

Strategyproof and Proportionally Fair Facility Location

Haris Aziz¹, Alexander Lam¹, Barton E. Lee², and Toby Walsh^{1,3}

¹ UNSW Sydney, Australia.

{haris.aziz, alexander.lam1, t.walsh}@unsw.edu.au

² ETH Zürich, Switzerland.

bartonlee@ethz.ch

³ Data61 CSIRO, Australia.

Abstract.

We focus on a simple, one-dimensional collective decision problem (often referred to as the facility location problem) and explore issues of strategyproofness and proportional fairness. We present several characterization results for mechanisms that satisfy strategyproofness and varying levels of proportional fairness. We also characterize one of the mechanisms as the unique equilibrium outcome for any mechanism that satisfies natural fairness and monotonicity properties. Finally, we identify strategyproof and proportionally fair mechanisms that provide the best welfare-optimal approximation among all mechanisms that satisfy the corresponding fairness axiom.

1 Introduction

Facility location problems are ubiquitous in society and capture various collective scenarios. Examples include electing political representatives (Border and Jordan, 1983; Feldman, Fiat and Golomb, 2016; Moulin, 1980), selecting policies (Barberà and Nicolò, 2021; Dragu and Laver, 2019; Kurz, Maaser and Napel, 2017), deciding how to allocate a public budget (Freeman, Pennock, Peters and Vaughan, 2021), and deciding the location or services provided by public facilities (Schummer and Vohra, 2002). Two key concerns in such problems are that the selection process may be vulnerable to strategic manipulations and/or fail to guarantee “fair” outcomes. In this paper, we simultaneously examine the issues of strategyproofness and fairness for the facility location problem.

In the facility location problem, each agent is viewed as a point on the unit interval. Depending on the motivating setting, the point could reflect the agent’s physical location, political position, or social preference. Each agent has single-peaked preferences and prefers the collective outcome to be near their own position. The goal of the collective decision problem is to take agents’ preferences (positions) into account to find a reasonable collective outcome (the location of the facility).

The facility location problem (or the one-dimensional collective decision problem) is one of the most fundamental problems in economics, computer science, and operations research. It takes a central place in social choice theory as single-peaked preferences are one of the key preference restrictions that circumvent the infamous Gibbard-Satterthwaite theorem⁴ (Gibbard, 1973; Satterthwaite, 1975)—this striking result was proven by Moulin (1980). When agents have single-peaked preferences, the mechanism that returns the median voter’s position is unanimous, non-dictatorial, and strategyproof. This seminal result has been discussed in hundreds of papers. Despite the importance of the median mechanism for the facility location problem, it does not satisfy several fairness concepts that are inspired from the theory of fair division and proportional representation. We focus on the following research questions.

⁴The Gibbard-Satterthwaite theorem says that in general social choice, no unanimous and non-dictatorial voting mechanism is strategyproof.

For the facility location problem, what are natural fairness concepts? How well can these fairness concepts be achieved by strategyproof mechanisms? For strategyproof mechanisms that satisfy one of these fairness concepts, which mechanism performs optimally in terms of social welfare? Which mechanisms achieve fairness in equilibrium?

Our contributions are four-fold. First, we consolidate a number of fairness axioms from the literature, explicitly describe their relations, and establish the compatibility of strategyproofness—and, in some cases, incompatibility—with these fairness concepts. We propose a new concept called *proportional fairness (PF)* that is based on the idea that the distance of a facility from a group of agents should depend both on the size of the group as well as how closely the agents are clustered. We also analyze existing axioms from the literature on fair division, participatory budgeting, and proportional representation such as *proportionality*, *unanimous fair share (UFS)*, *individual fair share (IFS)*, and unanimity. Our PF axiom is the strongest of these; Figure 1 describes the relationship between all the fairness axioms that we study.

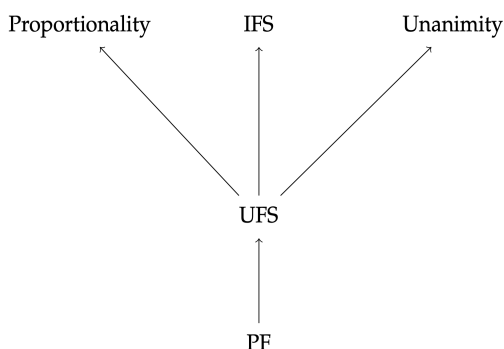


Fig. 1: Relations between axioms. An arrow from (A) to (B) denotes that (A) implies (B). All relations are strict.

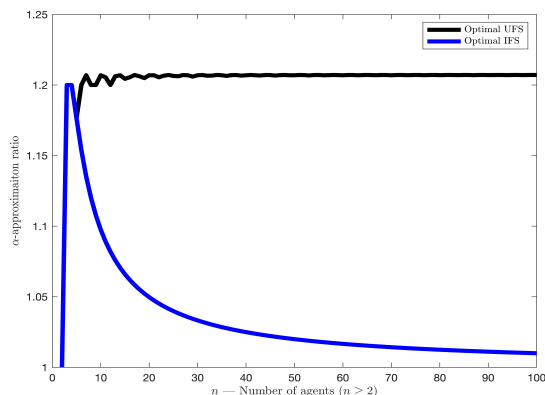


Fig. 2: Approximation ratio as a function of n . The welfare-optimal UFS (IFS) mechanism is also strategyproof, anonymous, and unanimous. The welfare-optimal UFS mechanism also satisfies proportionality and PF. The welfare-optimal IFS mechanism does not satisfy proportionality.

Second, we present two characterization results. We characterize the family of strategyproof mechanisms that satisfy unanimity, anonymity, and IFS. We then identify a specific mechanism, called the Uniform Phantom mechanism, that uniquely satisfies strategyproofness, unanimity, and proportionality. We also prove that the Uniform Phantom mechanism uniquely satisfies strategyproofness and UFS. Since we show that the Uniform Phantom mechanism also satisfies PF (and because PF implies UFS), we obtain as a corollary that the Uniform Phantom mechanism is the only strategyproof mechanism satisfying PF. Therefore, within the class of strategyproof mechanisms, PF and UFS collapse to the same property. In contrast, we show that within the class of strategyproof mechanisms, IFS is markedly weaker.

Third, we consider the fairness of outcomes under strategic behavior when a mechanism is not strategyproof. We prove that if a mechanism satisfies continuity, strict monotonicity, and UFS, then a pure Nash equilibrium exists, and every (pure) equilibrium under the mechanism satisfies UFS with respect to agents’

true locations. Furthermore, for such mechanisms,⁵ the equilibrium facility location is unique and coincides with the facility location of the Uniform Phantom mechanism. Thus, our equilibrium analysis of continuous, strictly monotonic, and UFS mechanisms provides an alternative characterization of the Uniform Phantom mechanism.

Lastly, we take an approximate mechanism design perspective (Nisan and Ronen, 2001; Procaccia and Tennenholtz, 2013) and explore how well the maximum social welfare can be approximated when fairness axioms are imposed. We identify “optimal and fair” mechanisms that provide the best possible approximation of the maximum social welfare among all mechanisms that satisfy the corresponding fairness axiom (such as IFS and UFS). The mechanisms we identify are also strategyproof; in this sense, the fairness axioms impose a greater cost on the approximation guarantees than the strategyproofness requirement. Figure 2 illustrates the approximation results.

1.1 Related literature

Facility location problems. The facility location problem has been studied extensively in operations research, economics, and computer science. As is common in the economics literature, our paper takes a mechanism design approach to the facility location. We assume an incomplete information setting, whereby agents have privately known location and can (mis)report their location. The problem is to design a mechanism that is strategyproof and achieves a “desirable” facility location with respect to the agents’ true locations.⁶ Moulin’s (1980) seminal work characterizes the family of strategyproof and Pareto efficient mechanisms when agents have single-peaked preferences.⁷ In our paper, agents have single-peaked preferences that are also *symmetric*, i.e., agents prefer the facility to be located closer to their location regardless of whether it is to left or right of their location; therefore, our setting is closer to Border and Jordan (1983). Border and Jordan characterize a strict subfamily of strategyproof mechanisms, which includes the family of strategyproof and unanimous mechanisms.⁸

Since Moulin (1980) and Border and Jordan (1983), numerous scholars have explored open-questions related to these characterizations (see, e.g., Barberà and Jackson, 1994; Barberà et al., 1998; Ching, 1997; Jennings et al., 2021; Massó and Moreno De Barreda, 2011; Peremans et al., 1997; Weymark, 2011). Others have explored extensions and variations of the facility location problem. For example, Nehring and Puppe (2006, 2007) relax the assumption that agents have single-peaked preferences; Miyagawa (1998, 2001) and Ehlers (2002, 2003) extend the facility location problem to consider locating multiple facilities; Aziz et al. (2020a,b) introduce capacity constraints into the problem; Jackson and Nicolò (2004) introduce interdependent utilities; Cantala (2004) introduce an outside option; and Schummer and Vohra (2002) extend the facility location problem to a network setting.⁹ Our paper contributes to this literature by formalizing a hierarchy

⁵One mechanism in this class is the Average mechanism, which locates the facility at the average of all agents’ reported locations.

⁶There is an extensive literature in operations research and computer science that studies the facility location problem within a complete information setting. These literatures largely focus on issues of computational complexity and approximation and, therefore, are not directly relevant to the present paper (for an overview, see Brandeau and Chiu, 1989; Zanjirani Farahani and Hekmatfar, 2009).

⁷More specifically, Moulin provides three characterizations. Via three (distinct) families of “Phantom mechanisms,” he characterizes: (i) all strategyproof and anonymous mechanisms, (ii) all strategyproof, anonymous, and Pareto efficient mechanisms, and (iii) all strategyproof mechanisms.

⁸Massó and Moreno De Barreda (2011) formalize the connection between the mechanism design problem in settings where agents have single-peaked preferences and where they must, in addition, be symmetric.

⁹For a recent survey of computational social choice literature on facility location problems, see Chan et al. (2021).

of “proportionally fair” axioms for the facility location problem and characterizing strategyproof and fair mechanisms. Additionally, in Section 5, we explore the equilibrium properties of non-strategyproof mechanisms.

Fairness in collective decision problems. Issues of fairness in collective decision problems have been studied in a variety of contexts (see, e.g., [Dummett, 1997](#); [Mill, 1861](#); [Nash, 1950, 1953](#); [Rawls, 1971](#); [Sen, 1980](#); [Shapley, 1953](#); [Yaari, 1981](#)). Most closely related to the present paper are the social choice and computational social choice literatures (for an overview, see [Arrow et al., 2010](#); [Aziz et al., 2019b](#); [Endriss, 2017](#); [Faliszewski et al., 2017](#); [Klamler, 2010](#); [Laslier and Sanver, 2010](#)). We formalize a hierarchy of fairness axioms for the facility location problem that are conceptually related to proportional representation. Two of our fairness axioms (IFS and UFS) are translations of the “individual fair share” and “unanimous fair share” axioms, which appear in fair division and participatory budgeting problems ([Aziz et al., 2019a](#); [Moulin, 2003](#)), into the facility location problem. In addition, we utilize a natural axiom of proportional representation, called “proportionality”, which is explored in the context of participatory budgeting by [Freeman et al. \(2021\)](#). Beyond translating existing notions of fairness into the facility location problem, we also introduce the new axiom of “Proportional Fairness” that is stronger than all of the aforementioned axioms.

Our approach contrasts with a number of facility location papers that attempt to obtain outcomes that achieve (or approximate) the egalitarian outcome, i.e., maximizing the utility of the worst off agent (see, e.g., [Procaccia and Tennenholtz, 2013](#)).¹⁰ [Mulligan \(1991\)](#) notes that the egalitarian objective is sensitive to extreme locations and recommends distributional equality as an underlying principle for considering equality measures. When placing multiple facilities, several new concepts have been proposed for capturing proportional fairness concerns (see, e.g., [Bigman and Fofack, 2000](#); [Jung et al., 2020](#)). However, these concepts are equivalent to weak Pareto optimality or unanimity when there is only one facility. For the single-facility problem, [Zhou et al. \(2022\)](#) recently examined the issue of welfare guarantees for groups of agents. Our approach and results differ in that we consider the classic facility location problem whereas [Zhou et al.](#) overlay it with additional information that places agents in predetermined groups.

In the context of the facility location problem, our paper characterizes strategyproof and “fair” mechanisms. Some of our results directly relate to those of [Freeman et al. \(2021\)](#). In the context of participatory budgeting setting, [Freeman et al.](#) explore the problem of designing strategyproof mechanisms that satisfy proportionality. One of their key results (Proposition 1) applies to the facility location problem and shows that there is a unique anonymous, continuous, strategyproof and proportional mechanism, which is called the Uniform Phantom mechanism.^{11,12} Our paper differs in focus and provides a broader treatment of issues of fairness and strategyproofness in facility location problems; for example, we characterize a larger family of strategyproof mechanisms that satisfy the weaker fairness axiom of IFS. In addition, one of our results strengthens [Freeman et al.’s](#) Proposition 1 by showing that the anonymity axiom is redundant in their characterization. We also provide an alternative characterization of the Uniform Phantom mechanism as the equilibrium outcome of any continuous, strictly monotonic, and UFS mechanism.

¹⁰The egalitarian approach also appears in more general collective choice problems (see, e.g., [Bogomolnaia and Moulin, 2004](#); [D’Aspremont and Gevers, 1977](#); [Hammond, 1976](#)).

¹¹Like our paper, [Freeman et al.’s \(2021\)](#) setting assumes that agents have single-peaked and symmetric preference. [Jennings et al. \(2021\)](#) provide a similar characterization of the Uniform Phantom mechanism in the setting where agents have single-peaked (and possibly asymmetric) preferences.

¹²[Jennings et al. \(2021\)](#) mention that the Uniform Phantom mechanism was first proposed by [Jennings \(2010\)](#) as the ‘linear median’ and it was independently discovered by [Caragiannis et al. \(2016\)](#). However, the Uniform Phantom mechanism appears indirectly in [Renault and Trannoy \(2005\)](#) (see also [Renault and Trannoy, 2011](#)).

Finally, we note that in more general mechanism design problems, “fairness” is often explored in a relatively minimal manner. For example, [Sprumont \(1991\)](#) interprets a mechanism to be fair if it satisfies anonymity and envy-freeness, and [Moulin \(2017\)](#) interprets a mechanism to be fair if it satisfies anonymity, envy-freeness, and a status-quo participation constraint. These minimal notions of fairness have persisted because of various impossibility results in the literature.¹³ Like [Sprumont \(1991\)](#) and [Moulin \(2017\)](#), the unidimensional facility location problem that we study escapes these impossibility results. Our paper contributes a complementary set of fairness axioms that go beyond the basic requirement of anonymity and connect to the notion of proportional representation. We do not consider envy-freeness since, in the context of the facility location problem, it is trivially satisfied by any facility location (see, e.g., Section 8.1 of [Moulin, 2017](#)). The status-quo participation constraint explored by [Moulin \(2017\)](#) requires that an agent weakly prefers the mechanism’s outcome to some status-quo outcome. This is distinct but has a similar flavor to our IFS axiom, which is one of our weakest fairness axioms. The IFS axiom requires that the facility location is not located too far from any agent. When reframed in terms of utility, IFS enforces a minimum utility guarantee for all agents, which could be viewed as an outside option.

Approximate mechanism design. The final section of our paper explores the performance of strategyproof and fair mechanisms with respect to maximizing utilitarian (or social) welfare. Adopting the approximation ratio approach of [Nisan and Ronen \(2001\)](#); [Procaccia and Tennenholtz \(2013\)](#), we measure the performance of these mechanisms by their worst-case performance over the domain of possible preferences profiles relative to the welfare-optimal mechanism. This is a common approach in the economics and computation literature (see, e.g., [Aziz et al., 2020b,a](#); [Feldman et al., 2016](#); [Nisan and Ronen, 2001](#)). For our main fairness axioms of Proportionality, IFS, UFS and PF, we identify the best performing strategyproof and fair mechanism. In particular, we find that the Uniform Phantom mechanism has the best welfare approximation ratio among all mechanisms satisfying UFS (including non-strategyproof mechanisms). In the participatory budgeting setting, [Caragiannis, Christodoulou and Protopapas \(2022\)](#) show a related result: when there are only 2 projects, the Uniform Phantom mechanism achieves the best cost approximation ratio among all strategyproof mechanisms.

2 Model

Let $N = \{1, \dots, n\}$ be a set of agents with $n \geq 2$, and let $X := [0, 1]$ be the domain of locations.¹⁴ Agent i ’s location is denoted by $x_i \in X$; the profile of agent locations is denoted by $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. A *mechanism* is a mapping $f : X^n \rightarrow X$ from a location profile $\hat{\mathbf{x}} \in X^n$ to a facility location $y \in X$.

Given a facility location $y \in X$, agent i ’s cost is $d(y, x_i) := |y - x_i|$. We take an agent’s *utility* to be $u(y, x_i) = 1 - d(y, x_i)$. This utility function is convenient but is not necessary for most of our results; the key assumption that we require is that agents’ utilities are symmetric (around their location) and single-peaked.¹⁵

¹³For example, Theorem 3 of [Border and Jordan \(1983\)](#) shows that, for the multi-dimensional facility location problem with not necessarily separable preferences, there is no strategyproof, unanimity-respecting, and anonymous mechanism (see also [Laffond, 1980](#)).

¹⁴Our results naturally extend to any compact interval on \mathbb{R} with the appropriate modification of axioms.

¹⁵With the exception of Section 6, which focuses on welfare approximation, all our results hold verbatim if agents have arbitrary single-peaked and symmetric preferences and the fairness axioms (IFS, UFS, PF) are defined with respect to the distance function (instead of utilities). In Appendix A, we discuss why the fairness axioms, when defined with respect to distance, may still be normatively desirable in cases where agents’ preferences are single-peaked (and possibly asymmetric).

A widely accepted—albeit minimal—fairness principle is that a mechanism should not depend on the agents’ labels. This is referred to as anonymity.

Definition 1 (Anonymous). A mechanism f is anonymous if, for every location profile \hat{x} and every bijection $\sigma : N \rightarrow N$,

$$f(\hat{x}_\sigma) = f(\hat{x}),$$

where $\hat{x}_\sigma := (\hat{x}_{\sigma(1)}, \hat{x}_{\sigma(2)}, \dots, \hat{x}_{\sigma(n)})$.

Given a location profile x , a facility location y is said to be *Pareto optimal* if there is no other facility location y' such that for all $i \in N$, $d(y', x_i) \leq d(y, x_i)$, with strict inequality holding for at least one agent. A mechanism f is said to be *Pareto efficient* if, for every location profile x , the facility location $f(x)$ is Pareto optimal. In our setting, Pareto optimality is equivalent to requiring that $y \in [\min_{i \in N} x_i, \max_{i \in N} x_i]$.

We are interested in mechanisms that are “strategyproof”, i.e., the mechanism never incentivizes an agent to misreport their location. Before providing a formal definition, we introduce some notation. Given a profile of locations (or reported locations) x' , the profile (x'_{-i}, x''_i) denotes the profile obtained by swapping x'_i with x''_i and leaving all other agent locations (or reports) unchanged.

Definition 2 (Strategyproof). A mechanism f is strategyproof if for every agent $i \in N$, we have for every x'_i, \hat{x}_{-i} and x_i ,

$$u(f(\hat{x}_{-i}, x_i), x_i) \geq u(f(\hat{x}_{-i}, x'_i), x_i) \quad \text{or, equivalently,} \quad d(f(\hat{x}_{-i}, x_i), x_i) \leq d(f(\hat{x}_{-i}, x'_i), x_i).$$

By [Barberà et al. \(2010\)](#), in our setting, a mechanism is strategyproof if and only if it is *group-strategyproof*: no subset of agents $N' \subseteq N$ can misreport their locations and obtain a facility location that is strictly preferred by all of the agents in N' compared to what is obtained by truthfully reporting their locations (see also [Massó and Moreno De Barreda, 2011](#), Remark 1).

Omitted proofs appear in the Appendix.

3 Proportional Fairness

We now introduce a hierarchy of proportional fairness axioms. The first three axioms have previously been proposed in the literature; the fourth axiom, Proportional Fairness, is a new concept that we propose. Whenever appropriate, we formulate our fairness axioms in two ways. We provide a formulation in terms of agents’ utilities (under the assumption that $u(y, x_i) = 1 - d(y, x_i)$) and also in terms of the distance function $d(y, x_i)$.

The first axiom, **Individual Fair Share (IFS)**, requires that the facility location provides every agent with at least $\frac{1}{n}$ of the maximum obtainable utility, i.e., 1. In the context of cake-cutting, IFS coincides with the axiom of [Steinhaus \(1948\)](#) commonly known as proportionality. It also appears as the “Fair Welfare Share” axiom in the context of participatory budgeting, as defined by [Bogomolnaia et al. \(2005\)](#).

Definition 3 (Individual Fair Share (IFS)). Given a profile of locations x , a facility location y satisfies Individual Fair Share (IFS) if each agent obtains at least $1/n$ utility, i.e. for all $i \in N$,

$$u(y, x_i) \geq 1/n \quad \text{or, equivalently,} \quad d(y, x_i) \leq 1 - 1/n.$$

The second axiom, **Unanimous Fair Share (UFS)**, is a strengthening of IFS; it requires that, for every group of agents that share the same location, say $S \subseteq N$, the facility location provides agents in S with at least $\frac{|S|}{n}$ utility. That is, the minimum-utility guarantee ensured by UFS increases proportionally with the group-size, $|S|$. In the context of participatory budgeting, UFS appears in [Aziz et al. \(2019a\)](#).

Definition 4 (Unanimous Fair Share (UFS)). *Given a profile of locations x such that a subset of $S \subseteq N$ agents share the same location, a facility location y satisfies Unanimous Fair Share (UFS) if for all $i \in S$,*

$$u(y, x_i) \geq \frac{|S|}{n} \quad \text{or, equivalently,} \quad d(y, x_i) \leq 1 - \frac{|S|}{n}.$$

The third axiom **Proportionality** requires that, if all agents are located at “extreme” locations (i.e., 0 or 1), the facility is located at the average of the agents’ locations. [Freeman et al. \(2021\)](#) focus on this axiom in a public budgeting setting.

Definition 5 (Proportionality). *Given a profile of locations x such that $x_i \in \{0, 1\}$ for all $i \in N$, a facility location y satisfies Proportionality if $y = \frac{|\{i \in N : x_i = 1\}|}{n}$.*

Finally, we propose a new fairness concept called **Proportional Fairness (PF)**. PF requires that the minimum-utility guarantee provided by the facility location to a group of agents depends on both the size of the group and how closely the agents are clustered. The idea behind the concept is similar in spirit to proportional representation axioms in voting which require that if a subset of agents is large enough and the agents in the subset have “similar” preferences, then the agents in the subset deserve an appropriate level of representation or utility (see, e.g., [Aziz et al., 2017](#); [Aziz and Lee, 2020, 2022](#); [Dummett, 1984](#); [Sánchez-Fernández et al., 2017](#)).

Definition 6 (Proportional Fairness (PF)). *Given a profile of locations x , a facility location y satisfies Proportional Fairness (PF) if, for any subset of agents $S \subseteq N$ within a range of distance $r := \max_{i \in S} \{x_i\} - \min_{i \in S} \{x_i\}$, the agents in S obtain at least $\frac{|S|}{n} - r$ utility, i.e. for all $i \in S$,*

$$u(y, x_i) \geq \frac{|S|}{n} - r \quad \text{or, equivalently,} \quad d(y, x_i) \leq 1 - \frac{|S|}{n} + r.$$

In the definition of PF, given a group S , r is non-negative and equals zero if and only if all agents in S share the same location. Hence, PF implies UFS. For any r that is larger, the corresponding fairness concept is weaker. For any r that is smaller, there may not exist any outcome that satisfies the corresponding definition.

A natural—albeit weak—notation of fairness is called **Unanimity**. It requires that, if all agents are unanimous in their most preferred location, then the facility is located at this same location. Notice that Pareto optimality implies unanimity.

Definition 7 (Unanimity). *Given a profile of locations x such that $x_i = c$ for some $c \in X$ and for all $i \in N$, a facility location y satisfies unanimity if $y = c$.*

Proposition 1 establishes the logical connection between the fairness axioms. Figure 1 provides an illustration of proposition. PF is the strongest fairness notion: it implies all of the other axioms (UFS, IFS, Proportionality, and Unanimity). The next strongest axiom is UFS: it implies IFS, proportionality, and unanimity. There is no relationship between proportionality, IFS, and unanimity; however, they are compatible with each other.

Proposition 1 (A hierarchy of axioms).

- (i) UFS implies proportionality, IFS, and unanimity
- (ii) PF implies UFS

All of the above relations are strict; there is no logical relation between proportionality, IFS, and unanimity. Figure 1 provides an illustration.

4 Strategyproof and Proportionally Fair Mechanisms

We begin by reviewing some prominent mechanisms from the literature. The **median mechanism** f_{med} places the facility at the median location (i.e., the $\lfloor n/2 \rfloor$ -th location when locations are placed in increasing order). The median mechanism is sometimes referred to as the utilitarian mechanism since it places the facility at a location that maximizes the sum of agent utilities.

The **midpoint mechanism** f_{mid} places the facility at the midpoint of the leftmost and rightmost agents, i.e.,

$$f_{\text{mid}}(\mathbf{x}) = \frac{1}{2} \left(\min_{i \in N} x_i + \max_{i \in N} x_i \right).$$

The midpoint mechanism is sometimes referred to as the egalitarian mechanism since it maximizes the minimum agent utility.

A **Nash mechanism** places the facility at a location that maximizes the product of agent utilities.¹⁶ Formally, a Nash mechanism f_{Nash} locates the facility at

$$f_{\text{Nash}}(\mathbf{x}) = \arg \max_{y \in [0,1]} \prod_{i \in N} u(y, x_i).$$

The Nash mechanism is described by [Moulin \(2003, p. 80\)](#) as achieving a “sensible compromise between utilitarianism and egalitarianism.”

Incompatibility results. All of the above mechanisms either fail to provide fair outcomes (per the axioms in Section 3) or fail to be strategyproof. The median mechanism fails Proportionality and IFS; however, it is strategyproof and satisfies unanimity. The midpoint mechanism—often heralded as a hallmark of fairness—fails to satisfy many of Section 3’s fairness axioms; it only satisfies the weakest axioms: IFS and unanimity. Furthermore, the midpoint mechanism is not strategyproof. Finally, the Nash mechanism obtains the strongest axiom of proportional fairness, PF—and, hence, satisfies the other fairness axioms: UFS, Proportionality, IFS, and unanimity. However, the Nash mechanism is not strategyproof ([Lam et al., 2021](#)). Proposition 2 summarizes these results.

Proposition 2 (Review of existing mechanisms).

- (i) The median mechanism satisfies unanimity and strategyproofness, but does not satisfy IFS, PF, UFS nor Proportionality.
- (ii) The midpoint mechanism satisfies IFS and unanimity, but it is not strategyproof. The midpoint mechanism does not satisfy PF, UFS, nor Proportionality.
- (iii) The Nash mechanism satisfies PF, but it is not strategyproof.

¹⁶The mechanism’s namesake is an allusion to the Nash bargaining solution ([Nash, 1950, 1953](#)). The product of utilities is often referred to as the *Nash social welfare* or *Nash welfare*.

4.1 Characterization of IFS and strategyproof mechanisms

We now characterize the family of strategyproof and IFS mechanisms. Our characterization leverages the class of Phantom mechanisms introduced by [Moulin \(1980\)](#) (see also [Border and Jordan, 1983](#)).¹⁷ Intuitively, Phantom mechanisms can be understood as locating the facility at the median of $2n - 1$ reports, where n reports correspond to the agents' reports and $n - 1$ reports are fixed (and pre-determined) at locations p_1, \dots, p_{n-1} . The fixed reports are referred to as “phantom” locations.

Definition 8 (Phantom Mechanisms). *Given $x \in X$ and $n - 1$ values $0 \leq p_1 \leq \dots \leq p_{n-1} \leq 1$, a Phantom mechanism locates the facility at $\text{Median}\{x_1, \dots, x_n, p_1, \dots, p_{n-1}\}$.*

The family of Phantom mechanisms is broad and captures many well-known mechanisms. To build intuition, we provide some examples below.

- (i) The classic median mechanism is obtained by locating $\lfloor (n - 1)/2 \rfloor$ phantoms at 0 and $\lceil (n - 1)/2 \rceil$ phantoms at 1.
- (ii) The “Maximum” (resp., “Minimum”) mechanism, which locates the facility at the maximum (resp., minimum) agent location, is obtained by locating all the phantoms at 1 (resp., 0).
- (iii) The “Moderate- $\frac{1}{2}$ ” mechanism, which locates the facility at the minimum (resp., maximum) agent reported location when all agents report below (resp., above) $1/2$ and otherwise (i.e., when some agent(s) report either side of $1/2$) the facility is located at $1/2$. This mechanism is obtained by locating all the phantoms at $1/2$.

On the other hand, mechanisms such as the midpoint mechanism and the Nash mechanism from Section 4 do not belong to the family of Phantom mechanisms. Similarly, the “Average” mechanism, which locates the facility at the average of all agents' reports, is not a Phantom mechanism. Figure 3 provides an illustration of these mechanisms (and also other mechanisms that will be defined later).

The family of Phantom mechanisms are known to characterize all strategyproof, anonymous, and Pareto efficient mechanisms (Corollary 2 of [Massó and Moreno De Barreda, 2011](#)).¹⁸ This characterization of Phantom mechanisms forms the foundation of our characterization results.

Theorem 1 says that the family of IFS, strategyproof, anonymous, and unanimous mechanisms are characterized by the subfamily of Phantom mechanisms that have their phantom locations contained in the interval $[\frac{1}{n}, 1 - \frac{1}{n}]$. Intuitively, when the facility is located in the interval $[\frac{1}{n}, 1 - \frac{1}{n}]$, IFS is satisfied regardless of the agents' locations. The restricted class of Phantom mechanisms in Theorem 1 satisfies IFS by preventing the facility from being located at an “extreme” point (i.e., beyond the interval $[\frac{1}{n}, 1 - \frac{1}{n}]$) unless all agents are located close together and at an extreme point.

Theorem 1 (Characterization: IFS, unanimous, anonymous, and strategyproof). *A mechanism is strategyproof, unanimous, anonymous and satisfies IFS if and only if it is a Phantom mechanism with $n - 1$ phantoms all contained in the interval $[\frac{1}{n}, 1 - \frac{1}{n}]$.*

¹⁷Although both [Moulin \(1980\)](#) and [Border and Jordan \(1983\)](#) deal with a setting where agents' locations are in \mathbb{R} rather than $[0, 1]$, it can be shown that their results extend naturally (see Footnote 8 of [Schummer and Vohra, 2002](#), for a sketch of the argument).

¹⁸This result differs to [Moulin's \(1980\)](#) results because [Massó and Moreno De Barreda's](#) corollary applies to the setting where agents have single-peaked and symmetric preferences. Moulin also shows that a broader class of phantom mechanisms (that uses $n + 1$ phantoms) characterizes the class of all anonymous and strategyproof but not necessarily Pareto-efficient mechanisms.

Proof. We start with the backwards direction. Let f be a Phantom mechanism with the $n - 1$ phantoms contained in $[\frac{1}{n}, 1 - \frac{1}{n}]$. First note that f is strategyproof because all Phantom mechanisms are strategyproof (see, e.g., Corollary 2 of [Massó and Moreno De Barreda, 2011](#)). Furthermore, it is immediate from the Phantom mechanism definition (Definition 8) that f satisfies unanimity. It remains to show that f satisfies IFS. To see this, notice that the facility is located above (resp., below) both of the endpoints of the interval $[\frac{1}{n}, 1 - \frac{1}{n}]$ if and only if all agents are located above (resp., below) of the interval. Therefore, in such cases, the facility is located within a distance of $\frac{1}{n}$ of all agents. Otherwise, the facility is located within the interval and the largest possible cost is $1 - \frac{1}{n}$, as required.

We now prove the forward direction. Let f be a mechanism that is strategyproof, unanimous, anonymous, and satisfies IFS. [Border and Jordan's \(1983\)](#) Lemma 3 says that any strategyproof and unanimous mechanism is Pareto efficient. Hence, f is strategyproof, IFS, unanimous, anonymous, and Pareto efficient. We now apply Corollary 2 of [Massó and Moreno De Barreda \(2011\)](#), which says that a mechanism is strategyproof, anonymous, and Pareto efficient if and only if it is a Phantom mechanism (Definition 8). We now show that $p_j \in [\frac{1}{n}, 1 - \frac{1}{n}]$ for all $j \in \{1, \dots, n - 1\}$. For sake of a contradiction, suppose $p_1 < \frac{1}{n}$ (the case of $p_{n-1} > 1 - \frac{1}{n}$ is dealt with similarly and, hence, is omitted). If $n - 1$ agents are located 0 and the remaining agent is located at 1, then the facility must be located at $p_1 < \frac{1}{n}$. But then the agent at location 1 experiences cost strictly greater than $1 - \frac{1}{n}$ —a contradiction of IFS. Therefore, $p_j \in [\frac{1}{n}, 1 - \frac{1}{n}]$ for all $j \in \{1, \dots, n - 1\}$, as required. \square

Theorem 1 is “tight” in the following sense: if any one of the requirements in theorem (i.e., strategyproofness, unanimity, anonymity, and IFS) is removed, then the theorem fails to hold. In Appendix B.4, for each smaller set of requirements, we identify a mechanism that satisfies them and does not belong to the family of mechanisms described in Theorem 1.

4.2 Characterization of PF, UFS, Proportional, and strategyproof mechanisms

We now show that strategyproofness and PF are compatible and can be achieved via the “Uniform Phantom” mechanism. By Proposition 1 this also implies that UFS and, hence, proportionality, IFS, and unanimity can be attained simultaneously. The Uniform Phantom mechanism is obtained from the general class of Phantom mechanisms (Definition 8) by locating the phantoms at $\frac{j}{n}$ for $j = 1, \dots, n - 1$. Figure 3 provides an illustration of the mechanism. This mechanism is the focus of [Freeman et al. \(2021\)](#); later we provide a discussion of the similarities and differences between our results and those of [Freeman et al.](#)

Definition 9 (Uniform Phantom Mechanism). *Given $x \in X$, the Uniform Phantom mechanism f_{Unif} locates the facility at*

$$\text{Median}\{x_1, \dots, x_n, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}.$$

It is immediate that the Uniform Phantom mechanism is strategyproof since it belongs to the family of Phantom mechanisms (Definition 8). However, in addition to strategyproofness, Proposition 3 says that the Uniform Phantom mechanism satisfies PF. Intuitively, the Uniform Phantom mechanism locates the facility at the n -th location of the $2n - 1$ phantom and agent locations. Given the phantom locations, for every $1/n$ units of distance, there is at least one phantom. Therefore, for any set of agents S , the distance between the most extreme agent in S and the facility is at most $\frac{n-|S|}{n}$ and, hence, the distance between any agent in S and the facility is at most $\frac{n-|S|}{n} + r$, where r is the range of the agents in S .

Proposition 3 (Uniform Phantom mechanism properties). *The Uniform Phantom mechanism is strategyproof and satisfies PF. Thus, it also satisfies UFS, IFS, proportionality, and unanimity.*

A natural question is whether there exist other strategyproof mechanisms satisfying UFS or proportionality and unanimity. It turns out that there are not: Theorem 2 says that the Uniform Phantom mechanism is the only strategyproof mechanism that is proportional and unanimous. A key challenge in the theorem is that anonymity is not supposed and hence, the well-known characterization of Phantom mechanisms cannot be immediately applied. In the appendix, we prove an auxiliary lemma that says anonymity is implied by strategyproofness, unanimity, and proportionality. With this in hand, the Phantom mechanism characterization can be utilized. Proportionality then implies the (unique) locations of the $n - 1$ phantoms. This is because of two observations. First, proportionality requires that, for any $k = 1, \dots, n - 1$, when k agents are located at 1 and $n - k$ agents at 0 the facility is located at k/n . Second, for such a profile of locations, any Phantom mechanism will locate the facility at the k th phantom. Therefore, the phantoms must be located at $\frac{k}{n}$ for $k = 1, \dots, n - 1$.

Theorem 2 (Characterization: proportional, unanimous, and strategyproof). *A mechanism satisfies strategyproofness, unanimity, and proportionality if and only if it is the Uniform Phantom mechanism.*

Proof. The backward direction follows immediately from Proposition 3 and Proposition 1. It remains to prove the forward direction. Suppose f is strategyproof and satisfies proportionality and unanimity. We utilize an auxiliary lemma, which says that any strategyproof, unanimous, and proportional mechanism must be anonymous (Lemma 4). The proof of Lemma 4 is quite involved and is proven in Appendix B.5. Given Lemma 4, we apply Border and Jordan's (1983) Lemma 3 (i.e., any strategyproof and unanimous mechanism is Pareto efficient). This tells us that f must also be anonymous and Pareto efficient. We now apply Corollary 2 of Massó and Moreno De Barreda (2011), which says that a mechanism is strategyproof, anonymous, and Pareto efficient if and only if it is a Phantom mechanism (Definition 8). We now show that $p_j = \frac{j}{n}$ for all $j \in \{1, \dots, n - 1\}$. To see this, take arbitrary $j \in \{1, \dots, n - 1\}$, and let \mathbf{x} be a profile of locations such that there are j agents at 1 and $n - j$ agents at 0. By definition of the Uniform Phantom mechanism, $f(\mathbf{x}) = p_j$. But proportionality requires that $f(\mathbf{x}) = \frac{j}{n}$; hence, $p_j = \frac{j}{n}$. This completes the proof. \square

Combining Proposition 1 and Proposition 3 with Theorem 2 provides two complementary characterizations. Corollary 1 says that the Uniform Phantom mechanism is the only strategyproof mechanism that satisfies UFS; similarly, the Uniform Phantom mechanism is the only strategyproof mechanism that satisfies PF.

Corollary 1 (Characterization: UFS/PF and strategyproof). *A mechanism satisfies strategyproofness and UFS (PF) if and only if it is the Uniform Phantom mechanism.*

UFS and PF are (strictly) stronger requirements than proportionality, so the characterization given by Corollary 1 does not hold if UFS or PF are replaced by proportionality. In other words, Theorem 2 does not hold if we remove unanimity. A simple example illustrating this can be found in Appendix B.6.

Theorem 2 and Corollary 1 gives the equivalence in Corollary 2. The statements are "tight": dropping any property in (i), (ii), or (iii) will break the equivalence with (iv).

Corollary 2. *The following are equivalent:*

- (i) f satisfies strategyproofness, proportionality, and unanimity

- (ii) f satisfies strategyproofness and UFS.
- (iii) f satisfies strategyproofness and PF.
- (iv) f is the Uniform Phantom mechanism.

A perhaps interesting implication of Corollary 2 is that, although combining proportionality and unanimity is a strictly weaker concept than UFS, when combined with strategyproofness the UFS concept is equivalent to requiring both proportionality and unanimity. Similarly, the UFS concept is strictly weaker concept than PF but, when combined with strategyproofness, PF is equivalent to UFS.

Comparing our results with Freeman et al. (2021). The Uniform Phantom mechanism appears in Freeman et al. (2021). Freeman et al.’s Proposition 1 shows that a mechanism is continuous, anonymous, proportional, and strategyproof if and only if it is the Uniform Phantom mechanism. Equivalently, by Border and Jordan’s (1983) Corollary 1, Freeman et al.’s characterization holds if continuity is replaced with unanimity. Our results complement Freeman et al.’s characterization. Firstly, we have shown (in Appendix B.6) that continuity (equivalently, unanimity) is essential for Freeman et al.’s characterization. Secondly, our Theorem 2 shows that the anonymity requirement can be removed.¹⁹ Finally, we provide a more general analysis of fairness axioms in facility location problems and show that the Uniform Phantom mechanism is the unique strategyproof mechanism that satisfies different combinations of these fairness axioms (Corollary 2).

5 Equilibria of non-strategyproof, UFS mechanisms

We now explore the equilibrium properties of non-strategyproof mechanisms. We begin with some terminology. Given two profiles of locations, $\mathbf{x} < \mathbf{x}'$ if and only if $x_i \leq x'_i$ for all $i \in N$ and $x_i < x'_i$ for some $i \in N$. We say a mechanism f is *strictly monotonic* if

$$f(\mathbf{x}) < f(\mathbf{x}') \text{ for all } \mathbf{x} < \mathbf{x}'.$$

An example of a strictly monotonic mechanism is the “Average” mechanism $f_{\text{avg}}(\mathbf{x}) := \frac{1}{n} \sum_{i \in N} x_i$. The Average mechanism is also continuous and satisfies UFS (see Proposition 4 in Appendix C). It is clearly not strategyproof. In contrast, the Uniform Phantom mechanism is not strictly monotonic.

Perhaps surprisingly, Theorem 3 says that the pure Nash equilibrium of any continuous, strictly monotonic, and UFS mechanism has the facility located at the same position as would have been attained by the (strategyproof) Uniform Phantom mechanism. Therefore, in the equilibrium outcome of such mechanisms, UFS with respect to the agents’ true location is satisfied—even if agents misreport their location in equilibrium. This provides an alternative characterization of the Uniform Phantom mechanism as the equilibrium outcome of any continuous, strictly monotonic, and UFS mechanism.²⁰

¹⁹ Studying a slightly different setting, where agents have single-peaked and (possibly) asymmetric preferences, Jennings et al. (2021) show that neither continuity nor anonymity is required for Freeman et al.’s characterization. The necessity of unanimity in Theorem 2 clarifies a key difference with the setting of symmetric preferences: continuity is required.

²⁰ In a slightly different setting, where agents have single-peaked (and possibly asymmetric) preferences, Yamamura and Kawasaki (2013) provide a general characterization of the equilibrium outcome of anonymous, continuous, strictly monotonic, and unrestricted-range mechanisms. Although Yamamura and Kawasaki’s results do not formally apply to our setting and do not focus on issues of fairness, our Theorem 3 is consistent with their characterization.

Theorem 3. *Suppose f is continuous, strictly monotonic, and satisfies UFS. There exists a pure Nash equilibrium, and the output of every (pure) equilibrium of f coincides with the facility location of the Uniform Phantom when agents report truthfully.*

Proof. The existence of a pure Nash equilibrium follows immediately (Debreu, 1952; Glicksberg, 1952; Fan, 1952).²¹ Now let \mathbf{x} be a profile of agents' (true) locations, and let \mathbf{x}^* be a pure Nash equilibrium of f . Denote by $s_{\text{unif}} := f_{\text{unif}}(\mathbf{x})$ the facility location under the Uniform Phantom mechanism when agents report truthfully. We wish to prove that $f(\mathbf{x}^*) = s_{\text{unif}}$. We consider two cases.

Case 1. Suppose $s_{\text{unif}} = k/n$ for some $k \in \{0, \dots, n\}$. By construction of the Uniform Phantom, it must be that at least $n - k$ agents have true location (weakly) below s_{unif} and at least k agents have true location (weakly) above. Now, for sake of a contradiction, suppose that $f(\mathbf{x}^*) < s_{\text{unif}} = k/n$ (the reverse inequality is treated similarly and therefore is omitted). Notice that there are at least k agents with true location strictly above than $f(\mathbf{x}^*)$; let $N' := \{i \in N : f(\mathbf{x}^*) < x_i\}$. If $x_i^* = 1$ for all $i \in N'$, then $f(\mathbf{x}^*) \geq k/n$ (since f satisfies UFS)—a contradiction because $f(\mathbf{x}^*) < s_{\text{unif}} = k/n$. Therefore, $x_i^* < 1$ for some agent $i \in N'$. But then \mathbf{x}^* cannot be an equilibrium: agent i can profitably deviate by reporting some $x'_i \in (x_i^*, 1]$, which—due to continuity and strict monotonicity of f —increases the facility location.

Case 2. Suppose $s_{\text{unif}} \in (\frac{k}{n}, \frac{k+1}{n})$ for some $k \in \{0, \dots, n-1\}$. By construction of the Uniform Phantom, it must be that at least $n - k$ agents have true location (weakly) below s_{unif} and at least $k + 1$ agents have true location (weakly) above—note that there are at least $k + 1$ agents weakly above s_{unif} because at least one agent is located at exactly s_{unif} . Now, for sake of a contradiction, suppose that $f(\mathbf{x}^*) < s_{\text{unif}}$ (the reverse inequality is treated similarly and therefore is omitted). Notice that there are at least $k + 1$ agents with location strictly above $f(\mathbf{x}^*)$; let $N'' := \{i \in N : f(\mathbf{x}^*) < x_i\}$. If $x_i^* = 1$ for all $i \in N''$, then $(k+1)/n \leq f(\mathbf{x}^*)$ (since f satisfies UFS)—a contradiction because $f(\mathbf{x}^*) < s_{\text{unif}} \in (\frac{k}{n}, \frac{k+1}{n})$. Therefore, $x_i^* < 1$ for some $i \in N''$. But \mathbf{x}^* cannot be an equilibrium: agent i can profitably deviate by reporting some $x'_i \in (x_i^*, 1]$, which—due to continuity and strict monotonicity of f —increases the facility location. \square

An immediate corollary of Theorem 3 is that the equilibrium outcome of any continuous, strictly monotonic, and UFS mechanism satisfies UFS with respect to the agents' true locations.

Corollary 3. *Suppose f is continuous, strictly monotonic, and satisfies UFS. The output of every (pure) equilibrium of f satisfies UFS with respect to the agents' true location profile.*

Another corollary of Theorem 3 is that the equilibrium outcome of the average mechanism coincides with the facility location of the Uniform Phantom mechanism when agents report truthfully.²²

Corollary 4. *Every (pure) equilibrium of the average mechanism coincides with the facility location of the Uniform Phantom when agents report truthfully.*

Unfortunately, Theorem 3 cannot be applied to the Nash mechanism's equilibrium outcome since the Nash mechanism is not strictly monotonic.²³

²¹Note that because f is continuous and strictly monotonic, agents have single-peaked utility with respect to their report and, hence, their utility is quasiconcave in their report.

²²In a slightly different setting, where agents have single-peaked (and possibly asymmetric) preferences, Renault and Trannoy (2005) obtain the same result (see also Renault and Trannoy, 2011).

²³This can be illustrated via a simple example with 3 agents. The Nash mechanism maps the location profiles $\mathbf{x} = (0, 0.5, 0.9)$ and $\mathbf{x}' = (0, 0.5, 1)$ to 0.5. However, strict monotonicity requires that \mathbf{x}' be mapped to a location strictly higher than 0.5.

6 Welfare approximation results

A common objective in collective decision making is to maximize (utilitarian or social) welfare. Therefore, given a profile of locations \mathbf{x} and a facility location y , the (utilitarian or social) *welfare* is defined as the sum of agents' utilities:

$$\sum_{i=1}^n u(y, x_i) = \sum_{i=1}^n (1 - d(y, x_i)).$$

In this section, we explore the performance of strategyproof and “fair” mechanisms with respect to *welfare maximization*. Rather than make distributional assumptions, we measure the performance of these mechanisms by their worst-case performance over the domain of preference profiles. Given a profile of agent locations, \mathbf{x} and facility location y , we define the *optimal welfare* by $\Phi^*(\mathbf{x}) := \max_{y \in X} \sum_{i=1}^n u(y, x_i)$, and given a mechanism f , let $\Phi_f(\mathbf{x})$ denote the welfare attained by the mechanism, i.e., $\Phi_f(\mathbf{x}) := \sum_{i=1}^n u(f(\mathbf{x}), x_i)$. The mechanism f is an α -approximation if

$$\max_{\mathbf{x} \in X^n} \left\{ \frac{\Phi^*(\mathbf{x})}{\Phi_f(\mathbf{x})} \right\} = \alpha. \quad (1)$$

Notice that $\alpha \geq 1$ for all mechanisms f . We refer to a mechanism f with 1-approximation ratio as a *welfare-optimal mechanism*.

Maximizing welfare vs minimizing cost. Another common objective in collective decision making is to minimize the total cost (i.e., $\sum_{i \in N} d(y, x_i)$). Minimizing the total cost and maximizing (utilitarian) welfare are equivalent optimization problems; hence, both problems have the same “optimal” mechanism. However, in general, when considering approximately-optimal mechanisms, the welfare approximation ratio of a mechanism will not equal the total cost approximation ratio.²⁴ In this paper, we focus on the welfare approximation ratio for two reasons. First, the welfare approximation ratio allows for a more detailed analysis that is not possible with the total cost approximation ratio. For example, *every* mechanism satisfying IFS has an identical approximation ratio of $(n - 1)$.²⁵ Second, the results we obtain from analyzing the welfare approximation ratio appear more intuitive than those obtained from the total cost approximation ratio.²⁶

We begin by defining the median mechanism, which is known to maximize (utilitarian) welfare (Procaccia and Tennenholtz, 2013).

Definition 10 (Median mechanism). *The median mechanism locates the facility at the median of all agents' locations. If there are an even number of agents, the facility is placed at the leftmost of the two middle agent locations.*

The median mechanism is strategyproof, anonymous, Pareto efficient, and satisfies unanimity. However, it does not satisfy IFS nor proportionality. Lemma 1 provides a welfare approximation lower bound for mechanisms that satisfy IFS.

Lemma 1. *Any mechanism satisfying IFS has a welfare approximation of at least $1 + \frac{n-2}{n^2-2n+2}$. As $n \rightarrow \infty$, this lower bound approaches 1.*

²⁴Formally, the total cost approximation ratio for a mechanism f is defined as $\max_{\mathbf{x} \in X^n} \left\{ \sum_{i \in N} d(f(\mathbf{x}), x_i) / \min_y \sum_{i \in N} d(y, x_i) \right\}$.

²⁵Results available from the authors upon request.

²⁶For example, the mechanism focused on in Theorem 4 outputs a facility location that converges to the (welfare-optimal) median mechanism as $n \rightarrow \infty$, and the welfare approximation ratio reflects this by converging to one. In contrast, the total cost approximation ratio is equal to $(n - 1)$ and grows unboundedly.

We now provide an example of an IFS mechanism, which we call the Constrained Median mechanism, that obtains the welfare approximation of Lemma 1. The Constrained Median mechanism locates the facility at the median location whenever the median location lies in the interval $[1/n, 1 - 1/n]$. When the median location is below $1/n$ (resp., above $1 - 1/n$), the facility is located at the minimum of $1/n$ and maximum-agent report (resp., maximum of $1 - 1/n$ and the minimum-agent report). Definition 11 provides a formal definition, and Figure 3 provides an illustration of the mechanism.

Definition 11 (Constrained Median). *The Constrained Median mechanism f_{CM} is a phantom mechanism that places $\lceil \frac{n-1}{2} \rceil$ phantoms at $1/n$ and the remaining phantoms at $1 - \frac{1}{n}$.*

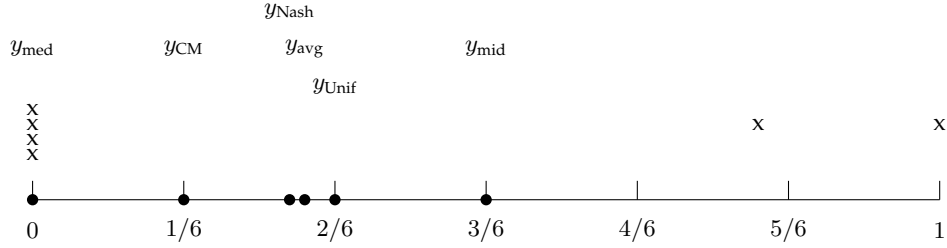


Fig. 3: Facility location problem with $n = 6$ agents, with location profile $(0, 0, 0, 0, 0.8, 1)$ represented by x . The facility locations (represented by \bullet) correspond to the: Median mechanism, $y_{med} = 0$; Constrained Median mechanism, $y_{CM} = \frac{1}{6}$; Nash mechanism, $y_{Nash} \approx 0.284$; Average mechanism, $y_{avg} = 0.3$; Uniform Phantom mechanism, $y_{Unif} = \frac{2}{6}$; and Midpoint mechanism, $y_{mid} = \frac{3}{6}$.

Theorem 4 says that the Constrained Median mechanism obtains the best approximation guarantee among all IFS mechanisms, including non-strategyproof mechanisms. Furthermore, the Constrained Median mechanism can easily be seen to not only satisfy IFS but also to be strategyproof, anonymous, and unanimous (Theorem 1).

Theorem 4. *Among all IFS mechanisms, the Constrained Median mechanism provides the best approximation guarantee, i.e., it achieves the approximation ratio in Lemma 1.*

Lemma 2 provides a minimum welfare approximation bound for mechanisms that satisfy UFS (or proportionality or PF).

Lemma 2. *Any mechanism satisfying UFS (or proportionality or PF) has a welfare approximation of at least*

$$\max_{k \in \mathbb{N} : 0 \leq k \leq n/2} \frac{n(n-k)}{k^2 + (n-k)^2}. \quad (2)$$

As $n \rightarrow \infty$, this lower bound approaches $\frac{\sqrt{2}+1}{2} \approx 1.207$.

We now show that the Uniform Phantom mechanism obtains the welfare approximation of Lemma 2. This means that the Uniform Phantom mechanism provides the best welfare approximation guarantee among all UFS (or proportional or PF) mechanisms, including non-strategyproof mechanisms. Furthermore, from Theorem 2, we know that the Uniform Phantom mechanism has the added benefit of being strategyproof, anonymous, and unanimous.

Mechanism	Strategyproof	PF	UFS	Proportionality	IFS	Util-approx (limit)
Uniform Phantom	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>	$\frac{\sqrt{2}+1}{2} \approx 1.207$
Median	<u>Yes</u>	No	No	No	No	1
Constrained Median	<u>Yes</u>	No	No	No	<u>Yes</u>	1
Nash mechanism	No	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>	$\in [\frac{\sqrt{2}+1}{2}, 2]$
Midpoint mechanism	No	No	No	No	<u>Yes</u>	$\frac{2}{2}$
Average mechanism	No	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>	$\frac{\sqrt{2}+1}{2}$

Table 1: Summary of results. All mechanisms are also unanimous, anonymous and Pareto efficient. Proofs of the results for the Average mechanism can be found in Appendix C. The approximation results for the Nash and Midpoint mechanisms are from Lam et al. (2021).

Theorem 5. *Among all UFS (or proportional or PF) mechanisms, the Uniform Phantom mechanism provides the best approximation guarantee, i.e., it achieves the approximation ratio in Lemma 2.*

Figure 2 illustrates the approximation results of this section.

7 Discussion

Facility location is a classical problem in economic design. In this paper, we provided a deeper understanding of strategyproof and proportionally fair mechanisms. Table 1 provides an overview of most of the mechanisms considered in the paper and the properties they satisfy. Our results provide strong support for the desirability of the Uniform Phantom mechanism in terms of satisfying fairness and strategyproofness.

Moving beyond the fairness axioms that we presented, one can also consider stronger notions of proportional fairness. For example, the following property, which we call Strong Proportional Fairness (SPF), is stronger than PF. Given a profile of locations x within range of distance R , a facility location y satisfies *Strong Proportional Fairness (SPF)* if, for any subset of voters $S \subseteq N$ within a range of distance r , the location should be at most $R \frac{n-|S|}{n} + r$ distance from each agent in S , i.e., $d(y, x_i) \leq R \frac{n-|S|}{n} + r$ for all $i \in S$.

However, it can be easily shown that the Uniform Phantom mechanism does not satisfy SPF. Our result (that the Uniform Phantom mechanism is the only SP and PF mechanism) then implies that there exists no strategyproof and SPF mechanism. In this sense, the compatibility between strategyproofness and fairness axioms ceases to hold when we move from PF to SPF.

There are several directions of future work including extensions to multiple facilities, multiple dimensions, handling facility capacities, alternative fairness concepts, considering weaker notions of strategyproofness, or more general utility functions.

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A Discussion of our fairness axioms

We discuss the relevance of our results and fairness axioms when agents’ preferences are single-peaked but not necessarily symmetric (or of the specific functional form: $u(y, x_i) = 1 - d(y, x_i)$).

In defining our fairness axioms in Section 3, we provided two formulations: one in terms of utility (assuming $u(y, x_i) = 1 - d(y, x_i)$) and one in terms of Euclidean distance. Suppose we adopt the Euclidean-based formulation. Then all our results in Sections 3, 4 and 5, hold verbatim if agents’ preferences are single-peaked and symmetric—and most of the results hold if agents’ preferences are single-peaked (and possibly asymmetric). However, the normative motivation for the fairness axioms, when formulated in terms of Euclidean distance, are less immediate since they no longer correspond to utility guarantees. Nonetheless, we argue that there are still reasons to believe that the axioms are reasonable notions of fairness. To see this, suppose that agents can have arbitrary single-peaked preferences. In our setting, it is known that a strategyproof mechanism must be “tops only” (i.e., the mechanism’s input can only consists of agent’s ideal points)—using any further information would cause the mechanism to be manipulable (Barberà and Jackson, 1994). Therefore, behind this veil of ignorance, it seems reasonable that a “fair” outcome should, at minimum:

- (i) impose conditions on the “closeness” of agents’ peaks and the facility location;
- (ii) “closeness” should be symmetric (since, for every asymmetric utility function, there is another asymmetric utility function that is a mirror image and skewed in the opposite direction);
- (iii) closeness should be “anonymous” (i.e., the same measure of closeness should be used for all agents).

These points imply that a single benchmark distance metric should be applied for each agent. We adopt the standard Euclidean distance for our axioms (IFS, UFS, PF), i.e., $d(y, x_i)$ equals $|y - x_i|$; this has desirable and natural features. For example, suppose $n = 2$ with one agent located at 0 and the other at 1. The absolute value $|y - x_i|$ is the only metric that requires the facility to be located at exactly $\frac{1}{2}$ via the IFS condition $d(y, x_i) \leq 1 - 1/n$ (the same is true for the UFS, PF conditions). Higher powers of $|y - x_i|$ could be considered (i.e., $|y - x_i|^p$ for $p > 1$) but this leads to non-existence. Lower powers could be considered (i.e., $|y - x_i|^p$ for $0 < p < 1$) but this leads to “fair” outcomes that asymmetrically favor one agent over the other.

B Omitted proofs

B.1 Proof of Proposition 1

Proof. Point (i): We wish to prove that UFS implies proportionality, IFS, and unanimity. Let x be an arbitrary location profile and let y be a facility location that satisfies UFS. From the definition of UFS, it is immediate that IFS and unanimity are satisfied. It remains to prove that proportionality is satisfied. For sake of a contradiction, suppose that proportionality is not satisfied. That is, x is such that $x_i \in \{0, 1\}$ for all $i \in N$ and $y \neq \frac{|\{i \in N : x_i = 1\}|}{n}$. Let $k = |\{i \in N : x_i = 1\}|$. If $k = 0$, then UFS requires that $y = 0$, and proportionality is satisfied—a contradiction. If $k > 0$, then UFS requires that:

$$|1 - y| \leq 1 - \frac{k}{n}, \quad \text{i.e., } y \geq \frac{k}{n} \quad \text{and} \quad |0 - y| \leq 1 - \frac{n - k}{n}, \quad \text{i.e., } y \leq \frac{k}{n}.$$

The inequalities above imply that $y = \frac{k}{n}$ and proportionality is satisfied—a contradiction.

Point (ii): We wish to prove that PF implies UFS. This follows immediately by noting that a set of agents all located at the same location are within a range of distance $r = 0$. Taking $r = 0$ in the PF definition shows that PF implies UFS.

It is straightforward to see that the relations in the proposition are strict and also that there is no logical relation between proportionality, IFS, and unanimity. We omit the proofs. \square

B.2 Proof of Proposition 2

Proof. Point (i): We wish to prove that the median mechanism satisfies unanimity and strategyproofness, but does not satisfy IFS, PF, UFS, nor Proportionality. The median mechanism is known to be strategyproof (Procaccia and Tennenholtz, 2013); it is also clearly unanimous. Finally, consider the agent location profile with $n - 1$ agents at 0 and 1 agent at 1. The median mechanism locates the facility at 0, which violates both IFS and Proportionality (and hence also UFS and PF).

Point (ii): We wish to prove that the midpoint mechanism satisfies IFS and unanimity, but does not satisfy strategyproofness, PF, UFS, nor Proportionality. The midpoint mechanism places the facility at the average of the leftmost and rightmost agent. It is therefore unanimous but not strategyproof. The maximum cost that can be incurred by an agent is $1/2$, which is obtained when the leftmost agent is at 0 and the rightmost agent is at 1. However, this IFS is satisfied since $n \geq 2$. To see that the midpoint mechanism does not satisfy Proportionality, consider the agent location profile with 2 agents at 0 and 1 agent at 1. The midpoint mechanism places the facility at $1/2$, but Proportionality requires that the mechanism is placed at $1/3$. Since Proportionality is not satisfied, UFS and PF are also not satisfied.

Point (iii): We wish to prove that the Nash mechanism satisfies PF but is not strategyproof. To this end, we first define a notion of monotonicity that requires that if a location profile is modified by an agent shifting its location, the facility placement under the modified profile will not shift in the opposite direction. Definition 12 formalizes this notion.

Definition 12 (Monotonic). *A mechanism f is monotonic if*

$$f(\mathbf{x}) \leq f(\mathbf{x}')$$

for all $f(\mathbf{x})$ and $f(\mathbf{x}')$ such that $x_i \leq x'_i$ for all $i \in N$ and $x_i < x'_i$ for some $i \in N$.

We next prove the following auxiliary lemma.

Lemma 3. *A mechanism that satisfies UFS and monotonicity also satisfies PF.*

Proof. Let \mathbf{x} be an arbitrary agent location profile, and f be a mechanism that satisfies UFS and monotonicity. Consider the set $S = \{1, \dots, m\} \subset N$ of m agents and denote $r := \max_{i \in S} \{x_i\} - \min_{i \in S} \{x_i\}$. We prove that the maximum distance of the facility from any agent in S is at most $\frac{n-m}{n} + r$.

Denote $f := \arg \max_{i \in S} \{d(f(\mathbf{x}), x_i)\}$ as the agent in S whose location x_f under \mathbf{x} is furthest from the respective facility location²⁷. Consider the modified profile \mathbf{x}' where $x'_i = \max_{i \in S} \{x_i\}$ for all $i \in S$ if $f(\mathbf{x}) \geq x_f$ and $x'_i = \min_{i \in S} \{x_i\}$ for all $i \in S$ if $f(\mathbf{x}) < x_f$. Also, $x'_i = x_i$ for all $i \in N \setminus S$. In other words, the agents in S have their locations moved to the rightmost agent in S if the facility is weakly right of the furthest agent of S under \mathbf{x} . If the facility is strictly left of the furthest agent of S under \mathbf{x} , the agents in S have their locations moved to the leftmost agent.

Due to monotonicity, the facility does not move closer to x_f when modifying \mathbf{x} to \mathbf{x}' , so we have

$$d(f(\mathbf{x}'), x_f) \geq d(f(\mathbf{x}), x_f).$$

Note that all m agents of S are at the same location under \mathbf{x}' . Denote this location as x'_S . Due to UFS, we also have

$$d(f(\mathbf{x}'), x'_S) \leq \frac{n-m}{n}.$$

We therefore have

$$d(f(\mathbf{x}), x_f) \leq d(f(\mathbf{x}'), x_f) \leq d(f(\mathbf{x}'), x'_S) + r \leq \frac{n-m}{n} + r.$$

□

The Nash mechanism is known to satisfy UFS and monotonicity (Lam et al., 2021). It therefore satisfies PF. □

B.3 Proof of Proposition 3

Proof. The Uniform Phantom mechanism is strategyproof since it is a Phantom mechanism and all Phantom mechanisms are strategyproof (see, e.g., Corollary 2 of Massó and Moreno De Barreda, 2011). We now

²⁷This is either $\max_{i \in S} \{x_i\}$ or $\min_{i \in S} \{x_i\}$.

prove that the Uniform Phantom mechanism satisfies PF. Let \mathbf{x} be an arbitrary location profile and let $S = \{1, \dots, s\} \subseteq N$ be a set of s agents; denote $r := \max_{i \in S} \{x_i\} - \min_{i \in S} \{x_i\}$. We prove $d(f(\mathbf{x}), x_i) \leq \frac{n-s}{n} + r$ for all $i \in S$. If $r = 1$, then the result is trivially true. Suppose that $r < 1$. If the location is within the range of the agents in S , PF is immediately satisfied. Next we consider the case there the location is outside the range of the agents in S . Recall that the Uniform Phantom mechanism places the facility at the n -th entity of the $2n - 1$ phantoms and agents. There are at least s agents in the range of the locations of the agents in S , so the facility is at most $n - s$ phantoms away from the nearest agent in S . Since the distance between adjacent phantoms is $1/n$, the facility is at most distance $(n - s)/n$ from the nearest agent in S . Hence, the maximum distance of the facility from any agent in S is $\frac{n-s}{n} + r$. \square

B.4 Tightness of Theorem 1

Proof. We wish to prove that each of the requirements in Theorem 1 are necessary for the theorem to hold. We show that if any one of the requirements (i.e., strategyproofness, unanimity, anonymity, and IFS) are removed, then the Theorem 1—not only fails to hold—but there exists such a mechanism. We do this by providing examples of mechanisms that fulfill all but one of the requirements of Theorem 1 but is not a phantom mechanism with the $n - 1$ phantoms contained in $[1/n, 1 - 1/n]$.

Strategyproofness. By Proposition 2, the midpoint mechanism is an example of a mechanism is not strategyproof, but satisfies unanimity, anonymity and IFS. However, the midpoint mechanism is not a phantom mechanism—this follows immediately because phantom mechanisms are necessarily strategyproof.

Unanimity. For $n \geq 2$, consider the constant- $\frac{1}{2}$ mechanism, whereby the facility is always located at $\frac{1}{2}$. This mechanism is clearly strategyproof and anonymous and does not satisfy unanimity. Furthermore, it satisfies IFS because the largest cost that any agent can experience is $\frac{1}{2}$, which is (weakly) lower than $1 - \frac{1}{n}$ for any $n \geq 2$. However, the constant- $\frac{1}{2}$ mechanism is not a phantom mechanism—this follows immediately because phantom mechanisms necessarily satisfy unanimity.²⁸

Anonymity. For simplicity take $n = 3$ and consider the mechanism f that locates the facility at

$$\begin{aligned} f(\mathbf{x}) := & \max\{\min\{x_1, \frac{1}{3} + \varepsilon\}, \min\{x_2, \frac{1}{3}\}, \min\{x_3, \frac{1}{3}\}, \\ & \min\{x_1, x_2, 1 - \frac{1}{3}\}, \min\{x_2, x_3, 1 - \frac{1}{3}\}, \min\{x_1, x_3, 1 - \frac{1}{3}\}, \\ & \min\{x_1, x_2, x_3, 1\}, 0\}, \end{aligned}$$

where $\varepsilon > 0$ is sufficiently small. This is a Generalized Median mechanism (Border and Jordan, 1983) and, hence, is strategyproof. Furthermore, it is easy to see that f is unanimous. Border and Jordan's (1983) Lemma 3 then says that f is Pareto efficient. However, the mechanism is not anonymous: for $\mathbf{x} = (0, 0, 1)$, $f(\mathbf{x}) = 1/3$, but for $\mathbf{x}' = (1, 0, 0)$, $f(\mathbf{x}') = 1/3 + \varepsilon$.

We now show that the mechanism satisfies IFS. Since the mechanism is Pareto efficient, IFS is trivially satisfied if $x_i \leq 1 - \frac{1}{3}$ for all $i \in N$ or if $x_i \geq \frac{1}{3}$ for all $i \in N$. Now consider some location profile \mathbf{x} that does

²⁸Recall that Definition 8 defines the family of Phantom mechanism as having $(n - 1)$ phantoms. In the literature, a broader class of mechanisms that has $(n + 1)$ phantoms is sometimes referred to as the class of phantom mechanism; unanimity is not necessarily satisfied by mechanisms in this broader class.

not belong to these trivial cases, i.e., there is at least one agent with location below $\frac{1}{3}$ (resp., $1 - \frac{1}{3}$) and at least one agent with location above $\frac{1}{3}$ (resp., $1 - \frac{1}{3}$). In these cases, IFS can only possibly be violated if $f(\mathbf{x}) > 1 - \frac{1}{3}$ or $f(\mathbf{x}) < \frac{1}{3}$. However, $f(\mathbf{x}) > 1 - \frac{1}{3}$ if and only if $x_i > 1 - \frac{1}{3}$ for all $i \in N$ —but the latter condition does not hold. Similarly, $f(\mathbf{x}) < \frac{1}{3}$ if and only if $x_i < \frac{1}{3}$ for all $i \in N$ —but, again, the latter condition does not hold. Therefore, we conclude that IFS is satisfied. However, the mechanism f is not a phantom mechanism—this follows immediately because phantom mechanisms are necessarily anonymous.

IFS. The Phantom mechanism that places all $n - 1$ phantoms at 0 is strategyproof, unanimous and anonymous. However, it does not satisfy IFS as the facility can be placed at 0 when there is an agent at 1. It is immediate that this Phantom mechanism violates the condition of the theorem that all phantoms are located in the interval $[1/n, 1 - 1/n]$. \square

B.5 Lemma 4 and proof of Lemma 4

Lemma 4. *A mechanism that is strategyproof, unanimous, and proportional must also be anonymous.*

Proof. Suppose f is strategyproof and satisfies proportionality and unanimity. We wish to show that f is anonymous (Definition 1). First we note that by [Border and Jordan's \(1983\) Proposition 2](#), any unanimous and strategyproof mechanism must satisfy the following uncompromising property.

Definition 13 (Uncompromising). *A mechanism f is uncompromising if, for every profile of locations \mathbf{x} , and each agent $i \in N$, if $f(\mathbf{x}) = y$ then*

$$x_i > y \implies f(x'_i, \mathbf{x}_{-i}) = y \quad \text{for all } x'_i \geq y \quad \text{and,} \quad (3)$$

$$x_i < y \implies f(x'_i, \mathbf{x}_{-i}) = y \quad \text{for all } x'_i \leq y. \quad (4)$$

Now consider an arbitrary profile of locations \mathbf{x} and an arbitrary permutation of the profile \mathbf{x} , which we denote by \mathbf{x}_σ . We will show that $f(\mathbf{x}) = f(\mathbf{x}_\sigma)$. First note that if \mathbf{x} is such that $x_i = c$ for some $c \in [0, 1]$, then $f(\mathbf{x}) = f(\mathbf{x}_\sigma)$ by unanimity. Therefore, we assume that \mathbf{x} is such that $x_i \neq x_j$ for some $i, j \in N$.

Case 1. Suppose that $f(\mathbf{x}) \neq x_i$ for any $i \in N$. Recall that [Border and Jordan's \(1983\) Lemma 3](#) says that any strategyproof and unanimous mechanism is Pareto efficient; therefore, $\min_{i \in N} x_i \leq f(\mathbf{x}) \leq \max_{i \in N} x_i$. Now if all agents strictly below (resp., above) $f(\mathbf{x})$ shift their location to 0 (resp., 1), then, by the uncompromising property, the facility location must be unchanged. Let \mathbf{x}' denote this augmented location profile and let k' denote the number of agents with $x'_i = 1$. By proportionality, it must be that $f(\mathbf{x}) = f(\mathbf{x}') = \frac{k'}{n}$. Now consider the permutation of the profile \mathbf{x}' , i.e., \mathbf{x}'_σ . The implication of the proportionality property is independent of agent labels; therefore, $f(\mathbf{x}'_\sigma) = f(\mathbf{x}')$. Now shift the agent locations in \mathbf{x}'_σ so that they replicate the permuted location profile \mathbf{x}_σ —note that this process only involves agents strictly above (resp., below) $f(\mathbf{x}'_\sigma)$ moving to a location above (resp., below) $f(\mathbf{x}'_\sigma)$. Therefore, by the uncompromising property, it must be that $f(\mathbf{x}'_\sigma) = f(\mathbf{x}_\sigma)$. Combining the three sets of equalities gives

$$f(\mathbf{x}) = f(\mathbf{x}') = f(\mathbf{x}'_\sigma) = f(\mathbf{x}_\sigma).$$

That is, the facility location is unchanged by permutations, i.e., anonymity is satisfied.

Case 2. Suppose that $f(\mathbf{x}) = x_i$ for some $i \in N$. Let $M \subseteq N$ be the subset of agents with $x_i = f(\mathbf{x})$. Let $M_0, M_1 \subseteq N$ correspond to the subset of agents with location strictly below and strictly above $f(\mathbf{x})$, respectively. Denote $|M_0| = k_0$ and $|M_1| = k_1$. We first show that

$$\frac{k_1}{n} \leq f(\mathbf{x}). \quad (5)$$

For sake of contradiction, suppose that (5) does not hold (i.e., $f(\mathbf{x}) < \frac{k_1}{n}$), and consider the location profile \mathbf{x}' obtained by modifying \mathbf{x} such that the M_0 (resp., M_1) agents' locations are shifted to 0 (resp., 1) and the other agents' (i.e., those in M) have location unchanged. By the uncompromising property, $f(\mathbf{x}') = f(\mathbf{x})$. Now consider the modified location profile \mathbf{x}'' such that $x''_i = x'_i$ for all $i \notin M$ and $x''_i = 0$ for all $i \in M$. By proportionality, $f(\mathbf{x}'') = \frac{k_1}{n}$ and, by supposition that $f(\mathbf{x}) < \frac{k_1}{n}$, we have

$$f(\mathbf{x}') = f(\mathbf{x}) < \frac{k_1}{n} = f(\mathbf{x}''). \quad (6)$$

Now notice that the profile \mathbf{x}' can be obtained from \mathbf{x}'' by shifting the subset of M agents' locations from 0 to $f(\mathbf{x})$, which is to the left of $f(\mathbf{x}'')$. The uncompromising property then requires that

$$f(\mathbf{x}') = f(\mathbf{x}'') = \frac{k_1}{n},$$

which contradicts (6). We conclude that (5) holds.

With condition (5) in hand, we can now proceed by considering two subcases.

Subcase 2a. Suppose that $f(\mathbf{x}) = \frac{k}{n}$ for some $k \in \{0, \dots, n\}$. Consider any profile of locations $\mathbf{x}' \in \{0, 1\}^n$ with k agents at location 1. By proportionality, $f(\mathbf{x}') = \frac{k}{n} = f(\mathbf{x})$. Note that the proportionality axiom is independent of agent labels and, by (5), $f(\mathbf{x}) \geq \frac{k_1}{n} \implies k \geq k_1$. Therefore, the permuted location of profile \mathbf{x}_σ can be attained by relabeling agents in \mathbf{x}' and then shifting their reports from 1 (resp., 0) to their original location that is weakly above (resp., below) $f(\mathbf{x}') = \frac{k}{n}$ —by the uncompromising property, the facility location will not change. Therefore, $f(\mathbf{x}_\sigma) = f(\mathbf{x}') = \frac{k}{n} = f(\mathbf{x})$, as required.

Subcase 2b. Suppose that $f(\mathbf{x}) \neq \frac{k}{n}$ for any $k \in \{0, \dots, n\}$. Let k^* be the smallest integer such that $f(\mathbf{x}) < \frac{k^*}{n}$; by (5), $k^* > k_1$. Now consider any location profile $\mathbf{x}' \in \{0, 1\}^n$ with exactly k^* agents at 1. By proportionality,

$$f(\mathbf{x}') = \frac{k^*}{n} > f(\mathbf{x}). \quad (7)$$

Now consider any subset $G \subseteq N$ that contains $|k^* - k_1| > 1$ agents who are located at 1. Let \mathbf{x}'' denote the profile obtained from \mathbf{x}' by shifting the G agents' locations to $\hat{x}_G = f(\mathbf{x})$. We shall prove that

$$f(\mathbf{x}'') = \hat{x}_G. \quad (8)$$

To see this, suppose that this is not the case. Then either

$$\hat{x}_G < f(\mathbf{x}'') \quad \text{or} \quad (9)$$

$$f(\mathbf{x}'') < \hat{x}_G. \quad (10)$$

In the former case, all agents in G have location strictly below $f(\mathbf{x}'')$ —namely, $\hat{x}_G = f(\mathbf{x})$. Therefore, by the uncompromising property, if all agents in G shift their location to 0 in the profile \mathbf{x}'' , then the facility location is unchanged and continues to be located at $f(\mathbf{x}'')$. Proportionality then requires that the facility then be located at $\frac{k_1}{n}$ and, hence, $f(\mathbf{x}'') = \frac{k_1}{n}$. But then (9) implies that $\hat{x}_G < \frac{k_1}{n}$, which in turn implies that $f(\mathbf{x}) = \hat{x}_G < \frac{k_1}{n}$ —this contradicts (5). In the latter case, all agents in G have location strictly above $f(\mathbf{x}'')$ —namely, $\hat{x}_G = f(\mathbf{x})$. Therefore, by the uncompromising property, if all agents in G shift their location back to 1 in the profile \mathbf{x}'' , then the facility location is unchanged and, by proportionality, is located at $\frac{k^*}{n}$. Hence, $f(\mathbf{x}'') = \frac{k^*}{n}$. Using (10), this implies that $\frac{k^*}{n} < f(\mathbf{x})$, which contradicts (7).

Now given (8), by the uncompromising property, shifting any agent with location at 0 (resp., 1) to any location weakly below (resp., above) \hat{x}_G must leave the facility's location unchanged. Therefore, for any profile with exactly k_0, k_1 agents strictly below \hat{x}_G and strictly above \hat{x}_G and $n - k_0 - k_1$ agents located at \hat{x}_G , the facility must be located at \hat{x}_G . But—since $\hat{x}_G = f(\mathbf{x})$ —it is immediate that any permutation of \mathbf{x} , say \mathbf{x}_σ , satisfies these 3 properties; hence,

$$f(\mathbf{x}_\sigma) = f(\mathbf{x}).$$

We conclude that any mechanism that satisfies strategyproofness, proportionality, unanimity must also satisfy anonymity. \square

B.6 Unanimity is Necessary for Theorem 2

Proof. We wish to prove that Theorem 2 does not hold if unanimity is removed. It suffices to consider the following mechanism for $n = 2$

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1, x_2 \leq 1/4, \\ 1 & \text{if } x_1, x_2 \geq 3/4, \\ 1/2 & \text{else.} \end{cases}$$

This mechanism is clearly anonymous, satisfies proportionality and is not the Uniform Phantom mechanism. It remains to show that it is strategyproof. Using a symmetry argument, we focus on deviations by agent 1 without loss of generality. Suppose $f(\mathbf{x}) = 0$, then it must be that $x_1 \leq 1/4$. But then agent 1 obtains the minimum possible distance to facility (given x_1 and given the mechanism's range); hence, no deviation can strictly decrease her distance. Suppose $f(\mathbf{x}) = 1/2$, then either $x_1 \in (1/4, 3/4)$ or $x_2 \in (1/4, 3/4)$. In the former case, agent 1 obtains the minimum possible distance to the facility (given x_1 and given the mechanism's range); hence, no deviation can strictly decrease her distance. In the latter case, no deviation by agent 1 can change the facility location. Finally, suppose $f(\mathbf{x}) = 1$, then it must be that $x_1 \geq 3/4$. But then agent 1 obtains the minimum possible distance to facility (given x_1 and given the mechanism's range); hence, no deviation can strictly decrease her distance. Therefore, the mechanism is SP. \square

B.7 Proof of Lemma 1

Proof. We wish to prove that a mechanism f that satisfies IFS has welfare approximation of at least $1 + \frac{n-2}{n^2-2n+2}$. To this end, suppose f satisfies IFS and consider the profile of locations $\mathbf{x} \in \{0, 1\}^n$ that places $n - 1$ agents at 0. IFS requires that $f(\mathbf{x}) = \frac{1}{n}$, which provides welfare

$$\Phi(f(\mathbf{x})) = (n-1)\left(1 - \frac{1}{n}\right) + \frac{1}{n} = \frac{(n-1)^2 + 1}{n}.$$

However, for this instance, the welfare-optimal welfare is $\Phi^*(\mathbf{x}) = n - 1$ (obtained by locating the facility at the median location, 0). Therefore, the approximation ratio of f is at least

$$\frac{\Phi^*(\mathbf{x})}{\Phi(f(\mathbf{x}))} = \frac{n(n-1)}{(n-1)^2 + 1} = 1 + \frac{n-2}{(n-1)^2 + 1}.$$

□

B.8 Proof of Theorem 4

Proof. We wish to prove that among all IFS mechanisms, the Constrained Median mechanism provides the best approximation guarantee i.e., it achieves the approximation ratio in Lemma 1. Let f_{CM} denote the Constrained Median mechanism. We shall prove that for any location profile $\mathbf{x} \in [0, 1]^n$ there exists some profile $\tilde{\mathbf{x}} \in \{0, 1\}^n$ such that

$$\frac{\Phi^*(\tilde{\mathbf{x}})}{\Phi(f_{\text{CM}}(\tilde{\mathbf{x}}))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{CM}}(\mathbf{x}))}, \quad (11)$$

which implies that

$$\max_{\mathbf{x} \in [0, 1]^n} \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{CM}}(\mathbf{x}))} = \max_{\mathbf{x} \in \{0, 1\}^n} \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{CM}}(\mathbf{x}))}.$$

We begin by noting that, whenever $f_{\text{CM}}(\mathbf{x}) \in (1/n, 1 - 1/n)$, the facility location coincides with the median location and f_{CM} obtains the maximum welfare. Thus, we can restrict our attention to profiles such that $f_{\text{CM}}(\mathbf{x}) \notin (1/n, 1 - 1/n)$. We proceed to prove (11) by considering a sequence of profiles that modify \mathbf{x} into some profile $\tilde{\mathbf{x}} \in \{0, 1\}^n$ such that each modified profile guarantees a weakly higher approximation ratio.

Let the agent labels be ordered such that $x_1 \leq \dots \leq x_n$; let $i = \text{med}$ denote the median agent. Without loss of generality, suppose $f_{\text{CM}}(\mathbf{x}) \in [0, 1/n]$. This implies that the median agent is weakly below $f_{\text{CM}}(\mathbf{x})$, i.e., $x_{\text{med}} \leq f_{\text{CM}}(\mathbf{x})$. To assist with visualizing the proof technique, we provide a running example with $n = 5$ agents. Figure 4 illustrates a profile \mathbf{x} such that $x_{\text{med}} \leq f_{\text{CM}}(\mathbf{x})$; in particular, $x_{\text{med}} = x_3$ and $f_{\text{CM}}(\mathbf{x}) = 1/5$.



Fig. 4: Running example. Profile \mathbf{x}

First, consider the modified profile \mathbf{x}' such that $x'_i = 0$ for $i \in N' := \{i : i < \text{med}\}$ and $x'_i = x_i$ for all $i \notin N'$. Applying this operation to the running example illustrated in Figure 4, we obtain the profile illustrated in Figure 5.


 Fig. 5: Running example. Profile \mathbf{x}'

In this modified profile, we have moved all agents strictly left of the median agent to 0, so neither the welfare-optimal (median) location nor the facility location under f_{CM} changes. Hence, relative to $\Phi^*(\mathbf{x})$ and $\Phi(f_{\text{CM}}(\mathbf{x}))$, the optimal welfare, $\Phi^*(\mathbf{x}')$, and the welfare provided by f_{CM} , $\Phi(f_{\text{CM}}(\mathbf{x}'))$, decrease by the same amount—namely, $\sum_{i \in N'} x_i \geq 0$. We conclude that

$$\frac{\Phi^*(\mathbf{x}')}{\Phi(f_{\text{CM}}(\mathbf{x}'))} = \frac{\Phi^*(\mathbf{x}) - \sum_{i \in N'} x_i}{\Phi(f_{\text{CM}}(\mathbf{x})) - \sum_{i \in N'} x_i} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{CM}}(\mathbf{x}))},$$

where the final inequality follows because $(x - a)/(y - a) \geq x/y$ for any $a \geq 0$ and $0 < y \leq x$.

Next we consider the modified profile \mathbf{x}'' such that $x''_{\text{med}} = 0$ and $x''_i = x'_i$ for all $i \neq \text{med}$. Applying this operation to the running example illustrated in Figure 5, we obtain the profile illustrated in Figure 6.


 Fig. 6: Running example. Profile \mathbf{x}''

In this modified profile, the welfare-optimal (median) location moves from x_{med} to 0, so the facility location under f_{CM} remains unchanged, i.e., $f_{\text{CM}}(\mathbf{x}'') = f(\mathbf{x}')$. Hence, relative to $\Phi^*(\mathbf{x}')$, the optimal welfare, $\Phi^*(\mathbf{x}'')$, decreases by x_{med} if n is even and decreases by 0 otherwise; relative to $\Phi(f_{\text{CM}}(\mathbf{x}'))$, the welfare under f_{CM} , $\Phi(f_{\text{CM}}(\mathbf{x}''))$, decreases by x_{med} . Defining the indicator function $\mathbb{I}_{n \text{ even}}$ as 1 if n is even and 0 otherwise, we conclude that

$$\frac{\Phi^*(\mathbf{x}'')}{\Phi(f_{\text{CM}}(\mathbf{x}''))} = \frac{\Phi^*(\mathbf{x}') - x_{\text{med}} \mathbb{I}_{n \text{ even}}}{\Phi(f_{\text{CM}}(\mathbf{x}')) - x_{\text{med}}} \geq \frac{\Phi^*(\mathbf{x}')}{\Phi(f_{\text{CM}}(\mathbf{x}'))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{CM}}(\mathbf{x}))}.$$

Now either $x_n \geq 1/n$ or $x_n < 1/n$. Suppose the former case holds, then

$$f_{\text{CM}}(\mathbf{x}) = 1/n = f_{\text{CM}}(\mathbf{x}') = f_{\text{CM}}(\mathbf{x}'').$$

Consider the modified profile $\mathbf{x}''' \in \{0, 1\}^n$ such that $x'''_i = 1$ for all $i \in N''' := \{i : x''_i \geq 1/n\}$ and $x'''_i = 0$ for all $i \notin N'''$. Applying this operation to the running example illustrated in Figure 6, we obtain the profile illustrated in Figure 7.

Fig. 7: Running example. Profile x'''

In this modified profile, we have moved all agent locations that were weakly right of $1/n$ in x'' to 1 and all other agents' locations are shifted to 0. Under x''' , the welfare-optimal (median) location remains unchanged (at $x''_{med} = 0$) and the facility location under f_{CM} remains at $1/n$. We conclude that

$$\frac{\Phi^*(x''')}{\Phi(f_{CM}(x'''))} = \frac{\Phi^*(x'') - \sum_{i \in N'''} (1 - x''_i) + \sum_{i \notin N'''} x''_i}{\Phi(f_{CM}(x'')) - \sum_{i \in N'''} (1 - x''_i) - \sum_{i \notin N'''} x''_i} \geq \frac{\Phi^*(x'')}{\Phi(f_{CM}(x''))} \geq \frac{\Phi^*(x)}{\Phi(f_{CM}(x))}.$$

Therefore, there exists $\tilde{x} \in \{0, 1\}^n$ —namely, x''' —with weakly higher approximation ratio than x .

Finally, suppose the latter case, $x_n < 1/n$, holds. In this case,

$$f_{CM}(x) = x_n = f_{CM}(x') = f_{CM}(x'') < 1/n.$$

Consider the modified profile x'''' such that $x''''_n = 1/n$ and $x''''_i = x''_i$ otherwise. In this modified profile, we have moved the last agent x''_n to $1/n$, so the welfare-optimal (median) location remains unchanged (at $x''_{med} = 0$) and the facility location under f_{CM} shifts to $1/n$, i.e., $f_{CM}(x''') = 1/n$. We conclude that

$$\frac{\Phi^*(x''')}{\Phi(f_{CM}(x'''))} = \frac{\Phi^*(x'') - (1/n - x_n)}{\Phi(f_{CM}(x'')) - (n-1)(1/n - x_n)} \geq \frac{\Phi^*(x'')}{\Phi(f_{CM}(x''))}.$$

Now the same steps from the former case can be used to show that there exists $\tilde{x} \in \{0, 1\}^n$ with weakly higher approximation ratio than x . Therefore, (11) holds.

It is straightforward to calculate the maximum approximation ratio among profiles $\tilde{x} \in \{0, 1\}^n$. The maximum is attained when \tilde{x} has $(n-1)$ agents at 0 and 1 agent at 1, which provides the required approximation ratio (see Proof of Lemma 1) \square

B.9 Proof of Lemma 2

Proof. We wish to prove that any mechanism satisfying UFS (or proportionality or PF) has a welfare approximation of at least (2). To this end, suppose f satisfies UFS. Consider the agent location profile $x \in \{0, 1\}^n$ that has $k \leq n/2$ agents at 1. The optimal welfare $\Phi^*(x) = n - k$ is obtained by placing the facility at the median location 0. UFS requires that $f(x) = \frac{k}{n}$, which provides welfare $\Phi(f(x)) = \frac{k^2 + (n-k)^2}{n}$. Therefore, the approximation ratio is

$$\frac{\Phi^*(x)}{\Phi(f(x))} = \frac{n(n-k)}{k^2 + (n-k)^2}.$$

Maximizing the above expression with respect to $k \in \mathbb{N} : 0 \leq k \leq n/2$ provides the approximation bound in the lemma statement. Defining $r := \frac{k}{n}$, this ratio is equal to

$$\frac{\Phi^*(\mathbf{x})}{\Phi(f(\mathbf{x}))} = \frac{1-r}{2r^2-2r+1}.$$

The derivative of this expression with respect to r is $\frac{2r^2-4r+1}{(2r^2-2r+1)^2}$, which is equal to 0 when $r = \frac{2-\sqrt{2}}{2}$ or $r = \frac{2+\sqrt{2}}{2}$. We ignore the latter as k cannot exceed n , and we note that $r = \frac{2-\sqrt{2}}{2}$ is a maximum point as the derivative is positive for $r \in [0, \frac{2-\sqrt{2}}{2})$ and negative for $r \in (\frac{2-\sqrt{2}}{2}, 1]$. We therefore deduce that $\frac{\Phi^*(\mathbf{x})}{\Phi(f(\mathbf{x}))}$ is maximized when $\frac{k}{n} = \frac{2-\sqrt{2}}{2}$, providing approximation ratio $\frac{\sqrt{2}+1}{2}$. This approximation ratio can be achieved asymptotically as $n \rightarrow \infty$. \square

B.10 Proof of Theorem 5

Proof. We wish to prove that among all UFS (or proportional or PF) mechanisms, the Uniform Phantom mechanism provides the best approximation guarantee, i.e., it achieves the approximation ratio in Lemma 2. To this end, let f_{Unif} denote the Uniform Phantom mechanism. We prove that for any location profile $\mathbf{x} \in [0, 1]^n$ there exists some profile $\tilde{\mathbf{x}} \in \{0, 1\}^n$ such that

$$\frac{\Phi^*(\tilde{\mathbf{x}})}{\Phi(f_{\text{Unif}}(\tilde{\mathbf{x}}))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{Unif}}(\mathbf{x}))}. \quad (12)$$

This implies that

$$\max_{\mathbf{x} \in [0, 1]^n} \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{Unif}}(\mathbf{x}))} = \max_{\mathbf{x} \in \{0, 1\}^n} \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{Unif}}(\mathbf{x}))}.$$

Let the agent labels be ordered such that $x_1 \leq \dots \leq x_n$; let $i = \text{med}$ denote the median agent. Suppose without loss of generality that $\mathbf{x} : x_{\text{med}} < f_{\text{Unif}}(\mathbf{x})$; if $x_{\text{med}} = f_{\text{Unif}}(\mathbf{x})$, then (12) is trivially satisfied. To assist with visualizing the proof technique, we provide a running example with $n = 6$ agents. Figure 8 illustrates a profile \mathbf{x} such that $x_{\text{med}} < f_{\text{Unif}}(\mathbf{x})$; in particular, $x_{\text{med}} = x_3$ and $f_{\text{Unif}}(\mathbf{x}) = x_5$.

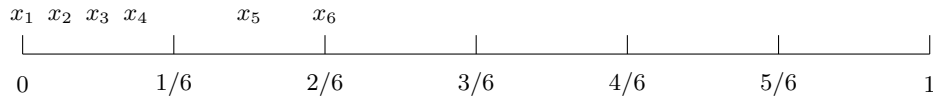
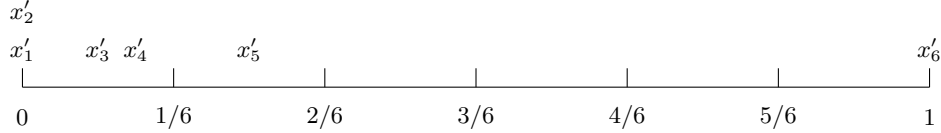


Fig. 8: Running example. Profile \mathbf{x}

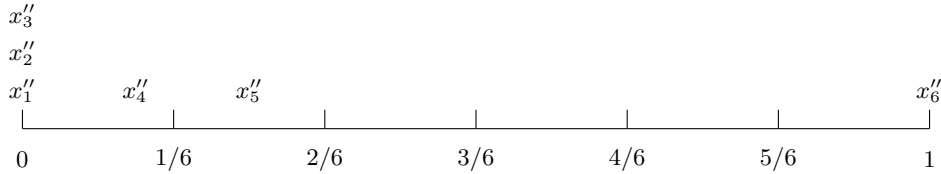
First, consider the modified profile \mathbf{x}' such that $x'_i = 1$ for all $i \in N' := \{i : f_{\text{Unif}}(\mathbf{x}) < x_i\}$, $x'_i = 0$ for all $i \in N'' := \{i : i < \text{med}\}$, and $x'_i = x_i$ for all $i \notin N' \cup N''$ —note that $N' \cap N'' = \emptyset$. Applying this operation to the running example illustrated in Figure 8, we obtain the profile illustrated in Figure 9.

Fig. 9: Running example. Profile \mathbf{x}'

In this modified profile, we have moved all agents with location strictly to the right of the Uniform Phantom location to 1, and all agents strictly left of the median to 0. Under \mathbf{x}' , neither the welfare-optimal (median) location nor the facility location under f_{Unif} changes. Therefore, relative to $\Phi^*(\mathbf{x})$ and $\Phi(f_{\text{Unif}}(\mathbf{x}))$, the optimal welfare, $\Phi^*(\mathbf{x}')$, and the welfare under f , $\Phi(f_{\text{Unif}}(\mathbf{x}'))$, decrease by the same amount—namely, $\sum_{i \in N'}(1 - x_i) + \sum_{i \in N''} x_i \geq 0$. We conclude that

$$\frac{\Phi^*(\mathbf{x}')}{\Phi(f_{\text{Unif}}(\mathbf{x}'))} = \frac{\Phi^*(\mathbf{x}) - \sum_{i \in N'}(1 - x_i) - \sum_{i \in N''} x_i}{\Phi(f_{\text{Unif}}(\mathbf{x})) - \sum_{i \in N'}(1 - x_i) - \sum_{i \in N''} x_i} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{Unif}}(\mathbf{x}))}.$$

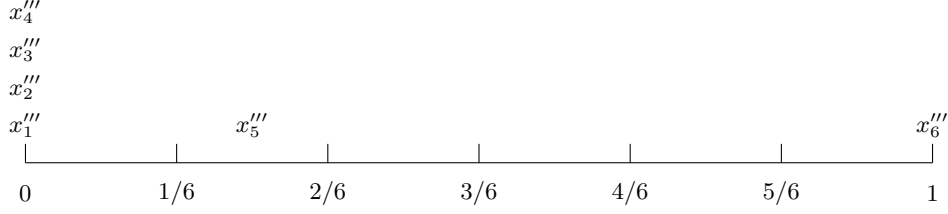
Next we consider the modified profile \mathbf{x}'' such that $x''_{\text{med}} = 0$ and $x''_i = x'_i$ for all $i \neq \text{med}$. Applying this operation to the running example illustrated in Figure 9, we obtain the profile illustrated in Figure 10.

Fig. 10: Running example. Profile \mathbf{x}''

In this modified profile, the welfare-optimal (median) location moves from x_{med} to 0 and the facility location under f_{Unif} remains unchanged, i.e., $f_{\text{Unif}}(\mathbf{x}'') = f_{\text{Unif}}(\mathbf{x}')$. Hence, relative to $\Phi^*(\mathbf{x}')$, the optimal welfare, $\Phi^*(\mathbf{x}'')$, decreases by x_{med} if n is even and decreases by 0 otherwise; relative to $\Phi(f_{\text{Unif}}(\mathbf{x}'))$, the welfare under f_{Unif} , $\Phi(f_{\text{Unif}}(\mathbf{x}''))$, decreases by x_{med} . We conclude that

$$\frac{\Phi^*(\mathbf{x}'')}{\Phi(f_{\text{Unif}}(\mathbf{x}''))} = \frac{\Phi^*(\mathbf{x}') - x_{\text{med}} \mathbb{I}_{n \text{ even.}}}{\Phi(f_{\text{Unif}}(\mathbf{x}')) - x_{\text{med}}} \geq \frac{\Phi^*(\mathbf{x}')}{\Phi(f_{\text{Unif}}(\mathbf{x}'))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{Unif}}(\mathbf{x}))}.$$

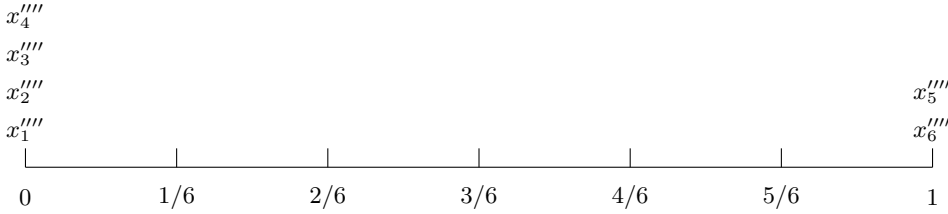
Now consider the modified profile \mathbf{x}''' such that $x'''_i = 0$ for all $i \in N''' := \{i : x''_i < f_{\text{Unif}}(\mathbf{x})\}$ and $x'''_i = x''_i$ for all $i \notin N'''$. Applying this operation to the running example illustrated in Figure 10, we obtain the profile illustrated in Figure 11.


 Fig. 11: Running example. Profile \mathbf{x}'''

In this modified profile, we move all agents strictly left of the Uniform Phantom facility location to 0, so neither the welfare-optimal (median) location of 0, nor the facility location under f_{Unif} changes. Hence, relative to $\Phi^*(\mathbf{x}'')$, the optimal welfare, $\Phi^*(\mathbf{x}''')$, increases by $\sum_{i \in N'''} x_i''$; relative to $\Phi(f_{\text{Unif}}(\mathbf{x}''))$, the welfare under f_{Unif} , $\Phi(f_{\text{Unif}}(\mathbf{x}'''))$, decreases by $\sum_{i \in N'''} x_i''$. We conclude that

$$\frac{\Phi^*(\mathbf{x}''')}{\Phi(f_{\text{Unif}}(\mathbf{x}'''))} = \frac{\Phi^*(\mathbf{x}'') + \sum_{i \in N'''} x_i''}{\Phi(f_{\text{Unif}}(\mathbf{x}'')) - \sum_{i \in N'''} x_i''} \geq \frac{\Phi^*(\mathbf{x}'')}{\Phi(f_{\text{Unif}}(\mathbf{x}''))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{Unif}}(\mathbf{x}))}.$$

Lastly, consider the modified profile \mathbf{x}'''' such that $x_i'''' = 1$ for all $i \in N'''' = \{i : f_{\text{Unif}}(\mathbf{x}''') \leq x_i\}$ and $x_i'''' = 0$ for all $i \notin N''''$. Applying this operation to the running example illustrated in Figure 11, we obtain the profile illustrated in Figure 12. In Figure 12, the Uniform Phantom location increases to $2/6$.


 Fig. 12: Running example. Profile \mathbf{x}''''

Under this modified profile, we have moved all agents weakly right of the Uniform Phantom location to 1, so the welfare-optimal (median) location does not change; the facility location under f_{Unif} moves to a (weakly) higher location, i.e., $f_{\text{Unif}}(\mathbf{x}''') : f_{\text{Unif}}(\mathbf{x}''') \leq f_{\text{Unif}}(\mathbf{x}'''')$.

Relative to $\Phi^*(\mathbf{x}''')$, the optimal welfare, $\Phi^*(\mathbf{x}'''')$, decreases by $\sum_{i \in N''''} (1 - x_i''')$. Relative to $\Phi(f_{\text{Unif}}(\mathbf{x}'''))$, the welfare under f_{Unif} , $\Phi(f_{\text{Unif}}(\mathbf{x}''''))$ also decreases by $\sum_{i \in N''''} (1 - x_i''')$ due to the movement in agents in N'''' . In addition, $\Phi(f_{\text{Unif}}(\mathbf{x}''''))$ decreases due to the movement in the facility location: this follows because the number of agents at location 0 is weakly higher than the number of agents at location 1. Let this additional decrease in $\Phi(f_{\text{Unif}}(\mathbf{x}''''))$ be denoted by $\Delta > 0$. We conclude that

$$\frac{\Phi^*(\mathbf{x}'''')}{\Phi(f_{\text{Unif}}(\mathbf{x}''''))} = \frac{\Phi^*(\mathbf{x}''') - \sum_{i \in N''''} (1 - x_i''')}{\Phi(f_{\text{Unif}}(\mathbf{x}''')) - \sum_{i \in N''''} (1 - x_i''') - \Delta} \geq \frac{\Phi^*(\mathbf{x}''')}{\Phi(f_{\text{Unif}}(\mathbf{x}'''))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{Unif}}(\mathbf{x}))}.$$

Therefore, there exists $\tilde{x} \in \{0, 1\}^n$ —namely, \mathbf{x}''' —with weakly higher approximation ratio than \mathbf{x} . Therefore, (12) holds. The theorem statement follows from the fact that the approximation ratio in Lemma 2 is constructed by restricting agents to locations $\{0, 1\}$. \square

C Average Mechanism Results

Proposition 4. *The average mechanism satisfies PF.*

Proof. The average mechanism satisfies UFS and monotonicity. By Lemma 3, it also satisfies PF. \square

Proposition 5. *The average mechanism achieves the approximation ratio in Lemma 2.*

Proof. Let f_{avg} denote the average mechanism. We prove that for any location profile $\mathbf{x} \in [0, 1]^n$ there exists some profile $\tilde{\mathbf{x}} \in \{0, 1\}^n$ such that

$$\frac{\Phi^*(\tilde{\mathbf{x}})}{\Phi(f_{\text{avg}}(\tilde{\mathbf{x}}))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{avg}}(\mathbf{x}))}. \quad (13)$$

This implies that

$$\max_{\mathbf{x} \in [0, 1]^n} \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{avg}}(\mathbf{x}))} = \max_{\mathbf{x} \in \{0, 1\}^n} \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{avg}}(\mathbf{x}))}.$$

Let the agent labels be ordered such that $x_1 \leq \dots \leq x_n$; let $i = \text{med}$ denote the median agent. Suppose without loss of generality that for odd n , we have $\mathbf{x} : x_{\text{med}} < f_{\text{avg}}(\mathbf{x})$ and for even n , we have $\mathbf{x} : x_{\frac{n}{2}+1} < f_{\text{avg}}(\mathbf{x})$. This is because (13) is trivially satisfied for odd n if $x_{\text{med}} = f_{\text{avg}}(\mathbf{x})$, and it is satisfied for even n if $x_{\text{med}} \leq f_{\text{avg}}(\mathbf{x}) \leq x_{\frac{n}{2}+1}$.

First, consider the modified profile \mathbf{x}' such that $x'_i = 1$ for all $i \in S := \{i : x_i \geq f_{\text{avg}}(\mathbf{x})\}$ and $x'_i = x_i$ for all $i \in \bar{S}$. In this modified profile, the welfare-optimal (median) location does not change, and the facility location under f_{avg} moves towards the agents in S . Denoting this change in facility location as $\Delta > 0$ and noting that $|S| < n - |S|$ due to the facility being located right of the welfare-optimal interval/median, the welfare under f_{avg} decreases by $((n - |S|) - |S|)\Delta > 0$ from the facility moving towards the $|S|$ agents at 1 and away from the remaining $n - |S|$ agents. Due to the agent movements, the optimal welfare $\Phi^*(\mathbf{x}')$ and the welfare under f , $\Phi(f_{\text{avg}}(\mathbf{x}'))$ both decrease by the same amount—namely, $\sum_{i \in S} (1 - x_i)$ —relative to $\Phi^*(\mathbf{x})$ and $\Phi(f_{\text{avg}}(\mathbf{x}))$. We conclude that

$$\frac{\Phi^*(\mathbf{x}')}{\Phi(f_{\text{avg}}(\mathbf{x}'))} = \frac{\Phi^*(\mathbf{x}) - \sum_{i \in S} (1 - x_i)}{\Phi(f_{\text{avg}}(\mathbf{x})) - \sum_{i \in S} (1 - x_i) - (n - 2|S|)\Delta} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{\text{avg}}(\mathbf{x}))}.$$

Now consider the modified profile \mathbf{x}'' such that $x''_i = 0$ for all $i \in S' := \{i : x'_i < x_{\text{med}}\}$, for all $i \in S'' := \{i : x_{\text{med}} < x_i < f_{\text{avg}}(\mathbf{x}')\}$ and for $i = \text{med}$, and $x''_i = x'_i$ otherwise. The change in optimal welfare, which we will denote as Δ'_{opt} , can be quantified by observing the agents' movements sequentially. The optimal welfare decreases by $\sum_{i \in S'} x_i$ from the agents of S' moving to 0. Next, the median agent (and welfare-optimal facility location) moving towards the S' agents at 0 causes the optimal welfare to decrease by $x_{\text{med}} \mathbb{1}_{n \text{ even}}$. Lastly, the remaining agents of S'' move towards the median at 0, increasing the optimal welfare by $\sum_{i \in S''} x_i$. We therefore have

$$\Delta'_{\text{opt}} = - \sum_{i \in S'} x_i - x_{\text{med}} \mathbb{1}_{n \text{ even}} + \sum_{i \in S''} x_i. \quad (14)$$

We next quantify the change in welfare corresponding to f_{avg} , which we denote as Δ'_{avg} . The welfare decreases by $\sum_{i \in S'} x_i + x_{med} + \sum_{i \in S''} x_i$ from the agent movements, and increases by $(n - 2|S|)\frac{1}{n} \sum_{i \in S' \cup \{med\} \cup S''} x_i$ from the facility moving towards the $n - |S|$ agents at 0 and away from the $|S|$ agents at 1. We therefore have

$$\Delta'_{avg} = - \sum_{i \in S'} x_i - x_{med} - \sum_{i \in S''} x_i + (n - 2|S|)\frac{1}{n} \sum_{i \in S' \cup \{med\} \cup S''} x_i. \quad (15)$$

We now show that $\Delta'_{opt} > \Delta'_{avg}$ by subtracting Equations (14) and (15). We first note that $|S''| = \frac{n}{2} - |S|$ for even n and $|S''| = \frac{n-1}{2} - |S|$ for odd n . If n is even, we have

$$\begin{aligned} \Delta'_{opt} - \Delta'_{avg} &= 2 \sum_{i \in S''} x_i - \frac{n - 2|S|}{n} \left(\sum_{i \in S' \cup \{med\}} x_i + \sum_{i \in S''} x_i \right) \\ &\geq 2 \sum_{i \in S''} x_i - \frac{2|S''|}{n} \left(\frac{n}{2} x_{med} + \sum_{i \in S''} x_i \right) \\ &= 2 \sum_{i \in S''} x_i - |S''| x_{med} - \frac{2|S''|}{n} \sum_{i \in S''} x_i \\ &= \left(\sum_{i \in S''} x_i - |S''| x_{med} \right) + \left(\sum_{i \in S''} x_i - \frac{2|S''|}{n} \sum_{i \in S''} x_i \right) \\ &\geq 0, \end{aligned}$$

where the first inequality is due to $x_{med} > x_i$ for all $i \in S'$, and we have $\sum_{i \in S''} x_i - |S''| x_{med} \geq 0$ due to $x_i > x_{med}$ for all $i \in S''$. Now if n is odd, we have

$$\begin{aligned} \Delta'_{opt} - \Delta'_{avg} &= 2 \sum_{i \in S''} x_i + x_{med} - \frac{n - 2|S|}{n} \left(\sum_{i \in S'} x_i + x_{med} + \sum_{i \in S''} x_i \right) \\ &\geq 2 \sum_{i \in S''} x_i + x_{med} - \frac{2|S''| + 1}{n} \left(\frac{n-1}{2} x_{med} + x_{med} + \sum_{i \in S''} x_i \right) \\ &= \left(\sum_{i \in S''} x_i - \frac{(2|S''| + 1)(n-1)}{2n} x_{med} \right) + \left(x_{med} + \sum_{i \in S''} x_i \right) \left(1 - \frac{2|S''| + 1}{n} \right) \\ &= \left(\sum_{i \in S''} x_i - |S''| x_{med} - \frac{|S|}{n} x_{med} \right) + \left(x_{med} + \sum_{i \in S''} x_i \right) \left(\frac{2|S|}{n} \right) \\ &\geq 0. \end{aligned}$$

We have shown that $\Delta'_{opt} > \Delta'_{avg}$, meaning that we have

$$\frac{\Phi^*(\mathbf{x}'')}{\Phi(f_{avg}(\mathbf{x}''))} = \frac{\Phi^*(\mathbf{x}') + \Delta'_{opt}}{\Phi(f_{avg}(\mathbf{x}')) + \Delta'_{avg}} \geq \frac{\Phi^*(\mathbf{x}')}{\Phi(f_{avg}(\mathbf{x}'))} \geq \frac{\Phi^*(\mathbf{x})}{\Phi(f_{avg}(\mathbf{x}))}.$$

Therefore, there exists $\tilde{\mathbf{x}} \in \{0, 1\}^n$ —namely, \mathbf{x}'' —with weakly higher approximation ratio than \mathbf{x} . Therefore, (13) holds. The proposition statement follows from the fact that the approximation ratio in Lemma 2 is constructed by restricting agents to locations $\{0, 1\}$. \square