Branching Bisimilarity for Processes with Time-outs

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Abstract

This paper provides an adaptation of branching bisimilarity to reactive systems with time-outs. Multiple equivalent definitions are procured, along with a modal characterisation and a proof of its congruence property for a standard process algebra with recursion. The last section presents a complete axiomatisation for guarded processes without infinite sequences of unobservable actions.

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1 Introduction

Strong bisimilarity [17] is the default semantic equivalence on labelled transition systems (LTSs), modelling systems that move from state to state by performing discrete, uninterpreted actions. In [11], it has been generalised, under the name *strong reactive bisimilarity*, to LTSs that feature, besides the hidden action τ [17], an unobservable *time-out* action t [9], modelling the end of a time-consuming activity from which we abstract. This addition significantly increases the expressiveness of the model [10, 11].

Applied to the verification of realistic distributed systems, strong bisimilarity is too fine an equivalence, especially because it does not cater to abstraction from internal activity. *Branching bisimilarity* [13] is a variant that does abstract from internal activity, and lies at the basis of many verification toolsets [3, 6]. The present paper generalises branching bisimilarity to LTSs with time-outs, thereby combining the virtues of [11] and [13]. It supports the resulting notion of *branching reactive bisimilarity* through a modal characterisation, congruence results for a standard process algebra with recursion, and a complete axiomatisation.

The addition of the time-out action t aims at modelling the passage of time while staying in the realm of *untimed* process algebra. Here, "untimed" means that our framework does not facilitate measuring time, even though it models whether a system can pause in some state or not. We assume that the execution of any action is instantaneous; thus, time elapses in states only. The amount of time spent in a state is dictated by the interaction of the system with an external entity called its *environment*.

We call a system *reactive* if it interacts with an environment able to allow or disallow visible actions. The environment represents a user or other systems, running in parallel, which has no control over τ or t actions. If X is the set of visible actions currently allowed by the environment and the system can perform any transition labelled by an element of $X \cup \{\tau\}$ then it will perform one of those transitions immediately. When a visible action is performed, it triggers the environment to choose a new set of allowed actions. If the environment is allowing X and the system cannot perform any transition labelled by τ or any allowed action, then the system is said to be *idling*. When the system idles, time-outs become executable, but the environment can also get impatient and choose a new X before any time-out occurs.

We have supposed that the environment cannot synchronise with the execution of a time-out, thus implying that, right after executing a time-out, the environment is still allowing the same set of allowed actions as before this execution. For example, the process $a.P + \mathrm{t}.(a.Q + \tau.R)$ will never reach Q because, for the time-out to happen, the environment has to block a and so $a.Q + \tau.R$ can only be reached when the environment blocks a. In this case, the τ-transition is always executed before the environment can allow a again.

Similarly, strong and branching reactive bisimilarity satisfy the process algebraic law $\tau.P + \mathrm{t}.Q = \tau.P$, essentially giving τ priority over t. Whereas this could have been formalised through an operational semantics in which the process $\tau.P + \mathrm{t}.Q$ lacks an outgoing t-transition, here, and in [11], we derive an LTS for a standard process algebra with time-outs in a way that treats t just like any other action. Instead, the priority of τ over t is implemented in the reactive bisimilarity: its says that even though the transition $\tau.P + \mathrm{t}.Q \xrightarrow{\mathrm{t}} Q$ is present in our LTS, it will never be taken. This approach is not only simpler, it also generalises better to choices like $b.P + \mathrm{t}.Q$, where the priority of b over t is conditional on the environment in which the system is placed, namely on whether or not this environment allows the b-action to occur.

From the system's perspective, the environment can be in two kinds of states: either allowing a specific set of actions, or being triggered to change. Our model does not stipulate how much time the environment takes to choose a new set of allowed actions once triggered, or even if it will ever make such a choice. Thus, the system could perform some transitions while the environment is triggered, especially those labelled τ. In our view, the most natural way to see the environment is as another system executed in parallel, while enforcing synchronisation on all visible actions. This implies that the environment allows a set X of actions when it idles in a state whose set of initial actions is X, and the environment is triggered when it is not idling, especially when it can perform a τ-transition. In this paradigm, while the environment is triggered, any action can be allowed for a brief amount of time. However, there is no reason to believe that it will necessarily settle down on a specific set. For instance, this can happen if the environment reaches a *divergence*: an infinite sequence of τ-transitions.

In [7], seven (or nine) forms of branching bisimilarity are classified; they differ only in the treatment of divergence. In the present paper we are chiefly interested in divergence-free processes, on grounds that in the intuition of [11] any sequence of τ-transitions could be executed in time zero; yet we do wish to allow infinite sequences of t-transitions. For divergence-free process all these forms of branching bisimilarity coincide. Nevertheless, we do not formally exclude divergences, and in their presence our branching reactive bisimilarity generalises the *stability respecting branching bisimilarity* of [7], which differs from the default version from [13] through the presence of Clause 2.e of Definition 1. There does not exist a plausible reactive generalisation of the default version.

Section 2 supplies the formal definition of branching reactive bisimilarity as well as its rooted version, which will be shown to be its congruence closure. It also provides equivalent definitions that reduce our bisimilarity to a non-reactive one and illustrate that branching reactive bisimilarity coincides with stability respecting branching bisimilarity in the absence of time-outs.

Section 3 gives a modal characterisation of branching reactive bisimilarity and its rooted version on an extension of the Hennessy-Milner logic. Section 4 introduces the process algebra CCSP_t^θ along with an alternative characterisation of branching reactive bisimilarity that will be used to prove that rooted branching reactive bisimilarity is a full congruence for CCSP_t^θ.

Section 5 displays a complete axiomatisation of our bisimilarity on different fragments of CCSP_t^θ. Most completeness proofs rely on standard techniques like equation merging, but the very last one uses a relatively new method called "canonical representatives".

2 Branching Reactive Bisimilarity

A *labelled transition system* (LTS) is a triple $(\mathbb{P}, Act, \rightarrow)$ with $\mathbb{P}$ a set (of *states* or *processes*), Act a set (of *actions*) and $\rightarrow \in \mathbb{P} \times Act \times \mathbb{P}$. In this paper we consider LTSs with $Act := A \uplus \{\tau, \mathrm{t}\}$, where A is a set of *visible actions*, τ is the *hidden or invisible action*, and t the *time-out action*. Let $A_\tau := A \cup \{\tau\}$. $P \xrightarrow{\alpha} P'$ stands for $(P, \alpha, P') \in \rightarrow$ and these triplets are called *transitions*. Moreover, $P \xrightarrow{(\alpha)} P'$ denotes that either $\alpha = \tau$ and $P = P'$, or $P \xrightarrow{\alpha} P'$. Furthermore, *paths* are sequences of connected transitions and $\Longrightarrow$ is the reflexive-transitive closure of $\xrightarrow{\tau}$. The set of *initial* actions of a process $P \in \mathbb{P}$ is $\mathcal{I}(P) := \{\alpha \in A_\tau \mid P \xrightarrow{\alpha}\}$. Here $P \xrightarrow{\alpha}$ means that there is a Q with $P \xrightarrow{\alpha} Q$.

▶ **Definition 1.** A *branching reactive bisimulation* is a symmetric[1] relation $\mathscr{R} \subseteq (\mathbb{P} \times \mathbb{P}) \cup (\mathbb{P} \times \mathscr{P}(A) \times \mathbb{P})$ such that, for all $P, Q \in \mathbb{P}$ and $X \subseteq A$,

1. if $\mathscr{R}(P, Q)$ then
  - **a.** if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ with $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$,
  - **b.** for all $Y \subseteq A$, $\mathscr{R}(P, Y, Q)$;

2. if $\mathscr{R}(P, X, Q)$ then
  - **a.** if $P \xrightarrow{\tau} P'$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ with $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', X, Q_2)$,
  - **b.** if $P \xrightarrow{a} P'$ with $a \in X$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ with $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', Q_2)$,
  - **c.** if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then there is a path $Q \Longrightarrow Q_0$ with $\mathscr{R}(P, Q_0)$,
  - **d.** if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there is a path $Q =: Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1], \mathscr{R}(P, X, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r})$,
  - **e.** if $P \stackrel{\tau}{\not\rightarrow}$ then there is a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.

For $P, Q \in \mathbb{P}$, if there exists a branching reactive bisimulation $\mathscr{R}$ with $\mathscr{R}(P, Q)$ (resp. $\mathscr{R}(P, X, Q)$) then P and Q are said to be *branching reactive bisimilar* (resp. *branching X-bisimilar*), which is denoted $P \leftrightarroweq_{br} Q$ (resp. $P \leftrightarroweq_{br}^X Q$).

To build the above definition, the definition of a strong reactive bisimulation [11] was modified in a branching manner [13]. Intuitively, a triplet $\mathscr{R}(P, X, Q)$ affirms that P and Q behave similarly when the environment allows (only) the set of actions in X to occur, whereas a couple $\mathscr{R}(P, Q)$ says that P and Q behave in the same way when the environment has been triggered to change. As said before, the environment can be seen as a system executed in parallel while enforcing the synchronisation of all visible actions.

Clause 1 captures the scenario of a triggered environment: if P can perform a visible or invisible action then Q has to be able to match it; and the environment can settle on a set Y of allowed actions at any moment. Time-outs are not considered because these can occur only when the system idles, and idling can happen only when the environment has stabilised on a set of allowed actions. One might notice that, in [11], the first clause was only required

[1] meaning that $(P, Q) \in \mathscr{R} \Leftrightarrow (Q, P) \in \mathscr{R}$ and $(P, X, Q) \in \mathscr{R} \Leftrightarrow (Q, X, P) \in \mathscr{R}$

for invisible actions. However, there the case $\alpha \neq \tau$ is actually implied by the other clauses. If in our definition Clause 1.a were restricted to invisible actions then $\leftrightarrow_{br}$ would not be a congruence for the parallel operator, as shown in Appendix A.

Clause 2 depicts the scenario of an environment allowing X. τ-transitions have to be matched since the environment cannot disallow them, and their execution does not trigger the environment to change. Visible actions have to be matched only if they are allowed, and their execution triggers the environment. Triggering the environment or not explains why Clause 2a matches Q_2 in a triplet and Clause 2b in a couple. If P idles (i.e. $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$) then the environment can be triggered, thus, Q has to be able to instantaneously reach a state Q_0 related to P in a triggered environment.[2] If P idles and has an outgoing time-out transition then Q has to be able to match it in a branching manner. This involves Q performing any sequence of τ and t-transitions, such that all states encountered prior to the last optional t are related to P.[3] Lastly, a stability respecting clause [7] was added for practical reasons. In Appendix A, an example shows that without it $\leftrightarrow_{br}$ would not even be an equivalence. For the important class of *divergence-free* systems, without infinite sequences $Q_0 \xrightarrow{\tau} Q_1 \xrightarrow{\tau} \dots$, Clause 2.e is easily seen to be redundant.

▶ **Lemma 2.** *Let $\mathscr{R}$ be a branching reactive bisimulation.*

1. *If $\mathscr{R}(P,X,Q)$, $P \stackrel{\tau}{\not\rightarrow}$ and $Q \Longrightarrow Q'$ then also $\mathscr{R}(P,X,Q')$.*
2. *If $\mathscr{R}(P,Q)$ or $\mathscr{R}(P,X,Q)$, $P \stackrel{\tau}{\not\rightarrow}$ and $Q \stackrel{\tau}{\not\rightarrow}$ then $\mathcal{I}(Q) = \mathcal{I}(P)$.*
3. *If $\mathscr{R}(P,X,Q)$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $Q \stackrel{\tau}{\not\rightarrow}$ then $\mathscr{R}(P,Q)$.*
4. *If $\mathscr{R}(P,X,Q)$ and $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then there is a path $Q \Longrightarrow Q_0$ with $\mathscr{R}(P,Q_0)$, $Q_0 \stackrel{\tau}{\not\rightarrow}$ and $\mathcal{I}(Q_0) = \mathcal{I}(P)$.*

Proof.
1. This is an immediate consequence of the symmetric counterpart of Clause 2.a (where Q takes a τ-step). When that clause yields $P \Longrightarrow P_1 \xrightarrow{(\tau)} P_2$ we have $P_2 = P$.
2. This is a direct consequence of Clause 1.a or 2.b and its symmetric counterpart.
3. By Clause 2.e there is path $Q \Longrightarrow Q_0$ with $Q_0 \stackrel{\tau}{\not\rightarrow}$. By Claim 1 of this lemma, $\mathscr{R}(P,X,Q_0)$. Thus, by Clause 2.c there is a path $Q_0 \Longrightarrow Q_1$ with $\mathscr{R}(P,Q_1)$, but $Q_1 = Q_0 = Q$ since $Q \stackrel{\tau}{\not\rightarrow}$.
4. By Clause 2.e there is path $Q \Longrightarrow Q_0$ with $Q_0 \stackrel{\tau}{\not\rightarrow}$. By Claim 1 of this lemma, $\mathscr{R}(P,X,Q_0)$. That $\mathcal{I}(Q_0) = \mathcal{I}(P)$ and $\mathscr{R}(P,Q_0)$ follows by Claims 2 and 3 of this lemma. ◀

Definition 1 enables us to elide some time-outs. Using the process algebra notation to be formally introduced in Section 4, the processes $a.\mathrm{t}.b.0$ and $a.\mathrm{t}.\mathrm{t}.b.0$ (as well as $a.\mathrm{t}.\tau.\mathrm{t}.b.0$) are branching reactive bisimilar. Both require an unquantified positive but finite amount of rest between the actions a and b. To support this example, Clause 2.d of Definition 1 must allow a single time-out transition of one process to be matched by either zero or multiple time-outs of the other. An alternative definition, treating time-outs more like visible transitions, is obtained by replacing Clause 2.d by

2. d. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2$ with $\mathscr{R}(P',X,Q_2)$.

Requiring that the matching time-out is executable (i.e. $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$) is not necessary here, as it is implied by the other clauses. Indeed, Lemma 2.3, which is not affected by changing Clause 2.d, implies the existence of a path $Q \Longrightarrow Q_1 \stackrel{\tau}{\not\rightarrow}$ such that $\mathscr{R}(P,Q_1)$ and $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$. Since $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$, Clause 2d yields

[2] By Lemma 2.4 we can even choose Q_0 such that $Q_0 \stackrel{\tau}{\not\rightarrow}$, so that $\mathcal{I}(Q_0) = \mathcal{I}(P)$.

[3] Clause 2.d requires this only for states of the form Q_{2i} with $i \in [0, r-1]$, but by Lemma 2.1 it holds for all of them. Clause 2.c further implies that in Clause 2.d we have $\mathscr{R}(P, Q_{2i+1})$ for all $i \in [0, r-1]$.

$Q_1 \xrightarrow{t} Q_2$ with $\mathscr{R}(P', X, Q_2)$. This version of the definition has been studied [21] and has properties similar to $\leftrightarrow_{br}$, which are recapped in Appendix B.

In [13], branching bisimilarity is expressed in multiple equivalent ways. For practical purposes, our definition uses the semi-branching format, which is equivalent to the branching format thanks to the following lemma.

▶ **Lemma 3** (Stuttering Lemma). *Let $P, P^\dagger, P^\ddagger, Q \in \mathbb{P}$, if $P \leftrightarrow_{br} Q$, $P^\ddagger \leftrightarrow_{br} Q$ (resp. $P \leftrightarrow_{br}^X Q$, $P^\ddagger \leftrightarrow_{br}^X Q$) and $P \xrightarrow{\tau} P^\dagger \xrightarrow{\tau} P^\ddagger$ then $P^\dagger \leftrightarrow_{br} Q$ (resp. $P^\dagger \leftrightarrow_{br}^X Q$).*

Proof. Let $\mathscr{R}$ be a branching reactive bisimulation. Let's define $\mathscr{R}' := \mathscr{R} \cup \{(P^\dagger, Q), (Q, P^\dagger) \mid \exists P, P^\ddagger \in \mathbb{P}, P \Longrightarrow P^\dagger \Longrightarrow P^\ddagger \wedge \mathscr{R}(P, Q) \wedge \mathscr{R}(P^\ddagger, Q)\} \cup \{(P^\dagger, X, Q), (Q, X, P^\dagger) \mid \exists P, P^\ddagger \in \mathbb{P}, P \Longrightarrow P^\dagger \Longrightarrow P^\ddagger \wedge \mathscr{R}(P, X, Q) \wedge \mathscr{R}(P^\ddagger, X, Q)\}$. $\mathscr{R}'$ is symmetric by definition and $\mathscr{R}'$ is a branching reactive bisimulation, as proven in Appendix E. ◀

▶ **Proposition 4.** *$\leftrightarrow_{br}$ and $(\leftrightarrow_{br}^X)_{X \subseteq A}$ are equivalence relations.*

Proof. Reflexivity and symmetry are trivial following the definition. For transitivity, consider two branching reactive bisimulations $\mathscr{R}_1$ and $\mathscr{R}_2$. Let's define $\mathscr{R} := (\mathscr{R}_1 \circ \mathscr{R}_2) \cup (\mathscr{R}_2 \circ \mathscr{R}_1)$. Here $\mathscr{R}_1 \circ \mathscr{R}_2 := \{(P, Q) \mid \exists R.\ \mathscr{R}(P, R) \wedge \mathscr{R}(R, Q)\} \cup \{(P, X, Q) \mid \exists R.\ \mathscr{R}(P, X, R) \wedge \mathscr{R}(R, X, Q)\}$. $\mathscr{R}$ is symmetric by definition and $\mathscr{R}$ is a branching reactive bisimulation, as proven in Appendix E. ◀

2.1 Rooted Version

A well-known limitation of branching bisimilarity $\leftrightarrow_b$ is that it fails to be a congruence for the choice operator $+$. For example, $a \leftrightarrow_b \tau.a$ but $a + b \not\leftrightarrow_b \tau.a + b$. Since the objective is to define a congruence, instead of $\leftrightarrow_{br}$ we use the *congruence closure* of $\leftrightarrow_{br}$, which is the coarsest congruence included in $\leftrightarrow_{br}$.

▶ **Definition 5.** A *rooted branching reactive bisimulation* is a symmetric relation $\mathscr{R} \subseteq (\mathbb{P} \times \mathbb{P}) \cup (\mathbb{P} \times \mathscr{P}(A) \times \mathbb{P})$ such that, for all $P, Q \in \mathbb{P}$ and $X \subseteq A$,

1. if $\mathscr{R}(P, Q)$
   - **a.** if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there is a transition $Q \xrightarrow{\alpha} Q'$ with $P' \leftrightarrow_{br} Q'$,
   - **b.** for all $Y \subseteq A$, $\mathscr{R}(P, Y, Q)$;
2. if $\mathscr{R}(P, X, Q)$
   - **a.** if $P \xrightarrow{\tau} P'$ then there is a transition $Q \xrightarrow{\tau} Q'$ with $P' \leftrightarrow_{br}^X Q'$,
   - **b.** if $P \xrightarrow{a} P'$ with $a \in X$ then there is a transition $Q \xrightarrow{a} Q'$ with $P' \leftrightarrow_{br} Q'$,
   - **c.** if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then $\mathscr{R}(P, Q)$,
   - **d.** if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then there is a transition $Q \xrightarrow{t} Q'$ with $P' \leftrightarrow_{br}^X Q'$.

For $P, Q \in \mathbb{P}$, if there exists a rooted branching reactive bisimulation $\mathscr{R}$ with $\mathscr{R}(P, Q)$ (resp. $\mathscr{R}(P, X, Q)$) then P and Q are said to be *rooted branching reactive bisimilar* (resp. rooted branching X-bisimilar), which is denoted $P \leftrightarrow_{br}^r Q$ (resp. $P \leftrightarrow_{br}^{rX} Q$).

A rooted version of a bisimulation consists in enforcing a stricter matching on the first transition of a system. In the branching case, the first transition is matched in the strong manner. The stability respecting clause can be removed, as it is now implied by the other clauses. Rooting the bisimilarity is the standard technique to obtain its congruence closure; later $\leftrightarrow_{br}^r$ will be proven to be a congruence. As any branching reactive bisimulation relating $P + b$ and $Q + b$, for a fresh action b, induces a rooted branching reactive bisimulation relating P and Q, it then follows that $\leftrightarrow_{br}^r$ is the coarsest included in $\leftrightarrow_{br}$. Since $\leftrightarrow_{br}$ is an equivalence, the proof of Proposition 4 can be adapted to $\leftrightarrow_{br}^r$ in a straightforward way.

▶ **Proposition 6.** *$\leftrightarrow_{br}^r$ and $(\leftrightarrow_{br}^{rX})_{X \subseteq A}$ are equivalence relations.*

2.2 Alternative Forms of Definition 1

Definition 1 can be rephrased in various ways. First of all, using Requirements 1.b and 2.c, one can move Requirement 2.d from Clause 2 (dealing with triples (P, X, Q)) to Clause 1 (dealing with pairs (P, Q)), now adding a universal quantifier over X to the requirement. Next, Requirement 2.e can be copied under Clause 1. This makes Clause 1.b unnecessary, thereby obtaining a definition in which the triples (P, X, Q) are encountered only after taking a t-transition. In this form it is obvious that branching reactive bisimilarity reduces to the classical stability respecting branching bisimilarity for systems without t-transitions. We have chosen the form of Definition 1 over the above alternatives, because we believe it comes with more natural intuitions for its plausibility.

In Appendix C a further modification of Definitions 1 and 5 is proposed, called *generalised [rooted] branching reactive bisimulation*. We show that each [rooted] branching reactive bisimulation is a generalised [rooted] branching reactive bisimulation, and two systems are [rooted] branching reactive bisimilar iff they are related by a generalised [rooted] branching reactive bisimulation. This characterisation of $\leftrightarrow_{br}$ and $\leftrightarrow_{br}^{r}$ will be used in the proofs of Theorem 11 and Proposition 15.

In [19], Pohlmann introduces an encoding which maps strong reactive bisimilarity to strong bisimilarity where time-outs are considered as any visible action. This encoding in essence places a given process in a most general environment, one that features environment time-out actions t_ε, as well as actions ε_X for settling in a state that allows exactly the actions in X. This proves that reactive equivalences can be expressed as non-reactive ones at the cost of increasing the processes' size. Thus, any tool set able to work on strong bisimulation could theoretically deal with its reactive counterpart.

In Appendix D, this encoding is slightly modified to yield a similar result for branching reactive bisimulation and its rooted version, for the latter result also employing actions t_X. It appears that these modifications do not impact its effect on strong reactive bisimilarity. Since our bisimilarity has some time-out eliding properties, it is not mapped to stability respecting branching bisimilarity, but to a new bisimilarity, defined below.

▶ **Definition 7.** A t-*branching bisimulation* is a symmetric relation $\mathscr{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$, if $\mathscr{R}(P, Q)$ then

1. if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau \cup \{\mathrm{t}_\varepsilon, \varepsilon_X \mid X \subseteq A\}$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ with $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$,
2. if $P \xrightarrow{\mathrm{t}} P'$ then there is a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} ... \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, 2r{-}1]$, $\mathscr{R}(P, Q_i)$ and $\mathscr{R}(P', Q_{2r})$,
3. if $P \stackrel{\tau}{\not\rightarrow}$ then there is a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.

For $P, Q \in \mathbb{P}$, if there exists a t-branching bisimulation $\mathscr{R}$ with $\mathscr{R}(P, Q)$ then P and Q are said to be t-*branching bisimilar*, which is denoted $P \leftrightarrow_{\mathrm{t}b} Q$.

The encoding also sends $\leftrightarrow_{br}^{r}$ to the rooted version of $\leftrightarrow_{\mathrm{t}b}$.

▶ **Definition 8.** A *rooted* t-*branching bisimulation* is a symmetric relation $\mathscr{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$, if $\mathscr{R}(P, Q)$ then

1. if $P \xrightarrow{\alpha} P'$ with $\alpha \in Act \cup \{\mathrm{t}_\varepsilon, \mathrm{t}_X, \varepsilon_X \mid X \subseteq A\}$ then there is a transition $Q \xrightarrow{\alpha} Q'$ with $P' \leftrightarrow_{\mathrm{t}b} Q'$.

For $P, Q \in \mathbb{P}$, if there exists a rooted t-branching bisimulation $\mathscr{R}$ with $\mathscr{R}(P, Q)$ then P and Q are said to be *rooted* t-*branching bisimilar*, which is denoted $P \leftrightarrow_{\mathrm{t}b}^{r} Q$.

Providing a complete axiomatisation of rooted t-branching bisimilarity will be useful in the proof of completeness of the axiomatisation of rooted branching reactive bisimilarity (Lemma 56).

3 Modal Characterisation

The Hennessy-Milner logic [15] expresses properties of the behaviour of processes in an LTS. In [11], the modality $\langle X\rangle\varphi$ was added to obtain a modal characterisation of strong reactive bisimilarity ($\leftrightarroweq_r$). In order to capture branching reactive bisimilarity we add another modality $X\varphi$. To avoid confusion, $\langle X\rangle\varphi$ is renamed $\langle \mathrm{t}_X\rangle\varphi$.

▶ **Definition 9.** The class $\mathbb{L}$ of *reactive Hennessy-Milner formulas* is defined as follows, where I is an index set, $\alpha \in Act$, $a \in A$ and $X \subseteq A$,

$$\varphi ::= \top \mid \bigwedge_{i\in I} \varphi_i \mid \neg\varphi \mid \langle\alpha\rangle\varphi \mid X\varphi$$

$P \models \top$			$P \models_Y \top$		
$P \models \bigwedge_{i\in I}\varphi_i$	iff	$\forall i \in I, P \models \varphi_i$	$P \models_Y \bigwedge_{i\in I}\varphi_i$	iff	$\forall i \in I, P \models_Y \varphi_i$
$P \models \neg\varphi$	iff	$P \not\models \varphi$	$P \models_Y \neg\varphi$	iff	$P \not\models_Y \varphi$
$P \models \langle\alpha\rangle\varphi$	iff	$\exists P \xrightarrow{\alpha} P', P' \models \varphi$	$P \models_Y \langle\tau\rangle\varphi$	iff	$\exists P \xrightarrow{\tau} P', P' \models_Y \varphi$
			$P \models_Y \langle\mathrm{t}\rangle\varphi$	iff	$\exists P \xrightarrow{\mathrm{t}} P', P' \models_Y \varphi$
$P \models_Y \langle a\rangle\varphi$	iff	$(a \in Y \vee \mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset) \wedge \exists P \xrightarrow{a} P', P' \models \varphi$			
$P \models X\varphi$	iff	$\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \wedge P \models_X \varphi$			
$P \models_Y X\varphi$	iff	$\mathcal{I}(P) \cap (X \cup Y \cup \{\tau\}) = \emptyset \wedge P \models_X \varphi$			

Table 1 Semantics of $\models$ and $(\models_Y)_{Y\subseteq A}$

The satisfaction rules of $\mathbb{L}$ are given in Table 1. $P \models \varphi$ means that P satisfies φ when the environment is triggered, and $P \models_Y \varphi$ indicates that P satisfies φ when the environment allows Y. The modality $X\varphi$ expresses that a process can idle in its current state during a period in which the environment allows the actions in X, after which it behaves according to φ.

The modality $\langle \mathrm{t}_X\rangle\varphi$ from [11] can now be defined as $\langle \mathrm{t}_X\rangle\varphi := X\langle \mathrm{t}\rangle\varphi$. Write $\mathbb{L}_s$ for the fragment of $\mathbb{L}$ from [11], which includes $\langle \mathrm{t}_X\rangle\varphi$ but does not feature $X\varphi$ or $\langle \mathrm{t}\rangle\varphi$. Then the modal characterisation theorem of [11] says $P \leftrightarroweq_r Q \;\Leftrightarrow\; \forall \varphi \in \mathbb{L}_s.\ (P \models \varphi \Leftrightarrow Q \models \varphi)$.

Here we restrict attention to the fragment of $\mathbb{L}$ that includes $\langle \mathrm{t}_X\rangle\varphi$ and $X\varphi$, but not $\langle \mathrm{t}\rangle\varphi$. On this fragment $\models_Y$ is defined such that whenever $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$ then $P \models_Y \varphi$ iff $P \models \varphi$. This is because the environment may choose to change during a period of idling.

To obtain a modal characterisation of [rooted] branching relative bisimilarity, we need a few other derived modalities. First of all, $\langle\varepsilon\rangle\varphi := \bigvee_{i\in\mathbb{N}}\langle\tau\rangle^i\varphi$. To lessen the notations, for all $\alpha \in A_\tau$, $\langle\hat{\alpha}\rangle\varphi$ denotes $\varphi \vee \langle\tau\rangle\varphi$ if $\alpha = \tau$, $\langle\alpha\rangle\varphi$ otherwise, and the modality $\langle\hat{\mathrm{t}}_X\rangle\varphi$ denotes $\langle \mathrm{t}_X\rangle\varphi \vee X\varphi$ or $X\langle\hat{\mathrm{t}}\rangle\varphi$. Moreover, $\varphi \wedge \langle\hat{\alpha}\rangle\varphi'$ is shortened to $\varphi\langle\hat{\alpha}\rangle\varphi'$. Furthermore, we define $\varphi\langle\varepsilon_X\rangle\varphi' := \bigvee_{i\in\mathbb{N}}\varphi\langle\varepsilon_X\rangle^{(i)}\varphi'$, where $\varphi\langle\varepsilon_X\rangle^{(0)}\varphi' := \langle\varepsilon\rangle(\varphi \wedge \langle\hat{\mathrm{t}}_X\rangle\varphi')$ and, for all $i > 0$, $\varphi\langle\varepsilon_X\rangle^{(i)}\varphi' := \langle\varepsilon\rangle(\varphi\wedge\langle \mathrm{t}_X\rangle(\varphi\wedge(\varphi\langle\varepsilon_X\rangle^{(i-1)}\varphi')))$. The satisfaction rules of these new modalities can be derived from the basic ones: see Table 2.

▶ **Definition 10.** The sub-classes $\mathbb{L}_b$ and $\mathbb{L}_b^r$ are defined as follows, where I is an index set, $\alpha \in A_\tau$, $X \subseteq A$, $\varphi, \varphi' \in \mathbb{L}_b$ and $\psi \in \mathbb{L}_b^r$,

$$\varphi ::= \top \mid \bigwedge_{i\in I}\varphi_i \mid \neg\varphi \mid \langle\varepsilon\rangle(\varphi\langle\hat{\alpha}\rangle\varphi') \mid \varphi\langle\varepsilon_X\rangle\varphi' \mid \langle\varepsilon\rangle\neg\langle\tau\rangle\top \qquad (\mathbb{L}_b)$$

$$\psi ::= \top \mid \bigwedge_{i\in I}\psi_i \mid \neg\psi \mid \langle\alpha\rangle\varphi \mid \langle \mathrm{t}_X\rangle\varphi \qquad (\mathbb{L}_b^r)$$

The last option for $\mathbb{L}_b$, inspired by [5], is used to encompass the stability respecting Clause 2.e of Definition 1.

$P \models \langle\hat{\alpha}\rangle\varphi$	iff	$\exists P \xrightarrow{(\alpha)} P', P' \models \varphi$	$P \models_Y \langle\hat{\tau}\rangle\varphi$	iff	$\exists P \xrightarrow{(\tau)} P', P' \models_Y \varphi$
$P \models \langle\varepsilon\rangle\varphi$	iff	$\exists P \Longrightarrow P', P' \models \varphi$	$P \models_Y \langle\varepsilon\rangle\varphi$	iff	$\exists P \Longrightarrow P', P' \models_Y \varphi$
$P \models \varphi\langle\varepsilon_X\rangle\varphi'$	iff	$\exists P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{(\mathrm{t})} P_{2r}$ with $r > 0$, such that $\forall i \in [1, 2r-1]\ P_i \models_X \varphi \wedge P_{2r} \models_X \varphi'$ and $\forall i \in [0, r-1]\ \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$			
$P \models_Y \varphi\langle\varepsilon_X\rangle\varphi'$	iff	$\exists P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{(\mathrm{t})} P_{2r}$ with $r > 0$, such that $\forall i \in [1, 2r-1]\ P_i \models_X \varphi \wedge P_{2r} \models_X \varphi'$ and $\mathcal{I}(P_1) \cap (Y \cup \{\tau\}) = \emptyset \wedge \forall i \in [0, r-1]\ \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$			
$P \models \langle\mathrm{t}_X\rangle\varphi$	iff	$\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \wedge \exists P \xrightarrow{\mathrm{t}} P', P' \models_X \varphi$			

Table 2 Semantics of $\models$ and $(\models_Y)_{Y \subseteq A}$ for the derived modalities

▶ **Theorem 11.** *Let $P, Q \in \mathbb{P}$. For all $X \subseteq A$,*
- $P \leftrightarrow_{br} Q$ *iff* $\forall \varphi \in \mathbb{L}_b, P \models \varphi \Leftrightarrow Q \models \varphi$,
- $P \leftrightarrow_{br}^{X} Q$ *iff* $\forall \varphi \in \mathbb{L}_b, P \models_X \varphi \Leftrightarrow Q \models_X \varphi$,
- $P \leftrightarrow_{br}^{r} Q$ *iff* $\forall \psi \in \mathbb{L}_b^r, P \models \psi \Leftrightarrow Q \models \psi$,
- $P \leftrightarrow_{br}^{rX} Q$ *iff* $\forall \psi \in \mathbb{L}_b^r, P \models_X \psi \Leftrightarrow Q \models_X \psi$.

Proof. ($\Rightarrow$) The four propositions are proven simultaneously by structural induction on $\mathbb{L}_b$ and $\mathbb{L}_b^r$ in Appendix F.

($\Leftarrow$) Let $\equiv := \{(P,Q) \mid \forall \varphi \in \mathbb{L}_b, P \models \varphi \Leftrightarrow Q \models \varphi\} \cup \{(P,X,Q) \mid \forall \varphi \in \mathbb{L}_b, P \models_X \varphi \Leftrightarrow Q \models_X \varphi\}$, and $\equiv^r := \{(P,Q) \mid \forall \psi \in \mathbb{L}_b^r, P \models \psi \Leftrightarrow Q \models \psi\} \cup \{(P,X,Q) \mid \forall \psi \in \mathbb{L}_b^r, P \models_X \psi \Leftrightarrow Q \models_X \psi\}$. It suffices to check that $\equiv$ [resp. $\equiv^r$] is a generalised [rooted] branching reactive bisimulation. This is done in Appendix F. ◀

4 Process Algebra and Congruence

The process algebra CCSP$_t^\theta$ is composed of classical operators from the well-known process algebras CCS [17], CSP [2, 18] and ACP [1, 4], as well as the time-out action t and two *environment operators* from [11], that were added in order to enable a complete axiomatisation.

▶ **Definition 12.** Let V be a countable set of variables, the *expressions* of CCSP$_t^\theta$ are recursively defined as follows:

$$E ::= 0 \mid x \mid \alpha.E \mid E + F \mid E \parallel_S F \mid \tau_I(E) \mid \mathcal{R}(E) \mid \theta_L^U(E) \mid \psi_X(E) \mid \langle y|\mathcal{S}\rangle$$

where $x \in V$, $\alpha \in Act$, $S, I, L, U, X \subseteq A$, $L \subseteq U$, $\mathcal{R} \subseteq A \times A$, $\mathcal{S}$ is a *recursive specification*: a set of equations $\{x = \mathcal{S}_x \mid x \in V_\mathcal{S}\}$ with $V_\mathcal{S} \subseteq V$ and each $\mathcal{S}_x$ a CCSP$_t^\theta$ expression, and $y \in V_\mathcal{S}$. We require that all sets $\{b \mid (a,b) \in \mathcal{R}\}$ for $a \in A$ are finite.

0 stands for a system which cannot perform any action. The expression $\alpha.E$ represents a system that first performs α and then E. The expression $E + F$ represents a choice to behave like E or F. The parallel composition $E \parallel_S F$ synchronises the execution of E and F, but only when performing actions in S. $\tau_I(E)$ represents the system E where all actions $a \in I$ are transformed into τ. The operator $\mathcal{R}$ renames a given action $a \in A$ into a choice between all actions b with $(a,b) \in \mathcal{R}$. $\langle y|\mathcal{S}\rangle$ is the y-component of a solution of $\mathcal{S}$.

CCSP$_t^\theta$ also has two environment operators that help to develop a complete axiomatisation (like the left merge for ACP). $\theta_L^U(E)$ is the expression E plunged into an environment X such that $L \subseteq X \subseteq U$. $\theta_X^X(E)$ is denoted $\theta_X(E)$. $\psi_X(E)$ plunges E into the environment X if a time-out occurs, but, has no effect if any other action is performed. The operational semantics of CCSP_t^θ is given in Figure 1. All operators except the environment ones follow the semantics of CCS, CSP or ACP. As $\theta_L^U(E)$ simulates the expression E plunged in an environment $L \subseteq X \subseteq U$, it has no effect on τ-transitions, which do not trigger the environment. Moreover, θ_L^U restricts the ability to perform visible actions to those allowed by the environment (i.e. included in U) and performing these actions triggers the environment. However, if the expression idles (i.e. $\mathcal{I}(E) \cap (L \cup \{\tau\}) = \emptyset$) then it might trigger the environment and $\theta_L^U(E)$ acts like E. $\psi_X(E)$ supposes that time-outs are performed while the environment allows X, thus, it has no effect on actions that are not t. However, if E can perform a time-out while the environment allows X (i.e. $\mathcal{I}(E) \cap (X \cup \{\tau\}) = \emptyset$) then $\psi_X(E)$ can perform the time-out while plunging the expression in the environment X.

$$\frac{}{\alpha.x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'} \qquad \frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$$

$$\frac{x \xrightarrow{a} x' \wedge \mathcal{R}(a,b)}{\mathcal{R}(x) \xrightarrow{b} \mathcal{R}(x')} \qquad \frac{x \xrightarrow{\tau} x'}{\mathcal{R}(x) \xrightarrow{\tau} \mathcal{R}(x')} \qquad \frac{x \xrightarrow{\mathrm{t}} x'}{\mathcal{R}(x) \xrightarrow{\mathrm{t}} \mathcal{R}(x')}$$

$$\frac{x \xrightarrow{\alpha} x' \wedge \alpha \notin S}{x \parallel_S y \xrightarrow{\alpha} x' \parallel_S y} \qquad \frac{y \xrightarrow{\alpha} y' \wedge \alpha \notin S}{x \parallel_S y \xrightarrow{\alpha} x \parallel_S y'} \qquad \frac{x \xrightarrow{a} x' \wedge y \xrightarrow{a} y' \wedge a \in S}{x \parallel_S y \xrightarrow{a} x' \parallel_S y'}$$

$$\frac{x \xrightarrow{\alpha} x' \wedge \alpha \notin I}{\tau_I(x) \xrightarrow{\alpha} \tau_I(x')} \qquad \frac{x \xrightarrow{a} x' \wedge a \in I}{\tau_I(x) \xrightarrow{\tau} \tau_I(x')} \qquad \frac{\langle \mathcal{S}_x | \mathcal{S} \rangle \xrightarrow{\alpha} x'}{\langle x | \mathcal{S} \rangle \xrightarrow{\alpha} x'}$$

$$\frac{x \xrightarrow{\tau} x'}{\theta_L^U(x) \xrightarrow{\tau} \theta_L^U(x')} \qquad \frac{x \xrightarrow{a} x' \wedge a \in U}{\theta_L^U(x) \xrightarrow{a} x'} \qquad \frac{x \xrightarrow{\alpha} x' \wedge \alpha \neq t}{\psi_X(x) \xrightarrow{\alpha} x'}$$

$$\frac{x \xrightarrow{\alpha} x' \wedge \mathcal{I}(x) \cap (L \cup \{\tau\}) = \emptyset}{\theta_L^U(x) \xrightarrow{\alpha} x'} \qquad \frac{x \xrightarrow{\mathrm{t}} x' \wedge \mathcal{I}(x) \cap (X \cup \{\tau\}) = \emptyset}{\psi_X(x) \xrightarrow{\mathrm{t}} \theta_X(x')}$$

Figure 1 Operational semantics of CCSP_t^θ

All $\mathcal{S}_x$ are considered to be sub-expressions of $\langle y | \mathcal{S} \rangle$. An occurrence of a variable x is *bound* in $E \in \text{CCSP}_t^\theta$ iff it occurs in a sub-expression $\langle y | \mathcal{S} \rangle$ of E such that $x \in V_\mathcal{S}$; otherwise it is *free*. An expression E is *invalid* if it has a sub-expression $\theta_L^U(F)$ or $\psi_X(F)$ such that a variable occurrence is free in F, but bound in E. An example justifying this condition can be found in [11]. The set of valid expressions of CCSP_t^θ is denoted $\mathbb{E}$. If an expression is valid and all of its variable occurrences are bound then it is *closed* and we call it a *process*; the set of processes is denoted $\mathbb{P}$.

A *substitution* is a partial function $\rho : V \rightharpoonup E$. The application $E[\rho]$ of a substitution ρ to an expression $E \in \mathbb{E}$ is the result of the simultaneous replacement, for all $x \in \text{dom}(\rho)$, of each free occurrence of x by the expression $\rho(x)$, while renaming bound variables to avoid name clashes. We write $\langle E | \mathcal{S} \rangle$ for the expression E where any $y \in V_\mathcal{S}$ is substituted by $\langle y | \mathcal{S} \rangle$.

4.1 Time-out Bisimulation

Thanks to the environment operator θ_L^U, it is possible to express our bisimilarity in a much more succinct way. Indeed, θ_X was defined so that $P \leftrightarrow_{br}^X Q$ if and only if $\theta_X(P) \leftrightarrow_{br} \theta_X(Q)$.

▶ **Definition 13.** A *branching time-out bisimulation* is a symmetric relation $\mathscr{B} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$, if $P \mathscr{B} Q$ then
1. if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ with $P \mathscr{B} Q_1$ and $P' \mathscr{B} Q_2$
2. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r-1]$, $\theta_X(P) \mathscr{B} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \mathscr{B} \theta_X(Q_{2r})$
3. if $P \stackrel{\tau}{\not\rightarrow}$ then there is a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.

Note that in Condition 2 above one also has $P \mathscr{B} Q_1$ and consequently $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$. A rooted version of branching time-out bisimulation can be defined in the same vein.

▶ **Definition 14.** A *rooted branching time-out bisimulation* is a symmetric relation $\mathscr{B} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$ such that $P \mathscr{B} Q$,
1. if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there is a step $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarrow_{br} Q'$
2. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there is a step $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow_{br} \theta_X(Q')$.

▶ **Proposition 15.** *Let $P, Q \in \mathbb{P}$,*
1. *$P \leftrightarrow_{br} Q$ (resp. $P \leftrightarrow_{br}^X Q$) iff there exists a branching time-out bisimulation $\mathscr{B}$ with $P \mathscr{B} Q$ (resp. $(\theta_X(P) \mathscr{B} \theta_X(Q))$),*
2. *$P \leftrightarrow_{br}^X Q$ if and only if $\theta_X(P) \leftrightarrow_{br} \theta_X(Q)$,*
3. *$P \leftrightarrow_{br}^r Q$ (resp. $P \leftrightarrow_{br}^{rX} Q$) iff there exists a rooted branching time-out bisimulation $\mathscr{B}$ with $P \mathscr{B} Q$ (resp. $(\theta_X(P) \mathscr{B} \theta_X(Q))$).*

Proof. Note that Proposition 15.2 is a trivial corollary of 15.1.

Let $\mathscr{R}$ be a [generalised rooted] branching reactive bisimulation, let's define $\mathscr{B} := \{(P, Q) \mid \mathscr{R}(P, Q)\} \cup \{(\theta_X(P), \theta_X(Q)) \mid \mathscr{R}(P, X, Q)\}$. $\mathscr{B}$ is a [rooted] branching time-out bisimulation, as proven in Appendix G. Let $\mathscr{B}$ be a [rooted] branching time-out bisimulation, let's define $\mathscr{R} = \{(P, Q) \mid P \mathscr{B} Q\} \cup \{(P, X, Q) \mid \theta_X(P) \mathscr{B} \theta_X(Q)\}$. $\mathscr{R}$ is a [rooted] generalised branching reactive bisimulation, as proven in Appendix G. ◀

Time-out bisimulations are very practical as there are no triplets to deal with anymore.

4.2 Congruence

Until now, bisimilarity was only defined between closed expressions, but any relation $\sim \subseteq \mathbb{P} \times \mathbb{P}$ can be extended to $\mathbb{E} \times \mathbb{E}$ in the following way: $E \sim F$ iff $\forall \rho : V \to \mathbb{P}$, $E[\rho] \sim F[\rho]$. It can be extended further to substitutions $\rho, \nu \in V \rightharpoonup \mathbb{E}$ by $\rho \sim \nu$ iff $\mathrm{dom}(\rho) = \mathrm{dom}(\nu)$ and $\forall x \in \mathrm{dom}(\rho)$, $\rho(x) \sim \nu(x)$.

▶ **Definition 16.** An equivalence $\sim \subseteq \mathbb{E} \times \mathbb{E}$ is a congruence for an n-ary operator f if $P_i \sim Q_i$ for all $i = 0, \ldots, n-1$ implies $f(P_0, \ldots, P_{n-1}) \sim f(Q_0, \ldots, Q_{n-1})$. It is a *lean congruence* if, for all $E \in \mathbb{E}$ and all $\rho, \nu \in V \rightharpoonup \mathbb{E}$ such that $\rho \sim \nu$, $E[\rho] \sim E[\nu]$. It is a *full congruence* if
1. it is a congruence for all operators in the language, and
2. for all recursive specifications $\mathcal{S}, \mathcal{S}'$ with $V_{\mathcal{S}} = V_{\mathcal{S}'}$ and $x \in V_{\mathcal{S}}$ such that $\langle x|\mathcal{S}\rangle, \langle x|\mathcal{S}'\rangle \in \mathbb{P}$, if $\forall y \in V_{\mathcal{S}}$, $\mathcal{S}_y \sim \mathcal{S}'_y$ then $\langle x|\mathcal{S}\rangle \sim \langle x|\mathcal{S}'\rangle$.

To show that $\sim$ is a lean congruence it suffices to restrict attention to closed substitutions $\rho, \nu \in V \to \mathbb{P}$, because the general property will then follow by composition of substitutions. A full congruence is a lean congruence, and a lean congruence is a congruence for all operators in the language, but both implications are strict, as shown in [8].

To show that $\leftrightarrow_{br}^r$ and $\leftrightarrow_{tb}^r$ are full congruences, it is first necessary to prove that $\leftrightarrow_{br}$ and $\leftrightarrow_{tb}$ are congruences for some of the operators of CCSP_t^θ.

▶ **Proposition 17.** *$\leftrightarrow_{br}$ and $\leftrightarrow_{tb}$ are congruences for action prefixing, parallel composition, abstraction, renaming and the environment operator θ_L^U, for all $L \subseteq U \subseteq A$.*

Proof. Let $\mathscr{B}$ be the smallest relation such that, for all $P, Q \in \mathbb{P}$,
- if $P \leftrightarrow_{br} Q$ then $P \mathscr{B} Q$;
- if $P \mathscr{B} Q$ then, for all $\alpha \in Act$, $I \subseteq A$, $\mathcal{R} \in A \times A$ and $L \subseteq U \subseteq A$, $\alpha.P \mathscr{B} \alpha.Q$, $\tau_I(P) \mathscr{B} \tau_I(Q)$, $\mathcal{R}(P) \mathscr{B} \mathcal{R}(Q)$ and $\theta_L^U(P) \mathscr{B} \theta_L^U(Q)$;
- if $P_1 \mathscr{B} Q_1$, $P_2 \mathscr{B} Q_2$ and $S \subseteq A$ then $P_1 \|_S P_2 \mathscr{B} Q_1 \|_S Q_2$.

It suffices to show that $\mathscr{B}$ is a branching time-out bisimulation up to $\leftrightarrow$, which implies $\mathscr{B} \subseteq \leftrightarrow_{br}$. A bisimulation "up to" is a notion introduced by Milner in [17]; it is commonly used when proving congruence properties. The proof uses some lemmas which were obtained in [11]. Details can be found in Appendix H. A similar proof yields the result for $\leftrightarrow_{tb}$. ◀

▶ **Theorem 18.** *$\leftrightarrow_{br}^r$ and $\leftrightarrow_{tb}^r$ are full congruences.*

Proof. Let $\mathscr{B} \subseteq \mathbb{P} \times \mathbb{P}$ be the smallest relation such that
- if $P \leftrightarrow_{br}^r Q$ then $P \mathscr{B} Q$;
- if $P_1 \mathscr{B} Q_1$ and $P_2 \mathscr{B} Q_2$ then $P_1 + P_2 \mathscr{B} Q_1 + Q_2$ and $\forall S \subseteq A,\ P_1 \|_S P_2 \mathscr{B} Q_1 \|_S Q_2$;
- if $P \mathscr{B} Q$ then $\forall \alpha \in Act,\ \alpha.P \mathscr{B} \alpha.Q$, $\forall I \subseteq A,\ \tau_I(P) \mathscr{B} \tau_I(Q)$, $\forall \mathcal{R} \subseteq A \times A,\ \mathcal{R}(P) \mathscr{B} \mathcal{R}(Q)$, $\forall L \subseteq U \subseteq A,\ \theta_L^U(P) \mathscr{B} \theta_L^U(Q)$ and $\forall X \subseteq A,\ \psi_X(P) \mathscr{B} \psi_X(Q)$;
- if $\mathcal{S}$ is a recursive specification with $z \in V_\mathcal{S}$ and $\rho, \nu \in V \setminus V_\mathcal{S} \to \mathbb{P}$ are substitutions such that $\forall x \in V \setminus V_\mathcal{S},\ \rho(x) \mathscr{B} \nu(x)$, then $\langle z|\mathcal{S}\rangle[\rho] \mathscr{B} \langle z|\mathcal{S}\rangle[\nu]$;
- if $\mathcal{S}$ and $\mathcal{S}'$ are recursive specifications and $x \in V_\mathcal{S} = V_{\mathcal{S}'}$ with $\langle x|\mathcal{S}\rangle, \langle x|\mathcal{S}'\rangle \in \mathbb{P}$ such that $\forall y \in V_\mathcal{S},\ \mathcal{S}_y \leftrightarrow_{br}^r \mathcal{S}'_y$, then $\langle x|\mathcal{S}\rangle \mathscr{B} \langle x|\mathcal{S}'\rangle$.

Since $\leftrightarrow_{br}^r \subseteq \mathscr{B}$, it suffices to prove that $\mathscr{B}$ is a rooted branching time-out bisimulation up to $\leftrightarrow_{br}$, as done in Appendix I. This implies $\mathscr{B} = \leftrightarrow_{br}^r$ and the definition will then give us that $\leftrightarrow_{br}^r$ is a lean congruence. Moreover, the last condition of $\mathscr{B}$ adds that it is a full congruence. A similar proof yields the result for $\leftrightarrow_{tb}^r$. ◀

5 Axiomatisation

We will provide complete axiomatisations for $\leftrightarrow_{br}^r$ and $\leftrightarrow_{tb}^r$ on various fragments of CCSP_t^θ.

5.1 Recursive Principles

The expression $\langle x|\mathcal{S}\rangle$ is intuitively defined as the x-component of the solution of $\mathcal{S}$. However, $\mathcal{S}$ could perfectly well have multiple solutions that are not bisimilar to each other. For instance, take $\mathcal{S} = \{x = x\}$; any expression is an x-component of a solution of $\mathcal{S}$. For our complete axiomatisation, we need to restrict attention to recursive specifications which have a unique solution with respect to our notion of bisimilarity. This property can be decomposed into two principles [1, 4]: the *recursive definition principle* (RDP) states that a system of recursive equations has at least one solution and the *recursive specification principle* (RSP) that it has at most one solution. The latter holds under a condition traditionally called *guardedness*.

▶ **Definition 19.** Let $\mathcal{S}$ be a recursive specification and $\sim \subseteq \mathbb{P} \times \mathbb{P}$, a *solution up to* $\sim$ of $\mathcal{S}$ is a substitution $\rho \in \mathbb{E}^{V_\mathcal{S}}$ such that $\rho \sim \mathcal{S}[\rho]$. Here ρ and $\mathcal{S} \in \mathbb{E}^{V_\mathcal{S}}$ are seen as $V_\mathcal{S}$-tuples.

In [1, 4] RDP was proven for the classical notion of strong bisimilarity $\leftrightarrow$. Since $\leftrightarrow_{br}^r$ and $\leftrightarrow_{tb}^r$ are included in $\leftrightarrow$, it holds for both of these relations as well.

▶ **Proposition 20** (RDP). *Let $\mathcal{S}$ be a recursive specification. The substitution $\rho : x \mapsto \langle x|\mathcal{S}\rangle$ for all $x \in V_\mathcal{S}$ is a solution of $\mathcal{S}$ up to $\leftrightarrow$. It is called the* default solution *of $\mathcal{S}$.*

An occurrence of a variable x in an expression E is *well-guarded* if x occurs in a subexpression $a.F$ of E, with $a \in A$. Here we do not allow τ and t as guards. An expression E is *well-guarded* if no operator τ_I occurs in E and all free occurrences of variables in E are well-guarded. A recursive specification $\mathcal{S}$ is *manifestly well-guarded* if no operator τ_I occurs in $\mathcal{S}$ and for all $x, y \in V_\mathcal{S}$ all occurrences of x in the expression $\mathcal{S}_y$ are well-guarded; it is *well-guarded* if it can be made manifestly well-guarded by repeated substitution of $\mathcal{S}_y$ for y within terms $\mathcal{S}_x$. A CCSP$_t^\theta$ process $P \in \mathbb{P}$ is *guarded* if each recursive specification occurring in E is well-guarded. It is *strongly guarded* if moreover there is no infinite path of τ and t-transitions $P_0 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} P_2 \xrightarrow{\alpha_1} \ldots$ with $\alpha_i \in \{\tau,\mathrm{t}\}$ for all $i > 0$, starting in a state P_0 reachable from P.

▶ **Proposition 21** (RSP). *Let $\mathcal{S}$ be a well-guarded recursive specification and $\rho, \nu \in \mathbb{E}^{V_\mathcal{S}}$. If ρ and ν are solutions of $\mathcal{S}$ up to $\leftrightarroweq_{br}^r$ (or $\leftrightarroweq_{\mathrm{tb}}^r$) then $\rho \leftrightarroweq_{br}^r \nu$ (resp. $\rho \leftrightarroweq_{\mathrm{tb}}^r \nu$).*

Proof. Modifying $\mathcal{S}$ by substituting $\mathcal{S}_y$ for y within terms $\mathcal{S}_x$ with $x, y \in V_\mathcal{S}$ does not affect the set of its solutions. Hence we can restrict attention to manifestly well-guarded $\mathcal{S}$.

Thanks to the composition of substitutions, it suffices to prove the proposition when $\rho, \nu \in \mathbb{P}^{V_\mathcal{S}}$ and only variables of $V_\mathcal{S}$ can occur in $\mathcal{S}_x$ for $x \in V_\mathcal{S}$. It suffices to show that the symmetric closure of $\mathscr{B} := \{(H[\mathcal{S}[\rho]], H[\mathcal{S}[\nu]]) \mid H \in \mathbb{E}$ is without τ_I operators and with free variables from $V_\mathcal{S}$ only$\}$ is a rooted branching time-out bisimulation up to $\leftrightarroweq_{br}$. Here $\mathcal{S}[\rho] \in \mathbb{P}^{V_\mathcal{S}}$ is seen as a substitution. Details can be found in Appendix J. An almost identical strategy can be applied to get RSP for $\leftrightarroweq_{\mathrm{tb}}^r$. ◀

The following lemma, whose proof can be found in Appendix K, states that, when considering strongly guarded processes, eliding a time-out is independent of the set of allowed actions.

▶ **Lemma 22.** *Let P be a strongly guarded CCSP$_t^\theta$ process and $X \subseteq A$. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $P \xrightarrow{\mathrm{t}} P'$ and $\theta_X(P) \leftrightarroweq_{br} \theta_X(P')$ then $\forall Y \subseteq A$, $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset \Rightarrow \theta_Y(P) \leftrightarroweq_{br} \theta_Y(P')$.*

Actually, this lemma holds because our restriction of strong guardedness is too strong. Indeed, the equation $x = t.(a + \tau.x)$ has a single solution, but it is not well-guarded. The process $P = \langle x \mid \{x = t.(a + \tau.x)\} \rangle$ is not guarded, yet satisfies $P \xrightarrow{\mathrm{t}} P' := a + \tau.P$ and $\theta_\emptyset(P) \leftrightarroweq_{br} \theta_\emptyset(P')$, while $\theta_{\{a\}}(P) \not\leftrightarroweq_{br} \theta_{\{a\}}(P')$. Even if we write P as $\tau_{\{b\}}(\langle x \mid \{x = t.(a + b.x)\} \rangle$ it fails to be strongly guarded. This restriction was kept because being more precise is very challenging. For instance, the equation $x = t.(a + \tau.x) + t.a$ has multiple solutions: the default one, $\langle x \mid \{x = t.(a + \tau.x) + t.a + t.(a + t.b)\} \rangle$ and others. Notice that adding a branch $t.a$ to an equation with one solution can lead to it having multiple ones. Intuitively, there are situations where time-out contraction enables to hide the existence of other time-outs. Characterising these situations requires the use of semantic conditions that are difficult to verify, thus, making them undesirable. Moreover, applying the Pohlmann encoding to the processes in order to, then, use the axiomatisation of $\leftrightarroweq_{tb}^r$ leads to the similar complications. This limitation deserves to be studied properly because it will appear for all bisimilarities authorising time-out contraction.

5.2 Axioms and Soundness

The set of axioms provided is composed of the axiomatisation of $\leftrightarroweq_r$ [11], together with three branching axioms. The *branching axiom* is well-known since it is used in the axiomatisation of rooted branching bisimilarity [13]. The t-*branching axiom* and the τ/t-*branching axiom* are newly introduced; they are the adaptation of the branching axiom to time-out contraction.

Let Ax^∞ be the set of all axioms in the first two rectangles in Table 3 and $Ax := Ax^\infty \setminus \{\mathrm{RDP}, \mathrm{RSP}\}$. Let Ax_r^∞ be the set of all axioms in Table 3 except the τ/t-branching

$x+(y+z)=(x+y)+z$	$\tau_I(x+y)=\tau_I(x)+\tau_I(y)$	$\mathcal{R}(x+y)=\mathcal{R}(x)+\mathcal{R}(y)$
$x+y=y+x$	$\tau_I(\alpha.x)=\alpha.\tau_I(x)$ if $\alpha\notin I$	$\mathcal{R}(\tau.x)=\tau.\mathcal{R}(x)$
$x+x=0$	$\tau_I(\alpha.x)=\tau.\tau_I(x)$ if $\alpha\in I$	$\mathcal{R}(\mathrm{t}.x)=\mathrm{t}.\mathcal{R}(x)$
$x+0=x$		$\mathcal{R}(a.x)=\sum_{\{b\mid\mathcal{R}(a,b)\}}b.\mathcal{R}(x)$

Expansion Theorem: if $P=\sum\limits_{i\in I}\alpha_i.P_i$ and $Q=\sum\limits_{j\in J}\beta_j.Q_j$ then

$$P\parallel_S Q=\sum_{i\in I,\alpha_i\notin S}(\alpha_i.P_i\parallel_S Q)+\sum_{j\in J,\beta_j\notin S}(P\parallel_S \beta_j.Q_j)+\sum_{i\in I,j\in J,\alpha_i=\beta_j\in S}\alpha_i.(P_i\parallel_S Q_j)$$

$$\alpha.(\tau.(x+y)+x)=\alpha.(x+y)\qquad\textbf{(Branching Axiom)}$$

$$\alpha.(\mathrm{t}.(x+\textstyle\sum_{i\in I}\mathrm{t}.y_i)+x)=\alpha.(x+\textstyle\sum_{i\in I}\mathrm{t}.y_i)\qquad\textbf{(t-Branching Axiom)}$$

$$\alpha.(\tau.(x+y)+\mathrm{t}.(x+y)+x)=\alpha.(x+y)\qquad\textbf{(τ/t-Branching Axiom)}$$

$\langle x|\mathcal{S}\rangle=\langle\mathcal{S}_x|\mathcal{S}\rangle$ **(RDP)** $\qquad\mathcal{S}\Rightarrow x=\langle x|\mathcal{S}\rangle$ with $\mathcal{S}$ well-guarded **(RSP)**

$\theta_L^U(\sum_{i\in I}\alpha_i.x_i)=\sum_{i\in I}\alpha_i.x_i$	$(\forall i\in I,\alpha_i\notin L\cup\{\tau\})$
$\theta_L^U(x+\alpha.y+\beta.z)=\theta_L^U(x+\alpha.y)$	$(\alpha\in L\cup\{\tau\}\wedge\beta\notin U\cup\{\tau\})$
$\theta_L^U(x+\alpha.y+\beta.z)=\theta_L^U(x+\alpha.y)+\theta_L^U(\beta.z)$	$(\alpha\in L\cup\{\tau\}\wedge\beta\in U\cup\{\tau\})$
$\theta_L^U(\alpha.x)=\alpha.x$	$(\alpha\neq\tau)$
$\theta_L^U(\tau.x)=\tau.\theta_L^U(x)$	
$\psi_X(x+\alpha.y)=\psi_X(x)+\alpha.y$	$(\alpha\notin X\cup\{\tau,t\})$
$\psi_X(x+\alpha.y+\mathrm{t}.z)=\psi_X(x+\alpha.y)$	$(\alpha\in X\cup\{\tau\})$
$\psi_X(x+\alpha.y+\beta.z)=\psi_X(x+\alpha.y)+\beta.z$	$(\alpha,\beta\in X\cup\{\tau\})$
$\psi_X(\alpha.x)=\alpha.x$	$(\alpha\neq\mathrm{t})$
$\psi_X(\sum_{i\in I}\mathrm{t}.y_i)=\sum_{i\in I}\mathrm{t}.\theta_X(y_i)$	

$$(\forall X\subseteq A,\ \psi_X(x)=\psi_X(y))\Rightarrow x=y\qquad\textbf{(Reactive Approximation Axiom)}$$

Table 3 Axiomatisation of $\underline{\leftrightarrow}_{br}^r$ and $\underline{\leftrightarrow}_{tb}^r$

one and $Ax_r:=Ax_r^\infty\setminus\{\mathrm{RDP},\mathrm{RSP}\}$. The τ/t-branching axiom is removed from Ax_r^∞ because the law $\mathbf{L}\tau$: $\tau.x+\mathrm{t}.y=\tau.x$ can be derived from the reactive approximation axiom [11], and applying $\mathbf{L}\tau$ to the branching axiom yields the τ/t-branching axiom, thus making it redundant.

▶ **Proposition 23.** *Let P,Q be two $CCSP_t^\theta$ processes.*

- *If $Ax^\infty\vdash P=Q$ then $P\underline{\leftrightarrow}_{tb}^r Q$.*
- *If $Ax_r^\infty\vdash P=Q$ then $P\underline{\leftrightarrow}_{br}^r Q$.*

Proof. Since $\underline{\leftrightarrow}_{br}^r$ and $\underline{\leftrightarrow}_{tb}^r$ are congruences, it suffices to prove that each axiom is sound, meaning that replacing, in each axiom, $=$ by the desired bisimilarity and each variable by a process produces a true statement. Most of these axioms were proven to be sound for the classical notion $\underline{\leftrightarrow}$ of strong bisimilarity [17] in [11]. Thus, since both $\underline{\leftrightarrow}_{br}^r$ and $\underline{\leftrightarrow}_{tb}^r$ are included in $\underline{\leftrightarrow}$, most of them are sound for $\underline{\leftrightarrow}_{br}^r$ and $\underline{\leftrightarrow}_{tb}^r$.

Only the branching axioms, RSP and the reactive approximation axiom remain to be proven sound. The soundness of the branching axioms is trivial and the soundness of RSP is exactly Proposition 21. For the reactive approximation axiom, it suffices to show that $\mathscr{B}:=\underline{\leftrightarrow}_{br}^r\cup\{(P,Q),(Q,P)\mid\forall X\subseteq A,\psi_X(P)\underline{\leftrightarrow}_{br}^r\psi_X(Q)\}$ is a rooted branching time-out bisimulation, as done in Appendix L. ◀

5.3 Completeness

A well-known feature of most process algebras is that the standard collection of axioms allows one to bring any guarded process expression in the following normal form [1, 4].

▶ **Definition 24.** Let P be a guarded CCSP_t^θ process. The *head-normal form* of P is $\hat{P} := \sum_{\{(\alpha,Q) \mid P \xrightarrow{\alpha} Q\}} \alpha.Q$.

In [11], it is proven that the axiomatisation of $\leftrightarroweq_r$ enables one to equate any guarded process with its head-normal form (using a definition of guardedness that is more liberal than the one employed here, with τ and t allowed as guards). Since the axiomatisation of $\leftrightarroweq_r$ is included in Ax^∞ and Ax_r^∞, this yields the property for them as well.

▶ **Lemma 25.** *Let P be a guarded $CCSP_t^\theta$ process. Then $Ax^\infty \vdash P = \hat{P}$ and $Ax_r^\infty \vdash P = \hat{P}$. Moreover, Ax or Ax_r are sufficient if P is recursion-free.* ◀

This lemma is used extensively in the proof of the following completeness results.

▶ **Proposition 26.** *Let P, Q be two recursion-free $CCSP_t^\theta$ processes. If $P \leftrightarroweq_{br} Q$ (resp. $P \leftrightarroweq_{\mathrm{t}b} Q$) then, for all $\alpha \in Act$, $Ax_r \vdash \alpha.\hat{P} = \alpha.\hat{Q}$ (resp. $Ax \vdash \alpha.\hat{P} = \alpha.\hat{Q}$).*

Proof. The *depth* $d(p)$ of a process P is the length of the longest path starting from P. Note that it is properly defined for recursion-free processes only. The proof proceeds by induction on $\max(d(P), d(Q))$. The technique is fairly standard and the details can be found in Appendix M. ◀

▶ **Theorem 27.** *Let P, Q be two recursion-free $CCSP_t^\theta$ processes. If $P \leftrightarroweq_{br}^r Q$ (resp. $P \leftrightarroweq_{\mathrm{t}b}^r Q$) then $Ax_r \vdash P = Q$ (resp. $Ax \vdash P = Q$).*

Proof. It suffices to express both processes in their head-normal form and then to equate each pair of matching branches using Proposition 26. Details are in Appendix M. ◀

The following theorem lifts this result for $\leftrightarroweq_{\mathrm{t}b}^r$ from finite (recursion-free) processes to arbitrary (infinite) ones, subject to the restriction of strong guardedness.

▶ **Theorem 28.** *Let P, Q be strongly guarded $CCSP_t^\theta$ processes.
If $P \leftrightarroweq_{\mathrm{t}b}^r Q$ then $Ax^\infty \vdash P = Q$.*

Proof. A well-known technique called *equation merging* can be applied. Details can be found in Appendix N. ◀

5.4 Canonical Representative

Unfortunately, equation merging does not work on reactive bisimulations [11]. Thus, another technique is used [14, 16], called *canonical representatives.* The idea is to build the simplest process for each equivalence class of $\leftrightarroweq_{br}^r$ and use them as intermediary to equate processes.

Let us denote with $\mathbb{P}^g$ the strongly guarded fragment of $\mathbb{P}$. For all $P \in \mathbb{P}^g$, $[P] := \{Q \in \mathbb{P}^g \mid P \leftrightarroweq_{br} Q\}$ is the $\leftrightarroweq_{br}$-equivalence class of P. $[\mathbb{P}^g]$ denotes the set of all $\leftrightarroweq_{br}$-equivalence classes. Using the axiom of choice, a choice function $\chi : [\mathbb{P}^g] \to \mathbb{P}^g$ can be defined such that $\forall R \in [\mathbb{P}^g], \chi(R) \in R$. A transition relation can be defined between $\leftrightarroweq_{br}$-equivalence classes:

$$\forall \alpha \in A_\tau, (R \xrightarrow{\alpha} R' \Leftrightarrow \chi(R) \Longrightarrow P_1 \xrightarrow{\alpha} P_2 \wedge P_1 \in R \wedge P_2 \in R' \wedge (\alpha \in A \vee R \neq R'))$$

$$R \xrightarrow{\mathrm{t}} R' \Leftrightarrow \exists X \subseteq A, r > 0, \chi(R) \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow P_{2r-1} \xrightarrow{\mathrm{t}} P_{2r}$$
$$\wedge \forall i \in [0, r{-}1], \theta_X(P_{2i}) \in [\theta_X(\chi(R))] \wedge \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$$
$$\wedge P_1 \in R \wedge P_{2r} \in R' \wedge [\theta_X(\chi(R))] \neq [\theta_X(\chi(R'))]$$

All bisimulations can be extended to $\leftrightarrow_{br}$-equivalence classes. It suffices to consider the set of states $\mathbb{P}^g \uplus [\mathbb{P}^g] \uplus \{\theta_X([P]) \mid X \subseteq A \wedge P \in \mathbb{P}^g\}$.

▶ **Proposition 29.** *Let $P \in \mathbb{P}^g$, $P \leftrightarrow_{br} [P]$.*

Proof. It suffices to prove that $\mathscr{B} := \{(P,[P]),([P],P) \mid P \in \mathbb{P}^g\}$ is a branching time-out bisimulation up to $\leftrightarrow_{br}$. Details can be found in Appendix O. ◀

▶ **Definition 30.** Let $P, Q \in \mathbb{P}^g$, the *canonical representative* of P and Q is a recursive specification $\mathcal{S}$ such that $V_\mathcal{S} := \{x_P, x_Q\} \cup \{x_R \mid R \in \bigcup_{P' \in \text{Reach}(P) \cup \text{Reach}(Q)} \text{Reach}([P'])\}$, and $\forall R \in \bigcup_{P' \in \text{Reach}(P) \cup \text{Reach}(Q)} \text{Reach}([P'])$,

$$\mathcal{S}_{x_P} := \sum_{\{(\alpha,P') \mid P \xrightarrow{\alpha} P'\}} \alpha.x_{[P']} \;;\; \mathcal{S}_{x_Q} := \sum_{\{(\alpha,Q') \mid Q \xrightarrow{\alpha} Q'\}} \alpha.x_{[Q']} \text{ and } \mathcal{S}_{x_R} := \sum_{\{(\alpha,R') \mid R \xrightarrow{\alpha} R'\}} \alpha.x_{R'}$$

The canonical representative is well-defined since P, Q, as well as processes $[P'] \in [\mathbb{P}^g]$ are finitely branching [11]. Additionally, $\bigcup_{P' \in \text{Reach}(P) \cup \text{Reach}(Q)} \text{Reach}([P'])$ is countable. Moreover, $\mathcal{S}$ is strongly guarded. Furthermore, by construction $\langle x_R | \mathcal{S} \rangle \leftrightarrow R$ for all $R \in \bigcup_{P' \in \text{Reach}(P) \cup \text{Reach}(Q)} \text{Reach}([P'])$.

▶ **Proposition 31.** *Let $P, Q \in \mathbb{P}^g$ and $\mathcal{S}$ be the canonical representative of P and Q. $Ax_r^\infty \vdash P = \langle x_P | \mathcal{S} \rangle$.*

Proof. It suffices to show that P and $\langle x_P | \mathcal{S} \rangle$ are y_P-components of solutions of $\{y_{P^\dagger} = \sum_{\{(\alpha,P^\ddagger) \mid P^\dagger \xrightarrow{\alpha} P^\ddagger\}} \alpha.y_{P^\ddagger} \mid P^\dagger \in \text{Reach}(P)\}$. Details can be found in Appendix P. ◀

▶ **Theorem 32.** *Let $P, Q \in \mathbb{P}^g$. If $P \leftrightarrow_{br}^r Q$ then $Ax_r^\infty \vdash P = Q$.*

Proof. It suffices to equate $\langle x_P | \mathcal{S} \rangle$ and $\langle x_Q | \mathcal{S} \rangle$ using RDP and the reactive approximation axiom. Details can be found in Appendix P. ◀

Conclusion

This paper defined a form of branching bisimilarity for processes with time-out transitions, and provided a modal characterisation, congruence results, and a complete axiomatisation for strongly guarded processes. The restriction to strongly guarded processes is rather severe; it rules out processes that may engage in an infinite sequence of time-out transitions, interspersed with τs. Relaxing this restriction is a suitable topic for further work. Another task is to combine this work with the ideas behind *justness* [12], a weaker form of fairness that allows the formulation and derivation of useful liveness properties. In a setting with time-outs, justness would demand that once a parallel component reaches a state in which a time-out transition is enabled, it cannot stay in that state forever after.

As an example of the use of branching reactive bisimulation, one could verify the correctness of a non-trivial system, such as Peterson's mutual exclusion protocol, as modelled in [10]. There it was argued that a similar model without time-out transitions is not possible. The model from [10] features eight visible actions of entering or leaving the critical or non-critical section of process A or B. Abstracting from all actions pertaining to process B yields a protocol that only deals with process A, and a correctness claim could be validated by showing it branching reactive bisimilar with a simple specification of the intended behaviour of A that would apply when B were not around. Although doing such a verification is entirely feasible, for now, it can not be achieved by algebraic means, using our complete axiomatisation. The reason is that abstraction from process B yields infinite sequences of unobservable actions, which are currently not covered by our work.

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A Examples

Scope of the First Clause of Definition 1

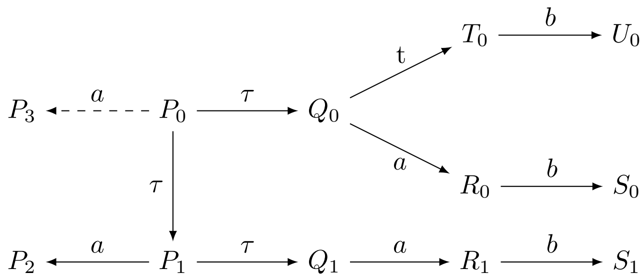

Figure 2 Counter-Example to a Naive Clause 1.a.

In Figure 2, the process $a.0 + \tau.(\mathrm{t}.b.0 + a.b.0) + \tau.(\tau.a.b.0 + a.0)$ is represented as an LTS. Let $A := \{a, b\}$. Removing the dashed a-transition generates the process $\tau.(\mathrm{t}.b.0 + a.b.0) + \tau.(\tau.a.b.0 + a.0)$.

First, we are going to show that these two processes are not branching reactive bisimilar. Let's try to build a branching reactive bisimulation between them. The only way to match the dashed a-transition of $a.0+\tau.(\mathrm{t}.0+a.b.0)+\tau.(\tau.a.b.0+a.0)$ is by the a-transition between P_1 and P_2, because all other a-transitions are followed by a b-transition. This requires to elide the τ-transition between P_0 and P_1, who must be branching reactive bisimilar. Since $P_0 \leftrightarrow_{br} P_1$, when considering the τ-transition between P_0 and Q_0, Q_0 has to be branching reactive bisimilar to P_1 or Q_1. Now, the a-transition between Q_0 and R_0 has to be matched by the a-transition between Q_1 and R_1 because of the following b-transition. This implies $Q_0 \leftrightarrow_{br} Q_1$, thus, $Q_0 \leftrightarrow^{\emptyset}_{br} Q_1$. One has $\mathcal{I}(Q_0) \cap (\emptyset \cup \{\tau\}) = \emptyset$ and $Q_0 \xrightarrow{\mathrm{t}} T_0$, i.e., when the environment temporary allows no visible actions, Q_0 can time-out into a state in which b is possible. This behaviour cannot be matched by Q_1—a contradiction.

Now, consider the alternative to Definition 1 in which the first clause has been changed to
1. a. if $P \xrightarrow{\tau} P'$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ with $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$.
In other words, the scope of the first clause is restricted to τ-transitions. This modification enables building a bisimulation between the two processes. Indeed, the dashed a-transition is only considered when the environment allows a. Thus, it is sufficient to get $P_0 \leftrightarrow^{A}_{br} P_1$ and $P_0 \leftrightarrow^{\{a\}}_{br} P_1$ and not $P_0 \leftrightarrow_{br} P_1$ anymore. Therefore, it is sufficient to match Q_0 and Q_1 in environments allowing a. As a result, the outgoing time-out transition of Q_0 is never considered when matching Q_0 with Q_1, solving our previous issue. Once this observation is made, building the bisimulation is trivial.

Finally, place both processes in the context $\_\_ \|_{\{a\}} (\tau.0 + a.0)$. It behaves like a one-way switch enabling to block all a-transitions forever as soon as the τ-transition is performed. Let's try to build a branching reactive bisimulation between the two processes. Following the same reasoning as before, it is necessary to get $P_0 \|_{\{a\}} (\tau.0 + a.0) \leftrightarrow^{A}_{br} P_1 \|_{\{a\}} (\tau.0 + a.0)$ because of the dashed a-transition, and then $Q_0 \|_{\{a\}} (\tau.0 + a.0) \leftrightarrow^{A}_{br} Q_1 \|_{\{a\}} (\tau.0 + a.0)$ because of the a-transition between Q_0 and R_0. Note that $Q_0 \|_{\{a\}} (\tau.0 + a.0) \xrightarrow{\tau} Q_0 \|_{\{a\}} 0 \xrightarrow{\mathrm{t}} T_0 \|_{\{a\}} 0 \xrightarrow{b} U_0 \|_{\{a\}} 0$ and $\mathcal{I}(Q_0 \|_{\{a\}} 0) \cap (A \cup \{\tau\}) = \emptyset$. As before, $Q_0 \|_{\{a\}} (\tau.0 + a.0)$ can time-out into a state in which b is executable, whereas this behaviour is impossible in $Q_1 \|_{\{a\}} (\tau.0 + a.0)$. As a result, restricting the scope of the first clause of Definition 1 to τ-transitions prevents $\leftrightarrow_{br}$ from being a congruence for parallel composition.

Necessity of the Stability Respecting Clause

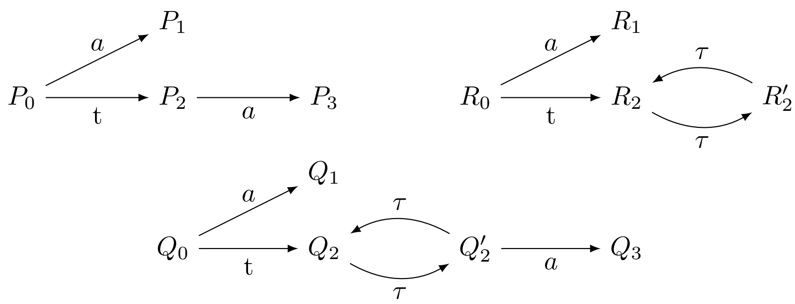

Figure 3 Counter-Example to the Absence of a Stability Respecting Clause

In Figure 3, three processes are represented as LTSs. Take $A := \{a\}$. According to Definition 1, $\neg(P_0 \leftrightarroweq_{br} Q_0)$ and $Q_0 \leftrightarroweq_{br} R_0$.

Let's try to build a branching reactive bisimulation between the top-left and bottom processes. Matching the time-out between Q_0 and Q_2 implies that $Q_2 \leftrightarroweq^{\emptyset}_{br} P_0$ or $Q_2 \leftrightarroweq^{\emptyset}_{br} P_2$. However, $P_0 \stackrel{\tau}{\not\rightarrow}$ and $P_2 \stackrel{\tau}{\not\rightarrow}$, thus, there should be a path $Q_2 \Longrightarrow Q_2' \stackrel{\tau}{\not\rightarrow}$, but this is not the case.

The symmetric closure of

$$\mathscr{R} := \{(Q_0, R_0), (Q_1, R_1), (Q_2, \emptyset, R_2), (Q_2', \emptyset, R_2')\} \cup \{(Q_0, X, R_0), (Q_1, X, R_1) \mid X \subseteq A\}$$

is a branching reactive bisimulation. The a-transition between Q_2' and Q_3 does not have to be matched since Q_2' is considered only when the environment disallows a.

Now, suppose that the stability respecting condition is removed from Definition 1. As a result, a branching reactive bisimulation can be built between the top-left and bottom processes. The symmetric closure of

$$\begin{aligned}\mathscr{R}' := &\{(P_0, Q_0), (P_1, Q_1), (P_2, Q_2), (P_2, Q_2'), (P_3, Q_3)\} \\ &\cup \{(P_0, X, Q_0), (P_1, X, Q_1), (P_2, X, Q_2), (P_2, X, Q_2'), (P_3, X, Q_3) \mid X \subseteq A\}\end{aligned}$$

would be a branching reactive bisimulation. Moreover, $\mathscr{R}$ would still be a branching reactive bisimulation, since Definition 1 has merely been weakened. Therefore, according to the modified Definition 1, $P_0 \leftrightarroweq_{br} Q_0$ and $Q_0 \leftrightarroweq_{br} R_0$. However, when trying to construct a branching reactive bisimulation between P_0 and R_0, because of the time-out transition, R_2 has to be matched to P_0 or P_2 and no a-transition is reachable from R_2; therefore, $\neg(P_0 \leftrightarroweq_{br} R_0)$. As a result, removing the stability respecting clause from Definition 1 prevents $\leftrightarroweq_{br}$ from being an equivalence relation.

B Concrete Time-out Version

Before studying $\leftrightarroweq_{br}$, we looked at another version which is not eliding any time-out transitions. More formally, it is defined by replacing Clause 2.d of Definition 1 by

2. d. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \stackrel{\mathrm{t}}{\rightarrow} P'$ then there exists a path $Q \Longrightarrow Q_1 \stackrel{\mathrm{t}}{\rightarrow} Q_2$ with $\mathscr{R}(P', X, Q_2)$.

It is not necessary to require to match with an executable time-out (i.e. $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$) since this is implied by the other clauses. It is also implied that $\mathscr{R}(P, X, Q_1)$ in the above clause. This bisimilarity has properties similar to $\leftrightarroweq_{br}$, to be recapped below. No proof will be provided here since they rely on the same strategies and are actually simpler because of the absence of time-out omission. However, a technical report [21] is available. In the remainder of this appendix, $\leftrightarroweq^{c}_{br}$ stands for the concrete time-out version.

The stuttering lemma (Lemma 3) still holds and $\leftrightarroweq^{c}_{br}$ and $(\leftrightarroweq^{cX}_{br})_{X\subseteq A}$ are still equivalence relations (Proposition 4). The rooted version $\leftrightarroweq^{cr}_{br}$ of $\leftrightarroweq^{c}_{br}$ is exactly Definition 5 and $\leftrightarroweq^{cr}_{br}$ and $(\leftrightarroweq^{crX}_{br})_{X\subseteq A}$ are still equivalence relations (Proposition 6). The Pohlmann encoding (Table 4) is simplified as the rooted variants are no longer needed. If $\leftrightarroweq^{s}_{b}$ stands for the classical stability respecting branching bisimulation [13, 7], and $\leftrightarroweq^{sr}_{b}$ for its rooted version, $P \leftrightarroweq^{c}_{br} Q \Leftrightarrow \vartheta(P) \leftrightarroweq^{s}_{b} \vartheta(Q)$; $P \leftrightarroweq^{cX}_{br} Q \Leftrightarrow \vartheta_X(P) \leftrightarroweq^{s}_{b} \vartheta_X(Q)$; $P \leftrightarroweq^{cr}_{br} Q \Leftrightarrow \vartheta(P) \leftrightarroweq^{sr}_{b} \vartheta(Q)$ and $P \leftrightarroweq^{crX}_{br} Q \Leftrightarrow \vartheta_X(P) \leftrightarroweq^{sr}_{b} \vartheta_X(Q)$.

The generalised definition of $\leftrightarroweq^{c}_{br}$ can be obtained by replacing Clause 1.b. and 2.c. in Definition 33 by

1. **b.** If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2$ with $\mathscr{R}(P', X, Q_2)$
2. **c.** If $\mathcal{I}(P) \cap (X \cup Y \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2$ with $\mathscr{R}(P', Y, Q_2)$

The rooted generalised version is exactly Definition 34 and they induce the same bisimilarities as the previous definitions (Proposition 35). In the modal characterisation, $X\varphi$ is not useful anymore, nor $\varphi\langle\varepsilon_X\rangle\varphi'$. Replacing the fifth induction rule of $\mathbb{L}_b$ by $\langle\varepsilon\rangle\langle \mathrm{t}_X\rangle\varphi$ yields the counterpart of Theorem 11.

The corresponding time-out bisimulation can be obtained by replacing Clause 2. of Definition 13 by

2. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there exists a path $P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2$ with $\theta_X(P') \mathrel{\mathscr{B}} \theta_X(Q_2)$.

The rooted time-out bisimulation is exactly Definition 14 and they agree with the previous definitions (Proposition 15). $\leftrightarroweq^{c}_{br}$ is a congruence for prefixing, parallel composition, abstraction, renaming and the operator θ^U_L (Proposition 17). $\leftrightarroweq^{cr}_{br}$ is a full congruence (Theorem 18).

As $\leftrightarroweq^{cr}_{br} \subseteq \leftrightarroweq$, RDP holds for $\leftrightarroweq^{cr}_{br}$. The definition of well-guarded recursion can be weakened by allowing t as a guard and RSP holds for $\leftrightarroweq^{cr}_{br}$ on processes that are guarded in this sense. Lemma 22 is not useful anymore since time-out omissions are not considered. The set of all axioms of Table 3 except the t-branching and τ/t-branching ones is a complete axiomatisation of $\leftrightarroweq^{c}_{br}$ (Theorem 32). Moreover, to obtain the complete axiomatisation of $\leftrightarroweq^{c}_{br}$ on recursion-free processes, it suffices to remove RDP and RSP.

C Generalised branching reactive bisimulation

The second clause of Definition 1 is quite tedious to check; thus, an equivalent definition of the bisimilarity would be useful. Actually, it is possible to define the exact same notion in a more general way at the cost of some clear motivations.

▶ **Definition 33.** A *generalised branching reactive bisimulation* is a symmetric relation $\mathscr{R} \subseteq (\mathbb{P} \times \mathbb{P}) \cup (\mathbb{P} \times \mathcal{P}(A) \times \mathbb{P})$ such that, for all $P, Q \in \mathbb{P}$ and $X \subseteq A$,

1. if $\mathscr{R}(P, Q)$
   - **a.** if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ with $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$,
   - **b.** if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there is a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r-1]$, $\mathscr{R}(P, X, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r})$,

   c. if $P \stackrel{\tau}{\not\rightarrow}$ then there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$;
2. if $\mathscr{R}(P,X,Q)$
   a. if $P \stackrel{\tau}{\rightarrow} P'$ then there is a path $Q \Longrightarrow Q_1 \stackrel{(\tau)}{\longrightarrow} Q_2$ with $\mathscr{R}(P,X,Q_1)$ and $\mathscr{R}(P',X,Q_2)$,
   b. if $P \stackrel{a}{\rightarrow} P'$ with $a \in X \vee \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then there is a path $Q \Longrightarrow Q_1 \stackrel{a}{\rightarrow} Q_2$ with $\mathscr{R}(P,X,Q_1)$ and $\mathscr{R}(P',Q_2)$,
   c. if $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P \stackrel{\mathrm{t}}{\rightarrow} P'$ then there is a path $Q = Q_0 \Longrightarrow Q_1 \stackrel{\mathrm{t}}{\rightarrow} Q_2 \Longrightarrow Q_3 \stackrel{\mathrm{t}}{\rightarrow} \ldots \Longrightarrow Q_{2r-1} \stackrel{(\mathrm{t})}{\longrightarrow} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r-1]$, $\mathscr{R}(P,Y,Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P',Y,Q_{2r})$,
   d. if $P \stackrel{\tau}{\not\rightarrow}$ then there is a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.

The strong point of the generalised definitions is the restriction on the use of triplets, making use of them only after performing a time-out. A generalised version of rooted branching reactive bisimulation can be defined in a similar fashion.

▶ **Definition 34.** A *generalised rooted branching reactive bisimulation* is a symmetric relation $\mathscr{R} \subseteq (\mathbb{P} \times \mathbb{P}) \cup (\mathbb{P} \times \mathcal{P}(A) \times \mathbb{P})$ such that, for all $P, Q \in \mathbb{P}$ and $X \subseteq A$,
1. if $\mathscr{R}(P,Q)$
   a. if $P \stackrel{\alpha}{\rightarrow} P'$ with $\alpha \in A_\tau$ then there is a transition $Q \stackrel{\alpha}{\rightarrow} Q'$ such that $P' \leftrightarroweq_{br} Q'$,
   b. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \stackrel{\mathrm{t}}{\rightarrow} P'$ then there is a transition $Q \stackrel{\mathrm{t}}{\rightarrow} Q'$ with $P' \leftrightarroweq_{br}^{X} Q'$,
2. if $\mathscr{R}(P,X,Q)$
   a. if $P \stackrel{\tau}{\rightarrow} P'$ then there is a transition $Q \stackrel{\tau}{\rightarrow} Q'$ such that $P' \leftrightarroweq_{br}^{X} Q'$,
   b. if $P \stackrel{a}{\rightarrow} P'$ with $a \in X \vee \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then there is a transition $Q \stackrel{a}{\rightarrow} Q'$ such that $P' \leftrightarroweq_{br} Q'$,
   c. if $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P \stackrel{\mathrm{t}}{\rightarrow} P'$ then there is a transition $Q \stackrel{\mathrm{t}}{\rightarrow} Q'$ such that $P' \leftrightarroweq_{br}^{Y} Q'$.

Note that if a system has no time-out, then a generalised [rooted] branching reactive bisimulation is a stability respecting [rooted] branching bisimulation, thus proving that [rooted] branching reactive bisimilarity is indeed an extension of stability respecting [rooted] branching bisimilarity to reactive systems with time-outs.

▶ **Proposition 35.** *Let* $P, Q \in \mathbb{P}$ *and* $X \subseteq A$*,*
- $P \leftrightarroweq_{br} Q$ *(resp.* $P \leftrightarroweq_{br}^{X} Q$*) iff there exists a generalised branching reactive bisimulation* $\mathscr{R}$ *with* $\mathscr{R}(P,Q)$ *(resp.* $\mathscr{R}(P,X,Q)$*),*
- $P \leftrightarroweq_{br}^{r} Q$ *(resp.* $P \leftrightarroweq_{br}^{rX} Q$*) iff there exists a rooted generalised branching reactive bisimulation* $\mathscr{R}$ *with* $\mathscr{R}(P,Q)$ *(resp.* $\mathscr{R}(P,X,Q)$*).*

Proof. Let $\mathscr{R}$ be a branching reactive bisimulation. Let's check that it is a generalised branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.
1. If $\mathscr{R}(P,Q)$
   a. this condition is shared by both definitions
   b. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \stackrel{\mathrm{t}}{\rightarrow} P'$ then, since $\mathscr{R}(P,Q)$, $\mathscr{R}(P,X,Q)$. Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \stackrel{\mathrm{t}}{\rightarrow} P'$, there exists a path $Q = Q_0 \Longrightarrow Q_1 \stackrel{\mathrm{t}}{\rightarrow} Q_2 \Longrightarrow Q_3 \stackrel{\mathrm{t}}{\rightarrow} \ldots \Longrightarrow Q_{2r-1} \stackrel{(\mathrm{t})}{\longrightarrow} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1], \mathscr{R}(P,X,Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P',X,Q_{2r})$. In particular, $Q_1 \stackrel{\tau}{\not\rightarrow}$.
   c. if $P \stackrel{\tau}{\not\rightarrow}$ then, since $\mathscr{R}(P,Q)$, $\mathscr{R}(P,\emptyset,Q)$, so there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.
2. If $\mathscr{R}(P,X,Q)$
   a. this condition is shared by both definitions

b. if $P \xrightarrow{a} P'$ with $a \in X \vee \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then if $a \in X$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', Q_2)$. Otherwise, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and so $P \stackrel{\tau}{\nrightarrow}$, thus there exists a path $Q \Longrightarrow Q_1 \stackrel{\tau}{\nrightarrow}$. Since $\mathscr{R}(P, X, Q)$, $P \stackrel{\tau}{\nrightarrow}$ and $Q \Longrightarrow Q_1$, $\mathscr{R}(P, X, Q_1)$. As $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $Q_1 \stackrel{\tau}{\nrightarrow}$, $\mathscr{R}(P, Q_1)$. Because $P \xrightarrow{a} P_1$ and $Q_1 \stackrel{\tau}{\nrightarrow}$, there exists a transition $Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P', Q_2)$. As a result, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', Q_2)$.

c. if $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, since $P \stackrel{\tau}{\nrightarrow}$, there exists a path $Q \Longrightarrow Q_1 \stackrel{\tau}{\nrightarrow}$. Furthermore, using Clause 2.a of Definition 1, $\mathscr{R}(P, X, Q_1)$. Moreover, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $Q_1 \stackrel{\tau}{\nrightarrow}$, thus, $\mathscr{R}(P, Q_1)$ and $\mathcal{I}(Q_1) = \mathcal{I}(P)$. Since $\mathscr{R}(P, Y, Q_1)$, $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$, there exists a path $Q_1 = Q'_0 \Longrightarrow Q'_1 \xrightarrow{\mathrm{t}} Q'_2 \Longrightarrow Q'_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q'_{2r-1} \xrightarrow{(\mathrm{t})} Q'_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1], \mathscr{R}(P, Y, Q'_{2i}) \wedge \mathcal{I}(Q'_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', Y, Q'_{2r})$. Since $Q_1 \stackrel{\tau}{\nrightarrow}$, $Q_1 = Q'_1$. As a result, there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ such that $Q_1 \stackrel{\tau}{\nrightarrow}$, $\forall i \in [1, r-1]$, $\mathscr{R}(P, Y, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', Y, Q_{2r})$.

d. this condition is shared by both definitions.

Let $\mathscr{R}$ be a generalised branching reactive bisimulation and define

$$\mathscr{R}' := \mathscr{R} \cup \{(P, X, Q) \mid \mathscr{R}(P, Q) \wedge X \subseteq A\} \cup \{(P, Y, Q), (P, Q) \mid \exists X \subseteq A,\ \mathscr{R}(P, X, Q) \\ \wedge (\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (X \cup \{\tau\}) = \emptyset \wedge Y \subseteq A\}$$

$\mathscr{R}'$ is symmetric by definition. Let's check that $\mathscr{R}'$ is a branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}'(P, Q)$ then $\mathscr{R}(P, Q)$ or there exists a set $Y \subseteq A$ such that $\mathscr{R}(P, Y, Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$.

a. If $P \xrightarrow{\alpha} P'$ then

- if $\mathscr{R}(P, Q)$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P, Q_2)$ and, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, Q_1)$ and $\mathscr{R}'(P', Q_2)$
- if there exists $Y \subseteq A$ such that $\mathscr{R}(P, Y, Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, since $\mathscr{R}(P, Y, Q)$ and $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$, $\alpha \neq \tau$, so there exists a path $Q \Longrightarrow Q_1 \xrightarrow{\alpha} Q_2$ such that $\mathscr{R}(P, Y, Q_1)$ and $\mathscr{R}(P, Q_2)$. Since $\mathcal{I}(Q) \cap (Y \cup \{\tau\}) = \emptyset$ and $\mathscr{R} \subseteq \mathscr{R}'$, $Q = Q_1$ so there exists a path $Q \xrightarrow{\alpha} Q_2$ such that $\mathscr{R}'(P, Q)$ and $\mathscr{R}'(P', Q_2)$.

b. For all $Z \subseteq A$,

- if $\mathscr{R}(P, Q)$ then, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, Z, Q)$
- if there exists $Y \subseteq A$ such that $\mathscr{R}(P, Y, Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, Z, Q)$.

2. If $\mathscr{R}'(P, X, Q)$ then $\mathscr{R}(P, X, Q)$, or $\mathscr{R}(P, Q)$, or there exists $Y \subseteq A$ such that $\mathscr{R}(P, Y, Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$.

a. If $P \xrightarrow{\tau} P'$ then

- if $\mathscr{R}(P, X, Q)$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P, X, Q_2)$ and, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, X, Q_1)$ and $\mathscr{R}'(P', X, Q_2)$
- if $\mathscr{R}(P, Q)$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P, Q_2)$ and, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, X, Q_1)$ and $\mathscr{R}'(P', X, Q_2)$
- if there exists $Y \subseteq A$ such that $\mathscr{R}(P, Y, Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then $P \stackrel{\tau}{\nrightarrow}$, so this case is impossible.

b. If $P \xrightarrow{a} P'$ with $a \in X$ then

- if $\mathscr{R}(P, X, Q)$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P, Q_2)$ and, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, X, Q_1)$ and $\mathscr{R}'(P', Q_2)$

- if $\mathscr{R}(P,Q)$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P,Q_1)$ and $\mathscr{R}(P,Q_2)$ and, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,X,Q_1)$ and $\mathscr{R}'(P',Q_2)$
- if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, since $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P,Y,Q_1)$ and $\mathscr{R}(P',Q_2)$. Since $\mathcal{I}(Q) \cap (Y \cup \{\tau\}) = \emptyset$, $Q = Q_1$ so there exists a path $Q \xrightarrow{a} Q_2$ such that $\mathscr{R}'(P,X,Q)$ and $\mathscr{R}'(P',Q_2)$.

c. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then

- if $\mathscr{R}(P,X,Q)$ then, since $P \stackrel{\tau}{\not\rightarrow}$, there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$. By Clause 2.a of Definition 33, $\mathscr{R}(P,X,Q_0)$. Since $\mathscr{R}(P,X,Q_0)$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $Q_0 \stackrel{\tau}{\not\rightarrow}$, $\mathcal{I}(Q) \cap (X \cup \{\tau\}) = \emptyset$, therefore, by definition, $\mathscr{R}'(P,Q_0)$.
- if $\mathscr{R}(P,Q)$ then, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P,Q)$.
- if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,Q)$.

d. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then

- if $\mathscr{R}(P,X,Q)$ then, there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} ... \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r{-}1]$, $\mathscr{R}(P,X,Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P',X,Q_{2r})$. Hence also $\mathscr{R}'(P,X,Q_{2i})$ and $\mathscr{R}'(P',X,Q_{2r})$. By Clause 2.a of Definition 33, $\mathscr{R}(P,X,Q_1)$, so $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$.
- if $\mathscr{R}(P,Q)$ then there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} ... \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r{-}1], \mathscr{R}(P,X,Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P',X,Q_{2r})$. Hence also $\mathscr{R}'(P,X,Q_{2i})$ and $\mathscr{R}'(P',X,Q_{2r})$. By Clause 1.a of Definition 33, $\mathscr{R}(P,Q_1)$, so $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$.
- if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then $\mathcal{I}(P) \cap ((Y \cup X) \cup \{\tau\}) = \emptyset$ so there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} ... \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r{-}1]$, $\mathscr{R}(P,X,Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P',X,Q_{2r})$. Hence also $\mathscr{R}'(P,X,Q_{2i})$ and $\mathscr{R}'(P',X,Q_{2r})$. Since $Q \stackrel{\tau}{\not\rightarrow}$, $Q = Q_1$. Since $P \stackrel{\tau}{\not\rightarrow}$, $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\mathscr{R}(P,Y,Q_1)$ and $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$, Clause 2.b of Definition 33 yields $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$. As a result, there is a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} ... \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r{-}1]$, $\mathscr{R}(P,X,Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P',X,Q_{2r})$.

e. If $P \stackrel{\tau}{\not\rightarrow}$ then

- if $\mathscr{R}(P,X,Q)$ then there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.
- if $\mathscr{R}(P,Q)$ then there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.
- if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then $Q \stackrel{\tau}{\not\rightarrow}$.

Let $\mathscr{R}$ be a rooted branching reactive bisimulation. Let's check that it is a generalised rooted branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}(P,Q)$
   - **a.** this condition is shared by both definitions
   - **b.** if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then, since $\mathscr{R}(P,Q)$, $\mathscr{R}(P,X,Q)$. Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, there exists a transition $Q \xrightarrow{t} Q'$ such that $P' \leftrightarrow_{br}^{X} Q'$.
2. If $\mathscr{R}(P,X,Q)$
   - **a.** this condition is shared by both definitions
   - **b.** if $a \in X$, this condition is shared by both definitions; otherwise, apply Clauses 2.c and 1.a of Definition 5

c. if $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then, since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $\mathscr{R}(P,Q)$ and so $\mathscr{R}(P,Y,Q)$. Since $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, there exists a transition $Q \xrightarrow{t} Q'$ such that $P' \leftrightarrow_{br}^{Y} Q'$.

Let $\mathscr{R}$ be a generalised rooted branching reactive bisimulation and define

$$\begin{aligned}\mathscr{R}' := \mathscr{R} \cup \{(P,X,Q) \mid \mathscr{R}(P,Q) \wedge X \subseteq A\} \cup \{(P,Y,Q),(P,Q) \mid \exists X \subseteq A, \mathscr{R}(P,X,Q) \\ \wedge (\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (X \cup \{\tau\}) = \emptyset \wedge Y \subseteq A\}\end{aligned}$$

$\mathscr{R}'$ is symmetric by definition. Let's check that $\mathscr{R}'$ is a rooted branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}'(P,Q)$ then $\mathscr{R}(P,Q)$ or there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$.
   a. If $P \xrightarrow{\alpha} P'$ then
   - if $\mathscr{R}(P,Q)$ then there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarrow_{br} Q'$.
   - if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, since $\mathscr{R}(P,Y,Q)$ and $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$, $\alpha \neq \tau$ so there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarrow_{br} Q'$.

   b. For all $Z \subseteq A$,
   - if $\mathscr{R}(P,Q)$ then, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,Z,Q)$
   - if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, by definition of $\mathscr{R}'$, $\mathscr{R}(P,Z,Q)$.
2. If $\mathscr{R}'(P,X,Q)$ then $\mathscr{R}(P,X,Q)$, or $\mathscr{R}(P,Q)$, or there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$.
   a. If $P \xrightarrow{\tau} P'$ then
   - if $\mathscr{R}(P,X,Q)$ then there exists a transition $Q \xrightarrow{\tau} Q'$ such that $P' \leftrightarrow_{br}^{X} Q'$,
   - if $\mathscr{R}(P,Q)$ then there exists a step $Q \xrightarrow{\tau} Q'$ such that $P' \leftrightarrow_{br} Q'$ and so $P' \leftrightarrow_{br}^{X} Q'$
   - if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then $P \stackrel{\tau}{\not\rightarrow}$, so this case is impossible.

   b. If $P \xrightarrow{a} P'$ with $a \in X$ then
   - if $\mathscr{R}(P,X,Q)$ then there exists a transition $Q \xrightarrow{a} Q'$ such that $P' \leftrightarrow_{br} Q'$
   - if $\mathscr{R}(P,Q)$ then there exists a transition $Q \xrightarrow{a} Q'$ such that $P' \leftrightarrow_{br} Q'$
   - if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, since $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$, there exists a transition $Q \xrightarrow{a} Q'$ such that $P' \leftrightarrow_{br} Q'$.

   c. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then
   - if $\mathscr{R}(P,X,Q)$ then, since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $\mathcal{I}(Q) \cap (X \cup \{\tau\}) = \emptyset$, therefore, by definition, $\mathscr{R}'(P,Q)$,
   - if $\mathscr{R}(P,Q)$ then, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,Q)$,
   - if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,Q)$,

   d. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then
   - if $\mathscr{R}(P,X,Q)$ then there exists a transition $Q \xrightarrow{t} Q'$ such that $P' \leftrightarrow_{br}^{X} Q'$.
   - if $\mathscr{R}(P,Q)$ then there exists a transition $Q \xrightarrow{t} Q'$ such that $P' \leftrightarrow_{br}^{X} Q'$.
   - if there exists $Y \subseteq A$ such that $\mathscr{R}(P,Y,Q)$ and $(\mathcal{I}(P) \cup \mathcal{I}(Q)) \cap (Y \cup \{\tau\}) = \emptyset$ then $\mathcal{I}(P) \cap ((Y \cup X) \cup \{\tau\}) = \emptyset$ so there exists a step $Q \xrightarrow{t} Q'$ such that $P' \leftrightarrow_{br}^{X} Q'$. ◀

D Pohlmann Encoding

Reactive bisimulations are sometimes complicated to check because of the large number of potential sets of allowed actions. In [19], Pohlmann introduces an encoding which reduces strong reactive bisimilarity to strong bisimilarity. To this end he introduces unary operators ϑ and ϑ_X for $X \subseteq A$ that model placing their argument process in an environment that is triggered to change, or allows exactly the actions in X, respectively. The actions $\mathrm{t}_\varepsilon \notin A$ and $\varepsilon_X \notin A$ for $X \subseteq A$ are generated by the new operators, but may not be used by processes substituted for their arguments P. They model a time-out action taken by the environment, and the stabilisation of an environment into one that allows exactly the set of actions X, respectively. After a slight modification of the encoding, a similar result can be obtained for branching reactive bisimilarity. We also introduce variants ϑ^r and ϑ^r_X of these operators that are targeting rooted branching reactive bisimilarity.

$$\begin{array}{lclcl}
\vartheta(P) \xrightarrow{\alpha} \vartheta(P') & \wedge & \vartheta^r(P) \xrightarrow{\alpha} \vartheta(P') & \Leftrightarrow & P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau \\
 & & \vartheta^r(P) \xrightarrow{\mathrm{t}_X} \vartheta_X(P') & \Leftrightarrow & \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \wedge P \xrightarrow{\mathrm{t}} P' \\
\vartheta(P) \xrightarrow{\varepsilon_X} \vartheta_X(P) & \wedge & \vartheta^r(P) \xrightarrow{\varepsilon_X} \vartheta_X(P) & & \\
\vartheta_X(P) \xrightarrow{\tau} \vartheta_X(P') & \wedge & \vartheta^r_X(P) \xrightarrow{\tau} \vartheta_X(P') & \Leftrightarrow & P \xrightarrow{\tau} P' \\
\vartheta_X(P) \xrightarrow{a} \vartheta(P') & \wedge & \vartheta^r_X(P) \xrightarrow{a} \vartheta(P') & \Leftrightarrow & P \xrightarrow{a} P' \wedge \alpha \in X \\
\vartheta_X(P) \xrightarrow{\mathrm{t}_\varepsilon} \vartheta(P) & \wedge & \vartheta^r_X(P) \xrightarrow{\mathrm{t}_\varepsilon} \vartheta^r(P) & \Leftrightarrow & \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \\
\vartheta_X(P) \xrightarrow{\mathrm{t}} \vartheta_X(P') & \wedge & \vartheta^r_X(P) \xrightarrow{\mathrm{t}} \vartheta_X(P') & \Leftrightarrow & \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \wedge P \xrightarrow{\mathrm{t}} P'
\end{array}$$

Table 4 Operational semantics of ϑ, ϑ^r, $(\vartheta_X)_{X\subseteq A}$ and $(\vartheta^r_X)_{X\subseteq A}$

In [19], the first rule only applies to τ-transitions; this echoes the previous remark about applying the first clause of Definition 1 only to invisible actions. As the intermediary actions t_ε and $(\varepsilon_X)_{X\subseteq A}$ interfere with rootedness, the actions $(\mathrm{t}_X)_{X\subseteq A}$ are added when rootedness has to be preserved. One can think of these as doing the actions ε_X and t in one (instead of two) steps. Note that the encoding rules mirror the clauses of Definition 1. The encoding transforms $\leftrightarrow_{br}$ into $\leftrightarrow_{tb}$ (see Definition 7), and $\leftrightarrow^r_{br}$ in $\leftrightarrow^r_{tb}$ (Definition 8).

▶ **Proposition 36.** *Let $P, Q \in \mathbb{P}$.*

- $P \leftrightarrow_{br} Q \Leftrightarrow \vartheta(P) \leftrightarrow_{tb} \vartheta(Q)$
- $P \leftrightarrow^r_{br} Q \Leftrightarrow \vartheta^r(P) \leftrightarrow^r_{tb} \vartheta^r(Q)$
- $P \leftrightarrow^X_{br} Q \Leftrightarrow \vartheta_X(P) \leftrightarrow_{tb} \vartheta_X(Q)$
- $P \leftrightarrow^{rX}_{br} Q \Leftrightarrow \vartheta^r_X(P) \leftrightarrow^r_{tb} \vartheta^r_X(Q)$

Proof. It suffices to prove that: if $\mathscr{R}$ is a branching reactive bisimulation then $\mathscr{R}' := \{(\vartheta(P), \vartheta(Q)) \mid \mathscr{R}(P, Q)\} \cup \{(\vartheta_X(P), \vartheta_X(Q)) \mid \mathscr{R}(P, X, Q)\}$ is a t-branching bisimulation; and if $\mathscr{R}$ is a t-branching bisimulation then $\mathscr{R}' := \{(P, Q), (P, X, Q) \mid \mathscr{R}(\vartheta(P), \vartheta(Q)) \wedge X \subseteq A\} \cup \{(P, X, Q) \mid \mathscr{R}(\vartheta_X(P), \vartheta_X(Q))\}$ is a branching reactive bisimulation. The rooted case is very similar.

Let $\mathscr{R}$ be a branching reactive bisimulation and define

$$\mathscr{R}' := \{(\vartheta(P), \vartheta(Q)) \mid \mathscr{R}(P, Q)\} \cup \{(\vartheta_X(P), \vartheta_X(Q)) \mid \mathscr{R}(P, X, Q)\}$$

We are going to check that $\mathscr{R}'$ is a t-branching bisimulation. Let $P, Q \in \mathbb{P}$ such that $\mathscr{R}'(P, Q)$.

- If $P = \vartheta(P^\dagger)$ and $Q = \vartheta(Q^\dagger)$ then, by definition of $\mathscr{R}'$, $\mathscr{R}(P^\dagger, Q^\dagger)$.
  1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau \cup \{\mathrm{t}_\varepsilon, \varepsilon_X \mid X \subseteq A\}$ then
     - if $\alpha \in A_\tau$ then, by the semantics of ϑ, $P' = \vartheta(P^\ddagger)$ and $P^\dagger \xrightarrow{\alpha} P^\ddagger$. Since $\mathscr{R}(P^\dagger, Q^\dagger)$, there exists a path $Q^\dagger \Longrightarrow Q^\star \xrightarrow{(\alpha)} Q^\ddagger$ such that $\mathscr{R}(P^\dagger, Q^\star)$ and $\mathscr{R}(P^\ddagger, Q^\ddagger)$. By the

semantics, there exists a path $Q \Longrightarrow \vartheta(Q^\star) \xrightarrow{(\alpha)} \vartheta(Q^\ddagger)$ such that, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, \vartheta(Q^\star))$ and $\mathscr{R}'(P', \vartheta(Q^\ddagger))$.
- if $\alpha = \mathrm{t}_\varepsilon$ then this case is not possible according to the semantics of ϑ.
- if $\alpha = \varepsilon_X$ with $X \subseteq A$ then, by the semantics of ϑ, $P' = \vartheta_X(P^\dagger)$. Since $\mathscr{R}(P^\dagger, Q^\dagger)$, $\mathscr{R}(P^\dagger, X, Q^\dagger)$. By the semantics, $Q \xrightarrow{\varepsilon_X} \vartheta_X(Q^\dagger)$ such that, by the definition of $\mathscr{R}'$, $\mathscr{R}'(P', \vartheta_X(Q^\dagger))$.

2. If $P \xrightarrow{\mathrm{t}} P'$ then, by the semantics, this is impossible.
3. If $P \stackrel{\tau}{\not\rightarrow}$ then, by the semantics of ϑ, $P^\dagger \stackrel{\tau}{\not\rightarrow}$. Since $\mathscr{R}(P^\dagger, Q^\dagger)$, $\mathscr{R}(P^\dagger, \emptyset, Q^\dagger)$, so there exists a path $Q^\dagger \Longrightarrow Q^\star \stackrel{\tau}{\not\rightarrow}$. By the semantics, there exists a path $Q \Longrightarrow \vartheta(Q^\star) \stackrel{\tau}{\not\rightarrow}$.

- If there exists $X \subseteq A$ such that $P = \vartheta_X(P^\dagger)$ and $Q = \vartheta_X(Q^\dagger)$ then, by definition of $\mathscr{R}'$, $\mathscr{R}(P^\dagger, X, Q^\dagger)$.
  1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau \cup \{\mathrm{t}_\varepsilon, \varepsilon_X \mid X \subseteq A\}$ then
     - if $P \xrightarrow{\tau} P'$ then, by the semantics, $P' = \vartheta_X(P^\ddagger)$ and $P^\dagger \xrightarrow{\tau} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger \Longrightarrow Q^\star \xrightarrow{(\tau)} Q^\ddagger$ such that $\mathscr{R}(P^\dagger, X, Q^\star)$ and $\mathscr{R}(P^\ddagger, X, Q^\ddagger)$. By the semantics, there exists a path $Q \Longrightarrow \vartheta_X(Q^\star) \xrightarrow{(\tau)} \vartheta_X(Q^\ddagger)$ such that, by the definition of $\mathscr{R}'$, $\mathscr{R}'(P, \vartheta_X(Q^\star))$ and $\mathscr{R}'(P', \vartheta_X(Q^\ddagger))$.
     - if $P \xrightarrow{a} P'$ with $a \in A$ then, by the semantics, $a \in X$, $P' = \vartheta(P^\ddagger)$ and $P^\dagger \xrightarrow{a} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger \Longrightarrow Q^\star \xrightarrow{a} Q^\ddagger$ such that $\mathscr{R}(P^\dagger, X, Q^\star)$ and $\mathscr{R}(P^\ddagger, Q^\ddagger)$. By the semantics, there exists a path $Q \Longrightarrow \vartheta_X(Q^\star) \xrightarrow{a} \vartheta(Q^\ddagger)$ such that, by the definition of $\mathscr{R}'$, $\mathscr{R}'(P, \vartheta_X(Q^\star))$ and $\mathscr{R}'(P', \vartheta(Q^\ddagger))$.
     - if $P \xrightarrow{\mathrm{t}_\varepsilon} P'$ then, by the semantics, $P' = \vartheta(P^\dagger)$ and $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$ and $P^\dagger \stackrel{\tau}{\not\rightarrow}$, there exists a path $Q^\dagger \Longrightarrow Q^\star \stackrel{\tau}{\not\rightarrow}$. Moreover, $\mathscr{R}(P^\dagger, X, Q^\star)$. Since $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$, $\mathscr{R}(P^\dagger, X, Q^\star)$ and $Q^\star \stackrel{\tau}{\not\rightarrow}$, $\mathcal{I}(Q^\star) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P^\dagger, Q^\star)$. By the semantics, there exists a path $Q \Longrightarrow \vartheta_X(Q^\star) \xrightarrow{\mathrm{t}_\varepsilon} \vartheta(Q^\star)$ such that, by the definition of $\mathscr{R}'$, $\mathscr{R}'(P, \vartheta_X(Q^\star))$ and $\mathscr{R}'(P', \vartheta(Q^\star))$.
     - if $\alpha = \varepsilon_X$ with $X \subseteq A$ then this case is impossible according to the semantics of ϑ_X.
  2. if $P \xrightarrow{\mathrm{t}} P'$ then, by the semantics, $P' = \vartheta_X(P^\ddagger)$, $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $P^\dagger \xrightarrow{\mathrm{t}} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger = Q_0^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{\mathrm{t}} Q_2^\dagger \Longrightarrow Q_3^\dagger \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1}^\dagger \xrightarrow{(\mathrm{t})} Q_{2r}^\ddagger$ with $r > 0$, such that $\forall i \in [0, r{-}1], \mathscr{R}(P^\dagger, X, Q_{2i}^\dagger) \wedge \mathcal{I}(Q_{2i+1}^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P^\ddagger, X, Q_{2r}^\ddagger)$. For all $i \in [0, r{-}1]$, since $P^\dagger \stackrel{\tau}{\not\rightarrow}$, $Q_{2i}^\dagger \Longrightarrow Q_{2i+1}^\dagger$ and $\mathscr{R}(P^\dagger, X, Q_{2i}^\dagger)$, $\mathscr{R}(P^\dagger, X, Q_{2i+1}^\dagger)$. By the semantics, there exists a path $Q \Longrightarrow \vartheta_X(Q_1^\dagger) \xrightarrow{\mathrm{t}} \vartheta_X(Q_2^\dagger) \Longrightarrow \vartheta_X(Q_3^\dagger) \xrightarrow{\mathrm{t}} \ldots \Longrightarrow \vartheta_X(Q_{2r-1}) \xrightarrow{(\mathrm{t})} \vartheta_X(Q_{2r})$ such that, by definition of $\mathscr{R}'$, $\forall i \in [0, 2r{-}1], \mathscr{R}'(P, \vartheta_X(Q_i^\dagger))$ and $\mathscr{R}'(P', \vartheta_X(Q_{2r}^\ddagger))$.
  3. if $P \stackrel{\tau}{\not\rightarrow}$ then, by the semantics of ϑ_X, $P^\dagger \stackrel{\tau}{\not\rightarrow}$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$. By the semantics, there exists a path $Q \Longrightarrow \vartheta_X(Q_0) \stackrel{\tau}{\not\rightarrow}$.

Let $\mathscr{R}$ be a t-branching bisimulation and define

$$\mathscr{R}' := \{(P, Q), (P, X, Q) \mid \mathscr{R}(\vartheta(P), \vartheta(Q)) \wedge X \subseteq A\} \cup \{(P, X, Q) \mid \mathscr{R}(\vartheta_X(P), \vartheta_X(Q))\}$$

We are going to show that $\mathscr{R}'$ is a branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}'(P, Q)$ then $\mathscr{R}(\vartheta(P), \vartheta(Q))$.
   a. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then, by the semantics, $\vartheta(P) \xrightarrow{\alpha} \vartheta(P')$. Since $\mathscr{R}(\vartheta(P), \vartheta(Q))$, there exists a path $\vartheta(Q) \Longrightarrow Q^\star \xrightarrow{(\alpha)} Q^\ddagger$ such that $\mathscr{R}(\vartheta(P), Q^\star)$ and $\mathscr{R}(\vartheta(P'), Q^\ddagger)$. By the semantics, $Q^\star = \vartheta(Q_1)$, $Q^\ddagger = \vartheta(Q_2)$ and $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, Q_1)$ and $\mathscr{R}'(P', Q_2)$.
   b. For all $Y \subseteq A$, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, Y, Q)$.

2. If $\mathscr{R}'(P,X,Q)$ then $\mathscr{R}(\vartheta(P),\vartheta(Q))$ or $\mathscr{R}(\vartheta_X(P),\vartheta_X(Q))$. If $\mathscr{R}(\vartheta(P),\vartheta(Q))$ then $\vartheta(P) \xrightarrow{\varepsilon_X} \vartheta_X(P)$, thus there exists a path $\vartheta(Q) \Longrightarrow Q^\star \xrightarrow{\varepsilon_X} Q^\ddagger$ such that $\mathscr{R}(\vartheta(P),Q^\star)$ and $\mathscr{R}(\vartheta_X(P),Q^\ddagger)$. By the semantics, $Q^\star = \vartheta(Q_0)$, $Q^\ddagger = \vartheta_X(Q_0)$ and $Q \Longrightarrow Q_0$. Therefore, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}(\vartheta_X(P),\vartheta_X(Q_0))$.
   a. If $P \xrightarrow{\tau} P'$ then, by the semantics, $\vartheta_X(P) \xrightarrow{\tau} \vartheta_X(P')$. Since $\mathscr{R}(\vartheta_X(P),\vartheta_X(Q_0))$, there exists a path $\vartheta_X(Q_0) \Longrightarrow Q^\star \xrightarrow{(\tau)} Q^\ddagger$ such that $\mathscr{R}(\vartheta_X(P),Q^\star)$ and $\mathscr{R}(\vartheta_X(P'),Q^\ddagger)$. By the semantics, $Q^\star = \vartheta_X(Q_1)$, $Q^\ddagger = \vartheta_X(Q_2)$ and $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,X,Q_1)$ and $\mathscr{R}'(P',X,Q_2)$.
   b. If $P \xrightarrow{a} P'$ with $a \in X$ then, by the semantics, $\vartheta_X(P) \xrightarrow{a} \vartheta(P')$. As $\mathscr{R}(\vartheta_X(P),\vartheta_X(Q_0))$, there exists a path $\vartheta_X(Q_0) \Longrightarrow Q^\star \xrightarrow{a} Q^\ddagger$ such that $\mathscr{R}(\vartheta_X(P),Q^\star)$ and $\mathscr{R}(\vartheta(P'),Q^\ddagger)$. By the semantics, $Q^\star = \vartheta_X(Q_1)$, $Q^\ddagger = \vartheta(Q_2)$ and $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,X,Q_1)$ and $\mathscr{R}'(P',Q_2)$.
   c. If $\mathcal{I}(P)\cap(X\cup\{\tau\}) = \emptyset$ then, by the semantics, $\vartheta_X(P) \xrightarrow{\mathrm{t}_\varepsilon} \vartheta(P)$. As $\mathscr{R}(\vartheta_X(P),\vartheta_X(Q_0))$, there exists a path $\vartheta_X(Q_0) \Longrightarrow Q^\star \xrightarrow{\mathrm{t}_\varepsilon} Q^\ddagger$ such that $\mathscr{R}(\vartheta_X(P),Q^\star)$ and $\mathscr{R}(\vartheta(P),Q^\ddagger)$. By the semantics, $Q^\star = \vartheta_X(Q'_0)$, $Q^\ddagger = \vartheta(Q'_0)$ and $Q \Longrightarrow Q'_0$ such that, by definition of $\mathscr{R}'$, $\mathscr{R}'(P,Q'_0)$.
   d. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, by the semantics, $\vartheta_X(P) \xrightarrow{\mathrm{t}} \vartheta_X(P')$. Since $\mathscr{R}(\vartheta_X(P),\vartheta_X(Q_0))$, there exists a path $\vartheta_X(Q_0) \Longrightarrow Q_1^\dagger \xrightarrow{\mathrm{t}} Q_2^\dagger \Longrightarrow Q_3^\dagger \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1}^\dagger \xrightarrow{(\mathrm{t})} Q_{2r}^\ddagger$ with $r > 0$, such that $\forall i \in [0,2r{-}1], \mathscr{R}(\vartheta_X(P),Q_i^\dagger)$ and $\mathscr{R}(\vartheta_X(P'),Q_{2r}^\ddagger)$. By the semantics, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ such that $\forall i \in [0,r{-}1], Q_{2i}^\dagger = \vartheta_X(Q_{2i}) \wedge Q_{2i+1}^\dagger = \vartheta_X(Q_{2i+1})$ and $Q_{2r}^\ddagger = \vartheta_X(Q_{2r})$. Thus, by definition of $\mathscr{R}'$, $\forall i \in [0,r{-}1], \mathscr{R}'(P,X,Q_{2i})$ and $\mathscr{R}'(P',X,Q_{2r})$. With the possible exception of $i = r{-}1$, for all $i \in [0,r{-}1]$ we have $\mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$. In the case that $Q_{2r} = Q_{2r-1}$ we can choose Q_{2r-1} such that $Q_{2r-1} \stackrel{\tau}{\not\rightarrow}$, and hence $Q_{2r-1}^\dagger \stackrel{\tau}{\not\rightarrow}$. Since $\mathscr{R}(\vartheta_X(P),Q_{2r-1}^\dagger)$ and $\vartheta_X(P) \xrightarrow{\mathrm{t}_\varepsilon}$, also $Q_{2r-1}^\dagger \xrightarrow{\mathrm{t}_\varepsilon}$. Thus $\mathcal{I}(Q_{2r-1}) \cap (X \cup \{\tau\}) = \emptyset$.
   e. If $P \stackrel{\tau}{\not\rightarrow}$ then, by the semantics, $\vartheta_X(P) \stackrel{\tau}{\not\rightarrow}$. Since $\mathscr{R}(\vartheta_X(P),\vartheta_X(Q_0))$, there exists a path $\vartheta_X(Q_0) \Longrightarrow Q^\star \stackrel{\tau}{\not\rightarrow}$. By the semantics, $Q^\star = \vartheta_X(Q_1)$ and $Q \Longrightarrow Q_1 \stackrel{\tau}{\not\rightarrow}$.

Let $\mathscr{R}$ be a rooted branching reactive bisimulation and define

$$\mathscr{R}' := \{(\vartheta^r(P),\vartheta^r(Q)) \mid \mathscr{R}(P,Q)\} \cup \{(\vartheta^r_X(P),\vartheta^r_X(Q)) \mid \mathscr{R}(P,X,Q)\}$$

We are going to check that $\mathscr{R}'$ is a rooted t-branching bisimulation. Let $P,Q \in \mathbb{P}$ such that $\mathscr{R}'(P,Q)$.

- If $P = \vartheta^r(P^\dagger)$ and $Q = \vartheta^r(Q^\dagger)$ then, by definition of $\mathscr{R}'$, $\mathscr{R}(P^\dagger,Q^\dagger)$.
  1. Let $P \xrightarrow{\alpha} P'$ with $\alpha \in Act \cup \{\mathrm{t}_\varepsilon,\mathrm{t}_X,\varepsilon_X \mid X \subseteq A\}$.
     - If $\alpha \in A_\tau$ then, by the semantics of ϑ^r, $P' = \vartheta(P^\ddagger)$ and $P^\dagger \xrightarrow{\alpha} P^\ddagger$. Since $\mathscr{R}(P^\dagger,Q^\dagger)$, there exists a transition $Q^\dagger \xrightarrow{\alpha} Q^\ddagger$ such that $P^\ddagger \leftrightarrow_{br} Q^\ddagger$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} \vartheta(Q^\ddagger)$ such that, by the first part of this proof, $\vartheta(P') \leftrightarrow_{tb} \vartheta(Q^\ddagger)$.
     - The case $\alpha = \mathrm{t}_\varepsilon$ is not possible according to the semantics of ϑ^r.
     - If $\alpha = \varepsilon_X$ with $X \subseteq A$ then, by the semantics of ϑ^r, $P' = \vartheta_X(P^\dagger)$. Since $\mathscr{R}(P^\dagger,Q^\dagger)$, $P^\dagger \leftrightarrow^r_{br} Q^\dagger$ so $P^\dagger \leftrightarrow^X_{br} Q^\dagger$. By the semantics, $Q \xrightarrow{\varepsilon_X} \vartheta_X(Q^\dagger)$ such that, by the first part of this proof, $P' \leftrightarrow_{tb} \vartheta_X(Q^\dagger)$.
     - The case $\alpha = \mathrm{t}$, by the semantics, is not possible.
     - If $P \xrightarrow{\mathrm{t}_X} P'$ then, by the semantics, $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$, $P^\dagger \xrightarrow{t} P^\ddagger$ and $P' = \vartheta_X(P^\ddagger)$. Since $\mathscr{R}(P^\dagger,Q^\dagger)$, $\mathcal{I}(Q^\ddagger) \cap (X \cup \{\tau\}) = \emptyset$ and there is a transition

$Q^\dagger \xrightarrow{t} Q^\ddagger$ such that $P^\ddagger \leftrightarrow_{br}^X Q^\ddagger$. By the semantics, $Q \xrightarrow{\mathrm{t}_X} \vartheta_X(Q^\ddagger)$ and, by the first part of this proof, $P' \leftrightarrow_{tb} \vartheta_X(Q^\ddagger)$.

- If there exists $X \subseteq A$ such that $P = \vartheta^r_X(P^\dagger)$ and $Q = \vartheta^r_X(Q^\dagger)$ then, by definition of $\mathscr{R}'$, $\mathscr{R}(P^\dagger, X, Q^\dagger)$.
  1. Let $P \xrightarrow{\alpha} P'$ with $\alpha \in Act \cup \{\mathrm{t}_\varepsilon, \mathrm{t}_X, \varepsilon_X \mid X \subseteq A\}$.
     - If $P \xrightarrow{\tau} P'$ then, by the semantics, $P' = \vartheta_X(P^\ddagger)$ and $P^\dagger \xrightarrow{\tau} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a transition $Q^\dagger \xrightarrow{\tau} Q^\ddagger$ such that $P^\ddagger \leftrightarrow_{br}^X Q^\ddagger$. By the semantics, there exists a transition $Q \xrightarrow{\tau} \vartheta_X(Q^\ddagger)$ such that, by the first part, $P' \leftrightarrow_{tb} \vartheta_X(Q^\ddagger)$.
     - If $P \xrightarrow{a} P'$ with $a \in A$ then, by the semantics, $a \in X$, $P' = \vartheta(P^\ddagger)$ and $P^\dagger \xrightarrow{a} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a transition $Q^\dagger \xrightarrow{a} Q^\ddagger$ such that $P^\ddagger \leftrightarrow_{br} Q^\ddagger$. By the semantics, there exists a transition $Q \xrightarrow{a} \vartheta(Q^\ddagger)$ such that, by the first part, $P' \leftrightarrow_{tb} \vartheta(Q^\ddagger)$.
     - If $P \xrightarrow{\mathrm{t}_\varepsilon} P'$ then, by the semantics, $P' = \vartheta^r(P^\dagger)$ and $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$ and $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$, $\mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P^\dagger, Q^\dagger)$. By the semantics, there exists a path $Q \xrightarrow{t_\varepsilon} \vartheta^r(Q^\dagger)$ such that, by the definition of $\mathscr{R}'$, $\mathscr{R}'(P', \vartheta^r(Q^\dagger))$. Considering the previous case, this implies that $P' \leftrightarrow^r_{tb} \vartheta^r(Q^\dagger)$ and so $P' \leftrightarrow_{tb} \vartheta^r(Q^\dagger)$.
     - The case $\alpha = \varepsilon_X$ or t_X with $X \subseteq A$ is impossible according to the semantics of ϑ_X.
     - If $P \xrightarrow{\mathrm{t}} P'$ then, by the semantics, $P' = \vartheta_X(P^\ddagger)$, $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $P^\dagger \xrightarrow{\mathrm{t}} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger \xrightarrow{\mathrm{t}} Q^\ddagger$ such that $P^\ddagger \leftrightarrow_{br}^X Q^\ddagger$. Moreover, $\mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. By the semantics, there exists a path $Q \xrightarrow{\mathrm{t}} \vartheta_X(Q^\ddagger)$ such that, by the first part, $P' \leftrightarrow_{tb} \vartheta_X(Q^\ddagger)$.

Let $\mathscr{R}$ be a rooted t-branching bisimulation and define

$$\mathscr{R}' := \{(P,Q),(P,X,Q) \mid \mathscr{R}(\vartheta^r(P), \vartheta^r(Q)) \wedge X \subseteq A\} \cup \{(P,X,Q) \mid \mathscr{R}(\vartheta^r_X(P), \vartheta^r_X(Q))\}$$
$$\cup \left\{ (P,Q),(P,X,Q) \;\middle|\; \begin{array}{l} \mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q)) \wedge X \subseteq A \wedge \\ (\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (Y \cup \{\tau\}) = \emptyset \end{array} \right\}$$

We are going to show that $\mathscr{R}'$ is a rooted branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}'(P,Q)$ then $\mathscr{R}(\vartheta^r(P), \vartheta^r(Q))$ or $\mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q))$ and $(\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (X \cup \{\tau\}) = \emptyset$.
   a. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then, by the semantics, $\vartheta^r(P) \xrightarrow{\alpha} \vartheta(P')$.
      - If $\mathscr{R}(\vartheta^r(P), \vartheta^r(Q))$, there exists a transition $\vartheta^r(Q) \xrightarrow{\alpha} Q^\ddagger$ such that $\vartheta(P') \leftrightarrow_{tb} Q^\ddagger$. By the semantics, $Q^\ddagger = \vartheta(Q')$ and $Q \xrightarrow{\alpha} Q'$ such that, by the second part, $P' \leftrightarrow_{br} Q'$.
      - If $\mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q))$ and $(\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (X \cup \{\tau\}) = \emptyset$ then $\vartheta_Y(P) \xrightarrow{\mathrm{t}_\varepsilon} \vartheta^r(P)$, thus there exists a transition $\vartheta^r_Y(Q) \xrightarrow{t_\varepsilon} \vartheta^r(Q)$ with $\vartheta^r(P) \leftrightarrow_{tb} \vartheta^r(Q)$. Since $\vartheta^r(P) \xrightarrow{\alpha} \vartheta(P')$, there exists a path $\vartheta^r(Q) \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $\vartheta^r(P) \leftrightarrow_{tb} Q_1$ and $\vartheta(P') \leftrightarrow_{tb} Q_2$. Since $\vartheta^r(Q) \stackrel{\tau}{\not\rightarrow}$, $\vartheta^r(Q) \xrightarrow{\alpha} Q_2$ so, by the semantics, $Q \xrightarrow{\alpha} Q'$ and $Q_2 = \vartheta(Q')$. As a result, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that, by the second part, $P' \leftrightarrow_{br} Q'$.
   b. For all $Y \subseteq A$, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, Y, Q)$.
2. If $\mathscr{R}'(P, X, Q)$ then $\mathscr{R}(\vartheta^r(P), \vartheta^r(Q))$, $\mathscr{R}(\vartheta^r_X(P), \vartheta^r_X(Q))$ or $\mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q))$ and $(\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (X \cup \{\tau\}) = \emptyset$.
   a. If $P \xrightarrow{\tau} P'$ then, by the semantics, $\vartheta^r(P) \xrightarrow{\tau} \vartheta(P')$ and $\vartheta^r_X(P) \xrightarrow{\tau} \vartheta_X(P')$.
      - If $\mathscr{R}(\vartheta^r(P), \vartheta^r(Q))$, there exists a path $\vartheta^r(Q) \xrightarrow{\tau} Q^\ddagger$ such that $\vartheta(P') \leftrightarrow_{tb} Q^\ddagger$. By the semantics, $Q^\ddagger = \vartheta(Q')$ and $Q \xrightarrow{\tau} Q'$ so that, by the second part, $P' \leftrightarrow_{br} Q'$ and thus $P' \leftrightarrow_{br}^X Q'$.

- If $\mathscr{R}(\vartheta^r_X(P), \vartheta^r_X(Q))$, there exists a path $\vartheta^r_X(Q) \xrightarrow{\tau} Q^\ddagger$ such that $\vartheta_X(P') \leftrightarroweq_{tb} Q^\ddagger$. By the semantics, $Q^\ddagger = \vartheta_X(Q')$ and $Q \xrightarrow{\tau} Q'$ so that, by the second part, $P' \leftrightarroweq^X_{br} Q'$.
- The case $\mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q))$ and $(\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (X \cup \{\tau\}) = \emptyset$ is impossible.

b. If $P \xrightarrow{a} P'$ with $a \in X$ then, by the semantics, $\vartheta^r(P) \xrightarrow{a} \vartheta(P')$, $\vartheta^r_X(P) \xrightarrow{a} \vartheta(P')$.

- If $\mathscr{R}(\vartheta^r(P), \vartheta^r(Q))$, there exists a step $\vartheta^r(Q) \xrightarrow{a} Q^\ddagger$ such that $\vartheta(P') \leftrightarroweq_{tb} Q^\ddagger$. By the semantics, $Q^\ddagger = \vartheta(Q')$ and $Q \xrightarrow{a} Q'$ such that, by the second part, $P' \leftrightarroweq_{br} Q'$.
- If $\mathscr{R}(\vartheta^r_X(P), \vartheta^r_X(Q))$, there exists a path $\vartheta^r_X(Q) \xrightarrow{a} Q^\ddagger$ such that $\vartheta(P') \leftrightarroweq_{tb} Q^\ddagger$. By the semantics, $Q^\ddagger = \vartheta(Q')$ and $Q \xrightarrow{a} Q'$ such that, by the second part, $P' \leftrightarroweq_{br} Q'$.
- If $\mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q))$ and $(\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (X \cup \{\tau\}) = \emptyset$ then $\vartheta^r_Y(P) \xrightarrow{t_\varepsilon} \vartheta^r(P)$, thus there exists a transition $\vartheta^r_Y(Q) \xrightarrow{t_\varepsilon} \vartheta^r(Q)$ with $\vartheta^r(P) \leftrightarroweq_{tb} \vartheta^r(Q)$. Since $\vartheta^r(P) \xrightarrow{a} \vartheta(P')$, therefore, there exists a path $\vartheta^r(Q) \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\vartheta^r(P) \leftrightarroweq_{tb} Q_1$ and $\vartheta(P') \leftrightarroweq_{tb} Q_2$. Since $\vartheta^r(Q) \overset{\tau}{\nrightarrow}$, $\vartheta^r(Q) \xrightarrow{a} Q_2$ so, by the semantics, $Q \xrightarrow{a} Q'$ and $Q_2 = \vartheta(Q')$. As a result, there exists a transition $Q \xrightarrow{a} Q'$ such that, by the second part, $P' \leftrightarroweq_{br} Q'$.

c. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then

- if $\mathscr{R}(\vartheta^r(P), \vartheta^r(Q))$ then, by definition of $\mathscr{R}'$, $\mathscr{R}'(P, Q)$.
- if $\mathscr{R}(\vartheta^r_X(P), \vartheta^r_X(Q))$ then $(\mathcal{I}(\vartheta^r_X(P)) \cup \mathcal{I}(\vartheta^r_X(Q))) \cap (X \cup \{\tau\}) = \emptyset$ thus, by definition of $\mathscr{R}'$, $\mathscr{R}(P, Q)$.
- If $\mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q))$ and $(\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (X \cup \{\tau\}) = \emptyset$ then, by definition of $\mathscr{R}'$, $\mathscr{R}(P, Q)$.

d. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then, by the semantics, $\vartheta^r(P) \xrightarrow{t_X} \vartheta_X(P')$, $\vartheta_X(P) \xrightarrow{t} \vartheta_X(P')$.

- If $\mathscr{R}(\vartheta^r(P), \vartheta^r(Q))$ then there exists a transition $\vartheta^r(Q) \xrightarrow{t_X} Q^\ddagger$ with $\vartheta_X(P') \leftrightarroweq_{tb} Q^\ddagger$. By the semantics, $Q^\ddagger = \vartheta_X(Q')$ and $Q \xrightarrow{t_X} Q'$. Moreover, by the second part, $P' \leftrightarroweq^X_{br} Q'$.
- if $\mathscr{R}(\vartheta^r_X(P), \vartheta^r_X(Q))$ then there exists a transition $\vartheta^r_X(Q) \xrightarrow{t} Q^\ddagger$ with $\vartheta_X(P') \leftrightarroweq_{tb} Q^\ddagger$. By the semantics, $Q^\ddagger = \vartheta_X(Q')$ and $Q \xrightarrow{t} Q'$. Moreover, by the second part, $P' \leftrightarroweq^X_{br} Q'$.
- if $\mathscr{R}(\vartheta^r_Y(P), \vartheta^r_Y(Q))$ and $(\mathcal{I}(\vartheta^r_Y(P)) \cup \mathcal{I}(\vartheta^r_Y(Q))) \cap (X \cup \{\tau\}) = \emptyset$ then $\vartheta^r_Y(P) \xrightarrow{t_\varepsilon} \vartheta^r(P)$, so there exists a transition $\vartheta^r_Y(Q) \xrightarrow{t_\varepsilon} \vartheta^r(Q)$ with $\vartheta^r(P) \leftrightarroweq_{tb} \vartheta^r(Q)$. Since $\vartheta^r(P) \xrightarrow{t_X} \vartheta_X(P')$, there exists a path $\vartheta^r(Q) \Longrightarrow Q_1 \xrightarrow{t_X} Q_2$ such that $\vartheta^r(P) \leftrightarroweq_{tb} Q_1$ and $\vartheta_X(P') \leftrightarroweq_{tb} Q_2$. As $\vartheta^r(Q) \overset{\tau}{\nrightarrow}$, $\vartheta^r(Q) \xrightarrow{t_X} Q_2$ so, by the semantics, $Q \xrightarrow{t} Q'$ and $Q_2 = \vartheta_X(Q')$. Moreover, by the second part, $P' \leftrightarroweq^X_{br} Q'$. ◀

It would have been possible to define the t-branching bisimilarity differently while preserving the same result. The encoded processes are part of a sub-class with specific properties. For instance, an encoded process cannot have an outgoing τ-transition and an outgoing time-out by definition of ϑ and $(\vartheta_X)_{X \subseteq A}$, i.e., for any encoded process P, $P \xrightarrow{t} \Rightarrow P \overset{\tau}{\nrightarrow}$. Thus, adding the condition $\forall i \in [0, r-1], Q_{2i+1} \overset{\tau}{\nrightarrow}$ in clause 2 of Definition 7 does not interfere with our result even though it obviously defines a different bisimilarity. We settled on Definition 7 because it is the one that yields the simplest proofs.

E Proofs of Stuttering Property and Transitivity

Proof of Lemma 3. Let $\mathscr{R}$ be a branching reactive bisimulation. Let's define

$$\mathscr{R}' := \{(P^\dagger, Q), (Q, P^\dagger) \mid \exists P, P^\ddagger \in \mathbb{P}, P \Longrightarrow P^\dagger \Longrightarrow P^\ddagger \wedge \mathscr{R}(P,Q) \wedge \mathscr{R}(P^\ddagger, Q)\} \cup \\ \{(P^\dagger, X, Q), (Q, X, P^\dagger) \mid \exists P, P^\ddagger \in \mathbb{P}, P \Longrightarrow P^\dagger \Longrightarrow P^\ddagger \wedge \mathscr{R}(P, X, Q) \wedge \mathscr{R}(P^\ddagger, X, Q)\}$$

$\mathscr{R}'$ is symmetric by definition and we are going to prove that $\mathscr{R}'$ is a branching reactive bisimulation. Note that $\mathscr{R} \subseteq \mathscr{R}'$ (by taking $P^\ddagger = P^\dagger$). Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. Let $\mathscr{R}'(P,Q)$.
   a. Suppose $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$.
      - Let there exist $P^\dagger, P^\ddagger \in \mathbb{P}$ such that $P^\dagger \Longrightarrow P \Longrightarrow P^\ddagger$, $\mathscr{R}(P^\dagger, Q)$ and $\mathscr{R}(P^\ddagger, Q)$. Since $P^\dagger \Longrightarrow P$ and $\mathscr{R}(P^\dagger, Q)$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}(P, Q_0)$. Since $P \xrightarrow{\alpha} P'$, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$. Thus, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, Q_1)$ and $\mathscr{R}'(P', Q_2)$.
      - Let there exist $Q^\dagger, Q^\ddagger \in \mathbb{P}$ such that $Q^\dagger \Longrightarrow Q \Longrightarrow Q^\ddagger$, $\mathscr{R}(P, Q^\dagger)$ and $\mathscr{R}(P, Q^\ddagger)$. Since $\mathscr{R}(P, Q^\ddagger)$, there exists a path $Q^\ddagger \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$. Since $Q \Longrightarrow Q^\ddagger$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, Q_1)$ and $\mathscr{R}'(P', Q_2)$.
   b. For all $Y \subseteq A$, $\mathscr{R}'(P, Y, Q)$ by definition of $\mathscr{R}'$.
2. Let $\mathscr{R}'(P, X, Q)$.
   a. Suppose $P \xrightarrow{\tau} P'$.
      - Let there exist $P^\dagger, P^\ddagger \in \mathbb{P}$ such that $P^\dagger \Longrightarrow P \Longrightarrow P^\ddagger$, $\mathscr{R}(P^\dagger, X, Q)$ and $\mathscr{R}(P^\ddagger, X, Q)$. Since $P^\dagger \Longrightarrow P$ and $\mathscr{R}(P^\dagger, X, Q)$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}(P, X, Q_0)$. Since $P \xrightarrow{\tau} P'$, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', X, Q_2)$. Thus, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, X, Q_1)$ and $\mathscr{R}'(P', X, Q_2)$.
      - Let there exist $Q^\dagger, Q^\ddagger \in \mathbb{P}$ such that $Q^\dagger \Longrightarrow Q \Longrightarrow Q^\ddagger$, $\mathscr{R}(P, X, Q^\dagger)$ and $\mathscr{R}(P, X, Q^\ddagger)$. Since $\mathscr{R}(P, X, Q^\ddagger)$, there exists a path $Q^\ddagger \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', X, Q_2)$. Since $Q \Longrightarrow Q^\ddagger$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, X, Q_1)$ and $\mathscr{R}'(P', X, Q_2)$.
   b. Suppose $P \xrightarrow{a} P'$ with $a \in X$.
      - Let there exist $P^\dagger, P^\ddagger \in \mathbb{P}$ such that $P^\dagger \Longrightarrow P \Longrightarrow P^\ddagger$, $\mathscr{R}(P^\dagger, X, Q)$ and $\mathscr{R}(P^\ddagger, X, Q)$. Since $P^\dagger \Longrightarrow P$ and $\mathscr{R}(P^\dagger, X, Q)$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}(P, X, Q_0)$. Since $P \xrightarrow{a} P'$, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', Q_2)$. Thus, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, X, Q_1)$ and $\mathscr{R}'(P', Q_2)$.
      - Let there exist $Q^\dagger, Q^\ddagger \in \mathbb{P}$ such that $Q^\dagger \Longrightarrow Q \Longrightarrow Q^\ddagger$, $\mathscr{R}(P, X, Q^\dagger)$ and $\mathscr{R}(P, X, Q^\ddagger)$. Since $\mathscr{R}(P, X, Q^\ddagger)$, there exists a path $Q^\ddagger \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', Q_2)$. Since $Q \Longrightarrow Q^\ddagger$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, X, Q_1)$ and $\mathscr{R}'(P', Q_2)$.
   c. Suppose $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$.
      - Let there exist $P^\dagger, P^\ddagger \in \mathbb{P}$ such that $P^\dagger \Longrightarrow P \Longrightarrow P^\ddagger$, $\mathscr{R}(P^\dagger, X, Q)$ and $\mathscr{R}(P^\ddagger, X, Q)$. Since $P^\dagger \Longrightarrow P$ and $\mathscr{R}(P^\dagger, X, Q)$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}(P, X, Q_0)$. Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, there exists a path $Q_0 \Longrightarrow Q_0'$ such that $\mathscr{R}(P, Q_0')$. Thus, there exists a path $Q \Longrightarrow Q_0'$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, Q_0')$.

- Let there exist $Q^\dagger, Q^\ddagger \in \mathbb{P}$ such that $Q^\dagger \Longrightarrow Q \Longrightarrow Q^\ddagger$, $\mathscr{R}(P, X, Q^\dagger)$ and $\mathscr{R}(P, X, Q^\ddagger)$. Since $\mathscr{R}(P, X, Q^\ddagger)$, there exists a path $Q^\ddagger \Longrightarrow Q_0$ such that $\mathscr{R}(P, Q_0)$. Since $Q \Longrightarrow Q^\ddagger$, there exists a path $Q \Longrightarrow Q_0$ such that, since $\mathscr{R} \subseteq \mathscr{R}'$, $\mathscr{R}'(P, Q_0)$.

d. Suppose $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$.

- Let there exist $P^\dagger, P^\ddagger \in \mathbb{P}$ such that $P^\dagger \Longrightarrow P \Longrightarrow P^\ddagger$, $\mathscr{R}(P^\dagger, X, Q)$ and $\mathscr{R}(P^\ddagger, X, Q)$. Since $P^\dagger \Longrightarrow P$ and $\mathscr{R}(P^\dagger, X, Q)$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}(P, X, Q_0)$. Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1]$, $\mathscr{R}(P, X, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r})$. Thus, there exists a path $Q =: Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1], \mathscr{R}(P, X, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r})$.
- Let there exist $Q^\dagger, Q^\ddagger \in \mathbb{P}$ such that $Q^\dagger \Longrightarrow Q \Longrightarrow Q^\ddagger$, $\mathscr{R}(P, X, Q^\dagger)$ and $\mathscr{R}(P, X, Q^\ddagger)$. Since $\mathscr{R}(P, X, Q^\ddagger)$, there exists a path $Q^\ddagger =: Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1], \mathscr{R}(P, X, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r})$. Since $Q \Longrightarrow Q^\ddagger$, there exists a path $Q =: Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1], \mathscr{R}(P, X, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r})$.

e. Suppose $P \stackrel{\tau}{\not\rightarrow}$.

- Let there exist $P^\dagger, P^\ddagger \in \mathbb{P}$ such that $P^\dagger \Longrightarrow P \Longrightarrow P^\ddagger$, $\mathscr{R}(P^\dagger, X, Q)$ and $\mathscr{R}(P^\ddagger, X, Q)$. Since $P^\dagger \Longrightarrow P$ and $\mathscr{R}(P^\dagger, X, Q)$, there exists a path $Q \Longrightarrow Q_0$. Since $P \stackrel{\tau}{\not\rightarrow}$, there exists a path $Q_0 \Longrightarrow Q_1 \stackrel{\tau}{\not\rightarrow}$. Thus, there exists a path $Q \Longrightarrow Q_1 \stackrel{\tau}{\not\rightarrow}$.
- Let there exist $Q^\dagger, Q^\ddagger \in \mathbb{P}$ such that $Q^\dagger \Longrightarrow Q \Longrightarrow Q^\ddagger$, $\mathscr{R}(P, X, Q^\dagger)$ and $\mathscr{R}(P, X, Q^\ddagger)$. Since $\mathscr{R}(P, X, Q^\ddagger)$, there exists a path $Q^\ddagger \Longrightarrow Q_1 \stackrel{\tau}{\not\rightarrow}$. Since $Q \Longrightarrow Q^\ddagger$, there exists a path $Q \Longrightarrow Q_1 \stackrel{\tau}{\not\rightarrow}$. ◀

Proof of Proposition 4. Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two branching reactive bisimulations and define

$$\mathscr{R} := (\mathscr{R}_1 \circ \mathscr{R}_2) \cup (\mathscr{R}_2 \circ \mathscr{R}_1)$$

$\mathscr{R}$ is clearly symmetric by definition. Let's check that $\mathscr{R}$ is a branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}(P, Q)$ then there exists $R \in \mathbb{P}$ such that $\mathscr{R}_1(P, R)$ and $\mathscr{R}_2(R, Q)$, or $\mathscr{R}_2(P, R)$ and $\mathscr{R}_1(R, Q)$. The two possibilities are similar; thus, suppose without loss of generality that $\mathscr{R}_1(P, R)$ and $\mathscr{R}_2(R, Q)$.
   - **a.** If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then, since $\mathscr{R}_1(P, R)$, there exists a path $R \Longrightarrow R_1 \xrightarrow{(\alpha)} R_2$ such that $\mathscr{R}_1(P, R_1)$ and $\mathscr{R}_1(P', R_2)$. Since $\mathscr{R}_2(R, Q)$ and $R \Longrightarrow R_1$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}_2(R_1, Q_0)$. Since $R_1 \xrightarrow{(\alpha)} R_2$, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $\mathscr{R}_2(R_1, Q_1)$ and $\mathscr{R}_2(R_2, Q_2)$. By definition of $\mathscr{R}$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$.
   - **b.** For all $Y \subseteq A$, since $\mathscr{R}_1(P, R)$ and $\mathscr{R}_2(R, Q)$, $\mathscr{R}_1(P, Y, R)$ and $\mathscr{R}_2(R, Y, Q)$, thus, $\mathscr{R}(P, Y, Q)$.
2. If $\mathscr{R}(P, X, Q)$ then there exists $R \in \mathbb{P}$ such that $\mathscr{R}_1(P, X, R)$ and $\mathscr{R}_2(R, X, Q)$, or $\mathscr{R}_2(P, X, R)$ and $\mathscr{R}_1(R, X, Q)$. The two possibilities are similar; thus, suppose without loss of generality that $\mathscr{R}_1(P, X, R)$ and $\mathscr{R}_2(R, X, Q)$.
   - **a.** If $P \xrightarrow{\tau} P'$ then, since $\mathscr{R}_1(P, X, R)$, there exists a path $R \Longrightarrow R_1 \xrightarrow{(\tau)} R_2$ such that $\mathscr{R}_1(P, X, R_1)$ and $\mathscr{R}_1(P', X, R_2)$. Since $\mathscr{R}_2(R, X, Q)$ and $R \Longrightarrow R_1$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}_2(R_1, X, Q_0)$. Since $R_1 \xrightarrow{(\tau)} R_2$, there exists a path

$Q_0 \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $\mathscr{R}_2(R_1, X, Q_1)$ and $\mathscr{R}_2(R_2, X, Q_2)$. By definition of $\mathscr{R}$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', X, Q_2)$.

b. If $P \xrightarrow{a} P'$ with $a \in X$ then, since $\mathscr{R}_1(P, X, R)$, there exists a path $R \Longrightarrow R_1 \xrightarrow{a} R_2$ such that $\mathscr{R}_1(P, X, R_1)$ and $\mathscr{R}_1(P', R_2)$. Since $\mathscr{R}_2(R, X, Q)$ and $R \Longrightarrow R_1$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}_2(R_1, X, Q_0)$. Since $R_1 \xrightarrow{a} R_2$, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}_2(R_1, X, Q_1)$ and $\mathscr{R}_2(R_2, Q_2)$. By definition of $\mathscr{R}$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $\mathscr{R}(P, X, Q_1)$ and $\mathscr{R}(P', Q_2)$.

c. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then, since $P \stackrel{\tau}{\not\rightarrow}$, there exists a path $R \Longrightarrow R_0 \stackrel{\tau}{\not\rightarrow}$. Moreover, using Clause 2.a, $\mathscr{R}_1(P, X, R_0)$. Moreover, there exists a path $R_0 \Longrightarrow R_0'$ such that $\mathscr{R}_1(P, R_0')$, but, since $R_0 \stackrel{\tau}{\not\rightarrow}$, $R_0 = R_0'$. By Clause 1.a, $\mathcal{I}(P) = \mathcal{I}(R_0)$, so $\mathcal{I}(R_0) \cap (X \cup \{\tau\}) = \emptyset$. Since $\mathscr{R}_2(R, X, Q)$ and $R \Longrightarrow R_0$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}_2(R_0, X, Q_0)$. Moreover, since $\mathcal{I}(R_0) \cap (X \cup \{\tau\}) = \emptyset$, there exists a path $Q_0 \Longrightarrow Q_0'$ such that $\mathscr{R}_2(R_0, Q_0')$. Thus, there exists a path $Q \Longrightarrow Q_0'$ such that, by definition of $\mathscr{R}$, $\mathscr{R}(P, Q_0')$.

d. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, since $\mathscr{R}_1(P, X, R)$, there exists a path $R = R_0 \Longrightarrow R_1 \xrightarrow{\mathrm{t}} R_2 \Longrightarrow R_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow R_{2r-1} \xrightarrow{(\mathrm{t})} R_{2r}$ with $r > 0$, such that $\forall i \in [0, r{-}1], \mathscr{R}_1(P, X, R_{2i}) \wedge \mathcal{I}(R_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}_1(P', X, R_{2r})$. For all $i \in [0, r{-}1]$, $\mathscr{R}_1(P, X, R_{2i})$, $R_{2i} \Longrightarrow R_{2i+1}$ and $P \stackrel{\tau}{\not\rightarrow}$, therefore, for all $i \in [0, r{-}1]$, $\mathscr{R}_1(P, X, R_{2i+1})$. Since $\mathscr{R}_2(R, X, Q)$, $\mathcal{I}(R_{2r-1}) \cap (X \cup \{\tau\}) = \emptyset$ and $R_{2r-1} \xrightarrow{(\mathrm{t})} R_{2r}$, there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2k-1} \xrightarrow{(\mathrm{t})} Q_{2k}$ with $k > 0$, such that $\forall j \in [0, k{-}1], \exists i \in [0, 2r{-}1], \mathscr{R}_2(R_i, X, Q_{2j}) \wedge \mathcal{I}(Q_{2j+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}_2(R_{2r}, X, Q_{2k})$. By definition of $\mathscr{R}$, $\forall j \in [0, k{-}1], \mathscr{R}(P, X, Q_{2j})$ and $\mathscr{R}(P', X, Q_{2k})$.

e. If $P \stackrel{\tau}{\not\rightarrow}$ then, since $\mathscr{R}_1(P, X, R)$, there exists a path $R \Longrightarrow R_0 \stackrel{\tau}{\not\rightarrow}$. Since $\mathscr{R}_2(R, X, Q)$ and $R \Longrightarrow R_0$, there exists a path $Q \Longrightarrow Q_0$ such that $\mathscr{R}_2(R_0, X, Q_0)$. Since $R_0 \stackrel{\tau}{\not\rightarrow}$, there exists a path $Q_0 \Longrightarrow Q_0' \stackrel{\tau}{\not\rightarrow}$. Hence there exists a path $Q \Longrightarrow Q_0' \stackrel{\tau}{\not\rightarrow}$. ◀

F Proof of Modal Characterisation

Proof of Theorem 11. ($\Rightarrow$) We are going to prove by structural induction on $\mathbb{L}_b$ and $\mathbb{L}_b^r$ that, for all $P, Q \in \mathbb{P}$, $X \subseteq A$, $\varphi \in \mathbb{L}_b$ and $\psi \in \mathbb{L}_b^r$,

- if $P \leftrightarroweq_{br} Q$ and $P \models \varphi$ then $Q \models \varphi$
- if $P \leftrightarroweq_{br}^X Q$ and $P \models_X \varphi$ then $Q \models_X \varphi$
- if $P \leftrightarroweq_{br}^r Q$ and $P \models \psi$ then $Q \models \psi$
- if $P \leftrightarroweq_{br}^{rX} Q$ and $P \models_X \psi$ then $Q \models_X \psi$

Note that, in the four cases, we dispose of the contraposition. Let $P, Q \in \mathbb{P}$, $X \subseteq A$, $\varphi \in \mathbb{L}_b$ and $\psi \in \mathbb{L}_b^r$.

- If $P \leftrightarroweq_{br} Q$ and $P \models \varphi$ then
  - if $\varphi = \top$ then $Q \models \top$.
  - if $\varphi = \bigwedge_{i \in I} \varphi_i$ with $(\varphi_i)_{i \in I} \in (\mathbb{L}_b)^I$ then, for all $i \in I$, $P \models \varphi_i$. Thus, by induction, for all $i \in I$, $Q \models \varphi_i$. Therefore, $Q \models \bigwedge_{i \in I} \varphi_i$.
  - if $\varphi = \neg\varphi'$ then $P \not\models \varphi'$. Thus, by induction, $Q \not\models \varphi'$. Therefore, $Q \models \neg\varphi'$.
  - if $\varphi = \langle\varepsilon\rangle(\varphi_1 \langle\hat{\alpha}\rangle \varphi_2)$ then there exists a path $P \Longrightarrow P_1 \xrightarrow{(\alpha)} P_2$ such that $P_1 \models \varphi_1$ and $P_2 \models \varphi_2$. Since $P \leftrightarroweq_{br} Q$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $P_1 \leftrightarroweq_{br} Q_1$ and $P_2 \leftrightarroweq_{br} Q_2$. By induction, $Q_1 \models \varphi_1$ and $Q_2 \models \varphi_2$. Therefore, $Q \models \varphi$.
  - if $\varphi = \varphi_1 \langle\varepsilon_X\rangle \varphi_2$ then there is a path $P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{(\mathrm{t})} P_{2r}$ with $r > 0$, such that $P \models \varphi_1 \wedge \forall i \in [1, 2r{-}1]\ P_i \models_X \varphi_1 \wedge P_{2r} \models_X \varphi_2$ and, moreover, $\forall i \in [0, r{-}1]\ \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$. Since $P \leftrightarroweq_{br} Q$, there exists a path $Q \Longrightarrow$

$Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2k-1} \xrightarrow{(t)} Q_{2k}$ with $k > 0$, such that $\forall j \in [0, k-1]$ $\mathcal{I}(Q_{2j+1}) \cap (X \cup \{\tau\}) = \emptyset$, $P_{2r} \leftrightarrow_{br}^{X} Q_{2k}$ and $\forall j \in [1, 2k-1]$, $\exists i \in [1, 2r-1]$, $P_i \leftrightarrow_{br}^{X} Q_j$. By induction, $Q \models \varphi_1$, $\forall j \in [1, 2k-1]$ $Q_j \models_X \varphi_1$ and $Q_{2k} \models_X \varphi_2$. Therefore, $Q \models \varphi_1\langle\varepsilon_X\rangle\varphi_2$.

- if $\varphi = \langle\varepsilon\rangle\neg\langle\tau\rangle\top$ then there exists a path $P \Longrightarrow P_0 \overset{\tau}{\not\rightarrow}$. Since $P \leftrightarrow_{br} Q$, there exists a path $Q \Longrightarrow Q_0 \overset{\tau}{\not\rightarrow}$. Therefore, $Q \models \varphi$.

- If $P \leftrightarrow_{br}^{X} Q$ and $P \models_X \varphi$ then
  - if $\varphi = \top$ then $Q \models_X \top$.
  - if $\varphi = \bigwedge_{i\in I} \varphi_i$ with $(\varphi_i)_{i\in I} \in (\mathbb{L}_b)^I$ then, for all $i \in I$, $P \models_X \varphi_i$. Thus, by induction, for all $i \in I$, $Q \models_X \varphi_i$. Therefore, $Q \models_X \bigwedge_{i\in I} \varphi_i$.
  - if $\varphi = \neg\varphi'$ then $P \not\models_X \varphi'$. Thus, by induction, $Q \not\models_X \varphi'$. Therefore, $Q \models_X \neg\varphi'$.
  - if $\varphi = \langle\varepsilon\rangle(\varphi_1\langle\hat{\alpha}\rangle\varphi_2)$ then
    * if $\alpha = \tau$ then there exists a path $P \Longrightarrow P_1 \xrightarrow{(\tau)} P_2$ such that $P_1 \models_X \varphi_1$ and $P_2 \models_X \varphi_2$. Since $P \leftrightarrow_{br}^{X} Q$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $P_1 \leftrightarrow_{br}^{X} Q_1$ and $P_2 \leftrightarrow_{br}^{X} Q_2$. By induction, $Q_1 \models_X \varphi_1$ and $Q_2 \models_X \varphi_2$. Therefore, $Q \models_X \varphi$.
    * if $\alpha \in A$ then $a \in X$ or $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and there exists a path $P \Longrightarrow P_1 \xrightarrow{a} P_2$ such that $P_1 \models_X \varphi_1$ and $P_2 \models \varphi_2$. Since $P \leftrightarrow_{br}^{X} Q$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$ such that $P_1 \leftrightarrow_{br}^{X} Q_1$ and $P_2 \leftrightarrow_{br} Q_2$. Moreover, with Lemma 2.4 we can get that $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \Leftrightarrow \mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$. By induction, $Q_1 \models_X \varphi_1$ and $Q_2 \models \varphi_2$. Therefore, $Q \models_X \varphi$.
  - if $\varphi = \varphi_1\langle\varepsilon_Y\rangle\varphi_2$ then there is a path $P \Longrightarrow P_1 \xrightarrow{t} P_2 \Longrightarrow P_3 \xrightarrow{t} \ldots \Longrightarrow P_{2r-1} \xrightarrow{(t)} P_{2r}$ with $r > 0$, such that $\mathcal{I}(P_1) \cap (X \cup \{\tau\}) = \emptyset$, $\forall i \in [1, 2r-1]$ $P_i \models_Y \varphi_1$, $P_{2r} \models_Y \varphi_2$ and $\forall i \in [0, r-1]$ $\mathcal{I}(P_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$. Since $P \leftrightarrow_{br}^{X} Q$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2k-1} \xrightarrow{(t)} Q_{2k}$ with $k > 0$, such that $\mathcal{I}(Q_1) \cap (X \cup Y \cup \{\tau\}) = \emptyset$, $\forall j \in [1, k-1]$ $\mathcal{I}(Q_{2j+1}) \cap (Y \cup \{\tau\}) = \emptyset$, $P_{2r} \leftrightarrow_{br}^{Y} Q_{2k}$ and $\forall j \in [1, 2k-1]$, $\exists i \in [1, 2r-1]$, $P_i \leftrightarrow_{br}^{Y} Q_j$. By induction, $\forall j \in [1, 2k-1]$ $Q_{2j} \models_Y \varphi_1$ and $Q_{2k} \models_Y \varphi_2$. Therefore, $Q \models_X \varphi_1\langle\varepsilon_Y\rangle\varphi_2$.
  - if $\varphi = \langle\varepsilon\rangle\neg\langle\tau\rangle\top$ then there exists a path $P \Longrightarrow P_0 \overset{\tau}{\not\rightarrow}$. Since $P \leftrightarrow_{br} Q$, there exists a path $Q \Longrightarrow Q_0 \overset{\tau}{\not\rightarrow}$. Therefore, $Q \models \varphi$.
- If $P \leftrightarrow_{br}^{r} Q$ and $P \models \psi$ then
  - if $\psi = \top$ then $Q \models \top$.
  - if $\psi = \bigwedge_{i\in I} \psi_i$ with $(\psi_i)_{i\in I} \in (\mathbb{L}_b^r)^I$ then, for all $i \in I$, $P \models \psi_i$. Thus, by induction, for all $i \in I$, $Q \models \psi_i$. Therefore, $Q \models \bigwedge_{i\in I} \psi_i$.
  - if $\psi = \neg\psi'$ then $P \not\models \psi'$. Thus, by induction, $Q \not\models \psi'$. Therefore, $Q \models \neg\psi'$.
  - if $\psi = \langle\alpha\rangle\varphi$ then there is a transition $P \xrightarrow{\alpha} P'$ such that $P' \models \varphi$. Since $P \leftrightarrow_{br}^{r} Q$, there exists a path $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarrow_{br} Q'$. By induction, $Q' \models \varphi$. Therefore, $Q \models \psi$.
  - if $\psi = \langle t_X\rangle\varphi$ then $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and there exists a transition $P \xrightarrow{t} P'$ such that $P' \models_X \varphi$. Since $P \leftrightarrow_{br}^{r} Q$, $\mathcal{I}(Q) \cap (X \cup \{\tau\}) = \emptyset$ and there exists a path $Q \xrightarrow{t} Q'$ such that $P' \leftrightarrow_{br}^{X} Q'$. By induction, $Q' \models_X \varphi$. Therefore, $Q \models \psi$.
- If $P \leftrightarrow_{br}^{rX} Q$ and $P \models_X \psi$ then
  - if $\psi = \top$ then $Q \models_X \top$.
  - if $\psi = \bigwedge_{i\in I} \psi_i$ with $(\psi_i)_{i\in I} \in (\mathbb{L}_b^r)^I$ then, for all $i \in I$, $P \models_X \psi_i$. Thus, by induction, for all $i \in I$, $Q \models_X \psi_i$. Therefore, $Q \models_X \bigwedge_{i\in I} \psi_i$.
  - if $\psi = \neg\psi'$ then $P \not\models_X \psi'$. Thus, by induction, $Q \not\models_X \psi'$. Therefore, $Q \models_X \neg\psi'$.
  - if $\psi = \langle\alpha\rangle\varphi$

  * if $\alpha = \tau$ then there exists a transition $P \xrightarrow{\tau} P'$ such that $P' \models_X \varphi$. Since $P \leftrightarroweq^{rX}_{br} Q$, there exists a transition $Q \xrightarrow{\tau} Q'$ such that $P' \leftrightarroweq^{X}_{br} Q'$. By induction, $Q' \models_X \varphi$. Therefore, $Q \models_X \psi$.
  * if $\alpha \in A$ then $a \in X$ or $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and there exists a transition $P \xrightarrow{a} P'$ such that $P' \models \varphi$. Since $P \leftrightarroweq^{rX}_{br} Q$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \Leftrightarrow \mathcal{I}(Q) \cap (X \cup \{\tau\}) = \emptyset$ and there exists a transition $Q \xrightarrow{a} Q'$ such that $P' \leftrightarroweq_{br} Q'$. By induction, $Q' \models \varphi$. Therefore, $Q \models_X \psi$.
- if $\psi = \langle t_Y \rangle \varphi$ then $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and there exists a transition $P \xrightarrow{t} P'$ such that $P_1 \models_Y \varphi$. Since $P \leftrightarroweq^{rX}_{br} Q$, $\mathcal{I}(Q) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and there exists a transition $Q \xrightarrow{t} Q'$ such that $P' \leftrightarroweq^{Y}_{br} Q'$. By induction, $Q' \models_Y \varphi$. Therefore, $Q \models_X \psi$.

($\Leftarrow$) Let $\equiv := \{(P,Q) \mid \forall \varphi \in \mathbb{L}_{tb}, P \models \varphi \Leftrightarrow Q \models \varphi\} \cup \{(P,X,Q) \mid \forall \varphi \in \mathbb{L}_{tb}, P \models_X \varphi \Leftrightarrow Q \models_X \varphi\}$, and $\equiv^r := \{(P,Q) \mid \forall \psi \in \mathbb{L}^r_{tb}, P \models \psi \Leftrightarrow Q \models \psi\} \cup \{(P,X,Q) \mid \forall \psi \in \mathbb{L}^r_{tb}, P \models_X \psi \Leftrightarrow Q \models_X \psi\}$. $(P,X,Q) \in \equiv$ will be denoted $P \equiv_X Q$ for clarity. Note that $\equiv^r \subseteq \equiv$. We are going to check that $\equiv$ is a generalised branching bisimulation and $\equiv^r$ a generalised rooted branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $P \equiv Q$
   a. if $P \xrightarrow{\alpha} P'$ then define $\mathcal{Q}^\dagger := \{Q^\dagger \mid Q \Longrightarrow Q^\dagger \wedge P \not\equiv Q^\dagger\}$ and $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \Longrightarrow Q^\dagger \xrightarrow{(\alpha)} Q^\ddagger \wedge P' \not\equiv Q^\ddagger\}$. Since $\mathbb{L}_b$ is closed under negation and conjunction, there exist two formulas $\varphi^\dagger, \varphi^\ddagger \in \mathbb{L}_b$ such that $P \models \varphi^\dagger$, $P' \models \varphi^\ddagger$, for all $Q^\dagger \in \mathcal{Q}^\dagger$, $Q^\dagger \not\models \varphi^\dagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models \varphi^\ddagger$. Note that $P \models \langle \varepsilon \rangle (\varphi^\dagger \langle \hat{\alpha} \rangle \varphi^\ddagger)$. Thus, $Q \models \langle \varepsilon \rangle (\varphi^\dagger \langle \hat{\alpha} \rangle \varphi^\ddagger)$. Therefore, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $Q_1 \models \varphi^\dagger$ and $Q_2 \models \varphi^\ddagger$. By definition of $\mathcal{Q}^\dagger$ and $\mathcal{Q}^\ddagger$, $P \equiv Q_1$ and $P' \equiv Q_2$.
   b. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then define $\mathcal{Q}^\dagger := \{Q^\dagger \mid Q \Longrightarrow Q^\dagger_1 \xrightarrow{t} Q^\dagger_2 \Longrightarrow Q^\dagger_3 \xrightarrow{t} ... \Longrightarrow Q^\dagger \wedge P \not\equiv_X Q^\dagger\}$ and $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \Longrightarrow Q^\dagger_1 \xrightarrow{t} Q^\dagger_2 \Longrightarrow Q^\dagger_3 \xrightarrow{t} ... \Longrightarrow Q^\dagger_{2r-1} \xrightarrow{(t)} Q^\ddagger \wedge P' \not\equiv_X Q^\ddagger\}$. Since $\mathbb{L}_b$ is closed under negation and conjunction, there exist two formulas $\varphi^\dagger, \varphi^\ddagger \in \mathbb{L}_b$ such that $P \models_X \varphi^\dagger$, $P' \models_X \varphi^\ddagger$, for all $Q^\dagger \in \mathcal{Q}^\dagger$, $Q^\dagger \not\models_X \varphi^\dagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models_X \varphi^\ddagger$. Note that $P \models \varphi^\dagger \langle \varepsilon_X \rangle \varphi^\ddagger$. Thus, $Q \models \varphi^\dagger \langle \varepsilon_X \rangle \varphi^\ddagger$. Therefore, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} ... \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1]$ $\mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$, $\forall i \in [1, 2r-1]$ $Q_i \models_X \varphi^\dagger$ and $Q_{2r} \models_X \varphi^\ddagger$. By definition of $\mathcal{Q}^\dagger$ and $\mathcal{Q}^\ddagger$, $\forall i \in [1, r-1]$, $P \equiv_X Q_{2i}$ and $P' \equiv_X Q_{2r}$.
   c. if $P \not\xrightarrow{\tau}$ then $P \models \langle \varepsilon \rangle \neg \langle \tau \rangle \top$. Thus $Q \models \langle \varepsilon \rangle \neg \langle \tau \rangle \top$. Therefore, $Q \Longrightarrow Q_1 \not\xrightarrow{\tau}$.
2. If $P \equiv_X Q$ then
   a. if $P \xrightarrow{\tau} P'$ then define $\mathcal{Q}^\dagger := \{Q^\dagger \mid Q \Longrightarrow Q^\dagger \wedge P \not\equiv_X Q^\dagger\}$ and $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \Longrightarrow Q^\ddagger \wedge P' \not\equiv_X Q^\ddagger\}$. Since $\mathbb{L}_b$ is closed under negation and conjunction, there exist two formulas $\varphi^\dagger, \varphi^\ddagger \in \mathbb{L}_b$ such that $P \models_X \varphi^\dagger$, $P' \models_X \varphi^\ddagger$, for all $Q^\dagger \in \mathcal{Q}^\dagger$, $Q^\dagger \not\models_X \varphi^\dagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models_X \varphi^\ddagger$. Note that $P \models_X \langle \varepsilon \rangle (\varphi^\dagger \langle \hat{\tau} \rangle \varphi^\ddagger)$. Thus, $Q \models_X \langle \varepsilon \rangle (\varphi^\dagger \langle \hat{\tau} \rangle \varphi^\ddagger)$. Therefore, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$ such that $Q_1 \models_X \varphi^\dagger$ and $Q_2 \models_X \varphi^\ddagger$. By definition of $\mathcal{Q}^\dagger$ and $\mathcal{Q}^\ddagger$, $P \equiv_X Q_1$ and $P' \equiv_X Q_2$.
   b. if $P \xrightarrow{a} P'$ with $a \in X$ or $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then define $\mathcal{Q}^\dagger := \{Q^\dagger \mid Q \Longrightarrow Q^\dagger \wedge P \not\equiv_X Q^\dagger\}$ and $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \Longrightarrow Q^\dagger \xrightarrow{a} Q^\ddagger \wedge P' \not\equiv Q^\ddagger\}$. Since $\mathbb{L}_b$ is closed under negation and conjunction, there exist two formulas $\varphi^\dagger, \varphi^\ddagger \in \mathbb{L}_b$ such that $P \models_X \varphi^\dagger$, $P' \models \varphi^\ddagger$, for all $Q^\dagger \in \mathcal{Q}^\dagger$, $Q^\dagger \not\models_X \varphi^\dagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models \varphi^\ddagger$. Note that $P \models_X \langle \varepsilon \rangle (\varphi^\dagger \langle \hat{\alpha} \rangle \varphi^\ddagger)$. Thus, $Q \models_X \langle \varepsilon \rangle (\varphi^\dagger \langle \hat{\alpha} \rangle \varphi^\ddagger)$. Therefore, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $a \in X \vee \mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$, $Q_1 \models_X \varphi^\dagger$ and $Q_2 \models \varphi^\ddagger$. By definition of $\mathcal{Q}^\dagger$ and $\mathcal{Q}^\ddagger$, $P \equiv_X Q_1$ and $P' \equiv Q_2$.
   c. if $\mathcal{I}(P) \cap (X \cup Y \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then define $\mathcal{Q}^\dagger := \{Q^\dagger \mid Q \Longrightarrow Q^\dagger_1 \xrightarrow{t} Q^\dagger_2 \Longrightarrow Q^\dagger_3 \xrightarrow{t} ... \Longrightarrow Q^\dagger \wedge P \not\equiv_Y Q^\dagger\}$ and $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \Longrightarrow Q^\dagger_1 \xrightarrow{t} Q^\dagger_2 \Longrightarrow Q^\dagger_3 \xrightarrow{t} ... \Longrightarrow$

$Q^\dagger_{2r-1} \xrightarrow{(t)} Q^\ddagger \wedge P' \not\equiv_Y Q^\ddagger\}$. Since $\mathbb{L}_b$ is closed under negation and conjunction, there exist two formulas $\varphi^\dagger, \varphi^\ddagger \in \mathbb{L}_b$ such that $P \models_Y \varphi^\dagger$, $P' \models_Y \varphi^\ddagger$, for all $Q^\dagger \in \mathcal{Q}^\dagger$, $Q^\dagger \not\models_Y \varphi^\dagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models_Y \varphi^\ddagger$. Note that $P \models_X \varphi^\dagger\langle\varepsilon_Y\rangle\varphi^\ddagger$. Thus, $Q \models_X \varphi^\dagger\langle\varepsilon_Y\rangle\varphi^\ddagger$. Therefore, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\mathcal{I}(Q_1) \cap (X \cup \{\tau\}) = \emptyset$, $\forall i \in [0, r-1]$ $\mathcal{I}(Q_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$, $\forall i \in [1, 2r-1]$ $Q_i \models_Y \varphi^\dagger$ and $Q_{2r} \models_Y \varphi^\ddagger$. By definition of $\mathcal{Q}^\dagger$ and $\mathcal{Q}^\ddagger$, $\forall i \in [1, r-1]$, $P \equiv_Y Q_{2i}$ and $P' \equiv_Y Q_{2r}$.

d. if $P \stackrel{\tau}{\not\rightarrow}$ then $P \models_X \langle\varepsilon\rangle\neg\langle\tau\rangle\top$. Thus $Q \models_X \langle\varepsilon\rangle\neg\langle\tau\rangle\top$. Therefore, $Q \Longrightarrow Q_1 \stackrel{\tau}{\not\rightarrow}$.

1. If $P \equiv^r Q$ then

a. if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then define $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \xrightarrow{\alpha} Q^\ddagger \wedge P' \not\equiv Q^\ddagger\}$. Since $\mathbb{L}^r_b$ is closed under negation and conjunction, there exist a formula $\varphi^\ddagger \in \mathbb{L}^r_b$ such that $P' \models \varphi^\ddagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models \varphi^\ddagger$. Note that $P \models \langle\alpha\rangle\varphi^\ddagger$. Thus, $Q \models \langle\alpha\rangle\varphi^\ddagger$. Therefore, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $Q' \models \varphi^\ddagger$. By definition of $\mathcal{Q}^\ddagger$, $P' \equiv Q'$.

b. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then define $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \xrightarrow{t} Q^\ddagger \wedge P' \not\equiv_X Q^\ddagger\}$. Since $\mathbb{L}^r_b$ is closed under negation and conjunction, there exist a formula $\varphi^\ddagger \in \mathbb{L}^r_b$ such that $P' \models_X \varphi^\ddagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models_X \varphi^\ddagger$. Note that $P \models \langle t_X\rangle\varphi^\ddagger$. Thus, $Q \models \langle t_X\rangle\varphi^\ddagger$. Therefore, there exists a transition $Q \xrightarrow{t} Q'$ such that $Q' \models_X \varphi^\ddagger$. By definition of $\mathcal{Q}^\ddagger$, $P' \equiv_X Q'$.

2. If $P \equiv^r_X Q$ then

a. if $P \xrightarrow{\tau} P'$ then define $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \xrightarrow{\tau} Q^\ddagger \wedge P' \not\equiv_X Q^\ddagger\}$. Since $\mathbb{L}^r_b$ is closed under negation and conjunction, there exist a formula $\varphi^\ddagger \in \mathbb{L}^r_b$ such that $P' \models_X \varphi^\ddagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models_X \varphi^\ddagger$. Note that $P \models_X \langle\tau\rangle\varphi^\ddagger$. Thus, $Q \models_X \langle\tau\rangle\varphi^\ddagger$. Therefore, there exists a transition $Q \xrightarrow{\tau} Q'$ such that $Q' \models_X \varphi^\ddagger$. By definition of $\mathcal{Q}^\ddagger$, $P' \equiv_X Q'$.

b. if $P \xrightarrow{a} P'$ with $a \in X \vee \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then define $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \xrightarrow{a} Q^\ddagger \wedge P' \not\equiv Q^\ddagger\}$. Since $\mathbb{L}^r_b$ is closed under negation and conjunction, there exist a formula $\varphi^\ddagger \in \mathbb{L}^r_b$ such that $P' \models \varphi^\ddagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models \varphi^\ddagger$. Note that $P \models_X \langle a\rangle\varphi^\ddagger$. Thus, $Q \models_X \langle a\rangle\varphi^\ddagger$. Therefore, there exists a path $Q \xrightarrow{\alpha} Q'$ such that $Q' \models \varphi^\ddagger$. By definition of $\mathcal{Q}^\ddagger$, $P' \equiv Q'$.

c. if $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then define $\mathcal{Q}^\ddagger := \{Q^\ddagger \mid Q \xrightarrow{t} Q^\ddagger \wedge P' \not\equiv_Y Q^\ddagger\}$. Since $\mathbb{L}^r_b$ is closed under negation and conjunction, there exist a formula $\varphi^\ddagger \in \mathbb{L}^r_b$ such that $P' \models_Y \varphi^\ddagger$ and, for all $Q^\ddagger \in \mathcal{Q}^\ddagger$, $Q^\ddagger \not\models_Y \varphi^\ddagger$. Note that $P \models_X \langle t_Y\rangle\varphi^\ddagger$. Thus, $Q \models_X \langle t_Y\rangle\varphi^\ddagger$. Therefore, there exists a path $Q \xrightarrow{t} Q'$ such that $Q' \models_Y \varphi^\ddagger$. By definition of $\mathcal{Q}^\ddagger$, $P' \equiv_Y Q'$. ◀

G Correctness of Time-out Bisimulation

Proof of Proposition 15. Let $\mathscr{R}$ be a branching reactive bisimulation, let's define

$$\mathscr{B} := \{(P, Q) \mid \mathscr{R}(P, Q)\} \cup \{(\theta_X(P), \theta_X(Q)) \mid \mathscr{R}(P, X, Q)\}$$

We are going to show that $\mathscr{B}$ is a branching time-out bisimulation. Let $P, Q \in \mathbb{P}$ such that $P \mathrel{\mathscr{B}} Q$. By definition of $\mathscr{B}$, $\mathscr{R}(P, Q)$ or $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $\mathscr{R}(^\dagger, X, Q^\dagger)$.

1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then

- if $\mathscr{R}(P, Q)$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$. Thus, by definition of $\mathscr{B}$, $P \mathrel{\mathscr{B}} Q_1$ and $P \mathrel{\mathscr{B}} Q_2$.
- if $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $\mathscr{R}(P^\dagger, X, Q^\dagger)$ then

- if $\alpha = \tau$ then, by the semantics, $P' = \theta_X(P^\ddagger)$ and $P^\dagger \xrightarrow{\tau} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{(\tau)} Q_2^\dagger$ such that $\mathscr{R}(P^\dagger, X, Q_1^\dagger)$ and $\mathscr{R}(P^\ddagger, X, Q_2^\dagger)$. By the semantics, there exists a path $Q \Longrightarrow \theta_X(Q_1^\dagger) \xrightarrow{(\tau)} \theta_X(Q_2^\dagger)$ such that, by the definition of $\mathscr{B}$, $P \mathrel{\mathscr{B}} \theta_X(Q_1^\dagger)$ and $P' \mathrel{\mathscr{B}} \theta_X(Q_2^\dagger)$.
- if $\alpha = a \in A$ then, by the semantics, $P^\dagger \xrightarrow{a} P'$ and $a \in X \vee \mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$.
  - if $a \in X$ then, since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{a} Q_2$ such that $\mathscr{R}(P^\dagger, X, Q_1^\dagger)$ and $\mathscr{R}(P', Q_2)$. By the semantics, there exists a path $Q \Longrightarrow \theta_X(Q_1^\dagger) \xrightarrow{a} Q_2$ such that, by the definition of $\mathscr{B}$, $P \mathrel{\mathscr{B}} \theta_X(Q_1^\dagger)$ and $P' \mathrel{\mathscr{B}} Q_2$.
  - if $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$, then there is a path $Q^\dagger \Longrightarrow Q_0^\dagger \stackrel{\tau}{\not\rightarrow}$ with $\mathscr{R}(P^\dagger, Q_0^\dagger)$. Now there exists a path $Q_0^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{a} Q_2$ such that $\mathscr{R}(P^\dagger, Q_1^\dagger)$ and $\mathscr{R}(P', Q_2)$. Moreover, we find that $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset \Leftrightarrow \mathcal{I}(Q_1^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. By the semantics, there exists a path $Q \Longrightarrow \theta_X(Q_1^\dagger) \xrightarrow{a} Q_2$ such that, by the definition of $\mathscr{B}$, $P \mathrel{\mathscr{B}} \theta_X(Q_1)$ and $P' \mathrel{\mathscr{B}} Q_2$.

2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then
   - if $\mathscr{R}(P, Q)$ then $\mathscr{R}(P, X, Q)$, so there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $\forall i \in [0, r{-}1]$, $\mathscr{R}(P, X, Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r})$. So $Q_1 \stackrel{\tau}{\not\rightarrow}$. By definition of $\mathscr{B}$, $\forall i \in [1, r{-}1]$, $\theta_X(P) \mathrel{\mathscr{B}} \theta_X(Q_{2i})$ and $\theta_X(P') \mathrel{\mathscr{B}} \theta_X(Q_{2r})$.
   - if $P = \theta_Y(P^\dagger)$, $Q = \theta_Y(Q^\dagger)$ and $\mathscr{R}(P^\dagger, Y, Q^\dagger)$ then, by the semantics of θ_Y, $\mathcal{I}(P^\dagger) \cap (Y \cup \{\tau\}) = \emptyset$ and $P^\dagger \xrightarrow{\mathrm{t}} P'$. By Clause 2.c of Definition 1, there is a path $Q^\dagger \Longrightarrow Q_0'$ with $\mathscr{R}(P^\dagger, Q_0')$, and thus also $\mathscr{R}(P^\dagger, X, Q_0')$. Therefore, since $P^\dagger \xrightarrow{\mathrm{t}} P'$, there exists a path $Q_0' \Longrightarrow Q_1' \xrightarrow{\mathrm{t}} Q_2' \Longrightarrow Q_3' \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2r-1}' \xrightarrow{(\mathrm{t})} Q_{2r}'$ with $r > 0$, such that $\forall i \in [0, r{-}1], \mathscr{R}(P^\dagger, X, Q_{2i}') \wedge \mathcal{I}(Q_{2i+1}') \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P', X, Q_{2r}')$. Write $Q_0 := Q = \theta_Y(Q^\dagger)$, $Q_1 := \theta_Y(Q_1')$ and $Q_j := Q_j'$ for $j \in [2, 2r]$. By the semantics, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2r+1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$ and, by the definition of $\mathscr{B}$, $\forall i \in [1, r{-}1]$, $\theta_X(P) \mathrel{\mathscr{B}} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \mathrel{\mathscr{B}} \theta_X(Q_{2r})$.
3. If $P \stackrel{\tau}{\not\rightarrow}$ then
   - if $\mathscr{R}(P, Q)$ then $\mathscr{R}(P, \emptyset, Q)$, so there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$.
   - if $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $\mathscr{R}(P^\dagger, X, Q^\dagger)$ then, by the semantics, $P^\dagger \stackrel{\tau}{\not\rightarrow}$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a path $Q^\dagger \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$. By the semantics, there exists a path $Q \Longrightarrow \theta_X(Q_0) \stackrel{\tau}{\not\rightarrow}$.

Let $\mathscr{B}$ be a branching time-out bisimulation, let's define

$$\mathscr{R} = \{(P, Q) \mid P \mathrel{\mathscr{B}} Q\} \cup \{(P, X, Q) \mid \theta_X(P) \mathrel{\mathscr{B}} \theta_X(Q)\}$$

We are going to show that $\mathscr{R}$ is a generalised branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}(P, Q)$ then $P \mathrel{\mathscr{B}} Q$.
   a. If $P \xrightarrow{\alpha} P'$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $P \mathrel{\mathscr{B}} Q_1$ and $P' \mathrel{\mathscr{B}} Q_2$, thus, by definition of $\mathscr{R}$, $\mathscr{R}(P, Q_1)$ and $\mathscr{R}(P', Q_2)$.
   b. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r{-}1]$, $\theta_X(P) \mathrel{\mathscr{B}} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \mathrel{\mathscr{B}} \theta_X(Q_{2r})$. Thus, by definition of $\mathscr{R}$, $\forall i \in [1, r{-}1]$, $\mathscr{R}(P, X, Q_{2i})$ and $\mathscr{R}(P', X, Q_{2r})$.
   c. If $P \stackrel{\tau}{\not\rightarrow}$ then there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$ such that $P \mathrel{\mathscr{B}} Q_0$, thus, by definition of $\mathscr{R}$, $\mathscr{R}(P, Q_0)$.

2. If $\mathscr{R}(P,X,Q)$ then $\theta_X(P)\ \mathscr{B}\ \theta_X(Q)$.
   a. If $P \xrightarrow{\tau} P'$ then, by the semantics, $\theta_X(P) \xrightarrow{\tau} \theta_X(P')$. Therefore, there exists a path $\theta_X(Q) \Longrightarrow Q^\dagger \xrightarrow{(\tau)} Q^\ddagger$ such that $\theta_X(P)\ \mathscr{B}\ Q^\dagger$ and $\theta_X(P')\ \mathscr{B}\ Q^\ddagger$. By the semantics, $Q^\dagger = \theta_X(Q_1)$, $Q^\ddagger = \theta_X(Q_2)$ and $Q \Longrightarrow Q_1 \xrightarrow{(\tau)} Q_2$. Moreover, by definition of $\mathscr{R}$, $\mathscr{R}(P,X,Q_1)$ and $\mathscr{R}(P',X,Q_2)$.
   b. If $P \xrightarrow{a} P'$ with $a \in X \vee \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then, by the semantics, $\theta_X(P) \xrightarrow{a} P'$. Therefore, there exists a path $\theta_X(Q) \Longrightarrow Q^\dagger \xrightarrow{a} Q_2$ such that $\theta_X(P)\ \mathscr{B}\ Q^\dagger$ and $P'\ \mathscr{B}\ Q_2$. By the semantics, $Q^\dagger = \theta_X(Q_1)$ and $Q \Longrightarrow Q_1 \xrightarrow{a} Q_2$. Moreover, by definition of $\mathscr{R}$, $\mathscr{R}(P,X,Q_1)$ and $\mathscr{R}(P',Q_2)$.
   c. If $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, by the semantics, $\mathcal{I}(\theta_X(P)) \cap (Y \cup \{\tau\}) = \emptyset$ and $\theta_X(P) \xrightarrow{\mathrm{t}} P'$. Therefore, $\theta_X(Q) \Longrightarrow Q_1^\dagger \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1^\dagger \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r-1]\ \theta_Y(P)\ \mathscr{B}\ \theta_Y(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$ and $\theta_Y(P')\ \mathscr{B}\ \theta_Y(Q_{2r})$. By the semantics, $Q_1^\dagger = \theta_X(Q_1)$ with $Q_1 \stackrel{\tau}{\not\rightarrow}$ and we have $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$. Moreover, by definition of $\mathscr{R}$, $\forall i \in [1, r-1]$, $\mathscr{R}(P,Y,Q_{2i}) \wedge \mathcal{I}(Q_{2r+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\mathscr{R}(P',Y,Q_{2r})$.
   d. If $P \stackrel{\tau}{\not\rightarrow}$ then, by the semantics, $\theta_X(P) \stackrel{\tau}{\not\rightarrow}$. Therefore, there exists a path $\theta_X(Q) \Longrightarrow Q^\dagger \stackrel{\tau}{\not\rightarrow}$ such that $\theta_X(P)\ \mathscr{B}\ Q^\dagger$. By the semantics, $Q^\dagger = \theta_X(Q_0)$ and $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$. Moreover, by definition of $\mathscr{R}$, $\mathscr{R}(P,X,Q_0)$.

This ends the proof of Proposition 15.1, and thereby its corollary 15.2.
Let $\mathscr{R}$ be a generalised rooted branching reactive bisimulation, let's define

$$\mathscr{B} := \{(P,Q) \mid \mathscr{R}(P,Q)\} \cup \{(\theta_X(P), \theta_X(Q)) \mid \mathscr{R}(P,X,Q)\}$$

We are going to show that $\mathscr{B}$ is a rooted branching time-out bisimulation. Let $P, Q \in \mathbb{P}$ such that $P\ \mathscr{B}\ Q$, by definition of $\mathscr{B}$, $\mathscr{R}(P,Q)$ or $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $\mathscr{R}(P,X,Q)$.

1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then
   - if $\mathscr{R}(P,Q)$ then there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarroweq_{br} Q'$.
   - if $P = \theta_X(P^\dagger)$, $Q = \theta_X(Q^\dagger)$ and $\mathscr{R}(P^\dagger, X, Q^\dagger)$ then
     - if $\alpha = \tau$ then, by the semantics, $P' = \theta_X(P^\ddagger)$ and $P^\dagger \xrightarrow{\tau} P^\ddagger$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a transition $Q^\dagger \xrightarrow{\tau} Q^\ddagger$ such that $P^\ddagger \leftrightarroweq_{br}^X Q^\ddagger$. By the semantics, there exists a transition $Q \xrightarrow{\tau} \theta_X(Q^\ddagger)$. Moreover, by Proposition 15.2, $P' \leftrightarroweq_{br} \theta_X(Q^\ddagger)$.
     - if $\alpha = a \in A$ then, by the semantics, $P^\dagger \xrightarrow{a} P'$ and $a \in X \vee \mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. Since $\mathscr{R}(P^\dagger, X, Q^\dagger)$, there exists a transition $Q^\dagger \xrightarrow{a} Q'$ such that $P' \leftrightarroweq_{br} Q'$. Moreover, $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset \Leftrightarrow \mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. By the semantics, there exists a transition $Q \xrightarrow{a} Q'$ such that $P' \leftrightarroweq_{br} Q'$.

2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then
   - if $\mathscr{R}(P,Q)$ then there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $P' \leftrightarroweq_{br}^X Q'$. Thus, $\theta_X(P') \leftrightarroweq_{br} \theta_X(Q')$ by Proposition 15.2.
   - if $P = \theta_Y(P^\dagger)$, $Q = \theta_Y(Q^\dagger)$ and $\mathscr{R}(P^\dagger, Y, Q^\dagger)$ then, by the semantics, $P^\dagger \xrightarrow{\mathrm{t}} P'$ and $\mathcal{I}(P^\dagger) \cap (Y \cup \{\tau\}) = \emptyset$. Since $\mathscr{R}(P^\dagger, Y, Q^\dagger)$, $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P^\dagger \xrightarrow{\mathrm{t}} P'$, there exists a transition $Q^\dagger \xrightarrow{\mathrm{t}} Q'$ such that $P' \leftrightarroweq_{br}^X Q'$. Thus, $\theta_X(P') \leftrightarroweq_{br} \theta_X(Q')$. Moreover, $\mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. By the semantics, there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarroweq_{br} \theta_X(Q')$.

Let $\mathscr{B}$ be a rooted branching time-out bisimulation, let's define

$$B := \{(P,Q) \mid P\ \mathscr{B}\ Q\} \cup \{(P,X,Q) \mid \theta_X(P)\ \mathscr{B}\ \theta_X(Q)\}$$

We are going to show that $\mathscr{R}$ is a generalised rooted branching reactive bisimulation. Let $P, Q \in \mathbb{P}$ and $X \subseteq A$.

1. If $\mathscr{R}(P,Q)$ then $P\ \mathscr{B}\ Q$.
   a. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarroweq_{br} Q'$.
   b. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarroweq_{br} \theta_X(Q')$. Thus, $P' \leftrightarroweq_{br}^X Q'$, by Proposition 15.2.
2. If $\mathscr{R}(P,X,Q)$ then $\theta_X(P)\ \mathscr{B}\ \theta_X(Q)$.
   a. If $P \xrightarrow{\tau} P'$ then, by the semantics, $\theta_X(P) \xrightarrow{\tau} \theta_X(P')$. Since $\theta_X(P)\ \mathscr{B}\ \theta_X(Q)$, there exists a transition $\theta_X(Q) \xrightarrow{\tau} Q^\ddagger$ such that $\theta_X(P') \leftrightarroweq_{br} Q^\ddagger$. By the semantics, $Q^\ddagger = \theta_X(Q')$ and there exists a transition $Q \xrightarrow{\tau} Q'$. By Proposition 15.2, $P' \leftrightarroweq_{br}^X Q'$.
   b. If $P \xrightarrow{a} P'$ with $a \in X \vee \mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then, by the semantics, $\theta_X(P) \xrightarrow{a} P'$. Since $\theta_X(P)\ \mathscr{B}\ \theta_X(Q)$, there exists a transition $\theta_X(Q) \xrightarrow{a} Q'$ such that $P' \leftrightarroweq_{br} Q'$. By the semantics, there exists a transition $Q \xrightarrow{a} Q'$ such that $P' \leftrightarroweq_{br} Q'$.
   c. If $\mathcal{I}(P) \cap ((X \cup Y) \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, by the semantics, $\theta_X(P) \xrightarrow{\mathrm{t}} P'$ and $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$. Since $\theta_X(P)\ \mathscr{B}\ \theta_X(Q)$, there exists a transition $\theta_X(Q) \xrightarrow{\mathrm{t}} Q'$ such that $\theta_Y(P') \leftrightarroweq_{br} \theta_Y(Q')$. By the semantics, there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$. By Proposition 15.2, $P' \leftrightarroweq_{br}^Y Q'$. ◀

H Congruence Proofs for $\leftrightarroweq_{br}$ and $\leftrightarroweq_{tb}$

To prove congruence properties, the notion of bisimulation *up to*, introduced by Milner in [17], is going to be helpful. Let $\leftrightarroweq$ denote the classical notion of strong bisimilarity [17]: A *(strong) bisimulation* is a symmetric relation $\mathscr{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$ with $P\ \mathscr{R}\ Q$, if $P \xrightarrow{\alpha} P'$ with $\alpha \in Act$ then there is a transition $Q \xrightarrow{\alpha} Q'$ such that $P'\ \mathscr{R}\ Q'$; write $P \leftrightarroweq Q$ if $P\ \mathscr{R}\ Q$ for some strong bisimulation $\mathscr{R}$.

▶ **Definition 37.** A *branching time-out bisimulation up to* $\leftrightarroweq$ is a symmetric relation $\mathscr{B} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$ with $P\ \mathscr{B}\ Q$,
1. if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $P \leftrightarroweq\mathscr{B}\leftrightarroweq Q_1$ and $P' \leftrightarroweq\mathscr{B}\leftrightarroweq Q_2$
2. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r-1]$, $\theta_X(P) \leftrightarroweq\mathscr{B}\leftrightarroweq \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarroweq\mathscr{B}\leftrightarroweq \theta_X(Q_{2r})$
3. if $P \stackrel{\tau}{\not\rightarrow}$ then there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\rightarrow}$,

where $\leftrightarroweq\mathscr{B}\leftrightarroweq$ stands for the relational composition $\leftrightarroweq \circ \mathscr{B} \circ \leftrightarroweq$.

▶ **Proposition 38.** *Let* $P, Q \in \mathbb{P}$. *Then* $P \leftrightarroweq_{br} Q$ *iff there exists a branching time-out bisimulation* $\mathcal{B}$ *up to* $\leftrightarroweq$ *such that* $P\ \mathscr{B}\ Q$.

Proof. First of all, a branching time-out bisimulation is a branching time-out bisimulation up to $\leftrightarroweq_{br}$ by reflexivity of $\leftrightarroweq$. Conversely, let $\mathscr{B}$ be a branching bisimulation up to $\leftrightarroweq$. We are going to show that $\leftrightarroweq\mathscr{B}\leftrightarroweq$ is a branching time-out bisimulation. By the reflexivity of $\leftrightarroweq_{br}$ this will suffice. Let $P, Q \in \mathbb{P}$ such that $P \leftrightarroweq\mathscr{B}\leftrightarroweq Q$. Then there exists $P^\dagger, Q^\dagger \in \mathbb{P}$ such that $P \leftrightarroweq P^\dagger\ \mathscr{B}\ Q^\dagger \leftrightarroweq Q$.

1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then, since $P \leftrightarroweq P^\dagger$, there exists a transition $P^\dagger \xrightarrow{\alpha} P^\ddagger$ such that $P' \leftrightarroweq P^\ddagger$. Since $P^\dagger\ \mathscr{B}\ Q^\dagger$, there exists a path $Q^\dagger \Longrightarrow Q^\star \xrightarrow{(\alpha)} Q^\ddagger$ such that $P^\dagger \leftrightarroweq\mathscr{B}\leftrightarroweq Q^\star$ and $P^\ddagger \leftrightarroweq\mathscr{B}\leftrightarroweq Q^\ddagger$. Since $Q^\dagger \leftrightarroweq Q$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $Q^\star \leftrightarroweq Q_1$ and $Q^\ddagger \leftrightarroweq Q_2$. Since $\leftrightarroweq$ is transitive, $P \leftrightarroweq\mathscr{B}\leftrightarroweq Q_1$ and $P' \leftrightarroweq\mathscr{B}\leftrightarroweq Q_2$.
2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, since $P \leftrightarroweq P^\dagger$, $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and there exists a transition $P^\dagger \xrightarrow{\mathrm{t}} P^\ddagger$ such that $P' \leftrightarroweq P^\ddagger$. Since $P^\dagger\ \mathscr{B}\ Q^\dagger$, there exists a path

$Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{t} Q_2^\dagger \Longrightarrow Q_3^\dagger \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1}^\dagger \xrightarrow{(t)} Q_{2r}^\dagger$ with $r > 0$, such that $Q_1^\dagger \stackrel{\tau}{\nrightarrow}$, $\forall i \in [1, r{-}1]$, $\theta_X(P^\dagger) \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2i}^\dagger) \wedge \mathcal{I}(Q_{2i+1}^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P^\ddagger) \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2r}^\dagger)$. Since $Q^\dagger \leftrightarrow Q$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ such that $Q_1 \stackrel{\tau}{\nrightarrow}$, $\forall i \in [1, r{-}1]$, $Q_{2i}^\dagger \leftrightarrow Q_{2i} \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $Q_{2r}^\dagger \leftrightarrow Q_{2r}$. Since $\leftrightarrow$ is transitive and a congruence for θ_X [11], $\forall i \in [1, r{-}1]$, $\theta_X(P) \mathrel{\mathscr{B}} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2r})$.

3. If $P \stackrel{\tau}{\nrightarrow}$ then, since $P \leftrightarrow P^\dagger$, $P^\dagger \stackrel{\tau}{\nrightarrow}$. Since $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$, there exists a path $Q^\dagger \Longrightarrow Q^\star \stackrel{\tau}{\nrightarrow}$. Since $Q^\dagger \leftrightarrow Q$, there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\nrightarrow}$ such that $Q^\star \leftrightarrow Q_0$. ◀

The following lemma was proven in [11, Appendix B]. It will be useful in the proof of Proposition 17.

▶ **Lemma 39.** *Let $P, Q \in \mathbb{P}$, $X, S, I \subseteq A$, $\mathcal{R} \subseteq A \times A$.*

- *If $P \stackrel{\tau}{\nrightarrow}$ and $\mathcal{I}(P) \cap X \subseteq S$ then $\theta_X(P \parallel_S Q) \leftrightarrow \theta_X(P \parallel_S \theta_{X\setminus(S\setminus\mathcal{I}(P))}(Q))$.*
- *$\theta_X(\tau_I(P)) \leftrightarrow \theta_X(\tau_I(\theta_{X\cup I}(P)))$.*
- *$\theta_X(\mathcal{R}(P)) \leftrightarrow \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P)))$.*

Proof of Proposition 17. Let $\mathscr{B}$ be the smallest relation satisfying, for all $P, Q \in \mathbb{P}$,

- if $P \leftrightarrow_{br} Q$ then $P \mathrel{\mathscr{B}} Q$
- if $P \mathrel{\mathscr{B}} Q$ and $\alpha \in Act$ then $\alpha.P \mathrel{\mathscr{B}} \alpha.Q$
- if $P_1 \mathrel{\mathscr{B}} Q_1$, $P_2 \mathrel{\mathscr{B}} Q_2$ and $S \subseteq A$ then $P_1 \parallel_S P_2 \mathrel{\mathscr{B}} Q_1 \parallel_S Q_2$
- if $P \mathrel{\mathscr{B}} Q$ and $I \subseteq A$ then $\tau_I(P) \mathrel{\mathscr{B}} \tau_I(Q)$
- if $P \mathrel{\mathscr{B}} Q$ and $\mathcal{R} \subseteq A \times A$ then $\mathcal{R}(P) \mathrel{\mathscr{B}} \mathcal{R}(Q)$
- if $P \mathrel{\mathscr{B}} Q$ and $L \subseteq U \subseteq A$ then $\theta_L^U(P) \mathrel{\mathscr{B}} \theta_L^U(Q)$.

We are going to show that $\mathscr{B}$ is a branching time-out bisimulation up to $\leftrightarrow$. This implies that $\mathscr{B} = \leftrightarrow_{br}$, using Proposition 38, and as $\mathscr{B}$ is a congruence for the operators of Proposition 17, so is $\leftrightarrow_{br}$. Before we do so, we show, by induction on the construction of $\mathscr{B}$, that

$$\text{if } P \mathrel{\mathscr{B}} Q \text{ and } P \stackrel{\tau}{\nrightarrow} \text{ then } Q \Longrightarrow Q' \text{ for some } Q' \text{ with } P \mathrel{\mathscr{B}} Q' \text{ and } \mathcal{I}(Q') = \mathcal{I}(P). \tag{1}$$

Let $P \mathrel{\mathscr{B}} Q$ and $P \stackrel{\tau}{\nrightarrow}$.

- If $P \leftrightarrow_{br} Q$ then, by Clause 3 of Definition 13, $Q \Longrightarrow Q'$ for some Q' with $Q' \stackrel{\tau}{\nrightarrow}$. By (the symmetric counterpart of) Clause 1, one obtains $P \mathrel{\mathscr{B}} Q'$. Clause 1 gives $\mathcal{I}(Q') = \mathcal{I}(P)$.
- If $P = \alpha.P^\dagger$ and $Q = \alpha.Q^\dagger$ with $\alpha \in Act$ then note that $\alpha \neq \tau$ and take $Q' := Q$. One has $\mathcal{I}(Q') = \mathcal{I}(P)$.
- If $P = P_1 \parallel_S P_1$ and $Q = Q_1 \parallel_S Q_2$ with $S \subseteq A$ and $P_i \mathrel{\mathscr{B}} Q_i$ for $i = 1, 2$, then, for $i = 1, 2$, $P_i \stackrel{\tau}{\nrightarrow}$, so by induction $Q_i \Longrightarrow Q_i'$ for some Q_i' with $P_i \mathrel{\mathscr{B}} Q_i'$ and $\mathcal{I}(Q_i') = \mathcal{I}(P_i)$. Now $Q \Longrightarrow Q_1' \|_S Q_2'$, $P \mathrel{\mathscr{B}} Q_1' \|_S Q_2'$ and $\mathcal{I}(Q_1' \|_S Q_2') = \mathcal{I}(P)$.
- If $P = \tau_I(P_1)$ and $Q = \tau_I(Q_1)$ with $I \subseteq A$ and $P_1 \mathrel{\mathscr{B}} Q_1$, then $P_1 \stackrel{\tau}{\nrightarrow}$, so by induction $Q_1 \Longrightarrow Q_1'$ for some Q_1' with $P_1 \mathrel{\mathscr{B}} Q_1'$ and $\mathcal{I}(Q_1') = \mathcal{I}(P_1)$. Now $Q \Longrightarrow \tau_I(Q_1')$, $P \mathrel{\mathscr{B}} \tau_I(Q_1')$ and $\mathcal{I}(\tau_I(Q_1')) = \mathcal{I}(P)$.
- If $P = \mathcal{R}(P_1)$ and $Q = \mathcal{R}(Q_1)$ with $\mathcal{R} \subseteq A \times A$ and $P_1 \mathrel{\mathscr{B}} Q_1$, then $P_1 \stackrel{\tau}{\nrightarrow}$, so by induction $Q_1 \Longrightarrow Q_1'$ for some Q_1' with $P_1 \mathrel{\mathscr{B}} Q_1'$ and $\mathcal{I}(Q_1') = \mathcal{I}(P_1)$. Now $Q \Longrightarrow \mathcal{R}(Q_1')$, $P \mathrel{\mathscr{B}} \mathcal{R}(Q_1')$ and $\mathcal{I}(\mathcal{R}(Q_1')) = \mathcal{I}(P)$.
- If $P = \theta_L^U(P_1)$ and $Q = \theta_L^U(Q_1)$ with $L \subseteq U \subseteq A$ and $P_1 \mathrel{\mathscr{B}} Q_1$, then $P_1 \stackrel{\tau}{\nrightarrow}$, so by induction $Q_1 \Longrightarrow Q_1'$ for some Q_1' with $P_1 \mathrel{\mathscr{B}} Q_1'$ and $\mathcal{I}(Q_1') = \mathcal{I}(P_1)$. Now $Q \Longrightarrow \theta_L^U(Q_1')$, $P \mathrel{\mathscr{B}} \theta_L^U(Q_1')$ and $\mathcal{I}(\theta_L^U(Q_1')) = \mathcal{I}(P)$.

We now check that $\mathscr{B}$ is a branching time-out bisimulation up to $\leftrightarrow$. Note that $\mathscr{B}$ is symmetric because $\leftrightarrow_{br}$ is. $\leftrightarrow$ was proven to be a congruence for CCSP_t^θ in [11]. Let $P, Q \in \mathbb{P}$ such that $P \mathrel{\mathscr{B}} Q$.

1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then we have to find a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $P \leftrightarrow\mathscr{B}\leftrightarrow Q_1$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow Q_2$. Remember that $\mathscr{B} \subseteq \leftrightarrow\mathscr{B}\leftrightarrow$. We are going to proceed by structural induction on P and by case distinction on the derivation of $P \mathrel{\mathscr{B}} Q$.
   - If $P \leftrightarrow_{br} Q$ then, by definition of $\leftrightarrow_{br}$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $P \leftrightarrow_{br} Q_1$ and $P' \leftrightarrow_{br} Q_2$, thus, by definition of $\mathscr{B}$, $P \mathrel{\mathscr{B}} Q_1$ and $P' \mathrel{\mathscr{B}} Q_2$.
   - If $P = \beta.P^\dagger$ and $Q = \beta.Q^\dagger$ with $\beta \in Act$ and $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$ then, by the semantics, $P' = P^\dagger$, $\beta = \alpha$, and thus there exists a path $Q \xrightarrow{\alpha} Q^\dagger$ such that $P \mathrel{\mathscr{B}} Q$ and $P' \mathrel{\mathscr{B}} Q^\dagger$.
   - If $P = P^\dagger \|_S P^\ddagger$ and $Q = Q^\dagger \|_S Q^\ddagger$ with $S \subseteq A$, $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$ and $P^\ddagger \mathrel{\mathscr{B}} Q^\ddagger$ then
     - if $\alpha \in S$ then, by the semantics, $P' = P'^\dagger \|_S P'^\ddagger$, $P^\dagger \xrightarrow{\alpha} P'^\dagger$ and $P^\ddagger \xrightarrow{\alpha} P'^\ddagger$. Note that $\alpha \neq \tau$ because $\alpha \in A$. Since $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$ and $P^\ddagger \mathrel{\mathscr{B}} Q^\ddagger$, by induction, there exist two paths $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{\alpha} Q_2^\dagger$ and $Q^\ddagger \Longrightarrow Q_1^\ddagger \xrightarrow{\alpha} Q_2^\ddagger$ such that $P^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger$, $P'^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger$, $P^\ddagger \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\ddagger$ and $P'^\ddagger \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\ddagger$. By the semantics, $Q \Longrightarrow Q_1^\dagger \|_S Q_1^\ddagger \xrightarrow{\alpha} Q_2^\dagger \|_S Q_2^\ddagger$. Moreover, by definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $P \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger \|_S Q_1^\ddagger$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger \|_S Q_2^\ddagger$.
     - if $\alpha \notin S$ then, by the semantics, two cases are possible. Suppose that $P' = P'^\dagger \|_S P^\ddagger$ and $P^\dagger \xrightarrow{\alpha} P'^\dagger$; the other case is symmetrical. Since $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$, by induction, there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{(\alpha)} Q_2^\dagger$ such that $P^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger$ and $P'^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger$. By the semantics, there exists a path $Q \Longrightarrow Q_1^\dagger \|_S Q^\ddagger \xrightarrow{(\alpha)} Q_2^\dagger \|_S Q^\ddagger$. Moreover, by definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $P \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger \|_S Q^\ddagger$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger \|_S Q^\ddagger$.
   - If $P = \tau_I(P^\dagger)$ and $Q = \tau_I(Q^\dagger)$ with $I \subseteq A$ and $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$ then, by the semantics, $P' = \tau_I(P'^\dagger)$, $P^\dagger \xrightarrow{\beta} P'^\dagger$ and $(\beta \in I \wedge \alpha = \tau) \vee \beta = \alpha$. Since $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$, by induction, there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{(\beta)} Q_2^\dagger$ such that $P^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger$ and $P'^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger$. By the semantics, $Q \Longrightarrow \tau_I(Q_1^\dagger) \xrightarrow{(\alpha)} \tau_I(Q_2^\dagger)$ such that, by definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $P \leftrightarrow\mathscr{B}\leftrightarrow \tau_I(Q_1^\dagger)$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow \tau_I(Q_2^\dagger)$.
   - If $P = \mathcal{R}(P^\dagger)$ and $Q = \mathcal{R}(Q^\dagger)$ with $\mathcal{R} \subseteq A \times A$ and $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$ then, by the semantics, $P' = \mathcal{R}(P'^\dagger)$, $P^\dagger \xrightarrow{\beta} P'^\dagger$ and $(\beta, \alpha) \in \mathcal{R} \vee \alpha = \beta = \tau$. Since $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$, by induction, there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{(\beta)} Q_2^\dagger$ such that $P^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger$ and $P'^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger$. By the semantics, $Q \Longrightarrow \mathcal{R}(Q_1^\dagger) \xrightarrow{(\alpha)} \mathcal{R}(Q_2^\dagger)$ such that, by definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $P \leftrightarrow\mathscr{B}\leftrightarrow \mathcal{R}(Q_1^\dagger)$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow \mathcal{R}(Q_2^\dagger)$.
   - If $P = \theta_L^U(P^\dagger)$ and $Q = \theta_L^U(Q^\dagger)$ with $L \subseteq U \subseteq A$ and $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$ then
     - if $\alpha = \tau$ then, by the semantics, $P' = \theta_X(P'^\dagger)$ and $P^\dagger \xrightarrow{\tau} P'^\dagger$. Since $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$, by induction, there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{(\tau)} Q_2^\dagger$ such that $P^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger$ and $P'^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger$. By the semantics, there exists a path $Q \Longrightarrow \theta_L^U(Q_1^\dagger) \xrightarrow{(\tau)} \theta_L^U(Q_2^\dagger)$ such that, by definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $P \leftrightarrow\mathscr{B}\leftrightarrow \theta_L^U(Q_1^\dagger)$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow \theta_L^U(Q_2^\dagger)$.
     - if $\alpha = a \in A$ then, by the semantics, $a \in U \vee \mathcal{I}(P^\dagger) \cap (L \cup \{\tau\}) = \emptyset$ and $P^\dagger \xrightarrow{a} P'$. Since $P^\dagger \mathrel{\mathscr{B}} Q^\dagger$, by induction there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{a} Q_2^\dagger$ such that $P^\dagger \leftrightarrow\mathscr{B}\leftrightarrow Q_1^\dagger$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger$. Moreover, in case $a \notin U$ we have $P^\dagger \stackrel{\tau}{\not\rightarrow}$ so (1) ensures that $Q^\dagger \Longrightarrow Q'$ for some Q' with $P^\dagger \mathrel{\mathscr{B}} Q'$ and $\mathcal{I}(Q') = \mathcal{I}(P)$. This implies that we may choose $Q_1^\dagger$ such that $Q' \Longrightarrow Q_1^\dagger$, and thus $Q' = Q_1^\dagger$. This gives us $\mathcal{I}(P^\dagger) \cap (L \cup \{\tau\}) = \emptyset \Leftrightarrow \mathcal{I}(Q_1^\dagger) \cap (L \cup \{\tau\}) = \emptyset$. By the semantics, there exists a path $Q \Longrightarrow \theta_L^U(Q_1^\dagger) \xrightarrow{a} Q_2^\dagger$ such that, by definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $P \leftrightarrow\mathscr{B}\leftrightarrow \theta_L^U(Q_1^\dagger)$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow Q_2^\dagger$.
2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then we have to find a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r-1]$, $\theta_X(P) \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2r})$. Remember that

$\mathscr{B} \subseteq \leftrightarrow \mathscr{B} \leftrightarrow$. We are going to proceed by by structural induction on P and by case distinction on the derivation of $P \mathscr{B} Q$.

- If $P \leftrightarrow_{br} Q$ then, by Definition 13, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\to}$, $\forall i \in [1, r-1]$, $\theta_X(P) \mathscr{B} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \mathscr{B} \theta_X(Q_{2r})$.
- If $P = \beta.P^\dagger$ and $Q = \beta.Q^\dagger$ with $\beta \in Act$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = P^\dagger$ and $\beta = t$. Thus, by the semantics, there exists a path $Q \xrightarrow{t} Q^\dagger$ such that $Q \stackrel{\tau}{\not\to}$ and, by definition of $\mathscr{B}$, $\theta_X(P') \mathscr{B} \theta_X(Q^\dagger)$.
- If $P = P^\dagger \|_S P^\ddagger$ and $Q = Q^\dagger \|_S Q^\dagger$ with $S \subseteq A$, $P^\dagger \mathscr{B} Q^\dagger$ and $Q^\dagger \mathscr{B} Q^\dagger$ then, since $t \notin S$, by the semantics, two cases are possible. Suppose that $P' = P^\dagger \|_S P'^\ddagger$ and $P^\ddagger \xrightarrow{t} P'^\ddagger$; the other case is symmetrical. Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $P^\dagger \stackrel{\tau}{\not\to}$ and $\mathcal{I}(P^\dagger) \cap X \subseteq S$. Moreover, $\mathcal{I}(P^\ddagger) \cap ((X \setminus S) \cup (X \cap S \cap \mathcal{I}(P^\dagger)) \cup \{\tau\}) = \emptyset$. Note that $(X \setminus S) \cup (X \cap S \cap \mathcal{I}(P^\dagger)) = X \setminus (S \setminus \mathcal{I}(P^\dagger))$. Since $P^\ddagger \mathscr{B} Q^\ddagger$, $P^\ddagger \xrightarrow{t} P'^\ddagger$ and $\mathcal{I}(P^\ddagger) \cap (X \setminus (S \setminus \mathcal{I}(P^\dagger)) \cup \{\tau\}) = \emptyset$, by induction, there exists a path $Q^\ddagger \Longrightarrow Q_1^\ddagger \xrightarrow{t} Q_2^\ddagger \Longrightarrow Q_3^\ddagger \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1}^\ddagger \xrightarrow{(t)} Q_{2r}^\ddagger$ with $r > 0$, such that $Q_1^\ddagger \stackrel{\tau}{\not\to}$, $\forall i \in [1, r-1]$, $\theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(P^\ddagger) \leftrightarrow \mathscr{B} \leftrightarrow \theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(Q_{2i}^\ddagger) \wedge \mathcal{I}(Q_{2i+1}^\ddagger) \cap (X \setminus (S \setminus \mathcal{I}(P^\dagger)) \cup \{\tau\}) = \emptyset$ and $\theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(P'^\ddagger) \leftrightarrow \mathscr{B} \leftrightarrow \theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(Q_{2r}^\ddagger)$. Moreover, by (1), since $P^\dagger \stackrel{\tau}{\not\to}$, there exists a path $Q^\dagger \Longrightarrow Q_0^\dagger \stackrel{\tau}{\not\to}$ such that $P^\dagger \mathscr{B} Q_0^\dagger$ and $\mathcal{I}(Q_0^\dagger) = \mathcal{I}(P^\dagger)$. By the semantics, there exists a path $Q \Longrightarrow Q_0^\dagger \|_S Q_1^\ddagger \xrightarrow{t} Q_0^\dagger \|_S Q_2^\ddagger \Longrightarrow Q_0^\dagger \|_S Q_3^\ddagger \xrightarrow{t} \ldots \Longrightarrow Q_0^\dagger \|_S Q_{2r-1}^\ddagger \xrightarrow{(t)} Q_0^\dagger \|_S Q_{2r}^\ddagger$. Since $Q_0^\dagger \stackrel{\tau}{\not\to}$ and $Q_1^\ddagger \stackrel{\tau}{\not\to}$ we have $Q_0^\dagger \|_S Q_1^\ddagger \stackrel{\tau}{\not\to}$. By Lemma 39, the definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $\forall i \in [1, r-1]$, $\theta_X(P) \leftrightarrow$

$$\theta_X(P^\dagger \|_S \theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(P^\ddagger)) \leftrightarrow \mathscr{B} \leftrightarrow \theta_X(Q_0^\dagger \|_S \theta_{X \setminus (S \setminus \mathcal{I}(Q_0^\dagger))}(Q_{2i}^\ddagger)) \leftrightarrow \theta_X(Q_0^\dagger \|_S Q_{2i}^\ddagger).$$

  Moreover, $\mathcal{I}(Q_0^\dagger \|_S Q_{2i+1}^\ddagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarrow$

$$\theta_X(P^\dagger \|_S \theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(P'^\ddagger)) \leftrightarrow \mathscr{B} \leftrightarrow \theta_X(Q_0^\dagger \|_S \theta_{X \setminus (S \setminus \mathcal{I}(Q_0^\dagger))}(Q_{2r}^\ddagger)) \leftrightarrow \theta_X(Q_0^\dagger \|_S Q_{2r}^\ddagger).$$

- If $P = \tau_I(P^\dagger)$ and $Q = \tau_I(Q^\dagger)$ with $I \subseteq A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = \tau_I(P'^\dagger)$, $P^\dagger \xrightarrow{t} P'^\dagger$ and $\mathcal{I}(P^\dagger) \cap ((X \cup I) \cup \{\tau\}) = \emptyset$. Since $P^\dagger \mathscr{B} Q^\dagger$ and $P \stackrel{\tau}{\not\to}$, also $P^\dagger \stackrel{\tau}{\not\to}$, so (1) ensures that $Q^\dagger \Longrightarrow Q_0^\dagger$ for some $Q_0^\dagger$ with $P^\dagger \mathscr{B} Q_0^\dagger$ and $\mathcal{I}(Q_0^\dagger) = \mathcal{I}(P^\dagger)$. Since $P^\dagger \mathscr{B} Q_0^\dagger$, by induction, there exists a path $Q_0^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{t} Q_2^\dagger \Longrightarrow Q_3^\dagger \xrightarrow{t} \ldots Q_{2r-1}^\dagger \xrightarrow{(t)} Q_{2r}^\dagger$ with $r > 0$, such that $\forall i \in [1, r-1]$, $\theta_{X \cup I}(P^\dagger) \leftrightarrow \mathscr{B} \leftrightarrow \theta_{X \cup I}(Q_{2i}^\dagger) \wedge \mathcal{I}(Q_{2i+1}^\dagger) \cap (X \cup I \cup \{\tau\}) = \emptyset$ and $\theta_{X \cup I}(P'^\dagger) \leftrightarrow \mathscr{B} \leftrightarrow \theta_{X \cup I}(Q_{2r}^\dagger)$. As $Q_0^\dagger \stackrel{\tau}{\not\to}$ we have $Q_0^\dagger = Q_1^\dagger$. By the semantics, $Q \Longrightarrow \tau_I(Q_1^\dagger) \xrightarrow{t} \tau_I(Q_2^\dagger) \Longrightarrow \tau_I(Q_3^\dagger) \xrightarrow{t} \ldots \Longrightarrow \tau_I(Q_{2r-1}^\dagger) \xrightarrow{(t)} \tau_I(Q_{2r}^\dagger)$. Since $\mathcal{I}(Q_1^\dagger) = \mathcal{I}(P^\dagger)$ and $\tau_I(P^\dagger) \stackrel{\tau}{\not\to}$, also $\tau_I(Q_1^\dagger) \stackrel{\tau}{\not\to}$. Lemma 39, the definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $\forall i \in [1, r-1]$,

$$\theta_X(P) \leftrightarrow \theta_X(\tau_I(\theta_{X \cup I}(P^\dagger))) \leftrightarrow \mathscr{B} \leftrightarrow \theta_X(\tau_I(\theta_{X \cup I}(Q_{2i}^\dagger))) \leftrightarrow \theta_X(\tau_I(Q_{2i}^\dagger))$$

  and $\mathcal{I}(\tau_I(Q_{2i+1}^\dagger)) \cap (X \cup \{\tau\}) = \emptyset$. Moreover,

$$\theta_X(P') \leftrightarrow \theta_X(\tau_I(\theta_{X \cup I}(P'^\dagger))) \leftrightarrow \mathscr{B} \leftrightarrow \theta_X(\tau_I(\theta_{X \cup I}(Q_{2r}^\dagger))) \leftrightarrow \theta_X(\tau_I(Q_{2r}^\dagger)).$$

- If $P = \mathcal{R}(P^\dagger)$ and $Q = \mathcal{R}(Q^\dagger)$ with $\mathcal{R} \subseteq A \times A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = \mathcal{R}(P'^\dagger)$, $P^\dagger \xrightarrow{t} P'^\dagger$ and $\mathcal{I}(P^\dagger) \cap (\mathcal{R}^{-1}(X) \cup \{\tau\}) = \emptyset$. Since $P^\dagger \mathscr{B} Q^\dagger$, by induction, there exists a path $Q^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{t} Q_2^\dagger \Longrightarrow Q_3^\dagger \xrightarrow{t} \ldots Q_{2r-1}^\dagger \xrightarrow{(t)} Q_{2r}^\dagger$ with $r > 0$, such that $Q_1^\dagger \stackrel{\tau}{\not\to}$, $\forall i \in [1, r-1]$, $\theta_{\mathcal{R}^{-1}(X)}(P^\dagger) \leftrightarrow \mathscr{B} \leftrightarrow \theta_{\mathcal{R}^{-1}(X)}(Q_{2i}^\dagger) \wedge \mathcal{I}(Q_{2i+1}^\dagger) \cap (\mathcal{R}^{-1}(X) \cup$

$\{\tau\}) = \emptyset$ and $\theta_{\mathcal{R}^{-1}(X)}(P'^{\dagger}) \leftrightarrow\mathscr{B}\leftrightarrow \theta_{\mathcal{R}^{-1}(X)}(Q_2^{\dagger})$. By the semantics, $Q \Longrightarrow \mathcal{R}(Q_1^{\dagger}) \xrightarrow{t} \mathcal{R}(Q_2^{\dagger}) \Longrightarrow \mathcal{R}(Q_3^{\dagger}) \xrightarrow{t} \dots \Longrightarrow \mathcal{R}(Q_{2r-1}^{\dagger}) \xrightarrow{(t)} \mathcal{R}(Q_{2r}^{\dagger})$ and $\mathcal{R}(Q_1^{\dagger}) \stackrel{\tau}{\not\rightarrow}$. By Lemma 39, the definition of $\mathscr{B}$ and the congruence property of $\leftrightarrow$, $\forall i \in [1, r-1]$,

$$\theta_X(P) \leftrightarrow \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P^{\dagger}))) \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}}(Q_{2i}^{\dagger}))) \leftrightarrow \theta_X(\tau_I(Q_{2i}^{\dagger}))$$

and $\mathcal{I}(\mathcal{R}(Q_{2i+1}^{\dagger})) \cap (X \cup \{\tau\}) = \emptyset$. Moreover,

$$\theta_X(P') \leftrightarrow \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P'^{\dagger}))) \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(Q_{2r}^{\dagger}))) \leftrightarrow \theta_X(\mathcal{R}(Q_{2r}^{\dagger})).$$

- If $P = \theta_L^U(P^{\dagger})$ and $Q = \theta_L^U(Q^{\dagger})$ with $L \subseteq U \subseteq A$ and $P^{\dagger} \mathscr{B} Q^{\dagger}$ then, by the semantics, $\mathcal{I}(P^{\dagger}) \cap (L \cup X \cup \{\tau\}) = \emptyset$ and $P^{\dagger} \xrightarrow{t} P'$. Since $P^{\dagger} \mathscr{B} Q^{\dagger}$ and $P \stackrel{\tau}{\not\rightarrow}$, also $P^{\dagger} \stackrel{\tau}{\not\rightarrow}$, so (1) ensures that $Q^{\dagger} \Longrightarrow Q_0^{\dagger}$ for some $Q_0^{\dagger}$ with $P^{\dagger} \mathscr{B} Q_0^{\dagger}$ and $\mathcal{I}(Q_0^{\dagger}) = \mathcal{I}(P^{\dagger})$. Since $P^{\dagger} \mathscr{B} Q_0^{\dagger}$, by induction, there exists a path $Q_0^{\dagger} \Longrightarrow Q_1^{\dagger} \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $Q_1^{\dagger} \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r-1]$, $\theta_X(P^{\dagger}) \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2r})$. As $Q_0^{\dagger} \stackrel{\tau}{\not\rightarrow}$ we have $Q_0^{\dagger} = Q_1^{\dagger}$. As $\mathcal{I}(Q_1^{\dagger}) = \mathcal{I}(P^{\dagger})$ one has $\mathcal{I}(Q_1^{\dagger}) \cap (L \cup \{\tau\}) = \emptyset$. By the semantics, there exists a path $Q \Longrightarrow \theta_L^U(Q_1^{\dagger}) =: Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \dots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$. Since $Q_1^{\dagger} \stackrel{\tau}{\not\rightarrow}$ we have $Q_1 \stackrel{\tau}{\not\rightarrow}$. Moreover, note that $P \leftrightarrow P^{\dagger}$ because $\mathcal{I}(P^{\dagger}) \cap (L \cup \{\tau\}) = \emptyset$. By the congruence property of $\leftrightarrow$, $\forall i \in [1, r-1]$,

$$\theta_X(P) \leftrightarrow \theta_X(P^{\dagger}) \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2i})$$

  and $\mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$. Moreover, $\theta_X(P') \leftrightarrow\mathscr{B}\leftrightarrow \theta_X(Q_{2r})$.

3. The last condition of Definition 37 is implied by (1). ◀

I Full Congruence Proofs for $\leftrightarrow_{br}^r$ and $\leftrightarrow_{tb}^r$

▶ **Definition 40.** Here, a *rooted branching time-out bisimulation up to* $\leftrightarrow_{br}$ is a symmetric relation $\mathscr{B} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$ with $P \mathscr{B} Q$,

1. if $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then there is a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarrow\mathscr{B}\leftrightarrow_{br} Q'$
2. if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then there is a transition $Q \xrightarrow{t} Q'$ such that $\theta_X(P') \leftrightarrow\mathscr{B}\leftrightarrow_{br} \theta_X(Q')$.

▶ **Proposition 41.** *Let $P, Q \in \mathbb{P}$. Then $P \leftrightarrow_{br}^r Q$ iff there exists a rooted branching time-out bisimulation $\mathscr{B}$ up to $\leftrightarrow_{br}$ such that $P \mathscr{B} Q$.*

Proof. First of all, a rooted branching time-out bisimulation is a rooted branching time-out bisimulation up to $\leftrightarrow_{br}$ by reflexivity of $\leftrightarrow$ and $\leftrightarrow_{br}$. Conversely, we are going to show that $\leftrightarrow\mathscr{B}\leftrightarrow_{br}$ is a branching time-out bisimulation. This implies that $\leftrightarrow\mathscr{B}\leftrightarrow_{br} \subseteq \leftrightarrow_{br}$, so that each rooted branching time-out bisimulation up to $\leftrightarrow_{br}$ is in fact a rooted branching time-out bisimulation. Let $P, Q \in \mathbb{P}$ such that $P \leftrightarrow\mathscr{B}\leftrightarrow_{br} Q$. There exists $P^{\dagger}, Q^{\dagger} \in \mathbb{P}$ such that $P \leftrightarrow P^{\dagger} \mathscr{B} Q^{\dagger} \leftrightarrow_{br} Q$.

1. If $P \xrightarrow{\alpha} P'$ then, since $P \leftrightarrow P^{\dagger}$, there is a transition $P^{\dagger} \xrightarrow{\alpha} P^{\ddagger}$ such that $P' \leftrightarrow P^{\ddagger}$. Thus, by Clause 1 of Definition 40, there is a transition $Q^{\dagger} \xrightarrow{\alpha} Q^{\ddagger}$ such that $P^{\ddagger} \leftrightarrow\mathscr{B}\leftrightarrow_{br} Q^{\ddagger}$. By Clause 1 of Definition 13 there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ with $Q^{\dagger} \leftrightarrow_{br} Q_1$ and $Q^{\ddagger} \leftrightarrow_{br} Q_2$. By the transitivity of $\leftrightarrow$ and $\leftrightarrow_{br}$ we obtain $P \leftrightarrow\mathscr{B}\leftrightarrow_{br} Q_1$ and $P' \leftrightarrow\mathscr{B}\leftrightarrow_{br} Q_2$.
2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$ then, since $P \leftrightarrow P^{\dagger}$, $\mathcal{I}(P^{\dagger}) \cap (X \cup \{\tau\}) = \emptyset$ and $P^{\dagger} \xrightarrow{t} P^{\ddagger}$ for some $P^{\ddagger}$ with $P' \leftrightarrow P^{\ddagger}$. Thus, by Clauses 1 and 2 of Definition 40

$\mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and there is a transition $Q^\dagger \xrightarrow{t} Q^\ddagger$ with $\theta_X(P^\ddagger) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q^\ddagger)$. By Clause 2 of Definition 13 there is a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $Q_1 \not\xrightarrow{\tau}$, $\forall i \in [1, r-1]$, $\theta_X(Q^\dagger) \leftrightarrow_{br} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(Q^\ddagger) \leftrightarrow_{br} \theta_X(Q_{2r})$. Since $\mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ we have $Q^\dagger \leftrightarrow \theta_X(Q^\dagger)$ and hence $Q^\dagger \leftrightarrow_{br} \theta_X(Q^\dagger)$. Thus $\theta_X(P) \leftrightarrow P \leftrightarrow P^\dagger \mathscr{B} Q^\dagger \leftrightarrow_{br} \theta_X(Q^\dagger) \leftrightarrow_{br} \theta_X(Q_{2i})$. Since $\leftrightarrow$ is a congruence for θ_X [11, Theorem 20], we have $\theta_X(P') \leftrightarrow \theta_X(P^\ddagger)$. By the transitivity of $\leftrightarrow$ and $\leftrightarrow_{br}$ we obtain $\theta_X(P) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q_{2i})$ and $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q_{2r})$.

3. If $P \not\xrightarrow{\tau}$ then, since $P \leftrightarrow P^\dagger$, $P^\dagger \not\xrightarrow{\tau}$, so by Clause 1 of Definition 40 $Q^\dagger \not\xrightarrow{\tau}$, and by Clause 3 of Definition 13 there is a path $Q \Longrightarrow Q_0 \not\xrightarrow{\tau}$. ◀

Proof of Theorem 18. Let $\mathscr{B} \subseteq \mathbb{P} \times \mathbb{P}$ be the smallest relation such that

- if $P \leftrightarrow^r_{br} Q$ then $P \mathscr{B} Q$
- if $P \mathscr{B} Q$ and $\alpha \in Act$ then $\alpha.P \mathscr{B} \alpha.Q$
- if $P_1 \mathscr{B} Q_1$ and $P_2 \mathscr{B} Q_2$ then $P_1 + P_2 \mathscr{B} Q_1 + Q_2$
- if $P_1 \mathscr{B} Q_1$, $P_2 \mathscr{B} Q_2$ and $S \subseteq A$ then $P_1 \|_S P_2 \mathscr{B} Q_1 \|_S Q_2$
- if $P \mathscr{B} Q$ and $I \subseteq A$ then $\tau_I(P) \mathscr{B} \tau_I(Q)$
- if $P \mathscr{B} Q$ and $\mathcal{R} \subseteq A \times A$ then $\mathcal{R}(P) \mathscr{B} \mathcal{R}(Q)$
- if $P \mathscr{B} Q$ and $L \subseteq U \subseteq A$ then $\theta^U_L(P) \mathscr{B} \theta^U_L(Q)$
- if $P \mathscr{B} Q$ and $X \subseteq A$ then $\psi_X(P) \mathscr{B} \psi_X(Q)$
- if $\mathcal{S}$ is a recursive specification with $z \in V_\mathcal{S}$ and $\rho, \nu \in V \setminus V_\mathcal{S} \to \mathbb{P}$ are substitutions such that $\forall x \in V \setminus V_\mathcal{S},\ \rho(x) \mathscr{B} \nu(x)$, then $\langle z|\mathcal{S}\rangle[\rho] \mathscr{B} \langle z|\mathcal{S}\rangle[\nu]$.
- if $\mathcal{S}$ and $\mathcal{S}'$ are recursive specifications and $x \in V_\mathcal{S} = V_{\mathcal{S}'}$ with $\langle x|\mathcal{S}\rangle, \langle x|\mathcal{S}'\rangle \in \mathbb{P}$ such that $\forall y \in V_\mathcal{S},\ \mathcal{S}_y \leftrightarrow^r_{br} \mathcal{S}'_y$, then $\langle x|\mathcal{S}\rangle \mathscr{B} \langle x|\mathcal{S}'\rangle$.

Note that since $\leftrightarrow$, $\mathscr{B}$ and $\leftrightarrow_{tb}$ are congruences for the operators listed in Proposition 17, so are the composed relations $\mathscr{B} \leftrightarrow_{tb}$ and $\leftrightarrow \mathscr{B} \leftrightarrow_{tb}$. (£)

Let $=_\mathcal{I} := \{(P, Q) \mid \mathcal{I}(P) = \mathcal{I}(Q)\}$. A trivial induction on the derivation of $P \mathscr{B} Q$, using the fact that $=_\mathcal{I}$ is a full congruence for CCSP^θ_t [11], shows that

$$P \mathscr{B} Q \Rightarrow \mathcal{I}(P) = \mathcal{I}(Q) \tag{@}$$

(For the second last case, the assumption that $\rho(x) \mathscr{B} \nu(x)$ for all $x \in V \setminus V_\mathcal{S}$ implies $\rho =_\mathcal{I} \nu$ by induction. Since $=_\mathcal{I}$ is a lean congruence, this implies $\langle z|\mathcal{S}\rangle[\rho] =_\mathcal{I} \langle z|\mathcal{S}\rangle[\nu]$.)
A trivial induction on $\mathbb{E}$ shows that

$$\forall E \in \mathbb{E}, \rho, \nu \in V \to \mathbb{P},\ (\forall x \in V, \rho(x) \mathscr{B} \nu(x)) \Rightarrow E[\rho] \mathscr{B} E[\nu] \tag{$\star$}$$

A useful corollary is

$$\begin{aligned} &\forall E \in \mathbb{E}, \mathcal{S} \text{ a recursive specification}, \rho, \nu \in V \setminus V_\mathcal{S} \to \mathbb{P}, \\ &\quad (\forall x \in V \setminus V_\mathcal{S},\ \rho(x) \mathscr{B} \nu(x)) \Rightarrow \langle E|\mathcal{S}\rangle[\rho] \mathscr{B} \langle E|\mathcal{S}\rangle[\nu] \end{aligned} \tag{\$}$$

Applied in the context of the last condition of $\mathscr{B}$, it implies

$$\forall E \in \mathbb{E}, \text{the variables of } E \text{ are in } V_\mathcal{S} \Rightarrow \langle E|\mathcal{S}\rangle \mathscr{B} \langle E|\mathcal{S}'\rangle \tag{\#}$$

Since $\leftrightarrow^r_{br} \subseteq \mathscr{B}$, it suffices to prove that $\mathscr{B}$ is a rooted branching time-out bisimulation up to $\leftrightarrow_{br}$ (so that $\mathscr{B} = \leftrightarrow^r_{br}$). Note that $\mathscr{B}$ is symmetric, since $\leftrightarrow^r_{br}$ is. Let $P, Q \in \mathbb{P}$ such that $P \mathscr{B} Q$.

1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then we need to find a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \mathscr{B} \leftrightarrow_{br} Q'$. This is sufficient as $\mathscr{B} \leftrightarrow_{br} \subseteq \leftrightarrow \mathscr{B} \leftrightarrow_{br}$. We are going to proceed by induction on the proof of $P \xrightarrow{\alpha} P'$ and by case distinction on the derivation of $P \mathscr{B} Q$.

- If $P \leftrightarroweq^r_{br} Q$ then there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarroweq_{br} Q'$, and so $P' \mathscr{B}\leftrightarroweq_{br} Q'$.
- If $P = \beta.P^\dagger$ and $Q = \beta.Q^\dagger$ such that $\beta \in Act$ and $P^\dagger \mathscr{B} Q^\dagger$ then $\alpha = \beta$ and $P' = P^\dagger$. Thus there exists a transition $Q \xrightarrow{\alpha} Q^\dagger$ such that $P^\dagger \mathscr{B} Q^\dagger$, and so $P^\dagger \mathscr{B}\leftrightarroweq_{br} Q^\dagger$.
- If $P = P^\dagger + P^\ddagger$ and $Q = Q^\dagger + Q^\ddagger$ such that $P^\dagger \mathscr{B} Q^\dagger$ and $P^\ddagger \mathscr{B} Q^\ddagger$ then, by the semantics, $P^\dagger \xrightarrow{\alpha} P'$ or $P^\ddagger \xrightarrow{\alpha} P'$. Suppose that $P^\dagger \xrightarrow{\alpha} P'$ (the other case proceeds symmetrically). Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\alpha} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$.
- If $P = P^\dagger \parallel_S P^\ddagger$ and $Q = Q^\dagger \parallel_S Q^\ddagger$ such that $S \subseteq A$, $P^\dagger \mathscr{B} Q^\dagger$ and $P^\ddagger \mathscr{B} Q^\ddagger$ then
  - if $\alpha \notin S$ then, by the semantics, $P' = P'^\dagger \parallel_S P^\ddagger$ and $P^\dagger \xrightarrow{\alpha} P'^\dagger$ or $P' = P^\dagger \parallel_S P'^\ddagger$ and $P^\ddagger \xrightarrow{\alpha} P'^\ddagger$. Suppose that $P^\dagger \xrightarrow{\alpha} P'^\dagger$ (the other case proceeds symmetrically). Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\alpha} Q'^\dagger$ such that $P'^\dagger \mathscr{B}\leftrightarroweq_{br} Q'^\dagger$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} Q'^\dagger \parallel_S Q^\ddagger$ such that, by (£), $P' \mathscr{B}\leftrightarroweq_{br} Q'^\dagger \parallel_S Q^\ddagger$.
  - if $\alpha \in S$ then, by the semantics, $P' = P'^\dagger \parallel_S P'^\ddagger$, $P^\dagger \xrightarrow{\alpha} P'^\dagger$ and $P^\ddagger \xrightarrow{\alpha} P'^\ddagger$. Since $P^\dagger \mathscr{B} Q^\dagger$ and $P^\ddagger \mathscr{B} Q^\ddagger$, there exists two transitions $Q^\dagger \xrightarrow{\alpha} Q'^\dagger$ and $Q^\ddagger \xrightarrow{\alpha} Q'^\ddagger$ such that $P'^\dagger \mathscr{B}\leftrightarroweq_{br} Q'^\dagger$ and $P'^\ddagger \mathscr{B}\leftrightarroweq_{br} Q'^\ddagger$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} Q'^\dagger \parallel_S Q'^\ddagger$ such that, by (£), $P' \mathscr{B}\leftrightarroweq_{br} Q'^\dagger \parallel_S Q'^\ddagger$.
- If $P = \tau_I(P^\dagger)$ and $Q = \tau_I(Q^\dagger)$ with $I \subseteq A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = \tau_I(P'^\dagger)$, $P^\dagger \xrightarrow{\beta} P'^\dagger$ and $(\beta \in I \cup \{\tau\} \wedge \alpha = \tau) \vee \beta = \alpha \notin I$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\beta} Q'^\dagger$ such that $P'^\dagger \mathscr{B}\leftrightarroweq_{br} Q'^\dagger$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} \tau_I(Q'^\dagger)$ such that, by (£), $P' \mathscr{B}\leftrightarroweq_{br} \tau_I(Q'^\dagger)$.
- If $P = \mathcal{R}(P^\dagger)$ and $Q = \mathcal{R}(Q^\dagger)$ with $\mathcal{R} \subseteq A \times A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = \mathcal{R}(P'^\dagger)$, $P^\dagger \xrightarrow{\beta} P'^\dagger$ and $(\beta, \alpha) \in \mathcal{R} \vee \beta = \alpha = \tau$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\beta} Q'^\dagger$ such that $P'^\dagger \mathscr{B}\leftrightarroweq_{br} Q'^\dagger$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} \mathcal{R}(Q'^\dagger)$ such that, by (£), $P' \mathscr{B}\leftrightarroweq_{br} \mathcal{R}(Q'^\dagger)$.
- If $P = \theta^U_L(P^\dagger)$ and $Q = \theta^U_L(Q^\dagger)$ with $L \subseteq U \subseteq A$ and $P^\dagger \mathscr{B} Q^\dagger$ then
  - if $\alpha = \tau$ then, by the semantics, $P' = \theta_X(P'^\dagger)$ and $P^\dagger \xrightarrow{\tau} P'^\dagger$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\tau} Q'^\dagger$ such that $P'^\dagger \mathscr{B}\leftrightarroweq_{br} Q'^\dagger$. By the semantics, there exists a transition $Q \xrightarrow{\tau} \theta^U_L(Q'^\dagger)$ such that, by (£), $P' \mathscr{B}\leftrightarroweq_{br} \theta^U_L(Q'^\dagger)$.
  - if $\alpha = a \in A$ then, by the semantics, $P^\dagger \xrightarrow{a} P'$ and $a \in U \vee \mathcal{I}(P^\dagger) \cap (L \cup \{\tau\}) = \emptyset$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{a} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$. According to (@), $\mathcal{I}(P^\dagger) \cap (L \cup \{\tau\}) = \emptyset \Leftrightarrow \mathcal{I}(Q^\dagger) \cap (L \cup \{\tau\}) = \emptyset$, thus, by the semantics, there exists a transition $Q \xrightarrow{a} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$.
- If $P = \psi_X(P^\dagger)$ and $Q = \psi_X(Q^\dagger)$ with $X \subseteq A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P^\dagger \xrightarrow{\alpha} P'$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\alpha} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$.
- Let $P = \langle z|\mathcal{S}\rangle[\rho]$ and $Q = \langle z|\mathcal{S}\rangle[\nu]$ with $\mathcal{S}$ a recursive specification, $z \in V_\mathcal{S}$ and $\rho, \nu \in V \setminus V_\mathcal{S} \to \mathbb{P}$ such that $\forall x \in V \setminus V_\mathcal{S},\ \rho(x) \mathscr{B} \nu(x)$. By the semantics, $\langle \mathcal{S}_z|\mathcal{S}\rangle[\rho] \xrightarrow{\alpha} P'$ is provable by a strict sub-proof of $P \xrightarrow{\alpha} P'$. Moreover, according to (\$), $\langle \mathcal{S}_z|\mathcal{S}\rangle[\rho] \mathscr{B} \langle \mathcal{S}_z|\mathcal{S}\rangle[\nu]$. By induction, there exists a transition $\langle \mathcal{S}_z|\mathcal{S}\rangle[\nu] \xrightarrow{\alpha} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$. By the semantics, there exists a transition $\langle z|\mathcal{S}\rangle[\nu] \xrightarrow{\alpha} Q'$ such that $P' \mathscr{B}\leftrightarroweq_{br} Q'$.
- Let $P = \langle x|\mathcal{S}\rangle$ and $Q = \langle x|\mathcal{S}'\rangle$ with $\mathcal{S}$ and $\mathcal{S}'$ two recursive specifications such that $\forall y \in V_\mathcal{S} = V_{\mathcal{S}'},\ \mathcal{S}_y \leftrightarroweq^r_{br} \mathcal{S}'_y$ and $x \in V_\mathcal{S}$. By the semantics, $\langle \mathcal{S}_x|\mathcal{S}\rangle \xrightarrow{\alpha} P'$ is provable by a strict sub-proof of $P \xrightarrow{\alpha} P'$. Moreover, according (#), $\langle \mathcal{S}_x|\mathcal{S}\rangle \mathscr{B} \langle \mathcal{S}_x|\mathcal{S}'\rangle$. By induction, there exists a transition $\langle \mathcal{S}_x|\mathcal{S}'\rangle \xrightarrow{\alpha} R'$ such that $P' \mathscr{B}\leftrightarroweq_{br} R'$. Since $\langle \_|\mathcal{S}'\rangle \in V_{\mathcal{S}'} \to \mathbb{P}$

and $\mathcal{S}_x \leftrightarrow^r_{br} \mathcal{S}'_x$, $\langle \mathcal{S}_x | \mathcal{S}' \rangle \leftrightarrow^r_{br} \langle \mathcal{S}'_x | \mathcal{S}' \rangle$. Therefore, there exists a transition $\langle \mathcal{S}'_x | \mathcal{S}' \rangle \xrightarrow{\alpha} Q'$ such that $R' \leftrightarrow_{br} Q'$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that, by transitivity of $\leftrightarrow_{br}$, $P' \mathscr{B} \leftrightarrow_{br} Q'$.

2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then we need to find a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$. We are going to proceed by induction on the proof of $P \xrightarrow{\mathrm{t}} P'$ and by case distinction on the derivation of $P \mathscr{B} Q$.

- If $P \leftrightarrow^r_{br} Q$ then there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow_{br} \theta_X(Q')$ and so $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$.
- If $P = \beta.P^\dagger$ and $Q = \beta.Q^\dagger$ such that $\beta \in \mathit{Act}$ and $P^\dagger \mathscr{B} Q^\dagger$ then $\beta = \mathrm{t}$ and $P' = P^\dagger$. Thus there is a transition $Q \xrightarrow{t} Q^\dagger$ such that, by definition of $\mathscr{B}$, $\theta_X(P^\dagger) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q^\dagger)$.
- If $P = P^\dagger + P^\ddagger$ and $Q = Q^\dagger + Q^\ddagger$ such that $P^\dagger \mathscr{B} Q^\dagger$ and $P^\ddagger \mathscr{B} Q^\ddagger$ then, by the semantics, $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$, $\mathcal{I}(P^\ddagger) \cap (X \cup \{\tau\}) = \emptyset$ and $P^\dagger \xrightarrow{\mathrm{t}} P'$ or $P^\ddagger \xrightarrow{\mathrm{t}} P'$. Suppose that $P^\dagger \xrightarrow{\mathrm{t}} P'$ (the other case is symmetrical). Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$. By the semantics, there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$.
- If $P = P^\dagger \|_S P^\ddagger$ and $Q = Q^\dagger \|_S Q^\ddagger$ such that $S \subseteq A$, $P^\dagger \mathscr{B} Q^\dagger$ and $P^\ddagger \mathscr{B} Q^\ddagger$ then, by the semantics, $P' = P'^\dagger \|_S P^\ddagger$ and $P^\dagger \xrightarrow{\mathrm{t}} P'^\dagger$ or $P' = P^\dagger \|_S P'^\ddagger$ and $P^\ddagger \xrightarrow{\mathrm{t}} P'^\ddagger$. Suppose that $P^\ddagger \xrightarrow{\mathrm{t}} P'^\ddagger$ (the other case is symmetrical). Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $P^\dagger \stackrel{\tau}{\not\rightarrow}$ and $\mathcal{I}(P^\dagger) \cap X \subseteq S$. Moreover, $\mathcal{I}(P^\ddagger) \cap (X \setminus S \cup (X \cap S \cap \mathcal{I}(P^\dagger)) \cup \{\tau\}) = \emptyset$. Note that $X \setminus S \cup (X \cap S \cap \mathcal{I}(P^\dagger)) = X \setminus (S \setminus \mathcal{I}(P^\dagger))$. Since $P^\ddagger \mathscr{B} Q^\ddagger$, there exists a transition $Q^\ddagger \xrightarrow{\mathrm{t}} Q'^\ddagger$ such that $\theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(P'^\ddagger) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(Q'^\ddagger)$. Since $P^\dagger \mathscr{B} Q^\dagger$ and $P^\ddagger \mathscr{B} Q^\ddagger$, $\mathcal{I}(P^\dagger) = \mathcal{I}(Q^\dagger)$ and $\mathcal{I}(P^\ddagger) = \mathcal{I}(Q^\ddagger)$. By the semantics, there exists a transition $Q \xrightarrow{\mathrm{t}} Q^\dagger \|_S Q'^\ddagger$. By (£) and Lemma 39, $\theta_X(P') = \theta_X(P^\dagger \|_S P'^\ddagger) \leftrightarrow \theta_X(P^\dagger \|_S \theta_{X \setminus (S \setminus \mathcal{I}(P^\dagger))}(P'^\ddagger)) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q^\dagger \|_S \theta_{X \setminus (S \setminus \mathcal{I}(Q^\dagger))}(Q'^\ddagger)) \leftrightarrow \theta_X(Q^\dagger \|_S Q'^\ddagger)$. In the last step we use that $Q^\dagger \stackrel{\tau}{\not\rightarrow}$, since $P^\dagger \stackrel{\tau}{\not\rightarrow}$ and $\mathcal{I}(P) = \mathcal{I}(Q)$, using (@). Now apply that $\leftrightarrow \subseteq \leftrightarrow_{br}$ and the transitivity of $\leftrightarrow$ and $\leftrightarrow_{br}$.
- If $P = \tau_I(P^\dagger)$ and $Q = \tau_I(Q^\dagger)$ with $I \subseteq A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = \tau_I(P'^\dagger)$ and $P^\dagger \xrightarrow{\mathrm{t}} P'^\dagger$. Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $\mathcal{I}(P^\dagger) \cap (X \cup I \cup \{\tau\}) = \emptyset$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\mathrm{t}} Q'^\dagger$ such that $\theta_{X \cup I}(P'^\dagger) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_{X \cup I}(Q'^\dagger)$. By the semantics, there exists a transition $Q \xrightarrow{\mathrm{t}} \tau_I(Q'^\dagger)$ such that, by (£) and Lemma 39, $\theta_X(P') \leftrightarrow \theta_X(\tau_I(\theta_{X \cup I}(P'^\dagger))) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(\tau_I(\theta_{X \cup I}(Q'^\dagger))) \leftrightarrow \tau_I(Q'^\dagger)$.
- If $P = \mathcal{R}(P^\dagger)$ and $Q = \mathcal{R}(Q^\dagger)$ with $\mathcal{R} \subseteq A \times A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = \mathcal{R}(P'^\dagger)$ and $P^\dagger \xrightarrow{\mathrm{t}} P'^\dagger$. Since $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $\mathcal{I}(P^\dagger) \cap (\mathcal{R}^{-1}(X) \cup \{\tau\}) = \emptyset$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\mathrm{t}} Q'^\dagger$ such that $\theta_{\mathcal{R}^{-1}(X)}(P'^\dagger) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_{\mathcal{R}^{-1}(X)}(Q'^\dagger)$. By the semantics, there exists a transition $Q \xrightarrow{\mathrm{t}} \mathcal{R}(Q'^\dagger)$ such that, by (£) and Lemma 39, $\theta_X(P') \leftrightarrow \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(P'^\dagger))) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(\mathcal{R}(\theta_{\mathcal{R}^{-1}(X)}(Q'^\dagger))) \leftrightarrow \mathcal{R}(Q'^\dagger)$.
- If $P = \theta_L^U(P^\dagger)$ and $Q = \theta_L^U(Q^\dagger)$ with $L \subseteq U \subseteq A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P^\dagger \xrightarrow{\mathrm{t}} P'$ and $\mathcal{I}(P^\dagger) \cap (L \cup \{\tau\}) = \emptyset$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$. According to (@), $\mathcal{I}(P^\dagger) \cap (L \cup \{\tau\}) = \emptyset \Leftrightarrow \mathcal{I}(Q^\dagger) \cap (L \cup \{\tau\}) = \emptyset$, thus, by the semantics, there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$.
- If $P = \psi_Y(P^\dagger)$ and $Q = \psi_Y(Q^\dagger)$ with $Y \subseteq A$ and $P^\dagger \mathscr{B} Q^\dagger$ then, by the semantics, $P' = \theta_Y(P'^\dagger)$, $P^\dagger \xrightarrow{\mathrm{t}} P'^\dagger$ and $\mathcal{I}(P^\dagger) \cap (Y \cup \{\tau\}) = \emptyset$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a transition $Q^\dagger \xrightarrow{\mathrm{t}} Q'^\dagger$ such that $\theta_Y(P'^\dagger) \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_Y(Q'^\dagger)$. Using (@), $\mathcal{I}(Q^\dagger) \cap$

$(Y \cup \{\tau\}) = \emptyset$, so by the semantics, there exists a transition $Q \xrightarrow{t} \theta_Y(Q'^{\dagger})$. By (£), $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(\theta_Y(Q'^{\dagger}))$.

- Let $P = \langle z|\mathcal{S}\rangle[\rho]$ and $Q = \langle z|\mathcal{S}\rangle[\nu]$ with $\mathcal{S}$ a recursive specification, $z \in V_{\mathcal{S}}$ and $\rho, \nu \in V \backslash V_{\mathcal{S}} \to \mathbb{P}$ such that $\forall x \in V \backslash V_{\mathcal{S}}, \rho(x) \ \mathscr{B}\ \nu(x)$. By the semantics, $\langle \mathcal{S}_z|\mathcal{S}\rangle[\rho] \xrightarrow{t} P'$ is provable by a strict sub-proof of $P \xrightarrow{t} P'$ and $\mathcal{I}(P) = \mathcal{I}(\langle \mathcal{S}_z|\mathcal{S}\rangle[\rho])$. Moreover, according to (\$), $\langle \mathcal{S}_z|\mathcal{S}\rangle[\rho] \ \mathscr{B}\ \langle \mathcal{S}_z|\mathcal{S}\rangle[\nu]$. By induction, there exists a transition $\langle \mathcal{S}_z|\mathcal{S}\rangle[\nu] \xrightarrow{t} Q'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$. By the semantics, there exists a transition $\langle z|\mathcal{S}\rangle[\nu] \xrightarrow{t} Q'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$.
- Let $P = \langle x|\mathcal{S}\rangle$ and $Q = \langle x|\mathcal{S}'\rangle$ with $\mathcal{S}$ and $\mathcal{S}'$ two recursive specifications such that $\forall y \in V_{\mathcal{S}} = V_{\mathcal{S}'}, \mathcal{S}_y \leftrightarrow^r_{br} \mathcal{S}'_y$ and $x \in V_{\mathcal{S}}$. By the semantics, $\langle \mathcal{S}_x|\mathcal{S}\rangle \xrightarrow{t} P'$ is provable be a strict sub-proof of $P \xrightarrow{\alpha} P'$ and $\mathcal{I}(P) = \mathcal{I}(\langle \mathcal{S}_x|\mathcal{S}\rangle)$. Moreover, according (#), $\langle \mathcal{S}_x|\mathcal{S}\rangle \ \mathscr{B}\ \langle \mathcal{S}_x|\mathcal{S}'\rangle$. By induction, there exists a transition $\langle \mathcal{S}_x|\mathcal{S}'\rangle \xrightarrow{t} R'$ such that $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(R')$. Since $\langle \_|\mathcal{S}'\rangle \in V_{\mathcal{S}'} \to \mathbb{P}$ and $\mathcal{S}_x \leftrightarrow^r_{br} \mathcal{S}'_x$, $\langle \mathcal{S}_x|\mathcal{S}'\rangle \leftrightarrow^r_{br} \langle \mathcal{S}'_x|\mathcal{S}'\rangle$. Moreover, according to (@), $\mathcal{I}(\langle \mathcal{S}_x|\mathcal{S}'\rangle) \cap (X \cup \{\tau\}) = \emptyset$. Therefore, there exists a transition $\langle \mathcal{S}'_x|\mathcal{S}'\rangle \xrightarrow{\alpha} Q'$ such that $\theta_X(R') \leftrightarrow_{br} \theta_X(Q')$. By the semantics, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that, by transitivity of $\leftrightarrow_{br}$, $\theta_X(P') \leftrightarrow \mathscr{B} \leftrightarrow_{br} \theta_X(Q')$.

As a result, $\mathcal{B}$ is a rooted branching time-out bisimulation up to $\leftrightarrow_{br}$, and $(\star)$ gives us that $\leftrightarrow^r_{br}$ is a lean congruence and the last condition of $\mathscr{B}$ adds that it is a full congruence. ◀

J Proof of RSP

To prove RSP, another version of $\leftrightarrow_{br}$ is needed, this time up to itself.

▶ **Definition 42.** A *branching time-out bisimulation up to* $\leftrightarrow_{br}$ is a symmetric relation $\mathscr{B} \subseteq \mathbb{P} \times \mathbb{P}$ such that, for all $P, Q \in \mathbb{P}$ such that $P \ \mathscr{B}\ Q$, and for all $X \subseteq A$,

1. if $P \Longrightarrow P' \xrightarrow{\alpha} P''$ with $\alpha \in A_\tau$ and $P \leftrightarrow_{br} P'$ then there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $P' \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} Q_1$ and $P'' \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} Q_2$
2. if $P = P_0 \Longrightarrow P_1 \xrightarrow{t} P_2 \Longrightarrow P_3 \xrightarrow{t} \ldots \Longrightarrow P_{2r-1} \xrightarrow{(t)} P_{2r}$ with $r > 0$, such that $\forall i \in [0, r{-}1]$, $\theta_X(P) \leftrightarrow_{br} \theta_X(P_{2i}) \wedge P \leftrightarrow_{br} P_{2i+1} \wedge \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$, then there exists a path $Q = Q_0 \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2n-1} \xrightarrow{(t)} Q_{2n}$ with $n > 0$, such that $\forall i \in [1, 2n{-}1]$ $\theta_X(P) \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} \theta_X(Q_i)$, $\forall j \in [0, n{-}1]$ $\mathcal{I}(Q_{2j+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_{2r}) \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} \theta_X(Q_{2n})$
3. if $P \Longrightarrow P_0 \stackrel{\tau}{\nrightarrow}$ with $P \leftrightarrow_{br} P_0$ then there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\nrightarrow}$.

▶ **Proposition 43.** *Let $P, Q \in \mathbb{P}$. Then $P \leftrightarrow_{br} Q$ iff there exists a branching time-out bisimulation $\mathcal{B}$ up to $\leftrightarrow_{br}$ such that $P \ \mathscr{B}\ Q$.*

Proof. Let $\mathscr{B}$ be a branching time-out bisimulation up to $\leftrightarrow_{br}$. We are going to show that $\leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br}$ is a branching time-out bisimulation. Let $P, Q \in \mathbb{P}$ such that $P \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} Q$. Then there exists $P^\dagger, Q^\dagger \in \mathbb{P}$ such that $P \leftrightarrow_{br} P^\dagger \ \mathscr{B}\ Q^\dagger \leftrightarrow_{br} Q$.

1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then, since $P \leftrightarrow_{br} P^\dagger$, there exists a path $P^\dagger \Longrightarrow P^\star \xrightarrow{(\alpha)} P^\ddagger$ such that $P \leftrightarrow_{br} P^\star$ and $P' \leftrightarrow_{br} P^\ddagger$. Since $P^\dagger \Longrightarrow P^\star \xrightarrow{(\alpha)} P^\ddagger$ and $P^\dagger \ \mathscr{B}\ Q^\dagger$, there exists a path $Q^\dagger \Longrightarrow Q^\star \xrightarrow{(\alpha)} Q^\ddagger$ such that $P^\star \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} Q^\star$ and $P^\ddagger \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} Q^\ddagger$. Since $Q^\dagger \Longrightarrow Q^\star$ and $Q^\dagger \leftrightarrow_{br} Q$, there exists a path $Q \Longrightarrow Q_0$ such that $Q^\star \leftrightarrow_{br} Q_0$; moreover, since $Q^\star \xrightarrow{(\alpha)} Q^\ddagger$, there exists a path $Q_0 \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that $Q^\star \leftrightarrow_{br} Q_1$ and $Q^\ddagger \leftrightarrow_{br} Q_2$. As a result, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ such that, by transitivity of $\leftrightarrow_{br}$, $P \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} Q_1$ and $P' \leftrightarrow_{br} \mathscr{B} \leftrightarrow_{br} Q_2$.

2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, since $P \leftrightarroweq_{br} P^\dagger$, there exists a path $P^\dagger = P_0^\dagger \Longrightarrow P_1^\dagger \xrightarrow{\mathrm{t}} P_2^\dagger \Longrightarrow P_3^\dagger \xrightarrow{\mathrm{t}} \dots \Longrightarrow P_{2r-1}^\dagger \xrightarrow{(\mathrm{t})} P_{2r}^\dagger$ with $r > 0$, such that $P_1^\dagger \stackrel{\tau}{\not\to}$, $\forall i \in [1, r{-}1]$, $\theta_X(P) \leftrightarroweq_{br} \theta_X(P_{2i}^\dagger) \wedge \mathcal{I}(P_{2i+1}^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarroweq_{br} \theta_X(P_{2r}^\dagger)$. As remarked in Section 2, we even have $P \leftrightarroweq_{br} P_{2i+1}^\dagger$ for all $i \in [1, r{-}1]$. As $\leftrightarroweq_{br}$ is a congruence, $\theta_X(P^\dagger) \leftrightarroweq_{br} \theta_X(P)$, so $\forall i \in [0, r{-}1]$, $\theta_X(P^\dagger) \leftrightarroweq_{br} \theta_X(P_{2i}^\dagger) \wedge P^\dagger \leftrightarroweq_{br} P_{2i+1}^\dagger$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a path $Q^\dagger = Q_0^\dagger \Longrightarrow Q_1^\dagger \xrightarrow{\mathrm{t}} Q_2^\dagger \Longrightarrow Q_3^\dagger \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2n-1}^\dagger \xrightarrow{(\mathrm{t})} Q_{2n}^\dagger$ with $n > 0$, such that $\forall i \in [1, 2n{-}1]$ $\theta_X(P^\dagger) \leftrightarroweq_{br} \mathscr{B} \leftrightarroweq_{br} \theta_X(Q_i^\dagger)$, $\forall j \in [0, n{-}1]$ $\mathcal{I}(Q_{2j+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_{2r}^\dagger) \leftrightarroweq_{br} \mathscr{B} \leftrightarroweq_{br} \theta_X(Q_{2n}^\dagger)$. Since $Q^\dagger \leftrightarroweq_{br} Q$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2m-1} \xrightarrow{(\mathrm{t})} Q_{2m}$ with $m > 0$, such that $Q_1 \stackrel{\tau}{\not\to}$, $\forall k \in [1, m{-}1], \exists j \in [0, 2n{-}1], \theta_X(Q_j^\dagger) \leftrightarroweq_{br} \theta_X(Q_{2k}) \wedge \mathcal{I}(Q_{2k+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(Q_{2n}^\dagger) \leftrightarroweq_{br} \theta_X(Q_{2m})$. As a result, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow Q_{2m-1} \xrightarrow{(\mathrm{t})} Q_{2m}$ with $m > 0$, such that $Q_1 \stackrel{\tau}{\not\to}$, and, by transitivity of $\leftrightarroweq_{br}$, $\forall i \in [1, m{-}1]$, $\theta_X(P) \leftrightarroweq_{br} \mathscr{B} \leftrightarroweq_{br} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarroweq_{br} \mathscr{B} \leftrightarroweq_{br} \theta_X(Q_{2m})$.
3. If $P \stackrel{\tau}{\not\to}$ then, since $P \leftrightarroweq_{br} P^\dagger$, there exists a path $P^\dagger \Longrightarrow P_0^\dagger \stackrel{\tau}{\not\to}$, and $P^\dagger \leftrightarroweq_{br} P_0^\dagger$. Since $P^\dagger \mathscr{B} Q^\dagger$, there exists a path $Q^\dagger \Longrightarrow Q_0^\dagger \stackrel{\tau}{\not\to}$. Since $Q^\dagger \leftrightarroweq_{br} Q$, there exists a path $Q \Longrightarrow Q_0 \stackrel{\tau}{\not\to}$. ◀

The following lemma will be useful to deal with the matching of paths.

▶ **Lemma 44.** *Let $H \in \mathbb{E}$ be well-guarded and have free variables from $W \subseteq V$ only, and let $\rho, \nu \in \mathbb{P}^W$.*
1. *$\mathcal{I}(H[\rho]) = \mathcal{I}(H[\nu])$.*
2. *If $H[\rho] \xrightarrow{\alpha} R$ with $\alpha \in Act$ then there exists $H' \in \mathbb{E}$ with free variables in W only such that $R = H'[\rho]$ and $H[\nu] \xrightarrow{\alpha} H'[\nu]$. Moreover, in case $\alpha \in \{\tau, \mathrm{t}\}$, also H' is well-guarded.*

Proof. 1. has been proven in [11]. We obtain 2. by induction on the derivation of $H[\rho] \xrightarrow{\alpha} R$, making a case distinction on the shape of H.

Let $H = \alpha.G$, so that $H[\rho] = \alpha.G[\rho]$. Then $R = G[\rho]$ and $H[\nu] \xrightarrow{\alpha} G[\nu]$. In case $\alpha \in \{\tau, \mathrm{t}\}$, also G is well-guarded.

The case $H = 0$ cannot occur. Nor can the case $H = x \in V$, as H is well-guarded.

Let $H = H_1 \|_S H_2$, so that $H[\rho] = H_1[\rho] \|_S H_2[\rho]$. Note that H_1 and H_2 are well-guarded and have free variables in W only. One possibility is that $a \notin S$, $H_1[\rho] \xrightarrow{\alpha} R_1$ and $R = R_1 \|_S H_2[\rho]$. By induction, R_1 has the form $H_1'[\rho]$ for some term $H_1' \in \mathbb{E}$ with free variables in W only, and in case $\alpha \in \{\tau, \mathrm{t}\}$, also H_1' is well-guarded. Moreover, $H_1[\nu] \xrightarrow{\alpha} H_1'[\nu]$. Thus $R = (H_1' \|_S H_2)[\rho]$, and $H' := H_1' \|_S H_2$ has free variables in W only. In case $\alpha \in \{\tau, \mathrm{t}\}$, H is well-guarded. Moreover, $H[\nu] = H_1[\nu] \|_S H_2[\nu] \xrightarrow{\alpha} H_1'[\nu] \|_S H_2[\nu] = H'[\nu]$.

The other two cases for $\|_S$, and the cases for the operators $+$ and $\mathcal{R}$, are equally trivial.

Let $H = \theta_L^U(H^\dagger)$, so that $H[\rho] = \theta_L^U(H^\dagger[\rho])$. Note that $H^\dagger$ is well-guarded and has free variables in W only. The case $\alpha = \tau$ is again trivial, so assume $\alpha \neq \tau$. Then $H^\dagger[\rho] \xrightarrow{\alpha} R$ and either $\alpha \in X$ or $\mathcal{I}(H^\dagger[\rho]) \cap (L \cup \{\tau\}) = \emptyset$. By induction, R has the form $H'[\rho]$ for some term $H' \in \mathbb{E}$ with free variables in W only, and in case $\alpha = \mathrm{t}$ this term is well-guarded. Moreover, $H^\dagger[\nu] \xrightarrow{\alpha} H'[\nu]$. Since $\mathcal{I}(H^\dagger[\rho]) = \mathcal{I}((H^\dagger[\nu])$ by Lemma 44.1, either $\alpha \in X$ or $\mathcal{I}(H^\dagger[\nu]) \cap (L \cup \{\tau\}) = \emptyset$. Consequently, $H[\nu] = \theta_L^U(H^\dagger[\nu]) \xrightarrow{\alpha} H'[\nu]$.

Let $H = \psi_X(H^\dagger)$, so that $H[\rho] = \psi_X(H^\dagger[\rho])$. Note that $H^\dagger$ is well-guarded and has free variables in W only. The case $\alpha \in A \cup \{\tau\}$ is trivial, so assume $\alpha = \mathrm{t}$. Then $H^\dagger[\rho] \xrightarrow{\mathrm{t}} R^\dagger$ for some $R^\dagger$ such that $R = \theta_X(R^\dagger)$. Moreover, $H^\dagger[\rho] \cap (X \cup \{\tau\}) = \emptyset$. By induction, $R^\dagger$ has the form $H'[\rho]$ for some well-guarded term $H' \in \mathbb{E}$ with free variables in W only. Moreover,

$H^\dagger[\nu] \xrightarrow{\mathrm{t}} H'[\nu]$. Thus $R = (\theta_X(H'))[\rho]$ and $\theta_X(H')$ is well-guarded and has free variables in W only. Since $\mathcal{I}(H^\dagger[\rho]) = \mathcal{I}((H^\dagger[\nu])$ by Lemma 44.1, $H^\dagger[\nu] \cap (X \cup \{\tau\}) = \emptyset$. Consequently, $H[\nu] = \psi_X(H^\dagger[\nu]) \xrightarrow{\mathrm{t}} \theta_X(H'[\nu]) = (\theta_X(H'))[\nu]$.

Finally, let $H = \langle x|\mathcal{S}\rangle$, so that $H[\rho] = \langle x|\mathcal{S}[\rho^\dagger]\rangle$, where $\rho^\dagger \in \mathbb{P}^{W\backslash V_\mathcal{S}}$ is the restriction of ρ to $W\backslash V_\mathcal{S}$. The transition $\langle \mathcal{S}_x[\rho^\dagger]|\mathcal{S}[\rho^\dagger]\rangle \xrightarrow{\alpha} R$ is derivable through a subderivation of the one for $\langle x|\mathcal{S}[\rho^\dagger]\rangle \xrightarrow{\alpha} R$. Moreover, $\langle \mathcal{S}_x[\rho^\dagger]|\mathcal{S}[\rho^\dagger]\rangle = \langle \mathcal{S}_x|\mathcal{S}\rangle[\rho]$. So by induction, R has the form $H'[\rho]$ for some term $H' \in \mathbb{E}$ with free variables in W only, and $\langle \mathcal{S}_x|\mathcal{S}\rangle[\nu] \xrightarrow{\alpha} H'[\nu]$. Moreover, in case $\alpha \in \{\tau, \mathrm{t}\}$, also H' is well-guarded. Since $\langle \mathcal{S}_x|\mathcal{S}\rangle[\nu] = \langle \mathcal{S}_x[\nu^\dagger]|\mathcal{S}[\nu^\dagger]\rangle$, it follows that $H[\nu] = \langle x|\mathcal{S}\rangle[\nu] = \langle x|\mathcal{S}[\nu^\dagger]\rangle \xrightarrow{\alpha} H'[\nu]$. ◀

▶ **Corollary 45.** *Let $H \in \mathbb{E}$ be well-guarded and have free variables from $W \subseteq V$ only, and let $\rho, \nu \in \mathbb{P}^W$.*

- *If $H[\rho] \Longrightarrow R \xrightarrow{\alpha} S$ with $\alpha \in Act$ then there exists $H', H'' \in \mathbb{E}$ with free variables in W only such that $R = H'[\rho]$, $S = H'[\rho]$ and $H[\nu] \Longrightarrow H'[\nu] \xrightarrow{\alpha} H'[\nu]$.*
- *if $H[\rho] \Longrightarrow R_1 \xrightarrow{\mathrm{t}} R_2 \Longrightarrow R_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow R_{2r-1} \xrightarrow{(\mathrm{t})} R_{2r}$ with $r > 0$ then there exists $(H_i)_{i\in[1,2r]} \in \mathbb{E}^{2r}$ with free variables in W only such that $\forall i \in [1, 2r]$, $R_i = H_i[\rho]$ and $H[\rho] \Longrightarrow H_1[\nu] \xrightarrow{\mathrm{t}} H_2[\nu] \Longrightarrow H_3[\nu] \xrightarrow{\mathrm{t}} \ldots \Longrightarrow H_{2r-1}[\nu] \xrightarrow{(\mathrm{t})} H_{2r}[\nu]$.* ◀

Proof of Proposition 21. It suffices to prove the proposition when $\rho, \nu \in \mathbb{P}^{V_\mathcal{S}}$ and only variables of $V_\mathcal{S}$ can occur in the expressions $\mathcal{S}_x$ for $x \in V_\mathcal{S}$. Indeed, the general case requires to prove that, for all $\sigma : V \to \mathbb{P}$, $\rho[\sigma] \leftrightarrow^r_{br} \nu[\sigma]$. Let $\hat{\sigma} : V \setminus V_\mathcal{S} \to \mathbb{P}$ be defined as $\forall x \in V \setminus V_\mathcal{S}, \hat{\sigma}(x) = \sigma(x)$. Since $\rho \leftrightarrow^r_{br} \mathcal{S}[\rho]$, $\rho[\sigma] \leftrightarrow^r_{br} \mathcal{S}[\rho][\sigma] = \mathcal{S}[\hat{\sigma}][\rho[\sigma]]$, therefore, proving the proposition with $\rho[\sigma]$, $\nu[\sigma]$ and $\mathcal{S}[\hat{\sigma}]$ is sufficient.

It also suffices to prove the proposition for the case that $\mathcal{S}$ is manifestly well-guarded. Indeed, if $\mathcal{S}$ is well-guarded, let $\mathcal{S}'$ be the manifestly well-guarded specification into which $\mathcal{S}$ can be converted. Since $\leftrightarrow^r_{br}$ is a lean congruence, a solution to $\mathcal{S}$ up to $\leftrightarrow^r_{br}$ is a solution to $\mathcal{S}'$ up to $\leftrightarrow^r_{br}$.

Let $\mathcal{S}$ be a manifestly well-guarded recursive specification with free variables from $V_\mathcal{S}$ only, and $\rho, \nu \in \mathbb{P}^{V_\mathcal{S}}$ two of its solutions up to $\leftrightarrow^r_{br}$. We are going to show that the symmetric closure of

$$\mathscr{B} := \{(H[\mathcal{S}[\rho]], H[\mathcal{S}[\nu]]) \mid H \in \mathbb{E} \text{ is without } \tau_I \text{ and with free variables from } V_\mathcal{S} \text{ only}\}$$

is a branching time-out bisimulation up to $\leftrightarrow^r_{br}$. Here $\mathcal{S}[\rho] := \{x = \mathcal{S}_x[\rho] \mid x \in V_\mathcal{S}\} \in \mathbb{P}^{V_\mathcal{S}}$ is employed as a substitution. Let $P, Q \in \mathbb{P}$ such that $P \mathrel{\mathscr{B}} Q$. Then there exists $H \in \mathbb{E}$ with free variables from $V_\mathcal{S}$ only such that $P = H[\mathcal{S}[\rho]]$ and $Q = H[\mathcal{S}[\nu]]$, the other case being symmetrical. Note that $H[\mathcal{S}[\rho]] = H[\mathcal{S}][\rho]$. Since H and $\mathcal{S}$ have free variables from $V_\mathcal{S}$ only, so does $H[\mathcal{S}]$. Moreover, since $\mathcal{S}$ is manifestly well-guarded, $H[\mathcal{S}]$ is well-guarded.

1. Let $H[\mathcal{S}[\rho]] \Longrightarrow P_1 \xrightarrow{\alpha} P_2$. By Corollary 45, there exists $H_1, H_2 \in \mathbb{E}$ with free variables from $V_\mathcal{S}$ only such that $P_1 = H_1[\rho]$, $P_2 = H_2[\rho]$ and $H[\mathcal{S}][\nu] \Longrightarrow H_1[\nu] \xrightarrow{\alpha} H_2[\nu]$. Furthermore, since $\leftrightarrow^r_{br}$ is a congruence and ρ and ν are solutions of $\mathcal{S}$ up to $\leftrightarrow^r_{br}$, $H_1[\rho] \leftrightarrow^r_{br} H_1[\mathcal{S}[\rho]]$, $H_1[\nu] \leftrightarrow^r_{br} H_1[\mathcal{S}[\nu]]$, $H_2[\rho] \leftrightarrow^r_{br} H_2[\mathcal{S}[\rho]]$ and $H_2[\nu] \leftrightarrow^r_{br} H_2[\mathcal{S}[\nu]]$, therefore, by definition of $\mathscr{B}$, $H_1[\rho] \leftrightarrow^r_{br} \mathscr{B} \leftrightarrow^r_{br} H_1[\nu]$ and $H_2[\rho] \leftrightarrow^r_{br} \mathscr{B} \leftrightarrow^r_{br} H_2[\nu]$.
2. Let $H[\mathcal{S}[\rho]] \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{(\mathrm{t})} P_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1]$, $\theta_X(P) \leftrightarrow_{br} \theta_X(P_{2i}) \wedge P \leftrightarrow_{br} P_{2i+1} \wedge \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$. By Corollary 45, there exists $(H_i)_{i\in[1,2r]} \in \mathbb{E}^{2r}$ with free variables from $V_\mathcal{S}$ only such that $\forall i \in [1, 2r]$, $P_i = H_i[\rho]$ and $H[\mathcal{S}][\nu] \Longrightarrow H_1[\nu] \xrightarrow{\mathrm{t}} H_2[\nu] \Longrightarrow H_3[\nu] \xrightarrow{\mathrm{t}} \ldots \Longrightarrow H_{2r-1} \xrightarrow{(\mathrm{t})} H_{2r}[\nu]$. Since all H_{2i+1} are well-guarded, by Lemma 44, $\forall i \in [0, r-1]$, $\mathcal{I}(H_{2i+1}[\nu]) \cap (X \cup \{\tau\}) = \emptyset$. Furthermore, since $\leftrightarrow^r_{br}$ is a congruence and ρ and ν are solutions of $\mathcal{S}$ up to $\leftrightarrow^r_{br}$, for all

$i \in [1,2r]$, $H_i[\rho] \leftrightarroweq^r_{br} H_i[\mathcal{S}[\rho]]$ and $H_i[\nu] \leftrightarroweq^r_{br} H_i[\mathcal{S}[\nu]]$; therefore, by definition of $\mathscr{B}$, for $i \in [1,2r]$, $\theta_X(H_i[\rho]) \leftrightarroweq^r_{br}\mathscr{B}\leftrightarroweq^r_{br} \theta_X(H_i[\nu])$ (notice that $\theta_X(H_i[\mathcal{S}][\rho]) = \theta_X(H_i)[\mathcal{S}][\rho]$). It follows that $\forall i \in [1,2r-1]\ \theta_X(P) \leftrightarroweq_{br}\mathscr{B}\leftrightarroweq_{br} \theta_X(H_i[\nu])$.

3. Let $H[\mathcal{S}[\rho]] \Longrightarrow P_0 \stackrel{\tau}{\nrightarrow}$. By Lemma 44, there exists a well-guarded $H_1 \in \mathbb{E}$ with free variables from $V_\mathcal{S}$ only such that $P_0 = H_0[\rho]$ and $H[\mathcal{S}][\nu] \Longrightarrow H_0[\nu]$. Since H_0 is well-guarded and $P_0 \stackrel{\tau}{\nrightarrow}$, according to Lemma 44.1, $H_0[\nu] \stackrel{\tau}{\nrightarrow}$.

Next, we will prove that $\mathscr{B}$ is a rooted branching time-out bisimulation. Let $P,Q \in \mathbb{P}$ such that $P\ \mathscr{B}\ Q$. Then there exists $H \in \mathbb{E}$ with free variables from $V_\mathcal{S}$ only such that $P = H[\mathcal{S}[\rho]]$ and $Q = H[\mathcal{S}[\nu]]$, the other case being symmetrical. Note that $H[\mathcal{S}[\rho]] = H[\mathcal{S}][\rho]$. Since H and $\mathcal{S}$ have free variables from $V_\mathcal{S}$ only, so does $H[\mathcal{S}]$. Moreover, since $\mathcal{S}$ is manifestly well-guarded, $H[\mathcal{S}]$ is well-guarded.

1. Let $P \xrightarrow{\alpha} P'$. By Lemma 44, there exists $H' \in \mathbb{E}$ with free variables from $V_\mathcal{S}$ only such that $P' = H'[\rho]$ and $Q = H[\mathcal{S}][\nu] \xrightarrow{\alpha} H'[\nu]$. Furthermore, since $\leftrightarroweq^r_{br}$ is a congruence and ρ and ν are solutions of $\mathcal{S}$ up to $\leftrightarroweq^r_{br}$, $H'[\rho] \leftrightarroweq^r_{br} H'[\mathcal{S}[\rho]]$ and $H'[\nu] \leftrightarroweq^r_{br} H'[\mathcal{S}[\nu]]$. Therefore, by definition of $\mathscr{B}$, $H'[\rho] \leftrightarroweq^r_{br}\mathscr{B}\leftrightarroweq^r_{br} H'[\nu]$. But, $\mathscr{B}$ is a branching time-out bisimulation up to $\leftrightarroweq^r_{br}$, thus, by Proposition 43, $H'[\rho] \leftrightarroweq^r_{br} H'[\nu]$.
2. Let $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$. By Lemma 44, there exists $H' \in \mathbb{E}$ with free variables from $V_\mathcal{S}$ only such that $P' = H'[\rho]$ and $Q = H[\mathcal{S}][\nu] \xrightarrow{t} H'[\nu]$. Exactly as above, not even using $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, this implies $H'[\rho] \leftrightarroweq^r_{br} H'[\nu]$. Thus, since $\leftrightarroweq^r_{br}$ is a congruence, $\theta_X(H'[\rho]) \leftrightarroweq^r_{br} \theta_X(H'[\nu])$.

By considering $H = x$ with $x \in V_\mathcal{S}$, this yields $\mathcal{S}_x[\rho] \leftrightarroweq^r_{br} \mathcal{S}_x[\nu]$ and so $\rho(x) \leftrightarroweq^r_{br} \mathcal{S}_x[\rho] \leftrightarroweq^r_{br} \mathcal{S}_x[\nu] \leftrightarroweq^r_{br} \nu(x)$. Consequently, $\rho \leftrightarroweq^r_{br} \nu$. ◀

K Proof of Lemma 22

Proof of Lemma 22. Suppose that there exists $X \subseteq A$ such that $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and there exists $P \xrightarrow{t} P'$ such that $\theta_X(P) \leftrightarroweq_{br} \theta_X(P')$, yet there exists $Y \subseteq A$ such that $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$ and $\neg(\theta_Y(P) \leftrightarroweq_{br} \theta_Y(P'))$.

Since $P \stackrel{\tau}{\nrightarrow}$ and $\theta_X(P) \leftrightarroweq_{br} \theta_X(P')$, By Definition 13.3 there exists a path $\theta_X(P') \Longrightarrow P^\dagger \stackrel{\tau}{\nrightarrow}$, such that $\theta_X(P') \leftrightarroweq_{br} P^\dagger$ by Definition 13.1. By the semantics, there exists a path $P' \Longrightarrow P'' \stackrel{\tau}{\nrightarrow}$ such that $P^\dagger = \theta_X(P'')$. Proposition 15.2 yields $P \leftrightarroweq^X_{br} P''$, so $P \leftrightarroweq_{br} P''$ by Lemma 2.3 and $\mathcal{I}(P) = \mathcal{I}(P'')$ by Lemma 2.2.

We are going to show that if $P \leftrightarroweq_{br} Q$ and $\mathcal{I}(P) = \mathcal{I}(Q)$ then we can find a nonempty path $Q(\Longrightarrow\xrightarrow{t})^* \Longrightarrow Q'$ such that $P \leftrightarroweq_{br} Q'$ and $\mathcal{I}(P) = \mathcal{I}(Q')$.

Since $P \leftrightarroweq_{br} Q$, $\mathcal{I}(P) \cap (Y \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{t} P'$, there exists a path $Q \Longrightarrow Q_1 \xrightarrow{t} Q_2 \Longrightarrow Q_3 \xrightarrow{t} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(t)} Q_{2r}$ with $r > 0$, such that $\forall i \in [1, r-1], \theta_Y(P) \leftrightarroweq_{br} \theta_Y(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$ and $\theta_Y(P') \leftrightarroweq_{br} \theta_Y(Q_{2r})$. Since $\neg(\theta_Y(P) \leftrightarroweq_{br} \theta_Y(P'))$, $Q \neq Q_{2r}$.

Since $\theta_Y(P') \Longrightarrow \theta_Y(P'') \stackrel{\tau}{\nrightarrow}$ and $\theta_Y(P') \leftrightarroweq_{br} \theta_Y(Q_{2r})$, there exists a path $\theta_Y(Q_{2r}) \Longrightarrow Q^\dagger \stackrel{\tau}{\nrightarrow}$ such that $Q^\dagger \leftrightarroweq_{br} \theta_Y(P'')$. According to the semantics, there is a path $Q_{2r} \Longrightarrow Q' \stackrel{\tau}{\nrightarrow}$ such that $Q^\dagger = \theta_Y(Q')$. Since $\mathcal{I}(P'') \cap (Y \cup \{\tau\}) = \emptyset$, $Q' \stackrel{\tau}{\nrightarrow}$ and $\theta_Y(P'') \leftrightarroweq_{br} \theta_Y(Q')$, Proposition 15.2 and Lemma 2 yields $Q' \leftrightarroweq_{br} P'' \leftrightarroweq_{br} P$ and $\mathcal{I}(Q') = \mathcal{I}(P'') = \mathcal{I}(P)$.

As a result, there exists an infinite τ/t-path starting in P'', but that contradicts the strong guardedness of P. ◀

L Soundness of the Reactive Approximation Axiom

▶ **Lemma 46.** $\forall P \in \mathbb{P}, \theta_X(P) \leftrightarroweq \theta_X(\theta_X(P))$

Proof. Trivial when considering the semantics of θ_X. ◀

Proof of Proposition 23. We show that $\mathscr{B}:= \{(P,Q),(Q,P) \mid \forall X \subseteq A,\ \psi_X(P) \leftrightarroweq^r_{br} \psi_X(Q)\}$ is a rooted branching time-out bisimulation. Let $P,Q \in \mathbb{P}$ such that $P \mathrel{\mathscr{B}} Q$. Thus, $\forall X \subseteq A, \psi_X(P) \leftrightarroweq^r_{br} \psi_X(Q)$.

1. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$ then, by the semantics, $\psi_A(P) \xrightarrow{\alpha} P'$. Since $\psi_A(P) \leftrightarroweq^r_{br} \psi_A(Q)$, there exists a transition $\psi_A(Q) \xrightarrow{\alpha} Q'$ such that $P' \leftrightarroweq_{br} Q'$. By the semantics, $Q \xrightarrow{\alpha} Q'$.
2. If $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then, by the semantics, $\psi_X(P) \xrightarrow{\mathrm{t}} \theta_X(P')$. Since $\psi_X(P) \leftrightarroweq^r_{br} \psi_X(Q)$, there exists a transition $\psi_X(Q) \xrightarrow{\mathrm{t}} Q^\ddagger$ with $\theta_X(\theta_X(P')) \leftrightarroweq_{br} \theta_X(Q^\ddagger)$. By the semantics, $Q^\ddagger = \theta_X(Q')$ and $Q \xrightarrow{\mathrm{t}} Q'$. By Lemma 46, $\theta_X(P') \leftrightarroweq \theta_X(\theta_X(P')) \leftrightarroweq_{br} \theta_X(\theta_X(Q')) \leftrightarroweq \theta_X(Q')$. ◀

M Proofs of Completeness for Finite Processes

▶ **Definition 47.** Call a process P *brb-stable* if (a) there is no process $P^\dagger$ with $P \xrightarrow{\tau} P^\dagger$ and $P \leftrightarroweq_{br} P^\dagger$, and (b) there is no set $X \subseteq A$ and process $P^\ddagger$ with $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, $P \xrightarrow{\mathrm{t}} P^\ddagger$ and $\theta_X(P) \leftrightarroweq_{br} \theta_X(P^\ddagger)$.

▶ **Lemma 48.** *If P and Q are brb-stable and $P \leftrightarroweq_{br} Q$ then $P \leftrightarroweq^r_{br} Q$.*

Proof. Assume that P and Q are brb-stable and $P \leftrightarroweq_{br} Q$. If $P \xrightarrow{\alpha} P'$ with $\alpha \in A_\tau$, then there is a path $Q \Longrightarrow Q_1 \xrightarrow{(\alpha)} Q_2$ with $P \leftrightarroweq_{br} Q_1$ and $P' \leftrightarroweq_{br} Q_2$. By symmetry and transitivity of $\leftrightarroweq_{br}$ we have $Q_1 \leftrightarroweq_{br} Q$, so by the brb-stability of Q it follows that $Q_1 = Q$. Moreover, if $\alpha = \tau$ and $Q_2 = Q_1$ then $P' \leftrightarroweq_{br} P$, contradicting the brb-stability of P. Thus $Q \xrightarrow{\alpha} Q_2$. This argument also yields that $\mathcal{I}(P) = \mathcal{I}(Q)$.

Furthermore, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there is a path $Q \Longrightarrow Q_1 \xrightarrow{\mathrm{t}} Q_2 \Longrightarrow Q_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow Q_{2r-1} \xrightarrow{(\mathrm{t})} Q_{2r}$ with $r > 0$, such that $Q_1 \stackrel{\tau}{\not\rightarrow}$, $\forall i \in [1, r{-}1]$, $\theta_X(P) \leftrightarroweq_{br} \theta_X(Q_{2i}) \wedge \mathcal{I}(Q_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarroweq_{br} \theta_X(Q_{2r})$. By Lemma 2.1, Lemma 2.3 and the transitivity of $\leftrightarroweq_{br}$, we have $P \leftrightarroweq^X_{br} Q_1$, $P \leftrightarroweq_{br} Q_1$ and $Q \leftrightarroweq_{br} Q_1$, respectively, so the brb-stability of Q yields $Q_1 = Q$. If $r > 1$ we would have $\theta_X(Q_2) \leftrightarroweq_{br} \theta_X(P) \leftrightarroweq_{br} \theta_X(Q)$, contradicting the brb-stability of Q. Thus $r = 1$ and $Q \xrightarrow{(\mathrm{t})} Q_2$. If $Q_2 = Q$ we would obtain $\theta_X(P') \leftrightarroweq_{br} \theta_X(Q_2) \leftrightarroweq_{br} \theta_X(P)$, contradicting the brb-stability of P. Hence $Q \xrightarrow{\mathrm{t}} Q_2$. So indeed $P \leftrightarroweq^r_{br} Q$. ◀

Proof of Proposition 26. We define the *length* of a path $P_0 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} P_n$ to be n and the *depth* of a process P, denoted $d(P)$, to be the length of the longest path starting from P. It is well defined because P is a recursion-free CCSP^θ_t process. Note that $d(\theta_X(P)) \leq d(P)$.

($\leftrightarroweq_{br}$) We will proceed by induction on $\max(d(P), d(Q))$. Let $n \in \mathbb{N}$ and suppose that the property holds for any recursion-free CCSP^θ_t processes P,Q such that $\max(d(P), d(Q)) < n$. Let P,Q be two recursion-free CCSP^θ_t processes such that $\max(d(P), d(Q)) = n$ and $P \leftrightarroweq_{br} Q$.

Since P is recursion-free, using Lemma 2.4, there exists a path $P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{\mathrm{t}} P_{2r} \Longrightarrow P_0$ with $r \in \mathbb{N}$, such that $P \leftrightarroweq_{br} P_1$ and $\forall i \in [1,r], \exists X_i \subseteq A, \mathcal{I}(P_{2i-1}) \cap (X_i \cup \{\tau\}) = \emptyset \wedge \theta_{X_i}(P) \leftrightarroweq_{br} \theta_X(P_{2r})$, $P \leftrightarroweq_{br} P_0$ and P_0 is brb-stable. We are going to show that, for all $\alpha \in Act$, $Ax_r \vdash \alpha.\hat{P} = \alpha.\hat{P}_0$. If P is brb-stable then $P = P_1$ and $r = 0$ so $P = P_0$ and this is trivial. Thus, suppose that P is not brb-stable. Then, as P_0 is brb-stable, $P \neq P_0$ and so $d(P_0) < d(P)$.

Let $I := \{(\alpha, P') \mid P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau \wedge (\alpha \neq \tau \vee P \not\leftrightarroweq_{br} P')\}$, listing the outgoing transitions of P not labelled by t and not elidable w.r.t. $\leftrightarroweq_{br}$. Let $(\alpha, P') \in I$. Since $P \leftrightarroweq_{br} P_0$, there exists a path $P_0 \Longrightarrow P_1 \xrightarrow{(\alpha)} P_2$ such that $P \leftrightarrow_{br} P_1$ and $P' \leftrightarrow_{br} P_2$. Since P_0 is brb-stable and $P_1 \leftrightarrow_{br} P \leftrightarrow_{br} P_0$, $P_0 = P_1$ and $P_0 \xrightarrow{(\alpha)} P_2$. If $\alpha = \tau$ then $P_0 \leftrightarrow_{br} P \not\leftrightarrow_{br} P' \leftrightarrow_{br} P_2$ so $P_0 \neq P_2$ and $P_0 \xrightarrow{\alpha} P_2$. Since $\max(d(P_2), d(P')) < d(P)$, by induction, $Ax_r \vdash \alpha.\hat{P}' = \alpha.\hat{P}_2$, so by Lemma 25 $Ax_r \vdash \alpha.P' = \alpha.P_2$. As a result, $Ax_r \vdash \hat{P}_0 = \sum_{(\alpha,P')\in I} \alpha.P' + \hat{P}_0$.

Let $J := \{(\tau, P') \mid P \xrightarrow{\tau} P' \wedge P \leftrightarrow_{br} P'\}$, listing the outgoing τ-transitions of P elidable w.r.t. $\leftrightarrow_{br}$. Let $(\tau, P') \in J$. Since $P' \leftrightarrow_{br} P \leftrightarrow_{br} P_0$ and $\max(d(P'), d(P_0)) < d(P)$, by induction, $Ax_r \vdash \tau.\hat{P}' = \tau.\hat{P}_0$, so by Lemma 25 $Ax_r \vdash \tau.P' = \tau.\hat{P}_0$.

Suppose that $P \xrightarrow{\tau}$. Since P is not brb-stable, there exists a transition $P \xrightarrow{\tau} P'$ such that $P \leftrightarrow_{br} P'$ (i.e. $J \neq \emptyset$). Now the following equality can be derived from Ax_r, for all $\beta \in Act$. Here the first step applies $\mathbf{L}\tau$.

$$\begin{aligned}
Ax_r \vdash \beta.\hat{P} &= \beta.(\sum_{\{(\alpha,P')\mid P\xrightarrow{\alpha}P'\wedge\alpha\in A_\tau\}} \alpha.P') = \beta.(\sum_{(\tau,P')\in J} \tau.P' + \sum_{(\alpha,P')\in I} \alpha.P') \\
&= \beta.(\tau.\hat{P}_0 + \sum_{(\alpha,P')\in I} \alpha.P') = \beta.(\tau.(\hat{P}_0 + \sum_{(\alpha,P')\in I} \alpha.P') + \sum_{(\alpha,P')\in I} \alpha.P') \\
&= \beta.(\hat{P}_0 + \sum_{(\alpha,P')\in I} \alpha.P') = \beta.\hat{P}_0
\end{aligned}$$

Now, suppose that $P \not\xrightarrow{\tau}$. Then $J = \emptyset$ and $I = \{(\alpha, P') \mid P \xrightarrow{\alpha} P' \wedge \alpha \in A\}$. Since P is not brb-stable and $P \not\xrightarrow{\tau}$, there exists $X \subseteq A$ and $P \xrightarrow{\mathrm{t}} P'$ such that $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P) \leftrightarrow_{br} \theta_X(P')$ $(\star)$. Moreover, since P_0 is brb-stable, $P \not\xrightarrow{\tau}$ and $P \leftrightarrow_{br} P_0$, $P_0 \not\xrightarrow{\tau}$. Let $I_0 := \{(\alpha, P_0') \mid P_0 \xrightarrow{\alpha} P_0' \wedge \alpha \in A_\tau\}$. Since $P \not\xrightarrow{\tau}$, for all $(\alpha, P_0') \in I_0$, there exists $P \xrightarrow{\alpha} P'$ with $P' \leftrightarrow_{br} P_0'$, thus, $\mathcal{I}(P) = \mathcal{I}(P_0)$ and, by induction, $Ax_r \vdash \sum_{(\alpha,P')\in I} \alpha.\hat{P}' = \sum_{(\alpha,P_0')\in I_0} \alpha.\hat{P}_0'$.

Let $H := \{(\mathrm{t}, P') \mid \forall X \subseteq A, (\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \Rightarrow \theta_X(P) \leftrightarrow_{br} \theta_X(P'))\}$, listing the outgoing time-outs of P that can be elided. Note that $(\star)$ implies $H \neq \emptyset$. Let $(\mathrm{t}, P') \in H$. Since $P \leftrightarrow_{br} P_0$, for all $X \subseteq A$, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then $\theta_X(P') \leftrightarrow_{br} \theta_X(P_0)$ and $\max(d(\theta_X(P')), d(\theta_X(P_0))) \leq \max(d(P'), d(P_0)) < d(P)$. Therefore, by induction, for all $X \subseteq A$, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then $Ax_r \vdash \mathrm{t}.\widehat{\theta_X(P')} = \mathrm{t}.\widehat{\theta_X(P_0)}$. As a result, using the reactive approximation axiom (RAA), $Ax_r \vdash \sum_{(\alpha,P')\in I} \alpha.\hat{P}' + \sum_{(\mathrm{t},P')\in H} \mathrm{t}.\hat{P}' = \sum_{(\alpha,P_0')\in I_0} \alpha.\hat{P}_0' + \mathrm{t}.\hat{P}_0$, so with Lemma 25

$$Ax_r \vdash \sum_{(\alpha,P')\in I} \alpha.P' + \sum_{(\mathrm{t},P')\in H} \mathrm{t}.P' = \sum_{(\alpha,P_0')\in I_0} \alpha.P' + \mathrm{t}.\hat{P}_0. \tag{2}$$

Let $K := \{(\mathrm{t}, P') \mid \forall X \subseteq A, (\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \Rightarrow \theta_X(P) \not\leftrightarrow_{br} \theta_X(P'))\}$, listing the outgoing time-outs of P that cannot be elided. Let $(\mathrm{t}, P') \in K$. Since $P \leftrightarrow_{br} P_0$, for all $X \subseteq A$, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then there exists a path $P_0 \Longrightarrow P_1^X \xrightarrow{\mathrm{t}} P_2^X \Longrightarrow P_3^X \xrightarrow{\mathrm{t}} \dots \Longrightarrow P_{2r-1}^X \xrightarrow{(\mathrm{t})} P_{2r}^X$ with $r > 0$, such that $Q_1 \not\xrightarrow{\tau}$, $\forall i \in [0, r-1]$, $\theta_X(P_{2i}^X) \leftrightarrow_{br} \theta_X(P) \wedge \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_{2r}^X) \leftrightarrow_{br} \theta_X(P')$. Since P_0 is brb-stable, $P_0 \xrightarrow{(\mathrm{t})} P_{2r}^X$ and, since $\theta_X(P) \not\leftrightarrow_{br} \theta_X(P')$, $P_0 \xrightarrow{\mathrm{t}} P_{2r}^X$ and $\max(d(\theta_X(P')), d(\theta_X(P_{2r}^X))) < n$. As a result, by induction, for all $X \subseteq A$, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ then there exists a transition $P_0 \xrightarrow{\mathrm{t}} P^X$ such that $Ax_r \vdash \mathrm{t}.\widehat{\theta_X(P')} = \mathrm{t}.\widehat{\theta_X(P^X)}$. Therefore, using RAA and Lemma 25,

$$Ax_r \vdash \hat{P}_0 = \hat{P}_0 + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P'. \tag{3}$$

According to Lemma 22, since P is recursion-free and thus strongly guarded, for all (t, P') such that $P \xrightarrow{\mathrm{t}} P'$, either, for all $X \subseteq A$ with $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ we have $\theta_X(P) \leftrightarrow_{br} \theta_X(P')$;

or, for all $X \subseteq A$ with $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ we have $\theta_X(P) \not\leftrightarrow_{br} \theta_X(P')$. As a result,

$$\hat{P} = \sum_{(\alpha,P')\in I} \alpha.P' + \sum_{(\mathrm{t},P')\in H} \mathrm{t}.P' + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P'. \tag{4}$$

Let $\alpha \in Act$. Then, using (4), (2) and (3), respectively, writing R for $\sum_{(\alpha,P_0')\in I_0} \alpha.P_0' + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P'$ and applying the t-branching axiom,

$$\begin{aligned}
Ax_r \vdash \alpha.\hat{P} &= \alpha.(\sum_{(\alpha,P')\in I} \alpha.P' + \sum_{(\mathrm{t},P')\in H} \mathrm{t}.P' + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P') \\
&= \alpha.(\sum_{(\alpha,P_0')\in I_0} \alpha.P_0' + \mathrm{t}.\hat{P}_0 + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P') \\
&= \alpha.(\sum_{(\alpha,P_0')\in I_0} \alpha.P_0' + \mathrm{t}.(\hat{P}_0 + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P') + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P') \\
&= \alpha.(R + \mathrm{t}.(\sum_{(\alpha,P_0')\in I_0} \alpha.\hat{P}_0' + \sum_{\{P_0''|P_0 \xrightarrow{\mathrm{t}} P_0''\}} \mathrm{t}.P_0'' + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P')) \\
&= \alpha.(R + \mathrm{t}.(R + \sum_{\{P_0''|P_0 \xrightarrow{\mathrm{t}} P_0''\}} \mathrm{t}.P_0'')) = \alpha.(R + \sum_{\{P_0''|P_0 \xrightarrow{\mathrm{t}} P_0''\}} \mathrm{t}.P_0'') \\
&= \alpha.(\hat{P}_0 + \sum_{(\mathrm{t},P')\in K} \mathrm{t}.P') = \alpha.\hat{P}_0
\end{aligned}$$

As a result, in any case, for all $\alpha \in Act$, $Ax_r \vdash \alpha.\hat{P} = \alpha.\hat{P}_0$.

Likewise, since Q is recursion-free, a similar brb-stable Q_0 can be defined. By the same reasoning, it can be proved that, for all $\alpha \in Act$, $Ax_r \vdash \alpha.\hat{Q} = \alpha.\hat{Q}_0$. Since P_0 and Q_0 are brb-stable and $P_0 \leftrightarrow_{br} P \leftrightarrow_{br} Q \leftrightarrow_{br} Q_0$, $P_0 \leftrightarrow^r_{br} Q_0$ according to Lemma 48.

To end the proof, it suffices to show that, for all $\alpha \in Act$, $Ax_r \vdash \alpha.\hat{P}_0 = \alpha.\hat{Q}_0$, but we are going to prove the stronger statement $Ax_r \vdash \hat{P}_0 = \hat{Q}_0$. Using RAA, it suffices to prove that, for all $X \subseteq A$, $Ax_r \vdash \psi_X(P_0) = \psi_X(Q_0)$.

Let $(\alpha, P_0') \in I_0$. Since $P_0 \leftrightarrow^r_{br} Q_0$, there exists a transition $Q_0 \xrightarrow{\alpha} Q_0'$ such that $P_0' \leftrightarrow_{br} Q_0'$. By induction, $Ax_r \vdash \alpha.\hat{P}_0' = \alpha.\hat{Q}_0'$.

Let $X \subseteq A$ and (t, P_0') such that $\mathcal{I}(P_0) \cap (X \cup \{\tau\}) = \emptyset$ and $P_0 \xrightarrow{\mathrm{t}} P_0'$. Since $P_0 \leftrightarrow^r_{br} Q_0$, there exists a transition $Q_0 \xrightarrow{\mathrm{t}} Q_0'$ such that $\theta_X(P_0') \leftrightarrow_{br} \theta_X(Q_0')$. By induction, $Ax_r \vdash \mathrm{t}.\widehat{\theta_X(P_0')} = \mathrm{t}.\widehat{\theta_X(Q_0')}$.

Let $X \subseteq A$. If $\mathcal{I}(P_0) \cap (X \cup \{\tau\}) \neq \emptyset$ then $\mathcal{I}(Q_0) \cap (X \cup \{\tau\}) \neq \emptyset$ and, using Lemma 25,

$$\begin{aligned}
Ax_r \vdash \psi_X(Q_0) &= \sum_{\{(\alpha,Q_0')|Q_0 \xrightarrow{\alpha} Q_0' \wedge \alpha \neq \mathrm{t}\}} \alpha.Q_0' \\
&= \sum_{\{(\alpha,Q_0')|Q_0 \xrightarrow{\alpha} Q_0' \wedge \alpha \neq \mathrm{t}\}} \alpha.Q_0' + \sum_{(\alpha,P_0')\in I_0} \alpha.P_0' = \psi_X(P_0 + Q_0)
\end{aligned}$$

If $\mathcal{I}(P_0) \cap (X \cup \{\tau\}) = \emptyset$ then $\mathcal{I}(Q_0) \cap (X \cup \{\tau\}) = \emptyset$ and

$$\begin{aligned}
Ax_r \vdash \psi_X(Q_0) &= \sum_{\{(\alpha,Q_0') \mid Q_0 \xrightarrow{\alpha} Q_0' \wedge \alpha \neq \mathrm{t}\}} \alpha.Q_0' + \sum_{\{(t,Q_0') \mid Q_0 \xrightarrow{\mathrm{t}} Q_0'\}} \mathrm{t}.\theta_X(Q_0') \\
&= \sum_{\{(\alpha,Q_0') \mid Q_0 \xrightarrow{\alpha} Q_0' \wedge \alpha \neq \mathrm{t}\}} \alpha.Q_0' + \sum_{(\alpha,P_0') \in I_0} \alpha.P_0' + \sum_{\{(\mathrm{t},Q_0') \mid Q_0 \xrightarrow{\mathrm{t}} Q_0'\}} \mathrm{t}.\theta_X(Q_0') \\
&\quad + \sum_{\{(\mathrm{t},P_0') \mid P_0 \xrightarrow{\mathrm{t}} P_0'\}} \mathrm{t}.\theta_X(P_0') = \psi_X(P_0 + Q_0)
\end{aligned}$$

As a result, for all $X \subseteq A$, $Ax_r \vdash \psi_X(Q_0) = \psi_X(P_0 + Q_0)$, and so, $Ax_r \vdash Q_0 = P_0 + Q_0$. Symmetrically, $Ax_r \vdash P_0 = P_0 + Q_0$. Therefore, $Ax_r \vdash P_0 = Q_0$.

($\leftrightarrow_{tb}$) We will proceed by induction on $\max(d(P), d(Q))$. Let $n \in \mathbb{N}$ and suppose that the property holds for any recursion-free CCSP$_t^\theta$ processes P, Q such that $\max(d(P), d(Q)) < n$. Let P, Q be two recursion-free CCSP$_t^\theta$ processes such that $\max(d(P), d(Q)) = n$ and $P \leftrightarrow_{tb} Q$.

Since P is recursion-free, there exists a path $P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow P_{2r-1} \xrightarrow{\mathrm{t}} P_{2r} \Longrightarrow P_0$ with $r \in \mathbb{N}$ such that $\forall i \in [0, 2r]$, $P \leftrightarrow_{tb} P_i$ and, for all $P_0 \xrightarrow{\tau/\mathrm{t}} P^\dagger$, $\neg(P_0 \leftrightarrow_{tb} P^\dagger)$. We are going to show that, for all $\alpha \in Act$, $Ax \vdash \alpha.\hat{P} = \alpha.\hat{P}_0$. If $P = P_0$ then it is trivial. Thus, suppose $P_0 \neq P$. Then $d(P_0) < d(P)$ and there exists $P \xrightarrow{\tau/\mathrm{t}} P'$ with $P \leftrightarrow_{tb} P'$.

Let $J := \{(\tau, P') \mid P \xrightarrow{\tau} P' \wedge P \leftrightarrow_{tb} P'\}$, listing the outgoing τ-transitions of P that can be elided w.r.t. $\leftrightarrow_{tb}$. Let $(\tau, P') \in J$. Since $P' \leftrightarrow_{tb} P \leftrightarrow_{tb} P_0$ and $\max(d(P'), d(P_0)) < d(P)$, by induction, $Ax \vdash \tau.\hat{P}' = \tau.\hat{P}_0$.

Let $K := \{(\mathrm{t}, P') \mid P \xrightarrow{\mathrm{t}} P' \wedge P \leftrightarrow_{tb} P'\}$, listing the outgoing time-outs of P that can be elided w.r.t. $\leftrightarrow_{tb}$. Let $(\mathrm{t}, P') \in J$. Since $P' \leftrightarrow_{tb} P \leftrightarrow_{tb} P_0$ and $\max(d(P'), d(P_0)) < d(P)$, by induction, $Ax \vdash \mathrm{t}.\hat{P}' = \mathrm{t}.\hat{P}_0$.

Let $I := \{(\alpha, P') \mid P \xrightarrow{\alpha} P'\} \setminus (J \cup K)$, listing the outgoing transitions of P that cannot be elided. Let $(\alpha, P') \in I$. If $\alpha \in A_\tau$ then, since $P \leftrightarrow_{tb} P_0$, there exists a path $P_0 \Longrightarrow P_1 \xrightarrow{(\alpha)} P_2$ such that $P \leftrightarrow_{tb} P_1$ and $P' \leftrightarrow_{tb} P_2$. Thus, $P_1 \leftrightarrow_{tb} P \leftrightarrow_{tb} P_0$, but, for all $P_0 \xrightarrow{\tau} P^\dagger$, $P_0 \not\leftrightarrow_{tb} P^\dagger$, so $P_0 = P_1$ and $P \xrightarrow{(\alpha)} P_2$. Since $(\alpha, P') \notin J$, $\alpha \in A$ or $P_0 \leftrightarrow_{tb} P \not\leftrightarrow_{tb} P' \leftrightarrow_{tb} P_2$ so $P_0 \xrightarrow{\alpha} P_2$ and $\max(d(P_2), d(P')) < d(P)$. Thus, by induction, $Ax \vdash \alpha.\hat{P}' = \alpha.\hat{P}_2$. If $\alpha = \mathrm{t}$ then, since $P \leftrightarrow_{tb} P_0$, there exists a path $P_0 \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \dots \Longrightarrow P_{2r-1} \xrightarrow{(\mathrm{t})} P_{2r}$ with $r > 0$, such that, for all $i \in [0, 2r-1]$, $P \leftrightarrow_{tb} P_i$ and $P' \leftrightarrow_{tb} P_{2r}$. Thus, $P_{2r-1} \leftrightarrow_{tb} P \leftrightarrow_{tb} P_0$, but, for all $P_0 \xrightarrow{\tau/t} P^\dagger$, $\neg(P_0 \leftrightarrow_{tb} P^\dagger)$, therefore, $P_0 = P_{2r-1}$ and $P_0 \xrightarrow{(\mathrm{t})} P_{2r}$. Since $(\mathrm{t}, P') \notin K$, $P_0 \leftrightarrow_{tb} P \not\leftrightarrow_{tb} P' \leftrightarrow_{tb} P_2$ so $P_0 \xrightarrow{\mathrm{t}} P_2$ and $\max(d(P_2), d(P')) < d(P)$. Thus, by induction, $Ax \vdash \mathrm{t}.\hat{P}' = \mathrm{t}.\hat{P}_2$. As a result, $Ax \vdash \hat{P}_0 + \sum_{(\alpha,P') \in I} \alpha.\hat{P}' = \hat{P}_0$.

Since there exists $P \xrightarrow{\tau/\mathrm{t}} P'$ with $P \leftrightarrow_{tb} P'$, $J \cup K \neq \emptyset$. We are going to perform a case distinction on the emptiness of J and K.

- If $J \neq \emptyset$ and $K \neq \emptyset$ then, employing the τ/t-branching axiom,

$$\begin{aligned}
Ax \vdash \alpha.\hat{P} &= \alpha.(\sum_{(\tau,P') \in J} \tau.\hat{P}' + \sum_{(t,P') \in K} \mathrm{t}.\hat{P}' + \sum_{(\alpha,P') \in I} \alpha.\hat{P}') \\
&= \alpha.(\tau.\hat{P}_0 + \mathrm{t}.\hat{P}_0 + \sum_{(\alpha,P') \in I} \alpha.\hat{P}') \\
&= \alpha.(\tau.(\hat{P}_0 + \sum_{(\alpha,P') \in I} \alpha.\hat{P}') + \mathrm{t}.(\hat{P}_0 + \sum_{(\alpha,P') \in I} \alpha.\hat{P}') + \sum_{(\alpha,P') \in I} \alpha.\hat{P}') = \alpha.\hat{P}_0
\end{aligned}$$

- If $J \neq \emptyset$ and $K = \emptyset$ then, employing the branching axiom,

$$Ax \vdash \alpha.\hat{P} = \alpha.(\sum_{(\tau,P')\in J} \tau.\hat{P}' + \sum_{(t,P')\in K} \mathrm{t}.\hat{P}' + \sum_{(\alpha,P')\in I} \alpha.\hat{P}') = \alpha.(\tau.\hat{P}_0 + \sum_{(\alpha,P')\in I} \alpha.\hat{P}')$$
$$= \alpha.(\tau.(\hat{P}_0 + \sum_{(\alpha,P')\in I} \alpha.\hat{P}') + \sum_{(\alpha,P')\in I} \alpha.\hat{P}') = \alpha.\hat{P}_0$$

- If $J = \emptyset$ and $K \neq \emptyset$ then $\{(\alpha, P') \mid P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau\} \subseteq I$. Let (α, P_2) such that $\alpha \in A_\tau$ and $P_0 \xrightarrow{\alpha} P_2$. Since $P \leftrightarrow_{tb} P_0$, there exists a path $P \Longrightarrow P' \xrightarrow{(\alpha)} P''$ such that $P' \leftrightarrow_{tb} P_0$ and $P'' \leftrightarrow_{tb} P_2$. Since $J = \emptyset$, $P = P'$. If $\alpha = \tau$ and $P = P''$ then $P_0 \leftrightarrow_{tb} P' \leftrightarrow_{tb} P_2$ and $P_0 \xrightarrow{\tau} P_2$, but that contradicts the definition of P_0. Thus, $P \xrightarrow{\alpha} P_2$. Therefore, by induction, for all $(\alpha, P_2) \in \{(\alpha, P_2) \mid P_0 \xrightarrow{\alpha} P_2 \wedge \alpha \in A_\tau\}$, there exists $(\alpha, P') \in \{(\alpha, P') \mid P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau\}$ such that $Ax \vdash \alpha.\hat{P}_2 = \alpha.\hat{P}'$. Symmetrically, for all $(\alpha, P') \in \{(\alpha, P') \mid P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau\}$, there exists $(\alpha, P_2) \in \{(\alpha, P_2) \mid P_0 \xrightarrow{\alpha} P_2 \wedge \alpha \in A_\tau\}$ such that $Ax \vdash \alpha.\hat{P}_2 = \alpha.\hat{P}'$. As a result,

$$Ax \vdash \sum_{\{(\alpha,P')\mid P\xrightarrow{\alpha}P'\wedge\alpha\in A_\tau\}} \alpha.\hat{P}' = \sum_{\{(\alpha,P_2)\mid P_0\xrightarrow{\alpha}P_2\wedge\alpha\in A_\tau\}} \alpha.\hat{P}_2.$$

Moreover, the reasoning that yields $Ax \vdash \hat{P}_0 + \sum_{(\alpha,P')\in I} \alpha.\hat{P}' = \hat{P}_0$ can be used to get $Ax \vdash \hat{P}_0 + \sum_{(\mathrm{t},P')\in I} \mathrm{t}.\hat{P}' = \hat{P}_0$. Thus, writing R for $\sum_{\{(\alpha,P_2)\mid P_0\xrightarrow{\alpha}P_2\wedge\alpha\in A_\tau\}} \alpha.\hat{P}_2 + \sum_{(\mathrm{t},P')\in I} \mathrm{t}.\hat{P}'$, and using the t-branching axiom,

$$Ax \vdash \alpha.\hat{P} = \alpha.(\sum_{(\tau,P')\in J} \tau.\hat{P}' + \sum_{(t,P')\in K} \mathrm{t}.\hat{P}' + \sum_{(\alpha,P')\in I} \alpha.\hat{P}')$$
$$= \alpha.(\mathrm{t}.\hat{P}_0 + \sum_{\{(\alpha,P')\mid P\xrightarrow{\alpha}P'\wedge\alpha\in A_\tau\}} \alpha.\hat{P}' + \sum_{(\mathrm{t},P')\in I} \mathrm{t}.\hat{P}')$$
$$= \alpha.(\mathrm{t}.(\hat{P}_0 + \sum_{(\mathrm{t},P')\in I} \mathrm{t}.\hat{P}') + \sum_{\{(\alpha,P_2)\mid P_0\xrightarrow{\alpha}P_2\wedge\alpha\in A_\tau\}} \alpha.\hat{P}_2 + \sum_{(\mathrm{t},P')\in I} \mathrm{t}.\hat{P}')$$
$$= \alpha.(\mathrm{t}.(R + \sum_{P_0\xrightarrow{\mathrm{t}}P_0'} \mathrm{t}.P_0') + R) = \alpha.(R + \sum_{P_0\xrightarrow{\mathrm{t}}P_0'} \mathrm{t}.P_0')$$
$$= \alpha.(\hat{P}_0 + \sum_{(\mathrm{t},P')\in I} \mathrm{t}.\hat{P}') = \alpha.\hat{P}_0$$

As a result, in any case, for all $\alpha \in Act$, $Ax \vdash \alpha.\hat{P} = \alpha.\hat{P}_0$. Similarly, since Q is recursion-free, there exists a recursion-free CCSP_t^θ process Q_0 such that $Q(\xrightarrow{\tau/\mathrm{t}})^* Q_0$, $Q \leftrightarrow_{tb} Q_0$ and, for all $Q_0 \xrightarrow{\tau/\mathrm{t}} Q^\dagger$, $\neg(Q_0 \leftrightarrow_{tb} Q^\dagger)$. Moreover, for all $\alpha \in Act$, $Ax \vdash \alpha.\hat{Q} = \alpha.\hat{Q}_0$. Notice that $P_0 \leftrightarrow_{tb} P \leftrightarrow_{tb} Q \leftrightarrow_{tb} Q_0$ and, since, for all $Q_0 \xrightarrow{\tau/\mathrm{t}} Q^\dagger$, $\neg(Q_0 \leftrightarrow_{tb} Q^\dagger)$ and, for all $P_0 \xrightarrow{\tau/\mathrm{t}} P^\dagger$, $\neg(P_0 \leftrightarrow_{tb} P^\dagger)$, $P_0 \leftrightarrow_{tb}^r Q_0$.

Let (α, P_0') such that $P_0 \xrightarrow{\alpha} P_0'$. Since $P_0 \leftrightarrow_{tb}^r Q_0$, there exists a path $Q_0 \xrightarrow{\alpha} Q_2$ such that $P_0' \leftrightarrow_{tb} Q_2$. Since $\max(d(P_0'), d(Q_2)) < n$, by induction, $Ax \vdash \alpha.P_0' = \alpha.Q_2$. As a result, $Ax \vdash \hat{P}_0 + \hat{Q}_0 = \hat{Q}_0$. Symmetrically, $Ax \vdash \hat{P}_0 + \hat{Q}_0 = \hat{P}_0$, and so, $Ax \vdash \hat{P}_0 = \hat{Q}_0$. Finally, for all $\alpha \in Act$, $Ax \vdash \alpha.\hat{P} = \alpha.\hat{Q}$. ◀

Proof of Theorem 27. Let $P, Q \in \mathbb{P}$ be two recursion-free CCSP_t^θ processes. Let $P \leftrightarrow_{br}^r Q$. P and Q can be equated in the same manner as P_0 and Q_0 in the proof of Proposition 26.

Suppose that $P \leftrightarroweq^r_{tb} Q$. Let (α, P') such that $P \xrightarrow{\alpha} P'$ with $\alpha \in Act$. Since $P \leftrightarroweq^r_{tb} Q$, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarroweq_{tb} Q'$. According to the previous proposition, $Ax \vdash \alpha.P' = \alpha.Q'$, thus,

$$Ax \vdash Q = \sum_{\{(\alpha,Q')|Q \xrightarrow{\alpha} Q'\}} \alpha.Q' = \sum_{\{(\alpha,Q')|Q \xrightarrow{\alpha} Q'\}} \alpha.Q' + \sum_{\{(\alpha,P')|P \xrightarrow{\alpha} P'\}} \alpha.P' = Q + P$$

As a result, $Ax \vdash Q = P + Q$. Symmetrically, $Ax \vdash P = P + Q$. Therefore, $Ax \vdash P = Q$. ◀

N Proof of Completeness by Equation Merging

Proof of Theorem 28. Let E_0 and F_0 two strongly guarded CCSP$^\theta_t$ processes such that $E_0 \leftrightarroweq^r_{tb} F_0$. We are going to build a recursive specification $\mathcal{S}$ such that E_0 and F_0 will be components of solutions of $\mathcal{S}$ in the same variable. Let $\mathcal{E}_{E_0}$ (resp. $\mathcal{E}_{F_0}$) be the set of reachable expressions from E_0 (resp. F_0). Let $V_\mathcal{S}$ be a set of new variables $\{x_{EF} \mid (E, F) \in \mathcal{E}_{E_0} \times \mathcal{E}_{F_0} \wedge E \leftrightarroweq_{tb} F\}$. We denote $x_0 = x_{E_0F_0} \in V_\mathcal{S}$ and we define the following set of equations $\mathcal{S}$, for all $x_{EF} \in V_\mathcal{S}$, with $\alpha \in A \cup \{\tau, \mathrm{t}\}$.

$$\begin{aligned}\mathcal{S}_{x_{EF}} := & \sum_{E \xrightarrow{\alpha} E', F \xrightarrow{\alpha} F', E' \leftrightarroweq_{tb} F'} \alpha.x_{E'F'} + \sum_{E \xrightarrow{\tau} E', E' \leftrightarroweq_{tb} F, x_{EF} \neq x_0} \tau.x_{E'F} \\ & + \sum_{F \xrightarrow{\tau} F', E \leftrightarroweq_{tb} F', x_{EF} \neq x_0} \tau.x_{EF'} + \sum_{E \xrightarrow{\mathrm{t}} E', E' \leftrightarroweq_{tb} F, x_{EF} \neq x_0} \mathrm{t}.x_{E'F} + \sum_{F \xrightarrow{\mathrm{t}} F', E \leftrightarroweq_{tb} F', x_{EF} \neq x_0} \mathrm{t}.x_{EF'}\end{aligned}$$

Note that $\mathcal{S}$ is well-guarded since E_0 and F_0 are strongly guarded CCSP$^\theta_t$ processes. For $x_{EF} \in V_\mathcal{S}$, we define $H_{EF}, G_{EF} \in \mathbb{E}$ such that

$$H_{EF} := \sum_{E \xrightarrow{\alpha} E', F \xrightarrow{\alpha} F', E' \leftrightarroweq_{tb} F'} \alpha.E' + \sum_{E \xrightarrow{\tau} E', E' \leftrightarroweq_{tb} F, x_{EF} \neq x_0} \tau.E' + \sum_{E \xrightarrow{\mathrm{t}} E', E' \leftrightarroweq_{tb} F, x_{EF} \neq x_0} \mathrm{t}.E'$$

$$G_{EF} := \begin{cases} H_{EF} + \tau.E + \mathrm{t}.E & \text{if } x_0 \neq x_{EF}, \exists F \xrightarrow{\tau} F', E \leftrightarroweq_{tb} F' \text{ and } \exists F \xrightarrow{\mathrm{t}} F', E \leftrightarroweq_{tb} F' \\ H_{EF} + \tau.E & \text{if } x_0 \neq x_{EF}, \exists F \xrightarrow{\tau} F', E \leftrightarroweq_{tb} F' \text{ and } \forall F \xrightarrow{\mathrm{t}} F', E \not\leftrightarroweq_{tb} F' \\ H_{EF} + \mathrm{t}.E & \text{if } x_0 \neq x_{EF}, \forall F \xrightarrow{\tau} F', E \not\leftrightarroweq_{tb} F' \text{ and } \exists F \xrightarrow{\mathrm{t}} F', E \leftrightarroweq_{tb} F' \\ E & \text{otherwise} \end{cases}$$

According to Lemma 25, for all $(E, F) \in \mathcal{E}_{E_0} \times \mathcal{E}_{F_0}$, $Ax^\infty \vdash E + H_{EF} = (\widehat{E + H_{EF}}) = \hat{E} = E$. Let $(E, F) \in \mathcal{E}_{E_0} \times \mathcal{E}_{F_0}$.

- If $x_0 \neq x_{EF}$ and $\exists F \xrightarrow{\tau} F', E \leftrightarroweq_{tb} F'$ and $\exists F \xrightarrow{\mathrm{t}} F', E \leftrightarroweq_{tb} F'$ then, for all $\alpha \in Act$, $Ax^\infty \vdash \alpha.G_{EF} = \alpha.(H_{EF} + \tau.E + \mathrm{t}.E) = \alpha.E$ using the τ/t-branching axiom.
- If $x_0 \neq x_{EF}$ and $\exists F \xrightarrow{\tau} F', E \leftrightarroweq_{tb} F'$ and $\forall F \xrightarrow{\mathrm{t}} F', E \not\leftrightarroweq_{tb} F'$ then, for all $\alpha \in Act$, $Ax^\infty \vdash \alpha.G_{EF} = \alpha.(H_{EF} + \tau.E) = \alpha.E$ using the branching axiom.
- If $x_0 \neq x_{EF}$ and $\forall F \xrightarrow{\tau} F', E \not\leftrightarroweq_{tb} F'$ and $\exists F \xrightarrow{\mathrm{t}} F', E \leftrightarroweq_{tb} F'$ then, for all $\alpha \in Act$, $Ax^\infty \vdash \alpha.G_{EF} = \alpha.(H_{EF} + \mathrm{t}.E)$. Let (α, E') such that $E \xrightarrow{\alpha} E'$ with $\alpha \in A_\tau$. Since $E \leftrightarroweq_{tb} F$, there exists a path $F \Longrightarrow F_1 \xrightarrow{(\alpha)} F_2$ such that $E \leftrightarroweq_{tb} F_1$ and $E' \leftrightarroweq_{tb} F_2$. Since $\forall F \xrightarrow{\tau} F', E \not\leftrightarroweq_{tb} F'$, $F = F_1$ and $F \xrightarrow{(\alpha)} F_2$. If $\alpha = \tau$ and $F = F_2$ then $E \xrightarrow{\tau} E'$ and $E' \leftrightarroweq_{tb} F$ so $H_{EF} \xrightarrow{\tau} E'$; else $F \xrightarrow{\alpha} F_2$ and $E' \leftrightarroweq_{tb} F'$ so $H_{EF} \xrightarrow{\alpha} E'$. Therefore, $Ax^\infty \vdash H_{EF} + \sum_{\{(\alpha,E')|E \xrightarrow{\alpha} E' \wedge \alpha \in A_\tau\}} \alpha.E' = H_{EF}$. Consequently, $Ax^\infty \vdash E = H_{EF} + E = H_{EF} + \sum_{\{E'|E \xrightarrow{\mathrm{t}} E'\}} \mathrm{t}.E'$. As a result, the t-branching axiom can be applied and so, for all $\alpha \in Act$, $Ax^\infty \vdash \alpha.G_{EF} = \alpha.(H_{EF} + \mathrm{t}.E) = \alpha.E$.

In any case, for all $\alpha \in Act$, $Ax^\infty \vdash \alpha.G_{EF} = \alpha.E$. $(*)$

If we prove that the family $(G_{EF})_{(E,F)\in\mathcal{E}_{E_0}\times\mathcal{E}_{F_0}}$ is a solution of $\mathcal{S}$ then, by definition of $G_{E_0F_0}$, there would exist a solution whose value for the variable x_0 is E. According to $(*)$, we need to prove that, for all $x_{EF} \in V_\mathcal{S}$,

$$\begin{aligned}
Ax^\infty \vdash G_{EF} = & \sum_{E\xrightarrow{\alpha}E',F\xrightarrow{\alpha}F',E'\leftrightarroweq_{tb}F'} \alpha.G_{E'F'} \\
& + \sum_{E\xrightarrow{\tau}E',E'\leftrightarroweq_{tb}F,x_{EF}\neq x_0} \tau.G_{E'F} + \sum_{F\xrightarrow{\tau}F',E\leftrightarroweq_{tb}F',x_{EF}\neq x_0} \tau.G_{EF'} \\
& + \sum_{E\xrightarrow{\mathrm{t}}E',E'\leftrightarroweq_{tb}F,x_{EF}\neq x_0} \mathrm{t}.G_{E'F} + \sum_{F\xrightarrow{\mathrm{t}}F',E\leftrightarroweq_{tb}F',x_{EF}\neq x_0} \mathrm{t}.G_{EF'} \\
= & \sum_{E\xrightarrow{\alpha}E',F\xrightarrow{\alpha}F',E'\leftrightarroweq_{tb}F'} \alpha.E' \\
& + \sum_{E\xrightarrow{\tau}E',E'\leftrightarroweq_{tb}F,x_{EF}\neq x_0} \tau.E' + \sum_{F\xrightarrow{\tau}F',E\leftrightarroweq_{tb}F',x_{EF}\neq x_0} \tau.E \\
& + \sum_{E\xrightarrow{\mathrm{t}}E',E'\leftrightarroweq_{tb}F,x_{EF}\neq x_0} \mathrm{t}.E' + \sum_{F\xrightarrow{\mathrm{t}}F',E\leftrightarroweq_{tb}F',x_{EF}\neq x_0} \mathrm{t}.E \\
= & \; H_{EF} + \sum_{x_{EF}\neq x_0,F\xrightarrow{\tau}F',E\leftrightarroweq_{tb}F'} \tau.E + \sum_{F\xrightarrow{\mathrm{t}}F',E\leftrightarroweq_{tb}F',x_{EF}\neq x_0} \mathrm{t}.E
\end{aligned}$$

- If $x_{EF} \neq x_0$ and $(\exists F \xrightarrow{\tau} F', E \leftrightarroweq_{tb} F') \vee (\exists F \xrightarrow{\mathrm{t}} F', E \leftrightarroweq_{tb} F')$ then this follows from the definition of G_{EF}.
- If $x_{EF} \neq x_0$ and $\forall F \xrightarrow{\tau} F', E \not\leftrightarroweq_{tb} F'$ and $\forall F \xrightarrow{\mathrm{t}} F', E \not\leftrightarroweq_{tb} F'$ then, by definition of G_{EF}, we have to prove $Ax^\infty \vdash E = H_{EF}$. Let (α, E') such that $E \xrightarrow{\alpha} E'$ and $\alpha \in A_\tau$. Since $E \leftrightarroweq_{tb} F$, there exists a path $F \Longrightarrow F_1 \xrightarrow{(\alpha)} F_2$ such that $E \leftrightarroweq_{tb} F_1$ and $E' \leftrightarroweq_{tb} F_2$. Since $\forall F \xrightarrow{\tau} F', \neg(E \leftrightarroweq_{tb} F')$, $F = F_1$, so there exists a transition $F \xrightarrow{(\alpha)} F_2$ such that either $F \xrightarrow{\alpha} F_2$ and $E' \leftrightarroweq_{tb} F_2$, or $\alpha = \tau$ and $E' \leftrightarroweq_{tb} F$. In either case, $H_{EF} \xrightarrow{\alpha} E'$.
  Let (t, E') such that $E \xrightarrow{\mathrm{t}} E'$. Since $E \leftrightarroweq_{tb} F$, there exists a path $F \Longrightarrow F_1 \xrightarrow{\mathrm{t}} F_2 \Longrightarrow F_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow F_{2r-1} \xrightarrow{(\mathrm{t})} F_{2r}$ with $r > 0$, such that, for all $i \in [0, 2r-1]$, $E \leftrightarroweq_{tb} F_i$ and $E' \leftrightarroweq_{tb} F_{2r}$. Since $\forall F \xrightarrow{\tau} F', E \not\leftrightarroweq_{tb} F'$ and $\forall F \xrightarrow{\mathrm{t}} F', E \not\leftrightarroweq_{tb} F'$, $F = F_{2r-1}$, so $F \xrightarrow{(\mathrm{t})} F_{2r}$. Thus either there exists a transition $F \xrightarrow{\mathrm{t}} F_{2r}$ such that $E' \leftrightarroweq_{tb} F_{2r}$ or $E' \leftrightarroweq_{tb} F$. In either case, $H_{EF} \xrightarrow{\mathrm{t}} E'$.
  As a result, $Ax^\infty \vdash E = \hat{E} = \widehat{E + H_{EF}} = \hat{H}_{EF} = H_{EF}$.
- If $x_{EF} = x_0$ then $E = E_0$, $F = F_0$ and we have to show that $Ax^\infty \vdash E_0 = H_{E_0F_0} = \sum_{E_0\xrightarrow{\alpha}E',F_0\xrightarrow{\alpha}F',E'\leftrightarroweq_{tb}F'} \alpha.E'$. Let (α, E') such that $E_0 \xrightarrow{\alpha} E'$. Since $E_0 \leftrightarroweq^r_{tb} F_0$, there exists a transition $F_0 \xrightarrow{\alpha} F'$ such that $E' \leftrightarroweq_{tb} F'$. $Ax^\infty \vdash E = \hat{E} = \hat{H}_{EF} = H_{EF}$. ◀

Note that we could define H'_{EF} and G'_{EF} by reverting the role of E and F and also get a solution whose value for the variable x_0 is F_0. Consequently, RSP yields $Ax^\infty \vdash E_0 = F_0$.

O The Canonical Representative

We are going to start by proving some lemmas facilitating the handling of classes.

▶ **Lemma 49.** *Let* $P \in \mathbb{P}^g$.

1. $\forall \alpha \in A_\tau, ([P] \xrightarrow{\alpha} R' \Leftrightarrow \exists P \Longrightarrow P_1 \xrightarrow{\alpha} P_2, (P_1, P_2) \in [P] \times R' \wedge (\alpha \in A \vee [P] \neq R'))$.
2. $[P] \not\xrightarrow{\tau} \Leftrightarrow \exists P_0 \in [P], P \Longrightarrow P_0 \not\xrightarrow{\tau}$.
3. *Let* $X \subseteq A$. *Then* $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset \Leftrightarrow \exists P \Longrightarrow P_0, P_0 \in [P] \wedge \mathcal{I}(P_0) \cap (X \cup \{\tau\}) = \emptyset$.
4. *If* $[P] \xrightarrow{\mathrm{t}} R'$ *and* $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$ *then* $\exists r > 0$, $\exists P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{\mathrm{t}} P_{2r}$, $P_1 \in [P] \wedge \forall i \in [0, 2r{-}1]$, $\theta_X(P_i) \in [\theta_X(P)] \wedge \forall j \in [0, r{-}1]$, $\mathcal{I}(P_{2j+1}) \cap (X \cup \{\tau\}) = \emptyset \wedge \theta_X(P_{2r}) \in [\theta_X(\chi(R'))] \wedge [\theta_X(P)] \neq [\theta_X(\chi(R'))]$.
5. *If* $\exists X \subseteq A, r > 0$, $\exists P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{\mathrm{t}} P_{2r}, P_1 \in [P] \wedge \forall i \in [0, r{-}1]$, $\theta_X(P_{2i}) \in [\theta_X(P)] \wedge \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset \wedge \theta_X(P_{2r}) \notin [\theta_X(P)]$ *then there exists an* R' *with* $[P] \xrightarrow{\mathrm{t}} R' \wedge \theta_X(P_{2r}) \in [\theta_X(\chi(R'))]$.

Proof. Let $P \in \mathbb{P}^g$.

1. Let $\alpha \in A_\tau$.
   - If $[P] \xrightarrow{\alpha} R'$ then, by definition of $\rightarrow$, there exists a path $\chi([P]) \Longrightarrow P_1 \xrightarrow{\alpha} P_2$ such that $P_1 \in [P]$, $P_2 \in R'$ and $\alpha \in A \vee [P] \neq R'$. Since $\chi([P]) \leftrightarrow_{br} P$, there exists a path $P \Longrightarrow P_1' \xrightarrow{(\alpha)} P_2'$ such that $P_1 \leftrightarrow_{br} P_1'$ and $P_2 \leftrightarrow_{br} P_2'$, thus, $P_1' \in [P]$ and $P_2' \in R'$. If $\alpha = \tau$ then $[P] \neq R'$, so $P_1' \not\leftrightarrow_{br} P_2'$ and so $P_1' \xrightarrow{\alpha} P_2$, otherwise, $P_1' \xrightarrow{\alpha} P_2'$.
   - If there exists a path $P \Longrightarrow P_1 \xrightarrow{\alpha} P_2$ such that $P_1 \in [P]$, $P_2 \in R'$ and $\alpha \in A \vee [P] \neq R'$ then, since $P \leftrightarrow_{br} \chi([P])$, there exists a path $\chi([P]) \Longrightarrow P_1' \xrightarrow{(\alpha)} P_2'$ such that $P_1 \leftrightarrow_{br} P_1'$ and $P_2 \leftrightarrow_{br} P_2'$, thus, $P_1' \in [P]$ and $P_2' \in R'$. If $\alpha = \tau$ then $[P] \neq R'$, therefore, $P_1' \not\leftrightarrow_{br} P_2'$ and so $P_1' \xrightarrow{\alpha} P_2'$, otherwise, $P_1' \xrightarrow{\alpha} P_2'$. By definition of $\rightarrow$, $[P] \xrightarrow{\alpha} R'$.
2. - If $[P] \xrightarrow{\tau}$ then, according to the previous point, there exists a path $P \Longrightarrow P_1 \xrightarrow{\tau} P_2$ such that $P_1 \in [P]$ and $P_2 \notin [P]$. Suppose that there is a path $P \Longrightarrow P_0 \not\xrightarrow{\tau}$ with $P_0 \in [P]$. Then $P_0 \leftrightarrow_{br} P$, so there exists a path $P_0 \Longrightarrow P^\dagger \xrightarrow{(\tau)} P^\ddagger$ such that $P_1 \leftrightarrow_{br} P^\dagger$ and $P_2 \leftrightarrow_{br} P^\ddagger$. Since $P_0 \not\xrightarrow{\tau}$, $P_0 = P^\ddagger \leftrightarrow_{br} P_2 \notin [P]$, but that's impossible.
   - Suppose that, for all paths $P \Longrightarrow P_0 \not\xrightarrow{\tau}$, $P_0 \notin [P]$. Since P is strongly guarded, there exists a path $P \Longrightarrow P_1$ such that $P_1 \in [P]$ and, for all $P_1 \xrightarrow{\tau} P'$, $P_1 \not\leftrightarrow_{br} P'$. Since $P \Longrightarrow P_1$ and $P_1 \in [P]$, there exists a transition $P_1 \xrightarrow{\tau} P_2$, and $P_1 \not\leftrightarrow_{br} P_2$. Thus, there exists a path $P \Longrightarrow P_1 \xrightarrow{\tau} P_2$ such that $P_1 \in [P]$ and $P_2 \notin [P]$. According to the previous point, $[P] \xrightarrow{\tau} [P_2]$.
3. This a corollary of the two previous points.
4. Suppose $[P] \xrightarrow{\mathrm{t}} R'$ and $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$. By definition of $\rightarrow$, there exists $Z \subseteq A$, $r > 0$ and a path $\chi([P]) = P_0 \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{\mathrm{t}} P_{2r}$ such that $P_1 \in [P]$, $P_{2r} \in R'$, $[\theta_Z(\chi([P]))] \neq [\theta_Z(\chi(R'))]$, and for all $i \in [0, r{-}1]$, $\theta_Z(P_{2i}) \in [\theta_Z(\chi([P]))]$ and $\mathcal{I}(P_{2i+1}) \cap (Z \cup \{\tau\}) = \emptyset$. For all $i \in [0, r{-}1]$, considering that $P_1 \leftrightarrow_{br} \chi([P]) \leftrightarrow_{br}^Z P_{2i}$, Lemma 2.1 gives $P_1 \leftrightarrow_{br}^Z P_{2i+1}$, and by Lemma 2.3 $P_1 \leftrightarrow_{br} P_{2i+1}$, i.e., $P_{2i+1} \in [P]$.
   By Statement 3 of this lemma, $\exists P \Longrightarrow P', P' \in [P] \wedge \mathcal{I}(P') \cap (X \cup \{\tau\}) = \emptyset$. According to Lemma 2.2, for $i \in [0, r{-}1]$, since $P' \leftrightarrow_{br} P_{2i+1}$, one obtains $\mathcal{I}([P_{2i+1}]) \cap (X \cup \{\tau\}) = \emptyset$. For all $i \in [1, r{-}1]$, since $P_{2i-1} \xrightarrow{\mathrm{t}} P_{2i}$, $\mathcal{I}(P_{2i-1}) \cap (Z \cup \{\tau\}) = \emptyset$, $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_Z(P_{2i-1}) \leftrightarrow_{br} \theta_Z(P_{2i})$, Lemma 22 yields $\theta_X(P_{2i-1}) \leftrightarrow_{br} \theta_X(P_{2i})$, i.e., $\theta_X(P_{2i}) \in [\theta_X(\chi([P]))]$. Moreover, $[\theta_X(P)] \neq [\theta_X(\chi(R'))]$, i.e., $\theta_X(P_{2r-1}) \not\leftrightarrow_{br} \theta_X(P_{2r})$, for if we had $\theta_X(P_{2r-1}) \leftrightarrow_{br} \theta_X(P_{2r})$ then the same reasoning would yield $\theta_Z(P_{2r-1}) \leftrightarrow_{br} \theta_Z(P_{2r})$, contradicting that $[\theta_Z(\chi([P]))] \neq [\theta_Z(\chi(R'))]$.
   Since $\chi([P]) \leftrightarrow_{br} P$, $[\theta_X(\chi([P]))] = [\theta_X(P)]$ and there exists a path $P = P_0' \Longrightarrow P_1' \xrightarrow{\mathrm{t}} P_2' \Longrightarrow P_3' \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2k-1}' \xrightarrow{(\mathrm{t})} P_{2k}'$ with $k > 0$, such that $P_1 \leftrightarrow_{br} P_1'$, $\forall j \in [0, k{-}1], \exists i \in [0, 2r{-}1]$, $\theta_X(P_i) \leftrightarrow_{br} \theta_X(P_{2j}') \wedge \mathcal{I}(P_{2j+1}') \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_{2r}) \leftrightarrow_{br} \theta_X(P_{2k}')$. With Lemma 2.1 we even have $\theta_X(P) \leftrightarrow_{br} \theta_X(\chi([P])) \leftrightarrow_{br} \theta_X(P_i) \leftrightarrow_{br} \theta_X(P_{2j}') \leftrightarrow_{br} \theta_X(P_{2j+1}')$. As a result, $P_1' \in [P] \wedge \forall i \in [0, 2k{-}1]$, $\theta_X(P_i') \in [\theta_X(P)] \wedge \forall j \in [0, k{-}1]$, $\mathcal{I}(P_{2j+1}') \cap (X \cup$

$\{\tau\}) = \emptyset \wedge \theta_X(P'_{2r}) \in [\theta_X(\chi(R'))] \wedge [\theta_X(P)] \neq [\theta_X(\chi(R'))]$. Since $[\theta_X(P)] \neq [\theta_X(\chi(R'))]$, $P'_{2k-1} \xrightarrow{\mathrm{t}} P'_{2k}$.

5. Suppose there exists $X \subseteq A$ and a path $P \Longrightarrow P_1 \xrightarrow{\mathrm{t}} P_2 \Longrightarrow P_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P_{2r-1} \xrightarrow{\mathrm{t}} P_{2r}$ such that $P_1 \in [P]$, $\forall i \in [0, r-1]$, $\theta_X(P_{2i}) \in [\theta_X(P)] \wedge \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_{2r}) \notin [\theta_X(P)]$. Since $P \leftrightarrow_{br} \chi([P])$, $[\theta_X(P)] = [\theta_X(\chi([P]))]$ and there exists a path $\chi([P]) \Longrightarrow P'_1 \xrightarrow{\mathrm{t}} P'_2 \Longrightarrow P'_3 \xrightarrow{\mathrm{t}} \ldots \Longrightarrow P'_{2k-1} \xrightarrow{(\mathrm{t})} P'_{2k}$ such that $P_1 \leftrightarrow_{br} P'_1$, $\forall j \in [0, k-1]$, $\exists i \in [0, 2r-1]$, $\theta_X(P_i) \leftrightarrow_{br} \theta_X(P'_{2j}) \wedge \mathcal{I}(P'_{2j+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_{2r}) \leftrightarrow_{br} \theta_X(P'_{2k})$. With Lemma 2.1 we even have $\theta_X(\chi([P])) \leftrightarrow_{br} \theta_X(P)) \leftrightarrow_{br} \theta_X(P_i) \leftrightarrow_{br} \theta_X(P'_{2j}) \leftrightarrow_{br} \theta_X(P'_{2j+1})$. As a a result, $P'_1 \in [P]$, $\forall j \in [0, k-1], \theta_X(P'_{2j}) \in [\theta_X(\chi([P]))]$. Notice that $\theta_X(P'_{2k}) \in [\theta_X(\chi([P'_{2k}]))]$ and so $[\theta_X(\chi([P]))] \neq [\theta_X(\chi([P'_{2k}]))]$ since $\theta_X(P_{2r}) \notin [\theta_X(\chi([P]))]$ but $\theta_X(P_{2r}) \in [\theta_X(\chi([P'_{2k}]))]$. Since $\theta_X(P'_{2k}) \notin [\theta_X(\chi([P]))]$, $P'_{2k-1} \xrightarrow{\mathrm{t}} P'_{2k}$. Therefore, by definition of $\rightarrow$, $[P] \xrightarrow{\mathrm{t}} [P'_{2k}]$ and $\theta_X(P_{2r}) \in [\theta_X(\chi([P'_{2k}]))]$. ◀

▶ **Corollary 50.** *Let $P \in \mathbb{P}^g$ and $X \subseteq A$.*

1. *If $[P] \Longrightarrow R'$ and $\mathcal{I}(R') \cap (X \cup \{\tau\}) = \emptyset$ then $\exists P \Longrightarrow P' \in R'$ with $\mathcal{I}(P') \cap (X \cup \{\tau\}) = \emptyset$.*
2. *If $\theta_X(P) \in [\theta_X(\chi(R))]$, $R \Longrightarrow R'$ and $\mathcal{I}(R') \cap (X \cup \{\tau\}) = \emptyset$ then $\exists P \Longrightarrow P' \in R'$ with $\mathcal{I}(P') \cap (X \cup \{\tau\}) = \emptyset$.*

Proof. The first statement follows directly from Lemma 49.1–3. For the second, suppose $\theta_X(P) \in [\theta_X(\chi(R))]$, $R \Longrightarrow R'$ and $\mathcal{I}(R') \cap (X \cup \{\tau\}) = \emptyset$. By the first statement, $\chi(R) \Longrightarrow Q' \in R'$ for some Q' with $\mathcal{I}(Q') \cap (X \cup \{\tau\}) = \emptyset$. By the semantics of θ_X, there is a path $\theta_X(\chi(R)) \Longrightarrow \theta_X(Q') \stackrel{\tau}{\not\rightarrow}$. Since $\theta_X(P) \leftrightarrow_{br} \theta_X(\chi(R))$, there is a path $\theta_X(P) \Longrightarrow P^\dagger \stackrel{\tau}{\not\rightarrow}$ with $P^\dagger \leftrightarrow_{br} \theta_X(Q')$. By the semantics, $P^\dagger = \theta_X(P')$ for some P' with $P \Longrightarrow P' \stackrel{\tau}{\not\rightarrow}$. So $P' \leftrightarrow^X_{br} Q'$ by Proposition 15.2, and Lemma 2.3 yields $P' \leftrightarrow_{br} Q'$. Thus $P' \in R'$ and Lemma 2.2 gives $\mathcal{I}(P') \cap (X \cup \{\tau\}) = \emptyset$. ◀

▶ Remark 51. Let $R \in [\mathbb{P}^g]$. If $R \xrightarrow{\mathrm{t}}$ then $R \stackrel{\tau}{\not\rightarrow}$.

Proof. Suppose $R \xrightarrow{\mathrm{t}} R'$. By the definition in Section 5.4, there is a path $\chi(R) \Longrightarrow P_1 \stackrel{\tau}{\not\rightarrow}$ with $P_1 \in R$. So by Lemma 49.2 $R \stackrel{\tau}{\not\rightarrow}$. ◀

▶ **Definition 52.** A *concrete branching time-out bisimulation up to reflexivity and transitivity* is a symmetric relation $\mathscr{B}$ on $\mathbb{P}^g \uplus [\mathbb{P}^g] \uplus \{\theta_X([P]) \mid X \subseteq A \wedge P \in \mathbb{P}^g\}$, such that, for all $P^\dagger \mathscr{B} Q$,

- if $P^\dagger \xrightarrow{\alpha} P^\ddagger$ with $\alpha \in A_\tau$, then $\exists$ path $Q \Longrightarrow Q^\dagger \xrightarrow{(\alpha)} Q^\ddagger$ with $P^\dagger \mathscr{B}^* Q^\dagger$ and $P^\ddagger \mathscr{B}^* Q^\dagger$,
- if $P^\dagger \xrightarrow{\mathrm{t}} P^\ddagger$ with $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$, then there is a path $Q \Longrightarrow Q^\dagger \xrightarrow{\mathrm{t}} Q^\ddagger$ with $\mathcal{I}(Q^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P^\ddagger) \mathscr{B}^* \theta_X(Q^\ddagger)$,
- if $P^\dagger \stackrel{\tau}{\not\rightarrow}$ then there is a path $Q \Longrightarrow Q^\dagger \stackrel{\tau}{\not\rightarrow}$.

Here $\mathscr{B}^* := \{(P^\dagger, Q^\dagger) \mid \exists n \geq 0.\ \exists P_0, \ldots, P_n.\ P^\dagger = P_0 \mathscr{B} P_1 \mathscr{B} \cdots \mathscr{B} P_n = Q^\dagger\}$.

▶ **Proposition 53.** *If $P \mathscr{B} Q$ for a concrete branching time-out bisimulation $\mathscr{B}$ up to reflexivity and transitivity, then $P \leftrightarrow_{br} Q$.*

Proof. It suffices to show that $\mathscr{B}^*$ is a branching time-out bisimulation. Clearly this relation is symmetric.

- Suppose $P_0 \mathscr{B} P_1 \mathscr{B} \cdots \mathscr{B} P_n$ for some $n \geq 0$ and $P \Longrightarrow P_0^\dagger \xrightarrow{(\alpha)} P_0^\ddagger$ with $\alpha \in A_\tau$. It suffices to find $P_n^\dagger, P_n^\ddagger$ such that $P_n \Longrightarrow P_n^\dagger \xrightarrow{(\alpha)} P_n^\ddagger$, $P_0^\dagger \mathscr{B}^* P_n^\dagger$ and $P_0^\ddagger \mathscr{B}^* P_n^\ddagger$. (In fact, we need this only in the special case where $P_0 = P_0^\dagger \neq P_0^\ddagger$, but establish the more general claim.) We proceed with induction on n. The case $n = 0$ is trivial.

  Fixing an $n > 0$, by Definition 52 there are $P_1^\dagger, P_1^\ddagger$ such that $P_1 \Longrightarrow P_1^\dagger \xrightarrow{(\alpha)} P_1^\ddagger$, $P_0^\dagger \mathscr{B}^* P_1^\dagger$ and $P_0^\ddagger \mathscr{B}^* P_1^\ddagger$. Now by induction there are $P_n^\dagger, P_n^\ddagger$ such that $P_n \Longrightarrow P_n^\dagger \xrightarrow{(\alpha)} P_n^\ddagger$, $P_1^\dagger \mathscr{B}^* P_n^\dagger$ and $P_1^\ddagger \mathscr{B}^* P_n^\ddagger$. Hence $P_0^\dagger \mathscr{B}^* P_n^\dagger$ and $P_0^\ddagger \mathscr{B}^* P_n^\ddagger$.

- Suppose $P_0 \mathscr{B} P_1 \mathscr{B} \cdots \mathscr{B} P_n$ for some $n \geq 0$ and there is a path $P_0 \Longrightarrow P_0^\dagger \stackrel{\mathrm{t}}{\rightarrow} P_0^\ddagger$ with $\mathcal{I}(P_0^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. It suffices to find $P_n^\dagger, P_n^\ddagger$ such that $P_n \Longrightarrow P_n^\dagger \stackrel{\mathrm{t}}{\rightarrow} P_n^\ddagger$, $\mathcal{I}(P_n^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_0^\ddagger) \mathscr{B}^* \theta_X(P_n^\ddagger)$. (In fact, we need this only in the special case where $P_0^\dagger = P_0$, but establish the more general claim.) We proceed with induction on n. The case $n = 0$ is trivial.
  Fixing an $n > 0$, by Definition 52 there exist $P_1^\dagger, P_1^\ddagger$ such that $P_1 \Longrightarrow P_1^\dagger \stackrel{\mathrm{t}}{\rightarrow} P_1^\ddagger$, $\mathcal{I}(P_1^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_0^\ddagger) \mathscr{B}^* \theta_X(P_1^\ddagger)$. By induction there are $P_n^\dagger, P_n^\ddagger$ with $P_n \Longrightarrow P_n^\dagger \stackrel{\mathrm{t}}{\rightarrow} P_n^\ddagger$, $\mathcal{I}(P_n^\dagger) \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P_1^\ddagger) \mathscr{B}^* \theta_X(P_n^\ddagger)$. Hence $\theta_X(P_0^\ddagger) \mathscr{B}^* \theta_X(P_n^\ddagger)$.
- Suppose $P_0 \mathscr{B} P_1 \mathscr{B} \cdots \mathscr{B} P_n$ for some $n \geq 0$ and there is a path $P_0 \Longrightarrow P_0^\dagger \stackrel{\tau}{\not\rightarrow}$. It suffices to find a path $P_n \Longrightarrow P_n^\dagger \stackrel{\tau}{\not\rightarrow}$. (In fact, we need this only in the special case where $P_0^\dagger = P_0$, but establish the more general claim.) We proceed with induction on n. The case $n = 0$ is trivial.
  Fixing an $n > 0$, by Definition 52 there exists a path $P_1 \Longrightarrow P_1^\dagger \stackrel{\tau}{\not\rightarrow}$. By induction, there exists a path $P_n \Longrightarrow P_n^\dagger \stackrel{\tau}{\not\rightarrow}$. ◀

▶ **Lemma 54.** *Let $P \in \mathbb{P}^g$ and $X \subseteq A$. If $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$ then $[P] \leftrightarroweq \theta_X([P])$.*

Proof. It suffices to see that $\mathscr{B} := \{([P], \theta_X([P])), (\theta_X([P]), [P])\} \cup Id$ is a strong bisimulation thanks to the semantics of θ_X. ◀

▶ **Lemma 55.** *Let $P \in \mathbb{P}^g$ and $X \subseteq A$. Then $\theta_X([P]) \leftrightarroweq_{br} [\theta_X(P)]$.*

Proof. We will show that $\mathscr{B} := \leftrightarroweq \cup \{(\theta_X([P]), [\theta_X(P)]), ([\theta_X(P)], \theta_X([P])) \mid P \in \mathbb{P}^g \wedge X \subseteq A\}$ is a concrete branching time-out bisimulation up to reflexivity and transitivity.

- If $\theta_X([P]) \stackrel{\tau}{\rightarrow} R'$ then $[P] \stackrel{\tau}{\rightarrow} R^\dagger$ with $R' = \theta_X(R^\dagger)$. According to Lemma 49.1, there exists a path $P \Longrightarrow P_1 \stackrel{\tau}{\rightarrow} P_2$ such that $P_1 \in [P]$ and $P_2 \in R^\dagger = [P_2]$ and $[P] \neq R^\dagger$. Thus, there exists a path $\theta_X(P) \Longrightarrow \theta_X(P_1) \stackrel{\tau}{\rightarrow} \theta_X(P_2)$ such that $\theta_X(P_1) \in [\theta_X(P)]$. If $\theta_X(P_1) \leftrightarroweq_{br} \theta_X(P_2)$ then $[\theta_X(P)] = [\theta_X(P_2)]$. Otherwise, $[\theta_X(P)] \neq [\theta_X(P_2)]$, thus, according to Lemma 49.1, $[\theta_X(P)] \stackrel{\tau}{\rightarrow} [\theta_X(P_2)]$. In either case, there exists a transition $[\theta_X(P)] \stackrel{(\tau)}{\longrightarrow} [\theta_X(P_2)]$ such that, by definition of $\mathscr{B}$, $[\theta_X(P_2)] \mathscr{B} \theta_X([P_2]) = R'$.
- If $[\theta_X(P)] \stackrel{\tau}{\rightarrow} R'$ then, according to Lemma 49.1, there exists a path $\theta_X(P) \Longrightarrow P_1 \stackrel{\tau}{\rightarrow} P_2$ such that $P_1 \in [\theta_X(P)]$, $P_2 \in R'$ and $[\theta_X(P)] \neq R'$. Thus, there exists a path $P \Longrightarrow P^\dagger \stackrel{\tau}{\rightarrow} P^\ddagger$ such that $P_1 = \theta_X(P^\dagger)$ and $P_2 = \theta_X(P^\ddagger)$. Notice that, since $[P_1] \neq [P_2]$, $[P^\dagger] \neq [P^\ddagger]$. According to Lemma 49.1, there exists a path $[P] \Longrightarrow [P^\dagger] \stackrel{\tau}{\rightarrow} [P^\ddagger]$. Thus, $\theta_X([P]) \Longrightarrow \theta_X([P^\dagger]) \stackrel{\tau}{\rightarrow} \theta_X([P^\ddagger])$. Moreover, by definition of $\mathscr{B}$, $\theta_X([P^\dagger]) \mathscr{B} [\theta_X(P^\dagger)] = [\theta_X(P)]$ and $\theta_X([P^\ddagger]) \mathscr{B} [\theta_X(P^\ddagger)] = R'$.
- If $\theta_X([P]) \stackrel{a}{\rightarrow} R'$ with $a \in A$ then $[P] \stackrel{a}{\rightarrow} R'$ and $a \in X \vee \mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$. If $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$, according to Lemma 49.3, there exists a path $P \Longrightarrow P_0$ such that $P_0 \in [P]$ and $\mathcal{I}(P_0) \cap (X \cup \{\tau\}) = \emptyset$. Otherwise, set $P_0 := P$. Since $[P_0] \stackrel{a}{\rightarrow} R'$, according to Lemma 49.1, there exists a path $P_0 \Longrightarrow P_1 \stackrel{a}{\rightarrow} P_2$ such that $P_1 \in [P_0]$ and $P_2 \in R'$. Notice that $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset \Rightarrow \mathcal{I}(P_0) \cap (X \cup \{\tau\}) = \emptyset \wedge P_0 = P_1$. Thus, there exists a path $\theta_X(P_0) \Longrightarrow \theta_X(P_1) \stackrel{a}{\rightarrow} P_2$ such that $\theta_X(P_1) \in [\theta_X(P)]$. According to Lemma 49.1, there exists a transition $[\theta_X(P)] \stackrel{a}{\rightarrow} R'$.
- If $[\theta_X(P)] \stackrel{a}{\rightarrow} R'$ with $a \in A$ then, according to Lemma 49.1, there exists a path $\theta_X(P) \Longrightarrow P_1 \stackrel{a}{\rightarrow} P_2$ such that $P_1 \in [\theta_X(P)]$ and $P_2 \in R'$. Thus, there exists a path $P \Longrightarrow P^\dagger \stackrel{a}{\rightarrow} P_2$ such that $P_1 = \theta_X(P^\dagger)$ and $a \in X \vee \mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset$. According to Lemma 49.1, there exists a path $[P] \Longrightarrow [P^\dagger] \stackrel{a}{\rightarrow} [P_2]$. Thus, $\theta_X([P]) \Longrightarrow \theta_X([P^\dagger]) \stackrel{a}{\rightarrow} [P_2]$ since $\mathcal{I}(P^\dagger) \cap (X \cup \{\tau\}) = \emptyset \Rightarrow \mathcal{I}([P^\dagger]) \cap (X \cup \{\tau\}) = \emptyset$ by Lemma 49.3. Moreover, by definition of $\mathscr{B}$, $\theta_X([P^\dagger]) \mathscr{B} [\theta_X(P^\dagger)] = [\theta_X(P)]$ and $[P_2] = R' \mathscr{B}^* R'$.

- If $\mathcal{I}(\theta_X([P]))\cap(Y\cup\{\tau\}) = \emptyset$ and $\theta_X([P]) \xrightarrow{t} R'$ then $\mathcal{I}([P])\cap(X\cup\{\tau\}) = \emptyset$; thus, according to Lemma 49.3, there exists a path $P \Longrightarrow P_0$ such that $P_0 \in [P]$ and $\mathcal{I}(P_0)\cap(X\cup\{\tau\}) = \emptyset$. Since $\mathcal{I}(P_0)\cap(X\cup\{\tau\}) = \emptyset$, $P \leftrightarroweq_{br} P_0 \leftrightarroweq \theta_X(P_0) \leftrightarroweq_{br} \theta_X(P)$ and so $[P] = [\theta_X(P)]$. Since $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$, according to Lemma 54, $\theta_X([P]) \leftrightarroweq [P] = [\theta_X(P)]$.
- If $\mathcal{I}([\theta_X(P)]) \cap (Y \cup \{\tau\}) = \emptyset$ and $[\theta_X(P)] \xrightarrow{t} R'$ then, according to Lemma 49.4, there exists $r > 0$ and a path $\theta_X(P) \Longrightarrow \theta_X(P_1) \xrightarrow{t} P_2 \Longrightarrow P_3 \xrightarrow{t} \ldots \Longrightarrow P_{2r-1} \xrightarrow{t} P_{2r}$ such that $\theta_X(P_1) \in [\theta_X(P)]$, $\mathcal{I}(\theta_X(P_1)) \cap (Y \cup \{\tau\}) = \emptyset$ and $\forall i \in [1, r-1]$, $\theta_Y(P_{2i}) \in [\theta_Y(\theta_X(P))] \wedge \mathcal{I}(P_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$, $\theta_Y(P_{2r}) \in [\theta_Y(\chi(R'))]$ and $[\theta_Y(\theta_X(P))] \neq [\theta_Y(\chi(R'))]$. Since $\theta_X(P_1) \xrightarrow{t} P_2$, $P_1 \xrightarrow{t} P_2$ and $\mathcal{I}(P_1) \cap (X \cup \{\tau\}) = \emptyset$, thus, $\theta_X(P) \leftrightarroweq_{br} \theta_X(P_1) \leftrightarroweq P_1$. Therefore, $P \Longrightarrow P_1$ and there exists a path $P_1 \xrightarrow{t} P_2 \Longrightarrow P_3 \xrightarrow{t} \ldots \Longrightarrow P_{2r-1} \xrightarrow{t} P_{2r}$ such that $P_1 \in [P_1]$, $\forall i \in [1, r-1]$, $\theta_Y(P_{2i}) \in [\theta_Y(P_1)] \wedge \mathcal{I}(P_{2i+1}) \cap (Y \cup \{\tau\}) = \emptyset$ and $[\theta_Y(P_1)] \neq [\theta_Y(\chi(R'))] = [\theta_Y(P_{2r})]$. According to Lemma 49.1 and 49.5, there exists a path $[P] \Longrightarrow [P_1] \xrightarrow{t} R''$ for some $R'' \in [\mathbb{P}^g]$ with $\theta_Y(P_{2r}) \in [\theta_Y(\chi(R''))]$. Thus, $\theta_X([P]) \Longrightarrow \theta_X([P_1]) \xrightarrow{t} R'$ since $\mathcal{I}([P_1]) \cap (X \cup \{\tau\}) = \emptyset$. Since $P_1 \leftrightarroweq \theta_X(P_1)$ and $\mathcal{I}(\theta_X(P_1)) \cap (Y \cup \{\tau\}) = \emptyset$, Lemma 49.3 yields $\mathcal{I}([P_1]) \cap (Y \cup \{\tau\}) = \emptyset$, and hence $\mathcal{I}(\theta_X([P_1])) \cap (Y \cup \{\tau\}) = \emptyset$. Moreover, by definition of $\mathscr{B}$, $\theta_Y(R') \mathrel{\mathscr{B}} [\theta_Y(\chi(R'))] = [\theta_Y(P_{2r})] = [\theta_Y(\chi(R''))] \mathrel{\mathscr{B}} \theta_Y(R'')$.
- If $\theta_X([P]) \stackrel{\tau}{\not\rightarrow}$ then $[P] \stackrel{\tau}{\not\rightarrow}$. According to Lemma 49.2, there exists a path $P \Longrightarrow P_0 \stackrel{\tau}{\not\rightarrow}$ such that $P_0 \in [P]$. Thus, there exists a path $\theta_X(P) \Longrightarrow \theta_X(P_0) \stackrel{\tau}{\not\rightarrow}$ such that $\theta_X(P_0) \in [\theta_X(P)]$. According to Lemma 49.2, $[\theta_X(P)] \stackrel{\tau}{\not\rightarrow}$.
- If $[\theta_X(P)] \stackrel{\tau}{\not\rightarrow}$ then, according to Lemma 49.2, there exists a path $\theta_X(P) \Longrightarrow P_0 \stackrel{\tau}{\not\rightarrow}$ such that $P_0 \in [\theta_X(P)]$. Thus, there exists a path $P \Longrightarrow P^\dagger \stackrel{\tau}{\not\rightarrow}$ such that $P_0 = \theta_X(P^\dagger)$. According to Lemma 49.1–2, there exists a path $[P] \Longrightarrow [P^\dagger] \stackrel{\tau}{\not\rightarrow}$. Thus, $\theta_X([P]) \Longrightarrow \theta_X([P^\dagger]) \stackrel{\tau}{\not\rightarrow}$. ◀

Proof of Proposition 29. We are going to show that $\mathscr{B} := \{(P, [P]), ([P], P) \mid P \in \mathbb{P}^g\}$ is a branching time-out bisimulation up to $\leftrightarroweq_{br}$ (see Definition 42).

1.
- Let $P \Longrightarrow P' \xrightarrow{\alpha} P''$ with $\alpha \in A_\tau$ and $P \leftrightarroweq_{br} P'$. If $\alpha \in A \vee P \not\leftrightarroweq_{br} P''$ then, according to Lemma 49.1, $[P] \xrightarrow{\alpha} [P'']$ and, by definition of $\mathscr{B}$, $P' \mathrel{\mathscr{B}} [P'] = [P]$ and $P'' \mathrel{\mathscr{B}} [P'']$. Otherwise, $\alpha = \tau \wedge P \leftrightarroweq_{br} P''$ thus, by definition of $\mathscr{B}$, $P'' \mathrel{\mathscr{B}} [P''] = [P]$. In either case, there exists a path $[P] \xrightarrow{(\alpha)} [P'']$ such that $P' \mathrel{\mathscr{B}} [P]$ and $[P''] \mathrel{\mathscr{B}} P''$.
- If $[P] \Longrightarrow R' \xrightarrow{\alpha} R''$ with $\alpha \in A_\tau$ and $[P] \leftrightarroweq_{br} R'$ then, according to Lemma 49.1, $P \Longrightarrow P_1 \xrightarrow{\alpha} P_2$ such that $P_1 \in R'$ and $P_2 \in R''$. Thus, by definition of $\mathscr{B}$, $P_1 \mathrel{\mathscr{B}} [P_1] = R'$ and $P_2 \mathrel{\mathscr{B}} [P_2] = R''$.

2.
- Let $P \Longrightarrow P_1 \xrightarrow{t} P_2 \Longrightarrow P_3 \xrightarrow{t} \ldots \Longrightarrow P_{2r-1} \xrightarrow{(t)} P_{2r}$ with $r > 0$, and $\forall i \in [0, r-1]$, $\theta_X(P) \leftrightarroweq_{br} \theta_X(P_{2i}) \wedge P \leftrightarroweq_{br} P_{2i+1} \wedge \mathcal{I}(P_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$. If $\theta_X(P) \leftrightarroweq_{br} \theta_X(P_{2r})$ then $\forall i \in [0, r]$, $\theta_X(P_{2i}) \mathrel{\mathscr{B}} [\theta_X(P_{2i})] = [\theta_X(P)] \leftrightarroweq_{br} \theta_X([P])$, in the last step applying Lemma 55. Otherwise, $P_{2r-1} \neq P_{2r}$ and by Lemma 49.5 there exists a transition $[P] \xrightarrow{t} R'$ such that $\theta_X(P_{2r}) \leftrightarroweq_{br} \theta_X(\chi(R')) \mathrel{\mathscr{B}} [\theta_X(\chi(R'))] \leftrightarroweq_{br} \theta_X(R')$ and $\forall i \in [0, r-1]$, $\theta_X(P_{2i}) \mathrel{\mathscr{B}} [\theta_X(P_{2i})] = [\theta_X(P)] \leftrightarroweq_{br} \theta_X([P])$. In either case, there exists a transition $[P] \xrightarrow{(t)} R'$ such that $\theta_X(P) \leftrightarroweq_{br} \mathscr{B} \leftrightarroweq_{br} \theta_X([P])$ and $\theta_X(P_{2r}) \leftrightarroweq_{br} \mathscr{B} \leftrightarroweq_{br} \theta_X(R')$. Moreover, by Lemma 49.3, $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$ since $\mathcal{I}(P_1) \cap (X \cup \{\tau\}) = \emptyset$ and $P_1 \in [P]$.
- Let $[P] = R_0 \Longrightarrow R_1 \xrightarrow{t} R_2 \Longrightarrow R_3 \xrightarrow{t} \ldots \Longrightarrow R_{2r-1} \xrightarrow{(t)} R_{2r}$ with $r > 0$, such that $\forall i \in [0, r-1]$, $\theta_X([P]) \leftrightarroweq_{br} \theta_X(R_{2i}) \wedge [P] \leftrightarroweq_{br} R_{2i+1} \wedge \mathcal{I}(R_{2i+1}) \cap (X \cup \{\tau\}) = \emptyset$. Then, according to Lemma 49.4 and Corollary 50, there exists a path $P = P_0 \Longrightarrow P_1 \xrightarrow{t} P_2 \Longrightarrow P_3 \xrightarrow{t} \ldots \Longrightarrow P_{2k-1} \xrightarrow{(t)} P_{2k}$ with $k > 0$, such that $\forall j \in [0, k-1]$,

$\mathcal{I}(P_{2j+1}) \cap (X \cup \{\tau\}) = \emptyset$ and $\forall j \in [0, 2k-1]$, $\exists i \in [0, 2r-1]$, $\theta_X(P_j) \in [\theta_X(\chi(R_i))]$ and $\theta_X(P_{2k}) \in [\theta_X(\chi(R_{2r}))]$. Thus, applying Lemma 55, $\forall j \in [0, 2k-1], \exists i \in [0, 2r-1]$, $\theta_X(P_j) \mathscr{B} [\theta_X(P_j)] = [\theta_X(\chi(R_i))] \leftrightarrow_{br} \theta_X(R_i) \leftrightarrow_{br} \theta_X([P])$ and $\theta_X(P_{2k}) \mathscr{B} [\theta_X(P_{2k})] = [\theta_X(\chi(R_{2r}))] \leftrightarrow_{br} \theta_X(R_{2r})$.

3.
- If $P \Longrightarrow P_0 \stackrel{\tau}{\not\to}$ with $P \leftrightarrow_{br} P_0$ then, according to Lemma 49.2, $[P] \stackrel{\tau}{\not\to}$.
- If $[P] \Longrightarrow R' \stackrel{\tau}{\not\to}$ with $[P] \leftrightarrow_{br} R'$ then, according to Lemma 49.1–2, there exists a path $P \Longrightarrow P' \Longrightarrow P_0 \stackrel{\tau}{\not\to}$ such that $P', P_0 \in R'$. ◀

P Completeness Proof by Canonical Representatives

▶ **Lemma 56.** *Let $P, Q \in \mathbb{P}^g$.*
- $[P] \leftrightarrow_{br} [Q] \Rightarrow [P] = [Q]$.
- $\theta_X([P]) \leftrightarrow_{br} \theta_X([Q]) \Rightarrow \theta_X([P]) \leftrightarrow_{tb} \theta_X([Q])$.

Proof.
- If $[P] \leftrightarrow_{br} [Q]$ then, by Proposition 29, $P \leftrightarrow_{br} [P] \leftrightarrow_{br} [Q] \leftrightarrow_{br} Q$. Thus, $[P] = [Q]$.
- We are going to show that $\mathscr{B} := Id \cup \{(\theta_X([P]), \theta_X([Q])) \mid \theta_X([P]) \leftrightarrow_{br} \theta_X([Q])\}$ is a t-branching bisimulation. Suppose $\theta_X([P]) \leftrightarrow_{br} \theta_X([Q])$. The first and third clause of Definition 7 are trivially satisfied, because Definition 13 features the same clauses. Towards the second clause, suppose $\theta_X([P]) \xrightarrow{\mathrm{t}} R'$. Then $\mathcal{I}([P]) \cap (X \cup \{\tau\}) = \emptyset$ and $[P] \xrightarrow{\mathrm{t}} R'$. As $[P] \leftrightarrow_{br}^X [Q]$, by Clause 2.c of Definition 1 there is a path $[Q] \Longrightarrow R$ for some $R \in [\mathbb{P}^g]$ with $[P] \leftrightarrow_{br} R$. By the previous statement of this lemma, $R = [P]$. Thus $\theta_X([Q]) \Longrightarrow \theta_X([P]) \xrightarrow{\mathrm{t}} R'$, which suffices to satisfy the second clause of Definition 7. ◀

Proof of Proposition 31. Let $\mathcal{S}'$ be a recursive specification such that $V_{\mathcal{S}'} := \{y_{P'} \mid P' \in \mathrm{Reach}(P)\}$ and, for all $P' \in \mathrm{Reach}(P)$, $\mathcal{S}_{y_{P'}} := \sum_{\{(\alpha,P'') \mid P' \xrightarrow{\alpha} P''\}} \alpha.y_{P''}$. Note that $\mathcal{S}'$ is strongly guarded since P is. We are going to show that P and $\langle x_P | \mathcal{S} \rangle$ are both y_P-components of solutions of $\mathcal{S}'$, so that the proposition follows by RSP.

First of all, consider $\rho : V_{\mathcal{S}'} \to \mathbb{P}$ such that $\forall P' \in \mathrm{Reach}(P)$, $\rho(y_{P'}) := P'$. For all $P' \in \mathrm{Reach}(P)$, $Ax_r^\infty \vdash \rho(y_{P'}) = P' = \sum_{\{(\alpha,P'') \mid P' \xrightarrow{\alpha} P''\}} \alpha.P'' = \sum_{\{(\alpha,P'') \mid P' \xrightarrow{\alpha} P''\}} \alpha.\rho(y_{P''})$ is a direct application of Lemma 25. Thus, for all $P' \in \mathrm{Reach}(P)$, $Ax_r^\infty \vdash \rho(y_{P'}) = \mathcal{S}'_{y_{P'}}[\rho]$, i.e., ρ is a solution of $\mathcal{S}'$ up to $\leftrightarrow_{br}$, and $\rho(y_P) = P$.

Next, consider $\nu : V_{\mathcal{S}'} \to \mathbb{P}$ such that, for all $P' \in \mathrm{Reach}(P)$,

$$\nu(y_{P'}) := \sum_{\{(\alpha,P'') \mid P' \xrightarrow{\alpha} P''\}} \alpha.\langle x_{[P'']} \mid \mathcal{S} \rangle$$

We are going to show that, for all $\alpha \in Act$ and all $P' \in \mathrm{Reach}(P)$, $Ax_r^\infty \vdash \alpha.\nu(y_{P'}) = \alpha.\langle x_{[P']} \mid \mathcal{S} \rangle$. Let $P' \in \mathrm{Reach}(P)$.
- If $\exists P' \xrightarrow{\tau} P''$, $P' \leftrightarrow_{br} P''$ then $\{(\alpha, [P'']) \mid P' \xrightarrow{\alpha} P'' \wedge (\alpha \in A \vee (\alpha = \tau \wedge P' \not\leftrightarrow_{br} P''))\} \subseteq \{(\alpha, R) \mid [P'] \xrightarrow{\alpha} R\}$. Thus,

$$\begin{aligned} Ax_r^\infty \vdash \alpha.\nu(y_{P'}) &= \alpha.(\sum_{\{(\alpha,P'') \mid P' \xrightarrow{\alpha} P'' \wedge \alpha \neq \mathrm{t}\}} \alpha.\langle x_{[P'']} \mid \mathcal{S} \rangle) && (\mathbf{L}\tau) \\ &= \alpha.(\sum_{\{(\alpha,P'') \mid P' \xrightarrow{\alpha} P'' \wedge (\alpha \in A \vee (\alpha = \tau \wedge P \not\leftrightarrow_{br} P'))\}} \alpha.\langle x_{[P'']} \mid \mathcal{S} \rangle + \tau.\langle x_{[P']} \mid \mathcal{S} \rangle) \\ &= \alpha.\langle x_{[P']} \mid \mathcal{S} \rangle && \text{(branching axiom and RDP)} \end{aligned}$$

- If there exists $X \subseteq A$ and a transition $P' \xrightarrow{\mathrm{t}} P''$ such that $\mathcal{I}(P') \cap (X \cup \{\tau\}) = \emptyset$ and $\theta_X(P') \leftrightarrow_{br} \theta_X(P'')$ then, since $P' \stackrel{\tau}{\not\rightarrow}$, for all $\alpha \in A_\tau$, $P' \xrightarrow{\alpha} P'' \wedge P'' \in R \iff [P'] \xrightarrow{\alpha} R$, using Lemma 49.1. Moreover, according to Lemma 22, $\{(\mathrm{t}, P'') \mid P' \xrightarrow{\mathrm{t}} P''\} =$

$$\{(\mathrm{t}, P'') \mid P' \xrightarrow{\mathrm{t}} P'' \wedge \forall Y \subseteq A, \mathcal{I}(P') \cap (Y \cup \{\tau\}) = \emptyset \Rightarrow \theta_Y(P') \leftrightarrow_{br} \theta_Y(P'')\}$$
$$\uplus \{(\mathrm{t}, P'') \mid P' \xrightarrow{\mathrm{t}} P'' \wedge \forall Y \subseteq A, \mathcal{I}(P') \cap (Y \cup \{\tau\}) = \emptyset \Rightarrow \theta_Y(P') \not\leftrightarrow_{br} \theta_Y(P'')\}.$$

For all (t, P'') in the first class, and for all $Z \subseteq A$ such that $\mathcal{I}(P') \cap (Z \cup \{\tau\}) = \emptyset$, recalling that $\langle x_R|\mathcal{S}\rangle \leftrightarrow R$ for all R, and applying Lemma 55 twice, we obtain

$$\theta_Z(\langle x_{[P'']}|\mathcal{S}\rangle) \leftrightarrow \theta_Z([P'']) \leftrightarrow_{br} [\theta_Z(P'')] = [\theta_Z(P')] \leftrightarrow_{br} \theta_Z([P']) \leftrightarrow \theta_Z(\langle x_{[P']}|\mathcal{S}\rangle).$$

Moreover, $\mathcal{I}([P']) = \mathcal{I}(P')$. Therefore the reactive approximation axiom yields

$$\sum_{\{(\alpha,R)|[P'] \xrightarrow{\alpha} R \wedge \alpha \in A_\tau\}} \alpha.\langle x_R \mid \mathcal{S}\rangle + \mathrm{t}.\langle x_{[P'']} \mid \mathcal{S}\rangle = \sum_{\{(\alpha,R)|[P'] \xrightarrow{\alpha} R \wedge \alpha \in A_\tau\}} \alpha.\langle x_R \mid \mathcal{S}\rangle + \mathrm{t}.\langle x_{[P']} \mid \mathcal{S}\rangle$$

According to Lemma 49.5, for all (t, P'') in the second class, and for all $Z \subseteq A$ such that $\mathcal{I}(P') \cap (Z \cup \{\tau\}) = \emptyset$, there exists a transition $[P'] \xrightarrow{\mathrm{t}} R'$ such that $\theta_Z(P'') \in [\theta_Z(\chi(R'))]$. Thus, applying Lemma 55 twice, $\theta_Z([P'']) \leftrightarrow_{br} [\theta_Z(P'')] = [\theta_Z(\chi(R'))] \leftrightarrow_{br} \theta_Z(R')$ and so, according to Lemma 56, $\theta_Z([P'']) \leftrightarrow_{tb} \theta_Z(R')$. Recalling that $\langle x_R|\mathcal{S}\rangle \leftrightarrow R$ for all R, we obtain $\theta_Z(\langle x_{[P'']}|\mathcal{S}\rangle) \leftrightarrow_{tb} \theta_Z(\langle x_{R'}|\mathcal{S}\rangle)$, and thus $\mathrm{t}.\theta_Z(\langle x_{[P'']}|\mathcal{S}\rangle) \leftrightarrow^r_{tb} \mathrm{t}.\theta_Z(\langle x_{R'}|\mathcal{S}\rangle)$. Therefore, thanks to Theorem 28, using that Ax^∞ can be derived from Ax^∞_r, $Ax^\infty_r \vdash \mathrm{t}.\theta_Z(\langle x_{[P'']} \mid \mathcal{S}\rangle) = \mathrm{t}.\theta_Z(\langle x_{R'} \mid \mathcal{S}\rangle)$. As a result, using RDP and the reactive approximation axiom,

$$Ax^\infty_r \vdash \sum_{\{(\mathrm{t},P'')|P' \xrightarrow{\mathrm{t}} P'' \wedge \forall X \subseteq A,\, \mathcal{I}(P') \cap (X \cup \{\tau\}) = \emptyset \;\Rightarrow\; \theta_X(P') \not\leftrightarrow_{br} \theta_X(P'')\}} \mathrm{t}.\langle x_{[P'']} \mid \mathcal{S}\rangle) + \langle x_{[P']} \mid \mathcal{S}\rangle = \langle x_{[P']} \mid \mathcal{S}\rangle$$

Therefore, for all $\alpha \in Act$,

$$\begin{aligned} Ax^\infty_r \vdash \alpha.\nu(y_{P'}) &= \alpha.(\sum_{\{(\alpha,P'')|P' \xrightarrow{\alpha} P''\}} \alpha.\langle x_{[P'']} \mid \mathcal{S}\rangle) \\ &= \alpha.(\sum_{\{(\alpha,R)|[P'] \xrightarrow{\alpha} R \wedge \alpha \in A_\tau\}} \alpha.\langle x_R \mid \mathcal{S}\rangle + \mathrm{t}.\langle x_{[P']} \mid \mathcal{S}\rangle \\ &+ \sum_{\{(\mathrm{t},P'')|P' \xrightarrow{\mathrm{t}} P'' \wedge \forall X \subseteq A,\, \mathcal{I}(P') \cap (X \cup \{\tau\}) = \emptyset \;\Rightarrow\; \theta_X(P') \not\leftrightarrow_{br} \theta_X(P'')\}} \mathrm{t}.\langle x_{[P'']} \mid \mathcal{S}\rangle) \\ &= \alpha.\langle x_{[P']} \mid \mathcal{S}\rangle \qquad \text{(t-branching axiom and RDP)} \end{aligned}$$

- If $\forall P \xrightarrow{\tau} P'$, $P \not\leftrightarrow_{br} P'$ and $\forall X \subseteq A$, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset \Rightarrow \forall P \xrightarrow{\mathrm{t}} P'$, $\theta_X(P) \not\leftrightarrow_{br} \theta_X(P')$ then, for all $\alpha \in A_\tau$, $P \xrightarrow{\alpha} P' \wedge P' \in R' \iff [P] \xrightarrow{\alpha} R'$ and $\mathcal{I}(P) = \mathcal{I}([P])$. Moreover, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $[P] \xrightarrow{\mathrm{t}} R'$ then there exists a transition $P \xrightarrow{\mathrm{t}} P'$ with $\theta_X(P') \in [\theta_X(\chi(R'))]$. Thus $\theta_X([P']) \leftrightarrow_{br} \theta_X(R')$ so $\theta_X([P']) \leftrightarrow_{tb} \theta_X(R')$ by Lemma 56, and therefore $Ax^\infty_r \vdash \mathrm{t}.\theta_X([P']) = \mathrm{t}.\theta_X(R')$. Conversely, if $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ and $P \xrightarrow{\mathrm{t}} P'$ then there exists a transition $[P] \xrightarrow{\mathrm{t}} R'$ such that $\theta_X(P') \in [\theta_X(\chi(R'))]$ and thus $Ax^\infty_r \vdash \mathrm{t}.\theta_X([P']) = \mathrm{t}.\theta_X(R')$. Using the reactive approximation axiom, $Ax^\infty_r \vdash \nu(y_{P'}) = \langle x_{[P']} \mid \mathcal{S}\rangle$ and so, for all $\alpha \in Act$, $Ax^\infty_r \vdash \alpha.\nu(y_{P'}) = \alpha.\langle x_{[P']} \mid \mathcal{S}\rangle$.

As a result, for all $P' \in \text{Reach}(P)$, $Ax_r^\infty \vdash \nu(y_{P'}) = \sum_{\{(\alpha,P'')|P' \xrightarrow{\alpha} P''\}} \alpha.\langle x_{[P'']}|\mathcal{S}\rangle = \sum_{\{(\alpha,P'')|P' \xrightarrow{\alpha} P''\}} \alpha.\nu(y_{P''}) = \mathcal{S}_{y_{P'}}[\nu]$, so ν is a solution of $\mathcal{S}'$ up to $\leftrightarrow_{br}$. Moreover, $\nu(y_P) = \sum_{\{(\alpha,P')|P \xrightarrow{\alpha} P'\}} \alpha.\langle x_{[P']}|\mathcal{S}\rangle$ which can be equated to $\langle x_P|\mathcal{S}\rangle$ by a single application of RDP. ◀

Proof of Theorem 32. According to Proposition 31, it suffices to establish that $Ax_r^\infty \vdash \langle x_P|\mathcal{S}\rangle = \langle x_Q|\mathcal{S}\rangle$. By applying RDP, this amounts to proving that

$$Ax_r^\infty \vdash \sum_{\{(\alpha,P')|P \xrightarrow{\alpha} P'\}} \alpha.\langle x_{[P']} \mid \mathcal{S}\rangle = \sum_{\{(\alpha,Q')|Q \xrightarrow{\alpha} Q'\}} \alpha.\langle x_{[Q']} \mid \mathcal{S}\rangle$$

Let (α, P') such that $P \xrightarrow{\alpha} P'$ and $\alpha \in A_\tau$. Since $P \leftrightarrow_{br}^r Q$, there exists a transition $Q \xrightarrow{\alpha} Q'$ such that $P' \leftrightarrow_{br} Q'$. Thus, $[P'] = [Q']$ and so $\langle x_{[P']} \mid \mathcal{S}\rangle = \langle x_{[Q']} \mid \mathcal{S}\rangle$. The same observation can be made for all (α, Q') such that $Q \xrightarrow{\alpha} Q'$ and $\alpha \in A_\tau$. As a result, $\mathcal{I}(P) = \mathcal{I}(Q)$ and

$$Ax_r^\infty \vdash \sum_{\{(\alpha,P')|P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau\}} \alpha.\langle x_{[P']} \mid \mathcal{S}\rangle = \sum_{\{(\alpha,Q')|Q \xrightarrow{\alpha} Q' \wedge \alpha \in A_\tau\}} \alpha.\langle x_{[Q']} \mid \mathcal{S}\rangle$$

Let (t, P') be such that $P \xrightarrow{\mathrm{t}} P'$. Since $P \leftrightarrow_{br}^r Q$, for all $X \subseteq A$ such that $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$, there exists a transition $Q \xrightarrow{\mathrm{t}} Q'$ such that $\theta_X(P') \leftrightarrow_{br} \theta_X(Q')$. Thus, $\theta_X([P']) \leftrightarrow_{br} \theta_X([Q'])$ by Proposition 29, and $\theta_X([P']) \leftrightarrow_{tb} \theta_X([Q'])$ by Lemma 56. Recalling that $\langle x_R|\mathcal{S}\rangle \leftrightarrow R$ for all R, $\theta_X(\langle x_{[P']} \mid \mathcal{S}\rangle) \leftrightarrow_{tb} \theta_X(\langle x_{[Q']} \mid \mathcal{S}\rangle)$ and hence $\mathrm{t}.\theta_X(\langle x_{[P']} \mid \mathcal{S}\rangle) \leftrightarrow_{tb}^r \mathrm{t}.\theta_X(\langle x_{[Q']} \mid \mathcal{S}\rangle)$. Since Ax^∞ can be derived from Ax_r^∞, according to Theorem 28, $Ax_r^\infty \vdash \mathrm{t}.\theta_X(\langle x_{[P']} \mid \mathcal{S}\rangle) = \mathrm{t}.\theta_X(\langle x_{[Q']} \mid \mathcal{S}\rangle)$. The same observation can be made for all (t, Q') such that $Q \xrightarrow{\mathrm{t}} Q'$. Let $X \subseteq A$. If $P \xrightarrow{\alpha}$ with $\alpha \in X \cup \{\tau\}$ then

$$\begin{aligned}
Ax_r^\infty \vdash \psi_X(\langle x_P|\mathcal{S}\rangle) &= \sum_{\{(\alpha,P')|P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau\}} \alpha.\langle x_{[P']} \mid \mathcal{S}\rangle \\
&= \sum_{\{(\alpha,Q')|Q \xrightarrow{\alpha} Q' \wedge \alpha \in A_\tau\}} \alpha.\langle x_{[Q']} \mid \mathcal{S}\rangle \\
&= \psi_X(\langle x_Q \mid \mathcal{S}\rangle)
\end{aligned}$$

Otherwise, $\mathcal{I}(P) \cap (X \cup \{\tau\}) = \emptyset$ so $\mathcal{I}(Q) \cap (X \cup \{\tau\}) = \emptyset$, thus,

$$\begin{aligned}
Ax_r^\infty \vdash \psi_X(\langle x_P|\mathcal{S}\rangle) &= \sum_{\{(\alpha,P')|P \xrightarrow{\alpha} P' \wedge \alpha \in A_\tau\}} \alpha.\langle x_{[P']} \mid \mathcal{S}\rangle + \sum_{\{(t,P')|P \xrightarrow{t} P'\}} \mathrm{t}.\theta_X(\langle x_{[P']} \mid \mathcal{S}\rangle) \\
&= \sum_{\{(\alpha,Q')|Q \xrightarrow{\alpha} Q' \wedge \alpha \in A_\tau\}} \alpha.\langle x_{[Q']} \mid \mathcal{S}\rangle + \sum_{\{(t,Q')|Q \xrightarrow{t} Q'\}} \mathrm{t}.\theta_X(\langle x_{[Q']} \mid \mathcal{S}\rangle) \\
&= \psi_X(\langle x_Q \mid \mathcal{S}\rangle)
\end{aligned}$$

Using the reactive approximation axiom, $Ax_r^\infty \vdash \langle x_P|\mathcal{S}\rangle = \langle x_Q|\mathcal{S}\rangle$. ◀