Male optimality and uniqueness in stable marriage problems with partial orders

(Extended Abstract)

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ABSTRACT

In this paper, we study the concepts of male optimality and uniqueness of stable marriages for partially ordered preferences. We give an algorithm to find a stable marriage that is male optimal, and a sufficient condition on the preferences, which guarantees the uniqueness of stable marriages.

Categories and Subject Descriptors

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General Terms

Algorithms, Theory

Keywords

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1. INTRODUCTION

The stable marriage problem [2] has a wide variety of practical applications, ranging from matching resident doctors to hospitals [4], to matching students to schools, or more generally to any two-sided market. In the classical stable marriage problem, both men and women express a strict order over the members of the other sex. Here we consider a potentially more realistic case, where both men and women can express their preferences via partial orders, i.e., by allowing ties and incomparability. This may be useful, for example, when preferences are elicited via compact preference representations like soft constraint or CP-net, that may produce partial orders, or also when preferences are obtained via multi-criteria reasoning.

We study the concepts of male optimality and uniqueness of stable marriages for partially ordered preferences. Male optimality allows us to give priority to one gender over the other (for example, in matching residents to hospitals in the US, priority is given to the residents). Uniqueness means that the solution is optimal, since it is as good as possible for all the participating agents. Uniqueness of solution is also a barrier against manipulation. We give an algorithm to find a stable marriage that is male optimal. Moreover, we give a sufficient condition on the preferences, that occurs often in real-life scenarios, which guarantees the uniqueness of stable marriages.

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2. BASIC NOTIONS

A *stable marriage problem* (SM) [2] of size n is the problem of finding a stable marriage between n men and n women. Such men and women each have a preference ordering over the members of the other sex.

A marriage is a one-to-one correspondence between and women and it is *stable* when there are no two people of opposite sex who would both rather be married to each other than their current partners.

The sequence of all preference ordering of men and women is usually called a *profile*. In the case of classical stable marriage problem, the profile is a sequence of strict total orders. Given a SM P, there is always at least one stable marriage for P. Morever, there may be several stable marriages.

Given an SM P, a *feasible partner* for a man m (resp., a woman w) is a woman w (resp., a man m) such that there is a stable marriage for P where m and w are married.

Given an SM P, a stable marriage is *male optimal* for P iff every man is paired with his highest ranked feasible partner.

The extended *Gale-Shapley* (GS) algorithm [2] is a well-known algorithm to solve the SM problem, that returns a male optimal stable marriage. At the beginning, each person is free. The following step is iterated until all men are engaged: choose a free man m, and let m propose to the most preferred woman w on his preference list. If w is free, then w and m become engaged. Moreover, in w's preference list all men less desirable than m are deleted, and w is deleted from the preference lists of all such men. Notice that this deletions from the preference lists implies that a proposal is always accepted. When all men are engaged, the engaged pairs form the male optimal stable marriage.

In SMs, each preference ordering is a strict total order over the members of the other sex. In this paper we consider the more general form where each preference ordering is a *partial order* [3]. We will denote with SMP a stable marriage problem with partially ordered profiles.

Given an SMP, we will sometimes use the notion of a *linearization* of such a problem, which is obtained by linearizing the preference orderings of the profile in a way that is compatible with the given partial orders.

A marriage for an SMP is *weakly stable* if there is no pair (man,woman) not married to each other such that each one strictly prefers the other to his/her current partner.

As in the classical case, a weakly stable marriage is *male optimal* if there is no man that can get a strictly better partner in some other weakly stable marriage. However, we may have several male optimal weakly stable marriages, as well as none.

3. MALE OPTIMALITY

We now present an algorithm that takes as input an SMP P and, either returns a male optimal weakly stable marriage for P, or the string 'I don't know'. This algorithm is sound but not complete: if the algorithm returns a marriage, then it is weakly stable and male optimal; however, it may fail to return a male optimal marriage even if there is one. We assume that the women express strict total orders over the men. If they don't, we simply pick any linearization. The algorithm exploits the *extended GS* algorithm [2], and at every step orders the free men by increasing number of their current top choices (i.e., the women that are undominated). At the beginning all men are unmarried, and thus L contains them all. Then, we continue to check the following cases on man m, which is the first element of L, until they do not occur any longer.

(i) If the set of top choices of m contains exactly one unmarried woman, say w, m proposes to w. Then, all men that are strictly worse than m in w's preferences are removed from w's preference list, and w is removed from the preference lists of these men. Also, m is removed from L and L is ordered again.

(ii) If m has a single top choice, say w, that is already married, m proposes to w, and w breaks the engagement with her current partner, say m'. Then, m is removed from L, m' becomes free and is put back in L, and L is ordered.

When we exit from this cycle, we check if L is empty. If L is empty, the algorithm returns the current marriage. The returned marriage, say (m_i, w_i), for $i = 1, \ldots, n$, is weakly stable, since it is the solution of a linearization of P where, for every m_i with ties or incomparability in current set of top choices, we have set w_i strictly better than all the other women in these top choices. Also, the returned marriage is male optimal since we have applied the extended GS algorithm.

If L is not empty, it means that the next free man in L has several current top choices and more than one is unmarried.

(i) If there is a way to assign to the men currently in L different unmarried women from their current top choices, then these men make these proposals, that are certainly accepted by the women, since every woman receives a proposal from a different man. Therefore, we add to the current marriage these new pairs and we return the resulting marriage. Such a marriage is weakly stable and male optimal by construction.

(ii) If it is not possible to make the above assignment, the algorithm removes unfeasible women from the current top choices of the men until it is possible to make the assignment or until all unfeasible women have been removed. If it is possible to make the assignment described above, the algorithm adds to the current marriage these new pairs and returns the resulting marriage; otherwise, if all unfeasible women have been removed from the current top choices, the algorithm stops returning the string 'I don't know'.

The time complexity of the algorithm is $O(n^{\frac{5}{2}})$.

4. UNIQUENESS

In [1] a sufficient condition on the preferences is provided, that guarantees uniqueness of stable marriages in classical SMs. Such a condition, called *horizontal heterogeneity*, identifies classes of preferences that occur frequently in many real-life scenarios. A profile satisfies this condition when the agents have different preferences over the other sex, each agent has a different most preferred mate and in addition he is the most preferred by the mate.

More precisely, in [1] it is shown that, given an instance of an SM, where $M = \{m_i, i = 1, \ldots n\}$ (resp., $W = \{w_i, i = 1, \ldots n\}$) is the ordered set of the men (resp., women), if the profile satisfies the following conditions:

- $\forall m_i \in M : w_i >_{m_i} w_j,\ \forall j$,
- $\forall w_i \in W : m_i >_{w_i} m_j,\ \forall j$,

there is a unique stable marriage: $\mu(w_i) = m_i$, $\forall i$.

We can show that the same result holds also when the preferences are partially ordered. Moreover, we can guarantee uniqueness of weakly stable marriages by relaxing the conditions above as follows.

In an SMP, assume to order men and women in increasing number of their top choices. Let us denote with m_k the first man in the ordered list with more than one top choice, if he exists. If the profile satisfies the following conditions:

- $\forall m_i \in M$ with $m_i < m_k$, $m_i : w_i >_{m_i} w_j$, $\forall j$;
- $\forall m_i \in M$ with $m_i \geq m_k$, $(m_i : w_i >_{m_i}$ $(or \bowtie_{m_i})$ $w_j)$, $\forall j < i$, and $(m_i : w_i >_{m_i} w_j)$, $\forall j > i$;
- $\forall w_i \in W$, with $w_i < w_k$, $m_i >_{w_i} m_j$, $\forall j$;
- $\forall w_i \in W$, with $w_i \geq w_k$, $(w_i : m_i >_{w_i}$ $(or \bowtie_{w_i})$ $m_j)$, $\forall j < i$, and $(w_i : m_i >_{w_i} m_j)$, $\forall j > i$,

there is a unique weakly stable marriage: $\mu(w_i) = m_i$, $\forall i$. These conditions require that every man m_i (resp., woman w_i) with a single alternative, i.e., w_i (resp., m_i) has as unique top choice w_i (resp., m_i), and every m_i (resp., w_i) with more than one top choice has exactly one alternative that must be chosen in every weakly stable marriage, that is, w_i (resp., m_i).

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