Symmetry Breaking using Value Precedence

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Abstract. We present a comprehensive study of the use of value precedence constraints to break value symmetry. We first give a simple encoding of value precedence into ternary constraints that is both efficient and effective at breaking symmetry. We then extend value precedence to deal with a number of generalizations like wreath value and partial interchangeability. We also show that value precedence is closely related to lexicographical ordering. Finally, we consider the interaction between value precedence and symmetry breaking constraints for variable symmetries.

1 INTRODUCTION

Symmetry is an important aspect of many search problems. Symmetry occurs naturally in many problems (e.g. if we have two identical machines to schedule, or two identical jobs to process). Symmetry can also be introduced when we model a problem (e.g. if we name the elements in a set, we introduce the possibility of permuting their order). We must deal with symmetry or we will waste much time visiting symmetric solutions, as well as parts of the search tree which are symmetric to already visited parts. One simple but highly effective mechanism to deal with symmetry is to add constraints which eliminate symmetric solutions [3].

Two common types of symmetry are variable symmetries (which act just on variables), and value symmetries (which act just on values). With variable symmetries, we have a number of well understood symmetry breaking constraints. For example, many problems can be modelled using matrices of decision variables, in which the rows and columns of the matrix are symmetric and can be freely permuted. We can break such symmetry by lexicographically ordering the rows and columns [4]. Efficient propagators have therefore been developed for such ordering constraints [5, 2].

Value symmetries are also common. However, symmetry breaking for value symmetry is less well understood. In this paper, we study a common type of value symmetry where the values for variables are interchangeable. For example, if we assign orders to machines and two orders are identical, we can swap them in any schedule. We show here how to deal with such symmetry. In particular, we give a simple encoding that breaks all the symmetry introduced by interchangeable values. We also show how this is closely related to the lexicographical ordering constraint.

2 BACKGROUND

A constraint satisfaction problem consists of a set of variables, each with a domain of values, and a set of constraints specifying allowed combinations of values for given subsets of variables. A solution is an assignment of values to variables satisfying the constraints. Finite domain variables take values which are integers, or tuples of integers taken from some given finite set. Set variables takes values which are sets of integers. A set variable S has a lower bound $lb(S)$ for its definite elements and an upper bound $ub(S)$ for its definite and potential elements.

Constraint solvers typically explore partial assignments enforcing a local consistency property. We consider the two most common local consistencies: arc consistency and bound consistency. Given a constraint C, a *bound support* on C is an assignment of a value to each finite domain variable between its minimum and maximum and of a set to each set variable between its lower and upper bounds which satisfies C. A constraint C is *bound consistent* (*BC*) iff for each finite domain variable, its minimum and maximum values belong to a bound support, and for each set variable S, the values in $ub(S)$ belong to S in at least one bound support and the values in $lb(S)$ belong to S in all bound supports. Given a constraint C on finite domain variables, a *support* is assignment to each variable of a value in its domain which satisfies C. A constraint C on finite domains variables is *generalized arc consistent* (*GAC*) iff for each variable, every value in its domain belongs to a support.

A *variable symmetry* is a bijection on variables that preserves solutions. For example, in a model of the rehearsal problem (prob039 in CSPLib) in which we assign musical pieces to time slots, we can always invert any schedule. This is a reflection symmetry on the variables. A *value symmetry* is a bijection on values that preserves solutions. For example, in our model of the rehearsal problem, two pieces requiring the same musicians are interchangeable and can be freely permuted in any solution. Note that some authors call these *global* value symmetries as they act globally on values [12]. Finally, a pair of values are *interchageable* if we can swap them in any solution.

3 VALUE PRECEDENCE

We can break all symmetry between a pair of interchangeable values using a global value precedence constraint [9].

$$\textsc{Precedence}([v_j, v_k], [X_1, .., X_n])$$

This holds iff $\min\{i \mid X_i = v_j \vee i = n+1\} < \min\{i \mid X_i = v_k \vee i = n+2\}$. Law and Lee give a specialized algorithm for enforcing GAC on such a global constraint [9]. We show here that this is unnecessary. We can encode the constraint efficiently and effectively into a simple sequence of ternary constraints.

We introduce a sequence of 0/1 variables, where $B_i = 1$ if $X_l = v_j$ for some $l < i$. Value precedence prevents us assigning $X_i = v_k$ unless $B_i = 1$. To ensure this, we post the sequence of ternary constraints, $C(X_i, B_i, B_{i+1})$ for $1 \leq i \leq n$ which hold iff $X_i = v_j$ implies $B_{i+1} = 1$, $X_i \neq v_j$ implies $B_i = B_{i+1}$, and $B_i = 0$ implies $X_i \neq v_k$. We also set $B_1 = 0$. We assume that we can enforce GAC on each individual ternary constraint C using a table constraint

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or using primitives like implication and equality constraints. As the constraint graph is Berge-acyclic, enforcing GAC on the ternary constraints achieves GAC on $\text{PRECEDENCE}([v_j, v_k], [X_1, .., X_n])$. Since C is functional in its first two arguments and there are $O(n)$ ternary constraints, this takes $O(nd)$ time where d is the maximum domain size of the X_i. This is therefore optimal (as was Law and Lee's specialized algorithm). The incremental behavior is also good. Law and Lee's algorithm maintains three pointers, α, β and γ to save re-traversing the vector. A constraint engine will also perform well incrementally on this encoding provided it ignores constraints once they are entailed. Our experimental results support this claim.

4 MULTIPLE VALUE INTERCHANGEABILITY

Many problems involve *multiple* interchangeable values. For example, in a finite domain model of the social golfer problem (prob010 in CSPLib) in which we assign groups to golfers in each week, all values are interchangeable. To break all such symmetry, Law and Lee [9] propose the global constraint:

$$\text{PRECEDENCE}([v_1, .., v_m], [X_1, .., X_n])$$

This holds iff $\min\{i \mid X_i = v_i \vee i = n+1\} < \min\{i \mid X_i = v_j \vee i = n+2\}$ for all $1 \leq i < j < m$. To propagate this constraint, Law and Lee suggest decomposing it into pairwise precedence constraints of the form $\text{PRECEDENCE}([v_i, v_j], [X_1, .., X_n])$ for all $i < j$ [9]. Law has conjectured (page 77 of [8]), that such a decomposition does not hinder GAC propagation. We prove this is not the case.

Theorem 1 *Enforcing GAC on* $\text{PRECEDENCE}([v_1, .., v_m], [X_1, .., X_n])$ *is strictly stronger than enforcing GAC on* $\text{PRECEDENCE}([v_i, v_j], [X_1, .., X_n])$ *for all* $1 \leq i < j \leq m$.

Proof: Clearly it is at least as strong. To show strictness, consider $\text{PRECEDENCE}([1, 2, 3, 4], [X_1, X_2, X_3, X_4])$ with $X_1 \in \{1\}$, $X_2 \in \{1, 2\}$, $X_3 \in \{1, 3\}$, and $X_4 \in \{3, 4\}$. Then enforcing GAC on $\text{PRECEDENCE}([1, 2, 3, 4], [X_1, X_2, X_3, X_4])$ prunes 1 from the domain of X_2. However, $\text{PRECEDENCE}([i, j], [X_1, X_2, X_3, X_4])$ is GAC for all $1 \leq i < j \leq 4$. $\diamond$

We propose instead a simple encoding of $\text{PRECEDENCE}([v_1, .., v_m], [X_1, .., X_n])$ into a sequence of ternary constraints. We introduce $n+1$ finite domain variables, where Y_i records the greatest index of the values used so far in the precedence order. We then post a sequence of ternary constraints, $D(X_i, Y_i, Y_{i+1})$ for $1 \leq i \leq n$ which hold iff $X_i \neq v_j$ for any $j > Y_i + 1$, $Y_{i+1} = Y_i + 1$ if $X_i = v_{1+Y_i}$ and $Y_{i+1} = Y_i$ otherwise. We set $Y_1 = 0$. Again, we achieve GAC on the global constraint simply by enforcing GAC on the individual ternary constraints. This takes $O(nmd)$ time. By comparison, enforcing GAC on the decomposition into precedence constraints between all pairs of interchangeable values takes $O(nm^2d)$ time.

5 PARTIAL INTERCHANGEABILITY

We may have a partition on the values, and values within each partition are interchangeable. For example, in the model of the rehearsal problem in which we assign musical pieces to time slots, we can partition the musical pieces into those requiring the same musicians. Suppose the values are divided into s equivalence classes, then we can break all symmetry with the global constraint:

$$\text{PRECEDENCE}([[v_{1,1}, .., v_{1,m_1}], .., [v_{s,1}, .., v_{s,m_s}]], [X_1, .., X_n])$$

This holds iff $\min\{i \mid X_i = v_{j,k} \vee i = n+1\} < \min\{i \mid X_i = v_{j,k+1} \vee i = n+2\}$ for all $1 \leq j \leq s$ and $1 \leq k < m_j$. This global constraint can be decomposed into s precedence constraints, one for each equivalence class. However, such decomposition hinders propagation.

Theorem 2 *Enforcing GAC on* $\text{PRECEDENCE}([[v_{1,1}, .., v_{1,m_1}], .., [v_{s,1}, .., v_{s,m_s}]], [X_1, .., X_n])$ *is strictly stronger than enforcing GAC on* $\text{PRECEDENCE}([v_{i,1}, .., v_{i,m_i}], [X_1, .., X_n])$ *for* $1 \leq i \leq s$.

Proof: Clearly it is at least as strong. To show strictness, consider $\text{PRECEDENCE}([[1, 2, 3], [4, 5, 6]], [X_1, .., X_5])$ with $X_1, X_2, X_3 \in \{1, 2, 3, 4, 5, 6\}$, $X_4 \in \{3\}$ and $X_5 \in \{6\}$. Then $\text{PRECEDENCE}([[1, 2, 3], [4, 5, 6]], [X_1, .., X_5])$ is unsatisfiable. However, $\text{PRECEDENCE}([1, 2, 3], [X_1, .., X_5])$ and $\text{PRECEDENCE}([4, 5, 6], [X_1, .., X_5])$ are both GAC. $\diamond$

Decomposition is again unnecessary as we can encode the global constraint into a sequence of ternary constraints. The idea is to keep a tuple recording the greatest value used so far within each equivalence class as we slide down the sequence. We introduce $n+1$ new finite-domain variables, Y_i whose values are s-tuples. We write $Y_i[j]$ to indicate the jth component of the tuple. We then post a sequence of ternary constraints, $E(X_i, Y_i, Y_{i+1})$ for $1 \leq i \leq n$ which hold iff for all $1 \leq j \leq s$ we have $X_i \neq v_{j,k}$ for all $k > Y_i[j] + 1$, $Y_{i+1}[j] = Y_i[j] + 1$ if $X_i = v_{j, Y_i[j]+1}$ and $Y_{i+1}[j] = Y_i[j]$ otherwise. The value taken by $Y_{i+1}[j]$ is the largest index within the jth equivalence class used up to X_i. Since E is functional in its third argument, this takes $O(nde)$ time where $e = \prod_{i \leq s} m_i$. Note that if all values are interchangeable with each other, then $s = 1$ and $m_1 = m$, and enforcing GAC takes $O(nmd)$ time. Similarly, for just one pair of interchangeable values, $s = n - 1$, $m_1 = 2$ and $m_i = 1$ for $i > 1$, and enforcing GAC takes $O(nd)$ time.

6 WREATH VALUE INTERCHANGEABILITY

Wreath value interchangeability [13] is a common type of symmetry in problems where variables are assigned a pair of values from $D_1 \times D_2$, values in D_1 are fully interchangeable, and for a fixed value in D_1, values in D_2 are interchangeable as well. For example, if we are scheduling a conference, the days of the week might be interchangeable, and given a particular day, the meeting rooms might then be interchangeable. For simplicity, we assume the same precedence ordering is used for the values in D_2 for every fixed value in D_1. However, we can relax this assumption without difficulty.

We can break all the symmetry of wreath-value interchangeability with the global constraint:

$$\text{PRECEDENCE}([u_1, .., u_m, [v_1, .., v_p]], [X_1, .., X_n])$$

This holds iff $\min\{i \mid X_i[1] = u_j \vee i = n+1\} < \min\{i \mid X_i[1] = u_{j+1} \vee i = n+2\}$ for all $1 \leq j < m$, and $\min\{i \mid X_i = \langle u_j, v_k \rangle \vee i = n+1\} < \min\{i \mid X_i = \langle u_j, v_{k+1} \rangle \vee i = n+2\}$ for all $1 \leq j \leq m$ and $1 \leq k < p$. This can be decomposed into precedence constraints of the form $\text{PRECEDENCE}([\langle u_i, v_j \rangle, \langle u_k, v_l \rangle], [X_1, .., X_n])$, but this hinders propagation.

Theorem 3 *Enforcing GAC on* $\text{PRECEDENCE}([u_1, .., u_m, [v_1, .., v_p]], [X_1, .., X_n])$ *is strictly stronger than enforcing GAC on* $\text{PRECEDENCE}([\langle u_i, v_j \rangle, \langle u_k, v_l \rangle], [X_1, .., X_n])$ *for all* $1 \leq i < k \leq m$*, and for all* $i = j$*,* $1 \leq i, j \leq m$*,* $1 \leq k < l \leq p$.

Proof: Clearly it is at least as strong. To show strictness, consider $\text{PRECEDENCE}([1, 2, [3, 4]], [X_1, X_2, X_3, X_4])$

with $X_1 \in \{\langle 1,3\rangle\}$, $X_2 \in \{\langle 1,3\rangle, \langle 1,4\rangle\}$, $X_3 \in \{\langle 1,3\rangle, \langle 2,3\rangle\}$, $X_4 \in \{\langle 2,3\rangle, \langle 2,4\rangle\}$, Then enforcing GAC on $\text{PRECEDENCE}([[1,2,[3,4]],[X_1,X_2,X_3,X_4])$ prunes $\langle 1,3\rangle$ from X_2. However, $\text{PRECEDENCE}([\langle u,v\rangle,\langle w,z\rangle],[X_1,X_2,X_3,X_4])$ is GAC for all $1 \leq u < w \leq 2$ and $3 \leq v,z \leq 4$, and for all $u = w$, $1 \leq u,w \leq 2$, $3 \leq v < z \leq 4$. $\diamond$

We again can propagate such a precedence constraint using a simple encoding into ternary constraints. We have a finite domain variable which records the greatest pair used so far down the sequence. We can then encode $\text{PRECEDENCE}([u_1,..,u_m,[v_1,..,v_p]],[X_1,..,X_n])$ by means of a sequence of ternary constraints, $F(X_i,Y_i,Y_{i+1})$ for $1 \leq i \leq n$ which hold iff $X_i[1] \neq u_j$ for all $j > Y_i[1]+1$, if $X_i[1] = u_{Y_i[1]}$ then $X_i[2] \neq v_j$ for all $j > Y_i[2]+1$, $Y_{i+1} = \langle Y_i[1]+1, 1\rangle$ if $X_i[1] = u_{Y_i[1]}+1$ and $Y_i[2] = m$, $Y_{i+1} = \langle Y_i[1], Y_i[2]+1\rangle$ if $X_i = \langle u_{Y_i[1]}, v_{Y_i[2]}+1\rangle$, and $Y_{i+1} = Y_i$ otherwise. Enforcing GAC using this encoding takes $O(ndmp)$ time. The extension to wreath value partial interchangeability and to wreath value interchangeability over k-tuples where $k > 2$ are both straight forwards.

7 MAPPING INTO VARIABLE SYMMETRY

An alternative way to deal with value symmetry is to convert it into variable symmetry [4, 10]. We introduce a matrix of 0/1 variables where $B_{ij} = 1$ iff $X_i = j$. We assume the columns of this 0/1 matrix represent the interchangeable values, and the rows represent the original finite domain variables. We now prove that value precedence is equivalent to lexicographically ordering the columns of this 0/1 matrix, but that channelling into 0/1 variables hinders propagation.

Theorem 4 $\text{PRECEDENCE}([v_1,..,v_m],[X_1,..,X_n])$ *is equivalent to* $X_i = v_j$ *iff* $B_{i,j} = 1$ *for* $0 < j \leq m$ *and* $0 < i \leq n$, *and* $[B_{1,j},..,B_{n,j}] \geq_{\text{lex}} [B_{1,j+1},..,B_{n,j+1}]$ *for* $0 < j < m$.

Proof: By induction on n. In the base case, $n = 1$, $X_1 = v_1$, $B_{1,1} = 1$ and $B_{1,j} = 0$ for $1 < j \leq m$. In the step case, suppose the value precedence constraint holds for a ground assignment in which v_k is the largest value used so far. Consider any extension with $X_{n+1} = v_l$. There are two cases. In the first, $l = k+1$. The lth column is thus $[0,..,0,1]$. This is lexicographically less than all previous columns. In the other case, $l \leq k$. Adding a new row with a single 1 in the lth column does not change the ordering between the $l-1$th, lth and $l+1$th columns. The proof reverses in a similar way. $\diamond$

We could thus impose value precedence by channelling into an 0/1 matrix model and using lexicographical ordering constraints [4]. However, this decomposition hinders propagation as the lexicographical ordering constraints do not exploit the fact that the 0/1 matrix has a single non-zero entry per row. Indeed, even if we add this implied constraint to the decomposition, propagation is hindered. It is thus worth developing a specialized propagator for the global PRECEDENCE constraint.

Theorem 5 *GAC on* $\text{PRECEDENCE}([v_1,..,v_m],[X_1,..,X_n])$ *is strictly stronger than GAC on* $X_i = v_j$ *iff* $B_{i,j} = 1$ *for* $0 < j \leq m$ *and* $0 < i \leq n$, *GAC on* $[B_{1,j},..,B_{n,j}] \geq_{\text{lex}} [B_{1,j+1},..,B_{n,j+1}]$ *for* $0 < j < m$, *and GAC on* $\sum_{j=1}^{m} B_{i,j} = 1$ *for* $0 < i \leq n$.

Proof: Clearly it is as strong. To show strictness, consider $X_1 = 1$, $X_2 \in \{1,2,3\}$, $X_3 = 3$, $B_{1,1} = B_{3,3} = 1$, $B_{1,2} = B_{1,3} = B_{3,1} = B_{3,2} = 0$, and $B_{2,1}, B_{2,2}, B_{2,3} \in \{0,1\}$. Then the decomposition is GAC. However, enforcing GAC on the value precedence constraint will prune 1 and 3 from X_2. $\diamond$

It is not hard to show that partial value interchangeability corresponds to partial column symmetry in the corresponding 0/1 matrix model. As with full interchangeability, we obtain more pruning with a specialized propagator than with lexicographical ordering constraints.

8 SURJECTION PROBLEMS

A surjection problem is one in which each value is used at least once. Puget converts value symmetries on surjection problems into variable symmetries by channelling into dual variables which record the first index using a value [11]. For interchangeable values, this gives $O(nm)$ binary symmetry breaking constraints: $X_i = j \rightarrow Z_j \leq i$, $Z_j = i \rightarrow X_i = j$, and $Z_k < Z_{k+1}$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k < m$. Any problem can be made into a surjection by introducing m additional new variables to ensure each value is used once. In this case, Puget's symmetry breaking constraints ensure value precedence. However, they may not prune all possible values. Consider $X_1 = 1$, $X_2 \in \{1,2\}$, $X_3 \in \{1,3\}$, $X_4 \in \{3,4\}$, $X_5 = 2$, $X_6 = 3$, $X_7 = 4$, $Z_1 = 1$, $Z_2 \in \{2,5\}$, $Z_3 \in \{3,4,6\}$, and $Z_4 \in \{4,7\}$. Then all the binary implications are AC. However, enforcing GAC on $\text{PRECEDENCE}([1,2,3,4],[X_1,\ldots,X_7])$ will prune 1 from X_2.

9 SET VARIABLES

We also meet interchangeable values in problems involving set variables. For example, in a set variable model of the social golfers problem in which we assign a set of golfers to the groups in each week, all values are interchangeable. We can break all such symmetry with value precedence constraints. For set variables, $\text{PRECEDENCE}([v_1,..,v_m],[S_1,..,S_n])$ holds iff $\min\{i \mid (v_j \in S_i \wedge v_k \notin S_i) \vee i = n+1\} < \min\{i \mid (v_k \in S_i \wedge v_j \notin S_i) \vee i = n+2\}$ for all $1 \leq j < k \leq m$ [9]. That is, the first time we distinguish between v_j and v_k (because both values don't occur in a given set variable), we have v_j occurring and not v_k. This breaks all symmetry as we cannot now swap v_j for v_k. Law and Lee again give a specialized propagator for enforcing BC on $\text{PRECEDENCE}([v_j,v_k],[S_1,..,S_n])$. We prove here that this decomposition hinders propagation.

Theorem 6 *Enforcing BC on* $\text{PRECEDENCE}([v_1,..,v_m],[S_1,..,S_n])$ *is strictly stronger than enforcing BC on* $\text{PRECEDENCE}([v_i,v_j],[S_1,..,S_n])$ *for all* $1 \leq i < j \leq m$.

Proof: Clearly it is at least as strong. To show strictness, consider $\text{PRECEDENCE}([0,1,2],[S_1,S_2,S_3,S_4,S_5])$ with $\{\} \subseteq S_1 \subseteq \{0\}$, $\{\} \subseteq S_2 \subseteq \{1\}$, $\{\} \subseteq S_3 \subseteq \{1\}$, $\{\} \subseteq S_4 \subseteq \{0\}$, and $S_5 = \{2\}$. Then enforcing BC on $\text{PRECEDENCE}([0,1,2],[S_1,S_2,S_3,S_4,S_5])$ sets S_1 to $\{0\}$. However, $\text{PRECEDENCE}([i,j],[S_1,S_2,S_3,S_4,S_5])$ is BC for all $0 \leq i < j \leq 2$. $\diamond$

As with finite domain variables, we do need to introduce a new propagator nor to decompose this global constraint. We view each set variable in terms of its characteristic function (a vector of 0/1 variables). This gives us an n by d matrix of 0/1 variables with column symmetry in the d dimension. Unlike the case with finite domain variables, rows can now have any sum. We can break all such column symmetry with a simple lexicographical ordering constraint [4]. If we use the lex chain propagator [2], we achieve BC on the original value precedence constraint in $O(nd)$ time.

In many constraint solvers, set variables also have restrictions on their cardinality. Unfortunately, adding such cardinality information makes value precedence intractable to propagate.

Theorem 7 *Enforcing BC on* PRECEDENCE($[v_1, .., v_m], [S_1, .., S_n]$) *where set variables have cardinality bounds is NP-hard.*

Proof: We give a reduction from a 1-in-3 SAT problem in N Boolean variables, x_1 to x_N and M positive clauses. We let $n = 2N + M$, $m = 2N$ and $v_i = i$. To encode the truth assignment which satisfies the 1-in-3 SAT problem, we have $S_{2i} = \{2i, 2i + 1\}$ and $\{2i\} \subseteq S_{2i+1} \subseteq \{2i, 2i+1\}$ for $1 \leq i \leq N$. S_{2i+1} will be $\{2i\}$ iff x_i is false and $\{2i, 2i+1\}$ otherwise. The remaining M CSP variables represent the M clauses. Suppose the ith clause is $x_j \vee x_k \vee x_l$, We let $S_{2N+i} \subseteq \{2j + 1, 2k + 1, 2l + 1\}$. Finally, we force S_{2N+i} to take two of the values $2j + 1$, $2k + 1$, $2l + 1$ from its upper bound. Value precedence only permits this if exactly two out of S_{2j}, S_{2k} and S_{2l} take the set value representing "false". The global value precedence constraint thus has bound support iff the corresponding 1-in-3 SAT problem is satisfiable. Hence, enforcing BC is NP-hard. $\diamond$

10 VALUE AND VARIABLE SYMMETRY

In many situations, we have both variable and value symmetry. Can we safely combine together symmetry breaking constraints for variable symmetries and value symmetries? Do we break all symmetry?

INTERCHANGEABLE VARIABLES

Suppose that all n variables and m values are interchangeable. We can safely combine a global value precedence constraint (which breaks all the value symmetry) with a simple ordering constraint (which breaks all the variable symmetry). However, this does not break all symmetry. For example, $[1, 2, 2]$ and $[1, 1, 2]$ are symmetric since inverting the first sequence and permuting 1 with 2 gives the second sequence. However, both sequences satisfy the value precedence and ordering constraints. We can break all symmetry with the global constraint INCREASINGSEQ($[X_1, .., X_n]$) which holds iff $X_1 = v_1$, $X_{i+1} = X_i$ or ($X_i = v_j$ and $X_{i+1} = v_{j+1}$) for all $0 < i < n$, and $|\{i \mid X_i = v_j\}| \leq |\{i \mid X_i = v_{j+1}\}|$ for all $0 < j < m$. That is, the values and the number of occurrences of each value increase monotonically. We can also have the values increasing but the number of occurrences decreasing. One way to propagate this constraint is to consider the corresponding 0/1 matrix model. The INCREASINGSEQ constraint lexicographically orders the rows and columns, as well as ordering the columns by their sums.

Consider, now, (partially) interchangeable set variables taking (partially) interchangeable values. This corresponds to an 0/1 matrix with (partial) row and (partial) column symmetry. Unfortunately, enforcing breaking all row and column symmetry is NP-hard [1]. We cannot expect therefore to break all symmetry when we have interchangeable set variables and interchangeable values. We can break symmetry partially by lexicographical ordering the rows and columns of the corresponding 0/1 matrix.

MATRIX SYMMETRY

Variables may be arranged in a matrix which has row and/or column symmetry [4]. Lexicographical ordering constraints will break such symmetries. Suppose that values are also (partially) interchangeable. As lexicographical ordering constraints can be combined in any number of dimensions [4], and as value precedence is equivalent to lexicographically ordering the 0/1 model, we can safely combine value precedence and row and column symmetry breaking constraints.

VARIABLE REFLECTION SYMMETRY

Suppose we have a sequence of $2n$ variables with a reflection symmetry. Then we can break all such symmetry with the lexicographical ordering constraint: $[X_1, \ldots, X_n] \leq_{\text{lex}} [X_{2n}, \ldots, X_{n+1}]$. For an odd length sequence, we just miss out the middle element. If values are also (partially) interchangeable, we can combine such a reflection symmetry breaking constraint with precedence constraints. Whilst these symmetry breaking constraints are compatible, they do not break all symmetry. For example, $[1, 2, 1, 1, 2]$ and $[1, 2, 2, 1, 2]$ are symmetric since inverting the first sequence and permuting 1 with 2 gives the second sequence. However, both sequences satisfy all symmetry breaking constraints.

VARIABLE ROTATION SYMMETRY

Suppose we have a sequence of n variables with a rotation symmetry. That is, if we rotate the sequence, we obtain a symmetric solution. We can break all such symmetry with the constraints: $[X_1, \ldots, X_n] \leq_{\text{lex}} [X_i, \ldots, X_n, X_1, \ldots X_{i-1}]$ for $1 < i \leq n$. If values are also (partially) interchangeable, then we can combine such symmetry breaking constraints with precedence constraints. Whilst these symmetry breaking constraints are compatible, they do not break all symmetry. For example, $[1, 1, 2, 1, 2]$ and $[1, 2, 1, 2, 2]$ are symmetric since rotating the first sequence by 2 elements and permuting 1 with 2 gives the second sequence. However, both sequences satisfy all symmetry breaking constraints.

11 EXPERIMENTAL RESULTS

To test the efficiency and effectiveness of these encodings of value precedence constraints, we ran a range of experiments. We report results here on Schur numbers (prob015 in CSPLib). This problem was used by Law and Lee in the first experimental study of value precedence [9]. We have observed similar results in other domains like the social golfers problem and Ramsey numbers (prob017 in CSPLib).

The Schur number $S(k)$ is the largest integer n for which the interval $[1, n]$ can be partitioned into k sum-free sets. S is sum-free iff $\forall a, b, c \in S \,.\, a \neq b + c$. Schur numbers are related to Ramsey numbers, $R(n)$ through the identity: $S(n) \leq R(n) - 2$. Schur numbers were proposed by the famous German mathematician Isaai Schur in 1918. $S(4)$ was open until 1961 when it was first calculated by computer. $S(3)$ is 13, $S(4)$ is 44, and $160 \leq S(5) \leq 315$. We consider the corresponding decision problem, $S(n, k)$ which asks if the interval $[1, n]$ can be partitioned into k sum-free sets. A simple model of this uses n finite domain variables with k interchangeable values.

Results are given in Table 1. The model *all* uses a single global precedence constraint to break all value symmetry. The model *adjacent* uses the method proposed by Law and Lee in [9] which posts $O(k)$ precedence constraints between adjacent interchangeable values. The model *none* use no precedence constraints. We coded the problem using the finite domain library in SICSTUS 3.12.3, and ran it on an AMD Athlon Dual Core 2.2GHz processor with 1 GB RAM.

The results show the benefits of a global value precedence constraint. With a few interchangeable values, we see the same pruning using *adjacent* as *all*. However, we observe better runtimes with the

problem	value symmetry breaking								
$S(n,k)$	none			adjacent values			all values		
	c	**b**	**t**	**c**	**b**	**t**	**c**	**b**	**t**
$S(13,3)$	126	276	0.02	360	**46**	**0.01**	243	**46**	**0.01**
$S(13,4)$	126	134400	16.43	477	**2,112**	1.48	243	**2,112**	**0.85**
$S(13,5)$				777	210,682	20.80	243	**6,606**	**11.88**
$S(13,6)$				879	309,917	79.60	243	**1,032**	**42.51**
$S(14,3)$	147	456	0.02	399	**76**	**0.02**	273	**76**	**0.02**
$S(14,4)$	147	46,1376	39.66	525	**8,299**	3.06	273	**8,299**	**1.96**
$S(14,5)$				816	813,552	66.83	273	**58,558**	**40.35**
$S(14,6)$				1,731	250,563	348.06	273	**57,108**	**197.39**
$S(15,3)$	168	600	0.03	438	**100**	**0.02**	303	**100**	**0.02**
$S(15,4)$	168	1,044,984	101.36	573	**17,913**	7.92	303	**17,913**	**4.73**
$S(15,5)$				855	1,047,710	259.15	303	**194,209**	**145.97**
$S(15,6)$									

Table 1. Decision problem associated with Schur numbers: **c**onstraints posted, **b**acktracks and **t**imes to find all solutions in seconds to $S(n,k)$. Blank entries are for problems not solved within the 10 minute cut off. Results are similar to find first solution.

all model as it encodes into fewer ternary constraints ($O(n)$ versus $O(nk)$). With more interchangeable values (e.g. $k > 4$), we observe both better runtimes and more pruning with the single global precedence constraint in the *all* model. The encoding of this global constraint into ternary constraints appears therefore to be an efficient and an effective mechanism to deal with interchangeable values.

12 RELATED WORK

Whilst there has been much work on symmetry breaking constraints for variable symmetries, there has been less on value symmetries. Law and Lee formally defined value precedence [9]. They also gave specialized propagators for breaking value precedence for a pair of interchangeable values. Gent proposed the first encoding of value precedence constraint [6]. However, it is uncertain what consistency is achieved as the encoding indexes with finite domain variables.

A number of methods that modify the underlying solver have been proposed to deal with value symmetry. Van Hentenryck *et al.* gave a labelling schema for breaking all symmetry with interchangeable values [7]. Inspired by this method, Roney-Dougal *et al.* gave a polynomial method to construct a GE-tree, a search tree without value symmetry [12]. Finally, Sellmann and van Hentenryck gave a $O(nd^{3.5} + n^2d^2)$ dominance detection algorithm for breaking all symmetry when both variables and values are interchangeable [13].

There are a number of earlier (and related) results about the tractability of symmetry breaking. Crawford *et al.* prove that breaking all symmetry in propositional problems is NP-hard in general [3]. Bessiere *et al.* prove that the special case of breaking all row and column symmetry for variables in a matrix model is NP-hard [1]. Sellmann and van Hentenryck prove a closely related result that dominance detection for breaking all symmetry with set variables and values that are interchangeable is NP-hard [13].

13 CONCLUSIONS

We have presented a detailed study of the use of value precedence constraints to break value symmetry. We first gave a simple encoding of value precedence into ternary constraints that is both efficient and effective. We then extended value precedence to deal with a number of generalizations like wreath value and partial interchangeability. We have also shown how value precedence is closely related to lexicographical ordering. Finally, we considered the interaction between value precedence and other symmetry breaking constraints. There are a number of interesting open questions. For example, how does value precedence interact with variable and value ordering heuristics?

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