Aggregating partially ordered preferences: impossibility and possibility results

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Abstract. We consider preferences which can be partially ordered and which need to be aggregated. We prove that, under certain conditions, if there are at least two agents and three outcomes, no aggregation system on partially ordered preferences can be fair. These result generalizes Arrow's impossibility theorem for combining total orders. We also provide two sufficient conditions which guarantee fairness for the majority rule over partial orders. This allows us to generalize Sen's theorem for total orders. Finally, we give a generalization of the Muller-Satterthwaite result for social choice functions over partial orders.

1 Introduction

Many problems require us to combine the preferences of different agents. For example, when planning a wedding, we must combine the preferences of the bride, the groom and possibly some or all of the in-laws. Incomparability is a useful mechanism to resolve conflict when aggregating such preferences. If half of the agents prefer a to b and the other half prefer b to a, then it may be best to say that a and b are incomparable.

In addition, an agent's preferences are not necessarily total. For example, while it is easy and reasonable to compare two apartments, it may be difficult to compare an apartment and a house. We may wish simply to declare them incomparable. Moreover, an agent may have several possibly conflicting preference criteria she wants to follow, and their combination can naturally lead to a partial order. For example, one may want a cheap but big apartment, so an 80 square meters apartment which costs 100.000 euros is incomparable to a 50 square meters apartment which costs 60.000 euros.

We assume therefore that both the preferences of an agent and the result of preference aggregation can be a partial order. In this context, it is natural to ask if we can combine partially ordered preferences *fairly*.

For total orders, Arrow's theorem shows this is impossible [1]. We show that this result can be generalized to partial orders under certain conditions. This is both disappointing and a little surprising. By moving from total orders to partial orders, we enrich greatly our ability to combine outcomes fairly. As in the example above, we can use incomparability to resolve conflict and thereby not contradict agents. Nevertheless, under the conditions identified here, we still do not escape the reach of Arrow's theorem.

These results assume that one is interested in obtaining a partial order over the different scenarios as the outcome of preference aggregation. One may wonder if the situation is easier when we are only interested in the most preferred outcomes in the aggregated preferences. However, we show that even in this case (that is, when considering social choice functions over partial orders) it is impossible to be fair. This is a generalization of the Muller-Satterthwaite theorem [10] for total orders.

We also identify two cases where fairness of social welfare functions over partial orders is possible, one of which is a generalization of Sen's theorem [12]. The two cases correspond to two extremes of the amount of partiality of the partial orders. In fact, one considers partially ordered profiles which are very ordered (that is, very close to be total orders), while the other one concerns profiles with no chain of ordered pairs, so where the ordering relation contains a very small number of pairs.

The paper is organized as follows. Section 2 gives the basic definitions about partial orders. Then, Section 3 defines social welfare functions on partially ordered profiles and introduces the notions of fairness in this context, Section 4 proves that it is impossible to be fair over partial orders under certain conditions, Section 5 shows cases in which the majority rule can be fair, and Section 6 defines social choice functions over partial orders and their properties, and proves that they cannot be fair. Finally, Section 7 describes the existing related work, and Section 8 summarizes the results and gives hints for dirctions for future work.

All the proofs of the results of this paper are contained in the appendix.

2 Partial orders

A binary relation R on a set S (that is, $R \subseteq S \times S$) is: **reflexive** iff for all $x \in S$, $(x, x) \in R$; **transitive** iff for all $x, y, z \in S$, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$; **antisymmetric** iff for all $x, y \in S$, $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$; **complete** iff for all $x, y \in S$, either $(x, y) \in R$ or $(y, x) \in R$.

A **total order** is a binary relation which is reflexive, transitive, antisymmetric, and complete. A total order has an unique **optimal** element, that is an element $o \in S$ such that $\forall x \in S, (o, x) \notin S$. We say that this element is **undominated**.

A **partial order** is a binary relation which is reflexive, transitive and antisymmetric but may be not complete. There may be pairs of elements (x, y) of S which are not in the partial order relation, that is, such that neither $(x, y) \in R$ nor $(y, x) \in R$. Such elements are **incomparable** (written $x \bowtie y$). Unlike a total order, a partial order can have several optimal and mutually incomparable elements. We say that these elements are **undominated**. Undominated elements will also be called *top* elements. The set of all top elements of a partial order o will be called $top(o)$. Elements which are below or incomparable to every other element will be called **bottom** elements.

Given any relation R which is either a total or a partial order, if $(x, y) \in R$, it can be that $x = y$ or that $x \neq y$. If R is such that $(x, y) \in R$ implies $x \neq y$, then R is said to be **strict**. This means that reflexivity does not hold.

Both total and partial orders can be extended to deal with ties, that is, sets of elements which are equally positioned in the ordering. Two elements which belong to a tie will be said to be *indifferent*. To summarize, in a total order with ties, two elements can be either ordered or indifferent. On the other hand, in a partial order with ties, two elements can be either ordered, indifferent, or incomparable. Notice that, while incomparability is not transitive in general, indifference is transitive, reflexive, and symmetric.

In the following we will sometimes need to consider partial orders with some restrictions. In particular, we will call a **rPO** a partial order where the top elements are all indifferent, or the bottom elements are all indifferent. In both POs and rPOs, ties are allowed everywhere, except when explicitly said otherwise.

3 Social welfare functions for partial orders

We assume that each agent's preference specify a partial order over the possible outcomes. We aggregate the preferences of a number of agents using a social welfare function. A **social welfare function** f is a function from profiles p to orderings over outcomes. A **profile** p is a sequence of n orderings $p_1, \ldots, p_n$ over outcomes, one for each agent $i \in \{1, \ldots, n\}$.

There are a number of properties that a social welfare function might be expected to possess. Except in the case of dictator, they are straightforward generalizations of the corresponding properties for social welfare functions for total orders [2]:

- **Freeness**: f is surjective, that is, it can produce any ordering.
- **Unanimity**: if all agents agree that a is preferable to b, then the resulting order must agree as well. That is, if $a >_{p_i} b$ for all agents i, then $a >_{f(p)} b$. Notice that a stronger notion could be defined, where unanimity over incomparability is also required. However, this is not needed for the results of our paper.
- **Independence to irrelevant alternatives**: the ordering between a and b in the result depends only on the relation between a and b given by the agents; that is, for all profiles p and p', for all a, b, for all agents i, if $p_i(a,b) = p'_i(a,b)$, then $f(p)(a,b) = f(p')(a,b)$, where, given an ordering o, $o(a,b)$ is the restriction of o on a and b.
- **Monotonicity**: We say that b improves with respect to a if the relationship between a and b does not move to the left along the following sequence: $>, \geq, (\bowtie \text{ or } =), \leq, <$. Given two profiles p and p', if passing from p to p' b improves with respect to a in one agent i and $p_j = p'_j$ for all $j \neq i$, then in passing from $f(p)$ to $f(p')$ b improves with respect to a.

Another desirable property of social welfare functions is the absence of a dictator. With partial orders, there are several possible notions of dictator:

Strong dictator: an agent i such that, in every profile p, $f(p) = p_i$, that is, her ordering is the result;
Dictator: an agent i such that, in every profile p, if $a \geq_{p_i} b$ then $a \geq_{f(p)} b$.
Weak dictator: an agent i such that, in every profile p, if $a \geq_{p_i} b$, then $a \not< _{f(p)} b$.

Nothing is said about the result if a is incomparable or indifferent to b for the dictator or weak dictator. Clearly a strong dictator is a dictator, and a dictator is a weak dictator. Note also that whilst there can only be one strong dictator or dictator, there can be any number of weak dictators.

We say a social welfare function is *strongly fair*, *fair* or *weakly fair* if its is unanimous, independent to irrelevant alternatives, and does not have a strong dictator, dictator or weak dictator respectively. Arrow's impossibility theorem [1, 9] shows that, if a social welfare function on total orders with ties is unanimous and independent to irrelevant alternatives and there are at least two voters and three outcomes, then there must be at least one dictator. It is possible to prove that freeness, monotonicity and independence to irrelevant alternatives imply unanimity. On the other hand, there are social welfare functions which are free, unanimous and independent to irrelevant alternatives but not monotonic [12]. Therefore a weaker version of Arrow's result on total orders with ties states that freeness, monotonicity and independence of irrelevant assumptions implies that there must be at least one dictator [7].

Proposition 1. *A social welfare function on partial orders can be fair.*

For example, the Pareto rule in which the outcome is ordered if every agent agrees, but is incomparable otherwise, is fair. Note that a social welfare function that is fair is also strongly fair. Hence a social welfare function on partial orders can be strongly fair. Actually strong fairness is a very weak property to demand. Even voting rules which appears very "unfair" may not have a strong dictator. For example, suppose we ask the agents in some fixed order, and order two outcomes according to the first agent who is not indifferent. This social welfare function (which we will call the Lex rule) is not fair, but it is strongly fair.

4 Impossibility results

We now show that, under certain conditions, it is impossible for a social welfare function over partial orders to be weakly fair. The conditions involve the shape of the partial orders. In fact, we assume the partial orders of the agents to be general (PO), but the resulting partial order must have all top or all bottom elements indifferent (rPO).

Theorem 1. *Given a social welfare function f over partial orders, assume the result is a rPO, there are at least 2 agents and 3 outcomes, and f is unanimous and independent to irrelevant alternatives. Then there is at least one weak dictator.*

As with total orders, we can also prove a weaker result in which we replace unanimity by monotonicity and freeness.

Corollary 1. *Given a social welfare function f over partial orders, assume the result is a rPO, there are at least 2 agents and 3 outcomes, and f is free, monotonic, and independent to irrelevant alternatives. Then there is at least one weak dictator.*

Consider, for example, the Pareto rule. With this rule, every agent is a weak dictator since no agent can be contradicted. Note that we could consider a social welfare function which modifies Pareto by applying the rule only to a strict subset of the agents, and ignores the rest. The agents in the subset will then all be weak dictators.

A number of results follow immediately from these theorems. If we denote the class of all social welfare functions from profiles made with orders of type A to orders of type B by $A^n \mapsto B$, then we have proved the impossibility of being weakly fair for functions in $PO^n \mapsto rPO$.

If all functions in $A^n \mapsto B$ are not weakly fair, then also functions in $A^n \mapsto B'$, where B' is a subtype of B, are not weakly fair, since by restricting the co-domain of the functions we are just considering a subset of them. Therefore this impossibility theorem implies also that the functions in $PO^n \mapsto O$, where O is anything more ordered than a rPO, cannot be weakly fair. For example, we can deduce that functions in $PO^n \mapsto TO$ cannot be weakly fair.

The same reasoning applies when we restrict the domain of the functions, that is, we pass from $A^n \mapsto B$ to $A'^n \mapsto B$ where A' is a subtype of A. In fact, by doing this, we restrict the kind of profiles we consider, so whatever is true in all the profiles of the larger set is also true in a subset of the profiles. In particular, if a function from A^n to B has a weak dictator, then the same function, restricted over the profiles in A'^n, also has the same weak dictator. Thus if the functions in $A^n \mapsto B$ cannot be weakly fair, also the functions in $A'^n \mapsto B$ cannot be weakly fair. In particular, from our result we can deduce that all functions in $TO^n \mapsto rPO$ cannot be weakly fair. Finally, we can deduce that all functions in $TO^n \mapsto TO$ cannot be weakly fair, which is exactly Arrow's theorem. In fact, we have a lattice of impossibility results for classes of social welfare functions, as described by Figure 1.

$$
\begin{array}{ccccc}
 & & PO^n \mapsto rPO & & \\
 & \swarrow & & \searrow & \\
PO^n \mapsto TO & & & & TO^n \mapsto rPO \\
 & \searrow & & \swarrow & \\
 & & TO^n \mapsto TO & &
\end{array}
$$

Fig. 1. Lattice of impossibility results. rPO stands for partial order where top elements or bottom elements are all indifferent, PO stands for partial order, TO stands for total order. Arrow's theorem applies to $TO^n \mapsto TO$. $\swarrow$ and $\searrow$ stand for the lattice ordering.

5 Possibility results

We now consider ways of assuring that a social welfare function is weakly fair. In fact, we will identify situations when the well known *majority rule* is transitive, which makes it weakly fair since it usually has all the other properties.

The majority rule we will consider says a is better than b iff the number of agents which say that $a > b$ is greater than the number of agents which say that $b > a$ plus the number of those that say that a and b are incomparable. Notice that ties are ignored by this rule.

We focus on the condition that Sen has proved sufficient for fairness in the case of total orders, namely **triplewise value-restriction** [12]. That is, for every triple of outcomes x_1, x_2, x_3, there exists $x_i \in \{x_1, x_2, x_3\}$ and $r \in \{1, 2, 3\}$ such that no agent ranks x_i as his r-th preference among x_1, x_2, x_3. To apply Sen's theorem to this context, we will consider linearizations of our profiles.

Note that, as we have partial orders, to assure transitivity in the resulting order, we must avoid both cycles (as in the total order case) and incomparability in the wrong places. More precisely, if the result has $a > b > c$, we cannot have $c > a$, which would create a cycle, and not even $a \bowtie c$, since in both cases transitivity would not hold. We will say that a profile p satisfies the **generalized triplewise value-restriction** if all the profiles obtained from p by linearizing any PO to a TO have the triplewise value-restriction property.

Theorem 2. *If all profiles satisfy the generalized triplewise value-restriction and are without ties, then the majority rule is transitive and thus weakly fair.*

We have therefore generalized Sen's theorem to partial orders without ties. This result is useful when the profiles are highly ordered, and, within each profile, the agents have similar orders. On the other extreme, we will now give another possibility result which can be applied to profiles which order few outcomes. This result assures transitivity of the resulting ordering by a more rough approach: it just avoids the presence of chains in the result. That is, for any triple x_1, x_2, x_3 of outcomes, it makes sure that the result cannot contain $x_i > x_j > x_k$ where i, j, k is any permutation of $\{1, 2, 3\}$. This is done by restricting the classes of orderings allowed for the agents.

A profile is **non-chaining** iff, for any triple of outcomes, only one of the following situations can happen:

- the outcomes are all incomparable,
- only two of them are ordered, or
- there is one of them, which is more preferred than the other two.

Or:

- the outcomes are all incomparable;
- only two of them are ordered, or

– there is one of them, which is less preferred than the other two.

Theorem 3. *If all profiles are non-chaining and without ties, then the majority rule is weakly fair.*

6 Social choice functions

In some situations, the result of aggregating the preferences of a number of agents might not need to be an order over outcomes. It might be enough to know the "most preferred" outcomes. For example, when aggregating the preferences of two people who want to buy an apartment, we don't need to know whether they prefer an 80 square meter apartment at the ground floor or a 50 square meter apartment at the 2nd floor, if they both prefer a 100 square meter apartment at the 3rd floor. They would just buy the 3rd floor apartment without trying to order the other two apartments. Social choice functions identify such most preferred outcomes, and do not care about the ordering on the other outcomes.

A social choice function on total orders is a mapping from a profile to the optimal outcome, or winner. With partial orders, there can be several outcomes which are incomparable and optimal. We can therefore consider a generalization in which a social choice function is a mapping from a profile to a non-empty set of outcomes, called the optimal outcomes, or the winners.

We need to modify slightly the usual notions to deal with this generalization. We say that a social choice function f is

- **unanimous** iff
  - given any profile p where outcome $a \in top(p_i)$ for every agent i, then $a \in f(p)$;
  - given any profile p where $\{a\} = top(p_i)$ for every agent i, then $f(p) = \{a\}$;
- **monotonic** iff, given two profiles p and p',
  - if $a \in f(p)$ and for any other alternative b, $a >_{p_i} b$ implies $a >_{p'_i} b$ and $a \bowtie_{p_i} b$ implies $a \bowtie_{p_i} b$ or $a >_{p'_i} b$, for all agents i, then we have $a \in f(p')$;
  - if $f(p) = S$ and for all $s \in S$, for all b, $s >_{p_i} b$ implies $s >_{p'_i} b$ and $a \bowtie_{p_i} b$ implies $a \bowtie_{p_i} b$ or $a >_{p'_i} b$, for all agents i, then $f(p') = S$.

As for social welfare functions, we will define three notions of dictators:

- a **strong dictator** is an agent i such that, for all profiles p, $f(p) = top(p_i)$;
- a **dictator** is an agent i such that, for all profiles p, $f(p) \subseteq top(p_i)$;
- a **weak dictator** is an agent i such that, for all profiles p, $f(p) \cap top(p_i) \neq \emptyset$.

Notice that, in any profile p, if a is the unique top of a weak dictator i, then $a \in f(p)$. However, this is not true if a is not the unique top of i.

Notice also that these three notions are consistent with the corresponding ones for social welfare functions. More precisely, a dictator (resp., weak, strong) for a social welfare function f is also a dictator (resp. weak, strong) for the social choice function f' obtained by f by $f'(p) = top(f(p))$ for every profile p.

Proposition 2. *A social choice function on partial orders can be at the same time unanimous, monotonic, and have no dictators.*

For example, the social choice function corresponding to the Pareto rule is unanimous, monotonic, and has no dictators. However, all the agents are weak dictators. Another example is the social choice function which returns the $\bigcup_i top(p_i)$, which again is unanimous, monotonic, and has no dictators (but all agents are weak dictators). On the other hand, the Lex rule has a strong dictator (which is the first agent).

The Muller-Satterthwaite theorem [10] can be generalized to social choice functions over partial orders without ties, for weak dictators.

Theorem 4. *If we have at least two agents and at least three outcomes, and the social choice function on partial order without ties is unanimous and monotonic, then there is at least one weak dictator.*

A further extension would be the generalization of the Gibbard-Satterthwaite theorem [8]. That is, are weak dictators inevitable if we have at least two agents and three outcomes, and the social choices function is strategy proof and onto? This is a the subject of our current work.

7 Related work

Since the original theorem by Arrow, some effort has been made to weaken its conditions. Both the domain and the codomain of a social welfare function have been the subject of more relaxed assumptions in several Arrow-like impossibility theorems:

- In [6], the codomain is a partial order, and profiles are allowed to be strict weak orders, which are negatively transitive and assymetric. This structure is more general than total orders but less general than partial orders, since, for example, it does not allow situations where $A > B$ and C is incomparable to both A and B.
- In [3], social orders can be partial, and agents are allowed to vote using a partial order. However, the set of profiles must be regular, meaning that for any three alternatives, every configuration of their orders must be present in a profile.
- In [13] agents must vote using total orders. However, the social order can be a quasi-ordering, which is reflexive and transitive. A similar setting is considered in [5], where agents use total orders with some additional requirements (such as the discrimination axiom which requires that each agent orders strictly at least one triple of candidates).
- In [4], each agent models her preferences using a non monotonic logic, giving a preorder (reflexive and transitive) on the outcomes. An additional hypothesis is required, called conflict resolution, which states that if a pair is ordered by any agent, then it must be ordered also in the social order. Conflict resolution is a very strong property to require. For example, the Pareto rule does not respect it.

With respect to all these approaches, our profiles are more general, since in our results a profile can be any set of partial orders. However, the resulting order of a social welfare function is required to be a restricted partial order, that is, a partial order with a unique top or a unique bottom. Thus our result is incomparable to all the Arrow's like theorems in the cited papers.

However, our possibility theorem for the majority rule is, to our knowledge, the first result of this kind for partially ordered profiles in social welfare functions. The same holds also for our impossibility result for social choice functions.

8 Conclusions and future work

We have proved that if there are at least two agents and three outcomes, social welfare functions on partial orders cannot be weakly fair (that is, they cannot be at the same time unanimous, independent to irrelevant alternatives, and with no weak dictator) if the result is a partial order with all top or all bottom elements indifferent. This result generalizes Arrow's impossibility theorem for combining total orders [1]. On the other hand, we also proved that some social welfare functions, such as the majority rule, can be weakly fair if we put some additional restrictions on the shape of the orderings of the agents. In particular, this happens when all the linearizations of the partial orders of the agents have the triplewise value-restriction, and also when they do not contain any chain of ordered pairs. We also proved that social choice functions over partial orders cannot be at the same time unanimous, monotonic, and without weak dictators if there are at least two agents and three outcomes, and ties are not allowed.

An interesting open question is whether voting systems on partial orders can have other desirable properties. For example, can they encourage non-tactical voting? Are they non-manipulable? We are currently studying whether the Gibbard-Satterwhite theorem can be extended to social choice functions over partially ordered profiles.

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Appendix: Proofs of main results

Theorem 1 *Given a social welfare function f over partial orders, assume the result is a rPO, there are at least 2 agents and 3 outcomes, and f is unanimous and independent to irrelevant alternatives. Then there is at least one weak dictator.*

Proof. The proof is similar in outline to one of the proof of Arrow's theorem [7]. However, we must adapt each step to this more general context of partial orders. We assume the resulting ordering is a rPO with all bottom elements indifferent. The proof can then be repeated very similarly for the other case in which the resulting ordering is a rPO with all top elements indifferent.

1. First we prove that, if a element b is at the very top or at the very bottom in all POs of the agents, then it must be a top or bottom element in the resulting rPO. If b is not a top or bottom element in the result, there must be other elements a and c such that $a > b > c$. We will now adjust the profile so that c is above a for all agents. Since we assumed just one top and one bottom for all agents, we can always do that while keeping b at the extreme position and not changing the ordering relation between a and b and between c and b.
   By unanimity, we must have c above a in the result. By independence to irrelevant alternatives, we still have $a > b$ and $b > c$. By transitivity, we have $a > c$ which is a contradiction.
2. We now prove that there is a pivotal agent n^* such that, when he moves b from bottom to top, the same change happens in the result.
   Assume all agents have b as the bottom. Then, b must be at the bottom in the result by unanimity. Let n^* be the first agent such that, when b moves from bottom to top, this happens in the result. Note that n^* must exist, because when all agents move b from bottom to top, by unanimity in the result we must have b at the top.
3. We continue by proving that n^* is a weak dictator for pairs of elements not involving b.
   Let us consider the following scenarios in the context of moving b from the bottom to the top of each agent's ranking.
   Π_1: b is still the bottom of n^*. In the result, b is the bottom element so we must have, $a > b$ for all a.
   Π_2: b has been moved to the top of n^*. In the result, b is a top element so we must have, $b \geq c$ or b incomparable to c for all c.
   Π_3: As in Π_2 but a has now been moved above b in n^* (and thus also above c), and all other agents move freely a and c leaving b in the top or bottom position.

   By independence to irrelevant alternatives, $a > b$ must be the result of Π_3, since all the ab preferences are the same as in Π_1. Also, $b \geq c$ or b incomparable to c must be the result of Π_3, since all $b - c$ preferences are the same as in Π_2. By transitivity, the result of Π_3 cannot have $c > a$ since it would imply $c > b$ which is contradictory with the assumption that b and c are either incomparable or $b \geq c$. Thus n^* is a weak dictator for pairs not involving b.
4. We now prove that there exists an agent n' which is a weak dictator for pairs with no c. To do this, it is enough to use the same construction as above but with c in place of b.
5. We show now that $n^* = n'$. On total orders, there can be only one dictator, so it follows immediately that $n^* = n'$. With partial orders, there can be more than one weak dictator. The argument that $n^* = n'$ is therefore much more elaborate.
   Without loss of generality, assume $n^* \leq n'$. Suppose that $n^* < n'$. Let us consider the following profiles: we start with all agents having b at the bottom and c at the top. Then we swap b and c in each orderings, going through the agents in the same order as in the previous constructions. When we move b up for n^*, b goes to the top in the result (by the previous part of the proof). c goes down for n^*, and in the result it can go down as well, in which case we can repeat the same construction as in points 1,2,3 starting with c at the top instead of b at the bottom, and we can prove that n^* is also a weak dictator for pairs not involving c. Thus we would have $n^* = n'$, which is a contradiction.
   Otherwise, c could stay at the top. However, since n^* is a weak dictator for pairs not involving b, and since c is the bottom for n^*, then all the elements must be at the top with c (as incomparable

or indifferent elements). This is true also in any other profile obtained from the current one by leaving all the other agents move freely a and b. Thus n^* is not contradicted on any pair. □

Corollary 1 *Given a social welfare function f over partial orders, assume the result is a rPO, there are at least 2 agents and 3 outcomes, and f is free, monotonic, and independent to irrelevant alternatives. Then there is at least one weak dictator.*

Proof. Suppose the social welfare function is free and monotonic, and that $a \geq b$ for all agents. If a is moved to the top of the ordering for all agents then, by independence to irrelevant alternatives, this leaves the result between a and b unchanged. Suppose in the result $a < b$ or a is incomparable to b. By monotonicity, any change to the votes of a over b will not help ensure $a \geq b$. Hence, the election cannot be free. Thus it must be $a \geq b$ in the result. The voting system is therefore unanimous. By Theorem 1, there must be at least one weak dictator. □

Theorem 2 *If all profiles satisfy the generalized triplewise value-restriction and are without ties, then the majority rule is transitive and thus weakly fair.*

Proof. Take any profile p' which is a linearization of p. Then p' has the triplewise value-restriction property, and the majority rule applied to p' produces an ordering without cycles, by Sen's theorem. Since p' is a linearization of p, p has a smaller or equal set of ordered pairs w.r.t. p'. Therefore, if the majority rule has not produced any cycle starting from p', it cannot produce any cycle if it starts from p. In fact, the majority rule counts the number of agents who order a pair, so if the profile has less ordered pairs, a smaller or equal number of pairs are ordered also in the result.

Assume now we have $a > b > c$ and a incomparable to c in the result. We will now show that, if this is the case, then there is a linearization which doesn't satisfy the triplewise value-restriction property.

In fact, since $a > b$ in the result we know that $|S_{a>b}| > |S_{a<b}| + |S_{a \bowtie b}|$. Similarly $|S_{b>c}| > |S_{b<c}| + |S_{b \bowtie c}|$. Since we are assuming that $a \bowtie c$ then $|S_{a>c}| \leq |S_{c>a}| + |S_{c \bowtie a}|$. We know that a majority of agents says that $a > b$. Let us now assume that no agent says that a is incomparable to b and that no agent says that b is incomparable to c. Let us consider the agents that say $a > b$, then for each such agent she must have one of the following orderings:(1) $a > b > c$; (2) $a > b \wedge c > b$; (3) $a > c > b$; (4) $c > a > b$. We want to prove that that there is at least one agent that says (1) and that there is at least one agent that says either (2) or (4). In fact it is not possible that all the agents which have said $a > b$ have all ordering (1), since that would mean that there is also a majority that says $a > c$, while $a \bowtie c$ by hypothesis. Moreover, it is not possible that all the agents that say $a > b$ vote as (2) or (3) or (4) but none using (1) since this would imply that $c > b$. Thus we can conclude that at least one agent must vote as in (1) and at least one agent votes either (2) or (4).

Now let us consider the agents in the majority that vote $b > c$. Each of them can have one of the following orderings:(i) $a > b > c$; (ii) $b > c \wedge b > a$; (iii)$b > c > a$; (iv) $b > a > c$. As before they cannot all vote (i), otherwise using the majority rule we would have $a > c$, and not $a \bowtie c$ as assumed. However at least one must vote (i), since (ii), (iii) and (iv) order b above a, while we know there is a majority saying $a > b$. Moreover, it is also not possible that all voters that say $b > c$ say (i) or say (iv) since again that would imply $a > c$. Thus there is at least an agent such that either she says (ii) or he says (iii).

To summarize, we have an agent that says (1) (or (i)) $a > b > c$, then we have an agent that says (3) or (4). Notice that if she says (3) we can linearize (3) into (4) by adding $c > a$. Thus for the second agent we have $c > a > b$. Finally we have an agent that says either (ii) or (iii). Again

(ii) can be linearized into (iii) by adding $c > a$. Thus for the third agent $b > c > a$. This is a linearization which violates the triplewise value-restriction property.

Notice that if we allow the agents to express incomparability between a and b and/or b and c this means that agents that voted (3) or (4) now could vote (2) and that agents that voted (iii) or (iv) now could vote (ii) or give even more incomparability. However this does not prevent the possibility to linearize the orderings as above. □

Theorem 3 *If all profiles are non-chaining and without ties, then the majority rule is weakly fair.*

Proof. We just need to show that the majority rule is transitive since it has all other properties of fairness. Let us first consider situation α. We will prove that for every triple a,b, and c it cannot be $a > b > c$, that there are no transitive chains in the ordering. Let us assume that $a > b > c$. Since $a > b$ then:

- (1) $|S_{a>b}| > |S_{a>b}| + |S_{a \bowtie b}|$

  Similarly from $b > c$ we have that

- (2) $|S_{b>c}| > |S_{c>b}| + |S_{b \bowtie c}|$

  From the fact that we are situation α we have that:

- (3) $|S_{a>b}| \leq |S_{b \bowtie c}|$
- (4) $|S_{b>c}| \leq |S_{b>a}| + |S_{b \bowtie a}|$

The reason for inequality (3) is that all the voters that say $a > b$ cannot order b and c. The reason for inequality (4) is that all the voters that put b above c must either put a below b or incomparable to b.

From the above inequalities we get the following inconsistency:

$$|S_{a>b}| \overset{(3)}{\leq} |S_{b \bowtie c}| \overset{(2)}{<} |S_{b>c}| \overset{(4)}{\leq} |S_{b>a}| + |S_{b \bowtie a}| \overset{(1)}{<} |S_{a>b}|$$

In situation β, (1) and (2) still hold while we have, from the fact that we are situation β:

- (3) $|S_{b>c}| \leq |S_{a \bowtie b}|$
- (4) $|S_{a>b}| \leq |S_{c>b}| + |S_{c \bowtie b}|$

The reason for inequality (3) is that all the voters that say $b > c$ cannot order a and b. The reason for inequality (4) is that all the voters that put a above b must either put b below c or incomparable to c.

From the above inequalities we get the following inconsistency:

$$|S_{b>c}| \overset{(3)}{\leq} |S_{a \bowtie b}| \overset{(1)}{<} |S_{a>b}| \overset{(4)}{\leq} |S_{c>b}| + |S_{c \bowtie b}| \overset{(2)}{\leq} |S_{c>b}|$$

□

Theorem 4 *If we have at least two agents and at least three outcomes, and the social choice function on partial order without ties is unanimous and monotonic, then there is at least one weak dictator.*

b	...	b	$a \bowtie h$	$a \bowtie h$	...	$a \bowtie h$
$a \bowtie h$	...	$a \bowtie h$	b	.	...	.
.		.	.	.		.
.		.	.	.		.
.		.	.	.		.
.	...	.	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

b	...	b	b	$a \bowtie h$	...	$a \bowtie h$
$a \bowtie h$	...	$a \bowtie h$	$a \bowtie h$	.	...	.
.		.	.	.		.
.		.	.	.		.
.		.	.	.		.
.	...	.	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

Fig. 2. Profiles P_1 and P_2.

b	...	b	$a \bowtie h$	.	...	.
.	...	.	b	.	...	.
.		.	.	.		.
.		.	.	.		.
.		.	.	$a \bowtie h$		$a \bowtie h$
$a \bowtie h$	...	$a \bowtie h$	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

b	...	b	b	.	...	.
.	...	.	$a \bowtie h$	.	...	.
.		.	.	.		.
.		.	.	.		.
.		.	.	$a \bowtie h$		$a \bowtie h$
$a \bowtie h$	...	$a \bowtie h$	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

Fig. 3. Profiles P'_1 and P'_2.

Proof. The proof follows the scheme of the proof of the Muller-Satterthwaite theorem that can be found in [11].

Consider three alternatives a, b, and h, and a profile P where $a \bowtie h$ (where $\bowtie$ means incomparability) are at the top above every other element, and b is the unique bottom, for all agents. By unanimity, $f(P)$ contains both a and h, and so $f(P)$ can be $\{a, h\}$ or $\{a, h, d\}$ where d is an alternative different from a, b, h, if any.

Let us now rise b one position at time in agent 1's ranking. By monotonicity, the set of winners still contains both a and h, as long as $b < a$ and $b < h$. When b is risen above a and h, by monotonicity the set of winners may contain b. If we continue this with the other agents in the order, at the end we must have b as the only winner by unanimity. Thus at some point b must appear in the set of winners.

Step 1. Consider profiles P_1 and P_2. P_1 is the last profile where the set of winners is still $\{a, h\}$ or $\{a, h, d\}$, where d is one or more other elements, whereas P_2 is the first profile such that the set of winners contain b.

If $d \in f(P_1)$, then by monotonicity on profiles P_1 and P_2, we have $d \in f(P_2)$. If instead $d \notin f(P_1)$, then $d \notin f(P_2)$. In fact, assume $d \in f(P_2)$; then, by monotonicity on P_2 and P_1, $d \in f(P_1)$ as well, which is a contradiction. Therefore,

- if $f(P_1) = \{a, h\}$ then $f(P_2)$ can be $\{b\}$, $\{a, b\}$, $\{h, b\}$ or $\{a, h, b\}$;
- if $f(P_1) = \{a, h, d\}$ then $f(P_2)$ can be $\{b, d\}$, $\{a, b, d\}$, $\{h, b, d\}$ or $\{a, h, b, d\}$.

Step 2. Consider the new profiles P'_1 and P'_2 in Figure 3.

Notice that $f(P'_2)$ must contain b, by monotonicity on P_2 and P'_2.

If $d \in f(P_2)$, then $d \in f(P'_2)$ by monotonicity on P_2 and P'_2. If $h \notin f(P_2)$, then $h \notin f(P'_2)$. In fact, assume $h \in f(P'_2)$; then monotonicity on P'_2 and P_2 implies $h \in f(P_2)$, that is a contradiction. Analogously, if $a \notin f(P_2)$, then $a \notin f(P'_2)$.

If $f(P_1) = \{a, h, d\}$, we know from Step 1 that $f(P_2)$ contains d. Then, for the reasoning above, $d \in f(P'_2)$. Hence $f(P'_2)$ can be $\{b, d\}$, $\{a, b, d\}$, $\{h, b, d\}$ or $\{a, h, b, d\}$. Whereas, if $f(P_1) = \{a, h\}$, then we know only that $f(P_2')$ must contain b, therefore $f(P_2')$ can be $\{b\}$, $\{b,d\}$, $\{a,b\}$, $\{a,b,d\}$, $\{h,b\}$, $\{h,b,d\}$, $\{a,h,b\}$ or $\{a,h,b,d\}$.

In particular, if $f(P_2)$ is $\{b,d\}$ or $\{b\}$, then by monotonicity on P_2 and P_2', $f(P_2')$ is resp. $\{b,d\}$ or $\{b\}$, and if $f(P_2) \neq \{b\}$, then $f(P_2') \neq \{b\}$. In fact, suppose $f(P_2') = \{b\}$. Then by monotonicity on P_2' and P_2, $f(P_2) = \{b\}$, that is a contradiction. Moreover, for the reasoning above, if $f(P_2)$ is $\{a,b,d\}$ or $\{a,b\}$, i.e., $h \notin f(P_2)$, then $h \notin f(P_2')$, and analogously, if $f(P_2)$ is $\{h,b,d\}$ or $\{h,b\}$, i.e., $a \notin f(P_2)$, then $a \notin f(P_2')$.

Summarizing,

- if $f(P_1) = \{a,h,d\}$,
  - if $f(P_2) = \{b,d\}$, then $f(P_2') = \{b,d\}$;
  - if $f(P_2) = \{a,b,d\}$, then $f(P_2') = \{b,d\}$ or $\{a,b,d\}$;
  - if $f(P_2) = \{h,b,d\}$, then $f(P_2') = \{b,d\}$ or $\{h,b,d\}$;
  - if $f(P_2) = \{a,h,b,d\}$, then $f(P_2')$ can be $\{b,d\}$, $\{a,b,d\}$, $\{h,b,d\}$ or $\{a,h,b,d\}$;
- if $f(P_1) = \{a,h\}$,
  - if $f(P_2) = \{b\}$, then $f(P_2') = \{b\}$;
  - if $f(P_2) = \{a,b\}$, then $f(P_2') = \{b,d\}$, $\{a,b\}$, $\{a,b,d\}$;
  - if $f(P_2) = \{h,b\}$, then $f(P_2') = \{b,d\}$, $\{h,b\}$, $\{h,b,d\}$;
  - if $f(P_2) = \{a,b,h\}$, then $f(P_2') = \{b,d\}$, $\{a,b\}$, $\{a,b,d\}$, $\{h,b\}$, $\{h,b,d\}$, $\{a,h,b\}$ or $\{a,h,b,d\}$.

Hence, $f(P_2')$ can be $\{b\}$, $\{b,d\}$, $\{a,b\}$, $\{a,b,d\}$, $\{h,b\}$, $\{h,b,d\}$, $\{a,h,b\}$ or $\{a,h,b,d\}$.

Notice that $f(P_1')$ doesn't contain b. In fact, if we suppose $b \in f(P_1')$, then by monotonicity on P_1' and P_1, also $f(P_1)$ should contain b. But this is a contradiction, since $f(P_1)$ doesn't contain b.

Moreover, $f(P_1') \neq \{d\}$. In fact, if $f(P_1') = \{d\}$, then by strict monotonicity on P_1' and P_2', $f(P_2') = \{d\}$, that is not one of the possible cases for $f(P_2')$. Hence, $f(P_1')$ can be $\{a\}$, $\{h\}$, $\{a,h\}$, $\{a,d\}$, $\{h,d\}$, $\{a,h,d\}$.

If $d \in f(P_2')$, then $d \in f(P_1')$ for monotononity on profiles P_2' and P_1'. If $d \notin f(P_2')$, then $d \notin f(P_1')$. In fact, if we suppose $d \in f(P_1')$, then monotonicity on P_1' and P_2', implies $d \in f(P_2')$, that is a contradiction.

Moreover, if $a \in f(P_2')$, then, for monotonicity on profile P_2' and P_1', $a \in f(P_1')$ and, for the same reason, if $h \in f(P_2')$, then for monotonicity on profile P_2' and P_1', $h \in f(P_1')$.

If $f(P_2') = \{b,d\}$, $\{a,b,d\}$, $\{h,b,d\}$ or $\{a,h,b,d\}$, then for the reasoning above, $f(P_1')$ must contain d and so it can be $\{a,d\}$, $\{h,d\}$, $\{a,h,d\}$, whereas if $f(P_2') = \{b\}$, $\{a,b\}$, $\{h,b\}$, or $\{a,h,b\}$, then $f(P_1')$ cannot contain d and so it can be $\{a\}$, $\{h\}$ or $\{a,h\}$.

More precisely,

- if $f(P_2') = \{b\}$, then $f(P_1')$ can be $\{a\}$, $\{h\}$ or $\{a,h\}$;
- if $f(P_2') = \{a,b\}$, then $f(P_1')$ can be $\{a\}$ or $\{a,h\}$;
- if $f(P_2') = \{h,b\}$, then $f(P_1')$ can be $\{h\}$ or $\{a,h\}$;
- if $f(P_2') = \{a,h,b\}$, then $f(P_1') = \{a,h\}$;
- if $f(P_2') = \{b,d\}$, then $f(P_1')$ can be $\{a,d\}$, $\{h,d\}$ or $\{a,h,d\}$;
- if $f(P_2') = \{a,b,d\}$, then $f(P_1')$ can be $\{a,d\}$ or $\{a,h,d\}$;
- if $f(P_2') = \{h,b,d\}$, then $f(P_1')$ can be $\{h,d\}$ or $\{a,h,d\}$;
- if $f(P_2') = \{a,h,b,d\}$, then $f(P_1') = \{a,h,d\}$;

Step 3. Consider an alternative e, distinct from a, h, and b, and the arbitrary profile P_3 in Figure 4, obtained from the profile P_1' without changing the ranking of a and h versus any other alternative in all agents' rankings, bringing b just above a and h (which are at the bottom) for agents $j < i$, and inserting the alternative e just above b for $j \leq i$ and just above a and h for $j > i$.

.	...	.	$a \bowtie h$	.	...	.
.	...	.	e	.	...	.
.		.	b	.		.
e		e	.	e		e
b		b	.	$a \bowtie h$		$a \bowtie h$
$a \bowtie h$	...	$a \bowtie h$	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

.	...	.	$a \bowtie h$	.	...	.
.	...	.	e	.	...	.
.		.	b	.		.
e		e	.	e		e
b		b	.	b		b
$a \bowtie h$	...	$a \bowtie h$	.	$a \bowtie h$	...	$a \bowtie h$
1	...	$i-1$	i	$i+1$	...	n

Fig. 4. Profiles P_3 and P_4.

Notice that $f(P_3)$ must not contain b, in fact if $b \in f(P_3)$ then, by monotonicity on P_3 and P_1, $b \in f(P_1)$, that is a contradiction. Hence $f(P_3)$ can be $\{a\}$, $\{h\}$, $\{d\}$, $\{a,h\}$, $\{a,d\}$, $\{h,d\}$, $\{a,h,d\}$.

By monotonicity on profiles P'_1 and P_3, if $a \in f(P'_1)$ then $a \in f(P_3)$ and if $h \in f(P'_1)$ then $h \in f(P_3)$. Moreover if $a \notin f(P'_1)$ then $a \notin f(P_3)$, in fact if we assume $a \in f(P_3)$, then monotonicity on profiles P_3 and P'_1 produces $a \in f(P'_1)$, that is a contradiction. Analogously, if $h \notin f(P'_1)$ then $h \notin f(P_3)$.

By monotonicity on profiles P'_1 and P_3, if $f(P'_1) = \{a\}$ then $f(P_3) = \{a\}$, if $f(P'_1) = \{h\}$ then $f(P_3) = \{h\}$ and if $f(P'_1) = \{a,h\}$ then $f(P_3) = \{a,h\}$. In particular, if $f(P'_1) \neq \{a\}$ then $f(P_3) \neq \{a\}$. In fact if $f(P_3) = \{a\}$, then by monotonicity on P_3 and P'_1, $f(P'_1)$ must be $\{a\}$, that is a contradiction. Analogously, if $f(P'_1) \neq \{h\}$ then $f(P_3) \neq \{h\}$ if $f(P'_1) \neq \{a,h\}$ then $f(P_3) \neq \{a,h\}$.

By Step 2 we know that $f(P'_1)$ can be $\{a\}$, $\{h\}$, $\{a,h\}$, $\{a,d\}$, $\{h,d\}$ or $\{a,h,d\}$, therefore, applying the reasoning above, we have that $f(P'_1) = f(P_3)$.

Step 4. Consider profile P_4 derived from profile P_3 by swapping the ranking of alternatives a and b for agents $j > i$, and profile $P_{4'}$ obtained from P_4 by bringing alternative e at the unique top of every agent's ranking. By unanimity, $f(P_{4'}) = \{e\}$. Note that $f(P_4)$ does not contain b. In fact, if $b \in f(P_4)$, by monotonicity on profiles P_4 and P'_4, $b \in f(P'_4)$, that is a contradiction since $f(P'_4) = \{e\}$. Also, if $d \notin f(P_3)$, then $d \notin f(P_4)$ and if $d \in f(P_3)$, then $d \in f(P_4)$. Moreover, if $h \notin f(P_3)$, then $h \notin f(P_4)$ and analogously if $a \notin f(P_3)$, then $a \notin f(P_4)$. In fact, if $a \in f(P_4)$, by monotonicity on P_4 and P_3, then $a \in f(P_3)$, that is a contradiction. Notice that if $f(P_3) \neq \{a\}$, then $f(P_4) \neq \{a\}$. In fact, if we assume $f(P_4) = \{a\}$, then by monotonicity on profiles P_4 and P_3, $f(P_3)$ must be $\{a\}$, that is a contradiction. Analogously, if $f(P_3) \neq \{h\}$ then $f(P_4) \neq \{h\}$, if $f(P_3) \neq \{a,h\}$ then $f(P_4) \neq \{a,h\}$, if $f(P_3) \neq \{a,d\}$ then $f(P_4) \neq \{a,d\}$ and if $f(P_3) \neq \{a,h,d\}$ then $f(P_4) \neq \{a,h,d\}$.

By Step 3, we know that $f(P_3)$ can be $\{a\}$, $\{h\}$, $\{a,h\}$, $\{a,d\}$, $\{h,d\}$ or $\{a,h,d\}$. Therefore, by reasoning above, $f(P_3) = f(P_4)$.

Step 5. Consider an arbitrary profile P_5, with a and h the only top elements of agent i's ranking. It can be obtained from profile P_4 without reducing the ranking of a and h versus any other alternative in any agent's ranking. Remember that, by step 4, $f(P_4)$ can be $\{a\}$, $\{h\}$, $\{a,h\}$, $\{a,d\}$, $\{h,d\}$ or $\{a,h,d\}$. By monotonicity on profiles P_4 and P_5, if $a \in f(P_4)$, then $a \in f(P_5)$, and if $h \in f(P_4)$, then $h \in f(P_5)$. Therefore, since in all possible cases $f(P_4)$ contains a or h (where *or* is not exclusive), the set of winner of an arbitrary profile, i.e. $f(P_5)$, must contain at least one (a or h) of the tops of the agent i. Thus agent i is a weak dictator.

It is easy to see that this proof can be easily generalized to the case of *more than two tops* for agent i. Moreover, the case of just one top for agent i can be proven via a simpler version of this proof. □