Feedback Vertex Sets in Tournaments*

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Abstract

We study combinatorial and algorithmic questions around minimal feedback vertex sets in tournament graphs.

On the combinatorial side, we derive strong upper and lower bounds on the maximum number of minimal feedback vertex sets in an n-vertex tournament. We prove that every tournament on n vertices has at most 1.6740^n minimal feedback vertex sets and that there is an infinite family of tournaments, all having at least 1.5448^n minimal feedback vertex sets. This improves and extends the bounds of Moon (1971).

On the algorithmic side, we design the first polynomial space algorithm that enumerates the minimal feedback vertex sets of a tournament with polynomial delay. The combination of our results yields the fastest known algorithm for finding a minimum size feedback vertex set in a tournament.

Keywords. Algorithms and data structures, tournaments, feedback vertex set, polynomial delay, combinatorial bounds.

1 Introduction

A tournament $T = (V, A)$ is a directed graph with exactly one arc between every pair of vertices. A feedback vertex set (FVS) of T is a subset of its vertices whose deletion makes T acyclic. A minimal FVS of T is a FVS of T that is minimal with respect to vertex-inclusion. The complement of a minimal FVS F induces a maximal acyclic subtournament whose unique vertex of in-degree zero is a "Banks winner" [1]: identifying the vertices of T with candidates in a voting scheme and arcs indicating preference of one candidate over another, the *Banks winner* of $T[V \setminus F]$ is the candidate collectively preferred to every other candidate in $V \setminus F$. Banks winners play an important role in social choice theory.

Extremal Combinatorics. We denote the number of minimal FVSs in a tournament T by $f(T)$, and the maximum $f(T)$ over all n-vertex tournaments by $M(n)$. The letter "M" was chosen in honor of Moon who in 1971 proved [19] that

$$1.4757^n \leq M(n) \leq 1.7170^n$$

for large n. Our combinatorial main result are the stronger bounds

$$1.5448^n \leq M(n) \leq 1.6740^n \ .$$

To prove our new lower bound on $M(n)$, we construct an infinite family of tournaments all having $21^{n/7} > 1.5448^n$ minimal FVSs. To prove our new upper bound on $M(n)$, we bound the maximum of a convex function bounding $M(n)$ from above, and otherwise rely on case distinctions and recurrence relations.

For general directed graphs, no non-trivial upper bound on the number of minimal FVSs is known. For undirected graphs, Fomin et al. [8] show that any undirected graph on n vertices contains at most

1.8638^n minimal FVSs, and that infinitely many graphs have $105^{n/10} > 1.5926^n$ minimal FVSs. Lower bounds of roughly $\log n$ on the size of a maximum-size acyclic subtournament have been obtained by Reid and Parker [25] and Neumann-Lara [21]. Other bounds on minimal or maximal sets with respect to vertex-inclusion have been obtained for dominating sets [9], bicliques [12], separators [10], potential maximal cliques [11], bipartite graphs [4], r-regular subgraphs [14], and, of course, independent sets [18, 20]. The increased interest in exponential time algorithms over the last few years has given new importance to such bounds, as the enumeration of the corresponding objects may be used in exponential time algorithms to solve various problems; see, for example [2, 3, 6, 11, 17, 22].

Enumeration. An algorithm by Schwikowski and Speckenmeyer [26] lists the minimal FVSs of a directed graph G with polynomial delay, by traversing a hypergraph whose vertices are bijectively mapped to minimal FVSs of G. Unfortunately the Schwikowski-Speckenmeyer-algorithm may use exponential space, and it is not known whether the minimal FVS problem allows a polynomial delay enumeration algorithm with polynomially bounded space complexity in directed graphs. Our algorithmic main result provides such an enumeration algorithm for the family of *tournaments*. Our algorithm is inspired from that by Tsukiyama et al. for the (conceptually simpler) enumeration of maximal independent sets [27]. It is based on iterative compression, a technique for parameterized [24] and exact algorithms [7]. We thereby positively answer Fomin et al.'s [7] question if the technique could be applied to other algorithmic areas.

Exact Algorithms. In the third [30] in a series [28, 29, 30] of very influential surveys on exact exponential time algorithms, Woeginger observes that Moon's upper bound on $M(n)$ provides an upper bound on the overall running time of the enumeration algorithm of Schwikowski and Speckenmeyer. He explicitly asks for a faster algorithm finding a feedback vertex set of a tournament of minimum size. Our new bound yields a time complexity of $O(1.6740^n)$. Unlike upper bound proofs on other [4, 8, 9, 10, 11, 12, 14, 18, 20] minimal or maximal sets with respect to vertex inclusion, for minimal FVSs in tournaments no known (non trivial) proof readily translates into a polynomial-space branching algorithm. Due to its space complexity, which differs from its time complexity by only a polynomial factor, the Schwikowski-Speckenmeyer-algorithm has only limited practicability [30]. With our new enumeration algorithm, we achieve however a polynomial-space $O(1.6740^n)$-time algorithm to find a minimum sized feedback vertex set in tournaments, and to even enumerate all minimal ones. Dom et al. [5] independently answered Woeginger's question by constructing an iterative–compression algorithm solving only the optimization version of the problem. However, the running time of their algorithm grows at least with 1.708^n and hence their result is inherently weaker than ours.

Organization of the paper. Preliminaries are provided in Section 2. In Section 3, we answer how many distinct minimal FVSs a (strong) tournament on n vertices has *at least*. Section 4 proves the lower bound on $M(n)$, and Section 5 gives the upper bound. We conclude with the polynomial-space polynomial-delay enumeration algorithm in Section 6. The main result of the paper is formulated in Corollary 2.

2 Preliminaries

Let $T = (V, A)$ be a tournament. For a vertex subset $V' \subseteq V$, the tournament $T[V']$ induced by V' is called a *subtournament* of T. For each vertex $v \in V$, its *in-neighborhood* and *out-neighborhood* are defined as $N^-(v) = \{u \in V \mid (u, v) \in A\}$ and $N^+(v) = \{u \in V \mid (v, u) \in A\}$, respectively. If there is an arc $(u, v) \in A$ then we say that u *beats* v and write $u \to v$. A tournament T is *strong* if there exists a directed path between any two vertices. A non-strong tournament T has a unique factorization $T = S_1 + \ldots + S_r$ into strong subtournaments $S_1, \ldots, S_r$, where every vertex $u \in V(S_k)$ beats all vertices $v \in V(S_\ell)$, for $1 \leq k < \ell \leq r$. For $n \in \mathbb{N}$ let $\mathcal{T}_n$ denote the set of tournaments with n vertices and let $\mathcal{T}_n^*$ denote the set of strong tournaments on n vertices.

The *score* of a vertex $v \in V$ is the size of its out-neighborhood, and denoted by $s_v(T)$ or s_v for short. Consider a labeling $1, \ldots, n$ of the vertices of T such that their scores are non-decreasing, and

associate with T the *score sequence* $s(T) = (s_1, \ldots, s_n)$. If T is strong then $s(T)$ satisfies the *Landau inequalities* [15, 16]:

$$\sum_{v=1}^{k} s_v \geq \binom{k}{2} + 1 \quad \text{for all} \quad k = 1, \ldots, n-1, \text{ and} \tag{1}$$

$$\sum_{v=1}^{n} s_v = \binom{n}{2}. \tag{2}$$

For every non-decreasing sequence s of positive integers satisfying conditions (1)–(2), there exists a tournament whose score sequence is s [16].

Let L be a set of non-zero elements from the ring $\mathbb{Z}_n$ of integers modulo n such that for all $i \in \mathbb{Z}_n$ exactly one of $+i$ and $-i$ belongs to L. The tournament $T_L = (V_L, A_L)$ with $V_L = \{1, \ldots, n\}$ and $A_L = \{(i, j) \in V_L \times V_L \mid (j - i) \bmod n \ \in L\}$ is *the circular n-tournament induced by L*. A *triangle* is a tournament of order 3. The cyclic triangle is denoted C_3.

A *FVS* F of a tournament $T = (V, A)$ is a subset of vertices, such that $T[V \setminus F]$ has no directed cycle. It is *minimal* if it does not contain a FVS of T as a proper subset. Let $\mathcal{F}(T)$ be the collection of minimal FVSs of T; its cardinality is denoted by $f(T)$. A *minimum FVS* is a FVS with a minimum number of vertices.

Acyclic tournaments are sometimes called *transitive*; the (up to isomorphism unique) transitive tournament on n vertices is denoted TT_n. Let τ be the unique topological order of the vertices of TT_n such that $\tau(u) < \tau(v)$ if and only if u beats v. For such an order τ and integer $i \in \{1, \ldots, n\}$ the subsequence of the first i values of τ is denoted $\tau_i(V(TT_n)) = (\tau^{-1}(1), \ldots, \tau^{-1}(i))$; call $\tau_1(V(TT_n))$ the *source* of TT_n. For a minimal FVS F of a tournament T the subtournament $T[V \setminus F]$ is a *maximal transitive subtournament* of T and $V \setminus F$ is a *maximal transitive vertex set*.

3 Minimum Number of Minimal FVSs

In this section we analyze the minimum number of minimal FVSs in tournaments.

Let the function $m : \mathbb{N} \to \mathbb{N}, n \mapsto \min_{T \in \mathcal{T}_n} f(T)$ count the minimum number of minimal FVSs over all tournaments of order n. Since a minimal FVS always exists, $m(n) \geq 1$ for all positive integers n. This bound is attained by the transitive tournaments TT_n of all orders n.

Observation 1 ([19]). *If $T = S_1 + \ldots + S_r$ is the factorization of a tournament T into strong subtournaments $S_1, \ldots, S_r$, then $f(T) = f(S_1) \cdot \ldots \cdot f(S_r)$.*

Hence from now on we consider only strong tournaments (on at least 3 vertices) and define $m^* : \mathbb{N} \setminus \{1, 2\} \to \mathbb{N}, n \mapsto \min_{T \in \mathcal{T}_n^*} f(T)$.

Lemma 2. *The function m^* is constant: $m^*(n) = 3$ for all $n \geq 3$.*

Proof. Let $T \in \mathcal{T}_n^*$ be a strong tournament. We show that $f(T) \geq 3$. As T is strong, it contains some cycle and thus some cyclic triangle C, with vertices v_1, v_2, v_3. For $i = 1, 2, 3$, define the vertex sets $W_i = \{v_i, v_{(i+1) \bmod 3}\}$. Every set W_i can be extended to a maximal transitive vertex set W_i' of T. Note that for $i = 1, 2, 3$ and $j \in \{1, 2, 3\} \setminus \{i\}$, we have $v_{(i+2) \bmod 3} \in W_j' \setminus W_i'$. Hence, there are three maximal transitive subtournaments of T whose complements form three minimal FVSs of T. Consequently, $m^*(n) \geq 3$ for all $n \geq 3$.

To complete the proof, construct a family $\{U_n \in \mathcal{T}_n^* \mid n \geq 3\}$ of strong tournaments with exactly three minimal FVSs. Set U_3 equal to the cyclic triangle. For $n \geq 4$, build the tournament U_n as follows: start with the transitive tournament TT_{n-2}, whose vertices are labeled $1, \ldots, n-2$ by decreasing scores. Then add two special vertices u_1, u_2 which are connected by an arbitrarily oriented arc. For $i \in \{1, 2\}$, add arcs from all vertices $2, \ldots, n-2$ to u_i. Finally, connect vertex 1 to u_i by an arc $(u_i, 1)$, for $i = 1, 2$. The resulting tournament U_n, depicted in Fig. 1, has exactly three minimal FVSs, namely $\{u_1, u_2\}, \{1\}$ and $\{2, \ldots, n-2\}$. $\qquad\square$
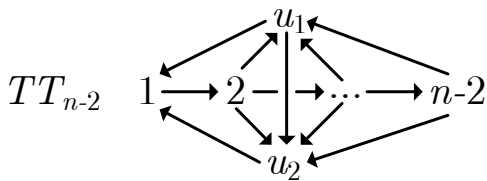
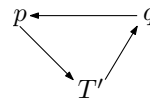Figure 1: Family U_n of strong tournaments with only three minimal FVSs.

Figure 2: A tournament $pq(T') \in \mathcal{T}_n^*$ with $f(pq(T')) = 2f(T') + 1$.

4 Lower Bound on the Maximum Number of Minimal FVSs

We prove a lower bound of $21^{n/7} > 1.5448^n$ on the maximum number of minimal FVSs of tournaments with n vertices.

Formally, we will bound from below the values of the function $M(n)$ mapping integers n to $\max_{T \in \mathcal{T}_n} f(T)$. By convention, set $M(0) = 1$. Note that M is monotonically non-decreasing on its domain: given any tournament $T \in \mathcal{T}_n$ and any vertex $v \in V(T)$, for every minimal FVS $F \in \mathcal{F}(T[V(T) \setminus \{v\}])$ either $F \in \mathcal{F}(T)$ or $F \cup \{v\} \in \mathcal{F}(T)$. As T and v are arbitrary it follows that $M(n) \geq M(n-1)$.

We will now show that there is an infinite family of tournaments on $n = 7k$ vertices, for any $k \in \mathbb{N}$, with $21^{n/7} > 1.5448^n$ minimal FVSs, improving upon Moon's [19] bound of 1.4757^n. Let ST_7 denote the Paley digraph of order 7, i.e. the circular 7-tournament induced by the set $L = \{1, 2, 4\}$ of quadratic residues modulo 7. All maximal transitive subtournaments of ST_7 are transitive triangles, of which there are exactly 21, as each vertex is the source of 3 distinct transitive triangles. Thus, all minimal FVSs for ST_7 are minimum FVSs. We remark that ST_7 is the unique 7-vertex tournament without any TT_4 as subtournament [25].

Lemma 3. *There exists an infinite family of tournaments with $21^{n/7}$ minimal FVSs.*

Proof. Let $k \in \mathbb{N}$ and form the tournament $T_0 = ST_7 + \ldots + ST_7$ from k copies of $ST_7 \in \mathcal{T}_7^*$. Then $T_0 \in \mathcal{T}_n$ for $n = 7k$, and the number of minimal FVSs in T_0 is $f(T_0) = f(ST_7)^k = 21^k = 21^{n/7}$. $\quad\square$

5 Upper Bound on the Maximum Number of Minimal FVSs

We give an upper bound of β^n, where $\beta = 1.6740$, on the maximum number of minimal FVSs in any tournament $T \in \mathcal{T}_n$, for any positive integer n. This improves the bound of 1.7170^n by Moon [19]. Instead of minimal FVSs we count maximal transitive subtournaments, and with respect to Observation 1 we count the maximal transitive subtournaments of *strong* tournaments.

We start with three properties of maximal transitive subtournaments. First, for a strong tournament $T = (V, A)$ with score sequence $s = (s_1, \ldots, s_n)$ the following holds: if $TT_k = (V', A')$ is a maximal transitive subtournament of T with $\tau_1(V') = (t)$ then $T[V' \setminus \{t\}]$ is a maximal transitive subtournament of $T[N^+(t)]$. Hence $f(T) \leq \sum_{v=1}^n M(s_v)$, where $s_v \leq n-2$ for all $v \in V$. This allows us to effectively bound $f(T)$ via a recurrence relation.

Second, there cannot be too many vertices with large score.

Lemma 4. *For $n \geq 8$ and $k \in \{0, 1, 2\}$, any strong tournament $T \in \mathcal{T}_n^*$ has at most $2(k+1)$ vertices of score at least $n - 2 - k$.*

Proof. Fix some strong tournament $T \in \mathcal{T}_n^*$ and $k \in \{0, 1, 2\}$. Suppose for contradiction that T contains $2k+3$ vertices with score at least $n - 2 - k$. Then the Landau inequalities (1) and (2) imply

Table 1: Extremal tournaments of up to 9 vertices

n	$M(n)$	$M(n)^{1/n} \approx$	$T \in \mathcal{T}_n : f(T) = M(n)$
1	1	1.00000	$T \in \mathcal{T}_1$
2	1	1.00000	$T \in \mathcal{T}_2$
3	3	1.44225	$T \in \mathcal{T}_3 \setminus \{TT_3\}$
4	3	1.31607	$T \in \mathcal{T}_4 \setminus \{TT_4\}$
5	7	1.47577	$QT_5 \cong pq(C_3), RT_5$
6	12	1.51309	$ST_6 \cong ST_7 - \{1\}$
7	21	1.54486	ST_7
8	25	1.49535	$ST_8 \cong pq(ST_6)$
9	43	1.51879	$ST_9 \cong pq(ST_7)$

the contradiction

$$
\begin{aligned}
2\binom{n}{2} &= 2\left(\sum_{v=1}^{n-(2k+3)} s_v + \sum_{v=n-(2k+2)}^{n} s_v \right) \\
&\geq 2\left(\binom{n-(2k+3)}{2} + 1 + (2k+3)(n-2-k) \right) = n^2 - n + 2.
\end{aligned}
$$

$\square$

For $n \leq 7$, we can explicitly list the strong n-vertex tournaments for which the Lemma fails: the cyclic triangle for $k = 0$, the tournaments RT_5, ST_6 for $k = 1$ and ST_7 for $k = 2$. RT_5 is the regular tournament of order 5 and ST_6 is the tournament obtained by arbitrarily removing some vertex from ST_7 (defined in the previous section) and all incident arcs.

Third, let T' be a tournament obtained from a tournament T by reversing all arcs of T. Then, $f(T) = f(T')$, whereas the score $s_v(T)$ of each vertex v turns into $s_v(T') = n - 1 - s_v(T)$. This implies that analyzing score sequences with maximum score $s_n \geq n - 1 - c$ for some constant c is symmetric to analyzing score sequences with minimum score $s_1 \leq c$.

Our proof that any tournament on n vertices has at most β^n maximal transitive subtournaments consists of several parts. We start by proving the bound for tournaments with few vertices. The inductive part of the proof first considers tournaments with large maximum score (and symmetrically small minimum score), and then all other tournaments.

We begin the proof by considering tournaments with up to 10 vertices. For $n \leq 4$ exact values for $M(n)$ were known before [19]. For $n = 5, \dots, 9$ we obtained exact values for $M(n)$ with the help of a computer. For these values the extremal tournaments obey the following structure: pick a strong tournament $T' \in \mathcal{T}_{n-2}^*$ and construct the strong tournament $pq(T') \in \mathcal{T}_n^*$ by attaching two vertices to T' as in Fig. 2; namely add vertices p and q to T', and arcs $q \to p$, and $p \to t$, $t \to q$ for each vertex t in T'. Then $f(pq(T')) = 2f(T') + 1$.

For $n = 5$, there are exactly two non-isomorphic strong tournaments $QT_5 \cong pq(C_3)$, $RT_5 \in \mathcal{T}_5^*$. For these, $f(QT_5) = f(RT_5) = M(5) = 2 \cdot 3 + 1 = 7$. For $n = 6$, ST_6 is the unique tournament from $\mathcal{T}_6$ with $f(ST_6) = M(6) = 12$ minimal FVSs. For $n = 7$ the previous section showed $f(ST_7) = 21$, and in fact ST_7 is the unique 7-vertex tournament with $M(7) = 21$ minimal FVSs. For $n \in \{8, 9\}$, $ST_n \cong pq(ST_{n-2})$; then $f(ST_n) = M(n)$. Table 1 summarizes that for $n \leq 9$, $M(n) \leq \beta^n$.

Next, we bound $M(10)$ by means of $M(n)$ for $n \leq 9$. Let W be a maximal transitive vertex set of $T \in \mathcal{T}_{10}^*$. Then either $v^* \in W$ or $v^* \notin W$, where v^* is a vertex with score s_{10}. There are at most $M(s_{10}) \leq M(9)$ maximal transitive vertex sets W such that $v^* \in W$ and at most $M(9)$ such sets W for which $v^* \notin W$. As $(2M(9))^{1/10} = 86^{1/10} < 1.5612$, the proof follows for all tournaments with at most 10 vertices.

For the rest of this section we consider tournaments with $n \geq 11$ vertices. Let $T = (V, A)$ be a strong tournament on $n \geq 11$ vertices; we will show that $f(T) \leq \beta^n$. The proof considers four main cases and several subcases with respect to the minimum and maximum score of the tournament.

We provide a complete proof of the upper bound on the maximum number of minimal feedback vertex set in tournaments.

Let $T = (V, A)$ be a strong tournament on $n \geq 11$ vertices and let $s = (s_1, \ldots, s_n)$ be the score sequence of T. We will show that $f(T) \leq \beta^n$. The proof considers four main cases and several subcases with respect to the minimum and maximum score of the tournament. To avoid a cumbersome nesting of cases, whenever inside a given case we assume that none of the earlier cases applies. By W we denote a maximal transitive vertex set of T.

Case 1: $s_n = n - 2$. Let b be the unique vertex beating vertex n.

If $b \notin W$ then $\tau_1(W) = (n)$; there are at most $M(s_n) = M(n-2)$ such W.

If $b \in W$ and $n \in W$, then $\tau_1(W \setminus \{b\}) = (n)$ as no vertex except b beats n. So, $\tau_2(W) = (b, n)$ and there are at most $M(s_b - 1)$ such W. For the last possibility, where $b \in W$ and $n \notin W$, note that W contains at least one in-neighbor of b, otherwise W were not maximal as n could be added. We consider 4 subcases depending on the score of b.

Case 1.1: $s_b = n - 2$. Let c be the unique vertex beating b. As at most 2 vertices have score $n - 2$ by Lemma 4, $s_c \leq n - 3$. We have that $c \in W$, otherwise W would not be maximal as $W \cup \{n\}$ induces a transitive subtournament of T. As b and its unique in-neighbor c are in W, $\tau_2(W) = (c, b)$. There are at most $M(s_c - 1) \leq M(n-4)$ such W. In total, $f(T) \leq M(n-2) + M(n-3) + M(n-4) \leq \beta^{n-4} + \beta^{n-3} + \beta^{n-2}$ which is at most β^n because $\beta \geq 1.4656$.

In the three remaining subcases, all in-neighbors of b have score at most $n - 3$: if $c_i \in N^-(b)$ had score $n - 2$, then Case 1.1 would apply with $n := c_i$ and $b := n$.

Case 1.2: $s_b = n - 3$. Let $N^-(b) := \{c_1, c_2\}$ such that $c_1 \to c_2$. Then either $\tau_1(W) = (c_1)$ or $\tau_1(W) = (c_2)$; there are at most $2M(n-3)$ such W. It follows $f(T) \leq M(n-2) + M(n-4) + 2M(n-3) \leq \beta^{n-4} + 2\beta^{n-3} + \beta^{n-2} \leq \beta^n$ as $\beta \geq 1.6181$.

Case 1.3: $s_b = n - 4$. Let $N^-(b) := \{c_1, c_2, c_3\}$. Observe that at most 2 vertices among $N^-(b)$ have score $n - 3$, otherwise T is not strong as $N^-(b) \cup \{b, n\}$ induce a strong component. Either $\tau_1(W) = (c_1)$ or $\tau_1(W) = (c_2)$ or $\tau_1(W) = (c_3)$; there are at most $2M(n-3) + M(n-4)$ such W. Thus, $f(T) \leq M(n-2) + M(n-5) + 2M(n-3) + M(n-4) \leq \beta^{n-5} + \beta^{n-4} + 2\beta^{n-3} + \beta^{n-2} \leq \beta^n$ as $\beta \geq 1.6664$.

Case 1.4: $s_b \leq n - 5$. Then there are at most $M(n-1)$ subtournaments not containing n. It follows $f(T) \leq M(n-2) + M(n-6) + M(n-1) \leq \beta^{n-6} + \beta^{n-2} + \beta^{n-1} \leq \beta^n$ as $\beta \geq 1.6737$.

Case 2: $s_n = n - 3$. Let b_1, b_2 be the two vertices beating n such that $b_1 \to b_2$. The tree in Fig. 3 pictures our case distinction. Its leaves correspond to six different cases, numbered (1)–(6), for membership or non-membership of n, b_1 and b_2 in some maximal transitive vertex set W of T. The cases corresponding to leafs (2) and (4) will be considered later. Let us now bound the number of possible W for the other cases (1), (3), (5) and (6).

Claim 1. *Among all maximal transitive vertex sets W of T,*

  (1) at most $M(n-3)$ are such that $b_1 \notin W$ and $b_2 \notin W$,

  (3) at most $M(s_{b_2} - 1)$ are such that $b_1 \notin W$, $b_2 \in W$ and $n \in W$,

  (5) at most $M(s_{b_1} - 2)$ are such that $b_1 \in W$, $b_2 \notin W$ and $n \in W$, and

  (6) at most $M(s_{b_1} - 2)$ are such that $b_1 \in W$, $b_2 \in W$ and $n \in W$.

Proof. If (1) $b_1 \notin W$ and $b_2 \notin W$, then $n \in W$ by maximality of W and n is the source of $T[W]$ as no vertex in W beats n. Thus, there are at most $M(s_n) = M(n-3)$ such W. If (3) $b_1 \notin W$, $b_2 \in W$ and $n \in W$, then $\tau_1(W \setminus \{b_2\}) = (n)$. Therefore, $\tau_2(W) = (b_2, n)$ and there are at most $M(s_{b_2} - 1)$ such W. If (5) $b_1 \in W$, $b_2 \notin W$ and $n \in W$, then $\tau_2(W) = (b_1, n)$, and as b_1 beats b_2, there are at most $M(s_{b_1} - 2)$ such W. If (6) $b_1 \in W$, $b_2 \in W$ and $n \in W$, then $\tau_3(W) = (b_1, b_2, n)$, and there are at most $M(s_{b_1} - 2)$ such W. $\qquad\square$
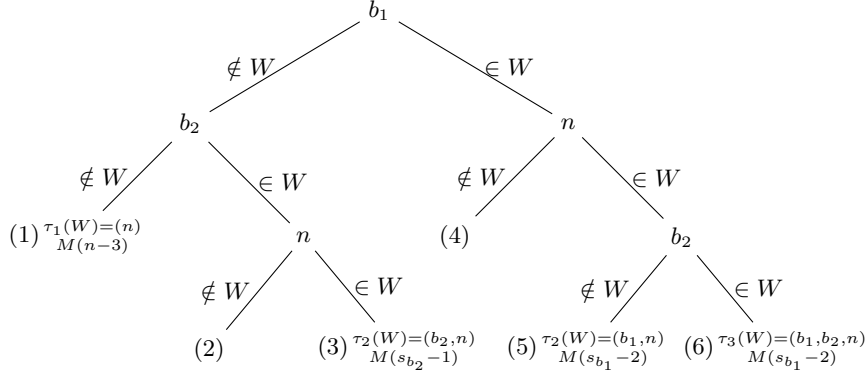
Figure 3: Different possibilities for a maximal transitive vertex set W.

To bound the number of subtournaments corresponding to the conditions in leafs (2) and (4), we will consider five subcases depending on the scores of b_1 and b_2. If b_1 and b_2 have low scores (Cases 2.4 and 2.5), there are few maximal transitive subtournaments of T corresponding to the conditions in the leafs (3), (5) and (6). Then, it will be sufficient to group the cases (2) and (4) into one case where $n \notin W$ and to note that there are at most $M(n-1)$ such subtournaments. Otherwise, if the scores of b_1 and b_2 are high (Cases 2.1 – 2.3), we use that in (2), some vertex of $N^-(b_2)$ is the source of W. If this were not the case, W would not be maximal as $W \cup \{n\}$ would induce a transitive tournament. Similarly, in (4) some vertex of $N^-(b_1)$ is the source of W if $b_2 \notin W$.

Let $c_1, \ldots, c_{|N^-(b_1)|}$ be the in-neighbors of b_1 such that $c_i \to c_{i+1}$ for all $i \in \{1, \ldots, |N^-(b_1)| - 1\}$ (every tournament has a Hamiltonian path [23]) and let $d_1, \ldots, d_{|N^-(b_2)|-1}$ be the in-neighbors of b_2 besides b_1 such that $d_i \to d_{i+1}$ for all $i \in \{1, \ldots, |N^-(b_2)| - 2\}$.

Let us first bound the number of subtournaments satisfying the conditions of (2) depending on s_{b_2}.

Claim 2. *If $s_{b_2} = n - 3$, there are at most $M(s_{d_1} - 1)$ maximal transitive vertex sets W such that $b_1 \notin W$, $b_2 \in W$ and $n \notin W$.*

Proof. As mentioned above, some in-neighbor of b_2 is the source of W. As $s_{b_2} = n-3$, $N^-(b_2) \setminus \{b_1\} = \{d_1\}$. Thus, $\tau_2(W) = (d_1, b_2)$ and there are at most $M(s_{d_1} - 1)$ such tournaments. $\square$

Claim 3. *If $s_{b_2} = n - 4$, there are at most $M(n-5) + 2M(s_{d_1} - 2)$ maximal transitive vertex sets W such that $b_1 \notin W$, $b_2 \in W$ and $n \notin W$.*

Proof. If $d_1 \notin W$ then $\tau_2(W) = (d_2, b_2)$ and there are at most $M(s_{b_2} - 1) = M(n-5)$ such W. Otherwise, $d_1 \in W$ and either $d_2 \notin W$ in which case $\tau_2(W) = (d_1, b_2)$, or $d_2 \in W$ in which case $\tau_3(W) = (d_1, d_2, b_2)$. There are at most $2M(s_{d_1} - 2)$ such W. $\square$

The next step is to bound the number of subtournaments satisfying the conditions of (4) depending on s_{b_1}.

Claim 4. *If $s_{b_1} = n - 3$, the number of maximal transitive vertex sets W such that $b_1 \in W$ and $n \notin W$ is at most $2M(n-5) + M(n-4)$ if b_2 beats no vertex of $N^-(b_1)$, and otherwise at most $2M(n-5) + M(n-4) + M(n-6)$ if $s_{b_2} = n - 3$ and at most $2M(n-5) + M(n-4) + 3M(n-7)$ if $s_{b_2} = n - 4$.*

Proof. If $N^-(b_1) \cap W \neq \emptyset$, then c_1 or c_2 is the source of W. The number of subsets W such that $c_1 \notin W$, and thus $\tau_2(W) = (c_2, b_1)$, is at most $M(s_{c_2} - 1) \leq M(n-4)$. The number of subsets W such that $c_1 \in W$, and thus $\tau_3(W) = (c_1, c_2, b_1)$ or $\tau_2(W) = (c_1, b_1)$, is at most $2M(s_{c_1} - 2) \leq 2M(n-5)$. If, on the other hand, $N^-(b_1) \cap W = \emptyset$, then $\tau_1(W) = (b_1)$ and some in-neighbor of b_2 is the source of $T[W \setminus \{b_1\}]$, otherwise W is not maximal as n can be added. Also note that b_2 beats some vertex of $N^-(b_1)$ (we have $N^-(b_2) \setminus N^-(b_1) \neq \emptyset$ as $N^-(b_1) \cap W = \emptyset$ but $N^-(b_2) \cap W \neq \emptyset$). If $s_{b_2} = n - 3$, we upper bound the number of such subsets W by $M(s_{b_1} - 3) = M(n-6)$ as $\tau_3(W) = (b_1, d_1, b_2)$. If

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$s_{b_2} = n - 4$, we have that $\tau_4(W) = (b_1, d_1, d_2, b_2)$, $\tau_3(W) = (b_1, d_2, b_2)$ or $\tau_3(W) = (b_1, d_1, b_2)$. Thus, there are at most $3M(s_{b_1} - 4) = 3M(n - 7)$ possible W such that $N^-(b_1) \cap W = \emptyset$ if $s_{b_1} = n - 3$ and $s_{b_2} = n - 4$. Summarizing, there are at most $2M(n - 5) + M(n - 4)$ subsets W if b_2 beats no vertex of $N^-(b_1)$, and otherwise at most $2M(n - 5) + M(n - 4) + M(n - 6)$ subsets W if $s_{b_2} = n - 3$ and at most $2M(n - 5) + M(n - 4) + 3M(n - 7)$ subsets W if $s_{b_2} = n - 4$. $\qquad\square$

Claim 5. *If $s_{b_1} = n - 4$ and $s_{b_2} = n - 3$, the number of maximal transitive vertex sets W such that $b_1 \in W$ and $n \notin W$ is*

- *at most $M(n - 7) + \sum_{c \in N^-(b_1)} 2M(s_c - 2)$ if $T[N^-(b_1)]$ is a directed cycle,*

- *at most $\max\{M(n-3) + M(n-4) + M(n-5); M(n-5) + 6M(n-6)\}$ if $T[N^-(b_1)]$ is transitive and $d_1 \in N^-(b_1)$, and*

- *at most $M(n-3) + M(n-4) + M(n-5) + M(n-7)$ if $T[N^-(b_1)]$ is transitive and $d_1 \notin N^-(b_1)$.*

Proof. If $c_3 \to c_1$, then W intersects $N^-(b_1)$ in at most $2^3 - 1 = 7$ possible ways ($N^-(b_1) \subseteq W$ would induce a cycle in $T[W]$). In one of them, $N^-(b_1) \cap W = \emptyset$, which implies $\tau_3(W) = (b_1, d_1, b_2)$; there are at most $M(s_{b_1} - 3) = M(n - 7)$ such W. For each $c \in N^-(b_1)$, there are 2 possibilities where $\tau_1(W) = (c)$; one where $\tau_2(W) = (c, b_1)$ and one where $\tau_3(W) = (c, y, b_1)$ where y is the out-neighbor of c in $N^-(b_1)$; there are $2M(s_c - 2)$ such W for each choice of c. In total, there are at most $M(n - 7) + \sum_{c \in N^-(b_1)} 2M(s_c - 2)$ possible W.

If, on the other hand, $c_1 \to c_3$, first assume that $s_{c_1} \le n - 3$, $s_{c_2} \le n - 4$, and $s_{c_3} \le n - 5$. Then either some vertex of $N^-(b_1)$ is the source of W (at most $M(n-3) + M(n-4) + M(n-5)$ possibilities for W), or $\tau_3(W) = (b_1, d_1, b_2)$ (at most $M(n-7)$ possibilities for W). Otherwise, it must be that $s_{c_1} \le n - 3$, $s_{c_2} \le n - 4$, $s_{c_3} = n - 4$ and that $d_1 = c_3$. Then, $\tau_2(W) = (c_3, b_1)$, $\tau_2(W) = (c_2, b_1)$, $\tau_3(W) = (c_2, c_3, b_1)$, $\tau_2(W) = (c_1, b_1)$, $\tau_3(W) = (c_1, c_2, b_1)$, $\tau_3(W) = (c_1, c_3, b_1)$, or $\tau_4(W) = (c_1, c_2, c_3, b_1)$; there are at most $M(n-5) + 6M(n-6)$ such W. In total, if $d_1 \in N^-(b_1)$, the number of possible W can be upper bounded by $\max\{M(n-3) + M(n-4) + M(n-5); M(n-5) + 6M(n-6)\}$, and if $d_1 \notin N^-(b_1)$, the number of possible W can be upper bounded by $M(n-3) + M(n-4) + M(n-5) + M(n-7)$. $\quad\square$

Armed with Claims 2–5, we now analyze the five subcases of Case 2, depending on the scores of b_1 and b_2.

Case 2.1: $\mathbf{s_{b_1} = n - 3, s_{b_2} = n - 3}$. By Claim 2, the number of maximal transitive vertex sets W such that $b_1, n \notin W$ and $b_2 \in W$ (leaf (2) in Fig. 3) is at most $M(n - 4)$. By Claim 4, the number of maximal transitive vertex sets W such that $b_1, n \notin W$ and $b_2 \in W$ (leaf (4) in Fig. 3) is at most $2M(n - 5) + M(n - 4)$, at most $2M(n - 5) + M(n - 4) + M(n - 6)$, or at most $2M(n - 5) + M(n - 4) + 3M(n - 7)$. Combined with Claim 1,

$$f(T) \le \max \begin{cases} M(n - 3) + M(n - 4) + M(n - 4) + (2M(n - 5) \\ \quad + M(n - 4)) + M(n - 5) + M(n - 5) \\ \qquad \le 4\beta^{n-5} + 3\beta^{n-4} + \beta^{n-3} \le \beta^n \text{ as } \beta \ge 1.6314 \ , \\ M(n - 3) + M(n - 4) + M(n - 4) + (2M(n - 5) \\ \quad + M(n - 4) + M(n - 6)) + M(n - 5) + M(n - 5) \\ \qquad \le \beta^{n-6} + 4\beta^{n-5} + 3\beta^{n-4} + \beta^{n-3} \le \beta^n \text{ as } \beta \ge 1.6516 \ , \\ M(n - 3) + M(n - 4) + M(n - 4) + (2M(n - 5) \\ \quad + M(n - 4) + 3M(n - 7)) + M(n - 5) + M(n - 5) \\ \qquad \le 3\beta^{n-7} + 4\beta^{n-5} + 3\beta^{n-4} + \beta^{n-3} \le \beta^n \text{ as } \beta \ge 1.6666 \ . \end{cases}$$

Case 2.2: $s_{b_1} = n - 3, s_{b_2} = n - 4$. If $c_1 \to b_2$ and $c_2 \to b_2$, then $b_1 \notin W$ and $b_2 \in W$ implies that some in-neighbor c of b_1 is in W, otherwise $W \cup \{b_1\}$ would induce a transitive tournament. But then, $n \notin W$, otherwise $\{c, b_2, n\}$ induces a directed cycle. This means that no maximal transitive

vertex set W satisfies the conditions of leaf (3) in Fig. 3. We bound the possible W corresponding to leafs (2)+(4) by $M(n-1)$ and obtain

$$f(T) \leq M(n-3) + M(n-1) + M(n-5) + M(n-5)$$
$$\leq 2\beta^{n-5} + \beta^{n-3} + \beta^{n-1} \leq \beta^n \text{ as } \beta \geq 1.6440 \ .$$

Otherwise, there is some vertex $c \in N^-(b_1)$ such that $b_2 \to c$. Then, the number of W in leaf (6) of Fig. 3 is upper bounded by $M(s_{b_2} - 2) = M(n-6)$, and by Claims 3 and 4 those in leafs (2) and (4) are upper bounded by $M(n-5) + 2M(s_{d_1} - 2)$ and $2M(n-5) + M(n-4) + 3M(n-7)$, respectively. Thus,

$$f(T) \leq M(n-3) + (M(n-5) + 2M(n-5)) + M(n-5) + (2M(n-5)$$
$$+ M(n-4) + 3M(n-7)) + M(n-5) + M(n-6)$$
$$\leq 3\beta^{n-7} + \beta^{n-6} + 7\beta^{n-5} + \beta^{n-4} + \beta^{n-3} \leq \beta^n \text{ as } \beta \geq 1.6740 \ .$$

Case 2.3: $s_{b_1} = n-4, s_{b_2} = n-3$. By Claim 2, at most $M(n-4)$ subsets W correspond to leaf (2) in Fig. 3. If $N^-(b_1)$ induces a directed cycle, Claim 5 upper bounds the number of subsets corresponding to leaf (4) by $M(n-7) + 2M(n-6) + 4M(n-5)$ as at most 2 vertices except b_2 and n have score $n-3$ by Lemma 4. Together with Claim 1, this gives

$$f(T) \leq M(n-3) + M(n-4) + M(n-4) + (M(n-7) + 2M(n-6)$$
$$+ 4M(n-5)) + M(n-6) + M(n-6)$$
$$\leq \beta^{n-7} + 4\beta^{n-6} + 4\beta^{n-5} + 2\beta^{n-4} + \beta^{n-3} \leq \beta^n \text{ as } \beta \geq 1.6670 \ .$$

Otherwise, $c_1 \to c_3$. If $d_1 \to b_1$, then Claim 5 upper bounds the number of subsets corresponding to leaf (4) by $M(n-3) + M(n-4) + M(n-5)$ or $M(n-5) + 6M(n-6)$. Then,

$$f(T) \leq \max \begin{cases} M(n-3) + M(n-4) + M(n-4) + (M(n-3) \\ \quad + M(n-4) + M(n-5)) + M(n-6) + M(n-6) \\ \leq 2\beta^{n-6} + \beta^{n-5} + 3\beta^{n-4} + 2\beta^{n-3} \leq \beta^n \text{ as } \beta \geq 1.6632, \\ M(n-3) + M(n-4) + M(n-4) + (M(n-5) \\ \quad + 6M(n-6)) + M(n-6) + M(n-6) \\ \leq 8\beta^{n-6} + \beta^{n-5} + 2\beta^{n-4} + \beta^{n-3} \leq \beta^n \text{ as } \beta \geq 1.6396 \ . \end{cases}$$

Otherwise, $b_1 \to d_1$. For the possible W with $b_1, b_2, n \in W$, none of $N^-(b_1) \cup \{d_1\}$ is in W as these vertices all create cycles with b_1, b_2, n. Thus, the number of possible subsets W corresponding to leaf (6) is upper bounded by $M(s_{b_1} - 3) = M(n-7)$. Then, by Claims 1 and 5,

$$f(T) \leq M(n-3) + M(n-4) + M(n-4) + (M(n-3) + M(n-4)$$
$$+ M(n-5) + M(n-7)) + M(n-6) + M(n-7)$$
$$\leq 2\beta^{n-7} + \beta^{n-6} + \beta^{n-5} + 3\beta^{n-4} + 2\beta^{n-3} \leq \beta^n \text{ as } \beta \geq 1.6672 \ .$$

Case 2.4: $s_{b_1} = n-4, s_{b_2} \leq n-4$. By grouping leafs (2) and (4) into one possibility where $n \notin W$, Claim 1 upper bounds the number of such maximal transitive vertex sets by

$$f(T) \leq M(n-3) + M(n-1) + M(n-5) + M(n-6) + M(n-6)$$
$$\leq 2\beta^{n-6} + \beta^{n-5} + \beta^{n-3} + \beta^{n-1} \leq \beta^n \text{ as } \beta \geq 1.6570 \ .$$

Case 2.5: $s_{b_1} \leq n-5$. By grouping leafs (2) and (4) into one possibility where $n \notin W$, Claim 1 upper bounds the number of such maximal transitive vertex sets by

$$f(T) \leq M(n-3) + M(n-1) + M(n-4) + M(n-7) + M(n-7)$$
$$\leq 2\beta^{n-7} + \beta^{n-4} + \beta^{n-3} + \beta^{n-1} \leq \beta^n \text{ as } \beta \geq 1.6679 \ .$$

Case 3: $s_n \leq n-4$. We may assume that the score sequence $s = s(T)$ satisfies

$$3 \leq s_1 \leq \ldots \leq s_n \leq n-4. \tag{3}$$

Let S_n be the set of all score sequences that are feasible for (1)–(3). The set S_n serves as domain of the linear map $G : S_n \to \mathbb{R}_+, s \mapsto \sum_{v=1}^n g(s_v)$ with the strictly convex terms $g : c \mapsto \beta^c$. Furthermore, for all $n \geq 11$, we define a special score sequence $\sigma(n)$, whose membership in S_n is easy to verify:

$$\sigma(n) := \begin{cases} (3,3,3,3,3,5,7,7,7,7,7) & \text{if } n = 11 \ , \\ (3,3,3,3,3,3,8,8,8,8,8,8) & \text{if } n = 12 \ , \\ (3,3,3,3,3,3,3,6,9,9,9,9,9,9) & \text{if } n = 13 \ , \text{ and} \\ (3,3,3,3,3,3,4,7,8,\ldots,n-9,n-8,n-5, \\ \quad n-4,n-4,n-4,n-4,n-4,n-4) & \text{if } n \geq 14 \ . \end{cases}$$

Lemma 5. *For $n \geq 11$, the sequence $\sigma(n)$ maximizes the value of G over all sequences in S_n: $G(s) \leq G(\sigma(n))$ for all $s \in S_n$.*

Once Lemma 5 is proved we can bound $f(T)$, for $s = s(T) \in S_n$, from above via

$$f(T) \leq G(s) \leq G(\sigma(n)) = \begin{cases} 5\beta^3 + \beta^5 + 5\beta^7, & \text{if } n = 11 \ , \\ 6\beta^3 + 6\beta^8, & \text{if } n = 12 \ , \\ 6\beta^3 + \beta^6 + 6\beta^9, & \text{if } n = 13 \ , \\ 6\beta^3 + \beta^4 + \frac{\beta^{n-7}-\beta^7}{\beta-1} + \beta^{n-5} + 6\beta^{n-4} \\ \quad \leq \frac{\beta^{n-7}}{\beta-1} + \beta^{n-5} + 6\beta^{n-4}, & \text{if } n \geq 14 \ , \end{cases} \tag{4}$$

which is at most β^n as $\beta \geq 1.6259$. To prove Lemma 5, we choose any sequence $s \in \operatorname{argmax}_{s' \in S_n} G(s')$ and then show that $s = \sigma(n)$. Recall that $s_1 \geq 3$ and $s_n \leq n-4$, and set $s_1^* = 3, s_n^* = n-4$.

Claim 6. *If some score c appears more than once in s, then $c \in \{s_1^*, s_n^*\}$.*

Proof. For contradiction, suppose that $s_1^* < s_u = s_v = c < s_n^*$ for two vertices u and v such that $1 \leq u < v \leq n$. First, suppose there exists an integer $k \in \{u, \ldots, v-1\}$ satisfying (1) with equality:

$$\sum_{v=1}^k s_v = \binom{k}{2} + 1 \ . \tag{5}$$

Then (1), (2) and Lemma 4 imply $8 \leq k \leq n-9$, so $k \notin \{s_1^*, s_n^*\}$. The choice of k among vertices of equal score c now yields

$$s_{k+1} = s_k = \sum_{v=1}^k s_v - \sum_{v=1}^{k-1} s_v \leq \binom{k}{2} + 1 - \binom{k-1}{2} - 1 = k-1 \ . \tag{6}$$

This however contradicts (1):

$$\sum_{v=1}^{k+1} s_v \leq \binom{k}{2} + 1 + (k-1) = \binom{k+1}{2} \ .$$

It is thus asserted that no vertex k with property (5) exists. The score sequence s' differing from s only in $s_u' = s_u - 1 = c-1, s_v' = s_v + 1 = c+1$, therefore belongs to S_n. So apply the function G to it, and use the strict convexity of g:

$$G(s') - G(s) = (g(c+1) - g(c)) - (g(c) - g(c-1)) > 0 \ .$$

This contradicts the choice of s as a maximizer of G, and establishes Claim 6. □

Claim 7. *The values $s_1^* = 3$ and $s_n^* = n - 4$ each appear between two and six times as scores in the sequence s.*

Proof. By Lemma 4, s_n^* is the score of no more than 6 vertices. By symmetry, s_1^* is the score of no more than 6 vertices. As a consequence of Claim 6, together s_1^* and s_n^* appear at least eight times in s. Hence there are at least two vertices of score s_1^* and at least two vertices of score s_n^*. $\square$

Claim 8. *If $n \geq 12$, each of s_1^* and s_n^* is the score of exactly six of the vertices.*

Proof. Assuming this were not the case for s_1^*, by Claim 7 it would be the score of two to five vertices. Hence there exists a vertex $a \in \{3, \ldots, 6\}$ with score $s_a > s_1^*$. It holds $s_n^* = n - 4 > a + 1$, which is obvious if $n \geq 13$ and follows from (2) if $n = 12$. So there must be two scores in s larger than s_a, precisely $s_a < s_{a+1} < s_{a+2}$. Observe that the sequence $s' = (s_1, \ldots, s_{a-1}, s_a - 1, s_{a+1} + 1, s_{a+2}, \ldots, s_n)$ is a member of S_n. The same argument on strict convexity of g as in Claim 6 gives

$$G(s') - G(s) = (g(y+1) - g(y)) - (g(x) - (g(x-1)) > 0$$

for $x = s_a < s_{a+1} = y$, again contradicting the choice of s as a maximizer of G. Consequently, the sequence s starts with six scores s_1^*. By symmetry, the same argumentation also applies for s_n^*, proving the claim. $\square$

Claim 9. *If $n = 11$, each of s_1^* and s_n^* is the score of exactly five of the vertices.*

Proof. As all scores are between 3 and 7, at most 5 vertices have score 3 and at most 5 vertices have score 7 by (2). Assume less than 5 vertices have score s_1^*. By Claim 7, s_1^* is the score of two to four vertices. Hence there exists a vertex $a \in \{3, 4, 5\}$ with score $s_a > s_1^*$. Thus, $s_n^* = 7 > a + 1$. So there must be two scores in s larger than s_a, precisely $s_a < s_{a+1} < s_{a+2}$. To conclude we construct a sequence s' with $G(s') > G(s)$ exactly as in the proof of Claim 8. $\square$

Claim 10. *It holds $s = \sigma(n)$.*

Proof. If $n = 11$, s has 5 vertices of score 3 and 5 vertices of score 7 by Claim 9. As, $\sigma(11)$ is the only such sequence not contradicting (2), the claim holds for $n = 11$. Similarly, $\sigma(n)$ is the only sequence not contradicting (1) and Claim 8 if $12 \leq n \leq 13$. Suppose now that $n \geq 14$. There are $n - 12$ elements of s being different from both s_1^* and s_n^*, which have a score equal to one of the $n - 8$ numbers in the range $4, \ldots, n - 5$. Symmetry of the map $d \mapsto \binom{n}{d}$ around $d = \frac{n}{2}$ together with (2) means that only pairs $\{h_1, n - 1 - h_1\}$ with $4 \leq h_1 < \frac{n-1}{2}$ and $\{h_2, n - 1 - h_2\}$ with $5 \leq h_2 < \frac{n-1}{2}$ of scores are missing in s. Moreover, (1) requires $h_1, h_2 < 7$, for otherwise $k = 8$ violates this relation. Since s was chosen to be a maximizer of G, this leaves $h_1 = 5$ and $h_2 = 6$. Thus $s = \sigma(n)$, completing the proof of the claim and of Lemma 5. $\square$

All cases taken together imply the following upper bound on the number of maximal transitive subtournaments.

Theorem 6. *Any strong tournament $T \in \mathcal{T}_n^*$ has at most 1.6740^n maximal transitive subtournaments.*

Moon [19] already observed that the following limit exists.

Corollary 1. *It holds $1.5448 \leq \lim_{n \to \infty} (M(n))^{1/n} \leq 1.6740$.*

We conjecture that the Paley digraph of order 7, ST_7, plays the same role for FVSs in tournaments as triangles play for independent sets in graphs, i.e. that the tournaments T maximizing $(f(T))^{1/|V(T)|}$ are exactly those whose factors are copies of ST_7.

6 Polynomial-Delay Enumeration in Polynomial Space

In this section, we give a polynomial-space algorithm for the enumeration of the minimal FVSs in a tournament with polynomial delay.

Let $T = (V, A)$ be a tournament with $V = \{v_1, \ldots, v_n\}$, and for each $i = 1, \ldots, n$ let $T_i = T[\{v_1, \ldots, v_i\}]$. For a vertex set X, we write $\chi_X(i) = 1$ if $v_i \in X$ and $\chi_X(i) = 0$ otherwise. Let $<$ denote the total order on V induced by the labels of the vertices. For vertex sets $X, Y \subseteq V$, say that X is *lexicographically smaller* than Y and write $X \prec Y$ if for the minimum index i for which $\chi_X(i) \neq \chi_Y(i)$ it holds that $v_i \in X$. Because X and Y are totally ordered by the restriction of $<$ to X and Y, respectively, $\prec$ is also a total order and each collection of subsets of V has a unique *lexicographically smallest* element.

The algorithm enumerates the maximal acyclic vertex sets of T. It performs a depth-first search in a tree $\mathcal{T}$ with the maximal acyclic vertex sets of T as leaves, whose forward and backward edges are constructed "on the fly". The depth of $\mathcal{T}$ is $|V|$, and we refer to the vertices of $\mathcal{T}$ as *nodes*. The algorithm only needs to keep in memory the path from the root to the current node in the tree and all the children of the nodes on this path. Each node at level j is labeled by a maximal acyclic vertex set J of T_j. As for its children, there are two cases. In case $J \cup \{v_{j+1}\}$ is acyclic then J's only child is $J \cup \{v_{j+1}\}$. In case $J \cup \{v_{j+1}\}$ is not acyclic then J has at least one and at most $\lfloor j/2 \rfloor + 1$ children. Let $L_J = (v^1, v^2, \ldots, v^{|J|})$ be a labeling of the vertices in J such that $(v^r, v^s) \in A$ for all $1 \leq r < s \leq j$; we view L_J as a sequence of vertices. The children of J are as follows. The first child J^0 is a copy of J, and is always present. The potential other children are, for $1 \leq z \leq |J| + 1$,

$$J^z = \{v^i \in J \mid i < z \wedge v^i \to v_{j+1}\} \cup \{v_{j+1}\} \cup \{v^i \in J \mid i \geq z \wedge v_{j+1} \to v^i\}$$

where set J^z is a potential child of J only if J^z is a maximal acyclic vertex set in T_{j+1} (the maximality of J^z can clearly be checked in polynomial time). Note how we try to insert v_{j+1} at every possible position in J. However, only at most $\lfloor j/2 \rfloor + 1$ positions make sense for v_{j+1}: before v^1 if $v_{j+1} \to v^1$, between v^i and v^{i+1} if $v^i \to v_{j+1} \to v^{i+1}$, where $1 \leq i \leq |J| - 1$, and after $v^{|J|}$ if $v^{|J|} \to v_{j+1}$; all other positions do not give maximal acyclic vertex sets and should not be generated in an actual implementation. Note that J^z may be a potential child of several sets on the same level in $\mathcal{T}$. Of all these sets, J^z is made the child only of the lexicographically smallest such set. To determine whether J is the lexicographically smallest such set, we compute by a greedy algorithm the lexicographically smallest maximal acyclic vertex set $H = H(J^z)$ of T_j which contains $J^z \setminus \{v_{j+1}\}$ as a subset. That is, we iteratively build the set H by setting

$$H_0 = J^z \setminus \{v_{j+1}\},$$
$$H_i = \begin{cases} H_{i-1} \cup \{v_i\}, & \text{if } H_{i-1} \cup \{v_i\} \text{ is acyclic,} \\ H_{i-1}, & \text{otherwise,} \end{cases} \quad i = 1, \ldots, j,$$
$$H = H_j \ .$$

Then we make J^z a child of the node labeled J only if $H = J$. This completes the description of the algorithm.

To show that the algorithm is correct, we prove that for every maximal acyclic vertex set W of T there is exactly one leaf in $\mathcal{T}$ labeled with W. By construction of the algorithm, it suffices to show that at least one leaf is labeled by W. The proof is by induction on the number $n = |V|$ of vertices in T. For $n = 1$ the claim clearly holds, so suppose that $n > 1$ and that the claim is true for all tournaments with fewer vertices. Then from the induction hypothesis we can conclude that for the induced subtournament $T' := T_{n-1}$ there is a tree $\mathcal{T}'$ constructed by the above algorithm and a bijection f' from the maximal acyclic vertex sets of T' to the leaves of $\mathcal{T}'$.

Let W be a maximal acyclic vertex set of T. If $v_n \notin W$ then W is an acyclic vertex set of T' as removing a vertex from a digraph does not introduce cycles. In fact, W is a maximal acyclic vertex set of T': for any vertex $v_\ell \in V \setminus (W \cup \{v_n\})$, $T'[W \cup \{v_\ell\}]$ has a cycle as W is a maximal acyclic vertex set for T and $T'[W \cup \{v_\ell\}] = T[W \cup \{v_\ell\}]$. Hence there exists a leaf $f'(W)$ in $\mathcal{T}'$ labeled by W. Since $W \cup \{v_n\}$ is not acyclic, by maximality of W for T, the algorithm constructs the child W^0 of $f'(W)$ labeled by W, and that child will be a leaf in the final tree constructed by the algorithm.

If $v_n \in W$, then let $W' = W \setminus \{v_n\}$. So, W' is an acyclic vertex set of T'. In case W' is maximal for T', there is a leaf $f'(W')$ in $\mathcal{T}'$ that is labeled by W'. Since $W' \cup \{v_n\}$ is acyclic, the algorithm will create a single child of $f'(W')$ labeled by $W' \cup \{v_n\} = W$, and that child will be a leaf in the final tree constructed by the algorithm. In case W' is not maximal for T', let N be the lexicographically smallest extension of W' to a maximal acyclic vertex set of T'. Hence there exists a leaf $f'(N)$ in the tree $\mathcal{T}'$ labeled by N. Observe that the sequence $L_{W'}$ is a subsequence of L_N, and that $N \cup \{v_n\}$ is not acyclic. Hence the algorithm creates children $N^1, N^2, \ldots$, one of which will be labeled by W.

To see that the algorithm runs with polynomial delay, note that the children and parent of a given node in $\mathcal{T}$ can all be computed in polynomial time. It follows that $\mathcal{T}$ can be traversed in a depth-first manner with polynomial delay per step of the traversal, and thus the leaves of $\mathcal{T}$ can be output with only a polynomial delay.

We show that the algorithm requires only polynomial space. We already observed that each node in $\mathcal{T}$ at level j has at most $\lfloor j/2 \rfloor + 1$ children. For each node we store the maximal acyclic vertex set by which it is labeled. Because we are traversing $\mathcal{T}$ in a depth-first-search manner, in each step of the algorithm we only need to save data of $O(n^2)$ nodes: those of the $O(n)$ nodes on the path from the root to the currently active node labeled by J, and the $O(n)$ children for each node on this path.

Theorem 7. *The described algorithm enumerates all FVSs of a tournament with polynomial delay and uses polynomial space.*

Corollary 2. *In a tournament with n vertices a minimum directed feedback vertex set can be found in $O(1.6740^n)$ time and polynomial space.*

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