truthful. It is also immediately envy free since each slice that agent  $i$  cuts (and sees allocated) has the same value,  $\frac{r}{k}$ .

To show that this procedure is not proportional, (weakly) envy free, equitable, (weakly) Pareto optimal, truthful or order monotonic consider four agents who value the cake as follows:  $v_1([0, \frac{1}{4}]) = \frac{1}{4}$ ,  $v_1([\frac{1}{4}, \frac{3}{4}]) = \frac{1}{12}$ ,  $v_1([\frac{3}{4}, 1]) = \frac{2}{3}$ ,  $v_2([\frac{1}{4}, \frac{1}{2}]) = \frac{1}{3}$ ,  $v_2([\frac{1}{2}, \frac{5}{8}]) = \frac{2}{3}$ ,  $v_3([0, \frac{1}{4}]) = \frac{1}{2}$ ,  $v_3([\frac{1}{2}, \frac{5}{8}]) = \frac{1}{12}$ ,  $v_3([\frac{5}{8}, \frac{3}{4}]) = \frac{1}{6}$ ,  $v_3([\frac{3}{4}, 1]) = \frac{1}{4}$ ,  $v_4([\frac{1}{4}, \frac{1}{2}]) = \frac{3}{4}$ ,  $v_4([\frac{1}{2}, \frac{3}{4}]) = \frac{1}{12}$ , and  $v_4([\frac{3}{4}, 1]) = \frac{1}{6}$ . All other slices have zero value. For instance,  $v_2([0, \frac{1}{4}]) = v_3([\frac{1}{4}, \frac{1}{2}]) = 0$ .

If we apply the online cut-and-choose procedure, agent 1 cuts off the slice  $[0, \frac{1}{4}]$  as  $v_1([0, \frac{1}{4}]) = \frac{1}{4}$  and 4 agents are to be allocated cake. Agent 2 places no value on this slice so agent 1 takes it. Agent 2 then cuts off the slice  $[\frac{1}{4}, \frac{1}{2}]$  as  $v_2([\frac{1}{4}, \frac{1}{2}]) = \frac{1}{3}v_2([\frac{1}{4}, 1])$  and 3 agents remain to be allocated cake. Agent 3 places no value on this slice so agent 2 takes it. Agent 3 then cuts the cake into two pieces of equal value:  $[\frac{1}{2}, \frac{3}{4}]$  and  $[\frac{3}{4}, 1]$ . Agent 4 takes the slice  $[\frac{3}{4}, 1]$  as it has greater value, leaving agent 3 with the slice  $[\frac{1}{2}, \frac{3}{4}]$ .

The procedure is not proportional as agent 4 receives the slice  $[\frac{3}{4}, 1]$  but  $v_4([\frac{3}{4}, 1]) = \frac{1}{6}$ . The procedure is not (weakly) envy free as agent 1 receives the slice  $[0, \frac{1}{4}]$  and agent 4 receives the slice  $[\frac{3}{4}, 1]$ , but  $v_1([0, \frac{1}{4}]) = \frac{1}{4}$  and  $v_1([\frac{3}{4}, 1]) = \frac{2}{3}$ . Hence agent 1 envies the slice allocated to agent 4. The procedure is not equitable as agents receive cake of different value. The procedure is not (weakly) Pareto optimal as allocating agent 1 with  $[\frac{3}{4}, 1]$ , agent 2 with  $[\frac{1}{2}, \frac{3}{4}]$ , agent 3 with  $[0, \frac{1}{4}]$ , and agent 4 with  $[\frac{1}{4}, \frac{1}{2}]$  gives all agents greater value.

The procedure is not truthful as agent 2 can get a more valuable slice by misrepresenting their preferences and cutting off the larger slice  $[\frac{1}{4}, \frac{5}{8}]$ . This slice contains all the cake of any value to agent 2. Agent 3 has  $v_3([\frac{1}{4}, \frac{5}{8}]) = \frac{1}{12}$  so lets agent 2 take this larger slice. Finally, the procedure is not order monotonic as the value of the cake allocated to agent 4 decreases from  $\frac{1}{6}$  to  $\frac{1}{8}$  when they arrive before agent 3.  $\square$

## 6 Online moving knife

Another class of cake cutting procedures uses one or more moving knives. For example, in the Dubins-Spanier procedure for  $n$  agents [Dubins and Spanier, 1961], a knife is moved across the cake from left to right. When an agent shouts “stop”, the cake is cut and this agent takes the piece to the left of the knife. The procedure continues with the remaining agents until one agent is left (who takes whatever remains). This procedure is proportional but is not envy-free. However, only the first  $n - 2$  agents allocated slices of cake can be envious.

We can use the Dubins-Spanier procedure as the basis of an online moving knife procedure. The first  $k$  agents ( $k \geq 2$ ) to arrive perform one round of a moving knife procedure to select a slice of the cake. Whoever chooses this slice, departs. At this point, if all agents have arrived, we continue the moving knife procedure with  $k - 1$  agents. Alternatively the next agent arrives and we start again a moving knife procedure with  $k$  agents.

*Example 2.* Consider again the example in which there are three agents, the first values only  $[\frac{1}{2}, 1]$ , the second values only  $[\frac{1}{3}, 1]$ , and the third values only  $[0, \frac{3}{4}]$ . If we operate the online moving knife procedure, the first two agents arrive and perform one round of the moving knife procedure. The second agent is the first to call “cut” and departs

with the slice  $[0, \frac{5}{9}]$  (as this has  $\frac{1}{3}$  of the total value of the cake for them). The third agent then arrives and performs a round of the moving knife procedure with the first agent using the remaining cake. The third agent is the first to call “cut” and departs with the slice  $[\frac{5}{9}, \frac{47}{72}]$  (as this has  $\frac{1}{2}$  the total value of the remaining cake for them). The first agent takes what remains, the slice  $[\frac{47}{72}, 1]$ . We can argue that everyone is happy as the second and third agents received a “fair” proportion of the cake that was left when they arrived, whilst the first agent received an even greater proportional value.

The online moving knife procedure has similar fairness properties as the online cut-and-choose procedure. However, as we shall show in the following sections, it is neither as resistant to collusion nor as fair in practice.

**Proposition 4** *The online moving knife procedure is weakly proportional, immediately envy free and weakly truthful. However, it is not proportional, (weakly) envy free, equitable, (weakly) Pareto optimal, truthful or order monotonic.*

**Proof:** Suppose  $j$  agents ( $j > 1$ ) have still to be allocated cake. Consider any agent who has arrived. They call “cut” as soon as the knife reaches  $\frac{1}{j}$  of the value of the cake left for fear that they will receive cake of less value at a later stage. Hence, the procedure is weakly truthful and weakly proportional. The procedure is also immediately envy free as they will assign less value to any slice that is allocated after their arrival and before their departure.

To show that this procedure is not proportional, (weakly) envy free, equitable, (weakly) Pareto optimal, or truthful consider again the example with four agents used in the last proof. Suppose  $k = 2$  so that two agents perform each round of the moving knife procedure. Agent 1 and 2 arrive and run a round of the moving knife procedure. Agent 1 calls “cut” and departs with the slice  $[0, \frac{1}{4}]$ . Agent 3 then arrives and agent 2 and 3 perform a second round of the moving knife procedure. Agent 2 calls “cut” and departs with the slice  $[\frac{1}{4}, \frac{1}{2}]$ . Agent 4 then arrives and agent 3 and 4 perform the third and final round of the moving knife procedure. Agent 3 calls “cut” and departs with the slice  $[\frac{1}{2}, \frac{3}{4}]$ , leaving agent 4 with the slice  $[\frac{3}{4}, 1]$ . This is the same allocation as the online cut-and-choose procedure. Hence, for the same reasons as before, the online moving knife procedure is not proportional, (weakly) envy free, (weakly) Pareto optimal or truthful.

Finally, to show that the online moving knife procedure is not order monotonic consider again  $k = 2$ , and three agents with valuation functions:  $v_1([0, \frac{1}{3}]) = v_1([\frac{1}{3}, \frac{2}{3}]) = v_1([\frac{2}{3}, 1]) = \frac{1}{3}$ ,  $v_2([0, \frac{1}{3}]) = 0$ ,  $v_2([\frac{1}{3}, \frac{2}{3}]) = v_2([\frac{2}{3}, 1]) = \frac{1}{2}$ ,  $v_3([0, \frac{1}{6}]) = \frac{1}{3}$ ,  $v_3([\frac{1}{6}, \frac{1}{3}]) = v_3([\frac{1}{3}, \frac{2}{3}]) = 0$ , and  $v_3([\frac{2}{3}, 1]) = \frac{2}{3}$ . Agent 1 and 2 arrive and run a round of the moving knife procedure. Agent 1 calls “cut” and departs with the slice  $[0, \frac{1}{3}]$ . Agent 3 then arrives and agent 2 and 3 perform a second and final round of the moving knife procedure. Agent 2 calls “cut” and departs with the slice  $[\frac{1}{3}, \frac{2}{3}]$ , leaving agent 3 with the slice  $[\frac{2}{3}, 1]$ . On the other hand, if agent 3 arrives ahead of agent 2 then the value of the interval allocated to agent 3 drops from  $\frac{2}{3}$  to  $\frac{1}{3}$ . Hence the procedure is not order monotonic.  $\square$



## 7 Online collusion

An important consideration in online cake cutting procedures is whether agents present together in the room can collude together to increase the amount of cake they receive. We shall show that this is a property that favours the online cut-and-choose procedure over the online moving knife procedure. We say that a cake cutting procedure is vulnerable (resistant) to *online collusion* iff there exists (does not exist) a protocol to which the colluding agents can agree which increases or keeps constant the value of the cake that each receives. We suppose that agents do not meet in advance so can only agree to a collusion when they meet during cake cutting. We also suppose that other agents can be present when agents are colluding. Note that colluding agents cannot change their arrival order and can only indirectly influence their departure order. The arrival order is fixed in advance, and the departure order is fixed by the online cake cutting procedure.

### 7.1 Online cut-and-choose

The online cut-and-choose procedure is resistant to online collusion. Consider, for instance, the first two agents to participate. The first agent cuts the cake before the second agent is present (and has agreed to any colluding protocol). As the first agent is risk averse, they will cut the cake proportionally for fear that the second agent will decline to collude. Suppose the second agent does not assign a proportional value to this slice. It would be risky for the second agent to agree to any protocol in which they accept this slice as they might assign less value to any cake which the first agent later offers in compensation. Similarly, suppose the second agent assigns a proportional or greater value to this slice. It would be risky for the second agent to agree to any protocol in which they reject this slice as they might assign less total value to the slice that they are later allocated and any cake which the first agent offers them in compensation. Hence, assuming that the second agent is risk averse, the second agent will follow the usual protocol of accepting the slice iff it is at least proportional. A similar argument can be given for the other agents.

### 7.2 Online moving knife

On the other hand, the online moving knife procedure is vulnerable to online collusion. Suppose four or more agents are cutting a cake using the online moving knife procedure, but the first two agents agree to the following protocol:

1. Each agent will (silently) indicate when the knife is over a slice worth  $\frac{3}{4}$  of the total.
2. Each will only call “stop” once the knife is over a slice worth  $\frac{3}{4}$  of the total and the other colluding agent has given their (silent) indication that the cake is also worth as much to them;
3. Away from the eyes of the other agents, the two colluding agents will share this slice of cake using a moving knife procedure.

Under this protocol, both agents will receive slices that they value more than  $\frac{1}{4}$  of the total. This is better than not colluding. Note that it is advantageous for the agents to

agree to a protocol in which they call “stop” later than this. For example, they could agree to call stop at  $\frac{(p-1)}{p}$  of the total value for some  $p > 3$ . In this way, they would receive more than  $\frac{(p-1)}{2p}$  of the total value of the cake (which tends to half the total value as  $p \rightsquigarrow \infty$ ).

## 8 Competitive analysis

An important tool to study online algorithms is competitive analysis. We say that an online algorithm is *competitive* iff the ratio between its performance and the performance of the corresponding offline algorithm is bounded. But how do we measure the performance of a cake cutting algorithm?

### 8.1 Egalitarian measure

An egalitarian measure of performance would be the reciprocal of the smallest value assigned by any agent to their slice of cake. We take the reciprocal so that the performance measure increases as agent gets less valuable slices of cake. Using such a measure of performance, neither the online cut-and-choose nor the online moving knife procedures are competitive. There exist examples with just 3 agents where the competitive ratio of either online procedure is unbounded. The problem is that the cake left to share between the late arriving agents may be of very little value to these agents.

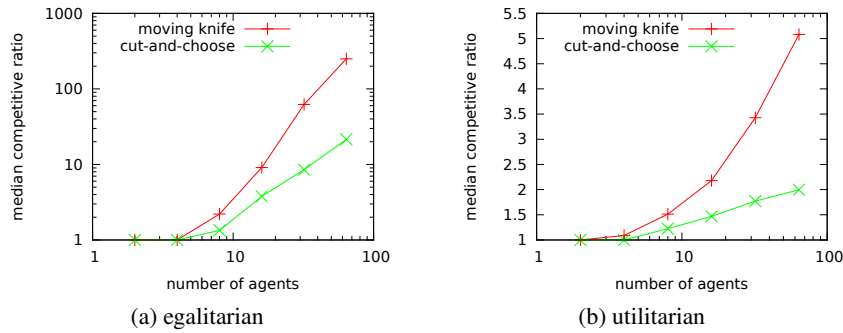
### 8.2 Utilitarian measure

An utilitarian measure of performance would be the reciprocal of the sum of the values assigned by the agents to their slices of cake (or equivalently the reciprocal of the mean value). With such a measure of performance, the online cut-and-choose and moving knife procedures are competitive provided the total number of agents,  $n$  is bounded. By construction, the first agent in the online cut-and-choose or moving knife procedure must receive cake of value at least  $\frac{1}{n}$  of the total. Hence, the sum of the valuations is at least  $\frac{1}{n}$ . On the other hand, the sum of the valuations of the corresponding offline algorithm cannot be more than  $n$ . Hence the competitive ratio cannot be more than  $n^2$ . In fact, there exist examples where the ratio is  $O(n^2)$ . Thus the utilitarian competitive ratio is bounded iff  $n$  itself is bounded.

## 9 Experimental results

To test the performance of these procedures in practice, we ran some experiments in which we computed the competitive ratio of the online moving knife and cut-and-choose procedures compared to their offline counterparts. We generated piecewise linear valuations for each agent by dividing the cake into  $k$  random segments, and assigning a random value to each segment, normalizing the total value of the cake. It is an interesting research question whether random valuations are more challenging than valuations which are more correlated. For instance, if all agents have the same valuation func-

tion (that is, if we have perfect correlation) then the online moving knife procedure performs identically to the offline. On the other hand, if the valuation functions are not correlated, online cake cutting procedures can struggle to be fair especially when late arriving agents more greatly value the slices of cake allocated to early departing agents. Results obtained uncorrelated instances need to be interpreted with some care as there are many pitfalls to using instances that are generated entirely at random [Gent *et al.*, 1997; MacIntyre *et al.*, 1998; Gent *et al.*, 2001].



**Fig. 1.** Competitive ratio between online and offline cake cutting procedures for (a) the egalitarian and (b) utilitarian performance measures. Note different scales to y-axes.

We generated cake cutting problems with between 2 and 64 agents, where each agent’s valuation function divides the cake into 8 random segments. At each problem size, we ran the online and offline moving knife and cut-and-choose procedures on the same 10,000 random problems. Overall, the online cut-and-choose procedure performed much better than the online moving knife procedure according to both the egalitarian and utilitarian performance measures. By comparison, the offline moving knife procedure performed slightly better than the offline cut-and-choose procedure according to both measures. See Figure 1 for plots of the competitive ratios between the performance of the online and offline procedures. Perhaps unsurprisingly, the egalitarian performance is rather disappointing when there are many agents since there is a high probability that one of the late arriving agents gets cake of little value. However, the utilitarian performance is reasonable, especially for the online cut-and-choose procedure. With 8 agents, the average value of cake assigned to an agent by the online cut-and-choose procedure is within about 20% of that assigned by the offline procedure. Even with 64 agents, the average value is within a factor of 2 of that assigned by the offline procedure.

## 10 Online Mark-and-Choose

A possible drawback of both of the online cake cutting procedures proposed so far is that the first agent to arrive can be the last to depart. What if we want a procedure in which agents can depart soon after they arrive? The next procedure has this property. Agents depart as soon as the next agent arrives (except for the last agent to arrive who takes whatever cake remains). However, the new procedure may not allocate cake from one end. In addition, the new procedure does not necessarily allocate continuous slices of cake.

In the online mark-and-choose procedure, the first agent to arrive marks the cake into  $n$  pieces. The second agent to arrive selects one piece to give to the first agent who then departs. The second agent then marks the remaining cake into  $n - 1$  pieces and waits for the third agent to arrive. The procedure repeats in this way until the last agent arrives. The last agent to arrive selects which of the two halves marked by the penultimate agent should be allocated to the penultimate agent, and takes whatever remains.

*Example 3.* Consider again the example in which there are three agents, the first values only  $[\frac{1}{2}, 1]$ , the second values only  $[\frac{1}{3}, 1]$ , and the third values only  $[0, \frac{3}{4}]$ . If we operate the online mark-and-choose procedure, the first agent arrives and marks the cake into 3 equally valued pieces:  $[0, \frac{2}{3}]$ ,  $[\frac{2}{3}, \frac{5}{6}]$ , and  $[\frac{5}{6}, 1]$ . The second agent then arrives and selects the least valuable piece for the first agent to take. In fact, both  $[\frac{2}{3}, \frac{5}{6}]$  and  $[\frac{5}{6}, 1]$  are each worth  $\frac{1}{4}$  of the total value of the cake to the second agent. The second agent therefore chooses between them arbitrarily. Suppose the second agent decides to give the slice  $[\frac{2}{3}, \frac{5}{6}]$  to the first agent. Note that the first agent assigns this slice with  $\frac{1}{3}$  of the total value of the cake. This leaves behind two sections of cake:  $[0, \frac{2}{3}]$  and  $[\frac{5}{6}, 1]$ . The second agent then marks what remains into two equally valuable pieces: the first is the interval  $[0, \frac{7}{12}]$  and the second contains the two intervals  $[\frac{7}{12}, \frac{2}{3}]$  and  $[\frac{5}{6}, 1]$ . The third agent then arrives and selects the least valuable piece for the second agent to take. The first piece is worth  $\frac{7}{12}$  of the total value of the cake to the third agent. As this is over half the total value, the other piece must be worth less. In fact, the second piece is worth  $\frac{1}{4}$  of the total value. The third agent therefore gives the second piece to the second agent. This leaves the third agent with the remaining slice  $[0, \frac{7}{12}]$ . It can again be claimed that everyone is happy as the first agents received a “fair” proportion of the cake that was left when they arrived, whilst both the second and third agent received an even greater proportional value.

This procedure again has the same fairness properties as the online cut-and-choose and moving knife procedures.

**Proposition 5** *The online mark-and-choose procedure is weakly proportional, immediately envy free and weakly truthful. However, it is not proportional, (weakly) envy free, equitable, (weakly) Pareto optimal, truthful, or order monotonic.*

**Proof:** Any agent marking the cake divides it into slices of equal value (for fear that they will be allocated one of the less valuable slices). Similarly, an agent selecting a slice for another agent selects the slice of least value to them (to maximize the value

that they receive). Hence, the procedure is weakly truthful and weakly proportional. The procedure is also immediately envy free as they will assign less value to the slice that they select for the departing agent than the value of the slices that they mark.

To show that this procedure is not proportional, (weakly) envy free, equitable, (weakly) Pareto optimal or truthful consider again the example with four agents used in earlier proofs. The first agent marks and is assigned the slice  $[0, \frac{1}{4}]$  by the second agent. The second agent then marks and is assigned the slice  $[\frac{1}{4}, \frac{1}{2}]$ . The third agent then marks and is assigned the slice  $[\frac{1}{2}, \frac{3}{4}]$ , leaving the fourth agent with the slice  $[\frac{3}{4}, 1]$ . The procedure is not proportional as the fourth agent only receives  $\frac{1}{6}$  of the total value, not (weakly) envy free as the first agent envies the fourth agent, and not equitable as agents receive cake of different value. The procedure is not (weakly) Pareto optimal as allocating the first agent with  $[\frac{3}{4}, 1]$ , the second with  $[\frac{1}{2}, \frac{3}{4}]$ , the third with  $[0, \frac{1}{4}]$ , and the fourth with  $[\frac{1}{4}, \frac{1}{2}]$  gives all agents greater value.

The procedure is not truthful as the second agent can get a larger and more valuable slice by misrepresenting their preferences and marking the cake into the slices  $[\frac{1}{4}, \frac{5}{8}]$ ,  $[\frac{5}{8}, \frac{3}{4}]$ , and  $[\frac{3}{4}, 1]$ . In this situation, the third agent allocates the second agent with the slice  $[\frac{1}{4}, \frac{5}{8}]$  which is of greater value to the second agent.

Finally, to show that the procedure is not order monotonic consider three agents and a cake in which the first agent places equal value on each of  $[0, \frac{1}{3}]$ ,  $[\frac{1}{3}, \frac{2}{3}]$  and  $[\frac{2}{3}, 1]$ , the second places no value on  $[0, \frac{1}{3}]$ , half the total value on  $[\frac{1}{3}, \frac{2}{3}]$ , and one quarter on each of  $[\frac{2}{3}, \frac{5}{6}]$ , and  $[\frac{5}{6}, 1]$ , and the third places a value of one sixth the total value on  $[0, \frac{1}{6}]$ , no value on  $[\frac{1}{6}, \frac{1}{3}]$  and  $[\frac{1}{3}, \frac{2}{3}]$ , and half the remaining value on  $[\frac{2}{3}, \frac{5}{6}]$  and  $[\frac{5}{6}, 1]$ . The first agent marks and is allocated the slice  $[0, \frac{1}{3}]$ . The second agent marks and is allocated the slice  $[\frac{1}{3}, \frac{2}{3}]$ , leaving the third agent with the slice  $[\frac{2}{3}, 1]$ . On the other hand, suppose the third agent arrives ahead of the second agent. In this case, the third agent marks the cake into two slice,  $[\frac{1}{3}, \frac{5}{6}]$  and  $[\frac{5}{6}, 1]$ . The second agent allocates the third agent the slice  $[\frac{5}{6}, 1]$ . Hence, the value of the interval allocated to the third agent halves when they go second in the arrival order. Hence the procedure is not order monotonic.  $\square$

## 11 Related work

There is an extensive literature on fair division and cake cutting procedures. See, for instance, [Brams and Taylor, 1996]. There has, however, been considerably less work on fair division problems similar to those considered here. Thomson considers a generalization where the number of agents may increase [Thomson, 1983]. He explores whether it is possible to have a procedure in which agents' allocations are monotonic (i.e. their values do not increase as the number of agents increase) combined with other common properties like weak Pareto optimality. Cloutier *et al.* consider a different generalization of the cake cutting problem in which the number of agents is fixed but there are multiple cakes [Cloutier *et al.*, 2010]. This models situations where, for example, agents wish to choose shifts across multiple days. This problem cannot be reduced to multiple single cake cutting problems if the agents' valuations across cakes are linked (e.g. you prefer the same shift each day). A number of authors have studied distributed mechanisms for fair division (see, for example, [Chevalerey *et al.*, 2009]). In such mech-

anisms, agents typically agree locally on deals to exchange goods. The usual goal is to identify conditions under which the system converges to a fair or envy free allocation.

## 12 Conclusions

We have proposed an online form of the cake cutting problem. This permits us to explore the concept of fair division when agents arrive and depart during the process of dividing a resource. It can be used to model situations, such as on the internet, when we need to divide resources asynchronously. There are many possible future directions for this work. One extension would be to undesirable goods (like chores) where we want as little of them as possible. It would also be interesting to consider the variation of the problem where agents have partial information about the valuation functions of the other agents. For voting and other forms of preference aggregation, there has been considerable interest of late in reasoning about preferences that are incomplete or partially known [Pini *et al.*, 2007; Walsh, 2007; Pini *et al.*, 2008]. With cake cutting, agents can act more strategically when they have such partial knowledge.

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