Fair Assignment Of Indivisible Objects Under Ordinal Preferences

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ABSTRACT

We consider the discrete assignment problem in which agents express ordinal preferences over objects and these objects are allocated to the agents in a fair manner. We use the stochastic dominance relation between fractional or randomized allocations to systematically define varying notions of proportionality and envy-freeness for discrete assignments. The computational complexity of checking whether a fair assignment exists is studied systematically for the fairness notions. We characterize the conditions under which a fair assignment is guaranteed to exist. For a number of fairness concepts, polynomial-time algorithms are presented to check whether a fair assignment exists or not. Our algorithmic results also extend to the case of variable entitlements of agents. Our NP-hardness result, which holds for several variants of envy-freeness, answers an open problem posed by Bouveret, Endriss, and Lang (ECAI 2010).

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences - Economics

General Terms

Economics, Theory and Algorithms

Keywords

Game theory (cooperative and non-cooperative), Social choice theory

1. INTRODUCTION

One of the most basic yet widely applicable problems in computer science and economics is to allocate discrete objects to agents given the preferences of the agents over the objects. The setting is referred to as the *assignment problem* or the *house allocation problem* (see, e.g., [10, 23, 12, 15]). In the setting, there is a set of agents $N = \{1, \dots, n\}$, a set of objects $O = \{o_1, \dots, o_m\}$ with each agent $i \in N$ expressing ordinal preferences $\succsim_i$ over O. The goal is to allocate the objects among the agents in a fair or optimal manner without allowing transfer of money.[1] The model is applicable to many resource allocation or fair division settings where the objects may be public houses, school seats, course enrollments, kidneys for transplant, car park spaces, chores, joint assets of a divorcing couple, or time slots in schedules.

In this paper, we consider the fair allocation of indivisible objects. Two of the most fundamental concepts of fairness are *envy-freeness* and *proportionality*. Envy-freeness requires that no agent considers that another agent's allocation would give him more utility than his own. Proportionality requires that each agent should get an allocation that gives him at least $1/n$ of the utility that he would get if he got all the objects. When agents' ordinal preferences are known but utility functions are not given, then ordinal notions of envy-freeness and proportionality need to be formulated. We consider a number of ordinal fairness concepts. Most of these concepts are based on the *stochastic dominance (SD)* relation which is a standard way of comparing fractional/ randomized allocations. An agent prefers one allocation over another with respect to the SD relation if he gets at least as much utility from the former allocation as the latter for all cardinal utilities consistent with the ordinal preferences. Although this paper is restricted to discrete assignments, using stochastic dominance to define fairness concepts for discrete assignments turns out to be fruitful. The fairness concepts we study include *SD envy-freeness, weak SD envy-freeness, possible envy-freeness, SD proportionality,* and *weak SD proportionality.* We consider the algorithmic problems of computing or verifying a discrete assignment that satisfies some ordinal notion of fairness.

Contributions.

We present a systematic way of formulating fairness properties in the context of the assignment problem. The logical relationships between the properties are proved. Interestingly, our framework leads to new solution concepts such as weak SD proportionality that have not been studied before. The motivation to study a range of fairness properties is that, depending on the situation, only some of them are achievable. In addition, only some of them can be computed efficiently.

We present a comprehensive study of the computational complexity of computing fair assignments under ordinal preferences. In particular, we present a polynomial-time algorithm to check whether an SD proportional assignment exists even when agents may express indifferences. As a corollary, for the case of two agents, we obtain a polynomial-time algorithm to check whether an SD envy-free assignment

[1]The assignment problem is a a fundamental setting within the wider domain of *fair division* or *multiagent resource allocation* [9].

exist. The results generalize those of (Proposition 2, [5]) and (Theorem 1, [18]). For a constant number of agents, we propose a polynomial-time algorithm to check whether a weak SD proportional assignment exists. As a corollary, for two agents, we obtain a polynomial-time algorithm to check whether a weak SD envy-free or a possible envy-free assignment exists. Even for an unbounded number of agents, if the preferences are strict, we characterize the conditions under which a weak SD proportional assignment exists. We show that the problems of checking whether possible envy-free, SD ency-free, or weak SD envy-free assignments exist are NP-complete. The statement regarding possible envy-freeness answers an open problem posed in [5]. All our computational results are summarized in Table 1.

Finally, we show that many of our algorithms also work when agents have different entitlements over the objects. Our study highlights the contrasts in the following settings: *i*) randomized/fractional versus discrete assignments, *ii*) strict versus non-strict preferences, and *iii*) multiple objects per agent versus a single object per agent

	Verify	Exists
All the concepts	in P	**in P for constant m (Remark 4)**
Weak SD proportional	in P	**in P for strict prefs (Th. 6)** **in P for constant n (Th. 7)**
SD proportional	in P	**in P (Th. 5)**
Weak SD envy-free	in P	**in P for strict prefs** **in P for $n = 2$ (Cor. 2)** **NP-complete (Th. 10)**
Possible envy-free	in P	in P for strict prefs [5] **in P for $n = 2$ (Cor. 2)** **NP-complete (Th. 10)**
SD envy-free	in P	NP-complete even for strict prefs [5] **in P for $n = 2$ (Cor. 1)**

Table 1: Complexity of fair assignment of indivisible goods for n agents and m objects. The results in bold are from this paper.

2. RELATED WORK

Proportionality and envy-freeness are two of the most established fairness concepts. Proportionality dates back to at least to the work of Steinhaus [21] in the context of cake-cutting. It is also referred to as *fair share guarantee* in the literature [16]. A formal study of envy-freeness in microeconomics can be traced back to the work of Foley [11].

Computation of fair discrete assignments has been intensely studied in the last decade within computer science. In many of the papers considered, agents express cardinal utilities for the objects and the goal is to compute fair assignments (see e.g., [10, 2, 14, 4]). The most prominent paper is that of Lipton et al. [14] in which algorithms for approximately envy-free assignments are discussed. A closely related problem is the *Santa Claus problem* in which the agents again express cardinal utilities for objects and the goal is to compute an assignment which maximizes the utility of the agent that gets the least utility (see e.g., [1, 2]). Just like in [5, 18, 17], we consider the setting in which agents do not express cardinal preferences. There are some merits of considering this setting. Firstly, ordinal preferences require elicitation of less information from the agents. Secondly, some of the weaker ordinal fairness concepts we consider may lead to positive existence or computational results. Thirdly, some of the stronger ordinal fairness concepts we consider are more robust than the standard fairness concepts. Fourthly, when the exchange of money is not possible, mechanisms that elicit cardinal preferences may be more susceptible to manipulation because of the larger strategy space. Finally, it may be the case that cardinal preferences are simply not available.

The ordinal fairness concepts we consider are SD envy-freeness; weak SD envy-freeness; possible envy-freeness; SD proportionality; and weak SD proportionality. Not all of these concepts are new but they have not been examined systematically for discrete assignments. SD envy-freeness and weak SD envy-freeness have been considered in the randomized assignment domain [3] but not the discrete domain. SD envy-freeness and weak SD envy-freeness have been considered implicitly for discrete assignments but the treatment was axiomatic [8, 7]. A mathematically equivalent version of SD envy-freeness and possible envy-freeness has been considered by Bouveret et al. [5] but only for strict preferences. A concept equivalent to SD proportionality was examined by Pruhs and Woeginger [18] but again only for strict preferences. Interestingly, weak SD or possible proportionality has not been studied in randomized or discrete settings (to the best of our knowledge).

Envy-freeness as a fairness concept is well-established in fair division especially cake-cutting. Fair assignment of indivisible goods has been extensively studied within economics but in most of the papers, either the goods are divisible or agents are allowed to use money to compensate each other (see e.g., [22]). In the model we consider, we do not allow money transfers. In a number of settings, exchange of money may *not* be admissible.

3. PRELIMINARIES

An assignment problem is a triple $(N, O, \succsim)$ such that $N = \{1, \dots, n\}$ is a set of agents, $O = \{o_1, \dots, o_m\}$ is a set of objects, and the preference profile $\succsim = (\succsim_1, \dots, \succsim_n)$ specifies for each agent i his preference $\succsim_i$ over O. Agents may be indifferent among objects. We denote $\succsim_i$: $E_i^1, \dots, E_i^{k_i}$ for each agent i with equivalence classes in decreasing order of preferences. Thus, each set E_i^j is a maximal equivalence class of objects among which agent i is indifferent, and k_i is the number of equivalence classes of agent i.

A fractional assignment p is a $(n \times m)$ matrix $[p(i)(o_j)]$ such that $p(i)(o_j) \in [0, 1]$ for all $i \in N$, and $o_j \in O$, and $\sum_{i \in N} p(i)(o_j) = 1$ for all $j \in \{1, \dots, n\}$. The value $p(i)(o_j)$ represents the probability of object o_j being allocated to agent i. Each row $p(i) = (p(i)(o_1), \dots, p(i)(o_m))$ represents the allocation of agent i. The set of columns correspond to the objects $o_1, \dots, o_m$. A feasible fractional assignment is *discrete* if $p(i)(o) \in \{0, 1\}$ for all $i \in N$ and $o \in O$.

A *uniform assignment* is a fractional assignment in which each agent gets $1/n$-th of each object. Although we will deal with discrete assignments, the fractional uniform assignment is useful in defining some fairness concepts. Similarly, we will use the SD relation to define relations between assignments. Our algorithmic focus will be on computing discrete assignments only even though concepts are defined using the framework of fractional assignments.

Given two fractional assignments p and q, $p(i) \succsim_i^{SD} q(i)$, i.e., agent i *SD prefers* allocation

$p(i)$ to allocation $q(i)$ if $\sum_{o_j \in \{o_k : o_k \succsim_i o\}} p(i)(o_j) \geq \sum_{o_j \in \{o_k : o_k \succsim_i o\}} q(i)(o_j)$ for all $o \in O$. He *strictly SD prefers* $p(i)$ to $q(i)$ if $p(i) \succsim_i^{SD} q(i)$ and $\neg[q(i) \succsim_i^{SD} p(i)]$. Although each agent i expresses ordinal preferences over objects, he could have a private cardinal utility u_i consistent with $\succsim_i$: $u_i(o) \geq u_i(o')$ if and only if $o \succsim_i o'$. The set of all utility functions consistent with $\succsim_i$ is denoted by $\mathcal{U}(\succsim_i)$.

An assignment p is *envy-free* if the total utility each agent i gets for his allocation is at least much as he would get if he had any another agent's allocation: $u_i(p(i)) \geq u_i(p(j))$ for all $j \in N$. An assignment is *proportional* if each agent gets at least $1/n$-th of the utility he would get if he got all the objects: $u_i(p(i)) \geq \sum_{o \in O} u_i(o)/n$.

We will require that the assignment is complete, that is, each object is allocated. In the absence of this requirement a null assignment is obviously envy-free. On the other hand a null assignment is not proportional.

4. FAIRNESS CONCEPTS UNDER ORDINAL PREFERENCES

We now define fairness notions that are independent of particular cardinal utilities of the agents.

Proportionality.

1. (a) *Weak SD proportionality*: An assignment p satisfies *weak SD proportionality* if no agent strictly SD prefers the uniform assignment to his allocation: $\neg[(1/n, \ldots, 1/n) \succ_i^{SD} p(i)]$ for all $i \in N$.
   (b) *Possible proportionality*: An assignment satisfies *possible proportionality* if for each agent, there are cardinal utilities consistent with his ordinal preferences such that his allocation yields him as at least as much utility as he would get under the uniform assignment. For each $i \in N$, there exists $u_i \in \mathcal{U}(\succsim_i)$ s.t. $u_i(p(i)) \geq \sum_{o \in O} u_i(o)/n$.
2. (a) *SD proportionality*: An assignment p satisfies *SD proportionality* if each agent SD prefers his allocation to the allocation under the uniform assignment: $p(i) \succsim_i^{SD} (1/n, \ldots, 1/n)$ for all $i \in N$.
   (b) *Necessary proportionality*: An assignment satisfies *necessary proportionality* if it is proportional for all cardinal utilities consistent with the agents' preferences. Pruhs and Woeginger [18] referred to *necessary proportionality* as "ordinal fairness". For each $i \in N$, and for each $u_i \in \mathcal{U}(\succsim_i), u_i(p(i)) \geq \sum_{o \in O} u_i(o)/n$.

Envy-freeness.

1. (a) *Weak SD envy-freeness*: An assignment p satisfies *weak SD envy-freeness* if no agent strictly SD prefers someone else's allocation to his: $\neg[p(j) \succ_i^{SD} p(i)]$ for all $i, j \in N$.
   (b) *Possible envy-freeness*: An assignment satisfies *possible envy-freeness* if for each agent, there are cardinal utilities consistent with his ordinal preferences such that his allocation yields him as at least as much utility as he would get if he was given some other agent's allocation. For each $i, j \in N$, there exists $u_i \in \mathcal{U}(\succsim_i)$ s.t. $u_i(p(i)) \geq u_i(p(j))$.
   (c) *Possible completion envy-freeness*: An assignment satisfies *possible completion envy-freeness* [5] if for each agent, there exists a preference relation of the agent over sets of objects that is consistent with his preferences over objects such that the agent prefers his allocation more than the allocations of other agents. The concept has also been referred to as *not envy-ensuring* [7].
2. (a) *SD envy-freeness*: An assignment p satisfies *SD envy-freeness* if each agent SD prefers his allocation to that of any other agent: $p(i) \succsim_i^{SD} p(j)$ for all $i, j \in N$.
   (b) *Necessary envy-freeness*: An assignment satisfies *necessary envy-freeness* if it is envy-free for all cardinal utilities consistent with the agents' preferences. For each $i, j \in N$, and for each $u_i \in \mathcal{U}(\succsim_i)$, $u_i(p(i)) \geq u_i(p(j))$.
   (c) *Necessary completion envy-freeness*: An assignment satisfies *necessary completion envy-freeness* [5] if for each agent, and each transitive closure of the *responsive set extension* of the agents [2], each agent prefers his allocation more than other agents' allocations. The concept has also been referred to as *not envy-possible* [7].

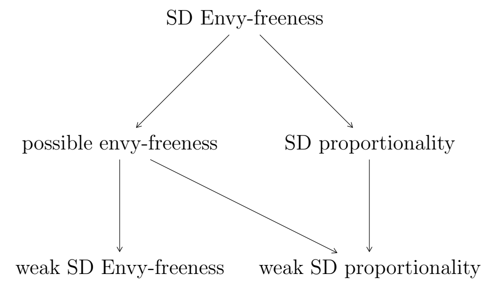

Figure 1: Inclusion relationships between fairness concepts. E.g., every SD envy-free outcome is also SD proportional. Any two concepts without any path between them are incomparable.

Possible completion envy-freeness and necessary completion envy-freeness were simply referred to as possible and necessary envy-freeness in [5]. We will use the former terms to avoid confusion.

5. RELATIONS BETWEEN FAIRNESS CONCEPTS

We note that $p(i) \succsim_i^{SD} p(j)$ if and only if for all utility functions u_i compatible with $\succsim_i$, $u_i(p(i)) \geq u_i(p(j))$. Based on this view of the SD relation we obtain the following equivalences.

[2]In the responsive set extension, preferences over objects are extended to preferences over sets of objects in such a way that a set in which an object is replaced by a more preferred object is more preferred.

Theorem 1. *The following equivalences hold between the fairness concepts defined. i) Weak SD proportionality and possible proportionality. ii) SD proportionality and necessary proportionality iii) possible envy-freeness and possible completion envy-freeness; iv) SD envy-freeness, necessary envy-freeness and necessary completion envy-freeness.*

It is well-known that when an allocation is complete and utilities are additive, envy-freeness implies proportionality. Assume that an assignment p is envy-free. Then for each $i \in N$, $u_i(p(i)) \geq u_i(p(j))$ for all $j \in N$. Thus, $n \times u_i(p(i)) \geq \sum_{j\in N} u_i(p(j)) = \sum_{o\in O} u_i(o)$. Hence $u_i(p(i)) \geq \frac{\sum_{o\in O} u_i(o)}{n}$. We can also get similar relations when we consider stronger and weaker notions of envy-freeness and proportionality.

Theorem 2. *The following relations hold between the fairness concepts defined. i) SD envy-freeness implies SD proportionality. ii) SD proportionality implies weak SD proportionality. iii) Possible envy-freeness implies weak SD proportionality. iv) For two agents, proportionality is equivalent to envy-freeness; SD proportionality is equivalent to SD envy-freeness; and weak SD proportionality, possible envy-freeness and weak SD envy-freeness are equivalent.*

In the next examples, we show that some of the inclusion relations do not hold in the opposite direction and that some of the solution concepts are incomparable.

Example 1. *SD proportionality does not imply weak SD envy-freeness. Consider the following preference profile:*

$$1: \quad \{a,b,c\},\{d,e,f\} \qquad 2: \quad \{a,b,c,d,e,f\}$$
$$3: \quad \{a,b,c,d,e,f\}$$

The allocation that gives $\{a,d\}$ to agent 1, $\{b,c\}$ to agent 2 and $\{e,f\}$ to agent 3 is SD proportional. However it is not weak SD envy-free since agent 1 is envious of agent 2. Hence it also follows that SD proportionality does not imply possible envy-freeness or SD envy-freeness.

Example 2. *Weak SD envy-freeness neither implies possible envy-freeness nor weak SD proportionality. Consider an assignment problem in which $N = \{1,2,3\}$, and there are 4 copies of A, 6 copies of B, 1 copy of C and 1 copy of D. Let the preference profile be as follows.*

$$1: \quad A,B,C,D$$
$$2: \quad \{A\},\{B,C,D\}$$
$$3: \quad \{B\},\{A,C,D\}.$$

	A	B	C	D
1	1	1	1	1
2	3	0	0	0
3	0	5	0	0

Table 2: Discrete assignment in Example 2

Clearly p, the assignment specified in Table 2 is weak SD envy-free. Assume that p is also possible envy-free. Let u_1 be the utility function of agent 1 for which he does not envy agent 2 or 3. Let $u_1(A) = a$; $u_1(B) = b$; $u_1(C) = c$; and $u_1(D) = d$. Since $A \succ_1 B \succ_1 C \succ_1 D$, we get that $a > b > c > d$. Since p is possible envy-free, $u_1(p(1)) \geq u_1(p(2)) \iff a+b+c+d \geq 3a \iff a \leq \frac{b+c+d}{2} \implies a < \frac{3b}{2}$. Since p is possible envy-free, $u_1(p(1)) \geq u_1(p(3)) \iff a+b+c+d \geq 5b \iff a+c+d \geq 4b \implies a > 2b$. This is a contradiction since both $a < \frac{3b}{2}$ and $a \geq 2b$ cannot hold.

Now we show that weak SD envy-freeness does not even imply weak SD proportionality. Assignment p is weak SD envy-free. If it were weak SD proportional then there exists utility function u_1 such that $u_1(a)+u_1(b)+u_1(c)+u_1(d) \geq \frac{4u_1(a)+6u_1(b)+u_1(c)+u_1(d)}{3}$ which means that $\frac{u_1(a)}{3} + u_1(b) \leq \frac{2u_1(c)}{3} + \frac{2u_1(d)}{3}$. But this is not possible.

Example 3. *Possible envy-freeness does not imply SD proportionality. Consider an assignment problem with two agents preferences $1: \{a\},\{b,c\}$ and $2: \{a,b,c\}$. Then the assignment in which 1 gets a and 2 gets b and c is possible envy-free. However it is not SD proportional, because agent 1's allocation does not SD dominate the uniform allocation. A necessary condition for SD proportionality is that m is a multiple of n and each agent gets exactly m/n objects.*

Finally, we note that all notions of proportionality and envy-freeness are trivially satisfied if randomized assignments are allowed by giving each agent $1/n$ of each object. As we show here, achieving any notion of proportionality is a challenge when outcomes need to be discrete.

Next, we study the existence and computation of fair assignments. Even the weakest fairness concepts like weak SD proportionality may not be possible to achieve: consider two agents with identical and strict preferences over two objects. This problem remains even if m is a multiple of n.

Example 4. *A weak SD proportional assignment may not exist even if m is a multiple of n. Consider the following preferences:*

$$1: \quad \{a_1,a_2,a_3,a_4\},\{b_1,b_2\}$$
$$2: \quad \{a_1,a_2,a_3,a_4\},\{b_1,b_2\}$$
$$3: \quad \{a_1,a_2,a_3,a_4\},\{b_1,b_2\}$$

If all agents get 2 objects, then those agents that have to get at least one object from $\{b_1,b_2\}$ will get an allocation that is strictly SD dominated by $(1/3,\ldots,1/3)$. Otherwise, at least one agent gets at most one object, and is therefore strictly SD dominated by the uniform assignment.

If m is not a multiple of n, then an even simpler example shows that a weak SD proportional assignment may not exist. Consider the case when all agents are indifferent among all objects. Then the agent who gets less objects than m/n will get an allocation that is strictly SD dominated by $(1/n,\ldots,1/n)$.

We consider the natural computational question of checking whether a fair assignment exists and if it does exist then to compute it. The problem of verifying whether an assignment is fair is easy for all the notions we defined.

Remark 3. *It can be verified in time polynomial in n and m whether an assignment is fair for all notions of fairness considered in the paper. For possible envy-freeness, a linear program can be used to find the 'witness' cardinal utilities of the agents.*

Remark 4. *For a constant number of objects, it can be checked in polynomial time whether a fair assignment exists for all notions of fairness considered in the paper. This is because the total number of discrete assignments is n^m.*

6. SD PROPORTIONALITY

In this section, we show that it can be checked in polynomial time whether an SD proportional assignment exists even for the case of indifferences. The algorithm is via a reduction to the problem of checking whether a bipartite graph admits a feasible b-matching.

For the algorithm, we also require the machinery of b-matchings. Let $H = (V_H, E_H, w)$ be an undirected graph with vertex capacities $b : V_H \to \mathbb{N}_0$ and edge capacities $c : E_H \to \mathbb{N}_0$. Then, a *b-matching* of H is a function $m : E_H \to \mathbb{N}_0$ such that $\sum_{e\in\{e'\in E_H : v\in e'\}} m(e) \leq b(v)$ for each $v \in V_H$, and $m(e) \leq c(e)$ for all $e \in E_H$. The *size* of b-matching m is defined as $\sum_{e\in E_H} m(e)$. We point out that if $b(v) = 1$ for all $v \in V_H$, and $c(e) = 1$ for all $e \in E_H$ then a maximum size b-matching is equivalent to a maximum cardinality matching. In a b-matching problem with upper *and* lower bounds, there further is a function $a : V_H \to \mathbb{N}_0$. A feasible b-matching then is a function $m : E_H \to \mathbb{N}_0$ such that $a(v) \leq \sum_{e\in\{e'\in E_H : v\in e'\}} m(e) \leq b(v)$. If H is bipartite, then the problem of computing a maximum weight feasible b-matching with lower and upper bounds can be solved in strongly polynomial time (Chapter 35, [19]).

THEOREM 5. *It can be checked in $O((n + m)^3 \log(n+m))$ time whether an SD proportional assignment exists.*

PROOF. Consider $(N, O, \succsim)$. If m is not a multiple of n, then no SD proportional assignment exists. In this case, in each discrete assignment p, there exists some agent $i \in N$ who gets less than m/n objects. Thus, the following does not hold: $p(i) \succsim_i^{SD} (1/n, \ldots, 1/n)$. Hence we can now assume that m is a multiple of n i.e., $m = nc$ where c is a constant.

We reduce the problem of checking the existence of an SD proportional assignment to checking whether a feasible b-matching exists for a graph $G = (V, E)$. For each agent i, and for each $\ell \in \{1, \ldots, k_i\}$ we introduce a vertex v_i^ℓ. For each $o \in O$, we create a corresponding vertex with the same name. Now, $V = \{v_i^1, \ldots, v_i^{k_i} : i \in N\} \cup O$.

The graph will be bipartite with independent sets O and $V \setminus O$. Let us now specify the edges of G.

- for each $i \in N$, $\ell \in \{1, \ldots, k_i\}$ and $o \in O$ we have that $\{v_i^\ell, o\} \in E$ if and only if $o \in \bigcup_{j=1}^{\ell} E_i^j$.

We specify the lower and upper bounds of each vertex.

- $a(v_i^\ell) = \left\lceil \frac{\sum_{j=1}^{\ell} |E_i^j|}{n} \right\rceil - \sum_{j=1}^{\ell-1} a(v_i^j)$ and $b(v_i^\ell) = \infty$ for each $i \in N$ and $\ell \in \{1, \ldots, k_i\}$;
- $a(o) = b(o) = 1$ for each $o \in O$.

For each edge $e \in E$, $c(e) = 1$.

Now that (V, E) has been specified, we then check whether a feasible b-matching exists or not. If so, we allocate an object o to an agent i if the edge incident to o that is assigned the value 1 is incident to a vertex corresponding to an equivalence class of agent i. We claim that an SD proportional assignment exists if and only if a feasible b-matching exists. If a feasible b-matching exists, then each $o \in O$ is matched so we have a complete assignment. For each agent $i \in N$, and for each E_i^ℓ, an agent is allocated at least $\left\lceil \sum_{j=1}^{\ell} |E_i^j|/n \right\rceil$ objects of the same or more preferred equivalence class. Thus, the assignment is SD proportional.

On the other hand if an SD proportional assignment p exists, then $p(i) \succsim_i^{SD} (1/n, \ldots, 1/n)$ implies that for each equivalence class E_i^ℓ, an agent is allocated at least $\left\lceil \sum_{j=1}^{\ell} |E_i^j|/n \right\rceil$ objects from the same or more preferred equivalence class as E_i^ℓ. Hence there is a b-matching in which the lower bound of each vertex of the type v_i^ℓ is met. For any remaining vertices $o \in O$ that have not been allocated, they may be allocated to any agent. Hence a feasible b-matching exists. □

COROLLARY 1. *For two agents, it can be checked in polynomial time $O(m^3 \log(m))$ whether an SD envy-free assignment exists for the case of indifferences.*

PROOF. For two agents, SD proportionality implies SD envy-freeness, and by Theorem 2, SD envy-freeness implies SD proportionality. □

Corollary 1 generalizes Proposition 10 of [5] which stated that for two agents and *strict* preferences, it can be checked in polynomial time whether a necessary envy-free assignment exists.

7. WEAK SD PROPORTIONALITY

In the previous section, we examined the complexity of SD proportional assignments. In this section we consider weak SD proportionality.

THEOREM 6. *For strict preferences, a weak SD proportional assignments exists if and only if one of two cases holds: i) $m = n$ and each agent gets an object that is not the least preferred; ii) $m > n$. Moreover, it can be checked in polynomial time whether a weak SD proportional assignment exists when agents have strict preferences.*

PROOF. If $m < n$, at least one agent will not get any object. Hence there exists no weak SD proportional assignment. Hence $m \geq n$ is a necessary condition for the existence of a weak SD proportional assignment.

Let us consider the case of $m = n$. Clearly each agent needs to get one object. If an agent i gets an object that is not the last preferred object o', then his allocation $p(i)$ is weak SD proportional. The reason is that $\sum_{o\succ o'} p(i)(o) = 1 > |\{o : o \succ o'\}|/m$. Hence the following does not hold $(1/m, \ldots, 1/m) \succsim_i^{SD} p(i)$. On the other hand, if i gets the least preferred object, his allocation is not weak SD proportional since $(1/m, \ldots, 1/m) \succ_i^{SD} p(i)$. Hence, we just need to check whether there exists an assignment in which each agent gets an object that is not least preferred. This can be solved as follows. We construct a graph (V, E) such that $V = N \cup O$ and for all $i \in N$ and $o \in O$, $\{i, o\} \in E$ if and only if $o \notin \min_{\succsim_i}(O)$. We just need to check whether (V, E) admits a perfect matching or not. If it does, the matching is a weak SD proportional assignment.

If $m \geq n+1$, we show that a weak SD proportional assignment exists. Note that a weak SD proportional assignment is least likely to exist when all agents have identical preferences. Allocate the most preferred object to the agents in the following order $1, 2, 3, \ldots n, n, n-1 \ldots, 1, \ldots$. Then each agent $i \in \{1, \ldots, n-1\}$ gets in the worst case his i-th most preferred object. Hence $1 > i/n$, thus the allocation of agents in $\{1, \ldots, n-1\}$ is weak SD proportional. As for agent n, in the worst case he get his n-th and $n+1$st most preferred objects. Since $2 \geq \frac{n}{n} + \frac{1}{n}$, the allocation of agent n is also weak SD proportional. This completes the proof. □

Indifferences result in all sorts of challenges. Some arguments that we used for the case for strict preferences do not work for the case of indifference. The case of strict preference may lead one to wrongly assume that given a sufficient number of objects, a weak SD proportional assignment is guaranteed to exist. However, if agents are allowed to express indifference, this is not the case. Consider the case where $m = nc + 1$ and each agent is indifferent among each of the objects. Then there exists no weak SD proportional assignment because some agent will get less than m/n objects.

We first present a helpful lemma which follows directly from the definition of weak SD proportionality.

LEMMA 1. *An assignment p is weak SD proportional if and only if for each $i \in N$, i) $\sum_{o' \succsim o} p(i)(o') > |\{o' \succsim o\}|/n$ for some $o \in O$; or ii) $\sum_{o' \succsim o} p(i)(o') \geq |\{o' \succsim o\}|/n$ for all $o \in O$.*

We will use Lemma 1 in designing an algorithm to check whether a weak SD proportional assignment exists when agents are allowed to express indifference.

THEOREM 7. *For a constant number of agents, it can be checked in polynomial time $O(m^3 \log(m))$ whether a weak SD proportional assignment exists.*

PROOF. Consider $(N, O, \succsim)$. We want to check whether a weak SD proportional assignment exists or not. By Lemma 1, this is equivalent to checking whether there exists a discrete assignment p, where for each $i \in N$, one of the following k_i conditions holds: for $l \in \{1, \ldots, k_i\}$,

$$\sum_{o \in \bigcup_{j=1}^{l} E_i^j} p(i)(o) > \frac{|\bigcup_{j=1}^{l} E_i^j|}{n} \quad (1)$$

or the following $k_i + 1$-st condition holds

$$p(i) \sim_i^{SD} (1/n, \ldots, 1/n). \quad (2)$$

The k_i+1-st condition only holds if each $|E_i^j|$ is a multiple of n for $j \in \{1, \ldots, k_i\}$.

We need to check whether there exists a discrete assignment in which for each agent one of the $k_i + 1$ conditions is satisfied. In total there are $\prod_{i=1}^{n}(k_i + 1)$ different ways in which the agents could be satisfied. We will now present an algorithm to check if there exists a feasible weakly SD proportional assignment in which for each agent i, a certain condition among the $k_i + 1$ conditions is satisfied. Since n is a constant, the total number of combinations of conditions is polynomial.

We define a bipartite graph $G = (V, E)$ whose vertex set is initially empty. For each agent i, if the condition number is $\ell \in \{1, \ldots, k_i\}$ then we add a vertex v_i^ℓ. If the condition number is $k_i + 1$, then we add k_i vertices — B_i^j for each E_i^j where $j \in \{1, \ldots, k_i\}$. For each $o \in O$, we add a corresponding vertex with the same name. The sets O and $V \setminus O$ will be independent sets in G. We now specify the edges of G.

- $\{v_i^\ell, o\} \in E$ if and only if $o \in \bigcup_{j=1}^{\ell} E_i^j$ for each $i \in N$, $\ell \in \{1, \ldots, k_i\}$ and $o \in O$.
- $\{B_i^j, o\} \in E$ if and only if $o \in E_i^j$ for each $i \in N$, $j \in \{1, \ldots, k_i\}$, and $o \in O$.

We specify the lower and upper bounds of each vertex.

- $a(v_i^\ell) = \left\lfloor \frac{|\bigcup_{j=1}^{\ell} E_i^j|}{n} \right\rfloor + 1$ and $b(v_i^\ell) = \infty$ for each $i \in N$ and $\ell \in \{1, \ldots, k_i\}$;
- $a(B_i^j) = b(B_i^j) = \frac{|E_i^j|}{n}$. for each B_i^j;
- $a(o) = b(o) = 1$ for each $o \in O$.

For each edge $e \in E$, $c(e) = 1$. For each n-tuple of satisfaction conditions, we construct the graph as specified above and then check whether there exists a feasible b-matching. A weak SD proportional assignment exists if and only if a feasible b-matching exists for the graph corresponding to at least one of the $\prod_{i=1}^{n}(k_i + 1)$ combinations of conditions. Since $\prod_{i=1}^{n}(k_i + 1)$ is polynomial if n is a constant and since a feasible b-matching can be checked in strongly polynomial time, we can check the existence of a weak SD proportional assignment in polynomial time. □

COROLLARY 2. *For two agents, it can be checked in polynomial time $O(m^3 \log(m))$ whether a weak SD envy-free or a possible envy-free assignment exists or not.*

PROOF. For two agents, weak SD proportional is equivalent to weak SD envy-free and possible envy-free. □

8. ENVY-FREENESS

In this section we prove that checking whether a (weak) SD envy-free or possible envy-free assignment exists is NP-complete. The complexity of the second problem was mentioned as an open problem in [5]. Bouveret et al. [5] showed that the problem of checking whether a necessary envy-free assignment exists is NP-complete. The statement carries over to the more general domain that allows for ties. We point out that if agents have identical preferences, it can be checked in linear time, whether an SD envy-free assignment exists even when preferences are not strict. Identical preferences have received special attention within fair division (see e.g., [6]).

THEOREM 8. *For agents with identical preferences, an SD envy-free assignment exists if and only if each equivalence class is a multiple of n.*

Next, we show that when preferences are strict, weak SD envy-freeness is computationally more tractable than SD envy-freeness.

THEOREM 9. *For strict preferences, it can be checked in time linear in n and m whether a complete weak SD envy-free assignment exists or not.*

PROOF. Let the number of distinct top-ranked objects be k. If $m < 2n - k$, then there is at least one agent who receives an object that is not his top-ranked o and no further items. Thus he necessarily envies the agent who received o. If $m \geq 2n - k$, then there exists a possible envy-free assignment [5]. A linear-time algorithm to compute such an assignment was outlined in [5]. By Remark 2, we know that possible envy-freeness implies weak SD envy-freeness. Hence there exists a weak SD envy-free assignment which can be computed by the same algorithm. □

Bouveret et al. [5] mentioned the case of indifferences for future work and mentioned the complexity of possible envy-freeness as an open problem. We present a reduction to prove that for *all* notions of envy-freeness considered in this paper, checking the existence of a fair assignment is NP-complete. The reason why it applies to all envy-freeness notions is that each agent has only two equivalence classes in his preferences.

THEOREM 10. *The following problems are NP-complete. i) Check whether a weak SD envy-free assignment exists. ii) Check whether a possible envy-free assignment exists. iii) Check whether an SD envy-free assignment exists.*

PROOF. Membership in NP is shown by Remark 3. To show hardness we use a reduction from X3C (exact cover by 3-sets). Exact cover by 3-sets consists of finding a subset X from a set of clauses C containing 3-sets of elements from S, such that each element of S is contained in exactly one of the clauses in X. X3C is known to be NP-complete [13]. Consider an instance of X3C$=(S,C)$ where $S=\{s_1,...,s_{3q}\}$ and $C=\{c_1,...,c_l\}$. Without loss of generality $l \geq q$. An easy way to see this is that if $l < q$ then elements from S will not be included in any of the clauses, implying there is no exact cover. For the instance, we associate the following assignment problem $(N,O,\succsim)$ where $N=\{a_1,...,a_{40l}\}$ is partitioned into 3 sets N_1, N_2 and N_3 with $|N_1|=l$, $|N_2|=30l$, $|N_3|=9l$ and $O=\{o_1,...,o_{120l}\}$ is partitioned into 3 sets O_1, O_2 and O_3 with $|O_1|=3l$, $|O_2|=90l$ and $|O_3|=27l$. The set O_1 is partitioned into 2 sets, O_1^S and O_1^B, the first one corresponding to the set of elements of S in the X3C instance and the second being a 'buffer' set. We have $|O_1^S|=3q$ and $|O_1^B|=3l-3q$. We associate each $c_j \in C$ with the j-th agent in N_1. With each $c_j \in C$ we also associate 9 consecutive agents in N_3. The preferences of the agents are defined as follows:

$i: O_2 \cup c_i, (O_1 \backslash c_i) \cup O_3$ for $i \in N_1$
$i: O_2, O_1 \cup O_3$ for all $i \in N_2$
$i: f(i), O \backslash f(i)$ for $i \in N_3$

The function $f: N_3 \to 2^O$ is such that it ensures the following properties: For each of the 3 elements e of c_j, 3 out of the 9 agents associated with c_j list e as a second choice object, and list $c_j \backslash \{e\}$ as first choice objects. Let us label these 3 agents a_1, a_2 and a_3. The sets of objects $f(a_1)$, $f(a_2)$ and $f(a_3)$ each exclude a distinct $\frac{1}{3}$ of the buffer objects O_1^B. For each $i \in N_3$, $f(i) \cap (O_2 \cup O_3) = O_f$. Let O_f contain $\frac{2}{3}$ of the elements of O_2 and $\frac{2}{3}$ of the elements of O_3. Consider an assignment that is weak SD-envy-free or possible envy-free or SD envy-free. We can make the following observations:

1. Agents in N_2 are allocated all objects from O_2 and none from $O \backslash O_2$. To show this, first consider the case where $30l$ or more objects from O_2 are assigned to $N \backslash N_2$. In this case, at least 1 agent in N_2 is envious of an agent from $N \backslash N_2$: there will be an agent b_1 in $N \backslash N_2$ with 3 or more objects from O_2, and there will be an agent b_2 in N_2 with at most 3 elements, at most 2 of which are from O_2. This is because if an agent has more than 3 objects, another has at most 2 and if they all have 3, some of those will be objects from O_1, and at least 1 agent from N_2 will have a second choice object. For all considered notions of envy-freeness b_2 will be envious of b_1.
If $0 < z_1 < 30l$ objects from O_2 are assigned to $N \backslash N_2$, we have 3 cases: i) $z_2 \leq z_1$ objects from $O \backslash O_2$ are assigned to N_2. In this case an agent from N_2 has 2 or less objects, which implies he will be envious of others in N_2 ii) $z_2 = z_1$ objects from $O \backslash O_2$ are assigned to N_2. To not be envious of each other agents from N_2 will each receive 2 first choice objects and one secondary choice object. At least one agent from N_1 will receive at least 3 objects from O_2, making agents in N_2 envious of him. iii) $z_2 > z_1$ objects from $O \backslash O_2$ are assigned to N_2. In this case all agents from N_2 are given 3 or 4 objects. If an agent has 2, he will be envious as before. There are not enough objects left for each agent in $N \backslash N_2$ to receive 3 or more objects. Therefore one of these agents, labelled b_1 only has 2 items. Even if those 2 items are most preferred items, he will be envious of at least one agent in N_1 because to any agent in $N \backslash N_2$ the ratio of most preferred items assigned to N_1 is higher than $\frac{1}{3}$. This implies at least one agent in N_1 will have 2 most preferred items according to b_1, and since all in N_2 have at least 3 objects, b_1 is envious of that agent.
2. Each agent in N_2 is allocated exactly 3 objects. Since as shown above all and only O_2 objects go to N_2, not all agents in N_2 can have 4 objects. Therefore if one has 4, those without 4 objects will envy him since they value all objects from O_2 the same.
3. Each agent in $N_2 \cup N_1$ have 3 objects. This is because if an agent in $N_2 \cup N_1$ has 4 or more objects, another has 2 or less. The argument in the first observation still applies, and therefore this agent will be envious of at least one agent from N_2.
4. Agents in N_1 will not be assigned any objects from O_3 since they all consider them to be second choices. To not envy agents in N_2 agents in N_1 have 3 of their preferred choices.
5. Each agent in $N_2 \cup N_3$ are given 2 of N_3's common preferred choices, and 1 of their second choices. This is the only way to avoid envy from an element of N_3 to at least 1 element of $N_2 \cup N_3$: if an element of N_2 has 2 or 3 of N_3 second choices, then another has 3 preferred choices, and therefore at least one of N_3 will be envious of him. If an agent in N_3 has 3 preferred choices, then at least one has only 1 preferred choice, and will be envious of the agent with 3 preferred choices.
6. An agent from N_1 does not have objects from O_1^S and also O_1^B, since otherwise at least 1 agent from N_3 will be envious of him. This is because of the conditions satisfied by f. There are at least 3 agents in N_3 who see the 1 or 2 selected elements from the 3-set associated to the N_1 agent as first choice objects. For any set of elements of size 2 or less in O_1^B, at least 1 of these 3 agents considers said set to be composed of first choice object. Therefore there is at least 1 agent in N_3 who will be envious of an agent in N_1 who selects both from O_1^S and O_1^B, since he sees this agent as having 3 preferred choices whilst he only has 2 (according to the previous observation).

If there exists an exact cover of S by a subset of C, then there is an envy free allocation since agents corresponding to elements of C used for the cover will be given their preferred items from O_1^S and the others will be given items from O_1^B. If there does not exist an exact cover of S by a subset of C then there does not exist an envy free allocation. This is because even if all the previous conditions are respected, at least one agent from N_1 gets a second choice object and is envious of agents from N_2. We can see this by considering the fact that no matter which agents of N_1 we assign buffer objects to, the remaining agents are not able to cover O_1^S with their sets of most preferred objects. This completes the proof. □

9. CONCLUSIONS

We have presented a taxonomy of fairness concepts under ordinal preferences, and identified the relationships between the concepts. Compared to transitive closures over sets of alternatives to define fairness concepts [5], using cardinal utilities and the SD relation to define fairness concepts not only gives more flexibility (for example reasoning about entitlements) but can also be convenient for algorithm design. We assumed that each agent has the same entitlement to the objects. However, it could also be the case if an agent i has entitlement e_i, then $u_i(p_i) \geq \frac{e_i}{\sum_{j\in N} e_j} \frac{\sum u_i(o)}{n}$ for proportionality and $u_i(p_i) \geq \frac{e_i}{e_j} u_i(p(j))$ for each $j \in N$ for envy-freeness. In the same way, possible and necessary fairness can also be defined. Our two algorithms for possible and necessary proportionality can also be modified to cater for entitlements by replacing $1/n$ with $\frac{e_i}{\sum_{j\in N} e_j}$ whenever a matching lower bound is specified for a vertex. We focussed on fairness and did not consider efficiency. A Pareto improvement over a weak SD proportional or SD proportional assignment does not lose its proportionality and can be implemented easily via trading cycles [20]. This may not be the case for different notions of envy-freeness when $n > 2$ since envy-freeness involves comparisons with other agents' allocations. Finally, none of the fairness concepts are guaranteed to be non-empty. However for existence, one can simply try to find the maximal (with respect to size or set inclusion) set of agents for which the fairness condition is satisfied. For maximal (SD or weak SD) proportionality (with respect to set inclusion), our algorithms can again be adapted.

The framework of ordinal fairness concepts leads to a number of new directions. It will be interesting to see how various approximation algorithms in the literature designed to reduce envy or maximize welfare fare in terms of satisfying ordinal notions of fairness (see e.g., [2, 14, 4]). Finally, strategic aspects of ordinal fairness is another interesting direction.

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