

Reactive Bisimulation Semantics for a Process Algebra with Time-Outs

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This paper introduces the counterpart of strong bisimilarity for labelled transition systems extended with time-out transitions. It supports this concept through a modal characterisation, congruence results for a standard process algebra with recursion, and a complete axiomatisation.

1 Introduction

This is a contribution to classic untimed non-probabilistic process algebra, modelling systems that move from state to state by performing discrete, uninterpreted actions. A system is modelled as a process-algebraic expression, whose standard semantics is a state in a labelled transition system (LTS). An LTS consists of a set of states, with action-labelled transitions between them. The execution of an action is assumed to be instantaneous, so when any time elapses the system must be in one of its states. With “untimed” I mean that I will refrain from quantifying the passage of time; however, whether a system can pause in some state or not will be part of my model.

Following [29], I consider *reactive* systems that interact with their environments through the synchronous execution of visible actions a, b, c, \dots taken from an alphabet A . At any time, the environment *allows* a set of actions $X \subseteq A$, while *blocking* all other actions. At discrete moments the environment can change the set of actions it allows. In a metaphor from [29], the environment of a system can be seen as a user interacting with it. This user has a button for each action $a \in A$, on which it can exercise pressure. When the user exercises pressure *and* the system is in a state where it can perform action a , the action occurs. For the system this involves taking an a -labelled transition to a following state; for the environment it entails the button going down, thus making the action occurrence observable. This can trigger the user to alter the set of buttons on which it exercises pressure.

The current paper considers two special actions that can occur as transition labels: the traditional *hidden action* τ [29], modelling the occurrence of an instantaneous action from which we abstract, and the *time-out* action t , modelling the end of a time-consuming activity from which we abstract. The latter was introduced in [16] and constitutes the main novelty of the present paper with respect to [29] and forty years of process algebra. Both special actions are assumed to be unobservable, in the sense that their occurrence cannot trigger any state-change in the environment. Conversely, the environment cannot cause or block the occurrence of these actions.

Following [16], I model the passage of time in the following way. When a system arrives in a state P , and at that time X is the set of actions allowed by the environment, there are two possibilities. If P has an outgoing transition $P \xrightarrow{\alpha} Q$ with $\alpha \in X \cup \{\tau\}$, the system immediately takes one of the outgoing transitions $P \xrightarrow{\alpha} Q$ with $\alpha \in X \cup \{\tau\}$, without spending any time in state P . The choice between these actions is entirely nondeterministic. The system cannot immediately take a transition \xrightarrow{b}

with $b \in A \setminus X$, because the action b is blocked by the environment. Neither can it immediately take a transition $P \xrightarrow{t} Q$, because such transitions model the end of an activity with a finite but positive duration that started when reaching state P .

In case P has no outgoing transition $P \xrightarrow{\alpha} Q$ with $\alpha \in X \cup \{\tau\}$, the system idles in state P for a positive amount of time. This idling can end in two possible ways. Either one of the time-out transitions $P \xrightarrow{t} Q$ occurs, or the environment spontaneously changes the set of actions it allows into a different set Y with the property that $P \xrightarrow{a} Q$ for some $a \in Y$. In the latter case a transition $P \xrightarrow{a} Q$ occurs, with $a \in Y$. The choice between the various ways to end a period of idling is entirely nondeterministic. It is possible to stay forever in state P only if there are no outgoing time-out transitions $P \xrightarrow{t} Q$.

The addition of time-outs enhances the expressive power of LTSs and process algebras. The process $a.P + t.b.Q$, for instance, models a choice between $a.P$ and $b.Q$ where the former has priority. In an environment where a is allowed it will always choose $a.P$ and never $b.Q$; but in an environment that blocks a the process will, after some delay, proceed with $b.Q$. Such a priority mechanism cannot be modelled in standard process algebras without time-outs, such as CCS [29], CSP [4, 24] and ACP [1, 8]. Additionally, mutual exclusion cannot be correctly modelled in any of these standard process algebras [17], but adding time-outs makes it possible—see Section 11 for a more precise statement.

In [16] I characterised the coarsest reasonable semantic equivalence on LTSs with time-outs—the one induced by *may testing*, as proposed by De Nicola & Hennessy [6]. In the absence of time-outs, may testing yields *weak trace equivalence*, where two processes are defined equivalent iff they have the same *weak traces*: sequence of actions the system can perform, while eliding hidden actions. In the presence of time-outs weak trace equivalence fails to be a congruence for common process algebraic operators, and may testing yields its congruence closure, characterised in [16] as *(rooted) failure trace equivalence*.

The present paper aims to characterise one of the finest reasonable semantic equivalences on LTSs with time-outs—the counterpart of strong bisimilarity for LTSs without time-outs. Naturally, strong bisimilarity can be applied verbatim to LTSs with time-outs—and has been in [16]—by treating t exactly like any visible action. Here, however, I aim to take into account the essence of time-outs, and propose an equivalence that satisfies some natural laws discussed in [16], such as $\tau.P + t.Q = \tau.P$ and $a.P + t.(Q + \tau.R + a.S) = a.P + t.(Q + \tau.R)$. To motivate the last law, note that the time-out transition $a.P + t.(Q + \tau.R + a.S) \xrightarrow{t} Q + \tau.R + a.S$ can occur only in an environment that blocks the action a , for otherwise a would have taken place before the time-out went off. The occurrence of this transition is not observable by the environment, so right afterwards the state of the environment is unchanged, and the action a is still blocked. Therefore, the process $Q + \tau.R + a.S$ will, without further ado, proceed with the τ -transition to R , or any action from Q , just as if the $a.S$ summand were not present.

Standard process algebras and LTSs without time-outs can model systems whose behaviour is triggered by input signals from the environment in which they operate. This is why they are called “reactive systems”. By means of time-outs one can additionally model systems whose behaviour is triggered by the *absence* of input signals from the environment, during a sufficiently long period. This creates a greater symmetry between a system and its environment, as it has always been understood that the environment or user of a system can change its behaviour as a result of sustained inactivity of the system it is interacting with. Hence one could say that process algebras and LTSs enriched with time-outs form a more faithful model of reactivity. It is for this reason that I use the name *reactive bisimilarity* for the appropriate form of bisimilarity on systems modelled in this fashion.

Section 2 introduces strong reactive bisimilarity as the proper counterpart of strong bisimilarity in the presence of time-out transitions. Naturally, it coincides with strong bisimilarity when there are no time-out transitions. Section 3 derives a modal characterisation; a reactive variant of the Hennessy-Milner logic. Section 4 offers an alternative characterisation of strong reactive bisimilarity that will be more

convenient in proofs, although it lacks the intuitive appeal to be used as the initial definition.

Section 5 recalls the process algebra CCSP, a common mix of CCS and CSP, and adds the time-out action, as well as two auxiliary operators that will be used in the forthcoming axiomatisation. Section 6 states that in this process algebra one can express all countably branching transition systems, and only those, or all and only the finitely branching ones when restricting to guarded recursion.

Section 7 recalls the concept of a congruence, focusing on the congruence property for the recursion operator, which is commonly the hardest to establish. It then shows that the simple *initials equivalence*, as well as Milner's strong bisimilarity, are congruences. Due to the presence of negative premises in the operational rules for the auxiliary operators, these proofs are not entirely trivial. Using these results as a stepping stone, Section 8 shows that strong reactive bisimilarity is a congruence for my extension of CCSP. Here the congruence property for one of the auxiliary operators with negative premises is needed in establishing the result for the common CCSP operators, such as parallel composition.

Section 9 shows that guarded recursive specifications have unique solutions up to strong reactive bisimilarity. Using this, Section 10 provides a sound and complete axiomatisation for processes with guarded recursion. My completeness proof combines three innovations in establishing completeness of process algebraic axiomatisations. First of all, following [19], it applies to *all* processes in a Turing powerful language like guarded CCSP, rather than the more common fragment merely employing finite sets of recursion equations featuring only choice and action prefixing. Secondly, instead of the classic technique of *merging guarded recursive equations* [27, 28, 35, 9, 26], which in essence proves two bisimilar systems P and Q equivalent by equating both to an intermediate variant that is essentially a *product* of P and Q , I employ the novel method of *canonical solutions* [25], which equates both P and Q to a canonical representative within the bisimulation equivalence class of P and Q —one that has only one reachable state for each bisimulation equivalence class of states of P and Q . In fact I tried so hard, and in vain, to apply the traditional technique of merging guarded recursive equations, that I came to believe that it fundamentally does not work for this axiomatisation. The third innovation is the use of the axiom of choice [36] in defining the transition relation on my canonical representative, in order to keep this process finitely branching.

Section 11 describes a worthwhile gain in expressiveness caused by the addition of time-outs, and presents an agenda for future work.

2 Reactive bisimilarity

A *labelled transition system* (LTS) is a triple $(\mathbb{P}, Act, \rightarrow)$ with \mathbb{P} a set (of *states* or *processes*), Act a set (of *actions*) and $\rightarrow \in \mathbb{P} \times Act \times \mathbb{P}$. In this paper I consider LTSs with $Act := A \uplus \{\tau, t\}$, where A is a set of *visible actions*, τ is the *hidden action*, and t the *time-out action*. The set of *initial actions* of a process $P \in \mathbb{P}$ is $\mathcal{I}(P) := \{\alpha \in A \cup \{\tau\} \mid P \xrightarrow{\alpha}\}$. Here $P \xrightarrow{\alpha}$ means that there is a Q with $P \xrightarrow{\alpha} Q$.

Definition 1 A *strong reactive bisimulation* is a symmetric relation $\mathcal{R} \subseteq (\mathbb{P} \times \mathcal{P}(A) \times \mathbb{P}) \cup (\mathbb{P} \times \mathbb{P})$ (meaning that $(P, X, Q) \in \mathcal{R} \Leftrightarrow (Q, X, P) \in \mathcal{R}$ and $(P, Q) \in \mathcal{R} \Leftrightarrow (Q, P) \in \mathcal{R}$), such that,

- if $(P, Q) \in \mathcal{R}$ and $P \xrightarrow{\tau} P'$, then there exists a Q' such that $Q \xrightarrow{\tau} Q'$ and $(P', Q') \in \mathcal{R}$,
- if $(P, Q) \in \mathcal{R}$ then $(P, X, Q) \in \mathcal{R}$ for all $X \subseteq A$,

and for all $(P, X, Q) \in \mathcal{R}$,

- if $P \xrightarrow{a} P'$ with $a \in X$, then there exists a Q' such that $Q \xrightarrow{a} Q'$ and $(P', Q') \in \mathcal{R}$,
- if $P \xrightarrow{\tau} P'$, then there exists a Q' such that $Q \xrightarrow{\tau} Q'$ and $(P', X, Q') \in \mathcal{R}$,

- if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, then $(P, Q) \in \mathcal{R}$, and
- if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, then $\exists Q'$ such that $Q \xrightarrow{t} Q'$ and $(P', X, Q') \in \mathcal{R}$.

Processes $P, Q \in \mathbb{P}$ are *strongly X -bisimilar*, denoted $P \leftrightarrow_r^X Q$, if $(P, X, Q) \in \mathcal{R}$ for some strong reactive bisimulation \mathcal{R} . They are *strongly reactive bisimilar*, denoted $P \leftrightarrow_r Q$, if $(P, Q) \in \mathcal{R}$ for some strong reactive bisimulation \mathcal{R} .

Intuitively, $(P, X, Q) \in \mathcal{R}$ says that processes P and Q behave the same way, as witnessed by the relation \mathcal{R} , when placed in the environment X —meaning any environment that allows exactly the actions in X to occur—whereas $(P, Q) \in \mathcal{R}$ says they behave the same way in an environment that has just been triggered to change. An environment can be thought of as an unknown process placed in parallel with P and Q , using the operator \parallel_A , enforcing synchronisation on all visible actions. The environment X can be seen as a process $\sum_{i \in I} a_i.R_i + t.R$ where $\{a_i \mid i \in I\} = X$. A triggered environment, on the other hand, can execute a sequence of instantaneous hidden actions before stabilising as an environment Y , for $Y \subseteq A$. During this execution, actions can be blocked and allowed in rapid succession. Since the environment is unknown, the bisimulation should be robust under any such environment.

The first clause for $(P, X, Q) \in \mathcal{R}$ is like the common transfer property of strong bisimilarity [29]: a visible a -transition of P can be matched by one of Q , such that the resulting processes P' and Q' are related again. However, I require it only for actions $a \in X$, because actions $b \in A \setminus X$ cannot happen at all in the environment X , and thus need not be matched by Q . Since the occurrence of a is observable by the environment, this can trigger the environment to change the set of actions it allows, so P' and Q' ought to be related in a triggered environment.

The second clause is the transfer property for τ -transitions. Since these are not observable by the environment, they cannot trigger a change in the set of actions allowed by it, so the resulting processes P' and Q' should be related only in the same environment X .

The first clause for $(P, Q) \in \mathcal{R}$ expresses the transfer property for τ -transitions in a triggered environment. Here it may happen that the τ -transition occurs before the environment stabilises, and hence P' and Q' will still be related in a triggered environment. A similar transfer property for a -transitions is already implied by the next two clauses.

The second clause allows a triggered environment to stabilise into any environment X .

The first two clauses for $(P, X, Q) \in \mathcal{R}$ imply that if $(P, X, Q) \in \mathcal{R}$ then $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \mathcal{I}(Q) \cap (X \cup \{\tau\})$. So $P \leftrightarrow_r^X Q$ implies $\mathcal{I}(P) = \mathcal{I}(Q)$. The condition $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ is necessary and sufficient for the system to remain a positive amount of time in state P when X is the set of allowed actions. The next clause says that during this time the environment may be triggered to change the set of actions it allows by an event outside our model, that is, by a time-out in the environment. So P and Q should be related in a triggered environment.

The last clause says that also a t -transition of P should be matched by one of Q . Naturally, the t -transition of P can be taken only when the system is idling in P , i.e., when $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$. The resulting processes P' and Q' should be related again, but only in the same environment allowing X .

Proposition 2 Strong X -bisimilarity and strong reactive bisimilarity are equivalence relations.

Proof: $\leftrightarrow_r^X, \leftrightarrow_r$ are reflexive, as $\{(P, X, P), (P, P) \mid P \in \mathbb{P} \wedge X \subseteq A\}$ is a strong reactive bisimulation. \leftrightarrow_r^X and \leftrightarrow_r are symmetric, since strong reactive bisimulations are symmetric by definition. \leftrightarrow_r^X and \leftrightarrow_r are transitive, for if \mathcal{R} and \mathcal{S} are strong reactive bisimulations, then so is

$$\mathcal{R}; \mathcal{S} = \{(P, X, R) \mid (P, X, Q) \in \mathcal{R} \wedge (Q, X, R) \in \mathcal{S}\} \cup \{(P, R) \mid (P, Q) \in \mathcal{R} \wedge (Q, R) \in \mathcal{S}\}. \quad \square$$

Note that the union of arbitrarily many strong reactive bisimulations is itself a strong reactive bisimulation. Therefore the family of relations $\leftrightarrow_r, \leftrightarrow_r^X$ for $X \subseteq A$ can be seen as a strong reactive bisimulation.

To get a firm grasp on strong reactive bisimilarity, the reader is invited to check the two laws mentioned in the introduction, and then to construct a strong reactive bisimulation between the two systems depicted on Page 28.

2.1 A more general form of reactive bisimulation

The following notion of a *generalised strong reactive bisimulation* (gsrb) generalises that of a strong reactive bisimulation; yet it induces the same concept of strong reactive bisimilarity. This makes the relation convenient to use for further analysis. I did not introduce it as the original definition, because it lacks a strong motivation.

Definition 3 A *gsrb* is a symmetric relation $\mathcal{R} \subseteq (\mathbb{P} \times \mathcal{P}(A) \times \mathbb{P}) \cup (\mathbb{P} \times \mathbb{P})$ such that, for all $(P, Q) \in \mathcal{R}$,

- if $P \xrightarrow{\alpha} P'$ with $a \in A \cup \{\tau\}$, then there exists a Q' such that $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$,
- if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ with $X \subseteq A$ and $P \xrightarrow{t} P'$, then $\exists Q'$ with $Q \xrightarrow{t} Q'$ and $(P', X, Q') \in \mathcal{R}$,

and for all $(P, Y, Q) \in \mathcal{R}$,

- if $P \xrightarrow{a} P'$ with either $a \in Y$ or $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$, then $\exists Q'$ with $Q \xrightarrow{a} Q'$ and $(P', Q') \in \mathcal{R}$,
- if $P \xrightarrow{\tau} P'$, then there exists a Q' such that $Q \xrightarrow{\tau} Q'$ and $(P', Y, Q') \in \mathcal{R}$,
- if $\mathcal{I}(P) \cap (X \cup Y \cup \{\tau\}) = \emptyset$ with $X \subseteq A$ and $P \xrightarrow{t} P'$ then $\exists Q'$ with $Q \xrightarrow{t} Q'$ and $(P', X, Q') \in \mathcal{R}$.

Proposition 4 $P \leftrightarrow_r Q$ iff there exists a *gsrb* \mathcal{R} with $(P, Q) \in \mathcal{R}$.

Likewise, $P \leftrightarrow_r^X Q$ iff there exists a *gsrb* \mathcal{R} with $(P, X, Q) \in \mathcal{R}$.

Proof: Clearly, each strong reactive bisimulation satisfies the five clauses of Definition 3 and thus is a *gsrb*. In the other direction, given a *gsrb* \mathcal{B} , let

$$\begin{aligned} \mathcal{R} := & \mathcal{B} \cup \{(P, X, Q) \mid (P, Q) \in \mathcal{B} \wedge X \subseteq A\} \\ & \cup \{(P, Q), (P, X, Q) \mid (P, Y, Q) \in \mathcal{B} \wedge \mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset\}. \end{aligned}$$

It is straightforward to check that \mathcal{R} satisfies the six clauses of Definition 1. □

Unlike Definition 1, a *gsrb* needs the triples (P, X, Q) only after encountering a t -transition; two systems without t -transitions can be related without using these triples at all.

3 A modal characterisation of strong reactive bisimilarity

The Hennessy-Milner logic [23] expresses properties on the behaviour of processes in an LTS.

Definition 5 The class \mathbb{O} of *infinitary HML formulas* is defined as follows, where I ranges over all index sets and α over $A \cup \{\tau\}$:

$$\varphi ::= \bigwedge_{i \in I} \varphi_i \mid \neg \varphi \mid \langle \alpha \rangle \varphi$$

\top abbreviates the empty conjunction, and $\varphi_1 \wedge \varphi_2$ stands for $\bigwedge_{i \in \{1,2\}} \varphi_i$.

$P \models \varphi$ denotes that process P satisfies formula φ . The first two operators represent the standard Boolean operators conjunction and negation. By definition, $P \models \langle \alpha \rangle \varphi$ iff $P \xrightarrow{\alpha} P'$ for some P' with $P' \models \varphi$.

A famous result stemming from [23] states that

$$P \simeq Q \Leftrightarrow \forall \varphi \in \mathbb{O}. (P \models \varphi \Leftrightarrow Q \models \varphi)$$

where \simeq denotes strong bisimilarity [29, 23], formally defined in Section 7.2. It states that the Hennessy-Milner logic yields a *modal characterisation* of strong bisimilarity. I will now adapt this into a modal characterisation of strong reactive bisimilarity.

To this end I extend the Hennessy-Milner logic with a new modality $\langle X \rangle$, for $X \subseteq A$, and auxiliary satisfaction relations $\models_X \subseteq \mathbb{P} \times \mathbb{O}$ for each $X \subseteq A$. The formula $P \models \langle X \rangle \varphi$ says that in an environment X , allowing exactly the actions in X , process P can perform a time-out transition to a process that satisfies φ . $P \models_X \varphi$ says that if P satisfies φ when placed in environment X . The relations \models and \models_X are the smallest ones satisfying:

$$\begin{array}{ll} P \models \bigwedge_{i \in I} \varphi_i & \text{if } \forall i \in I. P \models \varphi_i \\ P \models \neg \varphi & \text{if } P \not\models \varphi \\ P \models \langle \alpha \rangle \varphi \text{ with } \alpha \in A \cup \{\tau\} & \text{if } \exists P'. P \xrightarrow{\alpha} P' \wedge P' \models \varphi \\ P \models \langle X \rangle \varphi \text{ with } X \subseteq A & \text{if } \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \wedge \exists P'. P \xrightarrow{t} P' \wedge P' \models_X \varphi \\ \\ P \models_X \bigwedge_{i \in I} \varphi_i & \text{if } \forall i \in I. P \models_X \varphi_i \\ P \models_X \neg \varphi & \text{if } P \not\models_X \varphi \\ P \models_X \langle a \rangle \varphi \text{ with } a \in A & \text{if } a \in X \wedge \exists P'. P \xrightarrow{a} P' \wedge P' \models \varphi \\ P \models_X \langle \tau \rangle \varphi & \text{if } \exists P'. P \xrightarrow{\tau} P' \wedge P' \models_X \varphi \\ P \models_X \varphi & \text{if } \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \wedge P \models \varphi \end{array}$$

Note that a formula $\langle a \rangle \varphi$ is less often true under \models_X than under \models , due to the side condition $a \in X$. This reflects the fact that a cannot happen in an environment that blocks it. The last clause in the above definition reflects the fifth clause of Definition 1. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, then process P , operating in environment X , idles for a while, during which the environment can change. This ends the blocking of actions $a \notin X$ and makes any formula valid under \models also valid under \models_X .

Example 6 Both systems depicted on Page 28 satisfy $\langle \{a, b\} \rangle \langle \tau \rangle \langle b \rangle \top \wedge \langle \{a, b\} \rangle \langle \tau \rangle \neg \langle b \rangle \top \wedge \langle \{b\} \rangle \langle a \rangle \top \wedge \langle \{b\} \rangle \neg \langle a \rangle \top$ and neither satisfies $\langle \{a, b\} \rangle (\langle a \rangle \wedge \langle \tau \rangle \langle b \rangle)$ or $\langle \{b\} \rangle (\langle a \rangle \wedge \langle \tau \rangle \langle b \rangle)$.

Theorem 7 Let $P, Q \in \mathbb{P}$ and $X \subseteq A$. Then $P \simeq_r Q \Leftrightarrow \forall \varphi \in \mathbb{O}. (P \models \varphi \Leftrightarrow Q \models \varphi)$
and $P \simeq_r^X Q \Leftrightarrow \forall \varphi \in \mathbb{O}. (P \models_X \varphi \Leftrightarrow Q \models_X \varphi)$.

Proof: “ \Rightarrow ”: I prove by simultaneous structural induction on $\varphi \in \mathbb{O}$ that, for all $P, Q \in \mathbb{P}$ and $X \subseteq A$, $P \simeq_r Q \wedge P \models \varphi \Rightarrow Q \models \varphi$ and $P \simeq_r^X Q \wedge P \models_X \varphi \Rightarrow Q \models_X \varphi$. The converse implications ($Q \models \varphi \Rightarrow P \models \varphi$ and $Q \models_X \varphi \Rightarrow P \models_X \varphi$) follow by symmetry.

- Let $P \simeq_r Q \wedge P \models \varphi$.
 - Let $\varphi = \bigwedge_{i \in I} \varphi_i$. Then $P \models \varphi_i$ for all $i \in I$. By induction $Q \models \varphi_i$ for all i , so $Q \models \bigwedge_{i \in I} \varphi_i$.
 - Let $\varphi = \neg \psi$. Then $P \not\models \psi$. By induction $Q \not\models \psi$, so $Q \models \neg \psi$.
 - Let $\varphi = \langle \alpha \rangle \psi$ with $\alpha \in A \cup \{\tau\}$. Then $P \xrightarrow{\alpha} P'$ for some P' with $P' \models \psi$. By Definition 3, $Q \xrightarrow{\alpha} Q'$ for some Q' with $P' \simeq_r Q'$. So by induction $Q' \models \psi$, and thus $Q \models \langle \alpha \rangle \psi$.
 - Let $\varphi = \langle X \rangle \psi$ for some $X \subseteq A$. Then $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ for some P' with $P' \models_X \psi$. By Definition 3, $Q \xrightarrow{t} Q'$ for some Q' with $P' \simeq_r^X Q'$. So by induction $Q' \models_X \psi$. Moreover, $\mathcal{I}(Q) = \mathcal{I}(P)$, as $P \simeq_r Q$, so $\mathcal{I}(Q) \cap (X \cup \{\tau\}) = \emptyset$. Thus $Q \models \langle X \rangle \psi$.

- Let $P \leftrightarrow_r^X Q \wedge P \models_X \varphi$.
 - Let $\varphi = \bigwedge_{i \in I} \varphi_i$, and $P \models_X \varphi_i$ for all $i \in I$. By induction $Q \models_X \varphi_i$ for all $i \in I$, so $Q \models_X \bigwedge_{i \in I} \varphi_i$.
 - Let $\varphi = \neg\psi$, and $P \not\models_X \psi$. By induction $Q \not\models_X \psi$, so $Q \models_X \neg\psi$.
 - Let $\varphi = \langle a \rangle \psi$ with $a \in X$ and $P \xrightarrow{a} P'$ for some P' with $P' \models \psi$. By Definition 1, $Q \xrightarrow{a} Q'$ for some Q' with $P' \leftrightarrow_r Q'$. By induction $Q' \models \psi$, so $Q \models_X \langle a \rangle \psi$.
 - Let $\varphi = \langle \tau \rangle \psi$, and $P \xrightarrow{\tau} P'$ for some P' with $P' \models_X \psi$. By Definition 1, $Q \xrightarrow{\tau} Q'$ for some Q' with $P' \leftrightarrow_r^X Q'$. By induction $Q' \models_X \psi$, so $Q \models_X \langle \tau \rangle \psi$.
 - Let $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \models \varphi$. By the fifth clause of Definition 1, $P \leftrightarrow_r Q$. Hence, by the previous case in this proof, $Q \models \varphi$. Moreover, $\mathcal{I}(Q) \cap (X \cup \{\tau\}) = \mathcal{I}(P) \cap (X \cup \{\tau\})$, since $P \leftrightarrow_r^X Q$. Thus $Q \models_X \varphi$.

“ \Leftarrow ”: Write $P \equiv Q$ for $\forall \varphi \in \mathbb{O}$. ($P \models \varphi \Leftrightarrow Q \models \varphi$), and $P \equiv_X Q$ for $\forall \varphi \in \mathbb{O}$. ($P \models_X \varphi \Leftrightarrow Q \models_X \varphi$). I show that the family of relations \equiv, \equiv_X for $X \subseteq A$ constitutes a gsrp.

- Suppose $P \equiv Q$ and $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$. Let $\mathcal{Q}^\dagger := \{Q^\dagger \in \mathbb{P} \mid Q \xrightarrow{\alpha} Q^\dagger \wedge P' \not\equiv Q^\dagger\}$. For each $Q^\dagger \in \mathcal{Q}^\dagger$, let $\varphi_{Q^\dagger} \in \mathbb{O}$ be a formula such that $P' \models \varphi_{Q^\dagger}$ and $Q^\dagger \not\models \varphi_{Q^\dagger}$. (Such a formula always exists because \mathbb{O} is closed under negation.) Define $\varphi := \bigwedge_{Q^\dagger \in \mathcal{Q}^\dagger} \varphi_{Q^\dagger}$. Then $P' \models \varphi$, so $P \models \langle a \rangle \varphi$. Consequently, also $Q \models \langle a \rangle \varphi$. Hence there is a Q' with $Q \xrightarrow{\alpha} Q'$ and $Q' \models \varphi$. Since none of the $Q^\dagger \in \mathcal{Q}^\dagger$ satisfies φ , one obtains $Q' \notin \mathcal{Q}^\dagger$ and thus $P' \equiv Q'$.
- Let $X \subseteq A$ and suppose $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. Let

$$\mathcal{Q}^\dagger := \{Q^\dagger \in \mathbb{P} \mid Q \xrightarrow{t} Q^\dagger \wedge P' \not\equiv_X Q^\dagger\}.$$

For each $Q^\dagger \in \mathcal{Q}^\dagger$, let $\varphi_{Q^\dagger} \in \mathbb{O}$ be a formula such that $P' \models_X \varphi_{Q^\dagger}$ and $Q^\dagger \not\models_X \varphi_{Q^\dagger}$. Define $\varphi := \bigwedge_{Q^\dagger \in \mathcal{Q}^\dagger} \varphi_{Q^\dagger}$. Then $P' \models_X \varphi$, so $P \models \langle X \rangle \varphi$. Consequently, also $Q \models \langle X \rangle \varphi$. Hence there is a Q' with $Q \xrightarrow{t} Q'$ and $Q' \models_X \varphi$. Again $Q' \notin \mathcal{Q}^\dagger$ and thus $P' \equiv_X Q'$.

- Suppose $P \equiv_Y Q$ and $P \xrightarrow{a} P'$ with $a \in A$ and either $a \in Y$ or $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$. Let $\mathcal{Q}^\dagger := \{Q^\dagger \in \mathbb{P} \mid Q \xrightarrow{a} Q^\dagger \wedge P' \not\equiv Q^\dagger\}$. For each $Q^\dagger \in \mathcal{Q}^\dagger$, let $\varphi_{Q^\dagger} \in \mathbb{O}$ be a formula such that $P' \models \varphi_{Q^\dagger}$ and $Q^\dagger \not\models \varphi_{Q^\dagger}$. Define $\varphi := \bigwedge_{Q^\dagger \in \mathcal{Q}^\dagger} \varphi_{Q^\dagger}$. Then $P' \models \varphi$, so $P \models \langle a \rangle \varphi$, and also $P \models_Y \langle a \rangle \varphi$, using either the third or last clause in the definition of \models_X . Hence also $Q \models_Y \langle a \rangle \varphi$. Therefore there is a Q' with $Q \xrightarrow{a} Q'$ and $Q' \models \varphi$, using the third clause of either \models_X or \models . Since none of the $Q^\dagger \in \mathcal{Q}^\dagger$ satisfies φ , one obtains $Q' \notin \mathcal{Q}^\dagger$ and thus $P' \equiv Q'$.
- The fourth clause of Definition 3 is obtained exactly like the first, but using \models_Y instead of \models .
- Suppose $P \equiv_Y Q$, $P \xrightarrow{t} P'$ and $\mathcal{I}(P) \cap (X \cup Y \cup \{\tau\}) = \emptyset$, with $X \subseteq A$. Let

$$\mathcal{Q}^\dagger := \{Q^\dagger \in \mathbb{P} \mid Q \xrightarrow{t} Q^\dagger \wedge P' \not\equiv_X Q^\dagger\}.$$

For each $Q^\dagger \in \mathcal{Q}^\dagger$, let $\varphi_{Q^\dagger} \in \mathbb{O}$ be a formula such that $P' \models_X \varphi_{Q^\dagger}$ and $Q^\dagger \not\models_X \varphi_{Q^\dagger}$. Define $\varphi := \bigwedge_{Q^\dagger \in \mathcal{Q}^\dagger} \varphi_{Q^\dagger}$. Then $P' \models_X \varphi$, so $P \models \langle X \rangle \varphi$, and thus $P \models_Y \langle X \rangle \varphi$. Consequently, also $Q \models_Y \langle X \rangle \varphi$ and therefore $Q \models \langle X \rangle \varphi$. Hence there is a Q' with $Q \xrightarrow{t} Q'$ and $Q' \models_X \varphi$. Again $Q' \notin \mathcal{Q}^\dagger$ and thus $P' \equiv_X Q'$. \square

4 Time-out bisimulations

I will now present a characterisation of strong reactive bisimilarity in terms of a binary relation \mathcal{B} on processes—a *strong time-out bisimulation*—not parametrised by the set of allowed actions X . To this end I need a family of unary operators θ_X on processes, for $X \subseteq A$. These *environment* operators place a process in an environment that allows exactly the actions in X to occur. They are defined by the following structural operational rules.

$$\frac{x \xrightarrow{\tau} y}{\theta_X(x) \xrightarrow{\tau} \theta_X(y)} \quad \frac{x \xrightarrow{a} y}{\theta_X(x) \xrightarrow{a} y} \quad (a \in X) \quad \frac{x \xrightarrow{\alpha} y \quad x \not\xrightarrow{\beta} \text{ for all } \beta \in X \cup \{\tau\}}{\theta_X(x) \xrightarrow{\alpha} y} \quad (\alpha \in A \cup \{\tau\})$$

The operator θ_X modifies its argument by inhibiting all initial transitions (here including also those that occur after a τ -transition) that cannot occur in the specified environment. When an observable transition does occur, the environment may be triggered to change, and the inhibiting effect of the θ_X -operator comes to an end. The premises $x \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$ in the third rule guarantee that the process x will idle for a positive amount of time in its current state. During this time, the environment may be triggered to change, and again the inhibiting effect of the θ_X -operator comes to an end.

Below I assume that \mathbb{P} is closed under θ , that is, if $P \in \mathbb{P}$ and $X \subseteq A$ then $\theta_X(P) \in \mathbb{P}$.

Definition 8 A *strong time-out bisimulation* is a symmetric relation $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$, such that, for $P \mathcal{B} Q$,

- if $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$, then $\exists Q'$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \mathcal{B} Q'$,
- if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, then $\exists Q'$ such that $Q \xrightarrow{t} Q'$ and $\theta_X(P') \mathcal{B} \theta_X(Q')$.

Proposition 9 $P \leftrightarrow_r Q$ iff there exists a strong time-out bisimulation \mathcal{B} with $P \mathcal{B} Q$.

Proof: Let \mathcal{R} be a gsr on \mathbb{P} . Define $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$ by $P \mathcal{B} Q$ iff either $(P, Q) \in \mathcal{R}$ or $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $(P^\dagger, X, Q^\dagger) \in \mathcal{R}$. I show that \mathcal{B} is a strong time-out bisimulation.

- Let $P \mathcal{B} Q$ and $P \xrightarrow{a} P'$ with $a \in A$. First suppose $(P, Q) \in \mathcal{R}$. Then, by the first clause of Definition 3, there exists a Q' such that $Q \xrightarrow{a} Q'$ and $(P', Q') \in \mathcal{R}$. So $P' \mathcal{B} Q'$.
Next suppose $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $(P^\dagger, X, Q^\dagger) \in \mathcal{R}$. Since $\theta_X(P^\dagger) \xrightarrow{a} P'$ it must be that $P^\dagger \xrightarrow{a} P'$ and either $a \in X$ or $P^\dagger \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$. Hence there exists a Q' such that $Q^\dagger \xrightarrow{a} Q'$ and $(P', Q') \in \mathcal{R}$, using the third clause of Definition 3. Recall that $P^\dagger \leftrightarrow_r^X Q^\dagger$ implies $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\})$, and thus either $a \in X$ or $Q^\dagger \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$. It follows that $Q = \theta_X(Q^\dagger) \xrightarrow{a} Q'$ and $P' \mathcal{B} Q'$.
- Let $P \mathcal{B} Q$ and $P \xrightarrow{\tau} P'$. First suppose $(P, Q) \in \mathcal{R}$. Then, using the first clause of Definition 3, there is a Q' with $Q \xrightarrow{\tau} Q'$ and $(P', Q') \in \mathcal{R}$. So $P' \mathcal{B} Q'$.
Next suppose $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $(P^\dagger, X, Q^\dagger) \in \mathcal{R}$. Since $\theta_X(P^\dagger) \xrightarrow{\tau} P'$, it must be that P' has the form $\theta_X(P^\ddagger)$, and $P^\dagger \xrightarrow{\tau} P^\ddagger$. Thus, by the fourth clause of Definition 3, there is a Q^\ddagger with $Q^\dagger \xrightarrow{\tau} Q^\ddagger$ and $(P^\ddagger, X, Q^\ddagger) \in \mathcal{R}$. Now $Q = \theta_X(Q^\dagger) \xrightarrow{\tau} \theta_X(Q^\ddagger) =: Q'$ and $P' \mathcal{B} Q'$.
- Let $P \mathcal{B} Q$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. First suppose $(P, Q) \in \mathcal{R}$. Then, by the second clause of Definition 3, there is a Q' with $Q \xrightarrow{t} Q'$ and $(P', X, Q') \in \mathcal{R}$. So $\theta_X(P') \mathcal{B} \theta_X(Q')$.
Next suppose $P = \theta_Y(P^\dagger)$, $Q = \theta_Y(Q^\dagger)$ and $(P^\dagger, Y, Q^\dagger) \in \mathcal{R}$. Since $\theta_Y(P^\dagger) \xrightarrow{t} P'$, it must be that $P^\dagger \xrightarrow{t} P'$ and $P^\dagger \not\xrightarrow{\beta}$ for all $\beta \in Y \cup \{\tau\}$. Consequently, $\mathcal{I}(P^\dagger) = \mathcal{I}(P)$ and thus $\mathcal{I}(P^\dagger) \cap (X \cup Y \cup \{\tau\}) = \emptyset$. By the last clause of Definition 3 there is a Q' such that $Q^\dagger \xrightarrow{t} Q'$ and $(P', X, Q') \in \mathcal{R}$. So $\theta_X(P') \mathcal{B} \theta_X(Q')$. From $(P^\dagger, Y, Q^\dagger) \in \mathcal{R}$ and $\mathcal{I}(P^\dagger) \cap (Y \cup \{\tau\}) = \emptyset$, I infer $\mathcal{I}(Q^\dagger) \cap (Y \cup \{\tau\}) = \emptyset$. So $Q^\dagger \not\xrightarrow{\beta}$ for all $\beta \in Y \cup \{\tau\}$. This yields $Q = \theta_Y(Q^\dagger) \xrightarrow{t} Q'$.

Now let \mathcal{B} be a time-out bisimulation. Define $\mathcal{R} \subseteq \mathbb{P} \times \mathcal{P}(A) \times \mathbb{P}$ by $(P, Q) \in \mathcal{R}$ iff $P \mathcal{B} Q$, and $(P, X, Q) \in \mathcal{R}$ iff $\theta_X(P) \mathcal{B} \theta_X(Q)$. I need to show that \mathcal{R} is a gsrb.

- Suppose $(P, Q) \in \mathcal{R}$ and $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$. Then $P \mathcal{B} Q$, so there is a Q' such that $Q \xrightarrow{\alpha} Q'$ and $P' \mathcal{B} Q'$. Hence $(P', Q') \in \mathcal{R}$.
- Suppose $(P, Q) \in \mathcal{R}$, $X \subseteq A$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. Then $P \mathcal{B} Q$, so $\exists Q'$ such that $Q \xrightarrow{t} Q'$ and $\theta_X(P') \mathcal{B} \theta_X(Q')$. Thus $(P', X, Q') \in \mathcal{R}$.
- Suppose $(P, X, Q) \in \mathcal{R}$ and $P \xrightarrow{a} P'$ with either $a \in X$ or $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$. Then $\theta_X(P) \mathcal{B} \theta_X(Q)$. Moreover, $\theta_X(P) \xrightarrow{a} P'$. Hence there is a Q' such that $\theta_X(Q) \xrightarrow{a} Q'$ and $P' \mathcal{B} Q'$. It must be that $Q \xrightarrow{a} Q'$. Moreover, $(P', Q') \in \mathcal{R}$.
- Suppose $(P, X, Q) \in \mathcal{R}$ and $P \xrightarrow{\tau} P'$. Then $\theta_X(P) \mathcal{B} \theta_X(Q)$. Since $P \xrightarrow{\tau} P'$, one has $\theta_X(P) \xrightarrow{\tau} \theta_X(P')$. Hence there is an R such that $\theta_X(Q) \xrightarrow{\tau} R$ and $\theta_X(P') \mathcal{B} R$. The process R must have the form $\theta_X(Q')$ for some Q' with $Q \xrightarrow{\tau} Q'$. It follows that $(P', X, Q') \in \mathcal{R}$.
- Suppose $(P, Y, Q) \in \mathcal{R}$, $X \subseteq A$, $\mathcal{I}(P) \cap (X \cup Y \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. Then $\theta_Y(P) \mathcal{B} \theta_Y(Q)$ and $\theta_Y(P) \xrightarrow{t} P'$. Moreover, $\mathcal{I}(\theta_Y(P)) = \mathcal{I}(P)$, so by the second clause of Definition 8 there exists a Q' such that $\theta_Y(Q) \xrightarrow{t} Q'$ and $\theta_X(P') \mathcal{B} \theta_X(Q')$. So $Q \xrightarrow{t} Q'$ and $(P', X, Q') \in \mathcal{R}$. \square

Note that the union of arbitrarily many strong time-out bisimulations is itself a strong time-out bisimulation. Consequently, the relation $\xrightarrow{\tau}$ is a strong time-out bisimulation.

5 The process algebra CCSP_t^θ

Let A be a set of *visible actions* and Var an infinite set of *variables*. The syntax of CCSP_t^θ is given by

$$E ::= 0 \mid \alpha.E \mid E + E \mid E \parallel_S E \mid \tau_I(E) \mid \mathcal{R}(E) \mid \theta_L^U(E) \mid \psi_X(E) \mid x \mid \langle x | \mathcal{S} \rangle \text{ (with } x \in V_S)$$

with $\alpha \in Act := A \uplus \{\tau, t\}$, $S, I, U, L, X \subseteq A$, $L \subseteq U$, $\mathcal{R} \subseteq A \times A$, $x \in Var$ and \mathcal{S} a *recursive specification*: a set of equations $\{y = S_y \mid y \in V_S\}$ with $V_S \subseteq Var$ (the *bound variables* of \mathcal{S}) and each S_y a CCSP_t^θ expression. I require that all sets $\{b \mid (a, b) \in \mathcal{R}\}$ are finite.

The constant 0 represents a process that is unable to perform any action. The process $\alpha.E$ first performs the action α and then proceeds as E . The process $E + F$ behaves as either E or F . \parallel_S is a partially synchronous parallel composition operator; actions $a \in S$ must synchronise—they can occur only when both arguments are ready to perform them—whereas actions $\alpha \notin S$ from both arguments are interleaved. τ_I is an abstraction operator; it conceals the actions in I by renaming them into the hidden action τ . The operator \mathcal{R} is a relational renaming: it renames a given action $a \in A$ into a choice between all actions b with $(a, b) \in \mathcal{R}$. The *environment operators* θ_L^U and ψ_X are new in this paper and explained below. Finally, $\langle x | \mathcal{S} \rangle$ represents the x -component of a solution of the system of recursive equations \mathcal{S} .

The language CCSP is a common mix of the process algebras CCS [29] and CSP [4, 24]. It first appeared in [30], where it was named following a suggestion by M. Nielsen. The family of parallel composition operators \parallel_S stems from [31], and incorporates the two CSP parallel composition operators from [4]. The relation renaming operators $\mathcal{R}(-)$ stem from [34]; they combine both the (functional) renaming operators that are common to CCS and CSP , and the inverse image operators of CSP . The choice operator $+$ stems from CCS , and the abstraction operator from CSP , while the inaction constant 0, action prefixing operators $a._$ for $a \in A$, and the recursion construct are common to CCS and CSP . The time-out prefixing operator $t._$ was added by me in [16]. The syntactic form of inaction 0, action prefixing $\alpha.E$ and choice $E + F$ follows CCS , whereas the syntax of abstraction $\tau_I(-)$ and recursion $\langle x | \mathcal{S} \rangle$ follows ACP [1, 8]. The fragment of CCSP_t^θ without θ_L^U and ψ_X is called CCSP_t [16].

Table 1: Structural operational interleaving semantics of CCSP_t^θ

$\alpha.x \xrightarrow{\alpha} x$	$\frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'}$	$\frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$	$\frac{x \xrightarrow{\alpha} x'}{\mathcal{R}(x) \xrightarrow{\beta} \mathcal{R}(x')} \left(\begin{array}{l} \alpha = \beta = \tau \\ \vee \alpha = \beta = t \\ \vee (\alpha, \beta) \in \mathcal{R} \end{array} \right)$
$\frac{x \xrightarrow{\alpha} x'}{x \parallel_S y \xrightarrow{\alpha} x' \parallel_S y} (\alpha \notin S)$	$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{a} y'}{x \parallel_S y \xrightarrow{a} x' \parallel_S y'} (a \in S)$	$\frac{y \xrightarrow{\alpha} y'}{x \parallel_S y \xrightarrow{\alpha} x \parallel_S y'} (\alpha \notin S)$	
$\frac{x \xrightarrow{\alpha} x'}{\tau_I(x) \xrightarrow{\alpha} \tau_I(x')} (\alpha \notin I)$	$\frac{x \xrightarrow{a} x'}{\tau_I(x) \xrightarrow{\tau} \tau_I(x')} (a \in I)$	$\frac{\langle \mathcal{S}_x \mathcal{S} \rangle \xrightarrow{\alpha} y}{\langle x \mathcal{S} \rangle \xrightarrow{\alpha} y}$	
$\frac{x \xrightarrow{\tau} y}{\theta_L^U(x) \xrightarrow{\tau} \theta_L^U(y)}$	$\frac{x \xrightarrow{a} y}{\theta_L^U(x) \xrightarrow{a} y} (a \in U)$	$\frac{x \xrightarrow{\alpha} y \quad x \not\xrightarrow{\beta} \text{ for all } \beta \in L \cup \{\tau\}}{\theta_L^U(x) \xrightarrow{\alpha} y} (\alpha \in A \cup \{t\})$	
$\frac{x \xrightarrow{\alpha} y}{\psi_X(x) \xrightarrow{\alpha} y} (\alpha \in A \cup \{\tau\})$		$\frac{x \xrightarrow{t} y \quad x \not\xrightarrow{\beta} \text{ for all } \beta \in X \cup \{\tau\}}{\psi_X(x) \xrightarrow{t} \theta_X(y)}$	

An occurrence of a variable x in a CCSP_t^θ expression E is *bound* iff it occurs in a subexpression $\langle y | \mathcal{S} \rangle$ of E with $x \in V_S$; otherwise it is *free*. Here each \mathcal{S}_y for $y \in V_S$ counts as a subexpression of $\langle x | \mathcal{S} \rangle$. An expression E is *invalid* if it has a subexpression $\theta_L^U(F)$ or $\psi_X(F)$ such that a variable occurrence in F is free in F but bound in E . Let \mathbb{E} be the set of valid CCSP_t^θ expressions. Furthermore, $\mathbb{P} \subseteq \mathbb{E}$ is the set of *closed* valid CCSP_t^θ expressions, or *processes*; those in which every variable occurrence is bound.

A substitution is a partial function $\rho: \text{Var} \rightarrow \mathbb{E}$. The application $E[\rho]$ of a substitution ρ to an expression $E \in \mathbb{E}$ is the result of simultaneous replacement, for all $x \in \text{dom}(\rho)$, of each free occurrence of x in E by the expression $\rho(x)$, while renaming bound variables in E if necessary to prevent name clashes.

The semantics of CCSP_t^θ is given by the labelled transition relation $\rightarrow \subseteq \mathbb{P} \times \text{Act} \times \mathbb{P}$, where the transitions $P \xrightarrow{\alpha} Q$ are derived from the rules of Table 1. Here $\langle E | \mathcal{S} \rangle$ for $E \in \mathbb{E}$ and \mathcal{S} a recursive specification denotes the result of substituting $\langle y | \mathcal{S} \rangle$ for y in E , for all $y \in V_S$.

The auxiliary operators θ_L^U and ψ_X are added here to facilitate complete axiomatisation, similar to the left merge and communication merge of ACP [1, 8]. The operator θ_X^X is the same as what was called θ_X in Section 4. It inhibits those transitions of its argument that are blocked in the environment X , allowing only the actions from $X \subseteq A$. It stops inhibiting as soon as the system performs a visible action or takes a break, as this may trigger a change in the environment. The operator θ_L^U preserves those transitions that are allowed in some environment X with $L \subseteq X \subseteq U$. The letters L and U stand for *lower* and *upperbound*. The operator ψ_X places a process in the environment X when a time-out transition occurs; it is inert if any other transition occurs. If $P \xrightarrow{\beta}$ for $\beta \in A \cup \{\tau\}$, then a time-out transition $P \xrightarrow{t} Q$ cannot occur in an environment that allows β . Thus the transition $P \xrightarrow{t} Q$ survives only when considering an environments that blocks β , meaning $\beta \notin X \cup \{\tau\}$. Taking the contrapositive, $\beta \in X \cup \{\tau\}$ implies $P \not\xrightarrow{\beta}$.

The operator θ_0^U features in the forthcoming law L3, which is a convenient addition to my axiomatisation, although only ψ_X and $\theta_X (= \theta_X^X)$ are necessary for completeness.

Stratification. Even though negative premises occur in Table 1, the meaning of this transition system specification is well-defined, for instance by the method of *stratification* explained in [21, 13]. Assign

inductively to each expression $E \in \mathbb{E}$ an ordinal λ_E that counts the nesting depth of recursive specifications: if $E = \langle x | \mathcal{S} \rangle$ then λ_E is 1 more than the supremum of the λ_{S_y} for $y \in V_S$; otherwise λ_E is the supremum of $\lambda_{\langle x | \mathcal{S} \rangle}$ for all subterms $\langle x | \mathcal{S} \rangle$ of E . Moreover $\kappa_E \in \mathbb{N}$ is the nesting depth of θ_L^U and ψ_X operators in E that remain after replacing any subterm F of E with $\lambda_F < \lambda_E$ by 0. Now the ordered pair (λ_P, κ_P) constitutes a valid stratification for closed literals $P \xrightarrow{\alpha} P'$. Namely, whenever a transition $P \xrightarrow{\alpha} P'$ depends on a transition $Q \xrightarrow{\beta} Q'$, in the sense that there is a closed substitution instance τ of a rule from Table 1 with conclusion $P \xrightarrow{\alpha} P'$, and $Q \xrightarrow{\beta} Q'$ occurring in its premises, then either $\lambda_Q < \lambda_P$, or $\lambda_Q = \lambda_P$ and $\kappa_Q \leq \kappa_P$. Moreover, when $P \xrightarrow{\alpha} P'$ depends on a negative literal $Q \xrightarrow{\beta} \text{false}$, then $\lambda_Q = \lambda_P$ and $\kappa_Q < \kappa_P$.

The above argument hinges on the exclusion of invalid CCSP_t^θ expressions. The invalid expression $P := \langle x | \{x = \theta_{\{a\}}^{\{a\}}(b.0 + \mathcal{R}(x))\} \rangle$ for instance, with $\mathcal{R} = \{(b, a)\}$, does not have a well-defined meaning, since the transition $P \xrightarrow{b} 0$ is derivable iff one has the premise $P \xrightarrow{b} \text{false}$:

$$\frac{\frac{b.0 \xrightarrow{b} 0}{b.0 + \mathcal{R}(P) \xrightarrow{b} 0} \quad \frac{\frac{P \xrightarrow{b} \text{false}}{\mathcal{R}(P) \xrightarrow{a} \text{false}}}{b.0 + \mathcal{R}(P) \xrightarrow{a} \text{false}} \quad \frac{\frac{P \xrightarrow{\tau} \text{false} \text{ (OK)}}{\mathcal{R}(P) \xrightarrow{\tau} \text{false}}}{b.0 + \mathcal{R}(P) \xrightarrow{\tau} \text{false}}}{\frac{\theta_{\{a\}}^{\{a\}}(b.0 + \mathcal{R}(P)) \xrightarrow{b} 0}{P \xrightarrow{b} 0}}$$

However, the meaning of the valid expression $\langle x | \{x = \theta_{\{a\}}^{\{a\}}(\langle y | \{y = b.y\} \parallel_{\emptyset} \mathcal{R}(x)) \} \rangle$, for instance, is entirely unproblematic.

6 Guarded recursion and finitely branching processes

In many process algebraic specification approaches, only guarded recursive specifications are allowed.

Definition 10 An occurrence of a variable x in an expression E is *guarded* if x occurs in a subexpression $\alpha.F$ of E , with $\alpha \in \text{Act}$. An expression E is *guarded* if all free occurrences of variables in E are guarded. A recursive specification \mathcal{S} is *manifestly guarded* if all expressions S_y for $y \in V_S$ are guarded. It is *guarded* if it can be converted into a manifestly guarded recursive specification by repeated substitution of expressions S_y for variables $y \in V_S$ occurring in the expressions S_z for $z \in V_S$. Let *guarded* CCSP_t^θ be the fragment of CCSP_t^θ allowing only guarded recursion.

Definition 11 The set of processes *reachable* from a given process $P \in \mathbb{P}$ is inductively defined by

- (i) P is reachable from P , and
- (ii) if Q is reachable from P and $Q \xrightarrow{\alpha} R$ for some $\alpha \in \text{Act}$ then R is reachable from P .

A process P is *finitely branching* if for all $Q \in \mathbb{P}$ reachable from P there are only finitely many processes R such that $Q \xrightarrow{\alpha} R$ for some $\alpha \in \text{Act}$. Likewise, P is *countably branching* if there are countably many. A process is *finite* iff it is finitely branching, has finitely many reachable states, and is loop-free.

Proposition 12 Each CCSP_t^θ process is countably branching.

Proof: I show that for each CCSP_t^θ process Q there are only countably many transitions $Q \xrightarrow{\alpha} R$. Each such transition must be derivable from the rules of Table 1. So it suffices to show that for each Q there are only countably many derivations of transitions $Q \xrightarrow{\alpha} R$.

A derivation of a transition is a well-founded, upwardly branching tree, in which each node models an application of one of the rules of Table 1. Since each of these rules has finitely many positive premises,

such a proof tree is finitely branching, and thus finite. Let $d(\pi)$, the *depth* of π , be the length of the longest branch in a derivation π . If π derives a transition $Q \xrightarrow{\alpha} R$, then I call Q the *source* of π .

It suffices to show that for each $n \in \mathbb{N}$ there are only finitely many derivations of depth n with a given source. This I do by induction on n .

In case $Q = f(Q_1, \dots, Q_k)$, with f an k -ary CCSP_t^θ operator, a derivation π of depth n is completely determined by the concluding rule from Table 1, deriving a transition $Q \xrightarrow{\beta} R$, the subderivations of π with source Q_i for some of the $i \in \{1, \dots, k\}$, and the transition label β . The choice of the concluding rule depends on f , and for each f there are at most three choices. The subderivations of π with source Q_i have depth $< n$, so by induction there are only finitely many. When f is not a renaming operator \mathcal{R} , there is no further choice for the transition label β , as it is completely determined by the premises of the rule, and thus by the subderivations of those premises. In case $f = \mathcal{R}$, there are finitely many choices for β when faced with a given transition label α contributed by the premise of the rule for renaming. Here I use the requirement of Section 5 that all sets $\{b \mid (a, b) \in \mathcal{R}\}$ are finite. This shows there are only finitely many choices for π .

In case $Q = \langle x | \mathcal{S} \rangle$, the last step in π must be application of the rule for recursion, so π is completely determined by a subderivation π' of a transition with source $\langle \mathcal{S}_x | \mathcal{S} \rangle$. By induction there are only finitely many choices for π' , and hence also for π . \square

Proposition 13 Each CCSP_t^θ process with guarded recursion is finitely branching.

Proof: A trivial structural induction shows that if P is a CCSP_t^θ process with guarded recursion and Q is reachable from P , then also Q has with guarded recursion. Hence it suffices to show that for each CCSP_t^θ process Q with guarded recursion there are only finitely many derivations with source Q .

Let \xrightarrow{u} be the smallest binary relation on \mathbb{P} such that (i) $f(P_1, \dots, P_k) \xrightarrow{u} P_i$ for each k -ary CCSP_t^θ operator f except action prefixing, and each $i \in \{1, \dots, k\}$, and (ii) $\langle x | \mathcal{S} \rangle \xrightarrow{u} \langle \mathcal{S}_x | \mathcal{S} \rangle$. This relation is finitely branching. Moreover, on processes with guarded recursion, \xrightarrow{u} has no forward infinite chains $P_0 \xrightarrow{u} P_1 \xrightarrow{u} \dots$. In fact, this could have been used as an alternative definition of guarded recursion. Let, for any process Q with guarded recursion, $e(Q)$ be the length of the longest forward chain $Q \xrightarrow{u} P_1 \xrightarrow{u} \dots \xrightarrow{u} P_{e(Q)}$. I show with induction on $e(Q)$ that there are only finitely many derivations with source Q . In fact, this proceeds exactly as in the previous proof. \square

Proposition 14 ([11]) Each finitely branching processes in an LTS can be denoted by a closed CCSP_t expression with guarded recursion. Here I only need the operations inaction (0), action prefixing ($\alpha._$) and choice (+), as well as recursion $\langle x | \mathcal{S} \rangle$.

Proof: Let P be a finitely branching process in an LTS $(\mathbb{P}', \text{Act}, \rightarrow)$. Let

$$V_{\mathcal{S}} := \{x_Q \mid Q \in \mathbb{P}' \text{ is reachable from } P\} \subseteq \text{Var}.$$

For each Q reachable from P , let $\text{next}(Q)$ be the finite set of pairs $(\alpha, R) \in \text{Act} \times \mathbb{P}'$ such that there is a transition $Q \xrightarrow{\alpha} R$. Define the recursive specification \mathcal{S} as $\{x_Q = \sum_{(\alpha, R) \in \text{next}(Q)} \alpha.x_R \mid x_Q \in V_{\mathcal{S}}\}$. Here the finite choice operator $\sum_{i \in I} \alpha_i.P_i$ can easily be expressed in terms of inaction, action prefixing and choice. Now the CCSP_t process $\langle x_P | \mathcal{S} \rangle$ denotes P . \square

In fact, $\langle x_P | \mathcal{S} \rangle \Leftrightarrow P$, where \Leftrightarrow denotes strong bisimilarity [29], formally defined in the next section.

Likewise, recursion-free CCSP_t^θ processes are finite, and, up to strong bisimilarity, each finite process is denotable by a closed recursion-free CCSP_t^θ expression, using only 0, $\alpha._$ and +.

Proposition 15 ([11]) Each countably branching processes in an LTS can be denoted by a closed CCSP_t expression. Again I only need the CCSP_t operations inaction, action prefixing, choice and recursion.

Proof: The proof is the same as the previous one, except that $next(Q)$ now is a countable set, rather than a finite one, and consequently I need a countable choice operator $\sum_{i \in \mathbb{N}} \alpha_i.P_i$. The latter can be expressed in $CCSP_t$ with unguarded recursion by $\sum_{i \in \mathbb{N}} \alpha_i.P_i := \langle z_0 | \{z_i = \alpha_i.P_i + z_{i+1} \mid i \in \mathbb{N}\} \rangle$. \square

7 Congruence

Given an arbitrary process algebra with a collection of operators f , each with an arity n , and a recursion construct $\langle x | \mathcal{S} \rangle$ as in Section 5, let \mathbb{P} and \mathbb{E} be the sets of [closed] valid expressions, and let a substitution instance $E[\rho] \in \mathbb{E}$ for $E \in \mathbb{E}$ and $\rho : Var \rightarrow \mathbb{E}$ be defined as in Section 5. Any semantic equivalence $\sim \subseteq \mathbb{P} \times \mathbb{P}$ extends to $\sim \subseteq \mathbb{E} \times \mathbb{E}$ by defining $E \sim F$ iff $E[\rho] \sim F[\rho]$ for each closed substitution $\rho : Var \rightarrow \mathbb{P}$. It extends to substitutions $\rho, \nu : Var \rightarrow \mathbb{E}$ by $\rho \sim \nu$ iff $\rho(x) \sim \nu(x)$ for each $x \in \text{dom}(\rho)$.

Definition 16 ([14]) A semantic equivalence \sim is a *lean congruence* if $E[\rho] \sim E[\nu]$ for any expression $E \in \mathbb{E}$ and any substitutions ρ and ν with $\rho \sim \nu$. It is a *full congruence* if it satisfies

$$P_i \sim Q_i \text{ for all } i = 1, \dots, n \Rightarrow f(P_1, \dots, P_n) \sim f(Q_1, \dots, Q_n) \quad (1)$$

$$\mathcal{S}_y \sim \mathcal{S}'_y \text{ for all } y \in V_S \Rightarrow \langle x | \mathcal{S} \rangle \sim \langle x | \mathcal{S}' \rangle \quad (2)$$

for all functions f of arity n , processes $P_i, Q_i \in \mathbb{P}$, and recursive specifications $\mathcal{S}, \mathcal{S}'$ with $x \in V_S = V_{S'}$ and $\langle x | \mathcal{S} \rangle, \langle x | \mathcal{S}' \rangle \in \mathbb{P}$.

Clearly, each full congruence is also a lean congruence, and each lean congruence satisfies (1) above. Both implications are strict, as illustrated in [14].

A main result of the present paper will be that strong reactive bisimilarity is a full congruence for the process algebra $CCSP_t^\theta$. To achieve it I need to establish first that strong bisimilarity [29], \Leftrightarrow , and initials equivalence [12, Section 16], $=_{\mathcal{I}}$, are full congruences for $CCSP_t^\theta$.

7.1 Initials equivalence

Definition 17 Two $CCSP_t^\theta$ processes P and Q are *initials equivalent*, denoted $P =_{\mathcal{I}} Q$, if $\mathcal{I}(P) = \mathcal{I}(Q)$.

Theorem 18 Initials equivalence is a full congruence for $CCSP_t^\theta$.

Proof: In Appendix I. \square

7.2 Strong bisimilarity

Definition 19 A *strong bisimulation* is a symmetric relation \mathcal{B} on \mathbb{P} , such that, whenever $P \mathcal{B} Q$,

- if $P \xrightarrow{\alpha} P'$ with $\alpha \in Act$ then $Q \xrightarrow{\alpha} Q'$ for some Q' with $P' \mathcal{B} Q'$.

Two processes $P, Q \in \mathbb{P}$ are *strongly bisimilar*, $P \Leftrightarrow Q$, if $P \mathcal{B} Q$ for some strong bisimulation \mathcal{B} .

Contrary to reactive bisimilarity, strong bisimilarity treats the time-out action t , as well as the hidden action τ , just like any visible action. In the absence of time-out actions, there is no difference between a strong bisimulation and a time-out bisimulation, so \Leftrightarrow_r and \Leftrightarrow coincide. In general, strong bisimulation is a finer equivalence relation: $P \Leftrightarrow Q \Rightarrow P \Leftrightarrow_r Q \Rightarrow P =_{\mathcal{I}} Q$, and both implications are strict.

Lemma 1 For each $CCSP_t^\theta$ process P there exists a $CCSP_t^\theta$ process Q only built using inaction, action prefixing, choice and recursion, such that $P \Leftrightarrow Q$.

Proof: Immediately from Propositions 12 and 15. \square

Theorem 20 Strong bisimilarity is a full congruence for CCSP_t^θ .

Proof: The structural operational rules for CCSP_t (that is, CCSP_t^θ without the operators θ_L^U and ψ_X) fit the *tyft/tyxt format with recursion* of [14]. By [14, Theorem 3] this implies that \Leftrightarrow is a full congruence for CCSP_t . (In fact, when omitting the recursion construct, the operational rules for CCSP_t fit the *tyft/tyxt format* of [22], and by the main theorem of [22], \Leftrightarrow is a congruence *for the operators of* CCSP_t , that is, it satisfies (1) in Definition 16. The work of [14] extends this result of [22] with recursion.)

The structural operational rules for all of CCSP_t^θ fit the *ntyft/ntyxt format with recursion* of [14]. By [14, Theorem 2] this implies that \Leftrightarrow is a lean congruence for CCSP_t^θ . (In fact, when omitting the recursion construct, the operational rules for CCSP_t^θ fit the *ntyft/ntyxt format* of [21], and by the main theorem of [21], \Leftrightarrow is a congruence for the operators of CCSP_t^θ . The work of [14] extends this result of [21] with recursion.)

To verify (2) for the whole language CCSP_t^θ , let \mathcal{S} and \mathcal{S}' be recursive specifications with $x \in V_S = V_{S'}$, such that $\langle x|\mathcal{S}\rangle, \langle x|\mathcal{S}'\rangle \in \mathbb{P}$ and $\mathcal{S}_y \Leftrightarrow \mathcal{S}'_y$ for all $y \in V_S$. Let $\{P_i \mid i \in I\}$ be the collection of processes of the form $\theta_L^U(Q)$ or $\psi_X(Q)$, for some L, U, X , that occur as a closed subexpression of \mathcal{S}_y or \mathcal{S}'_y for one of the $y \in V_S$, not counting strict subexpressions of a closed subexpression R of \mathcal{S}_y or \mathcal{S}'_y that is itself of the form $\theta_L^U(Q)$ or $\psi_X(Q)$. Pick a fresh variable $z_i \notin V_S$ for each $i \in I$, and let, for $y \in Y$, $\widehat{\mathcal{S}}_y$ be the result of replacing each occurrence of P_i in \mathcal{S}_y by z_i . Then $\widehat{\mathcal{S}}_y$ does not contain the operators $\theta_L^U(Q)$ or $\psi_X(Q)$. In deriving this conclusion it is essential that $\langle x|\mathcal{S}\rangle$ is a valid expression, for this implies that the term $\mathcal{S}_y \in \mathbb{E}$, which may contain free occurrences of the variables $y \in V_S$, does not have a subterm of the form $\theta_L^U(F)$ or $\psi_X(F)$ that contains free occurrences of these variables. Let $\widehat{\mathcal{S}} := \{y = \widehat{\mathcal{S}}_y \mid y \in V_S\}$; it is a recursive specification in the language CCSP_t . The recursive specification $\widehat{\mathcal{S}}'$ is defined in the same way.

For each $i \in I$ there is, by Lemma 1, a process Q_i in the language CCSP_t such that $P_i \Leftrightarrow Q_i$. Now let $\rho, \eta : \{z_i \mid i \in I\} \rightarrow \mathbb{P}$ be the substitutions defined by $\rho(z_i) = P_i$ and $\eta(z_i) = Q_i$ for all $i \in I$. Then $\rho \Leftrightarrow \eta$. Since \Leftrightarrow is a lean congruence for CCSP_t^θ , one has $\langle x|\widehat{\mathcal{S}}\rangle[\rho] \Leftrightarrow \langle x|\widehat{\mathcal{S}}\rangle[\eta]$ and likewise $\langle x|\widehat{\mathcal{S}}'\rangle[\rho] \Leftrightarrow \langle x|\widehat{\mathcal{S}}'\rangle[\eta]$. For the same reason one has $\widehat{\mathcal{S}}_y[\eta] \Leftrightarrow \widehat{\mathcal{S}}_y[\rho] = \mathcal{S}_y \Leftrightarrow \mathcal{S}'_y \Leftrightarrow \widehat{\mathcal{S}}'_y[\rho] \Leftrightarrow \widehat{\mathcal{S}}'_y[\eta]$ for all $y \in V_S$. Since $\widehat{\mathcal{S}}[\eta]$ and $\widehat{\mathcal{S}}'[\eta]$ are recursive specifications over CCSP_t , $\langle x|\widehat{\mathcal{S}}[\eta]\rangle \Leftrightarrow \langle x|\widehat{\mathcal{S}}'[\eta]\rangle$. Hence $\langle x|\mathcal{S}\rangle = \langle x|\widehat{\mathcal{S}}[\rho]\rangle = \langle x|\widehat{\mathcal{S}}\rangle[\rho] \Leftrightarrow \langle x|\widehat{\mathcal{S}}\rangle[\eta] = \langle x|\widehat{\mathcal{S}}[\eta]\rangle \Leftrightarrow \langle x|\widehat{\mathcal{S}}'[\eta]\rangle \Leftrightarrow \langle x|\widehat{\mathcal{S}}'[\rho]\rangle = \langle x|\mathcal{S}'\rangle$. \square

7.3 Lemmas

The following lemmas on the relation between θ_X and the other operators of CCSP_t^θ deal with strong bisimilarity, but are needed in the congruence proof for strong reactive bisimilarity.

Lemma 2 If $\mathcal{I}(Q) \cap (Y \cup \{\tau\}) = \emptyset$ then $\theta_Y(Q) \Leftrightarrow Q$.

Proof: This follows immediately from the operational rules for θ_X . \square

Lemma 3 If $P \not\stackrel{\tau}{\rightarrow}, \mathcal{I}(P) \cap X \subseteq S$ and $Y = X \setminus (S \setminus \mathcal{I}(P))$, then $\theta_X(P \parallel_S Q) \Leftrightarrow \theta_X(P \parallel_S \theta_Y(Q))$.

Proof: Let $P \in \mathbb{P}$ and $S, X, Y \subseteq A$ be as indicated in the lemma. Let

$$\mathcal{B} := \Leftrightarrow \cup \{(\theta_X(P \parallel_S Q), \theta_X(P \parallel_S \theta_Y(Q))) \mid Q \in \mathbb{P}\}$$

It suffices to show that the symmetric closure $\widetilde{\mathcal{B}}$ of \mathcal{B} is a strong bisimulation.

So let $R \widetilde{\mathcal{B}} T$ and $R \xrightarrow{\alpha} R'$ with $\alpha \in A \cup \{\tau, \text{t}\}$. I have to find a T' with $T \xrightarrow{\alpha} T'$ and $R' \widetilde{\mathcal{B}} T'$.

- The case that $R \Leftrightarrow T$ is trivial.

- Let $R = \theta_X(P \parallel_S Q)$ and $T = \theta_X(P \parallel_S \theta_Y(Q))$, for some $Q \in \mathbb{P}$.

First assume $\alpha = \tau$. Then $Q \xrightarrow{\tau} Q'$ for some Q' with $R' = \theta_X(P \parallel_S Q')$. Consequently, $T = \theta_X(P \parallel_S \theta_Y(Q)) \xrightarrow{\tau} \theta_X(P \parallel_S \theta_Y(Q')) =: T'$ and $R' \mathcal{B} T'$.

Now assume $\alpha \in A \cup \{t\}$. Then $P \parallel_S Q \xrightarrow{\alpha} R'$. I first deal with the case that $\alpha \in X$, and consider the three rules from Table 1 that could have derived $P \parallel_S Q \xrightarrow{\alpha} R'$.

- The case that $\alpha \notin S$ and $P \xrightarrow{\alpha} P'$ cannot occur, because $\mathcal{I}(P) \cap X \subseteq S$.
- Let $\alpha \in S$, $P \xrightarrow{\alpha} P'$, $Q \xrightarrow{\alpha} Q'$ and $R' = P' \parallel_S Q'$. Then $\alpha \in \mathcal{I}(P)$, so $\alpha \notin S \setminus \mathcal{I}(P)$ and thus $\alpha \in Y$. Hence $\theta_Y(Q) \xrightarrow{\alpha} Q'$. Now $T = \theta_X(P \parallel_S \theta_Y(Q)) \xrightarrow{\alpha} P' \parallel_S Q' = R'$.
- Let $\alpha \notin S$, $Q \xrightarrow{\alpha} Q'$ and $R' = P \parallel_S Q'$. Then $\alpha \in Y$, so $\theta_Y(Q) \xrightarrow{\alpha} Q'$. Therefore, $P \parallel_S \theta_Y(Q) \xrightarrow{\alpha} P \parallel_S Q'$ and thus $T = \theta_X(P \parallel_S \theta_Y(Q)) \xrightarrow{\alpha} P \parallel_S Q' = R'$.

Finally, assume $\alpha \in (A \cup \{t\}) \setminus X$. In that case $P \parallel_S Q \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$. Therefore, $Q \not\xrightarrow{\beta}$ for all $\beta \in (X \setminus S) \cup \{\tau\}$, and for all $\beta \in X \cap S \cap \mathcal{I}(P)$, and thus for all $\beta \in Y \cup \{\tau\}$. By Lemma 2, $\theta_Y(Q) =_{\mathcal{I}} Q$, and hence $P \parallel_S \theta_Y(Q) \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$. Again, I consider the three rules from Table 1 that could have derived $P \parallel_S Q \xrightarrow{\alpha} R'$.

- Let $\alpha \notin S$, $P \xrightarrow{\alpha} P'$ and $R' = P' \parallel_S Q$. Then $P \parallel_S \theta_Y(Q) \xrightarrow{\alpha} P' \parallel_S \theta_Y(Q)$ and thus $T = \theta_X(P \parallel_S \theta_Y(Q)) \xrightarrow{\alpha} P' \parallel_S \theta_Y(Q) =: T'$. By Lemma 2, $\theta_Y(Q) \stackrel{\simeq}{=} Q$. Since $\stackrel{\simeq}{=}$ is a congruence for \parallel_S , it follows that $R' = P' \parallel_S Q \stackrel{\simeq}{=} P' \parallel_S \theta_Y(Q) = T'$.
- Let $\alpha \in S$, $P \xrightarrow{\alpha} P'$, $Q \xrightarrow{\alpha} Q'$ and $R' = P' \parallel_S Q'$. Then $\theta_Y(Q) \xrightarrow{\alpha} Q'$ and therefore $P \parallel_S \theta_Y(Q) \xrightarrow{\alpha} P' \parallel_S Q'$ and $T = \theta_X(P \parallel_S \theta_Y(Q)) \xrightarrow{\alpha} P' \parallel_S Q' = R'$.
- Let $\alpha \notin S$, $Q \xrightarrow{\alpha} Q'$ and $R' = P \parallel_S Q'$. Then $\theta_Y(Q) \xrightarrow{\alpha} Q'$, so $P \parallel_S \theta_Y(Q) \xrightarrow{\alpha} P \parallel_S Q'$ and thus $T = \theta_X(P \parallel_S \theta_Y(Q)) \xrightarrow{\alpha} P \parallel_S Q' = R'$.

- Let $R = \theta_X(P \parallel_S \theta_Y(Q))$ and $T = \theta_X(P \parallel_S Q)$, for some $Q \in \mathbb{P}$.

First assume $\alpha = \tau$. Then $Q \xrightarrow{\tau} Q'$ for some Q' with $R' = \theta_X(P \parallel_S \theta_Y(Q'))$. Consequently, $T = \theta_X(P \parallel_S Q) \xrightarrow{\tau} \theta_X(P \parallel_S Q') =: T'$ and $R' \mathcal{B} T'$.

Now assume $\alpha \in A \cup \{t\}$. Then $P \parallel_S \theta_Y(Q) \xrightarrow{\alpha} R'$ and either $\alpha \in X$ or $P \parallel_S \theta_Y(Q) \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$. In the latter case one obtains $\theta_Y(Q) \not\xrightarrow{\beta}$ for all $\beta \in Y \cup \{\tau\}$ (as above), and thus $Q \not\xrightarrow{\beta}$ for all $\beta \in Y \cup \{\tau\}$, that is, $\mathcal{I}(Q) \cap (Y \cup \{\tau\}) = \emptyset$. Furthermore, this implies that $P \parallel_S Q \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$.

I consider the three rules from Table 1 that could have derived $P \parallel_S Q \xrightarrow{\alpha} R'$.

- Let $\alpha \notin S$, $P \xrightarrow{\alpha} P'$ and $R' = P' \parallel_S \theta_Y(Q)$. Then $\alpha \notin X$, because $\mathcal{I}(P) \cap X \subseteq S$. Hence $P \parallel_S \theta_Y(Q) \not\xrightarrow{\beta}$ for all $\beta \in X \cup \{\tau\}$, so $\mathcal{I}(Q) \cap (Y \cup \{\tau\}) = \emptyset$. Now $T = \theta_X(P \parallel_S Q) \xrightarrow{\alpha} P' \parallel_S Q =: T'$ and $R' \stackrel{\simeq}{=} T'$, using Lemma 2.
- Let $\alpha \in S$, $P \xrightarrow{\alpha} P'$, $\theta_Y(Q) \xrightarrow{\alpha} Q'$ and $R' = P' \parallel_S Q'$. Then $Q \xrightarrow{\alpha} Q'$. Hence $P \parallel_S Q \xrightarrow{\alpha} P' \parallel_S Q'$ and thus $T = \theta_X(P \parallel_S Q) \xrightarrow{\alpha} P' \parallel_S Q' = R'$.
- Let $\alpha \notin S$, $\theta_Y(Q) \xrightarrow{\alpha} Q'$ and $R' = P \parallel_S Q'$. Then $Q \xrightarrow{\alpha} Q'$. Consequently, $P \parallel_S Q \xrightarrow{\alpha} P \parallel_S Q'$ and thus $T = \theta_X(P \parallel_S Q) \xrightarrow{\alpha} P \parallel_S Q' = R'$. \square

Lemma 4 $\theta_X(\tau_I(P)) \stackrel{\simeq}{=} \theta_X(\tau_I(\theta_{X \cup I}(P)))$.

Proof: For given X and I , let $\mathcal{B} := Id \cup \{(\theta_X(\tau_I(P)), \theta_X(\tau_I(\theta_{X \cup I}(P)))) \mid P \in \mathbb{P}\}$. It suffices to show that the symmetric closure $\tilde{\mathcal{B}}$ of \mathcal{B} is a strong bisimulation. So let $R \mathcal{B} T$ and $R \xrightarrow{\alpha} R'$ with $\alpha \in A \cup \{\tau, t\}$. I have to find a T' with $T \xrightarrow{\alpha} T'$ and $R' \tilde{\mathcal{B}} T'$.

- The case that $R = T$ is trivial.

- Let $R = \theta_X(\tau_I(P))$ and $T = \theta_X(\tau_I(\theta_{X \cup I}(P)))$, for some $P \in \mathbb{P}$.

First assume $\alpha = \tau$. Then $\tau_I(P) \xrightarrow{\tau} R''$ for some R'' such that $R' = \theta_X(R'')$. Therefore, $P \xrightarrow{\beta} P'$ for some $\beta \in I \cup \{\tau\}$ and some P' with $R'' = \tau_I(P')$. In case $\beta = \tau$, it turns out that $T = \theta_X(\tau_I(\theta_{X \cup I}(P))) \xrightarrow{\tau} \theta_X(\tau_I(\theta_{X \cup I}(P'))) =: T'$. Moreover, $R' \mathcal{B} T'$. In case $\beta \in I$, $\theta_{X \cup I}(P) \xrightarrow{\beta} P'$, so $\tau_I(\theta_{X \cup I}(P)) \xrightarrow{\tau} \tau_I(P')$ and $T = \theta_X(\tau_I(\theta_{X \cup I}(P))) \xrightarrow{\tau} \theta_X(\tau_I(P')) = R'$. Now assume $\alpha \in A \cup \{t\}$. Then $\tau_I(P) \xrightarrow{\alpha} R'$ and either $\alpha \in X$ or $\tau_I(P) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. It follows that $\alpha \notin I$ and $P \xrightarrow{\alpha} P'$ for some P' with $R' = \tau_I(P')$. Moreover, in case $\alpha \notin X$ one has $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup I \cup \{\tau\}$, and hence also $\theta_{X \cup I}(P) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup I \cup \{\tau\}$, and thus $\tau_I(\theta_{X \cup I}(P)) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Now $\theta_{X \cup I}(P) \xrightarrow{\alpha} P'$, so $\tau_I(\theta_{X \cup I}(P)) \xrightarrow{\alpha} \tau_I(P')$ and thus $T = \theta_X(\tau_I(\theta_{X \cup I}(P))) \xrightarrow{\alpha} \tau_I(P') = R'$.

- Let $R = \theta_X(\tau_I(\theta_{X \cup I}(P)))$ and $T = \theta_X(\tau_I(P))$, for some $P \in \mathbb{P}$.

First assume $\alpha = \tau$. Then $\tau_I(\theta_{X \cup I}(P)) \xrightarrow{\tau} R''$ for some R'' such that $R' = \theta_X(R'')$. Therefore, $\theta_{X \cup I}(P) \xrightarrow{\beta} P'$ for some $\beta \in I \cup \{\tau\}$ and some P' with $R'' = \tau_I(P')$. In case $\beta = \tau$, it turns out that $P \xrightarrow{\tau} P''$ for some P'' such that $P' = \theta_{X \cup I}(P'')$. So $T = \theta_X(\tau_I(P)) \xrightarrow{\tau} \theta_X(\tau_I(P'')) =: T'$, and $R' \mathcal{B} T'$. In case $\beta \in I$, one has $P \xrightarrow{\beta} P'$, so $\tau_I(P) \xrightarrow{\tau} \tau_I(P')$ and $T = \theta_X(\tau_I(P)) \xrightarrow{\tau} \theta_X(\tau_I(P')) = R'$.

Now assume $\alpha \in A \cup \{t\}$. Then $\tau_I(\theta_{X \cup I}(P)) \xrightarrow{\alpha} R'$, so $\alpha \notin I$ and $\theta_{X \cup I}(P) \xrightarrow{\alpha} P'$ for some P' such that $R' = \tau_I(P')$. Thus $P \xrightarrow{\alpha} P'$ and either $\alpha \in X$ or $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup I \cup \{\tau\}$. In the latter case $\tau_I(P) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Now $\tau_I(P) \xrightarrow{\alpha} \tau_I(P')$ and consequently $T = \theta_X(\tau_I(P)) \xrightarrow{\alpha} \tau_I(P') = R'$. \square

Lemma 5 $\theta_X(\mathcal{R}(P)) \stackrel{\sim}{\leftrightarrow} \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)))$.

Proof: For given $X \subseteq A$ and $\mathcal{R} \subseteq A \times A$, let $\mathcal{B} := \text{Id} \cup \{(\theta_X(\mathcal{R}(P)), \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)))) \mid P \in \mathbb{P}\}$. It suffices to show that the symmetric closure $\tilde{\mathcal{B}}$ of \mathcal{B} is a strong bisimulation. So let $R \tilde{\mathcal{B}} T$ and $R \xrightarrow{\alpha} R'$ with $\alpha \in A \cup \{\tau, t\}$. I have to find a T' with $T \xrightarrow{\alpha} T'$ and $R' \tilde{\mathcal{B}} T'$.

- The case that $R = T$ is trivial.
- Let $R = \theta_X(\mathcal{R}(P))$ and $T = \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)))$, for some $P \in \mathbb{P}$.

First assume $\alpha = \tau$. Then $P \xrightarrow{\tau} P'$ for some P' such that $R' = \theta_X(\mathcal{R}(P'))$.

Hence $T = \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P))) \xrightarrow{\tau} \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P'))) =: T'$, and $R' \mathcal{B} T'$.

Now assume $\alpha \in A \cup \{t\}$. Then $\mathcal{R}(P) \xrightarrow{\alpha} R'$, and either $\alpha \in X$ or $\mathcal{R}(P) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. In the latter case, $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in \mathcal{R}^{-1}(X) \cup \{\tau\}$. Moreover, $P \xrightarrow{\gamma} P'$, for some γ with $\gamma = t = \alpha$ or $(\gamma, \alpha) \in \mathcal{R}$, and some P' with $R' = \mathcal{R}(P')$. In case $\alpha \in X$, one has $\gamma \in \mathcal{R}^{-1}(X)$. Therefore, $\theta_{\mathcal{R}^{-1}(X)}(P) \xrightarrow{\gamma} P'$, and thus $\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)) \xrightarrow{\alpha} \mathcal{R}(P')$.

Either $\alpha \in X$ or $\theta_{\mathcal{R}^{-1}(X)}(P) \xrightarrow{\beta} \text{fail}$ for all $\beta \in \mathcal{R}^{-1}(X) \cup \{\tau\}$, in which case $\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Consequently, $T = \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P))) \xrightarrow{\alpha} \mathcal{R}(P') = R'$.

- Let $R = \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)))$ and $T = \theta_X(\mathcal{R}(P))$, for some $P \in \mathbb{P}$.

First assume $\alpha = \tau$. Then $P \xrightarrow{\tau} P'$ for some P' such that $R' = \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P')))$.

Hence $T = \theta_X(\mathcal{R}(P)) \xrightarrow{\tau} \theta_X(\mathcal{R}(P')) =: T'$, and $R' \tilde{\mathcal{B}} T'$.

Now assume $\alpha \in A \cup \{t\}$. Then $\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)) \xrightarrow{\alpha} R'$ and either $\alpha \in X$ or $\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Therefore, $\theta_{\mathcal{R}^{-1}(X)}(P) \xrightarrow{\gamma} P'$ for some γ with $\gamma = t = \alpha$ or $(\gamma, \alpha) \in \mathcal{R}$, and some P' such that $R' = \mathcal{R}(P')$. Hence $P \xrightarrow{\gamma} P'$, and thus $\mathcal{R}(P) \xrightarrow{\alpha} \mathcal{R}(P')$. In case $\alpha \notin X$, one has $\theta_{\mathcal{R}^{-1}(X)}(P) \xrightarrow{\beta} \text{fail}$ for all $\beta \in \mathcal{R}^{-1}(X) \cup \{\tau\}$, and thus $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in \mathcal{R}^{-1}(X) \cup \{\tau\}$, so $\mathcal{R}(P) \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Hence $T = \theta_X(\mathcal{R}(P)) \xrightarrow{\alpha} \mathcal{R}(P') = R'$. \square

8 Strong reactive bisimilarity is a full congruence for CCSP_t^θ

The forthcoming proofs showing that \Leftrightarrow_r is a full congruence for CCSP_t^θ follow the lines of Milner [29], but are more complicated due to the nature of reactive bisimilarity. A crucial tool is Milner's notion of *bisimilarity up-to*. The lemmas from Section 7.3 play an essential rôle. Even if we would not be interested in the operators θ_L^U and ψ_X , the proof needs to take the operator $\theta_X (= \theta_X^X)$ along in order to deal with the other operators. This is a consequence of the occurrence of θ_X in Definition 8.

Definition 21 Given a relation $\sim \subseteq \mathbb{P} \times \mathbb{P}$, a *strong time-out bisimulation up to* \sim is a symmetric relation $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$, such that, for $P \mathcal{B} Q$,

- if $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$, then $\exists Q'$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \sim \mathcal{B} \sim Q'$,
- if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, then $\exists Q'$ with $Q \xrightarrow{t} Q'$ and $\theta_X(P') \sim \mathcal{B} \sim \theta_X(Q')$.

Here $\sim \mathcal{B} \sim := \{(R, T) \mid \exists R', T'. R \sim R' \mathcal{B} T' \sim T'\}$.

Proposition 22 If $P \mathcal{B} Q$ for some strong time-out bisimulation \mathcal{B} up to \Leftrightarrow , then $P \Leftrightarrow_r Q$.

Proof: Using the reflexivity of \Leftrightarrow it suffices to show that $\Leftrightarrow \mathcal{B} \Leftrightarrow$ is a strong time-out bisimulation. Clearly this relation is symmetric, and that it satisfies the first clause of Definition 8 is straightforward, using transitivity of \Leftrightarrow . So assume $P \Leftrightarrow R \mathcal{B} T \Leftrightarrow Q$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. Then $\mathcal{I}(R) \cap (X \cup \{\tau\}) = \emptyset$. By the transfer property of \Leftrightarrow , there exists an R' with $R \xrightarrow{t} R'$ and $P' \Leftrightarrow R'$. Since \Leftrightarrow is a congruence for θ_X it follows that $\theta_X(P') \Leftrightarrow \theta_X(R')$. By Definition 21, there exists a T' with $T \xrightarrow{t} T'$ and $\theta_X(R') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(T')$. Again using the transfer property of \Leftrightarrow , there exists an Q' with $Q \xrightarrow{t} Q'$ and $\theta_X(T') \Leftrightarrow \theta_X(Q')$. Thus, $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$. \square

Theorem 23 Strong reactive bisimilarity is a lean congruence for CCSP_t^θ . In other words, if $\rho, \nu : \text{Var} \rightarrow \mathbb{E}$ are substitutions with $\rho \Leftrightarrow_r \nu$, then $E[\rho] \Leftrightarrow_r E[\nu]$ for any expression $E \in \mathbb{E}$.

Proof: It suffices to prove this theorem for the special case that $\rho, \nu : \text{Var} \rightarrow \mathbb{P}$ are closed substitutions; the general case then follows by means of composition of substitutions. Let $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$ be the smallest relation satisfying

- if $P \Leftrightarrow_r Q$, then $P \mathcal{B} Q$,
- if $P \mathcal{B} Q$ and $\alpha \in A \cup \{\tau, t\}$, then $\alpha.P \mathcal{B} \alpha.Q$,
- if $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$, then $P_1 + P_2 \mathcal{B} Q_1 + Q_2$,
- if $P \mathcal{B} Q$, $L \subseteq U \subseteq A$ and $X \subseteq A$, then $\theta_L^U(P) \mathcal{B} \theta_L^U(Q)$ and $\psi_X(P) \mathcal{B} \psi_X(Q)$,
- if $P_1 \mathcal{B} Q_1$, $P_2 \mathcal{B} Q_2$ and $S \subseteq A$, then $P_1 \parallel_S P_2 \mathcal{B} Q_1 \parallel_S Q_2$,
- if $P \mathcal{B} Q$ and $I \subseteq A$, then $\tau_I(P) \mathcal{B} \tau_I(Q)$,
- if $P \mathcal{B} Q$ and $\mathcal{R} \subseteq A \times A$, then $\mathcal{R}(P) \mathcal{B} \mathcal{R}(Q)$,
- if \mathcal{S} is a recursive specification with $z \in V_{\mathcal{S}}$, and $\rho, \nu : \text{Var} \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$ are substitutions satisfying $\rho(x) \mathcal{B} \nu(x)$ for all $x \in \text{Var} \setminus V_{\mathcal{S}}$, then $\langle z | \mathcal{S} \rangle[\rho] \mathcal{B} \langle z | \mathcal{S} \rangle[\nu]$.

A straightforward induction on the derivation of $P \mathcal{B} Q$, employing Theorem 18, yields that

$$\text{if } P \mathcal{B} Q \text{ then } \mathcal{I}(P) = \mathcal{I}(Q), \text{ i.e., } P =_{\mathcal{I}} Q. \quad (\text{e})$$

(For the last case, the assumption that $\rho(x) \mathcal{B} \nu(x)$ for all $x \in \text{Var} \setminus V_{\mathcal{S}}$ implies $\rho =_{\mathcal{I}} \nu$ by induction. Since $=_{\mathcal{I}}$ is a lean congruence by Theorem 18, this implies $\langle z | \mathcal{S} \rangle[\rho] =_{\mathcal{I}} \langle z | \mathcal{S} \rangle[\nu]$.)

A trivial structural induction on $E \in \mathbb{E}$ shows that

$$\text{if } \rho, \nu : \text{Var} \rightarrow \mathbb{P} \text{ satisfy } \rho(x) \mathcal{B} \nu(x) \text{ for all } x \in \text{Var}, \text{ then } E[\rho] \mathcal{B} E[\nu]. \quad (*)$$

For \mathcal{S} a recursive specification and $\rho : \text{Var} \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$, let $\rho_{\mathcal{S}} : \text{Var} \rightarrow \mathbb{P}$ be the closed substitution given by $\rho_{\mathcal{S}}(x) := \langle x | \mathcal{S} \rangle [\rho]$ if $x \in V_{\mathcal{S}}$ and $\rho_{\mathcal{S}}(x) := \rho(x)$ otherwise. Then $\langle E | \mathcal{S} \rangle [\rho] = E[\rho_{\mathcal{S}}]$ for all $E \in \mathbb{E}$. Hence an application of (*) with $\rho_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ yields that under the conditions of the last clause for \mathcal{B} above one even has $\langle E | \mathcal{S} \rangle [\rho] \mathcal{B} \langle E | \mathcal{S} \rangle [\nu]$ for all expressions $E \in \mathbb{E}$. (\\$)

It suffices to show that \mathcal{B} is a strong time-out bisimulation up to \leftrightarrow , because then $P \leftrightarrow_r Q \Leftrightarrow P \mathcal{B} Q$, and (*) implies that \mathcal{B} is a lean congruence. Because \leftrightarrow_r is symmetric, so is \mathcal{B} . So I need to show that \mathcal{B} satisfies the two clauses of Definition 21.

- Let $P \mathcal{B} Q$ and $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$. I have to find a Q' with $Q \xrightarrow{\alpha} Q'$ and $P' \leftrightarrow \mathcal{B} \leftrightarrow Q'$. In fact, I show that even $P' \mathcal{B} Q'$. This I will do by structural induction on the proof π of $P \xrightarrow{\alpha} P'$ from the rules of Table 1. I make a case distinction based on the derivation of $P \mathcal{B} Q$.
 - Let $P \leftrightarrow_r Q$. Using that the relation \leftrightarrow_r is a strong time-out bisimulation, there must be a process Q' such that $Q \xrightarrow{\alpha} Q'$ and $P' \leftrightarrow_r Q'$. Hence $P' \mathcal{B} Q'$.
 - Let $P = \beta.P^\dagger$ and $Q = \beta.Q^\dagger$ with $\beta \in A \cup \{\tau, t\}$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $\alpha = \beta$ and $P' = P^\dagger$. Take $Q' := Q^\dagger$. Then $Q \xrightarrow{\alpha} Q'$ and $P' \mathcal{B} Q'$.
 - Let $P = P_1 + P_2$ and $Q = Q_1 + Q_2$ with $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$. I consider the first rule from Table 1 that could have been responsible for the derivation of $P \xrightarrow{\alpha} P'$; the other proceeds symmetrically. So suppose that $P_1 \xrightarrow{\alpha} P'$. Then by induction $Q_1 \xrightarrow{\alpha} Q'$ for some Q' with $P' \mathcal{B} Q'$. By the same rule from Table 1, $Q \xrightarrow{\alpha} Q'$.
 - Let $P = \theta_L^U(P^\dagger)$, $Q = \theta_L^U(Q^\dagger)$ and $P^\dagger \mathcal{B} Q^\dagger$. First suppose $\alpha \in A$. Since $\theta_L^U(P^\dagger) \xrightarrow{\alpha} P'$, it must be that $P^\dagger \xrightarrow{\alpha} P'$ and either $\alpha \in U$ or $P^\dagger \xrightarrow{\beta}$ for all $\beta \in L \cup \{\tau\}$. In the latter case, (⊖) yields $\mathcal{I}(P^\dagger) = \mathcal{I}(Q^\dagger)$, and thus $Q^\dagger \xrightarrow{\beta}$ for all $\beta \in L \cup \{\tau\}$. By induction there exists a Q' such that $Q^\dagger \xrightarrow{\alpha} Q'$ and $P' \mathcal{B} Q'$. So, in both cases, $Q = \theta_L^U(Q^\dagger) \xrightarrow{\alpha} Q'$.
Now suppose $\alpha = \tau$. Since $\theta_L^U(P^\dagger) \xrightarrow{\tau} P'$ it must be that P' has the form $\theta_L^U(P^\ddagger)$, and $P^\dagger \xrightarrow{\tau} P^\ddagger$. By induction, there exists a Q^\ddagger such that $Q^\dagger \xrightarrow{\tau} Q^\ddagger$ and $P^\ddagger \mathcal{B} Q^\ddagger$. Now $Q = \theta_L^U(Q^\dagger) \xrightarrow{\tau} \theta_L^U(Q^\ddagger) =: Q'$ and $P' \mathcal{B} Q'$.
 - Let $P = \psi_X(P^\dagger)$, $Q = \psi_X(Q^\dagger)$ and $P^\dagger \mathcal{B} Q^\dagger$. Since $\psi_X(P^\dagger) \xrightarrow{\alpha} P'$, one has $P^\dagger \xrightarrow{\alpha} P'$. By induction there exists a Q' with $Q^\dagger \xrightarrow{\alpha} Q'$ and $P' \mathcal{B} Q'$. So $Q = \psi_X(Q^\dagger) \xrightarrow{\alpha} Q'$.
 - Let $P = P_1 \parallel_S P_2$ and $Q = Q_1 \parallel_S Q_2$ with $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$. I consider the three rules from Table 1 that could have been responsible for the derivation of $P \xrightarrow{\alpha} P'$.
First suppose that $\alpha \notin S$, $P_1 \xrightarrow{\alpha} P'_1$ and $P' = P'_1 \parallel_S P_2$. By induction, $Q_1 \xrightarrow{\alpha} Q'_1$ for some Q'_1 with $P'_1 \mathcal{B} Q'_1$. Consequently, $Q_1 \parallel_S Q_2 \xrightarrow{\alpha} Q'_1 \parallel_S Q_2$, and $P' = P'_1 \parallel_S P_2 \mathcal{B} Q'_1 \parallel_S Q_2$.
Next suppose that $\alpha \in S$, $P_1 \xrightarrow{\alpha} P'_1$, $P_2 \xrightarrow{\alpha} P'_2$ and $P' = P'_1 \parallel_S P'_2$. By induction, $Q_1 \xrightarrow{\alpha} Q'_1$ for some Q'_1 with $P'_1 \mathcal{B} Q'_1$, and $Q_2 \xrightarrow{\alpha} Q'_2$ for some Q'_2 with $P'_2 \mathcal{B} Q'_2$. Consequently, $Q_1 \parallel_S Q_2 \xrightarrow{\alpha} Q'_1 \parallel_S Q'_2$, and $P' = P'_1 \parallel_S P'_2 \mathcal{B} Q'_1 \parallel_S Q'_2$.
The remaining case proceeds symmetrically to the first.
 - Let $P = \tau_I(P^\dagger)$ and $Q = \tau_I(Q^\dagger)$ with $I \subseteq A$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $P^\dagger \xrightarrow{\beta} P^\ddagger$ for some P^\ddagger with $P' = \tau_I(P^\ddagger)$, and either $\beta = \alpha \notin I$, or $\beta \in I$ and $\alpha = \tau$. By induction, $Q^\dagger \xrightarrow{\beta} Q^\ddagger$ for some Q^\ddagger with $P^\ddagger \mathcal{B} Q^\ddagger$. Consequently, $Q = \tau_I(Q^\dagger) \xrightarrow{\alpha} \tau_I(Q^\ddagger) =: Q'$ and $P' \mathcal{B} Q'$.
 - Let $P = \mathcal{R}(P^\dagger)$ and $Q = \mathcal{R}(Q^\dagger)$ with $\mathcal{R} \subseteq A \times A$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $P^\dagger \xrightarrow{\beta} P^\ddagger$ for some P^\ddagger with $P' = \mathcal{R}(P^\ddagger)$, and either $(\beta, \alpha) \in \mathcal{R}$ or $\beta = \alpha = \tau$. By induction, $Q^\dagger \xrightarrow{\beta} Q^\ddagger$ for some Q^\ddagger with $P^\ddagger \mathcal{B} Q^\ddagger$. Consequently, $Q = \mathcal{R}(Q^\dagger) \xrightarrow{\alpha} \mathcal{R}(Q^\ddagger) =: Q'$ and $P' \mathcal{B} Q'$.

- Let $P = \langle z|\mathcal{S} \rangle[\rho] = \langle z|\mathcal{S}[\rho] \rangle$ and $Q = \langle z|\mathcal{S} \rangle[\nu] = \langle z|\mathcal{S}[\nu] \rangle$ where \mathcal{S} is a recursive specification with $z \in V_{\mathcal{S}}$, and $\rho, \nu : \text{Var} \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$ satisfy $\rho(x) \mathcal{B} \nu(x)$ for all $x \in \text{Var} \setminus V_{\mathcal{S}}$. By Table 1 the transition $\langle \mathcal{S}_z|\mathcal{S}[\rho] \rangle \xrightarrow{\alpha} P'$ is provable by means of a strict subproof of the proof π of $\langle z|\mathcal{S} \rangle[\rho] \xrightarrow{\alpha} P'$. By (\$) above one has $\langle \mathcal{S}_z|\mathcal{S}[\rho] \rangle \mathcal{B} \langle \mathcal{S}_z|\mathcal{S}[\nu] \rangle$. So by induction there is a Q' such that $\langle \mathcal{S}_z|\mathcal{S}[\nu] \rangle \xrightarrow{\alpha} Q'$ and $P' \mathcal{B} Q'$. By Table 1, $Q = \langle z|\mathcal{S}[\nu] \rangle \xrightarrow{\alpha} Q'$.
- Let $P \mathcal{B} Q$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. I have to find a Q' such that $Q \xrightarrow{t} Q'$ and $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$. This I will do by structural induction on the proof π of $P \xrightarrow{t} P'$ from the rules of Table 1. I make a case distinction based on the derivation of $P \mathcal{B} Q$.
 - Let $P \Leftrightarrow_r Q$. Using that the relation \Leftrightarrow_r is a strong time-out bisimulation, there must be a process Q' such that $Q \xrightarrow{t} Q'$ and $\theta_X(P') \Leftrightarrow_r \theta_X(Q')$. Thus $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$.
 - Let $P = \beta.P^\dagger$ and $Q = \beta.Q^\dagger$ with $\beta \in A \cup \{\tau, t\}$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $\beta = t$ and $P' = P^\dagger$. Take $Q' := Q^\dagger$. Then $Q \xrightarrow{t} Q'$ and $P' \mathcal{B} Q'$. Thus $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$.
 - Let $P = P_1 + P_2$ and $Q = Q_1 + Q_2$ with $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$. I consider the first rule from Table 1 that could have been responsible for the derivation of $P \xrightarrow{t} P'$; the other proceeds symmetrically. So suppose that $P_1 \xrightarrow{t} P'$. Since $\mathcal{I}(P_1) \cap (X \cup \{\tau\}) \subseteq \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, by induction $Q_1 \xrightarrow{t} Q'$ for some Q' with $P' \Leftrightarrow \mathcal{B} \Leftrightarrow Q'$. Hence $Q \xrightarrow{t} Q'$.
 - Let $P = \theta_L^U(P^\dagger)$, $Q = \theta_L^U(Q^\dagger)$ and $P^\dagger \mathcal{B} Q^\dagger$. Since $\theta_L^U(P^\dagger) \xrightarrow{t} P'$ it must be that $P^\dagger \xrightarrow{t} P'$ and $P^\dagger \xrightarrow{\beta} \not\mathcal{B}$ for all $\beta \in L \cup \{\tau\}$. Consequently, $P \xrightarrow{\alpha} P'$ iff $P^\dagger \xrightarrow{\alpha} P'$, for all $\alpha \in A \cup \{t\}$. So $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. By induction, $Q^\dagger \xrightarrow{t} Q'$ for some Q' with $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$. By (@), $Q^\dagger \xrightarrow{\beta} \not\mathcal{B}$ for all $\beta \in L \cup \{\tau\}$. Hence $Q = \theta_L^U(Q^\dagger) \xrightarrow{t} Q'$.
 - Let $P = \psi_Y(P^\dagger)$, $Q = \psi_Y(Q^\dagger)$ and $P^\dagger \mathcal{B} Q^\dagger$. Since $\psi_Y(P^\dagger) \xrightarrow{t} P'$ one has $P^\dagger \xrightarrow{t} P'$ for some P' with $P' = \theta_Y(P^\dagger)$, and $P^\dagger \xrightarrow{\beta} \not\mathcal{B}$ for all $\beta \in Y \cup \{\tau\}$, i.e., $\mathcal{I}(P^\dagger) \cap (Y \cup \{\tau\}) = \emptyset$. By induction, $Q^\dagger \xrightarrow{t} Q'$ for a Q' with $\theta_Y(P^\dagger) \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_Y(Q')$. By (@), $\mathcal{I}(P^\dagger) = \mathcal{I}(Q^\dagger)$, so $Q^\dagger \xrightarrow{\beta} \not\mathcal{B}$ for all $\beta \in Y \cup \{\tau\}$. Let $Q' := \theta_Y(Q^\dagger)$, so that $Q = \psi_Y(Q^\dagger) \xrightarrow{t} \theta_Y(Q^\dagger) = Q'$. From $\theta_Y(P^\dagger) \Leftrightarrow P'' \mathcal{B} Q'' \Leftrightarrow \theta_Y(Q^\dagger)$ one obtains

$$\theta_X(\theta_Y(P^\dagger)) \Leftrightarrow \theta_X(P'') \mathcal{B} \theta_X(Q'') \Leftrightarrow \theta_X(\theta_Y(Q^\dagger)),$$

using that \Leftrightarrow is a congruence for $\theta_X (= \theta_X^X)$. Thus $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$.

- Let $P = P_1 \parallel_S P_2$ and $Q = Q_1 \parallel_S Q_2$ with $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$. I consider the last rule from Table 1 that could have been responsible for the derivation of $P \xrightarrow{t} P'$. The other proceeds symmetrically. So suppose that $P_2 \xrightarrow{t} P'_2$ and $P' = P_1 \parallel_S P'_2$. Let $Y := X \setminus (S \setminus \mathcal{I}(P_1)) = (X \setminus S) \cup (X \cap S \cap \mathcal{I}(P_1))$. Then $\mathcal{I}(P_2) \cap (Y \cup \{\tau\}) = \emptyset$. By induction, $Q_2 \xrightarrow{t} Q'_2$ for some Q'_2 with $\theta_Y(P'_2) \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_Y(Q'_2)$. Let $Q' := Q_1 \parallel_S Q'_2$, so that $Q = Q_1 \parallel_S Q_2 \xrightarrow{t} Q_1 \parallel_S Q'_2 = Q'$. From $\theta_Y(P'_2) \Leftrightarrow P''_2 \mathcal{B} Q''_2 \Leftrightarrow \theta_Y(Q'_2)$ and $P_1 \mathcal{B} Q_1$ one obtains $P_1 \parallel_S \theta_Y(P'_2) \Leftrightarrow P_1 \parallel_S P''_2 \mathcal{B} Q_1 \parallel_S Q''_2 \Leftrightarrow P_1 \parallel_S \theta_Y(Q'_2)$, using that \Leftrightarrow is a congruence for \parallel_S . Therefore, since \Leftrightarrow is also a congruence for $\theta_X (= \theta_X^X)$,

$$\theta_X(P_1 \parallel_S \theta_Y(P'_2)) \Leftrightarrow \theta_X(P_1 \parallel_S P''_2) \mathcal{B} \theta_X(Q_1 \parallel_S Q''_2) \Leftrightarrow \theta_X(Q_1 \parallel_S \theta_Y(Q'_2)).$$

Since $\mathcal{I}(P_1 \parallel_S P_2) \cap (X \cup \{\tau\}) = \emptyset$, one has $P_1 \xrightarrow{\tau} \not\mathcal{B}$ and $\mathcal{I}(P_1) \cap X \subseteq S$. Moreover, since $P_1 \mathcal{B} Q_1$, one has $\mathcal{I}(P_1) = \mathcal{I}(Q_1)$. Hence $\theta_X(P') = \theta_X(P_1 \parallel_S P'_2) \Leftrightarrow \theta_X(P_1 \parallel_S \theta_Y(P'_2)) \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q_1 \parallel_S \theta_Y(Q'_2)) \Leftrightarrow \theta_X(Q_1 \parallel_S Q'_2) = \theta_X(Q')$ by Lemma 3.

- Let $P = \tau_I(P^\dagger)$ and $Q = \tau_I(Q^\dagger)$ with $I \subseteq A$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $P^\dagger \xrightarrow{t} P'$ for some P' with $P' = \tau_I(P^\dagger)$. Moreover, $\mathcal{I}(P^\dagger) \cap (X \cup I \cup \{\tau\}) = \emptyset$. By induction,

$Q^\dagger \xrightarrow{t} Q^\ddagger$ for some Q^\ddagger with $\theta_{X \cup I}(P^\ddagger) \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_{X \cup I}(Q^\ddagger)$. Let $Q' := \tau_I(Q^\ddagger)$, so that $Q = \tau_I(Q^\dagger) \xrightarrow{t} \tau_I(Q^\ddagger) = Q'$. From $\theta_{X \cup I}(P^\ddagger) \Leftrightarrow P'' \mathcal{B} Q'' \Leftrightarrow \theta_{X \cup I}(Q^\ddagger)$ one obtains

$$\theta_X(\tau_I(P^\ddagger)) \Leftrightarrow \theta_X(\tau_I(\theta_{X \cup I}(P^\ddagger))) \Leftrightarrow \theta_X(\tau_I(P'')) \mathcal{B} \theta_X(\tau_I(Q'')) \Leftrightarrow \dots \Leftrightarrow \theta_X(\tau_I(Q^\ddagger)),$$

using Lemma 4 and that \Leftrightarrow is a congruence for τ_I and θ_X . Thus $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$.

- Let $P = \mathcal{R}(P^\dagger)$ and $Q = \mathcal{R}(Q^\dagger)$ with $\mathcal{R} \subseteq A \times A$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $P^\dagger \xrightarrow{t} P^\ddagger$ for some P^\ddagger with $P' = \mathcal{R}(P^\ddagger)$. Moreover, $\mathcal{I}(P^\dagger) \cap (\mathcal{R}^{-1}(X) \cup \{\tau\}) = \emptyset$. By induction, $Q^\dagger \xrightarrow{t} Q^\ddagger$ for some Q^\ddagger with $\theta_{\mathcal{R}^{-1}(X)}(P^\ddagger) \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_{\mathcal{R}^{-1}(X)}(Q^\ddagger)$. Let $Q' := \mathcal{R}(Q^\ddagger)$, so that $Q = \mathcal{R}(Q^\dagger) \xrightarrow{t} \mathcal{R}(Q^\ddagger) = Q'$. From $\theta_{\mathcal{R}^{-1}(X)}(P^\ddagger) \Leftrightarrow P'' \mathcal{B} Q'' \Leftrightarrow \theta_{\mathcal{R}^{-1}(X)}(Q^\ddagger)$ one obtains

$$\theta_X(\mathcal{R}(P^\ddagger)) \Leftrightarrow \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P^\ddagger))) \Leftrightarrow \theta_X(\mathcal{R}(P'')) \mathcal{B} \theta_X(\mathcal{R}(Q'')) \Leftrightarrow \dots \Leftrightarrow \theta_X(\mathcal{R}(Q^\ddagger)),$$

using Lemma 5 and that \Leftrightarrow is a congruence for \mathcal{R} and θ_X . Thus $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$.

- Let $P = \langle z | \mathcal{S} \rangle [\rho] = \langle z | \mathcal{S} \rangle [\rho]$ and $Q = \langle z | \mathcal{S} \rangle [\nu] = \langle z | \mathcal{S} \rangle [\nu]$ where \mathcal{S} is a recursive specification with $z \in V_{\mathcal{S}}$, and $\rho, \nu : \text{Var} \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$ satisfy $\rho(x) \mathcal{B} \nu(x)$ for all $x \in \text{Var} \setminus V_{\mathcal{S}}$. By Table 1 the transition $\langle \mathcal{S}_z | \mathcal{S} \rangle [\rho] \xrightarrow{t} P'$ is provable by means of a strict subproof of the proof π of $\langle z | \mathcal{S} \rangle [\rho] \xrightarrow{t} P'$. The rule for recursion in Table 1 also implies that $\mathcal{I}(\langle z | \mathcal{S} \rangle [\rho]) = \mathcal{I}(\langle \mathcal{S}_z | \mathcal{S} \rangle [\rho])$. Therefore, $\mathcal{I}(\langle \mathcal{S}_z | \mathcal{S} \rangle [\rho]) \cap (X \cup \{\tau\}) = \emptyset$. By (\$) above one has $\langle \mathcal{S}_z | \mathcal{S} \rangle [\rho] \mathcal{B} \langle \mathcal{S}_z | \mathcal{S} \rangle [\nu]$. So by induction there is a Q' such that $\langle \mathcal{S}_z | \mathcal{S} \rangle [\nu] \xrightarrow{t} Q'$ and $\theta_X(P') \Leftrightarrow \mathcal{B} \Leftrightarrow \theta_X(Q')$. By Table 1, $Q = \langle z | \mathcal{S} \rangle [\nu] \xrightarrow{t} Q'$. \square

Proposition 24 If $P \mathcal{B} Q$ for some strong time-out bisimulation \mathcal{B} up to \Leftrightarrow_r , then $P \Leftrightarrow_r Q$.

Proof: Exactly as the proof of Proposition 22, now using that \Leftrightarrow_r is a congruence for θ_X . \square

Theorem 25 Strong reactive bisimilarity is a full congruence for CCSP_t^θ .

Proof: Let $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$ be the smallest relation satisfying

- if \mathcal{S} and \mathcal{S}' are recursive specifications with $x \in V_{\mathcal{S}} = V_{\mathcal{S}'}$ and $\langle x | \mathcal{S} \rangle, \langle x | \mathcal{S}' \rangle \in \mathbb{P}$, such that $\mathcal{S}_y \Leftrightarrow \mathcal{S}'_y$ for all $y \in V_{\mathcal{S}}$, then $\langle x | \mathcal{S} \rangle \mathcal{B} \langle x | \mathcal{S}' \rangle$,

in addition to the eight or nine clauses listed in the proof of Theorem 23. Again, a straightforward induction on the derivation of $P \mathcal{B} Q$, employing Theorem 18, yields that

$$\text{if } P \mathcal{B} Q \text{ then } \mathcal{I}(P) = \mathcal{I}(Q), \text{ i.e., } P =_{\mathcal{I}} Q. \quad (\textcircled{c})$$

(For the new case, the assumption that $\mathcal{S}_y \Leftrightarrow \mathcal{S}'_y$ for all $y \in V_{\mathcal{S}}$ implies $\mathcal{S}_y =_{\mathcal{I}} \mathcal{S}'_y$ for all $y \in V_{\mathcal{S}}$. So by Theorem 18, $\langle x | \mathcal{S} \rangle =_{\mathcal{I}} \langle x | \mathcal{S}' \rangle$.) A trivial structural induction on $E \in \mathbb{E}$ shows again that

$$\text{if } \rho, \nu : \text{Var} \rightarrow \mathbb{P} \text{ satisfy } \rho(x) \mathcal{B} \nu(x) \text{ for all } x \in \text{Var}, \text{ then } E[\rho] \mathcal{B} E[\nu]. \quad (*)$$

This again implies that in the last clause for \mathcal{B} one even has $\langle E | \mathcal{S} \rangle [\rho] \mathcal{B} \langle E | \mathcal{S}' \rangle [\nu]$ for all $E \in \mathbb{E}$, (\$) and likewise, in the new clause, $\langle E | \mathcal{S} \rangle \mathcal{B} \langle E | \mathcal{S}' \rangle$ for all $E \in \mathbb{E}$ with variables from $V_{\mathcal{S}}$. (#)

It suffices to show that \mathcal{B} is a strong time-out bisimulation up to \Leftrightarrow_r , because then $\mathcal{B} \subseteq \Leftrightarrow_r$ with Proposition 24, and the new clause for \mathcal{B} implies (2). By construction \mathcal{B} is symmetric.

- Let $P \mathcal{B} Q$ and $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$. I have to find a Q' with $Q \xrightarrow{\alpha} Q'$ and $P' \Leftrightarrow_r \mathcal{B} \Leftrightarrow_r Q'$. In fact, I show that even $P' \mathcal{B} \Leftrightarrow_r Q'$. This I will do by structural induction on the proof π of $P \xrightarrow{\alpha} P'$ from the rules of Table 1. I make a case distinction based on the derivation of $P \mathcal{B} Q$.

- Let $P = \langle x|\mathcal{S} \rangle \in \mathbb{P}$ and $Q = \langle x|\mathcal{S}' \rangle \in \mathbb{P}$ where \mathcal{S} and \mathcal{S}' are recursive specifications with $x \in V_{\mathcal{S}} = V_{\mathcal{S}'}$, such that $\mathcal{S}_y \dot{\leftrightarrow} \mathcal{S}'_y$ for all $y \in V_{\mathcal{S}}$, meaning that for all $y \in W$ and $\sigma : V_{\mathcal{S}} \rightarrow \mathbb{P}$ one has $\mathcal{S}_y[\sigma] \dot{\leftrightarrow}_r \mathcal{S}'_y[\sigma]$.
By Table 1 the transition $\langle \mathcal{S}_x|\mathcal{S} \rangle \xrightarrow{\alpha} P'$ is provable by means of a strict subproof of π . By (#) above one has $\langle \mathcal{S}_x|\mathcal{S} \rangle \mathcal{B} \langle \mathcal{S}_x|\mathcal{S}' \rangle$. So by induction there is an $R' \in \mathbb{P}$ such that $\langle \mathcal{S}_x|\mathcal{S}' \rangle \xrightarrow{\alpha} R'$ and $P' \mathcal{B} \dot{\leftrightarrow}_r R'$. Since $\langle _|\mathcal{S}' \rangle$ is the application of a substitution of the form $\sigma : V_{\mathcal{S}'} \rightarrow \mathbb{P}$, one has $\langle \mathcal{S}_x|\mathcal{S}' \rangle \dot{\leftrightarrow}_r \langle \mathcal{S}'_x|\mathcal{S}' \rangle$. Hence there is a Q' with $P \vdash \langle \mathcal{S}'_x|\mathcal{S}' \rangle \xrightarrow{\alpha} Q'$ and $R' \dot{\leftrightarrow}_r Q'$. So $P' \mathcal{B} \dot{\leftrightarrow}_r Q'$. By Table 1, $Q = \langle x|\mathcal{S}' \rangle \xrightarrow{\alpha} Q'$.
- The remaining nine cases proceed just as in the proof of Theorem 23, but with $\mathcal{B} \dot{\leftrightarrow}_r$ substituted for the blue occurrences of \mathcal{B} . In the case for θ_L^U with $\alpha = \tau$, I conclude from $P^\ddagger \mathcal{B} \dot{\leftrightarrow}_r Q^\ddagger$ that $\theta_L^U(P^\ddagger) \mathcal{B} \dot{\leftrightarrow}_r \theta_L^U(Q^\ddagger)$. Besides applying the definition of \mathcal{B} , this also involves the application of Theorem 23 that $\dot{\leftrightarrow}_r$ is already known to be a congruence for θ_L^U . The same reasoning applies in the cases for \parallel_S, τ_I and \mathcal{R} .
- Let $P \mathcal{B} Q$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. I will find a Q' such that $Q \xrightarrow{t} Q'$ and $\theta_X(P') \dot{\leftrightarrow} \mathcal{B} \dot{\leftrightarrow}_r \theta_X(Q')$. This I will do by structural induction on the proof π of $P \xrightarrow{t} P'$ from the rules of Table 1. I make a case distinction based on the derivation of $P \mathcal{B} Q$.
 - Let $P = \langle x|\mathcal{S} \rangle \in \mathbb{P}$ and $Q = \langle x|\mathcal{S}' \rangle \in \mathbb{P}$ where \mathcal{S} and \mathcal{S}' are recursive specifications with $x \in V_{\mathcal{S}} = V_{\mathcal{S}'}$, such that for all $y \in W$ and $\sigma : V_{\mathcal{S}} \rightarrow \mathbb{P}$ one has $\mathcal{S}_y[\sigma] \dot{\leftrightarrow}_r \mathcal{S}'_y[\sigma]$. By Table 1 the transition $\langle \mathcal{S}_x|\mathcal{S} \rangle \xrightarrow{t} P'$ is provable by means of a strict subproof of the proof π of $\langle x|\mathcal{S} \rangle \xrightarrow{t} P'$. The rule for recursion in Table 1 also implies that $\mathcal{I}(\langle x|\mathcal{S} \rangle) = \mathcal{I}(\langle \mathcal{S}_x|\mathcal{S} \rangle)$. Therefore, $\mathcal{I}(\langle \mathcal{S}_x|\mathcal{S} \rangle) \cap (X \cup \{\tau\}) = \emptyset$. By (#) above one has $\langle \mathcal{S}_x|\mathcal{S} \rangle \mathcal{B} \langle \mathcal{S}_x|\mathcal{S}' \rangle$. So by induction there is an $R' \in \mathbb{P}$ such that $\langle \mathcal{S}_x|\mathcal{S}' \rangle \xrightarrow{t} R'$ and $\theta_X(P') \dot{\leftrightarrow} \mathcal{B} \dot{\leftrightarrow}_r \theta_X(R')$. Since $\langle _|\mathcal{S}' \rangle$ is the application of a substitution of the form $\sigma : V_{\mathcal{S}'} \rightarrow \mathbb{P}$, $\langle \mathcal{S}_x|\mathcal{S}' \rangle \dot{\leftrightarrow}_r \langle \mathcal{S}'_x|\mathcal{S}' \rangle$. Using (©), $\mathcal{I}(\langle \mathcal{S}_x|\mathcal{S}' \rangle) \cap (X \cup \{\tau\}) = \emptyset$. Hence $\exists Q'$ with $P \vdash \langle \mathcal{S}'_x|\mathcal{S}' \rangle \xrightarrow{t} Q'$ and $R' \dot{\leftrightarrow}_r Q'$, and thus $\theta_X(R') \dot{\leftrightarrow}_r \theta_X(Q')$, using Theorem 23. So $\theta_X(P') \dot{\leftrightarrow} \mathcal{B} \dot{\leftrightarrow}_r \theta_X(Q')$. By Table 1, $Q = \langle x|\mathcal{S}' \rangle \xrightarrow{t} Q'$.
 - The remaining eight cases proceed just as in the proof of Theorem 23, but with $\mathcal{B} \dot{\leftrightarrow}_r$ substituted for the blue occurrences of $\mathcal{B} \dot{\leftrightarrow}$. □

9 The Recursive Specification Principle

For $W \subseteq \text{Var}$ a set of variables, a W -tuple of expressions is a function $\vec{E} \in \mathbb{E}^W$. It has a component $\vec{E}(x)$ for each variable $x \in W$. Note that a W -tuple of expressions is nothing else than a substitution. Let id_W be the identity function, given by $id_W(x) = x$ for all $x \in W$. If $G \in \mathbb{E}$ and $\vec{E} \in \mathbb{E}^W$ then $G[\vec{E}]$ denotes the result of simultaneous substitution of $\vec{E}(x)$ for x in G , for all $x \in W$. Likewise, if $\vec{G} \in \mathbb{E}^V$ and $\vec{E} \in \mathbb{E}^W$ then $\vec{G}[\vec{E}] \in \mathbb{E}^V$ denotes the V -tuple with components $G(y)[\vec{E}]$ for $y \in V$. Henceforth, I regard a recursive specification \mathcal{S} as a $V_{\mathcal{S}}$ -tuple with components $\mathcal{S}(y) = \mathcal{S}_y$ for $y \in V_{\mathcal{S}}$. If $\vec{E} \in \mathbb{E}^W$ and $\mathcal{S} \in \mathbb{E}^V$, then $\langle \vec{E}|\mathcal{S} \rangle \in \mathbb{E}^W$ is the W -tuple with components $\langle \vec{E}(x)|\mathcal{S} \rangle \in \mathbb{E}^W$ for $x \in W$.

For \mathcal{S} a recursive specification and $\vec{E} \in \mathbb{E}^{V_{\mathcal{S}}}$ a $V_{\mathcal{S}}$ -tuple of expressions, $\vec{E} \dot{\leftrightarrow}_r \mathcal{S}[\vec{E}]$ states that \vec{E} is a *solution* of \mathcal{S} , up to strong reactive bisimilarity. The tuple $\langle id_{V_{\mathcal{S}}}|_{\mathcal{S}} \rangle \in \mathbb{E}^{V_{\mathcal{S}}}$ is called the *default solution*.

In [1, 8] two requirements occur for process algebras with recursion. The *recursive definition principle* (RDP) says that each recursive specification must have a solution, and the *recursive specification principle* (RSP) says that guarded recursive specifications have at most one solution. When dealing with process algebras where the meaning of a closed expression is a semantic equivalence class of processes, these principles become requirements on the semantic equivalence employed.

Proposition 26 Let \mathcal{S} be a guarded recursive specification, and $x \in V_{\mathcal{S}}$. Then $\langle x|\mathcal{S} \rangle \Leftrightarrow_r \langle \mathcal{S}_x|\mathcal{S} \rangle$.

Proof: Let $\sigma : Var \rightarrow \mathbb{P}$ be closed substitution. I have to show that $\langle x|\mathcal{S} \rangle[\sigma] \Leftrightarrow_r \langle \mathcal{S}_x|\mathcal{S} \rangle[\sigma]$. Equivalently I may show this for $\sigma : Var \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$. Now $\langle x|\mathcal{S} \rangle[\sigma] = \langle x|\mathcal{S}[\sigma] \rangle \in \mathbb{P}$ and $\langle \mathcal{S}_x|\mathcal{S} \rangle[\sigma] = \langle \mathcal{S}_x[\sigma]|\mathcal{S}[\sigma] \rangle \in \mathbb{P}$. Consequently, it suffices to prove the proposition under the assumption that $\langle x|\mathcal{S} \rangle, \langle \mathcal{S}_x|\mathcal{S} \rangle \in \mathbb{P}$. This follows immediately from the rule for recursion in Table 1 and Definition 8. \square

Proposition 26 says that the recursive definition principle holds for strong reactive bisimulation semantics. The “default solution” of a recursive specification is in fact a solution. Note that the conclusion of Proposition 26 can be restated as $\langle id_{V_{\mathcal{S}}}\mathcal{S} \rangle \Leftrightarrow_r \langle \mathcal{S}|\mathcal{S} \rangle$, and that $\mathcal{S}[\langle id_{V_{\mathcal{S}}}\mathcal{S} \rangle] = \langle \mathcal{S}|\mathcal{S} \rangle$.

The following theorem establishes the recursive specification principle for strong reactive bisimulation semantics. Some aspects of the proof that are independent of the notion of bisimilarity employed are delegated to the following two lemmas.

Lemma 6 Let $H \in \mathbb{E}$ be guarded and have free variables from $W \subseteq Var$ only, and let $\vec{P}, \vec{Q} \in \mathbb{P}^W$. Then $\mathcal{I}(H[\vec{P}]) = \mathcal{I}(H[\vec{Q}])$.

Proof: In Appendix I. \square

Lemma 7 Let $H \in \mathbb{E}$ be guarded and have free variables from $W \subseteq Var$ only, and let $\vec{P}, \vec{Q} \in \mathbb{P}^W$. If $H[\vec{P}] \xrightarrow{\alpha} R'$ with $\alpha \in Act$, then R' has the form $H'[\vec{P}]$ for some term $H' \in \mathbb{E}$ with free variables in W only. Moreover $H[\vec{Q}] \xrightarrow{\alpha} H'[\vec{Q}]$.

Proof: By induction on the derivation of $H[\vec{P}] \xrightarrow{\alpha} R'$, making a case distinction on the shape of H .

Let $H = \alpha.G$, so that $H[\vec{P}] = \alpha.G[\vec{P}]$. Then $R' = G[\vec{P}]$ and $H[\vec{Q}] \xrightarrow{\alpha} G[\vec{Q}]$.

The case $H = 0$ cannot occur. Nor can the case $H = x \in Var$, as H is guarded.

Let $H = H_1 \parallel_S H_2$, so that $H[\vec{P}] = H_1[\vec{P}] \parallel_S H_2[\vec{P}]$. Note that H_1 and H_2 are guarded and have free variables in W only. One possibility is that $a \notin S$, $H_1[\vec{P}] \xrightarrow{\alpha} R_1$ and $R' = R_1 \parallel_S H_2[\vec{P}]$. By induction, R_1 has the form $H'_1[\vec{P}]$ for some term $H'_1 \in \mathbb{E}$ with free variables in W only. Moreover, $H_1[\vec{Q}] \xrightarrow{\alpha} H'_1[\vec{Q}]$. Thus $R' = (H'_1 \parallel_S H_2)[\vec{P}]$, and $H' := H'_1 \parallel_S H_2$ has free variables in W only. Moreover, $H[\vec{Q}] = H_1[\vec{Q}] \parallel_S H_2[\vec{Q}] \xrightarrow{\alpha} H'_1[\vec{Q}] \parallel_S H_2[\vec{Q}] = H'[\vec{Q}]$.

The other two cases for \parallel_S , and the cases for the operators $+$, τ_I and \mathcal{R} , are equally trivial.

Let $H = \theta_L^U(H^\dagger)$, so that $H[\vec{P}] = \theta_L^U(H^\dagger[\vec{P}])$. Note that H^\dagger is guarded and has free variables in W only. The case $\alpha = \tau$ is again trivial, so assume $\alpha \neq \tau$. Then $H^\dagger[\vec{P}] \xrightarrow{\alpha} R'$ and either $\alpha \in X$ or $H^\dagger[\vec{P}] \xrightarrow{\beta} \text{fail}$ for all $\beta \in L \cup \{\tau\}$. By induction, R' has the form $H'[\vec{P}]$ for some term $H' \in \mathbb{E}$ with free variables in W only. Moreover, $H^\dagger[\vec{Q}] \xrightarrow{\alpha} H'[\vec{Q}]$. Since $\mathcal{I}(H^\dagger[\vec{P}]) = \mathcal{I}(H^\dagger[\vec{Q}])$ by Lemma 6, either $\alpha \in X$ or $H^\dagger[\vec{Q}] \xrightarrow{\beta} \text{fail}$ for all $\beta \in L \cup \{\tau\}$. Consequently, $H[\vec{Q}] = \theta_L^U(H^\dagger[\vec{Q}]) \xrightarrow{\alpha} H'[\vec{Q}]$.

Let $H = \psi_X(H^\dagger)$, so that $H[\vec{P}] = \psi_X(H^\dagger[\vec{P}])$. Note that H^\dagger is guarded and has free variables in W only. The case $\alpha \in A \cup \{\tau\}$ is trivial, so assume $\alpha = t$. Then $H^\dagger[\vec{P}] \xrightarrow{t} R^\dagger$ for some R^\dagger such that $R' = \theta_X(R^\dagger)$. Moreover, $H^\dagger[\vec{P}] \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. By induction, R^\dagger has the form $H'[\vec{P}]$ for some term $H' \in \mathbb{E}$ with free variables in W only. Moreover, $H^\dagger[\vec{Q}] \xrightarrow{t} H'[\vec{Q}]$. Since $\mathcal{I}(H^\dagger[\vec{P}]) = \mathcal{I}(H^\dagger[\vec{Q}])$ by Lemma 6, $H^\dagger[\vec{Q}] \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Consequently, $H[\vec{Q}] = \psi_X(H^\dagger[\vec{Q}]) \xrightarrow{t} \theta_X(H'[\vec{Q}])$.

Finally, let $H = \langle x|\mathcal{S} \rangle$, so that $H[\vec{P}] = \langle x|\mathcal{S}[\vec{P}^\dagger] \rangle$, where \vec{P}^\dagger is the $W \setminus V_{\mathcal{S}}$ -tuple that is left of \vec{P} after deleting the y -components, for $y \in V_{\mathcal{S}}$. The transition $\langle \mathcal{S}_x[\vec{P}^\dagger]|\mathcal{S}[\vec{P}^\dagger] \rangle \xrightarrow{\alpha} R'$ is derivable through a subderivation of the one for $\langle x|\mathcal{S}[\vec{P}^\dagger] \rangle \xrightarrow{\alpha} R'$. Moreover, $\langle \mathcal{S}_x[\vec{P}^\dagger]|\mathcal{S}[\vec{P}^\dagger] \rangle = \langle \mathcal{S}_x|\mathcal{S} \rangle[\vec{P}]$. So by induction, R' has the form $H'[\vec{P}]$ for some term $H' \in \mathbb{E}$ with free variables in W only, and $\langle \mathcal{S}_x|\mathcal{S} \rangle[\vec{Q}] \xrightarrow{\alpha} H'[\vec{Q}]$. Since $\langle \mathcal{S}_x|\mathcal{S} \rangle[\vec{Q}] = \langle \mathcal{S}_x[\vec{Q}^\dagger]|\mathcal{S}[\vec{Q}^\dagger] \rangle$, it follows that $H[\vec{Q}] = \langle x|\mathcal{S} \rangle[\vec{Q}] = \langle x|\mathcal{S}[\vec{Q}^\dagger] \rangle \xrightarrow{\alpha} H'[\vec{Q}]$. \square

Theorem 27 Let \mathcal{S} be a guarded recursive specification. If $\vec{E} \leftrightarrow_r \mathcal{S}[\vec{E}]$ and $\vec{F} \leftrightarrow_r \mathcal{S}[\vec{F}]$ with $\vec{E}, \vec{F} \in \mathbb{E}^{V_S}$, then $\vec{E} \leftrightarrow_r \vec{F}$.

Proof: It suffices to prove Theorem 27 under the assumptions that $\vec{E}, \vec{F} \in \mathbb{P}^{V_S}$ and only the variables from V_S occur free in the expressions \mathcal{S}_x for $x \in V_S$. For in the general case I have to establish that $\vec{E}[\sigma] \leftrightarrow_r \vec{F}[\sigma]$ for an arbitrary closed substitution $\sigma : \text{Var} \rightarrow \mathbb{P}$. Let $\hat{\sigma} : \text{Var} \setminus V_S \rightarrow \mathbb{P}$ be given by $\hat{\sigma}(x) = \sigma(x)$ for all $x \in \text{Var} \setminus V_S$. Then $\vec{E} \leftrightarrow_r \mathcal{S}[\vec{E}]$ implies $\vec{E}[\sigma] \leftrightarrow_r \mathcal{S}[\vec{E}][\sigma] = \mathcal{S}[\hat{\sigma}][\vec{E}[\sigma]]$. Hence, I merely have to prove the theorem with $\vec{E}[\sigma], \vec{F}[\sigma]$ and $\mathcal{S}[\hat{\sigma}]$ in place of \vec{E}, \vec{F} and \mathcal{S} .

It also suffices to prove Theorem 27 under the assumption that \mathcal{S} is a manifestly guarded recursive specification. Namely, for a general guarded recursive specification \mathcal{S} , let \mathcal{S}' be the manifestly guarded specification into which \mathcal{S} can be converted. Then $\vec{E} \leftrightarrow_r \mathcal{S}[\vec{E}]$ implies $\vec{E} \leftrightarrow_r \mathcal{S}'[\vec{E}]$ by Theorem 23.

So let \mathcal{S} be manifestly guarded with free variable from V_S only, and let $\vec{P}, \vec{Q} \in \mathbb{P}^{V_S}$ be two of its solutions, that is, $\vec{P} \leftrightarrow_r \mathcal{S}[\vec{P}]$ and $\vec{Q} \leftrightarrow_r \mathcal{S}[\vec{Q}]$. I will show that the symmetric closure of

$$\mathcal{B} := \{H[\mathcal{S}[\vec{P}]], H[\mathcal{S}[\vec{Q}]] \mid H \in \mathbb{E} \text{ has free variables in } V_S \text{ only}\}$$

is a strong time-out bisimulation up to \leftrightarrow_r . Once I have that, taking $H := x \in V_S$ yields $\mathcal{S}_x[\vec{P}] \leftrightarrow_r \mathcal{S}_x[\vec{Q}]$ by Proposition 24, and thus $P(x) \leftrightarrow_r \mathcal{S}_x[\vec{P}] \leftrightarrow_r \mathcal{S}_x[\vec{Q}] \leftrightarrow_r Q(x)$ for all $x \in V_S$. So $\vec{P} \leftrightarrow_r \vec{Q}$.

- Let $R \mathcal{B} T$ and $R \xrightarrow{\alpha} R'$ with $\alpha \in A \cup \{\tau\}$. I have to find a T' with $T \xrightarrow{\alpha} T'$ and $P' \leftrightarrow_r \mathcal{B} \leftrightarrow_r Q'$. Assume that $R = H[\mathcal{S}[\vec{P}]]$ and $T = H[\mathcal{S}[\vec{Q}]]$ —the case that $R = H[\mathcal{S}[\vec{Q}]]$ will follow by symmetry. Note that $H[\mathcal{S}[\vec{P}]]$ can also be written as $H[\mathcal{S}][\vec{P}]$. Since the expressions \mathcal{S}_x for $x \in V_S$ have free variables from V_S only, so does $H[\mathcal{S}]$. Moreover, since \mathcal{S} is manifestly guarded, the expression $H[\mathcal{S}]$ must be guarded. By Lemma 7, R' must have the form $H'[\vec{P}]$, where $H' \in \mathbb{E}$ has free variables in V_S only. Moreover, $T = H[\mathcal{S}[\vec{Q}]] = H[\mathcal{S}][\vec{Q}] \xrightarrow{\alpha} H'[\vec{Q}] =: T'$. Furthermore, by Theorem 23, $H'[\vec{P}] \leftrightarrow_r H'[\mathcal{S}[\vec{P}]]$ and $H'[\mathcal{S}[\vec{Q}]] \leftrightarrow_r H'[\vec{Q}]$. Thus, $R' = H'[\vec{P}] \leftrightarrow_r \mathcal{B} \leftrightarrow_r H'[\vec{Q}] = T'$.
- Let $R \mathcal{B} T$, $\mathcal{I}(R) \cap (X \cup \{\tau\}) = \emptyset$ and $R \xrightarrow{t} R'$. I have to find a T' such that $T \xrightarrow{t} T'$ and $\theta_X(R') \leftrightarrow_r \mathcal{B} \leftrightarrow_r \theta_X(T')$. The proof for this case proceeds exactly as that of the previous case, up to the last sentence; the condition $\mathcal{I}(R) \cap (X \cup \{\tau\}) = \emptyset$ is not even used. Now from $R' = H'[\vec{P}] \leftrightarrow_r H'[\mathcal{S}[\vec{P}]] \mathcal{B} H'[\mathcal{S}[\vec{Q}]] \leftrightarrow_r H'[\vec{Q}] = T'$ it follows that

$$\theta_X(R') \leftrightarrow_r \theta_X(H'[\mathcal{S}][\vec{P}]) \mathcal{B} \theta_X(H'[\mathcal{S}][\vec{Q}]) \leftrightarrow_r \theta_X(T')$$

using Theorem 23 and the observation that $\theta_X(H'[\mathcal{S}[\vec{P}]]) = \theta_X(H')[\mathcal{S}[\vec{P}]]$. \square

10 Complete axiomatisations

Let Ax denote the collection of axioms from Tables 2, 3 and 4, Ax' the ones from Tables 2 and 3, and Ax'' merely the ones from Table 2. Moreover, let Ax_f , resp. Ax'_f and Ax''_f , be same collections without the two axioms using the recursion construct $\langle x | \mathcal{S} \rangle$, RDP and RSP. In this section I establish the following.

Let P and Q be recursion-free CCSP_t processes. Then $P \leftrightarrow Q \Leftrightarrow Ax''_f \vdash P = Q$. (3)

Let P and Q CCSP_t processes with guarded recursion. Then $P \leftrightarrow Q \Leftrightarrow Ax'' \vdash P = Q$. (4)

Let P and Q be recursion-free CCSP_t^θ processes. Then $P \leftrightarrow Q \Leftrightarrow Ax'_f \vdash P = Q$. (5)

Let P and Q CCSP_t^θ processes with guarded recursion. Then $P \leftrightarrow Q \Leftrightarrow Ax' \vdash P = Q$. (6)

Let P and Q be recursion-free CCSP_t^θ processes. Then $P \leftrightarrow_r Q \Leftrightarrow Ax_f \vdash P = Q$. (7)

Let P and Q CCSP_t^θ processes with guarded recursion. Then $P \leftrightarrow_r Q \Leftrightarrow Ax \vdash P = Q$. (8)

Table 2: A complete axiomatisation of strong bisimilarity on guarded CCSP_t

$x + (y + z) = (x + y) + z$	$\tau_I(x + y) = \tau_I(x) + \tau_I(y)$	$\mathcal{R}(x + y) = \mathcal{R}(x) + \mathcal{R}(y)$
$x + y = y + x$	$\tau_I(\alpha.x) = \alpha.\tau_I(x)$ if $\alpha \notin I$	$\mathcal{R}(\tau.x) = \tau.\mathcal{R}(x)$
$x + x = x$	$\tau_I(\alpha.x) = \tau.\tau_I(x)$ if $\alpha \in I$	$\mathcal{R}(t.x) = t.\mathcal{R}(x)$
$x + 0 = 0$	$\langle x \mathcal{S} \rangle = \langle \mathcal{S}_x \mathcal{S} \rangle$ (RDP)	$\mathcal{R}(a.x) = \sum_{\{b (a,b) \in \mathcal{R}\}} b.\mathcal{R}(x)$
If $P = \sum_{i \in I} \alpha_i.P_i$ and $Q = \sum_{j \in J} \beta_j.Q_j$ then		
$P \parallel_S Q = \sum_{i \in I, \alpha_i \notin \mathcal{S}} (\alpha_i.P_i \parallel_S Q) + \sum_{j \in J, \beta_j \notin \mathcal{S}} (P \parallel_S \beta_j.Q_j) + \sum_{i \in I, j \in J, \alpha_i = \beta_j \in \mathcal{S}} \alpha_i.(P_i \parallel_S Q_j)$		
Recursive Specification Principle (RSP)		$\mathcal{S} \Rightarrow x = \langle x \mathcal{S} \rangle$ (\mathcal{S} guarded)

In each of these cases “ \Leftarrow ” states the soundness of the axiomatisation and “ \Rightarrow ” completeness.

Section 10.1 recalls (4), which stems from [19], and (3), which is folklore. Then Section 10.2 extends the existing proofs of (4) and (3) to obtain (6) and (5). In Section 10.3 I move from strong bisimilarity to strong reactive bisimilarity; I discuss the merits of the axiom RA from Table 4, and establish its soundness, thereby obtaining direction “ \Leftarrow ” of (8) and (7). I prove the completeness of Ax_f for recursion-free processes—direction “ \Rightarrow ” of (7)—in Section 10.4. Sections 10.5–10.7 deal with the completeness of Ax for guarded CCSP_t ^{θ} —direction “ \Rightarrow ” of (8). Section 10.8 explains why I need the axiom of choice for the latter result.

10.1 A complete axiomatisation of strong bisimilarity on guarded CCSP_t

The well-known axioms of Table 2 are *sound* for strong bisimilarity, meaning that writing \Leftrightarrow for =, and substituting arbitrary expressions for the free variables x, y, z , or the meta-variables P_i and Q_j , turns them into true statements. In these axioms α, β range over Act and a, b over A . All axioms involving variables are equations. The axiom involving P and Q is a template that stands for a family of equations, one for each fitting choice of P and Q . This is the CCSP_t version of the *expansion law* from [29]. The axiom RDP ($\langle x | \mathcal{S} \rangle = \langle \mathcal{S}_x | \mathcal{S} \rangle$) says that recursively defined processes $\langle x | \mathcal{S} \rangle$ satisfy their set of defining equations \mathcal{S} . As discussed in the previous section, this entails that each recursive specification has a solution. The axiom RSP [1, 8] is a conditional equation, with as antecedents the equations of a guarded recursive specification \mathcal{S} . It says that the x -component of any solution of \mathcal{S} —a vector of processes substituted for the variables V_S —equals $\langle x | \mathcal{S} \rangle$. In other words, each solution of \mathcal{S} equals the default solution. This is a compact way of saying that solutions of guarded recursive specifications are unique.

Theorem 28 For CCSP_t processes $P, Q \in \mathbb{P}$ with guarded recursion, one has $P \Leftrightarrow Q$, that is, P and Q are strongly bisimilar, iff $P = Q$ is derivable from the axioms of Table 2.

In this theorem, “if”, the *soundness* of the axiomatisation of Table 2, is an immediate consequence of the soundness of the individual axioms. “Only if” states the *completeness* of the axiomatisation.

A crucial tool in its proof is the simple observation that the axioms from the first box of Table 2 allow any CCSP_t process with guarded recursion to be brought in the form $\sum_{i \in I} \alpha_i.P_i$ —a *head normal form*. Using this, the rest of the proof is a standard argument employing RSP, independent of the choice of the specific process algebra. It can be found in [27, 29], [1], [8] and many other places. However, in the literature this completeness theorem was always stated and proved for a small fragment of the process

Table 3: A complete axiomatisation of strong bisimilarity on guarded CCSP_t^θ

$\theta_L^U(\sum_{i \in I} \alpha_i.x_i) = \sum_{i \in I} \alpha_i.x_i$	$(\alpha_i \notin L \cup \{\tau\} \text{ for all } i \in I)$
$\theta_L^U(x + \alpha.y + \beta.z) = \theta_L^U(x + \alpha.y)$	$(\alpha \in L \cup \{\tau\} \wedge \beta \notin U \cup \{\tau\})$
$\theta_L^U(x + \alpha.y + \beta.z) = \theta_L^U(x + \alpha.y) + \theta_L^U(\beta.z)$	$(\alpha \in L \cup \{\tau\} \wedge \beta \in U \cup \{\tau\})$
$\theta_L^U(\beta.x) = \beta.x$	$(\beta \neq \tau)$
$\theta_L^U(\tau.x) = \tau.\theta_L^U(x)$	
$\psi_X(x + \alpha.z) = \psi_X(x) + \alpha.z$	$(\alpha \notin X \cup \{\tau, t\})$
$\psi_X(x + \alpha.y + t.z) = \psi_X(x + \alpha.y)$	$(\alpha \in X \cup \{\tau\})$
$\psi_X(x + \alpha.y + \beta.z) = \psi_X(x + \alpha.y) + \beta.z$	$(\alpha, \beta \in X \cup \{\tau\})$
$\psi_X(\alpha.x) = \alpha.x$	$(\alpha \neq t)$
$\psi_X(\sum_{j \in I} t.y_j) = \sum_{j \in I} t.\psi_X(y_j)$	

algebra, allowing only guarded recursive specifications with a finite number of equations, and whose right-hand sides \mathcal{S}_y involve only the basic operators inaction, action prefixing and choice. Since the set of true statements $P \Leftrightarrow Q$, with P and Q processes in a process algebra like guarded CCSP_t , is well-known to be undecidable, and even not recursively enumerable, it was widely believed that no sound and complete axiomatisation of strong bisimilarity could exist. Only in March 2017, Kees Middelburg observed (in the setting of the process algebra ACP [1, 8]) that the standard proof applies almost verbatim to arbitrary processes with guarded recursion, although one has to be a bit careful in dealing with the infinite nature of recursive specifications. The argument has been carefully documented in [19], in the setting of the process algebra ACP. This result does not contradict the non-enumerability of the set of true statements $P \Leftrightarrow Q$, due to the fact that RSP is a proof rule with infinitely many premises.

A well-known simplification of Theorem 28 and its proof also yields completeness without recursion:

Theorem 29 For CCSP_t processes $P, Q \in \mathbb{P}$ without recursion, one has $P \Leftrightarrow Q$ iff $P = Q$ is derivable from the axioms of Table 2 minus RDP and RSP.

10.2 A complete axiomatisation of strong bisimilarity on guarded CCSP_t^θ

Table 3 extends Table 2 with axioms for the auxiliary operators θ_L^U and ψ_X . With Table 1 it is straightforward to check the soundness of these axioms. The fourth axiom, for instance, follows from the second or third rule for θ_L^U in Table 1, depending on whether $\beta \in L \cup \{t\}$. Moreover, a straightforward induction shows that these axioms suffice to convert each CCSP_t^θ process with guarded recursion into the form $\sum_{i \in I} \alpha_i.P_i$ —a head normal form. The below proposition sharpens this observation by pointing out that one can take the processes P_i for $i \in I$ to be exactly the ones that are reachable by one α_i -labelled transition from P .

Definition 30 Given a CCSP_t^θ process $P \in \mathbb{P}$, let $\widehat{P} := \sum_{\{(\alpha, Q) | P \xrightarrow{\alpha} Q\}} \alpha.Q$.

By Proposition 12, P is countably branching, so using Proposition 15 \widehat{P} is a valid CCSP_t^θ process. In case $P \in \mathbb{P}$ is a process with only guarded recursion, then P is finitely branching by Proposition 13, so also \widehat{P} is a valid CCSP_t^θ process with only guarded recursion.

Proposition 31 Let $P \in \mathbb{P}$ have guarded recursion only. Then $Ax' \vdash P = \widehat{P}$. The conditional equation RSP is not even needed here.

Proof: The proof is by induction on the measure $e(P)$, defined in the proof of Proposition 13.

Let $P = \langle x | \mathcal{S} \rangle$. Axiom RDP yields $Ax \vdash P = \langle x | \mathcal{S} \rangle = \langle \mathcal{S}_x | \mathcal{S} \rangle$. Moreover, $e(\langle \mathcal{S}_x | \mathcal{S} \rangle) < e(\langle x | \mathcal{S} \rangle)$. So by induction, $Ax \vdash \langle x | \mathcal{S} \rangle = \widehat{\langle x | \mathcal{S} \rangle}$. Moreover, $\{(\alpha, Q) \mid \langle \mathcal{S}_x | \mathcal{S} \rangle \xrightarrow{\alpha} Q\} = \{(\alpha, Q) \mid \langle x | \mathcal{S} \rangle \xrightarrow{\alpha} Q\}$, so $\widehat{\langle \mathcal{S}_x | \mathcal{S} \rangle} = \widehat{\langle x | \mathcal{S} \rangle} = \widehat{P}$. Thus $Ax \vdash P = \widehat{P}$.

Let $P = \theta_L^U(P')$. Using that $e(P') < e(P)$, by induction $Ax \vdash P' = \widehat{P'}$ so $Ax \vdash P = \theta_L^U(\widehat{P'})$. Let

$$\widehat{P'} = \sum_{h \in H} \tau.P_h + \sum_{i \in I} a_i.Q_i + \sum_{j \in J} b_j.R_j + \sum_{k \in K} \gamma_k.T_k,$$

where $a_i \in L$ for all $i \in I$, $b_j \in U \setminus L$ for all $j \in J$, and $\gamma_k \notin U \cup \{\tau\}$ for all $k \in K$. (So γ_k may be t .) In case $H \cup I = \emptyset$, one has $Ax \vdash P = \theta_L^U(\widehat{P'}) = \widehat{P'} = \widehat{P}$, using the first axiom for θ_L^U . Otherwise

$$Ax \vdash P = \sum_{h \in H} \tau.\theta_L^U(P_h) + \sum_{i \in I} a_i.Q_i + \sum_{j \in J} b_j.R_j$$

by the remaining four axioms for θ_L^U . The right-hand side is \widehat{P} .

The cases for the remaining operators are equally straightforward. \square

In the special case that P is a recursion-free process, also the axiom RDP is not needed for this result.

Once we have head normalisation, the proofs of Theorems 28 and 29 are independent of the precise syntax of the process algebra in question. Using Proposition 31 we immediately obtain (6) and (5):

Theorem 32 For CCSP_t^θ processes $P, Q \in \mathbb{P}$ with guarded recursion, one has $P \Leftrightarrow Q$ iff $P = Q$ is derivable from the axioms of Tables 2 and 3. \square

Theorem 33 For CCSP_t^θ processes $P, Q \in \mathbb{P}$ without recursion, one has $P \Leftrightarrow Q$ iff $P = Q$ is derivable from the axioms of Tables 2 and 3 minus RDP and RSP.

A law that turns out to be particularly useful in verifications modulo strong reactive bisimilarity is

$$\boxed{\theta_K^V(\theta_L^U(x)) \Leftrightarrow \theta_{K \cup L}^{V \cap U}(x) \quad \text{provided } U = V \text{ or } K = L \text{ or } K \subseteq L \subseteq U \subseteq V \text{ or } L \subseteq K \subseteq V \subseteq U \quad (\text{L1})}. \square$$

Note that the right-hand side only exists if $(K \cup L) \subseteq (V \cap U)$. This law is sound for strong bisimilarity, as demonstrated by the following proposition. Yet it is not needed to add it to Table 3, as all its closed instances are derivable. In fact, this is a consequence of the above completeness theorems.

Proposition 34 $\theta_K^V(\theta_L^U(P)) \Leftrightarrow \theta_{K \cup L}^{V \cap U}(P)$, provided $(K \cup L) \subseteq (V \cap U)$ and either $U = V$ or $K = L$ or $K \subseteq L \subseteq U \subseteq V$ or $L \subseteq K \subseteq V \subseteq U$.

Proof: For given $K, L, U, V \subseteq A$ with $(K \cup L) \subseteq (V \cap U)$ and either $U = V$ or $K = L$ or $K \subseteq L \subseteq U \subseteq V$ or $L \subseteq K \subseteq V \subseteq U$, let

$$\mathcal{B} := \text{Id} \cup \{(\theta_K^V(\theta_L^U(P)), \theta_{K \cup L}^{V \cap U}(P)) \mid P \in \mathbb{P}\}.$$

It suffices to show that the symmetric closure $\widetilde{\mathcal{B}}$ of \mathcal{B} is a strong bisimulation. So let $R \widetilde{\mathcal{B}} T$ and $R \xrightarrow{\alpha} R'$ with $\alpha \in A \cup \{\tau, t\}$. I have to find a T' with $T \xrightarrow{\alpha} T'$ and $R' \widetilde{\mathcal{B}} T'$.

- The case that $R = T$ is trivial.

Table 4: A complete axiomatisation of strong reactive bisimilarity on guarded CCSP_t^θ

$$\boxed{\frac{\psi_X(x) = \psi_X(y) \text{ for all } X \subseteq A}{x = y} \quad (\text{RA})}$$

- Let $R = \theta_K^V(\theta_L^U(P))$ and $T = \theta_{K \cup L}^{V \cap U}(P)$.
 First assume $\alpha = \tau$. Then $P \xrightarrow{\tau} P'$ for some P' such that $R' = \theta_K^V(\theta_L^U(P'))$.
 Hence $T = \theta_{K \cup L}^{V \cap U}(P) \xrightarrow{\tau} \theta_{K \cup L}^{V \cap U}(P') =: T'$, and $R' \mathcal{B} T'$.
 Now assume $\alpha \in A \cup \{t\}$. Then $\theta_L^U(P) \xrightarrow{\alpha} R'$ and either $\alpha \in V$ or $\theta_L^U(P) \not\xrightarrow{\beta}$ for all $\beta \in K \cup \{\tau\}$.
 Using that $K \subseteq U$, this implies that either $\alpha \in V$ or $P \not\xrightarrow{\beta}$ for all $\beta \in K \cup \{\tau\}$. Moreover,
 $P \xrightarrow{\alpha} R'$ and either $\alpha \in U$ or $P \not\xrightarrow{\beta}$ for all $\beta \in L \cup \{\tau\}$. It follows that either $\alpha \in V \cap U$ or
 $P \not\xrightarrow{\beta}$ for all $\beta \in K \cup L \cup \{\tau\}$. (Here I use that either $U = V$ or $K = L$ or $K \subseteq L \subseteq U \subseteq V$ or
 $L \subseteq K \subseteq V \subseteq U$.) Consequently, $T = \theta_{K \cup L}^{V \cap U}(P) \xrightarrow{\alpha} R'$.
- Let $R = \theta_{K \cup L}^{V \cap U}(P)$ and $T = \theta_K^V(\theta_L^U(P))$.
 First assume $\alpha = \tau$. Then $P \xrightarrow{\tau} P'$ for some P' such that $R' = \theta_{K \cup L}^{V \cap U}(P')$.
 Hence $T = \theta_K^V(\theta_L^U(P)) \xrightarrow{\tau} \theta_K^V(\theta_L^U(P')) =: T'$, and $R' \mathcal{B} T'$.
 Now assume $\alpha \in A \cup \{t\}$. Then $P \xrightarrow{\alpha} R'$ and either $\alpha \in V \cap U$ or $P \not\xrightarrow{\beta}$ for all $\beta \in K \cup L \cup \{\tau\}$.
 Consequently, $\theta_L^U(P) \xrightarrow{\alpha} R'$ and thus $T = \theta_K^V(\theta_L^U(P)) \xrightarrow{\alpha} R'$. \square

The side condition to L1 cannot be dropped, for $\theta_{\{c\}}^{\{a,c\}} \theta_{\emptyset}^{\{c\}}(a.0 + c.0) \xrightarrow{a} 0$, yet $\theta_{\{c\}}^{\{c\}}(a.0 + c.0) \not\xrightarrow{a}$.

10.3 A complete axiomatisation of strong reactive bisimilarity on guarded CCSP_t^θ

To obtain a sound and complete axiomatisation of strong reactive bisimilarity for CCSP_t^θ with guarded recursion, one needs to combine the axioms of Tables 2, 3 and 4. These axioms are useful only in combination with the full congruence property of strong reactive bisimilarity, Theorem 25. This is what allows us to apply these axioms within subexpressions of a given expression. Since $\Leftrightarrow \subseteq \Leftrightarrow_r$, the soundness of all equational axioms for strong reactive bisimilarity follows from their soundness for strong bisimilarity. The soundness of RSP has been established as Theorem 27. The soundness of RA, the *reactive approximation axiom*, is contributed by the following proposition.

Proposition 35 Let $P, Q \in \mathbb{P}$. If $\psi_X(P) \Leftrightarrow_r \psi_X(Q)$ for all $X \subseteq A$, then $P \Leftrightarrow_r Q$.

Proof: Given $P, Q \in \mathbb{P}$ with $\psi_X(P) \Leftrightarrow_r \psi_X(Q)$ for all $X \subseteq A$, I show that $\mathcal{B} := \Leftrightarrow_r \cup \{(P, Q), (Q, P)\}$ is a strong time-out bisimulation.

Let $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$. Take any $X \subseteq A$. Then $\psi_X(P) \xrightarrow{\alpha} P'$. Since $\psi_X(P) \Leftrightarrow_r \psi_X(Q)$, this implies $\psi_X(Q) \xrightarrow{\alpha} Q'$ for some Q' with $P' \Leftrightarrow_r Q'$, and hence $Q \xrightarrow{\alpha} Q'$.

Let $P \xrightarrow{t} P'$ and $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$. Then $\psi_X(P) \xrightarrow{t} \theta_X(P')$ and $\mathcal{I}(\psi_X(P)) \cap (X \cup \{\tau\}) = \emptyset$. Since $\psi_X(P) \Leftrightarrow_r \psi_X(Q)$, this implies $\psi_X(Q) \xrightarrow{t} Q''$ for some Q'' with $\theta_X(\theta_X(P')) \Leftrightarrow_r \theta_X(Q'')$. It must be that $Q \xrightarrow{t} Q'$ for some Q' with $Q'' = \theta_X(Q')$. By Proposition 34, $\theta_X(\theta_X(R)) \Leftrightarrow_r \theta_X(R)$ for all $R \in \mathbb{P}$. Thus $\theta_X(P') \Leftrightarrow_r \theta_X(\theta_X(P')) \Leftrightarrow_r \theta_X(\theta_X(Q')) \Leftrightarrow_r \theta_X(Q')$, which had to be shown. \square

At first sight it appears that axiom RA is not very handy, as the number of premises to verify is exponential in the size of the alphabet A of visible actions. In case A is infinite, there are even uncountably many premises. However, in practical verifications this is hardly an issue, as one uses a partition of the

premises into a small number of equivalence classes, each of which requires only one common proof. This technique will be illustrated on three examples below. Furthermore, one could calculate the set of visible actions $\mathcal{J}(P)$ of a process P that can be encountered as initial actions after one t -transition followed by a sequence of τ -transitions. For large classes of processes, $\mathcal{J}(P)$ will be a finite set. Now axiom RA can be modified by changing $X \subseteq A$ into $X \subseteq \mathcal{J}(P) \cup \mathcal{J}(Q)$. This preserves the soundness of the axiom, because only the actions in $\mathcal{J}(P)$ play any rôle in evaluating $\psi_X(P)$.

A crucial property of strong reactive bisimilarity was mentioned in the introduction:

$$\boxed{\tau.P + t.Q = \tau.P \quad (\text{L2})}$$

It is an immediate consequence of RA, since $\psi_X(\tau.P + t.Q) = \psi_X(\tau.P)$ for any $X \subseteq A$, by Table 3. Another useful law in verifications modulo strong reactive bisimilarity is

$$\boxed{\sum_{i \in I} a_i.x_i + t.y = \sum_{i \in I} a_i.x_i + t.\theta_\emptyset^{A \setminus I_n}(y), \text{ where } I_n = \{a_i \mid i \in I\}. \quad (\text{L3})}$$

Its soundness is intuitively obvious: the t -transition to y will be taken only in an environment X with $X \cap I_n = \emptyset$. Hence one can just as well restrict the behaviour of y to those transitions that are allowed in one such environment. This law was one of the prime reasons for extending the family of operators $\theta_X (= \theta_X^X)$, which were needed to establish the key theorems of this paper, to the larger family θ_\emptyset^U . Law L3 for finite I is effortlessly derivable from its simple instance

$$\boxed{a.x + t.y = a.x + t.\theta_\emptyset^{A \setminus \{a\}}(y). \quad (\text{L3'})}$$

in combination with L1. I now show how to derive L3 from RA. For this proof I need to partition the set of premises of RA in only two equivalence classes.

First let $X \cap I_n \neq \emptyset$. Then $\psi_X(\sum_{i \in I} a_i.x_i + t.y) = \sum_{i \in I} a_i.x_i = \psi_X(\sum_{i \in I} a_i.x_i + t.\theta_\emptyset^{A \setminus I_n}(y))$.

Next let $X \cap I_n = \emptyset$. Then $\psi_X(\sum_{i \in I} a_i.x_i + t.y) = \sum_{i \in I} a_i.x_i + t.\theta_X(y)$
 $= \sum_{i \in I} a_i.x_i + t.\theta_X(\theta_\emptyset^{A \setminus I_n}(y))$
 $= \psi_X(\sum_{i \in I} a_i.x_i + t.\theta_\emptyset^{A \setminus I_n}(y))$,

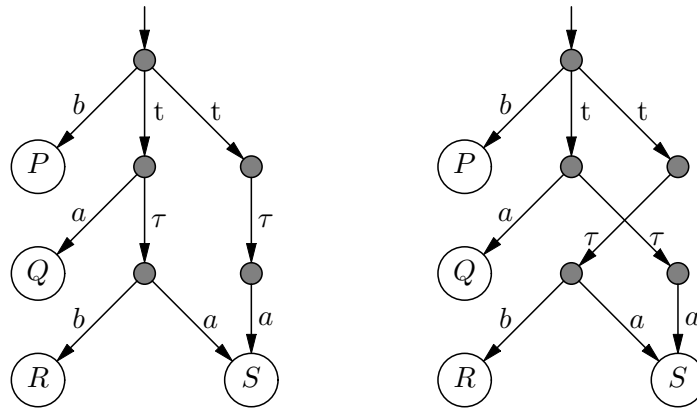
where the second step is an application of L1.

As an application of L3' one obtains the law from [16] that was justified in the introduction:

$$\begin{aligned} a.P + t.(Q + \tau.R + a.S) &= a.P + t.\theta_\emptyset^{A \setminus \{a\}}(Q + \tau.R + a.S) \\ &= a.P + t.\theta_\emptyset^{A \setminus \{a\}}(Q + \tau.R) \\ &= a.P + t.(Q + \tau.R). \end{aligned}$$

As a third illustration of the use of RA I derive an equational law that does not follow from L1, L2 and L3, namely

$$b.P + t.(a.Q + \tau.(b.R + a.S)) + t.\tau.a.S = b.P + t.(a.Q + \tau.a.S) + t.\tau.(b.R + a.S)$$



These systems are surely not strongly bisimilar. Moreover, L3 does not help in proving them equivalent, as applying $\theta_\emptyset^{A \setminus \{b\}}$ to any of the four targets of a t-transition does not kill any of the transitions of those processes. In particular, $\theta_\emptyset^{A \setminus \{b\}}(b.R + a.S) = b.R + a.S$. To derive this law from RA, I partition $\mathcal{P}(A)$ into three equivalence classes.

First let $b \in X$. Then $\psi_X(b.P + t.(a.Q + \tau.(b.R + a.S)) + t.\tau.a.S)$
 $= b.P$
 $= \psi_X(b.P + t.(a.Q + \tau.a.S) + t.\tau.(b.R + a.S)).$

Next let $b \notin X$ and $a \in X$. Then

$$\begin{aligned} & \psi_X(b.P + t.(a.Q + \tau.(b.R + a.S)) + t.\tau.a.S) \\ &= b.P + t.\theta_X(a.Q + \tau.(b.R + a.S)) + t.\theta_X(\tau.a.S) \\ &= b.P + t.(a.Q + \tau.\theta_X(b.R + a.S)) + t.\tau.\theta_X(a.S) \\ &= b.P + t.(a.Q + \tau.a.S) + t.\tau.a.S \\ &= b.P + t.(a.Q + \tau.\theta_X(a.S)) + t.\tau.\theta_X(b.R + a.S) \\ &= b.P + t.\theta_X(a.Q + \tau.a.S) + t.\theta_X(\tau.(b.R + a.S)) \\ &= \psi_X(b.P + t.(a.Q + \tau.a.S) + t.\tau.(b.R + a.S)). \end{aligned}$$

Finally let $a, b \notin X$. Then

$$\begin{aligned} & \psi_X(b.P + t.(a.Q + \tau.(b.R + a.S)) + t.\tau.a.S) \\ &= b.P + t.\theta_X(a.Q + \tau.(b.R + a.S)) + t.\theta_X(\tau.a.S) \\ &= b.P + t.\tau.\theta_X(b.R + a.S) + t.\tau.\theta_X(a.S) \\ &= b.P + t.\tau.(b.R + a.S) + t.\tau.a.S \\ &= b.P + t.\tau.a.S + t.\tau.(b.R + a.S) \\ &= b.P + t.\tau.\theta_X(a.S) + t.\tau.\theta_X(b.R + a.S) \\ &= b.P + t.\theta_X(a.Q + \tau.a.S) + t.\theta_X(\tau.(b.R + a.S)) \\ &= \psi_X(b.P + t.(a.Q + \tau.a.S) + t.\tau.(b.R + a.S)). \end{aligned}$$

In words, the 4 processes that are targets of t-transitions always run in an environment that blocks b . In an environment that allows a , the branch $b.R$ disappears, so that the left branch of the first process can be matched with the left branch of the second process, and similarly for the two right branches. In an environment that blocks a , this matching won't fly, as the branch $b.R$ now survives. However, the branches $a.Q$ will disappear, so that the left branch of the first process can be matched with the right branch of the second, and vice versa.

10.4 Completeness for finite processes

Theorem 36 Let P and Q be closed recursion-free CCSP_t^θ expressions. Then $P \dot{\simeq}_r Q \Rightarrow Ax_f \vdash P = Q$.

Proof: Let the *length* of a path $P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} P_n$ of a processes P be n . Let $d(P)$, the *depth* of P , be the length of its longest path; it is guaranteed to exist when P is a closed recursion-free CCSP_t^θ expression. I prove the theorem with induction on $\max(d(P), d(Q))$.

Suppose $P \dot{\simeq}_r Q$. By Proposition 31 one has $Ax_f \vdash P = \widehat{P}$ and $Ax_f \vdash Q = \widehat{Q}$. I will show that $Ax_f \vdash \psi_X(\widehat{P}) = \psi_X(\widehat{Q})$ for all $X \subseteq A$. This will suffice, as then Axiom RA yields $Ax_f \vdash \widehat{P} = \widehat{Q}$ and thus $Ax_f \vdash P = Q$. So pick $X \subseteq A$. Let

$$\widehat{P} = \sum_{i \in I} \alpha_i.P'_i + \sum_{j \in J} t.P''_j \quad \text{and} \quad \widehat{Q} = \sum_{k \in K} \beta_k.Q'_k + \sum_{h \in H} t.Q''_h$$

with $\alpha_j, \beta_k \in A \cup \{\tau\}$ for all $i \in I$ and $k \in K$. The following two claims are the crucial part of the proof.

Claim 1: For each $i \in I$ there is a $k \in K$ with $\alpha_i = \beta_k$ and $Ax_f \vdash P'_i = Q'_k$.

Claim 2: If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, then for each $j \in J$ there is a $h \in H$ with $Ax_f \vdash \theta_X(P''_j) = \theta_X(Q''_h)$.

With these claims, the rest of the proof is straightforward. Since $P \leftrightarrow_r Q$, one has $\mathcal{I}(P) = \mathcal{I}(\widehat{P}) = \{\alpha_i \mid i \in I\} = \{\beta_k \mid k \in K\} = \mathcal{I}(\widehat{Q}) = \mathcal{I}(Q)$. First suppose that $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$. Then

$$\psi_X(\widehat{P}) = \sum_{i \in I} \alpha_i.P'_i + \sum_{j \in J} t.\theta_X(P''_j) \quad \text{and} \quad \psi_X(\widehat{Q}) = \sum_{k \in K} \beta_k.Q'_k + \sum_{h \in H} t.\theta_X(Q''_h).$$

Claim 1 yields $Ax_f \vdash \psi_X(\widehat{Q}) = \psi_X(\widehat{Q}) + \alpha_i.P'_i$ for each $i \in I$. Likewise, Claim 2 yields $Ax_f \vdash \psi_X(\widehat{Q}) = \psi_X(\widehat{Q}) + t.\theta_X(P''_j)$ for each $j \in J$. Together this yields $Ax_f \vdash \psi_X(\widehat{Q}) = \psi_X(\widehat{Q}) + \psi_X(\widehat{P})$. By symmetry one obtains $Ax_f \vdash \psi_X(\widehat{P}) = \psi_X(\widehat{P}) + \psi_X(\widehat{Q})$ and thus $Ax_f \vdash \psi_X(\widehat{P}) = \psi_X(\widehat{Q})$.

Next suppose $\mathcal{I}(P) \cap (X \cup \{\tau\}) \neq \emptyset$. Then $\psi_X(\widehat{P}) = \sum_{i \in I} \alpha_i.P'_i$ and $\psi_X(\widehat{Q}) = \sum_{k \in K} \beta_k.Q'_k$. The proof proceeds just as above, but without the need for Claim 2.

Proof of Claim 1: Pick $i \in I$. Then $\widehat{P} \xrightarrow{\alpha_i} P'_i$. So $\widehat{Q} \xrightarrow{\alpha_i} Q'$ for some Q' with $P'_i \leftrightarrow_r Q'$. Hence there is a $k \in K$ with $\alpha_i = \beta_k$ and $Q' = Q'_k$. Using that $d(P'_i) < d(P)$ and $d(Q'_i) < d(Q)$, by induction $Ax_f \vdash P'_i = Q'_k$.

Proof of Claim 2: Pick $j \in J$. Then $\widehat{P} \xrightarrow{t} P''_j$. Since $\mathcal{I}(\widehat{P}) \cap (X \cup \{\tau\}) = \emptyset$, there is a Q'' such that $\widehat{Q} \xrightarrow{t} Q''$ and $\theta_X(P''_j) \leftrightarrow_r \theta_X(Q'')$. Hence there is a $h \in H$ with $Q'' = Q''_h$. Using that $d(\theta_X(P''_j)) \leq d(P''_j) < d(P)$ and $d(\theta_X(Q''_h)) \leq d(Q''_h) < d(Q)$, by induction $Ax_f \vdash \theta_X(P''_j) = \theta_X(Q''_h)$. \square

10.5 The method of canonical representatives

The classic technique of proving completeness axiomatisations for process algebras with recursion involves *merging guarded recursive equations* [27, 28, 35, 9, 26]. In essence it proves two bisimilar systems P and Q equivalent by equating both to an intermediate variant that is essentially a *product* of P and Q . I tried so hard, and in vain, to apply this technique to obtain (8), that I came to believe that it fundamentally does not work for this axiomatisation.

Therefore, I here employ the novel method of *canonical solutions* [25], which equates both P and Q to a canonical representative within the bisimulation equivalence class of P and Q —one that has only one reachable state for each bisimulation equivalence class of states of P and Q . Moreover, my proof employs the axiom of choice [36] in defining the transition relation on my canonical representative, in order to keep this process finitely branching.

10.6 The canonical representative

Let \mathbb{P}^g denote the set of CCSP_t^θ processes with guarded recursion. Let $[P] := \{Q \in \mathbb{P}^g \mid Q \leftrightarrow_r P\}$ be the strong reactive bisimulation equivalence class of a process $P \in \mathbb{P}^g$. Below, by “abstract process” I will mean such an equivalence class. Choose a function χ that selects an element out of each \leftrightarrow_r -equivalence class of CCSP_t^θ processes with guarded recursion—this is possible by the axiom of choice [36]. Define the transition relations $\xrightarrow{\alpha}$, for $\alpha \in \text{Act}$, between abstract processes by

$$R \xrightarrow{\alpha} R' \iff \exists P' \in R'. \chi(R) \xrightarrow{\alpha} P'. \quad (9)$$

I will show that $P \leftrightarrow_r [P]$ for all $P \in \mathbb{P}^g$. Formally, \leftrightarrow_r has been defined only between processes belonging to the same TSS \mathbb{P} , and here $[P] \notin \mathbb{P}$. However, this restriction is not material: two processes $P \in \mathbb{P}$ and $Q \in \mathbb{Q}$ from different LTSs can be compared by considering \leftrightarrow_r on the disjoint union $\mathbb{P} \uplus \mathbb{Q}$.

Lemma 8 Let $\alpha \in A \cup \{\tau\}$. Then $[P] \xrightarrow{\alpha} R'$ iff $P \xrightarrow{\alpha} P'$ for some P' with $R' = [P']$.

Proof: Let $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$. Since $P \leftrightarrow_r \chi([P])$, by Definition 8 there is a Q' such that $\chi([P]) \xrightarrow{\alpha} Q'$ and $P' \leftrightarrow_r Q'$. Hence $[P] \xrightarrow{\alpha} [Q']$ by (9). Moreover, $P' \in [Q'] = [P']$.

Let $[P] \xrightarrow{\alpha} R'$ with $\alpha \in A \cup \{\tau\}$. Then $\chi([P]) \xrightarrow{\alpha} Q'$ for some $Q' \in R'$. Since $\chi([P]) \leftrightarrow_r P$, there is a P' such that $P \xrightarrow{\alpha} P'$ and $Q' \leftrightarrow_r P'$. Hence $P' \in R'$ and thus $R' = [P']$. \square

Corollary 37 $\mathcal{I}([P]) = \mathcal{I}(P)$ for all $P \in \mathbb{P}^g$. \square

Lemma 9 If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then $[P] \xrightarrow{t} [Q']$ for a Q' with $\theta_X(P') \leftrightarrow_r \theta_X(Q')$. Moreover, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $[P] \xrightarrow{t} [Q']$ then $P \xrightarrow{t} P'$ for a P' with $\theta_X(Q') \leftrightarrow_r \theta_X(P')$.

Proof: Let $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. Since $P \leftrightarrow_r \chi([P])$, by Definition 8 there is a Q' such that $\chi([P]) \xrightarrow{t} Q'$ and $\theta_X(P') \leftrightarrow_r \theta_X(Q')$. Hence $[P] \xrightarrow{t} [Q']$ by (9).

Let $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $[P] \xrightarrow{t} [Q']$. Then $\chi([P]) \xrightarrow{t} R'$ for some $R' \in [Q']$. Since $\chi([P]) \leftrightarrow_r P$ (so $\mathcal{I}(\chi([P])) = \mathcal{I}(P)$), there is a P' such that $P \xrightarrow{t} P'$ and $\theta_X(R') \leftrightarrow_r \theta_X(P')$. As \leftrightarrow_r is a congruence for θ_X , one has $\theta_X(Q') \leftrightarrow_r \theta_X(R')$, and thus $\theta_X(Q') \leftrightarrow_r \theta_X(P')$. \square

Definition 38 A *strong time-out bisimulation up to reflexivity and transitivity* is a symmetric relation $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$, such that, for $P \mathcal{B} Q$,

- if $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$, then $\exists Q'$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \mathcal{B}^* Q'$,
- if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, then $\exists Q'$ with $Q \xrightarrow{t} Q'$ and $\theta_X(P') \mathcal{B}^* \theta_X(Q')$.

Here $\mathcal{B}^* := \{(R, T) \mid \exists n \geq 0. \exists R_0, \dots, R_n. R = R_0 \mathcal{B} R_1 \mathcal{B} \dots \mathcal{B} R_n = T\}$.

Proposition 39 If $P \mathcal{B} Q$ for a strong time-out bisimulation \mathcal{B} up to refl. and trans., then $P \leftrightarrow_r Q$.

Proof: It suffices to show that \mathcal{B}^* is a strong time-out bisimulation. Clearly this relation is symmetric.

- Suppose $R_0 \mathcal{B} R_1 \mathcal{B} \dots \mathcal{B} R_n$ for some $n \geq 0$ and $R_0 \xrightarrow{\alpha} R'_0$ with $\alpha \in A \cup \{\tau\}$. I have to find an R'_n such that $R_n \xrightarrow{\alpha} R'_n$ and $R'_0 \mathcal{B}^* R'_n$. I proceed with induction on n . The case $n = 0$ is trivial. Fixing an $n > 0$, by Definition 38 there is an R'_1 such that $R_1 \xrightarrow{\alpha} R'_1$ and $R'_0 \mathcal{B}^* R'_1$. Now by induction there is an R'_n such that $R_n \xrightarrow{\alpha} R'_n$ and $R'_1 \mathcal{B}^* R'_n$. Hence $R'_0 \mathcal{B}^* R'_n$.
- Suppose $R_0 \mathcal{B} R_1 \mathcal{B} \dots \mathcal{B} R_n$ for some $n \geq 0$, $\mathcal{I}(R_0) \cap (X \cup \{\tau\}) = \emptyset$ and $R_0 \xrightarrow{t} R'_0$. By Definition 38 $\mathcal{I}(R_0) = \mathcal{I}(R_1) = \dots = \mathcal{I}(R_n)$. I have to find an R'_n such that $R_n \xrightarrow{t} R'_n$ and $\theta_X(R'_0) \mathcal{B}^* \theta_X(R'_n)$. This proceeds exactly as for the case above. \square

Lemma 10 $\theta_X([P]) \leftrightarrow_r [\theta_X(P)]$ for all $P \in \mathbb{P}^g$ and $X \subseteq A$.

Proof: I show that the symmetric closure of $\mathcal{B} := \{(\theta_X([P]), [\theta_X(P)]) \mid P \in \mathbb{P}^g \wedge X \subseteq A\}$ is a strong time-out bisimulation up to reflexivity and transitivity.

- Let $\theta_X([P]) \xrightarrow{\tau} R'$. Then $[P] \xrightarrow{\tau} Q'$ for some Q' with $R' = \theta_X(Q')$. By Lemma 8, $P \xrightarrow{\tau} P'$ for some P' with $Q' = [P']$. Hence $\theta_X(P) \xrightarrow{\tau} \theta_X(P')$ and thus $[\theta_X(P)] \xrightarrow{\tau} [\theta_X(P')]$ by Lemma 8. Moreover, $R' = \theta_X([P']) \mathcal{B} [\theta_X(P')]$.
- Let $[\theta_X(P)] \xrightarrow{\tau} R'$. By Lemma 8, $\theta_X(P) \xrightarrow{\tau} Q'$ for some Q' with $R' = [Q']$. Thus $P \xrightarrow{\tau} P'$ for some P' with $Q' = \theta_X(P')$. Now $[P] \xrightarrow{\tau} [P']$ by Lemma 8, and thus $\theta_X([P]) \xrightarrow{\tau} \theta_X([P'])$. Moreover, $R' = [\theta_X(P')] \mathcal{B}^{-1} \theta_X([P'])$.

- Let $\theta_X([P]) \xrightarrow{a} R'$ with $a \in A$. Then $[P] \xrightarrow{a} R'$ and either $a \in \mathcal{I}([P])$ or $[P] \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Thus either $a \in \mathcal{I}(P)$ or $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$, using Corollary 37. By Lemma 8, $P \xrightarrow{a} P'$ for some P' with $R' = [P']$. Hence $\theta_X(P) \xrightarrow{a} P'$ and thus $[\theta_X(P)] \xrightarrow{a} [P'] = R'$.
- Let $[\theta_X(P)] \xrightarrow{a} R'$ with $a \in A$. By Lemma 8, $\theta_X(P) \xrightarrow{a} P'$ for some P' with $R' = [P']$. Thus $P \xrightarrow{a} P'$ and either $a \in \mathcal{I}(P)$ or $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Therefore either $a \in \mathcal{I}([P])$ or $[P] \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$, using Corollary 37. Moreover, $[P] \xrightarrow{a} [P']$ by Lemma 8. It follows that $\theta_X([P]) \xrightarrow{a} [P'] = R'$.
- Let $\mathcal{I}(\theta_X([P])) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X([P]) \xrightarrow{t} R' = [Q']$. Then $[P] \xrightarrow{t} [Q']$ and $[P] \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. Thus $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$, using Corollary 37, so $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathcal{I}(\theta_X(P)) \cap (X \cup \{\tau\}) = \emptyset$. By Lemma 9, $P \xrightarrow{t} P'$ for some P' with $\theta_X(Q') \dot{\leftrightarrow}_r \theta_X(P')$. Hence $\theta_X(P) \xrightarrow{t} P'$ and thus, again applying Lemma 9, $[\theta_X(P)] \xrightarrow{t} [T']$ for some T' with $\theta_X(P') \dot{\leftrightarrow}_r \theta_X(T')$. Moreover, $\theta_X(R') = \theta_X([Q']) \mathcal{B} [\theta_X(Q')] = [\theta_X(T')] \mathcal{B}^{-1} \theta_X([T'])$.
- Let $\mathcal{I}([\theta_X(P)]) \cap (X \cup \{\tau\}) = \emptyset$ and $[\theta_X(P)] \xrightarrow{t} [Q']$. By Lemma 9, $\theta_X(P) \xrightarrow{t} P'$ for a P' with $\theta_X(Q') \dot{\leftrightarrow}_r \theta_X(P')$. Hence $P \xrightarrow{t} P'$ and $P \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$, so $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$. Hence, by Lemma 9, $[P] \xrightarrow{t} [T']$ for a T' with $\theta_X(P') \dot{\leftrightarrow}_r \theta_X(T')$. By Corollary 37, $[P] \xrightarrow{\beta} \text{fail}$ for all $\beta \in X \cup \{\tau\}$. So $\theta_X([P]) \xrightarrow{t} [T']$. Moreover, $\theta_X([Q']) \mathcal{B} [\theta_X(Q')] = [\theta_X(T')] \mathcal{B}^{-1} \theta_X([T'])$. \square

Proposition 40 $P \dot{\leftrightarrow}_r [P]$ for all $P \in \mathbb{P}^g$.

Proof: Using Proposition 24, I show that the symmetric closure of the relation $\mathcal{B} := \{(P, [P]) \mid P \in \mathbb{P}^g\}$ is a strong time-out bisimulation up to $\dot{\leftrightarrow}_r$. Here the right-hand side processes come from an LTS that is closed under θ and contains the processes $[P]$ for $P \in \mathbb{P}^g$.

- Let $P \xrightarrow{\alpha} P'$ with $\alpha \in A \cup \{\tau\}$. Then $[P] \xrightarrow{\alpha} [P']$ by Lemma 8, and $P' \mathcal{B} [P']$.
- Let $[P] \xrightarrow{\alpha} R'$ with $\alpha \in A \cup \{\tau\}$. Then, by Lemma 8, $P \xrightarrow{\alpha} P'$ for some P' with $R' = [P']$. Moreover, $R' \mathcal{B}^{-1} P'$.
- Let $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. By Lemma 9, $[P] \xrightarrow{t} [Q']$ for some Q' such that $\theta_X(P') \dot{\leftrightarrow}_r \theta_X(Q')$. Moreover, using Lemma 10, $\theta_X(P') \mathcal{B} [\theta_X(P')] = [\theta_X(Q')] \dot{\leftrightarrow}_r \theta_X([Q'])$.
- Let $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$ and $[P] \xrightarrow{t} [Q']$. Then, by Lemma 9, $P \xrightarrow{t} P'$ for some P' with $\theta_X(Q') \dot{\leftrightarrow}_r \theta_X(P')$. By Lemma 10, $\theta_X([Q']) \dot{\leftrightarrow}_r [\theta_X(Q')] = [\theta_X(P')] \mathcal{B}^{-1} \theta_X([P'])$. \square

By Proposition 13 each $P \in \mathbb{P}^g$ is finitely branching. By construction, so is $[P]$.

No two states reachable from $[P]$ are strongly reactive bisimilar. Hence the process $[P]$ with its above-generated transition relation can be seen as a version of P where each equivalence class of reachable states is collapsed into a single state—a kind of minimisation. But it is not exactly a minimisation, as not all states reachable from $[P]$ need be strongly reactive bisimilar with reachable states of P .

10.7 Completeness for finitely branching processes

I will now give a syntactic representation of each process $[P]$, for $P \in \mathbb{P}^g$, as a CCSP_t^θ process with guarded recursion. Take a different variable x_R for each $\dot{\leftrightarrow}_r$ -equivalence class R of CCSP_t^θ processes with guarded recursion. Let V_S be the set of all those variables, and define the recursive specification \mathcal{S} by

$$x_R = \sum_{R \xrightarrow{\alpha} R'} \alpha \cdot x_{R'} .$$

By construction, $R \dot{\leftrightarrow} \langle x_R | \mathcal{S} \rangle$, that is, the process $[P] := \langle x_{[P]} | \mathcal{S} \rangle \in \mathbb{P}^g$ is strongly bisimilar to $[P]$. In fact, the symmetric closure of the relation $\{([P], [P]) \mid P \in \mathbb{P}^g\}$ is a strong bisimulation. Thus, $[P]$ serves as a normal form within the $\dot{\leftrightarrow}_r$ -equivalence class of $P \in \mathbb{P}^g$.

The above construction will not work when there are not as many variables as equivalence classes of CCSP_t^θ processes with guarded recursion. Note that each real number in the interval $[0, 1)$ can be represented as an infinite sequence of 0s and 1s, and thus as a CCSP_t^θ processes with guarded recursion employing the finite alphabet $A = \{0, 1\}$. Hence there are uncountably many equivalence classes of CCSP_t^θ processes with guarded recursion.

To solve this problem, one starts here already with the proof of (8), and fixes two process P_0 and $Q_0 \in \mathbb{P}^g$ with $P_0 \dot{\leftrightarrow}_r Q_0$. The task is to prove $Ax \vdash P_0 = Q_0$. Now call an equivalence class R of CCSP_t^θ processes with guarded recursion *relevant* if either R is reachable from $[P_0] = [Q_0]$, or a member of R is reachable from P_0 or Q_0 . There are only countably many relevant equivalence classes. It suffices to take a variable x_R only for relevant R . Below, I will call a process $P \in \mathbb{P}^g$ *relevant* if it is a member of a relevant equivalence class; in case we had enough variables to start with, all processes $P \in \mathbb{P}^g$ may be called relevant.

Lemma 11 Let $P, Q \in \mathbb{P}^g$ be relevant. Then $[P] \dot{\leftrightarrow}_r [Q] \Rightarrow [P] = [Q]$.

Proof: Suppose $[P] \dot{\leftrightarrow}_r [Q]$. Then $P \dot{\leftrightarrow}_r [P] \dot{\leftrightarrow}_r [Q] \dot{\leftrightarrow}_r Q$, so $[P] = [Q]$, and hence $[P] = [Q]$. \square

Lemma 12 Let $P, Q \in \mathbb{P}^g$ be relevant. Then $\theta_X([P]) \dot{\leftrightarrow}_r \theta_X([Q]) \Rightarrow \theta_X([P]) \dot{\leftrightarrow} \theta_X([Q])$.

Proof: I show that $\mathcal{B} := \text{Id} \cup \{(\theta_X([P]), \theta_X([Q])) \mid \theta_X([P]) \dot{\leftrightarrow}_r \theta_X([Q])\}$ is a strong bisimulation.

Suppose $\theta_X([P]) \dot{\leftrightarrow}_r \theta_X([Q])$. Then $[P] \dot{\dashv} \beta$ for all $\beta \in X \cup \{\tau\}$ iff $[Q] \dot{\dashv} \beta$ for all $\beta \in X \cup \{\tau\}$, since $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \mathcal{I}(\theta_X([P])) \cap (X \cup \{\tau\}) = \mathcal{I}(\theta_X([Q])) \cap (X \cup \{\tau\}) = \mathcal{I}([Q]) \cap (X \cup \{\tau\})$.

First consider the case that $[P] \dot{\dashv} \beta$ for all $\beta \in X \cup \{\tau\}$. Then $\theta_X([P]) \dot{\leftrightarrow} [P]$ and $\theta_X([Q]) \dot{\leftrightarrow} [Q]$. Hence $[P] \dot{\leftrightarrow} \theta_X([P]) \dot{\leftrightarrow}_r \theta_X([Q]) \dot{\leftrightarrow} [Q]$. So by Lemma 11, $[P] = [Q]$, and thus $\theta_X([P]) \dot{\leftrightarrow} \theta_X([Q])$. Henceforth I suppose that $[P] \dot{\dashv} \beta$ for some $\beta \in X \cup \{\tau\}$. So $[P] \dot{\dashv} \beta$ and $[Q] \dot{\dashv} \beta$.

- Let $\theta_X([P]) \xrightarrow{a} P''$ with $a \in A$. Then $\theta_X([Q]) \xrightarrow{a} Q''$ for some Q'' with $P'' \dot{\leftrightarrow}_r Q''$. One has $[P] \xrightarrow{a} P''$ and $[Q] \xrightarrow{a} Q''$. The process P'' must have the form $[P']$, and likewise $Q'' = [Q']$. Since $[P'] \dot{\leftrightarrow}_r [Q']$, Lemma 11 yields $[P'] = [Q']$.
- Let $\theta_X([P]) \xrightarrow{\tau} P''$. Then $\theta_X([Q]) \xrightarrow{\tau} Q''$ for some Q'' with $P'' \dot{\leftrightarrow}_r Q''$. The process P'' must have the form $\theta_X([P'])$, and likewise $Q'' = \theta_X([Q'])$. Hence $P'' \mathcal{B} Q''$. \square

Definition 41 Given a relevant CCSP_t^θ process $P \in \mathbb{P}^g$, let $\tilde{P} := \sum_{\{(\alpha, Q) \mid P \xrightarrow{\alpha} Q\}} \alpha.[Q]$.

Thus, \tilde{P} is defined like the head-normal form \hat{P} of $P \in \mathbb{P}^g$, except that all processes Q reachable from P by performing one transitions, are replaced by the normal form within their $\dot{\leftrightarrow}_r$ -equivalence class. So $P \dot{\leftrightarrow} \hat{P} \dot{\leftrightarrow}_r \tilde{P}$. Note that $[P] = \chi([P])$ is provable through a single application of the axiom RDP.

The following step is the only one where the reactive approximation axiom (RA) is used.

Proposition 42 Let $P, Q \in \mathbb{P}^g$ be relevant. Then $P \dot{\leftrightarrow}_r Q \Rightarrow Ax \vdash \tilde{P} = \tilde{Q}$.

Proof: Suppose $P \dot{\leftrightarrow}_r Q$. Then $\tilde{P} \dot{\leftrightarrow}_r P \dot{\leftrightarrow}_r Q \dot{\leftrightarrow}_r \tilde{Q}$. With Axiom RA it suffices to show that $Ax_f \vdash \psi_X(\tilde{P}) = \psi_X(\tilde{Q})$ for all $X \subseteq A$. So pick $X \subseteq A$. Let

$$\tilde{P} = \sum_{i \in I} \alpha_i.P'_i + \sum_{j \in J} t.P''_j \quad \text{and} \quad \tilde{Q} = \sum_{k \in K} \beta_k.Q'_k + \sum_{h \in H} t.Q''_h$$

with $\alpha_j, \beta_k \in A \cup \{\tau\}$ for all $i \in I$ and $k \in K$. As for Theorem 36, the following two claims are crucial.

Claim 1: For each $i \in I$ there is a $k \in K$ with $\alpha_i = \beta_k$ and $Ax \vdash P'_i = Q'_k$.

Claim 2: If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, then for each $j \in J$ there is a $h \in H$ with $Ax \vdash \theta_X(P''_j) = \theta_X(Q''_h)$.

With these claims the proof proceeds exactly as the one of Theorem 36.

Proof of Claim 1: Pick $i \in I$. Then $\tilde{P} \xrightarrow{\alpha_i} P'_i$. So $\tilde{Q} \xrightarrow{\alpha_i} Q'$ for some Q' with $P'_i \xleftrightarrow{r} Q'$. Hence there is a $k \in K$ with $\alpha_i = \beta_k$ and $Q' = Q'_k$. The processes P'_i and Q'_k must have the form $[P']$ and $[Q']$ for some $P', Q' \in \mathbb{P}^g$. Hence, by Lemma 11, $P'_i = Q'_k$, and thus certainly $Ax \vdash P'_i = Q'_k$.

Proof of Claim 2: Pick $j \in J$. Then $\tilde{P} \xrightarrow{t} P''_j$. Since $\mathcal{I}(\tilde{P}) \cap (X \cup \{\tau\}) = \emptyset$, there is a Q'' such that $\tilde{Q} \xrightarrow{t} Q''$ and $\theta_X(P''_j) \xleftrightarrow{r} \theta_X(Q'')$. Hence there is a $h \in H$ with $Q'' = Q''_h$. The processes P''_j and Q''_h have the form $[P'']$ and $[Q'']$ for some $P'', Q'' \in \mathbb{P}^g$. So by Lemma 12, $\theta_X(P''_j) \xleftrightarrow{r} \theta_X(Q'')$. The completeness of Ax for strong bisimilarity (Theorem 32) now yields $Ax \vdash \theta_X(P''_j) = \theta_X(Q'')$. \square

Theorem 43 Let $P \in \mathbb{P}^g$ be relevant. Then $Ax \vdash P = [P]$.

Proof: Let $reach(P)$ be the set of processes reachable from P . Take a different variable z_R for each $R \in reach(P)$, and define the recursive specification S' by $V_{S'} := \{z_R \mid R \in reach(P)\}$ and

$$z_R = \sum_{R \xrightarrow{\alpha} R'} \alpha.z_{R'}.$$

By construction, $R \xleftrightarrow{r} \langle x_R | S' \rangle$. In fact, the symmetric closure of $\{(R, \langle x_R | S' \rangle) \mid R \in reach(P)\}$ is a strong bisimulation. To establish Theorem 43 through an application of RSP, I show that both P and $[P]$ are x_P -components of solutions of S' . So I show

$$Ax \vdash R = \sum_{R \xrightarrow{\alpha} R'} \alpha.R' \quad \text{and} \quad Ax \vdash [R] = \sum_{R \xrightarrow{\alpha} R'} \alpha.[R']$$

for all $R \in reach(P)$. The first of these statements is a direct application of Proposition 31. The second statement can be reformulated as $Ax \vdash [R] = \tilde{R}$. As remarked above, $Ax \vdash [R] = \widetilde{\chi([R])}$ through a single application of RDP. Hence I need to show that $Ax \vdash \widetilde{\chi([R])} = \tilde{R}$. Considering that $\chi([R]) \xleftrightarrow{r} R$, this is a consequence of Proposition 42. \square

Corollary 44 Let $P, Q \in \mathbb{P}^g$ be relevant. Then $P \xleftrightarrow{r} Q \Rightarrow Ax \vdash P = Q$.

Proof: Let $P \xleftrightarrow{r} Q$. Then $[P] = [Q]$ by Lemma 11, so $Ax \vdash P = [P] = [Q] = Q$. \square

10.8 Necessity of the axiom of choice

At first glance it may look like the above proof can be simplified so as to avoid calling the axiom of choice, namely by changing (9) into

$$R \xrightarrow{\alpha} R' \Leftrightarrow \exists P \in R, P' \in R'. P \xrightarrow{\alpha} P'.$$

However, this would make some processes $[P]$ infinitely branching, even when P is finitely branching. Figure 10.8 shows an uncountable collection of strongly reactive bisimilar finitely branching processes. Here each pair of a dashed b -transition and the dotted one right below it constitutes a design choice: either the dashed or the dotted b -transition is present, but not both. Since there is this binary choice for infinitely

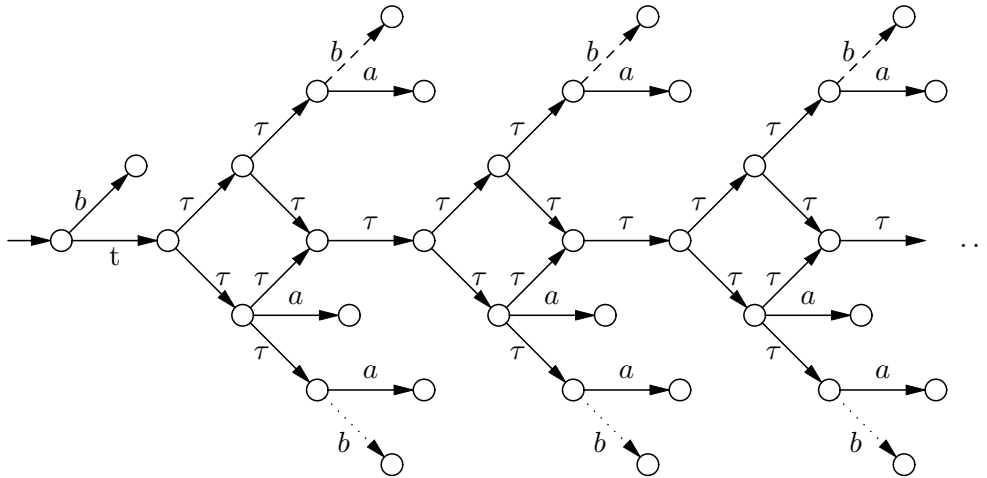


Figure 1: An uncountable variety of strongly reactive bisimilar processes

many pairs of b -transitions, this figure represents an uncountable collection of processes. All of them are strongly reactive bisimilar, because the t -transition will only be taken in an environment that blocks b . In case a is blocked as well, all the a -transitions from a state with an outgoing τ -transition can be dropped, and the difference between these processes disappears. In case a is allowed by the environment, all b transitions can be dropped, and again the difference between these processes disappears. Hence the above alternative definition would yield uncountable many outgoing t -transitions from the equivalence class of all these processes. This would make it impossible to represent such a “minimised” process in CCSP_t^θ .

11 Concluding remarks

This paper laid the foundations of the proper analogue of strong bisimulation semantics for a process algebra with time-outs. This makes it possible to specify systems in this setting and verify their correctness properties. The addition of time-outs comes with considerable gains in expressive power. An illustration of this is mutual exclusion.

As shown in [17], it is fundamentally impossible to correctly specify mutual exclusion protocols in standard process algebras, such as CCS [29], CSP [4, 24], ACP [1, 8] or CCSP, unless the correctness of the specified protocol hinges on a fairness assumption. The latter, in the view of [17], does not provide an adequate solution, as fairness assumptions are in many situations unwarranted and lead to false conclusions. In [7] a correct process-algebraic rendering of mutual exclusion is given, but only after making two important modifications to standard process algebra. The first involves making a justness assumption. Here *justness* [18] is an alternative to fairness, in some sense a much weaker form of fairness—meaning weaker than weak fairness. Unlike (strong or weak) fairness, its use typically is warranted and does not lead to false conclusions. The second modification is the addition of a new construct—*signals*—to CCS, or any other standard process algebra. Interestingly, both modifications are necessary; just using justness, or just adding signals, is insufficient. Bouwman [2, 3] points out that since the justness requirement was fairly new, and needed to be carefully defined to describe its interaction with signals anyway, it is possible to specify mutual exclusion without adding signals to the language

at all, instead reformulating the justness requirement in such a way that it effectively turns some actions into signals. Yet justness is essential in all these approaches. This may be seen as problematic, because large parts of the foundations of process algebra are incompatible with justness, and hence need to be thoroughly reformulated in a justness-friendly way. This is pointed out in [15].

The addition of time-outs to standard process algebra makes it possible to specify mutual exclusion without assuming justness! Instead, one should make the assumption called *progress* in [18], which is weaker than justness, uncontroversial, unproblematic, and made (explicitly or implicitly) in virtually all papers dealing with issues like mutual exclusion. I will defer substantiation of this claim to a future occasion.

Besides applications to protocol verification, future work includes adapting the work done here to a form of reactive bisimilarity that abstracts from hidden actions, that is, to provide a counterpart for process algebras with time-outs of, for instance, branching bisimilarity [20], weak bisimilarity [29] or coupled similarity [32, 10]. Other topics worth exploring are the extension to probabilistic processes, and especially the relations with timed process algebras. Davies & Schneider in [5], for instance, added a construct with a quantified time-out to the process algebra CSP [4, 24], elaborating the timed model of CSP presented by Reed & Roscoe in [33].

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I Initials Congruence

This appendix contains the proofs of two facts about initials equivalence I need in this paper, namely that it is a full congruence for CCSP_t^θ , and that it is not affected by which processes are substituted for variables whose free occurrences are guarded.

Theorem 18 Initials equivalence is a full congruence for CCSP_t^θ .

Proof: Let $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$ be the smallest relation satisfying

- if \mathcal{S} and \mathcal{S}' are recursive specifications with $x \in V_{\mathcal{S}} = V_{\mathcal{S}'}$ and $\langle x|\mathcal{S}\rangle, \langle x|\mathcal{S}'\rangle \in \mathbb{P}$, such that $\mathcal{S}_y =_{\mathcal{I}} \mathcal{S}'_y$ for all $y \in V_{\mathcal{S}}$, then $\langle x|\mathcal{S}\rangle \mathcal{B} \langle x|\mathcal{S}'\rangle$,
- if $P =_{\mathcal{I}} Q$, then $P \mathcal{B} Q$,
- if $P \mathcal{B} Q$ and $\alpha \in A \cup \{\tau, t\}$, then $\alpha.P \mathcal{B} \alpha.Q$,
- if $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$, then $P_1 + P_2 \mathcal{B} Q_1 + Q_2$,
- if $P_1 \mathcal{B} Q_1$, $P_2 \mathcal{B} Q_2$ and $S \subseteq A$, then $P_1 \parallel_S P_2 \mathcal{B} Q_1 \parallel_S Q_2$,
- if $P \mathcal{B} Q$ and $I \subseteq A$, then $\tau_I(P) \mathcal{B} \tau_I(Q)$,
- if $P \mathcal{B} Q$ and $\mathcal{R} \subseteq A \times A$, then $\mathcal{R}(P) \mathcal{B} \mathcal{R}(Q)$,
- if $P \mathcal{B} Q$, $L \subseteq U \subseteq A$ and $X \subseteq A$, then $\theta_L^U(P) \mathcal{B} \theta_L^U(Q)$ and $\psi_X(P) \mathcal{B} \psi_X(Q)$,
- if \mathcal{S} is a recursive specification with $z \in V_{\mathcal{S}}$, and $\rho, \nu : \text{Var} \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$ are substitutions satisfying $\rho(x) \mathcal{B} \nu(x)$ for all $x \in \text{Var} \setminus V_{\mathcal{S}}$, then $\langle z|\mathcal{S}\rangle[\rho] \mathcal{B} \langle z|\mathcal{S}\rangle[\nu]$.

A trivial structural induction on $E \in \mathbb{E}$ (not using the first two clauses) shows that

$$\text{if } \rho, \nu : \text{Var} \rightarrow \mathbb{P} \text{ satisfy } \rho(x) \mathcal{B} \nu(x) \text{ for all } x \in \text{Var}, \text{ then } E[\rho] \mathcal{B} E[\nu]. \quad (*)$$

For \mathcal{S} a recursive specification and $\rho : \text{Var} \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$, let $\rho_{\mathcal{S}} : \text{Var} \rightarrow \mathbb{P}$ be the closed substitution given by $\rho_{\mathcal{S}}(x) := \langle x|\mathcal{S}\rangle[\rho]$ if $x \in V_{\mathcal{S}}$ and $\rho_{\mathcal{S}}(x) := \rho(x)$ otherwise. Then $\langle E|\mathcal{S}\rangle[\rho] = E[\rho_{\mathcal{S}}]$ for all $E \in \mathbb{E}$. Hence an application of (*) with $\rho_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ yields that under the conditions of the last clause for \mathcal{B} above one even has $\langle E|\mathcal{S}\rangle[\rho] \mathcal{B} \langle E|\mathcal{S}\rangle[\nu]$ for all expressions $E \in \mathbb{E}$, (\$)

and likewise, in the first clause, $\langle E|\mathcal{S}\rangle \mathcal{B} \langle E|\mathcal{S}'\rangle$ for all $E \in \mathbb{E}$ with variables from $V_{\mathcal{S}}$. (#)

It suffices to show that $P \mathcal{B} Q \Rightarrow P =_{\mathcal{I}} Q$, because then $\mathcal{B} = =_{\mathcal{I}}$, and (*) implies that \mathcal{B} is a lean congruence. Moreover, the clauses for \mathcal{B} (not needing the last) then imply that $=_{\mathcal{I}}$ is a full congruence. This I will do by induction on the *stratum* (λ_R, κ_R) of processes $R \in \mathbb{P}$, as defined in Section 5. So pick a stratum (λ, κ) and assume that $P' \mathcal{B} Q' \Rightarrow P' =_{\mathcal{I}} Q'$ for all $P', Q' \in \mathbb{P}$ with $(\lambda_P, \kappa_P) < (\lambda, \kappa)$ and $(\lambda_Q, \kappa_Q) < (\lambda, \kappa)$. I need to show that $P \mathcal{B} Q \Rightarrow P =_{\mathcal{I}} Q$ for all $P, Q \in \mathbb{P}$ with $(\lambda_P, \kappa_P) \leq (\lambda, \kappa)$ and $(\lambda_Q, \kappa_Q) \leq (\lambda, \kappa)$.

Because $=_{\mathcal{I}}$ is symmetric, so is \mathcal{B} . Hence, it suffices to show that $P \mathcal{B} Q \wedge P \xrightarrow{\alpha} \Rightarrow Q \xrightarrow{\alpha}$ for all $P, Q \in \mathbb{P}$ with $(\lambda_P, \kappa_P), (\lambda_Q, \kappa_Q) \leq (\lambda, \kappa)$ and all $\alpha \in A \cup \{\tau\}$. This I will do by structural induction on the proof π of $P \xrightarrow{\alpha}$ from the rules of Table 1. I make a case distinction based on the derivation of $P \mathcal{B} Q$. So assume $P \mathcal{B} Q$, $(\lambda_P, \kappa_P), (\lambda_Q, \kappa_Q) \leq (\lambda, \kappa)$, and $P \xrightarrow{\alpha}$ with $\alpha \in A \cup \{\tau\}$.

- Let $P = \langle x|\mathcal{S}\rangle \in \mathbb{P}$ and $Q = \langle x|\mathcal{S}'\rangle \in \mathbb{P}$ where \mathcal{S} and \mathcal{S}' are recursive specifications with $x \in V_{\mathcal{S}} = V_{\mathcal{S}'}$, such that $\mathcal{S}_y =_{\mathcal{I}} \mathcal{S}'_y$ for all $y \in V_{\mathcal{S}}$, meaning that for all $y \in V_{\mathcal{S}}$ and $\sigma : V_{\mathcal{S}} \rightarrow \mathbb{P}$ one has $\mathcal{S}_y[\sigma] =_{\mathcal{I}} \mathcal{S}'_y[\sigma]$.

By Table 1 the transition $\langle \mathcal{S}_x|\mathcal{S}\rangle \xrightarrow{\alpha}$ is provable by means of a strict subproof of π . By (#) above one has $\langle \mathcal{S}_x|\mathcal{S}\rangle \mathcal{B} \langle \mathcal{S}_x|\mathcal{S}'\rangle$. So by induction $\langle \mathcal{S}_x|\mathcal{S}'\rangle \xrightarrow{\alpha}$. Since $\langle _|\mathcal{S}'\rangle$ is the application of a substitution of the form $\sigma : V_{\mathcal{S}'} \rightarrow \mathbb{P}$, one has $\langle \mathcal{S}_x|\mathcal{S}'\rangle =_{\mathcal{I}} \langle \mathcal{S}'_x|\mathcal{S}'\rangle$. Hence there is a Q' with $P \vdash \langle \mathcal{S}'_x|\mathcal{S}'\rangle \xrightarrow{\alpha}$. By Table 1, $Q = \langle x|\mathcal{S}'\rangle \xrightarrow{\alpha}$.

- The case $P =_{\mathcal{I}} Q$ is trivial.
- Let $P = \beta.P^\dagger$ and $Q = \beta.Q^\dagger$ with $\beta \in A \cup \{\tau, t\}$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $\alpha = \beta$ and $Q \xrightarrow{\alpha}$.
- Let $P = P_1 + P_2$ and $Q = Q_1 + Q_2$ with $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$. I consider the first rule from Table 1 that could have been responsible for the derivation of $P \xrightarrow{\alpha}$; the other proceeds symmetrically. So suppose that $P_1 \xrightarrow{\alpha}$. Then by induction $Q_1 \xrightarrow{\alpha}$. By the same rule, $Q \xrightarrow{\alpha}$.
- Let $P = P_1 \parallel_S P_2$ and $Q = Q_1 \parallel_S Q_2$ with $P_1 \mathcal{B} Q_1$ and $P_2 \mathcal{B} Q_2$. I consider the three rules from Table 1 that could have been responsible for the derivation of $P \xrightarrow{\alpha}$.
First suppose that $\alpha \notin S$, and $P_1 \xrightarrow{\alpha}$. By induction, $Q_1 \xrightarrow{\alpha}$. Consequently, $Q_1 \parallel_S Q_2 \xrightarrow{\alpha}$.
Next suppose that $\alpha \in S$, $P_1 \xrightarrow{\alpha}$ and $P_2 \xrightarrow{\alpha}$. By induction, $Q_1 \xrightarrow{\alpha}$ and $Q_2 \xrightarrow{\alpha}$. So $Q_1 \parallel_S Q_2 \xrightarrow{\alpha}$.
The remaining case proceeds symmetrically to the first.
- Let $P = \tau_I(P^\dagger)$ and $Q = \tau_I(Q^\dagger)$ with $I \subseteq A$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $P^\dagger \xrightarrow{\beta}$ and either $\beta = \alpha \notin I$, or $\beta \in I$ and $\alpha = \tau$. By induction, $Q^\dagger \xrightarrow{\beta}$. Consequently, $Q = \tau_I(Q^\dagger) \xrightarrow{\alpha}$.
- Let $P = \mathcal{R}(P^\dagger)$ and $Q = \mathcal{R}(Q^\dagger)$ with $\mathcal{R} \subseteq A \times A$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $P^\dagger \xrightarrow{\beta}$ and either $(\beta, \alpha) \in \mathcal{R}$ or $\beta = \alpha = \tau$. By induction, $Q^\dagger \xrightarrow{\beta}$. Consequently, $Q = \mathcal{R}(Q^\dagger) \xrightarrow{\alpha}$.
- Let $P = \theta_L^U(P^\dagger)$, $Q = \theta_L^U(Q^\dagger)$ and $P^\dagger \mathcal{B} Q^\dagger$. Then $(\lambda_{P^\dagger}, \kappa_{P^\dagger}) < (\lambda, \kappa)$ and $(\lambda_{Q^\dagger}, \kappa_{Q^\dagger}) < (\lambda, \kappa)$, as remarked in Section 5. So by induction $P^\dagger =_{\mathcal{I}} Q^\dagger$. (This is the only use of stratum induction.)
Since $\theta_L^U(P^\dagger) \xrightarrow{\alpha}$, it must be that $P^\dagger \xrightarrow{\alpha}$ and either $\alpha \in U \cup \{\tau\}$ or $P^\dagger \xrightarrow{\beta}$ for all $\beta \in L \cup \{\tau\}$. In the latter case, $Q^\dagger \xrightarrow{\beta}$ for all $\beta \in L \cup \{\tau\}$. Moreover, $Q^\dagger \xrightarrow{\alpha}$. So, in both cases, $Q = \theta_L^U(Q^\dagger) \xrightarrow{\alpha}$.
- Let $P = \psi_X(P^\dagger)$, $Q = \psi_X(Q^\dagger)$ and $P^\dagger \mathcal{B} Q^\dagger$. Since $\psi_X(P^\dagger) \xrightarrow{\alpha}$, one has $P^\dagger \xrightarrow{\alpha}$. By induction $Q^\dagger \xrightarrow{\alpha}$. So $Q = \psi_X(Q^\dagger) \xrightarrow{\alpha}$.
- Let $P = \langle z | \mathcal{S} \rangle [\rho] = \langle z | \mathcal{S} \rangle [\rho]$ and $Q = \langle z | \mathcal{S} \rangle [\nu] = \langle z | \mathcal{S} \rangle [\nu]$ where \mathcal{S} is a recursive specification with $z \in V_{\mathcal{S}}$, and $\rho, \nu : \text{Var} \setminus V_{\mathcal{S}} \rightarrow \mathbb{P}$ satisfy $\rho(x) \mathcal{B} \nu(x)$ for all $x \in \text{Var} \setminus V_{\mathcal{S}}$. By Table 1 the transition $\langle \mathcal{S}_z | \mathcal{S} \rangle [\rho] \xrightarrow{\alpha}$ is provable by means of a strict subproof of the proof π of $\langle z | \mathcal{S} \rangle [\rho] \xrightarrow{\alpha}$. By (\$) above one has $\langle \mathcal{S}_z | \mathcal{S} \rangle [\rho] \mathcal{B} \langle \mathcal{S}_z | \mathcal{S} \rangle [\nu]$. So by induction, $\langle \mathcal{S}_z | \mathcal{S} \rangle [\nu] \xrightarrow{\alpha}$. By Table 1, $Q = \langle z | \mathcal{S} \rangle [\nu] \xrightarrow{\alpha}$. \square

Lemma 6 Let $H \in \mathbb{E}$ be guarded and have free variables from $W \subseteq \text{Var}$ only, and let $\vec{P}, \vec{Q} \in \mathbb{P}^W$. Then $\mathcal{I}(H[\vec{P}]) = \mathcal{I}(H[\vec{Q}])$.

Proof: Lemma 6 can be strengthened as follows.

Let $H \in \mathbb{E}$ be such that all free occurrences of variables from $W \subseteq \text{Var}$ in H are guarded, and let $\vec{P}, \vec{Q} \in \mathbb{P}^W$. Then $H[\vec{P}] =_{\mathcal{I}} H[\vec{Q}]$.

The proof proceeds with structural induction on H .

- Let $H = \langle x | \mathcal{S} \rangle$, so that $H[\vec{P}] = \langle x | \mathcal{S}[\vec{P}^\dagger] \rangle$, where \vec{P}^\dagger is the $W \setminus V_{\mathcal{S}}$ -tuple that is left of \vec{P} after deleting the y -components, for $y \in V_{\mathcal{S}}$, and $H[\vec{Q}] = \langle x | \mathcal{S}[\vec{Q}^\dagger] \rangle$. For each $y \in V_{\mathcal{S}}$, all free occurrences of variables from $W \setminus V_{\mathcal{S}}$ in \mathcal{S}_y are guarded. Thus, by induction, $\mathcal{S}_y[\vec{P}^\dagger] =_{\mathcal{I}} \mathcal{S}_y[\vec{Q}^\dagger]$. Since $=_{\mathcal{I}}$ is a full congruence for CCSP_t^θ , it follows that $H[\vec{P}] = \langle x | \mathcal{S}[\vec{P}^\dagger] \rangle =_{\mathcal{I}} \langle x | \mathcal{S}[\vec{Q}^\dagger] \rangle = H[\vec{Q}]$.
- Let $H = \alpha.H'$ for some $\alpha \in \text{Act}$. Then $\mathcal{I}(H[\vec{P}]) = \mathcal{I}(H[\vec{Q}])$ (namely \emptyset if $\alpha = t$ and $\{\alpha\}$ otherwise).
- Let $H = H_1 \parallel_S H_2$. Since all free occurrences of variables from $W \subseteq \text{Var}$ in H are guarded, so are those in H_1 and H_2 . Thus, by induction, $H_1[\vec{P}] =_{\mathcal{I}} H_1[\vec{Q}]$ and $H_2[\vec{P}] =_{\mathcal{I}} H_2[\vec{Q}]$. Since $=_{\mathcal{I}}$ is a full congruence for S it follows that $H[\vec{P}] =_{\mathcal{I}} H[\vec{Q}]$.
- The cases for all other operators go exactly like the case for \parallel_S . \square