

# Symmetries of Symmetry Breaking Constraints

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## Abstract

Symmetry is an important feature of many constraint programs. We show that *any* symmetry acting on a set of symmetry breaking constraints can be used to break symmetry. Different symmetries pick out different solutions in each symmetry class. We use these observations in two methods for eliminating symmetry from a problem. These methods are designed to have many of the advantages of symmetry breaking methods that post static symmetry breaking constraint without some of the disadvantages. In particular, the two methods prune the search space using fast and efficient propagation of posted constraints, whilst reducing the conflict between symmetry breaking and branching heuristics. Experimental results show that the two methods perform well on some standard benchmarks.

## 1 Introduction

Symmetry occurs in many problems. For instance, certain workers in a staff rostering problem might have the same skills and availability. If we have a valid schedule, we may be able to permute these workers and still have a valid schedule. We typically need to factor such symmetry out of the search space to be able to find solutions efficiently. One popular way to deal with symmetry is to add constraints which eliminate symmetric solutions (see, for instance, [1; 2; 3; 4; 5; 6; 7]). A general method is to add constraints which limit search to the lexicographically least solution in each symmetry class. Such symmetry breaking is usually simple to implement and is often highly efficient and effective [8; 9]. Even for problems with many symmetries, a small number of symmetry breaking constraints can often eliminate much or all of the symmetry.

One problem with posting symmetry breaking constraints is that they pick out particular solutions in each symmetry class, and branching heuristics may conflict with this choice. In this paper, we consider two methods for posting symmetry breaking constraints that tackle this conflict. The two methods exploit the observations that *any* symmetry acting on a set of symmetry breaking constraints can be used to break symmetry, and that different symmetries pick out different

solutions. The first method is *model restarts* which was proposed in [10]. We periodically restart search with a new model which contains a different symmetry of the symmetry breaking constraints. The second method posts a symmetry of the symmetry breaking constraint dynamically during search. The symmetry is incrementally chosen to be consistent with the branching heuristic. Our experimental results show that both methods are effective at reducing the conflict between branching heuristics and symmetry breaking.

## 2 Background

A constraint satisfaction problem (CSP) consists of a set of variables, each with a domain of values, and a set of constraints specifying allowed combinations of values for subsets of variables. A solution is an assignment to the variables satisfying the constraints. We write  $sol(C)$  for the set of all solutions to the constraints  $C$ . A common method to find a solution of a CSP is backtracking search. Constraint solvers typically prune the backtracking search space by enforcing a local consistency property like domain consistency. A constraint is *domain consistent* iff for each variable, every value in its domain can be extended to an assignment satisfying the constraint. We make a constraint domain consistent by pruning values for variables which cannot be in any satisfying assignment. During the search for a solution, a constraint can become entailed. A constraint is *entailed* when any assignment of values from the respective domains satisfies the constraint. For instance,  $X_1 < X_n$  is entailed iff the largest value in the domain of  $X_1$  is smaller than the smallest value in the domain of  $X_n$ . A constraint is *dis-entailed* when its negation is entailed. For instance,  $X < Y$  is dis-entailed if and only if the smallest value in the domain of  $X$  is larger than or equal to the largest value in the domain of  $Y$ .

CSPs can contain symmetry. We consider two common types of symmetry (see [11] for more discussion). A *variable symmetry* is a permutation of the variables that preserves solutions. Formally, a variable symmetry is a bijection  $\sigma$  on the indices of variables such that if  $X_1 = d_1, \dots, X_n = d_n$  is a solution then  $X_{\sigma(1)} = d_1, \dots, X_{\sigma(n)} = d_n$  is also. A *value symmetry*, on the other hand, is a permutation of the values that preserves solutions. Formally, a value symmetry is a bijection  $\theta$  on the values such that if  $X_1 = d_1, \dots, X_n = d_n$  is a solution then  $X_1 = \theta(d_1), \dots, X_n = \theta(d_n)$  is also. Symmetries can more generally act on both variables and values.

Our methods also work with such symmetries. As the inverse of a symmetry and the identity mapping are symmetries, the set of symmetries of a problem forms a group under composition. We will use a simple running example which has a small number of variable and value symmetries. This example demonstrates that we can use symmetry itself to pick out different solutions in each symmetry class.

**Running Example.** *The all interval series problem (prob007 in CSPLib.org [12]) asks for a permutation of 0 to  $n - 1$  so that neighbouring differences form a permutation of 1 to  $n - 1$ . We model this as a CSP with  $X_i = j$  iff the  $i$ th number is  $j$ , and auxiliary variables for the neighbouring differences. One solution for  $n = 11$  is:*

$$X_1, X_2, \dots, X_{11} = 3, 7, 4, 6, 5, 0, 10, 1, 9, 2, 8 \quad (a)$$

*This model has a number of different symmetries. First, there is a variable symmetry  $\sigma_{rev}$  that reverses any solution:*

$$X_1, X_2, \dots, X_{11} = 8, 2, 9, 1, 10, 0, 5, 6, 4, 7, 3 \quad (b)$$

*Second, there is a value symmetry  $\theta_{inv}$  that inverts values. If we subtract all values in (a) from 10, we generate a second (but symmetric) solution:*

$$X_1, X_2, \dots, X_{11} = 7, 3, 6, 4, 5, 10, 0, 9, 1, 8, 2 \quad (c)$$

*Third, we can do both. By reversing and inverting (a), we generate a fourth (but symmetric) solution:*

$$X_1, X_2, \dots, X_{11} = 2, 8, 1, 9, 0, 10, 5, 4, 6, 3, 7 \quad (d)$$

*The model thus has four symmetries in total:  $\sigma_{id}$  (the identity mapping),  $\sigma_{rev}$ ,  $\theta_{inv}$ , and  $\theta_{inv} \circ \sigma_{rev}$ . ♣*

### 3 Symmetry breaking

One common way to deal with symmetry is to add constraints to eliminate symmetric solutions [1]. Two important properties of symmetry breaking constraints are soundness and completeness. A set of symmetry breaking constraint is sound iff it leaves at least one solution in each symmetry class, and complete iff it leaves exactly one solution.

**Running Example.** *Consider again the all interval series problem. To eliminate the reversal symmetry  $\sigma_{rev}$ , we can post the constraint:*

$$X_1 < X_{11} \quad (1)$$

*This eliminates solution (b) as it is the reversal of (a). To eliminate the value symmetry  $\theta_{inv}$ , we can post:*

$$X_1 \leq 5, \quad X_1 = 5 \Rightarrow X_2 < 5 \quad (2)$$

*This eliminates solution (c) as it is the inversion of (a). Finally, to eliminate the third symmetry  $\theta_{inv} \circ \sigma_{rev}$  where we both reverse and invert the solution, we can post:*

$$\langle X_1, \dots, X_6 \rangle \leq_{lex} \langle 10 - X_{11}, \dots, 10 - X_6 \rangle \quad (3)$$

*This eliminates solution (a) as it is the reversal and inversion of (d). Note that of the four symmetric solutions given earlier, only (d) with  $X_1 = 2$  and  $X_{11} = 7$  satisfies all these symmetry breaking constraints. The other three solutions are eliminated. Thus (1) to (3) are a sound and complete set of symmetry breaking constraints. ♣*

We now show that *any symmetry* acting on a set of symmetry breaking constraints itself breaks the symmetry in a problem. Different symmetries pick out different solutions in each symmetry class. To prove this, we need to consider the action of a symmetry on a symmetry breaking constraint. Symmetry has been defined acting on assignments. We lift this definition to constraints. The action of a variable symmetry on a constraint changes the variables on which the constraint acts. More precisely, a variable symmetry  $\sigma$  applied to the constraint  $C(X_j, \dots, X_k)$  gives  $C(X_{\sigma(j)}, \dots, X_{\sigma(k)})$ . The action of a value symmetry is also easy to compute. A value symmetry  $\theta$  applied to the constraint  $C(X_j, \dots, X_k)$  gives  $C(\theta(X_j), \dots, \theta(X_k))$ .

**Running Example.** *To illustrate how we can break symmetry with the symmetry of a set of symmetry breaking constraints, we consider symmetries of (1), (2) and (3).*

*If we apply  $\sigma_{rev}$  to (1), we get an ordering constraint that again breaks the reversal symmetry:*

$$X_{\sigma_{rev}(1)} < X_{\sigma_{rev}(11)}$$

*This simplifies to:*

$$X_{11} < X_1$$

*If we apply  $\sigma_{rev}$  to (2), we get constraints that again breaks the inversion symmetry:*

$$X_{11} \leq 5, \quad X_{11} = 5 \Rightarrow X_{10} < 5$$

*Finally, if we apply  $\sigma_{rev}$  to (3), we get a constraint that again breaks the combined reversal and inversion symmetry:*

$$\langle X_{11}, \dots, X_6 \rangle \leq_{lex} \langle 10 - X_1, \dots, 10 - X_6 \rangle$$

*Note that of the four symmetric solutions given earlier, only (c) satisfies  $\sigma_{rev}$  of (1), (2) and (3).*

*We can also break symmetry with any other symmetry of the symmetry breaking constraints. For instance, if we apply  $\theta_{inv} \circ \sigma_{rev}$  to (1), we get a constraint that again breaks the reversal symmetry:*

$$10 - X_{11} < 10 - X_1$$

*This simplifies to:*

$$X_1 < X_{11}$$

*If we apply  $\theta_{inv} \circ \sigma_{rev}$  to (2), we get constraints that again breaks the inversion symmetry:*

$$10 - X_{11} \leq 5, \quad 10 - X_{11} = 5 \Rightarrow 10 - X_{10} < 5$$

*This simplifies to:*

$$X_{11} \geq 5, \quad X_{11} = 5 \Rightarrow X_{10} > 5$$

*Finally, if we apply  $\theta_{inv} \circ \sigma_{rev}$  to (3), we get a constraint that again breaks the combined reversal and inversion symmetry:*

$$\langle 10 - X_{11}, \dots, 10 - X_6 \rangle \leq_{lex} \langle X_1, \dots, X_6 \rangle$$

*Note that of the four symmetric solutions given earlier, only (a) satisfies  $\theta_{inv} \circ \sigma_{rev}$  of (1), (2) and (3). ♣*

The running example illustrates that we can break symmetry with a symmetry of a set of symmetry breaking constraints. We now prove that this holds in general:

Any symmetry acting on a set of symmetry breaking constraints itself breaks symmetry.

More precisely, if a set of symmetry breaking constraints is sound, then any symmetry of these constraints is also sound. Similarly, if a set of symmetry breaking constraints is complete, then any symmetry of these constraints is also complete.

**Observation 1.** Given a set of symmetries  $\Sigma$  of  $C$ , if  $S$  is a sound (complete) set of symmetry breaking constraints for  $\Sigma$  then  $\sigma(S)$  for any  $\sigma \in \Sigma$  is also a sound (complete) set of symmetry breaking constraints for  $\Sigma$ .

**Proof:** (Soundness) Consider  $s \in \text{sol}(C \cup S)$ . Then  $s \in \text{sol}(C)$  and  $s \in \text{sol}(S)$ . Hence  $\sigma(s) \in \text{sol}(C)$ . Since  $s \in \text{sol}(S)$ , it follows that  $\sigma(s) \in \text{sol}(\sigma(S))$ . Thus,  $\sigma(s) \in \text{sol}(C \cup \sigma(S))$ . Hence, there is at least one solution,  $\sigma(s)$  in every symmetry class of  $C \cup \sigma(S)$ . That is,  $\sigma(S)$  is a sound set of symmetry breaking constraints for  $\Sigma$ .

(Completeness) Consider  $s \in \text{sol}(C \cup \sigma(S))$ . By a similar argument to soundness,  $\sigma^{-1}(s) \in \text{sol}(C \cup S)$ . Hence, there is at most one solution in every symmetry class of  $C \cup \sigma(S)$ . That is,  $\sigma(S)$  is a complete set of symmetry breaking constraints for  $\Sigma$ .  $\square$

Different symmetries of the symmetry breaking constraints pick out different solutions in each symmetry class. Thus, if the branching heuristic is going towards a particular solution, there is a symmetry of the symmetry breaking constraints which does not conflict with this.

**Observation 2.** Given a symmetry group  $\Sigma$ , a sound set  $S$  of symmetry breaking constraints for  $\Sigma$ , and any complete assignment  $A$ , then there exists a symmetry  $\sigma$  in  $\Sigma$  such that  $A$  satisfies  $\sigma(S)$ .

**Proof:** Since the set of symmetry breaking constraints  $S$  is sound, it leaves at least one solution (call it  $B$ ) in the same symmetry class as  $A$ . That is,  $B$  satisfies  $S$ . Since  $A$  and  $B$  are in the same symmetry class, there exists a symmetry  $\sigma$  in  $\Sigma$  with  $\sigma(A) = B$ .  $\Sigma$  forms a group so also contains the inverse symmetry  $\sigma^{-1}$ . Since  $B$  satisfies  $S$  and  $\sigma(A) = B$ , it follows that  $\sigma(A)$  satisfies  $S$ . Hence  $\sigma^{-1}(\sigma(A))$  satisfies  $\sigma^{-1}(S)$ . That is,  $A$  satisfies  $\sigma^{-1}(S)$ .  $\square$

## 4 Model restarts

We start with a simple application of these observations. To tackle conflict between branching heuristics and symmetry breaking constraints, Heller *et al.* propose using *model restarts* [10]. In this method, backtracking search is restarted periodically, using a new model which contains different symmetry breaking constraints. By posting different symmetry breaking constraints, we hope at some point for the branching heuristic and symmetry breaking not to conflict. Our observations that any symmetry acting on a set of symmetry breaking constraints can be used to break symmetry, and that different symmetries pick out different solutions, provide us with precisely the tools we need to perform model restarts to any domain (and not just to interchangeable variables and values as in [10]). When we restart search, we simply post a different symmetry of the symmetry breaking constraints. We experimented with several possibilities. The

simplest is to choose a symmetry at random from the symmetry group. We also tried various heuristics like using the symmetry most consistent or most inconsistent with previous choices of the branching heuristic. However, we observed the best performance of model restarts with a random choice of symmetry so we only report results here with such a choice.

**Running Example.** Consider again the all interval series problem and posting symmetries of the symmetry breaking constraints (1), (2) and (3). The following table gives the amount of search needed to find an all interval series of size  $n = 11$  using a branching heuristic that branches in order on the variables introduced to represent neighbouring differences in the series, trying values in numerical order. This would seem to be a good branching heuristic since it can find a solution without backtracking.

Symmetry posted of (1) to (3)	Branches	Time to solve/s
$\sigma_{id}$	1	0.00
$\sigma_{rev}$	222,758	13.74
$\theta_{inv}$	425,765	24.99
$\theta_{inv} \circ \sigma_{rev}$	170,425	10.23

It is clear from this table that the different symmetries of the symmetry breaking constraints interact differently with the branching heuristic. In particular, the identity symmetry does not conflict in any way as the branching heuristic goes directly to the following solution at the end of the first branch:

$$X_1, X_2, \dots, X_{11} = 0, 10, 1, 9, 2, 8, 3, 7, 4, 6, 5$$

This solution is consistent with  $\sigma_{id}$  of (1), (2) and (3).

The other symmetries of the symmetry breaking constraint conflict with the branching heuristic. In particular, the following symmetry breaking constraints conflict with this solution:  $\sigma_{rev}$  of (1) as  $X_{11} = 5 > X_1 = 0$ ,  $\theta_{inv}$  of (1) as  $10 - X_1 = 10 > 10 - X_{11} = 5$  and  $\theta_{inv} \circ \sigma_{rev}$  of (3) as  $10 - X_{11} = 5 > X_1 = 0$ . As a result, posting these symmetries of the symmetry breaking constraints increases the search needed to find a solution.

Model restarts will help overcome this conflict. Suppose we restart search every 100 branches and choose to post a random symmetry of (1), (2) and (3). Let  $t$  be the average number of branches to find a solution. There is  $\frac{1}{4}$  chance that the first restart will post  $\sigma_{id}$  of (1), (2) and (3). In this situation, we find a solution after 1 branch. Otherwise we post one of the other symmetries of (1), (2) and (3). We then explore 100 branches, reach the cutoff and fail to find a solution. As each restart is independent, we restart and explore on average another  $t$  more branches. Hence:

$$t = \frac{1}{4} \cdot 1 + \frac{3}{4} (100 + t)$$

Solving for  $t$  gives  $t = 301$ . Thus, using model restarts with a random symmetry of the symmetry breaking constraints, we take just 301 branches on average to find an all interval series of size  $n = 11$ . ♣

Note that posting random symmetries of the symmetry breaking constraints is not equivalent to fixing the symmetry

breaking and randomly branching. As we saw in the example, different symmetries of the symmetry breaking constraints interact in different ways with the problem constraints. Although the problem constraints are themselves initially symmetrical, branching decisions quickly break the symmetries.

## 5 Posting constraints dynamically

We now consider a more sophisticated use of the observations that any symmetry acting on a set of symmetry breaking constraints itself breaks symmetry, and that different symmetries pick out different solutions in each symmetry class. We will incrementally and dynamically post a symmetry of the symmetry breaking constraints which is consistent with the branching decisions made so far. Thus, if the branching heuristic is smart or lucky enough to branch immediately to a solution, symmetry breaking will not interfere with this.

**Running Example.** Consider again the all interval series problem. Suppose we begin by trying  $X_1 = 10$ . Since the  $X_i$  are all different,  $X_{11} \in [0, 9]$ . Hence, the symmetry breaking constraint  $X_{11} < X_1$  is entailed. This is  $\sigma_{rev}$  of (1). It is also  $\theta_{inv}$  of (1). We do not yet need to commit to which of these two symmetries of the symmetry breaking constraints we will post. We are sure, however, that we are not posting  $\sigma_{id}$  or  $\theta_{inv} \circ \sigma_{rev}$  of the constraints (1) to (3). These two symmetries would require  $X_1 > X_{11}$ , and this is dis-entailed. We therefore post  $X_{11} < X_1$  and continue search. ♣

In the example, we posted symmetry breaking constraint once they are entailed. When there are only a few symmetries, we can easily implement this with non-backtrackable variables and reification. Suppose we reify the two ordering constraints:

$$B_1 \Leftrightarrow (X_1 < X_{11}), \quad B_2 \Leftrightarrow (X_{11} < X_1)$$

We then make the Boolean variables,  $B_1$  and  $B_2$  non-backtrackable so that, once they are instantiated, their value remains on backtracking. We assume that our solver posts the conclusion of an implication when its hypothesis is entailed. Suppose  $X_1 < X_{11}$  is entailed. Then  $B_1$  will be set *true*. As  $B_1$  is non-backtrackable,  $X_1 < X_{11}$  will be posted. Unfortunately, posting symmetry breaking constraints like this as soon as they are entailed may be a little eager.

**Running Example.** Consider again the all interval series problem. As before, suppose the branching heuristic has set  $X_1 = 10$ , and we have posted the entailed symmetry breaking constraint  $X_{11} < X_1$ . Now  $X_1 \geq 5$  is also entailed. This is  $\theta_{inv}$  of the first inequality in (2). If we post this, we commit to breaking symmetry with  $\theta_{inv}$  of (1) to (3). However, this would rule out breaking symmetry with  $\sigma_{rev}$  of (1) to (3) which are also still consistent with the branching decisions so far.

Suppose we next branch on  $X_{11} = 5$ . The assignments to  $X_1$  and  $X_{11}$  are only consistent with  $\theta_{inv}$  of (2) and of (3). In fact, both of these constraints are now entailed. However,  $X_1 = 10$  and  $X_{11} = 5$  are not consistent with posting  $\sigma_{rev}$  of (3). This would require that:

$$\langle X_{11}, \dots, X_6 \rangle \leq_{lex} \langle 10 - X_1, \dots, 10 - X_6 \rangle$$

This is dis-entailed. Hence, our branching decisions have committed us to break symmetry with  $\theta_{inv}$  of (1) to (3). We therefore post these constraints. If search continues, we will discover the unique solution consistent with symmetry breaking and the initial branching decisions:

$$X_1, X_2, \dots, X_{11} = 10, 0, 9, 1, 8, 2, 7, 3, 6, 4, 5$$

♣

There is a tradeoff between posting symmetry breaking constraints early (so propagation prunes the search space) and late (so we do not conflict with future branching decisions). We propose the following rule for when to post symmetry breaking constraints. The rule only posts symmetry breaking constraints once the branching heuristic has forced their choice. It would, however, be interesting to explore other more eager or lazy rules. Suppose  $S$  is a set of symmetry breaking constraints for  $\Sigma$ , and we have posted  $T$ , a symmetry of a subset of  $S$ . A symmetry  $\sigma \in \Sigma$  is *consistent* with  $T$  iff  $T$  is entailed by  $\sigma(S)$  and *inconsistent* otherwise. A symmetry  $\sigma \in \Sigma$  is *eliminated* by posting some symmetry breaking constraint  $c$  iff  $\sigma$  is consistent with  $T$  but inconsistent with  $T \cup \{c\}$ . The *forced symmetry* rule is defined as follows:

*Given a set of symmetry breaking constraints, if during backtracking search a symmetry of one of these constraints is entailed, this symmetry is consistent with previously posted symmetry breaking constraints, and all symmetries eliminated by this entailed constraint are inconsistent with the current state then we post the entailed constraint.*

We first show that this rule is sound.

**Observation 3.** Given a set of symmetries  $\Sigma$  of  $C$ , if  $S$  is a sound set of symmetry breaking constraints for  $\Sigma$  then the forced symmetry rule using  $S$  is a sound symmetry breaking method.

**Proof:** The rule only permits constraints of a particular symmetry to be posted. By Observation 1, this is sound.  $\square$

In general, this rule may not be complete even when given a complete set of symmetry breaking constraints. However, it is easy to modify the rule so that it is complete. Whenever we reach a solution, we simply pick a consistent symmetry and post all the symmetry breaking constraints associated with this symmetry. We can also define a common property of many symmetry breaking constraints for which the unmodified rule is complete. A set of symmetry breaking constraints  $S$  for the symmetries  $\Sigma$  of  $C$  is *proper* iff  $S$  is sound and complete for  $\Sigma$  and every non-identity symmetry in  $\Sigma$  maps any solution of  $S \cup C$  onto a different solution. With a proper set of symmetry breaking constraints, each solution within a symmetry class is associated with a different symmetry. For instance, constraints (1) to (3) form a proper set of symmetry

We now prove that with a proper set of symmetry breaking constraints, the forced symmetry rule is a sound and complete symmetry breaking method. That is, it will find exactly one solution in each symmetry class.

**Observation 4.** Given a set of symmetries  $\Sigma$  of  $C$ , if  $S$  is a proper set of symmetry breaking constraints for  $\Sigma$  then the forced symmetry rule is both sound and complete.

**Proof:** (Soundness) Immediate as a proper set is sound.

(Completeness) Consider the first solution visited. As the set of symmetry breaking constraints is proper, only one symmetry of these constraints will be entailed. All other symmetries are inconsistent with the current state and are eliminated. The forced symmetry rule therefore post this symmetry of the symmetry breaking constraints. By Observation 1, as the symmetry breaking constraints are complete, this eliminates all other solutions in the same symmetry class.  $\square$

Finally, we observe that with certain symmetry breaking constraints, the forced symmetry rule is equivalent to posting symmetry breaking constraints as soon as they are entailed. For symmetry breaking constraints like  $X_1 < X_{11}$ , as soon as the constraint or its negation is entailed, all variable symmetries are either consistent or they are eliminated.

## 6 Interchangeable variables and values

To test these two symmetry breaking methods, we consider a common type of symmetry where variables and values partition into interchangeable sets [13; 14]. This is sometimes called piecewise variable and value symmetry. We chose this class of symmetry over the many other types of symmetry studied in the past as it was used in the previous experimental study of model restarts [10]. Suppose that the  $n$  variables partition into  $a$  disjoint sets and variables within each set are interchangeable. Similarly, suppose that the  $m$  values partition into  $b$  disjoint sets and values within each set are interchangeable. We will order variable indices so that  $X_{p(i)}$  to  $X_{p(i+1)-1}$  is the  $i$ th partition of variables for  $1 \leq i \leq a$ , and value indices so that  $d_{q(j)}$  to  $d_{q(j+1)-1}$  is the  $j$ th partition of values for  $1 \leq j \leq b$ .

Flener *et al.* [14] proved that we can eliminate the exponential number of symmetries due to such interchangeability with a polynomial number of symmetry breaking constraints:

$$X_{p(i)} \leq \dots \leq X_{p(i+1)-1}$$

$$\text{GCC}([X_{p(i)}, \dots, X_{p(i+1)-1}], [d_1, \dots, d_m], [O_1^i, \dots, O_m^i]) \\ (O_{q(j)}^1, \dots, O_{q(j)}^a) \geq_{\text{lex}} \dots \geq_{\text{lex}} (O_{q(j+1)-1}^1, \dots, O_{q(j+1)-1}^a)$$

Where  $i \in [1, a]$  and  $j \in [1, b]$ , and GCC counts the number of occurrences of the values in each equivalence class of variables. That is,  $O_j^i = |\{k | X_k = d_j, p(i) \leq k < p(i+1)\}|$ . The *signature* of  $d_k$  is  $(O_k^1, \dots, O_k^a)$ , the number of occurrences of  $d_k$  in each variable partition. The signature is invariant to the permutation of variables within each equivalence class. By ordering variables within each equivalence class, we prevent permutation of interchangeable variables. Similarly, by ordering the signatures, we prevent permutation of interchangeable values.

We will post symmetries of these symmetry breaking constraints. We consider symmetries that act along two degrees of freedom: the order of interchangeable variables within a variable partition, and the order of the signatures of interchangeable values within a value partition. Let  $\sigma$  be some permutation of the indices of interchangeable variables. Then we can break the symmetry of variable interchangeability with the following symmetry of the variable ordering constraints:

$$X_{\sigma(p(i))} \leq \dots \leq X_{\sigma(p(i+1)-1)}$$

Similarly let  $\theta$  be some permutation of the indices of interchangeable values. Then we can break the symmetry of value interchangeability with this symmetry of the signature ordering constraints:

$$(O_{\theta(q(j))}^1, \dots, O_{\theta(q(j))}^a) \geq_{\text{lex}} \dots \geq_{\text{lex}} (O_{\theta(q(j+1)-1)}^1, \dots)$$

## 7 Experiments

We used model restarts and the forced symmetry rule to post symmetries dynamically of the symmetry breaking constraints of Flener *et al.* [14]. Problems are coded into Gecode 2.2.0. We evaluated the two methods on the same two benchmark domains used in previous studies of symmetry breaking for interchangeable variables and values [15]. Experiments were run on an 2-way Intel Xeon with 6MB of cache and 4 cores in each processor running at 2GHz. All instances were terminated after 10 minutes. We used smallest domain as a variable ordering heuristic in each experiment. For value ordering heuristic, we used lexicographical, anti-lexicographical and random orderings.

Our experiments are designed to test two hypotheses. The first hypothesis is that these two methods are less prone to conflict between branching heuristics and symmetry breaking. The second hypothesis is that these two methods explore a smaller search tree than dynamic methods like SBDS. This is due to both the propagation of the posted symmetry breaking constraints and the need to limit SBDS to just generators to make it computationally tractable. We limit our comparison of dynamic methods to comparison against SBDS. Whilst there is a specialized dynamic symmetry breaking method for interchangeable variables and values, experiments in [10] show that this is several orders of magnitude slower than static methods. In addition, dominance detection methods like SBDD are shown to be three orders of magnitude slower than static methods in [10]. Finally, we used SBDS to break just generators of the symmetry group as breaking the full symmetry group quickly ran out of memory. We used SBDS with two different sets of generators: one set has a generator that exchanges each pair of consecutive variables/values in each partition; the other has a generator that exchanges the first two variables/values in each partition and one that rotate all variables/values. We got similar results with both and thus here report only results for the first set, denoted SBDS-pair in Table 1.

The first set of experiments uses random graph coloring problems generated in the same way as the previous experimental study in [15]. There is a variable for each vertex and not-equals constraints between variables corresponding to connected vertices. All values in this model are interchangeable. In addition, we introduce variable symmetry by partitioning variables into interchangeable sets of size at most 8. We randomly connect the vertices within each partition with either a complete graph or an empty graph, and choose each option with equal probability. Similarly, between any two partitions there is equal probability that the partitions are completely connected or independent. Results for graphs with 40 vertices are shown in the top half of Table 1.

The second set of experiments uses a more structured benchmark which is again taken from a previous experimen-

#	Lex		Static posting		Dynamic posting		SBDS-pair		Model Restarts			
	opt	t/b	Antiflex	Random	Lex/Antiflex	Random	Lex/Antiflex	Random	Lex	Random		
			opt	t/b	opt	t/b	opt	t/b	opt	t/b		
Graph Coloring												
1	<b>13</b>	<b>0.11</b>	-	119.64	<b>13</b>	0.18	<b>13</b>	0	15*	1.14	<b>13</b>	5.13
		387	-	424 K		<b>315</b>	4746	13253 K	14606 K	2087		13 K
2	<b>14</b>	6.29	<b>14</b>	16.1	<b>14</b>	12.36	20.7	0.01	20*	<b>5.22</b>	<b>14</b>	23.22
		25 K		76 K		25 K	46 K	7918 K	13719 K	<b>13 K</b>		54 K
3	<b>16</b>	<b>0.32</b>	40*	-	<b>16</b>	0.52	7.03	0.01	16*	27.01	<b>16</b>	6.18
		<b>730</b>		-		801	1516 K	6746 K	6492 K	38 K		12 K
5	<b>13</b>	254.93	-	-	<b>13*</b>	119.44	0.08	0.01	16*	<b>117.9</b>	<b>13</b>	208.4
		1001 K		-		1368 K	1150 K	15710 K	12505 K	<b>410 K</b>		693 K
6	<b>8</b>	<b>0.03</b>	-	-	<b>8</b>	0.07	27*	4.13	<b>8</b>	0.74	<b>8</b>	0.77
		60		-		<b>53</b>	1535 K	40 K	40 K	1980		1794
7	<b>17</b>	<b>0.11</b>	-	-	<b>17</b>	0.18	-	0.01	17*	20.67	<b>17</b>	100.77
		<b>170</b>		-		185	-	4697 K	4522 K	59 K		284 K
9	<b>8*</b>	<b>0.01</b>	16*	25.88	<b>8*</b>	0.05	21*	0.01	<b>8*</b>	1.83	<b>8*</b>	5.61
		3850 K		4550 K		1548 K	1394 K	13795 K	13710 K	3361 K		3463 K
10	<b>10</b>	<b>0.03</b>	<b>10</b>	368.25	<b>10</b>	0.08	<b>10</b>	1.77	<b>10</b>	2.75	<b>10</b>	-
		31		1364 K		56	4219	3629 K	3629 K	6503		-
11	<b>18</b>	<b>166.89</b>	-	-	<b>18</b>	353.8	-	0	18*	296.81	-	-
		<b>511 K</b>		-		772 K	-	8998 K	8691 K	619 K		-
12	<b>15</b>	34.7	-	-	<b>15</b>	<b>23.91</b>	-	0.01	15*	574.25	<b>15</b>	436.82
		115 K		-		<b>50 K</b>	44.51	5953 K	5874 K	847 K		654 K
13	<b>14*</b>	0.01	-	-	<b>14*</b>	0.04	27*	0	14*	<b>211.28</b>	-	-
		1652 K		-		1231 K	1083 K	6680 K	6283 K	<b>528 K</b>		-
14	<b>12*</b>	0.02	-	-	<b>12*</b>	0.04	25*	2.58	12*	<b>3.56</b>	<b>12</b>	140.81
		2003 K		-		1236 K	1176 K	7754 K	6953 K	<b>7958</b>		386 K
15	<b>11</b>	<b>0.04</b>	-	-	<b>11</b>	0.06	26*	0	11*	-	<b>11</b>	27.97
		<b>33</b>		-		<b>33</b>	1501 K	5483 K	5220 K	-		41 K
Concert Hall Scheduling												
1	<b>2894</b>	<b>2.2</b>	<b>2894</b>	7.56	<b>2894</b>	2.76	<b>2894</b>	542.37	1804*	128.28	<b>2894</b>	134.08
		2128		17 K		4186	3.88	912 K	568.56	149 K		169 K
2	<b>2245</b>	<b>1.99</b>	<b>2245</b>	12.36	<b>2245</b>	3.79	<b>2245</b>	417.05	2194*	136.87	<b>2245</b>	89.82
		<b>1836</b>		32 K		5716	11.36	1258 K	1493 K	165 K		127 K
3	<b>2639</b>	<b>15 K</b>	<b>2639</b>	71.37	<b>2639</b>	27.92	<b>2639</b>	167.32	1873*	15.92	2639*	192.82
		<b>2093</b>		131 K		31 K	36 K	827 K	423.7	2587*		544 K
5	<b>3634</b>	<b>3.86</b>	<b>3634</b>	18.28	<b>3634</b>	7.9	<b>3634</b>	108.68	3487*	166.03	<b>3634</b>	128.15
		<b>3797</b>		44 K		13 K	11 K	1201 K	471.78	206 K		171 K
7	<b>3262</b>	<b>1.46</b>	<b>3262</b>	8.88	<b>3262</b>	2.88	<b>3262</b>	594.55	3224*	109.35	<b>3262</b>	114.58
		<b>1102</b>		19 K		3444	6.61	1088 K	1359 K	115 K		139 K
8	<b>3288</b>	<b>3.61</b>	<b>3288</b>	17.42	<b>3288</b>	5.22	<b>3288</b>	475.48	1725*	199.97	<b>3288</b>	156.35
		<b>2606</b>		36 K		5781	12 K	928 K	1199 K	199 K		169 K
9	<b>3434</b>	<b>16.36</b>	<b>3434</b>	57.81	<b>3434</b>	24.36	<b>3434</b>	105.84	2335*	486.67	3434*	112.34
		<b>14 K</b>		122 K		32 K	49.16	943 K	1362 K	441 K		628 K
10	<b>2847</b>	<b>4.69</b>	<b>2847</b>	19.41	<b>2847</b>	7.52	<b>2847</b>	153.95	2647*	231.1	<b>2847</b>	241.09
		<b>4888</b>		50 K		13 K	12 K	1177 K	473.14	297 K		332 K
11	<b>3295</b>	<b>5.31</b>	<b>3295</b>	33.07	<b>3295</b>	10.16	<b>3295</b>	241.42	3295*	250.43	<b>3295</b>	261.13
		<b>3451</b>		62 K		12 K	34.56	672 K	161.71	223 K		273 K
12	<b>1197</b>	<b>11.1</b>	<b>1197</b>	38.43	<b>1197</b>	15.05	<b>1197</b>	112.34	958*	479.43	<b>1197</b>	241.41
		<b>10 K</b>		74 K		21 K	27.67	736 K	419.76	475 K		271 K
13	<b>2565</b>	<b>2.84</b>	<b>2565</b>	18.03	<b>2565</b>	5.25	<b>2565</b>	101.34	2565*	156.49	<b>2565</b>	130.58
		<b>2411</b>		43 K		7266	13.45	993 K	1187 K	186 K		170 K
14	<b>3235</b>	<b>6.91</b>	<b>3235</b>	25.84	<b>3235</b>	9.57	<b>3235</b>	157.02	2385*	349.81	<b>3235</b>	373.23
		<b>5650</b>		50 K		12 K	15.69	819 K	1168 K	324 K		398 K
15	<b>3234</b>	<b>17.95</b>	<b>3234</b>	63.28	<b>3234</b>	24.32	<b>3234</b>	252.34	2331*	449.52	3234*	281.55
		<b>15 K</b>		138 K		32 K	44.64	1122 K	1622 K	592 K		733 K

Table 1: Static vs Dynamic posting of symmetry breaking constraints on Graph Coloring and Concert Hall Scheduling. “opt” is the quality of the solution found (\* indicates optimality was not proved), “t” is the runtime in seconds, “b” is the number of backtracks. The best method for a problem instance is in **bold font**.

tal study [15]. In the concert hall scheduling problem, we have  $n$  applications to use one of  $m$  identical concert halls. Each application has a start and end time as well as an offer for the hall. We accept applications so that their intervals do not overlap and the profit (the sum of the offers of accepted applications) is maximized. We randomly generate instances so that applications are split into partitions of size at most 8 and within each partition all applications have the same start and end time and offer. Our model assigns  $X_i = j$  if the  $i^{\text{th}}$  application is accepted and placed in hall  $j$ , and  $X_i = m + 1$  if it is rejected. Variables corresponding to applications in the same partition are interchangeable. Values divide into two partitions: the values 1 to  $m$  are interchangeable, while the value  $m + 1$  is in a separate partition. Results for instances with 40 applications and 10 halls are shown in the bottom half of Table 1.

The results support both our hypotheses. Both methods are less prone to conflict between symmetry breaking and the branching heuristic. With both SBDS and our forced symmetry rule for dynamically posting symmetry breaking constraints, we obtained the same results with the lexicographical and the (inverse) anti-lexicographical value ordering heuristic. With model restarts, results with the lexicographical and the anti-lexicographical value ordering heuristic are sufficiently similar that we only report the former. Our second hypothesis, that the two methods explore a smaller search tree than SBDS is also confirmed. SBDS was unable to prove optimality in all but one instance. In addition, on the harder graph coloring instances, both methods tend to outperform the static method. It is hard, however, to choose between the two methods. The model restarts method offers slightly better performance on the graph coloring instances, whilst our method of dynamically posting static symmetry breaking constraints offers better performance on the concert hall scheduling instances.

## 8 Other related work

Closest in spirit to our forced symmetry rule for dynamically posting symmetry breaking constraints is SBDS [16; 17; 18]. SBDS can work with any type of branching decision but for simplicity we assume that branching decisions are of the form  $Var = val$ . All current implementations of SBDS make this assumption. If we have a symmetry  $\sigma$ , the partial assignment  $A$  and have explored and rejected  $Var = val$  then on backtracking, SBDS posts:

$$\sigma(A) \rightarrow \sigma(Var \neq val)$$

This ensures that we never explore the symmetric state to the one that has just been excluded. Our forced symmetry rule also posts static symmetry breaking dynamically during search. However, the two methods differ along three important dimensions. First, SBDS posts symmetry breaking constraints when backtracking and exploring the second branch of the search tree; the forced symmetry rule, on the other hand, can post symmetry breaking constraints down either branch. Second, SBDS posts symmetries of the current nogood; the forced symmetry rule, on the other hand, can post *any* type of symmetry breaking constraint. Here, for instance, it posts ordering constraints on the signatures. Third,

whilst neither method conflicts with the branching heuristic if the branching heuristic goes directly to a solution, the forced symmetry rule may conflict with the branching heuristic later in search. Constraint propagation on constraints posted by the forced symmetry rule can prune values that branching might have taken.

There are a number of other related methods. Jefferson *et al.* have proposed GAPLex, a hybrid method that also combines together static and dynamic symmetry breaking [19]. However, GAPLex is limited to dynamically posting lexicographical ordering constraints, and to searching with a fixed variable ordering. As a consequence, GAPLex performs poorly when there are large numbers of symmetries. In addition, GAPLex is unable to profit from effective dynamic variable ordering heuristics. Puget has also proposed “Dynamic Lex”, a hybrid method that dynamically posts static symmetry breaking constraints during search which works with dynamic variable ordering heuristics [20]. This method adds lexicographical ordering symmetry breaking constraints dynamically during search that are compatible with the current partial assignment. In this way, the first solution found during tree search is not pruned by symmetry breaking. Unfortunately Dynamic Lex needs to compute the stabilizers of the current partial assignment. This requires a potentially expensive graph isomorphism problem to be solved at each node of the search tree. Whilst Dynamic Lex works with dynamic variable ordering heuristics, it assumes that values are tried in order. Finally Dynamic Lex is limited to posting lexicographical ordering constraints. This is problematic when there are many symmetries. A direct comparison of our methods with Dynamic Lex would be interesting but poses some challenges. For instance, Heller *et al.* [10] do not compare model restarts with Dynamic Lex, arguing:

*“It is not clear how this method [Dynamic Lex] can be generalized, though, and for the case of piecewise variable and value symmetry, no method with similar properties is known yet.”*

## 9 Conclusions

We proved that any symmetry acting on a set of symmetry breaking constraints itself breaks symmetry, and that different symmetries pick out different solutions in each symmetry class. These observations can be used to reduce the conflict between symmetry breaking and branching heuristics. We have studied two methods for breaking symmetry that tackle this conflict. The first method uses *model restarts* which was proposed in [10]. We periodically restart search with a new model which contains a random symmetry of the symmetry breaking constraints. The second method posts a symmetry of the symmetry breaking constraint dynamically during search. The symmetry is incrementally chosen to be consistent with the branching heuristic. The two methods benefit from propagation of the posted symmetry breaking constraints, whilst reducing the conflict between symmetry breaking and branching heuristics. Experimental results demonstrated that the two methods perform well on some standard benchmarks.

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