Rooted Divergence-Preserving Branching Bisimilarity is a Congruence

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Abstract

We prove that rooted divergence-preserving branching bisimilarity is a congruence for the process specification language consisting of $\mathbf{0}$, action prefix, choice, and the recursion construct $\mu X.\_$.

Keywords: Process algebra; Recursion; Branching bisimulation; Divergence; Congruence.

1 Introduction

Branching bisimilarity [13] is a behavioural equivalence on processes that is compatible with abstraction from internal activity, while at the same time preserving the branching structure of processes in a strong sense [9]. Branching bisimilarity abstracts to a large extent from *divergence* (i.e., infinite internal activity). For instance, it identifies a process, say P, that may perform some internal activity after which it returns to its initial state (i.e., P has a τ-loop) with a process, say P', that admits the same behaviour as P except that it cannot perform the internal activity leading to the initial state (i.e., P' is P without the τ-loop).

In situations where fairness principles apply, abstraction from divergence is often desirable. But there are circumstances in which abstraction from divergence is undesirable: A behavioural equivalence that abstracts from divergence is not compatible with any temporal logic featuring an *eventually* modality: for any desired state that P' will eventually reach, the mentioned internal activity of P may be performed forever, and thus prevent P from reaching this desired state. It is also generally not compatible with a process-algebraic priority operator (cf. [23, pp. 130–132]) or sequencing operator [3]. Since a divergence may be exploited to simulate recursively enumerable branching in a computable transition system [21], a divergence-insensitive behavioural equivalence may be considered too coarse for a theory that integrates computability and concurrency [2]. Preservation of divergence is widely considered an important correctness criterion when studying the relative expressiveness of process calculi [14, 25, 6].

The notion of *branching bisimilarity with explicit divergence*, also stemming from [13], is a suitable refinement of branching bisimilarity that is compatible with the well-known branching-time temporal logic CTL* without the nexttime operator X (which is known to be incompatible with abstraction from internal activity). In fact, in [12] we have proved that it is the coarsest semantic equivalence on labelled transition systems with silent moves that is a congruence for parallel composition (as found in process algebras like CCS, CSP or ACP) and only equates processes satisfying the same CTL$^*_{-X}$ formulas. In [2], for stylistic reasons, *branching bisimilarity with explicit divergence* was named *divergence-preserving branching bisimilarity*; we shall henceforth use this term.

Divergence-preserving branching bisimilarity is the finest behavioural equivalence in the linear time — branching time spectrum [8]. It is the principal behavioural equivalence underlying the theory of executability [1, 2, 16, 17]. Reduction modulo divergence-preserving branching bisimilarity is a part of methods for formal verification and analysis of the behaviour of systems [18, 24, 22, 26]. In [5] a game-based characterisation of divergence-preserving branching bisimilarity is presented.

Processes are usually specified in some process specification language. For compositional reasoning it is then important that the behavioural equivalence used is a congruence with respect to the constructs of that language. Following Milner [19], we consider the language *basic CCS with recursion*, i.e., the language consisting of $\mathbf{0}$, action prefix, and choice, extended with the recursion construct $\mu X.\_$; this language precisely allows the specification of finite-state behaviours. It is easy to see that divergence-preserving branching bisimilarity is not a congruence for that language; in fact, it is not a congruence for any language that includes choice. The goal of this paper is to prove that adding the usual root condition suffices to obtain a congruence—and, in fact, the coarsest congruence—for the language under consideration that is included in divergence-preserving branching bisimilarity.

Recently, a congruence format was proposed for (rooted) divergence-preserving branching bisimilarity [4]. The operational rules for action prefix and choice are in this format. Unfortunately, however, this format does not support the recursion construct $\mu X.\_$. Interestingly, as far as we know, the recursion construct has not been covered at all in the rich literature on congruence formats, with the recent exception of [10]. (The article [10] differentiates between *lean* and *full* congruences for recursion; in this article we consider the full congruence.)

The congruence result obtained in this paper should serve as a stepping stone towards a complete axiomatisation of divergence-preserving branching bisimilarity for basic CCS with recursion. Such work, inspired by Milner's complete axiomatisation of weak bisimilarity [19], would combine the adaptations of [7] to branching bisimilarity, and of [15] to several divergence-sensitive variants of weak bisimilarity.

2 Rooted divergence-preserving branching bisimilarity

Let $\mathcal{A}$ be a non-empty set of *actions*, and let τ be a special action not in $\mathcal{A}$. Let $\mathcal{A}_\tau = \mathcal{A} \cup \{\tau\}$. Furthermore, let $\mathcal{V}$ be a set of *recursion variables*. The set of *process expressions* $\mathcal{E}$ is generated by the following grammar:

$$E ::= \mathbf{0} \ | \ X \ | \ \alpha.E \ | \ \mu X.E \ | \ E + E \qquad (\alpha \in \mathcal{A}_\tau,\ X \in \mathcal{V}) \ .$$

An occurrence of a recursion variable X in a process expression E is *bound* if it is in de scope of a $\mu X.\_$, and otherwise it is *free*. We denote by $FV(E)$ the set of variables with a free occurrence in E. If $\vec{X} = X_0, \ldots, X_n$ is a sequence of variables, and $\vec{F} = F_0, \ldots, F_n$ is a sequence of process expressions of the same length, then we write $E[\vec{F}/\vec{X}]$ for the process expression obtained from E by replacing all free occurrences of X_i in E by F_i $(i = 0, \ldots, n)$, applying α-conversion to E if necessary to avoid capture.

On $\mathcal{E}$ we define an $\mathcal{A}_\tau$-labelled transition relation $\longrightarrow \subseteq \mathcal{E} \times \mathcal{A}_\tau \times \mathcal{E}$ as the least ternary relation satisfying the following rules for all $\alpha \in \mathcal{A}_\tau$, $X \in \mathcal{V}$, and process expressions E, E', F and F':

$$1\,\frac{}{\alpha.E \xrightarrow{\alpha} E} \qquad 2\,\frac{E[\mu X.E/X] \xrightarrow{\alpha} E'}{\mu X.E \xrightarrow{\alpha} E'} \qquad 3\,\frac{E \xrightarrow{\alpha} E'}{E + F \xrightarrow{\alpha} E'} \qquad 4\,\frac{F \xrightarrow{\alpha} F'}{E + F \xrightarrow{\alpha} F'}$$

We write $E \xrightarrow{\alpha} E'$ for $(E, \alpha, E') \in \longrightarrow$ and we abbreviate the statement '$E \xrightarrow{\alpha} E'$ or ($\alpha = \tau$ and $E = E'$)' by $E \xrightarrow{(\alpha)} E'$. Furthermore, we write $\twoheadrightarrow$ for the reflexive-transitive closure of $\xrightarrow{\tau}$, i.e., $E \twoheadrightarrow E'$ if there exist $E_0, E_1, \ldots, E_n \in \mathcal{E}$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} E_n = E'$.

A process expression is *closed* if it contains no free occurrences of recursion variables; we denote by $\mathcal{P}$ the subset of $\mathcal{E}$ consisting of all closed process expressions. It is easy to check that if P is a closed process expression and $P \xrightarrow{\alpha} E$, then E is a closed process expression too. Hence, the transition relation restricts in a natural way to closed process expressions, and thus associates with every closed process expression a behaviour. We proceed to define when two process expressions may be considered to represent the same behaviour.

Definition 1. A symmetric binary relation $\mathcal{R}$ on $\mathcal{P}$ is a *branching bisimulation* if it satisfies the following condition for all $P, Q \in \mathcal{P}$ and $\alpha \in \mathcal{A}_\tau$:

(T) if $P \mathrel{\mathcal{R}} Q$ and $P \xrightarrow{\alpha} P'$ for some closed process expression P', then there exist closed process expressions Q' and Q'' such that $Q \longrightarrow\!\!\!\!\rightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P \mathrel{\mathcal{R}} Q''$ and $P' \mathrel{\mathcal{R}} Q'$.

We write $P \leftrightarroweq_b Q$ if there exists a branching bisimulation $\mathcal{R}$ such that $P \mathrel{\mathcal{R}} Q$. The relation $\leftrightarroweq_b$ is referred to as branching bisimilarity.

We say that a branching bisimulation $\mathcal{R}$ *preserves (internal) divergence* if

(D) if $P \mathrel{\mathcal{R}} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k \mathrel{\mathcal{R}} Q$ for all $k \in \omega$, then there is an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ such that $Q = Q_0$, $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_k \mathrel{\mathcal{R}} Q_\ell$ for all $k, \ell \in \omega$.

We write $P \leftrightarroweq_b^\Delta Q$ if there exists a divergence-preserving branching bisimulation $\mathcal{R}$ such that $P \mathrel{\mathcal{R}} Q$. The relation $\leftrightarroweq_b^\Delta$ was introduced in [13] under the name branching bisimilarity with explicit divergence and is here referred to as divergence-preserving branching bisimilarity.

The relation $\leftrightarroweq_b^\Delta$ was studied in detail in [11]; we recap some of the facts established *ibidem*.

First, the relation $\leftrightarroweq_b^\Delta$ is an equivalence relation. Second, the relation $\leftrightarroweq_b^\Delta$ satisfies the condition (T), with the following generalisation as a straightforward consequence.

Lemma 2. *Let P and Q be closed process expressions. If $P \leftrightarroweq_b^\Delta Q$ and $P \longrightarrow\!\!\!\!\rightarrow P'' \xrightarrow{\alpha} P'$ for some closed process expressions P' and P'', then there exist closed process expressions Q' and Q'' such that $Q \longrightarrow\!\!\!\!\rightarrow Q'' \xrightarrow{\alpha} Q'$, $P'' \leftrightarroweq_b^\Delta Q''$ and $P' \leftrightarroweq_b^\Delta Q'$.*

Third, $\leftrightarroweq_b^\Delta$ also satisfies (D). In [11] several alternative definitions of divergence preservation are studied, which, in the end, all give rise to the same notion of divergence-preserving branching bisimilarity. In particular, the following alternative relational characterisations will be useful tools in the remainder.

Proposition 3. *Let P and Q be closed process expressions. Then*

- *$P \leftrightarroweq_b^\Delta Q$ if, and only if, P and Q are related by some branching bisimulation $\mathcal{R}$ satisfying*

  (D′) if $P \mathrel{\mathcal{R}} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$, then there is an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $Q = Q_0$, $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_{\sigma(\ell)} \mathrel{\mathcal{R}} Q_\ell$ for all $\ell \in \omega$; and

- *$P \leftrightarroweq_b^\Delta Q$ if, and only if, P and Q are related by some branching bisimulation $\mathcal{R}$ satisfying*

  (D″) if $P \mathrel{\mathcal{R}} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$, then there exists a closed process expression Q' such that $Q \xrightarrow{\tau} Q'$ and $P_k \mathrel{\mathcal{R}} Q'$ for some $k \in \omega$.

Moreover, $\leftrightarroweq_b^\Delta$ itself satisfies (D′) and (D″).

Proof. See [11]; condition (D′) is (D3) and condition (D″) is (D2). □

And finally, it was proved in [11] that $\leftrightarroweq_b^\Delta$ satisfies the following so-called stuttering property.

Proposition 4. *Let P be a closed process expression and let $Q_0, \ldots, Q_k$ be closed process expressions such that $Q_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} Q_k$. If $P \leftrightarroweq_b^\Delta Q_0$ and $P \leftrightarroweq_b^\Delta Q_k$, then $P \leftrightarroweq_b^\Delta Q_i$ for all $0 \leq i \leq k$.*

The relation $\leftrightarroweq_b^\Delta$ is not compatible with $+$ ($\mathbf{0} \leftrightarroweq_b^\Delta \tau.\mathbf{0}$ but $\mathbf{0} + a.\mathbf{0} \not\leftrightarroweq_b^\Delta \tau.\mathbf{0} + a.\mathbf{0}$), and hence not a congruence for the language we are considering. We proceed to define a relation for which we shall prove that it is the coarsest congruence for our language that is contained in $\leftrightarroweq_b^\Delta$.

Definition 5. Let P and Q be closed process expressions. We say that P and Q are *rooted divergence-preserving branching bisimilar* (notation: $P \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} Q$) if for all $\alpha \in \mathcal{A}_\tau$ the following holds:

(R1) if $P \xrightarrow{\alpha} P'$, then there exists a Q' such that $Q \xrightarrow{\alpha} Q'$ and $P' \mathrel{\underline{\leftrightarrow}}_{b}^{\Delta} Q'$; and

(R2) if $Q \xrightarrow{\alpha} Q'$, then there exists a P' such that $P \xrightarrow{\alpha} P'$ and $P' \mathrel{\underline{\leftrightarrow}}_{b}^{\Delta} Q'$.

The following proposition is a straightforward consequence of the fact that $\mathrel{\underline{\leftrightarrow}}_{b}^{\Delta}$ is an equivalence.

Proposition 6. *The relation $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$ is an equivalence relation on $\mathcal{P}$.* □

Moreover, it is easy to verify that $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} \subseteq \mathrel{\underline{\leftrightarrow}}_{b}^{\Delta}$.

We have defined the notions of $\mathrel{\underline{\leftrightarrow}}_{b}^{\Delta}$ and $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$ on closed process expressions because those are thought of as directly representing behaviour. Due to the presence of the binding construct $\mu X.\_$ it is, however, convenient to lift these notions to expressions with free variables even if the goal is simply to establish behavioural equivalence of closed process expressions.

Definition 7. Let E and F be process expressions, and let the sequence $\vec{X}$ of variables at least include all the variables with a free occurrence in E or F. We write $E \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} F$ ($E \mathrel{\underline{\leftrightarrow}}_{b}^{\Delta} F$) if $E[\vec{P}/\vec{X}] \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} F[\vec{P}/\vec{X}]$ ($E[\vec{P}/\vec{X}] \mathrel{\underline{\leftrightarrow}}_{b}^{\Delta} F[\vec{P}/\vec{X}]$) for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{X}$.

It is clear from the definition above that, since $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$ is an equivalence relation on $\mathcal{P}$, its lifted version is an equivalence relation on $\mathcal{E}$. We shall prove that it is also compatible with the constructs of the syntax, i.e., if $E \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} F$, then $\alpha.E \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} \alpha.F$ for all $\alpha \in \mathcal{A}_\tau$, $\mu X.E \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} \mu X.F$ for all $X \in \mathcal{V}$, $E + H \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} F + H$ and $H + E \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} H + F$ for all process expressions H. To prove that $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$ is compatible with $\alpha.$ and $+$ is straightforward, but for $\mu X.\_$ this is considerably more work.

3 The congruence proof

Our proof that $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$ is compatible with $\mu X.\_$ relies on the following observation: If $\vec{Y}$ is some sequence of variables and $\vec{P}$ is a sequence of closed terms of the same length, then, on the one hand, $E \mathrel{\underline{\leftrightarrow}}_{rb} F$ implies $E[\vec{P}/\vec{Y}] \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} F[\vec{P}/\vec{Y}]$, and, on the other hand, if X does not occur in $\vec{Y}$, then from $\mu X.E[\vec{P}/\vec{Y}] \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} \mu X.F[\vec{P}/\vec{Y}]$ it follows that $(\mu X.E)[\vec{P}/\vec{Y}] \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} (\mu X.F)[\vec{P}/\vec{Y}]$. Therefore, it is enough to establish that $E \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} F$ implies $\mu X.E \mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta} \mu X.F$ in the special case that E and F are process expressions with no other free variables than X; such process expressions will be called X-closed.

The rest of this section is organised as follows.

We shall first characterise, in Section 3.1, the relation $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$ on X-closed process expressions in terms of the transition relation on X-closed process expressions.

Then, in Section 3.2, we shall present a suitable notion of rooted divergence-preserving branching bisimulation up to $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$, and we shall prove that every pair of rooted divergence-preserving branching bisimilar X-closed process expressions (E, F) gives rise to a relation $\mathcal{R}^u$ of which we can show that it is a rooted divergence-preserving branching bisimulation up to $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$. The relation $\mathcal{R}^u$ will be defined in such a way that it relates $\mu X.E$ and $\mu X.F$ and thus allows us to conclude that these process expressions are rooted divergence-preserving bisimilar.

In Section 3.3, we shall then put the pieces together and prove $\mathrel{\underline{\leftrightarrow}}_{rb}^{\Delta}$ is the coarsest congruence contained in $\mathrel{\underline{\leftrightarrow}}_{b}^{\Delta}$ for basic CCS with recursion.

3.1 $\mathrel{\underline{\leftrightarrow}}_{b}^{\Delta}$ on X-closed process expressions

We say that a process expression E is X-*closed* if $FV(E) \subseteq \{X\}$; the set of all X-closed process expressions is denoted by $\mathcal{P}_X$. Note that if E is X-closed and $E \xrightarrow{\alpha} E'$, then E' is X-closed too, and so the $\mathcal{A}_\tau$-labelled transition relation restricts in a natural way to X-closed process expressions.

Definition 8. We define when X is *exposed* in a (not necessarily X-closed) process expression E with induction on the structure of E:

i. if $E = X$, then X is exposed in E;

ii. if $E = \mu Y.E'$, Y is a recursion variable distinct from X and X is exposed in E', then X is exposed in E;

iii. if $E = E_1 + E_2$ and X is exposed in E_1 or E_2, then X is exposed in E.

Note that the variable X is exposed in E if, and only if, E has an unguarded occurrence of X in the sense of [19].

We establish a relationship between the transitions of a closed process expression $E[P/X]$ that is obtained by substituting a closed process expression P for the variable X in an X-closed process expression E, and the transitions of E and P.

Lemma 9. *Let E be an X-closed process expression, and let P be a closed process expression.*

1. *If $E \xrightarrow{\alpha} E'$, then $E[P/X] \xrightarrow{\alpha} E'[P/X]$, and if X is exposed in E and $P \xrightarrow{\alpha} P'$, then $E[P/X] \xrightarrow{\alpha} P'$.*

2. *If $E[P/X] \xrightarrow{\alpha} P'$ for some closed process expression P', then either there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E, $P \xrightarrow{\alpha} P'$ and every derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.*

Proof. Statement 1 of the lemma is established with straightforward inductions on a derivation of $E \xrightarrow{\alpha} E'$ and on the structure of E.

We proceed to establish with induction on a derivation of $E[P/X] \xrightarrow{\alpha} P'$ that there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E, $P \xrightarrow{\alpha} P'$ and a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$. This implies statement 2.

We distinguish cases according to the structure of E:

- Clearly, E cannot be $\mathbf{0}$, for if $E = \mathbf{0}$, then $E[P/X] = \mathbf{0}$, and $\mathbf{0}$ does not admit any transitions.

- If $E = X$, then X is exposed in E and $P = E[P/X] \xrightarrow{\alpha} P'$. It is then also immediate that the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.

- If $E = \beta.E'$ for some $\beta \in \mathcal{A}_\tau$ and some X-closed process expression E', then $\beta = \alpha$ and $E \xrightarrow{\beta} E'$. Since $E[P/X] = \beta.(E'[P/X])$, rule 1 is the last rule applied in the derivation of the transition $E[P/X] \xrightarrow{\alpha} P'$, so $P' = E'[P/X]$.

- If $E = \mu Y.F$ for some process expression F with $FV(F) \subseteq \{X, Y\}$, then there are two subcases:

  On the one hand, if $Y = X$, then, since X has no free occurrence in E, we have $E = E[P/X] \xrightarrow{\alpha} P'$. Furthermore, since P' is closed we have that $P' = P'[P/X]$.

  On the other hand, if $Y \neq X$, then $E[P/X] = \mu Y.(F[P/X])$, and therefore the last rule applied in the considered derivation of the transition $E[P/X] \xrightarrow{\alpha} P'$ is rule 2. Consequently, the considered derivation has a proper subderivation of the transition $F[P/X][\mu Y.(F[P/X])/Y] \xrightarrow{\alpha} P'$. Note that $F[P/X][\mu Y.(F[P/X])/Y] = (F[\mu Y.F/Y])[P/X]$. Hence, by the induction hypothesis, either there exists an E' such that $F[\mu Y.F/Y] \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in $F[\mu Y.F/Y]$, $P \xrightarrow{\alpha} P'$, and the derivation of $F[\mu Y.F/Y][P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation. In the first case, it follows from $F[\mu Y.F/Y] \xrightarrow{\alpha} E'$, by rule 2, that $E = \mu Y.F \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$. In the second case, it suffices to note that X is exposed in F, hence also in E, and that a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$.

- If $E = E_1 + E_2$, then $[E][P/X] = E_1[P/X] + E_2[P/X]$.

  The last rule applied in the considered derivation of the transition $[E][P/X] \stackrel{\alpha}{\longrightarrow} P'$ is either rule 3 or rule 4.

  If it is rule 3, then $E_1[P/X] \stackrel{\alpha}{\longrightarrow} P'$, and since this transition has a derivation that is a proper subderivation of the considered derivation of $[E][P/X] \stackrel{\alpha}{\longrightarrow} P'$, by the induction hypothesis it follows that either $E_1 \stackrel{\alpha}{\longrightarrow} E'$ and $P' = E'[P/X]$, or X is exposed in E_1, $P \stackrel{\alpha}{\longrightarrow} P'$, and a derivation of $P \stackrel{\alpha}{\longrightarrow} P'$ appears as a subderivation the derivation of $E_1[P/X] \stackrel{\alpha}{\longrightarrow} P'$.

  In the first case, it remains to note that then also $E \stackrel{\alpha}{\longrightarrow} E'$, and in the second case, it remains to note that X is also exposed in E.

  If the last rule applied in the considered derivation is rule 4, then the proof is analogous. □

Corollary 10. *Let E be an X-closed process expression. If $E[\mu X.E/X] \stackrel{\alpha}{\longrightarrow} P'$ for some closed process expression P', then there exists an X-closed process expression E' such that $E \stackrel{\alpha}{\longrightarrow} E'$ and $P' = E'[\mu X.E/X]$.*

Proof. Consider a derivation of $E[\mu X.E/X] \stackrel{\alpha}{\longrightarrow} P'$ that is minimal in the sense that it does not have a derivation of $E[\mu X.E/X] \stackrel{\alpha}{\longrightarrow} P'$ as proper subderivation. Let $P = \mu X.E$. Since every derivation of $P \stackrel{\alpha}{\longrightarrow} P'$ has a derivation of $E[P/X] \stackrel{a}{\longrightarrow} P'$ as a proper subderivation (see the operational rules, and rule 2 in particular), it follows that the considered derivation of $E[\mu X.E/X] \stackrel{\alpha}{\longrightarrow} P'$ does not have a subderivation of $P \stackrel{\alpha}{\longrightarrow} P'$. Hence, by Lemma 9.2 there exists an X-closed process expression E' such that $E \stackrel{\alpha}{\longrightarrow} E'$ and $P' = E'[\mu X.E/X]$. □

Corollary 11. *Let G_0 and E be X-closed process expressions. If there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $G_0[\mu X.E/X]{=}P_0$ and $P_k \stackrel{\tau}{\longrightarrow} P_{k+1}$ for all $k \in \omega$, then there is an infinite sequence of X-closed process expressions $(G_k)_{k\in\omega}$ such that $P_k = G_k[\mu X.E/X]$ and, for all $k \in \omega$, either $G_k \stackrel{\tau}{\longrightarrow} G_{k+1}$ or X is exposed in G_k and $E \stackrel{\tau}{\longrightarrow} G_{k+1}$.*

Proof. We construct $(G_k)_{k\in\omega}$ with induction on k. Suppose that G_k with $G_k[\mu X.E/X] = P_k$ has already been constructed. Since $P_k \stackrel{\tau}{\longrightarrow} P_{k+1}$, by Lemma 9.2 there are two cases: either there is a G_{k+1} with $G_k \stackrel{\tau}{\longrightarrow} G_{k+1}$ and $P_{k+1} = G_{k+1}[\mu X.E/X]$, in which case we are done, or X is exposed in G_k and $\mu X.E \stackrel{\tau}{\longrightarrow} P_{k+1}$. In the latter case $E[\mu X.E/X] \stackrel{\tau}{\longrightarrow} P_{k+1}$ (see the operational rules, and rule 2 in particular). By Corollary 10 there exists an X-closed process expression G_{k+1} such that $E \stackrel{\tau}{\longrightarrow} G_{k+1}$ and $P_{k+1} = G_{k+1}[\mu X.E/X]$. □

Let E and E' be process expressions. We write $E \longrightarrow E'$ if there exists an $\alpha \in \mathcal{A}_\tau$ such that $E \stackrel{\alpha}{\longrightarrow} E'$, and denote by $\longrightarrow^*$ the reflexive-transitive closure of $\longrightarrow$. If $E \longrightarrow^* E'$, then we say that E' is reachable from E.

Proposition 12 ([7, Proposition 1])**.** *If E is a process expression, then the set of all expressions reachable from E is finite.*

On $\mathcal{E}$ we define an $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation $\longrightarrow \subseteq \mathcal{E} \times (\mathcal{V} \uplus \mathcal{A}_\tau) \times \mathcal{E}$ as the least ternary relation satisfying, besides the four rules of Section 2, also the rule

$$5\ \frac{}{X \stackrel{X}{\longrightarrow} \mathbf{0}}$$

for each $X \in \mathcal{V}$. Intuitively, the $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation treats a process expression E as the closed term obtained from E by replacing all free occurrences of the variable X by the closed process expression $X.\mathbf{0}$ in which X is interpreted as an action instead of as a recursion variable. Note that a variable X is exposed in an expression E according to Definition 8 iff $\exists F.\ E \stackrel{X}{\longrightarrow} F$, which is the case iff $E \stackrel{X}{\longrightarrow} \mathbf{0}$. Now let $\underline{\leftrightarrow}^{\Delta}_{bX}$ and $\underline{\leftrightarrow}^{\Delta}_{rbX}$ be defined exactly like $\underline{\leftrightarrow}^{\Delta}_{b}$ and $\underline{\leftrightarrow}^{\Delta}_{rb}$, but using the $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation instead of the $\mathcal{A}_\tau$-labelled one, and applying all definitions directly to expressions with free variables, instead of applying the lifting of Definition 7.

We proceed to show that on X-closed process expressions $\leftrightarrow_{bX}^{\Delta}$ coincides with $\leftrightarrow_{b}^{\Delta}$, and $\leftrightarrow_{rbX}^{\Delta}$ with $\leftrightarrow_{rb}^{\Delta}$. This characterisation, for weak and branching bisimilarity without preservation of divergence, stems from [19] and [7]. Here we use it solely to obtain Corollaries 15 and 16.

Lemma 13. *The relation*

$$\mathcal{B} = \{(E[P/X], F[P/X]) \mid E, F \text{ are } X\text{-closed, } E \leftrightarrow_{bX}^{\Delta} F,\ P \text{ is closed}\}$$

is a branching bisimulation satisfying (D$''$) of Proposition 3.

Proof. It is immediate from its definition that $\mathcal{B}$ is symmetric.

We show it satisfies (T). Suppose E, F are X-closed, $E \leftrightarrow_{bX}^{\Delta} F$ and P closed. Let $E[P/X] \xrightarrow{\alpha} P'$ for some $\alpha \in \mathcal{A}_\tau$. By Lemma 9.2 either there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E and $P \xrightarrow{\alpha} P'$. In the first case, since $E \leftrightarrow_{bX}^{\Delta} F$, there exist process expressions F' and F'' such that $F \twoheadrightarrow F'' \xrightarrow{(\alpha)} F'$, $E \leftrightarrow_{bX}^{\Delta} F''$ and $E' \leftrightarrow_{bX}^{\Delta} F'$. By Lemma 9.1 $F[P/X] \twoheadrightarrow F''[P/X] \xrightarrow{(\alpha)} F'[P/X]$. Furthermore, $E[P/X]\ \mathcal{B}\ F''[P/X]$ and $P' = E'[P/X]\ \mathcal{B}\ F'[P/X]$. In the second case, since X is exposed in E, we have that $E \xrightarrow{X} \mathbf{0}$ and hence, since $E \leftrightarrow_{bX}^{\Delta} F$, there exist process expressions F' and F'' such that $F \twoheadrightarrow F'' \xrightarrow{X} F'$, $E \leftrightarrow_{bX}^{\Delta} F''$ and $\mathbf{0} \leftrightarrow_{bX}^{\Delta} F'$. Moreover, since $F'' \xrightarrow{X} F'$, X is exposed in F'', so by Lemma 9.1 $F[P/X] \twoheadrightarrow F''[P/X] \xrightarrow{\alpha} P'$. Furthermore, $E[P/X]\ \mathcal{B}\ F''[P/X]$ and $P'\ \mathcal{B}\ P'$.

It remains to show that $\mathcal{B}$ satisfies (D$''$). Suppose E, F are X-closed, $E \leftrightarrow_{bX}^{\Delta} F$ and P is closed, and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $E[P/X] = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$. By Lemma 9.2 either there exists an infinite sequence of X-closed process expressions $(E_k)_{k\in\omega}$ such that $E_0 = E$, $E_k \xrightarrow{\tau} E_{k+1}$ and $P_{k+1} = E_{k+1}[P/X]$ for all $k \in \omega$, or there exists a finite sequence of X-closed process expressions $(E_i)_{i\leq k}$ for some $k \in \omega$ such that $E_0 = E$, $E_i \xrightarrow{\tau} E_{i+1}$ and $P_{i+1} = E_{i+1}[P/X]$ for all $i < k$, $E_k \xrightarrow{X} \mathbf{0}$ and $P \xrightarrow{\tau} P_{k+1}$. In the first case, since $E \leftrightarrow_{bX}^{\Delta} F$, using (D$''$), there exist a process expression F' such that $F \xrightarrow{\tau} F'$ and $E_k \leftrightarrow_{bX}^{\Delta} F'$ for some $k \in N$. By Lemma 9.1 $F[P/X] \xrightarrow{\tau} F'[P/X]$. Furthermore, $E_k[P/X]\ \mathcal{B}\ F'[P/X]$. In the second case, since $E \leftrightarrow_{bX}^{\Delta} F$, with induction on i there exists a sequence $F_0, \ldots, F_m, F_{m+1}$ and a mapping $\rho : \{0, \ldots, m\} \to \{0, \ldots, k\}$ with $\rho(m) = k$ such that $F = F_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_m \xrightarrow{X} F_{m+1}$ and $E_{\rho(j)} \leftrightarrow_{b}^{\Delta} F_j$ for all $j = 0, \ldots, m$. If $m = 0$, then X is exposed in F, so by Lemma 9.1 $F[P/X] \xrightarrow{\tau} P_{k+1}$. Furthermore, $P_{k+1}\ \mathcal{B}\ P_{k+1}$. If $m > 0$, then let $F' = F_1$. By Lemma 9.1 $F[P/X] \xrightarrow{\tau} F'[P/X]$. Furthermore, $E_{\rho(1)}[P/X]\ \mathcal{B}\ F'[P/X]$. □

For every $\alpha \in \mathcal{A}_\tau$ and $n \in \omega$, we define the closed process expression α^n inductively by $\alpha^0 = \mathbf{0}$ and $\alpha^{n+1} = \alpha.\alpha^n$. Note that, if $\alpha \neq \tau$, then $\alpha^i \leftrightarrow_{b}^{\Delta} \alpha^j$ implies $i = j$. Recall that we have assumed that $\mathcal{A}$ is non-empty; we now fix, for the remainder of this section, a particular action $a \in \mathcal{A}$.

Proposition 14. *Let E and F be X-closed process expressions. Then $E \leftrightarrow_{bX}^{\Delta} F$ iff $E \leftrightarrow_{b}^{\Delta} F$, and $E \leftrightarrow_{rbX}^{\Delta} F$ iff $E \leftrightarrow_{rb}^{\Delta} F$.*

Proof. We need to show that $E \leftrightarrow_{bX}^{\Delta} F$ iff $E[P/X] \leftrightarrow_{b}^{\Delta} F[P/X]$ for each closed process expression P, and likewise $E \leftrightarrow_{rbX}^{\Delta} F$ iff $E[P/X] \leftrightarrow_{rb}^{\Delta} F[P/X]$ for each closed process expression P.

"Only if": Lemma 13 immediately yields that $E \leftrightarrow_{bX}^{\Delta} F$ implies $E[P/X] \leftrightarrow_{b}^{\Delta} F[P/X]$ for each closed process expression P. Now let $E \leftrightarrow_{rbX}^{\Delta} F$ and $E[P/X] \xrightarrow{\alpha} P'$. By Lemma 9.2 either there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E and $P \xrightarrow{\alpha} P'$. In the first case, since $E \leftrightarrow_{rbX}^{\Delta} F$, there exist a process expression F' such that $F \xrightarrow{\alpha} F'$ and $E' \leftrightarrow_{bX}^{\Delta} F'$. By Lemma 9.1 $F[P/X] \xrightarrow{\alpha} F'[P/X]$. Furthermore, by Lemma 13 $P' = E'[P/X] \leftrightarrow_{b}^{\Delta} F'[P/X]$. In the second case, since X is exposed in E we have that $E \xrightarrow{X} \mathbf{0}$, and hence, since $E \leftrightarrow_{rbX}^{\Delta} F$, there exists a process expression F' such that $F \xrightarrow{X} F'$. By Lemma 9.1 $F[P/X] \xrightarrow{\alpha} P'$. Furthermore, $P' \leftrightarrow_{b}^{\Delta} P'$. The other clause follows by symmetry, thus yielding $E[P/X] \leftrightarrow_{rb}^{\Delta} F[P/X]$.

"If": Let E and F be X-closed process expressions. Since by Proposition 12 the set of all process expressions reachable from E and F is finite, there exists a natural number $n \in \omega$ such that for all G reachable from E or F it holds that $G \not\leftrightarrow_{b}^{\Delta} a^n$, and thus $G[a^{n+1}/X] \not\leftrightarrow_{b}^{\Delta} a^n$. Let

$$\mathcal{R} = \{(E', F') \mid E \longrightarrow^* E',\ F \longrightarrow^* F',\ E'[a^{n+1}/X] \leftrightarrow_{b}^{\Delta} F'[a^{n+1}/X]\}.$$

Claim: The symmetric closure of $\mathcal{R}$ is a branching bisimulation satisfying (D$''$) w.r.t. the $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation.

Proof of the claim: To prove that $\mathcal{R}$ satisfies condition (T) of Definition 1, let E' and F' be such that $E' \mathrel{\mathcal{R}} F'$, and suppose that $E' \xrightarrow{\alpha} E''$. Then $E'[a^{n+1}/X] \leftrightarroweq_b^\Delta F'[a^{n+1}/X]$ and, using Lemma 9.1, $E'[a^{n+1}/X] \xrightarrow{\alpha} E''[a^{n+1}/X]$. Since $E'[a^{n+1}/X] \leftrightarroweq_b^\Delta F'[a^{n+1}/X]$ there exist closed process expressions Q''' and Q'' such that $F'[a^{n+1}/X] \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'''$, $E'[a^{n+1}/X] \leftrightarroweq_b^\Delta Q''$ and $E''[a^{n+1}/X] \leftrightarroweq_b^\Delta Q'''$. By Lemma 9.2, using that $a \neq \tau$, there exists a X-closed process expression F'' such that $F' \twoheadrightarrow F''$ and $Q'' = F''[a^{n+1}/X]$; moreover, either there exists an X-closed process expression F''' such that $F'' \xrightarrow{(\alpha)} F'''$ and $Q''' = F'''[a^{n+1}/X]$, or X is exposed in F'' and $a^{n+1} \xrightarrow{\alpha} Q'''$. In the latter case we would have $E''[a^{n+1}/X] \leftrightarroweq_b^\Delta Q''' = a^n$, which is impossible by the choice of n. So the former case applies: we have $F' \twoheadrightarrow F'' \xrightarrow{(\alpha)} F'''$, $E' \mathrel{\mathcal{R}} F''$ and $E'' \mathrel{\mathcal{R}} F'''$. The case that $F' \xrightarrow{\alpha} F''$ proceeds by symmetry, so the symmetric closure of $\mathcal{R}$ satisfies condition (T).

To show that $\mathcal{R}$ (and its symmetric closure) satisfies (D$''$), let $(E_k)_{k\in\omega}$ be an infinite sequence of X-closed process expressions such that $E_k \xrightarrow{\tau} E_{k+1}$ for all $k \in \omega$, and let F_0 be such that $E_0 \mathrel{\mathcal{R}} F_0$. Then $E_0[a^{n+1}/X] \leftrightarroweq_b^\Delta F_0[a^{n+1}/X]$ and by Lemma 9.1 $E_k[a^{n+1}/X] \xrightarrow{\tau} E_{k+1}[a^{n+1}/X]$ for all $k\in\omega$. Using (D$''$), there exist a process expression Q' such that $F_0 \xrightarrow{\tau} Q'$ and $E_k[a^{n+1}/X] \leftrightarroweq_b^\Delta Q'$ for some $k \in N$. By Lemma 9.2, using that $a \neq \tau$, there exists a X-closed process expression F' such that $F_0 \xrightarrow{\tau} F'$ and $Q' = F'[a^{n+1}/X]$. Furthermore, $E_k \mathrel{\mathcal{R}} F'$.

Application of the claim: Let $E[P/X] \leftrightarroweq_b^\Delta F[P/X]$ for each closed process expression P. Then $E[a^{n+1}/X] \leftrightarroweq_b^\Delta F[a^{n+1}/X]$. The claim yields $E \leftrightarroweq_{bX}^\Delta F$.

Now let $E[P/X] \leftrightarroweq_{rb}^\Delta F[P/X]$ for each closed P. Then $E[a^{n+1}/X] \leftrightarroweq_{rb}^\Delta F[a^{n+1}/X]$. Suppose that $E \xrightarrow{\alpha} E'$ with $\alpha \in \mathcal{A}_\tau$. Then $E[a^{n+1}/X] \xrightarrow{\alpha} E'[a^{n+1}/X]$ by Lemma 9.1. So there exists a Q' with $F[a^{n+1}/X] \xrightarrow{\alpha} Q'$ and $E'[a^{n+1}/X] \leftrightarroweq_b^\Delta Q'$. By Lemma 9.2 either there exists an X-closed process expression F' such that $F \xrightarrow{\alpha} F'$ and $Q' = F'[a^{n+1}/X]$, or X is exposed in F and $a^{n+1} \xrightarrow{\alpha} Q'$. In the latter case we would have $E'[a^{n+1}/X] \leftrightarroweq_b^\Delta Q' = a^n$, which is impossible by the choice of n. So the former case applies, and $E' \mathrel{\mathcal{R}} F'$. The claim yields $E' \leftrightarroweq_{bX}^\Delta F'$. The other clause follows by symmetry, so $E \leftrightarroweq_{rbX}^\Delta F$. □

The following is an immediate corollary of Propositions 14, 3 and 4.

Corollary 15. *Let E and F be X-closed process expressions such that $E \leftrightarroweq_b^\Delta F$.*

1. *If $E \xrightarrow{\alpha} E'$, then there exist X-closed process expressions $F_0, \ldots, F_n$ and F' such that $F = F_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_n \xrightarrow{(\alpha)} F'$ such that $E \leftrightarroweq_b^\Delta F_i$ $(0 \leq i \leq n)$ and $E' \leftrightarroweq_b^\Delta F'$.*

2. *If X is exposed in E, then there exist $k \geq 0$ and X-closed process expressions $F_0, \ldots, F_k$ such that $F = F_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_k$, $E \leftrightarroweq_b^\Delta F_i$ $(0 \leq i \leq k)$, and X is exposed in F_k.*

3. *If there is an infinite sequence of X-closed process expressions $(E_k)_{k\in\omega}$ such that $E = E_0$ and $E_k \xrightarrow{\tau} E_{k+1}$, then there exists an X-closed process expression F' such that $F \xrightarrow{\tau} F'$ and $E_k \leftrightarroweq_b^\Delta F'$ for some $k \in \omega$.*

Similarly, by combining Propositions 14 and Definition 5 we get the following corollary.

Corollary 16. *Let E and F be X-closed process expressions such that $E \leftrightarroweq_{rb}^\Delta F$. If $E \xrightarrow{\alpha} E'$, then there exists an X-closed process expression F' such that $F \xrightarrow{\alpha} F'$ and $E' \leftrightarroweq_b^\Delta F'$.*

3.2 Rooted divergence-preserving branching bisimulation up to $\leftrightarroweq_b^\Delta$

As was already illustrated by Milner [20], a suitable up-to relation is a crucial tool in the proof that a behavioural equivalence is compatible with the recursion construct. In [7], Milner's notion of weak bisimulation up to weak bisimilarity is adapted to branching bisimulation up to branching bisimilarity. Here we make two further modifications. Not only do we add a divergence condition; we also incorporate rootedness into the relation.

Definition 17. Let $\mathcal{R}$ be a symmetric binary relation on $\mathcal{P}$, and denote by $\mathcal{R}^u$ the relation $\leftrightarrow_b^\Delta ; \mathcal{R} ; \leftrightarrow_b^\Delta$. We say that $\mathcal{R}$ is a *rooted divergence-preserving branching bisimulation up to* $\leftrightarrow_b^\Delta$ if for all $P, Q \in \mathcal{P}$ such that $P \mathcal{R} Q$ the following three conditions are satisfied:

(U1) if $P \xrightarrow{\alpha} P'$, then there exist Q' such that $Q \xrightarrow{\alpha} Q'$, and $P' \mathcal{R}^u Q'$.

(U2) if $P \twoheadrightarrow P'' \xrightarrow{(\alpha)} P'$, then there exist Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P'' \mathcal{R}^u Q''$ and $P' \mathcal{R}^u Q'$.

(U3) if there exists an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$, and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there also exists an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $Q = Q_0$, and $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_{\sigma(\ell)} \mathcal{R}^u Q_\ell$ for all $\ell \in \omega$.

Proposition 18. *Let P and Q be closed process expressions and let $\mathcal{R}$ be a rooted divergence-preserving branching bisimulation up to $\leftrightarrow_b^\Delta$. If $P \mathcal{R} Q$, then $P \leftrightarrow_{rb}^\Delta Q$.*

Proof. If $P \mathcal{R} Q$ and $P \xrightarrow{\alpha} P'$, then since $\mathcal{R}$ satisfies condition (U1) of Definition 17, there exists a Q' such that $Q \xrightarrow{\alpha} Q'$ and $P' \mathcal{R}^u Q'$. Furthermore, since $\mathcal{R}$ is symmetric, whenever $P \mathcal{R} Q$ also $Q \mathcal{R} P$, so if $Q \xrightarrow{\alpha} Q'$, then by condition (U1) of Definition 17 there exists a P' such that $P \xrightarrow{\alpha} P'$ and $Q' \mathcal{R}^u P'$. It remains to establish that $P' \leftrightarrow_b^\Delta Q'$, and for this, it suffices by Proposition 3 to prove that $\mathcal{R}^u$ is a branching bisimulation satisfying (D$'$).

Note that, since $\leftrightarrow_b^\Delta$ and $\mathcal{R}$ are both symmetric, also $\mathcal{R}^u$ is symmetric.

To prove that $\mathcal{R}^u$ satisfies (T), let P_0, P_1, Q_0 and Q_1 be closed process expressions such that $P_1 \leftrightarrow_b^\Delta P_0 \mathcal{R} Q_0 \leftrightarrow_b^\Delta Q_1$, and suppose that $P_1 \xrightarrow{\alpha} P_1'$. Since $P_1 \leftrightarrow_b^\Delta P_0$ and $\leftrightarrow_b^\Delta$ satisfies (T), there exist P_0' and P_0'' such that $P_0 \twoheadrightarrow P_0'' \xrightarrow{(\alpha)} P_0'$, $P_1 \leftrightarrow_b^\Delta P_0''$ and $P_1' \leftrightarrow_b^\Delta P_0'$. So it follows by condition (U2) of Definition 17 that there exist Q_0' and Q_0'' such that $Q_0 \twoheadrightarrow Q_0'' \xrightarrow{(\alpha)} Q_0'$, $P_0'' \mathcal{R}^u Q_0''$ and $P_0' \mathcal{R}^u Q_0'$. Hence, since $Q_0 \leftrightarrow_b^\Delta Q_1$, by Lemma 2 there exist closed process expressions Q_1' and Q_1'' such that $Q_1 \twoheadrightarrow Q_1'' \xrightarrow{(\alpha)} Q_1'$, $Q_0'' \leftrightarrow_b^\Delta Q_1''$ and $Q_0' \leftrightarrow_b^\Delta Q_1'$. Note, moreover, that $P_1 \leftrightarrow_b^\Delta P_0'' \mathcal{R}^u Q_0'' \leftrightarrow_b^\Delta Q_1''$ whence $P_1 \mathcal{R}^u Q_1''$, and $P_1' \leftrightarrow_b^\Delta P_0' \mathcal{R}^u Q_0' \leftrightarrow_b^\Delta Q_1'$ whence $P_1' \mathcal{R}^u Q_1'$.

It remains to prove that $\mathcal{R}^u$ satisfies (D$'$) of Proposition 3. To this end, let P_0, P_1, Q_0 and Q_1 be closed process expressions such that $P_1 \leftrightarrow_b^\Delta P_0 \mathcal{R} Q_0 \leftrightarrow_b^\Delta Q_1$, and suppose that there exists an infinite sequence of closed process expressions $(P_{1,k})_{k\in\omega}$ such that $P_1 = P_{1,0}$ and $P_{1,k} \xrightarrow{\tau} P_{1,k+1}$. Then, since $P_1 \leftrightarrow_b^\Delta P_0$, by Proposition 3, there exists an infinite sequence of closed process expressions $(P_{0,k})_{\ell\in\omega}$ and a mapping $\sigma_P : \omega \to \omega$ such that $P_0 = P_{0,0}$, $P_{0,\ell} \xrightarrow{\tau} P_{0,\ell+1}$ and $P_{1,\sigma_P(\ell)} \leftrightarrow_b^\Delta P_{0,\ell}$ for all $\ell \in \omega$. Hence, since $P_0 \mathcal{R} Q_0$ and $\mathcal{R}$ is a divergence-preserving branching bisimulation up to $\leftrightarrow_b^\Delta$, there exists an infinite sequence of closed process expressions $(Q_{0,m})_{m\in\omega}$ and a mapping $\sigma_{P,Q} : \omega \to \omega$ such that $Q_0 = Q_{0,0}$, $Q_{0,m} \xrightarrow{\tau} Q_{0,m+1}$ and $P_{0,\sigma_{P,Q}(m)} \mathcal{R}^u Q_{0,m}$ for all $m \in \omega$. Hence, since $Q_0 \leftrightarrow_b^\Delta Q_1$, by Proposition 3, there exists an infinite sequence of closed process expressions $(Q_{1,n})_{n\in\omega}$ and a mapping $\sigma_Q : \omega \to \omega$ such that $Q_1 = Q_{1,0}$, $Q_{1,n} \xrightarrow{\tau} Q_{0,n+1}$ and $Q_{0,\sigma_Q(n)} \leftrightarrow_b^\Delta Q_{1,n}$ for all $n \in \omega$. We define

$$\sigma = \sigma_P \circ \sigma_{P,Q} \circ \sigma_Q \quad ,$$

and then we have that $P_{1,\sigma(n)} \leftrightarrow_b^\Delta ; \mathcal{R}^u ; \leftrightarrow_b^\Delta Q_{1,n}$, and hence $P_{1,\sigma(n)} \mathcal{R}^u Q_{1,n}$ for all $n \in \omega$. □

To prove that $\leftrightarrow_{rb}^\Delta$ is compatible with $\mu X.\_$ means to prove that if $E \leftrightarrow_{rb}^\Delta F$, then $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$. We first do this in the special case that E and F are X-closed. A crucial step in this proof will be to show that if $E \leftrightarrow_{rb}^\Delta F$ for X-closed process expressions E and F, then the symmetric closure $\mathcal{R}_{E,F}$ of the relation

$$\{(G[\mu X.E/X], G[\mu X.F/X]) \mid G \in \mathcal{E},\ FV(G) \subseteq \{X\}\} \tag{1}$$

is a rooted branching bisimulation up to $\leftrightarrow_b^\Delta$. The result then follows by taking $G := X$. Until Corollary 25 we fix X-closed process expressions E and F such that $E \leftrightarrow_{rb}^\Delta F$.

Lemma 19. *For all X-closed process expressions G, if $G[\mu X.E/X] \xrightarrow{\alpha} P$, then there exists a Q such that $G[\mu X.F/X] \xrightarrow{\alpha} Q$ and $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q$.*

Proof. Let G be an X-closed process expression, and suppose that $G[\mu X.E/X] \xrightarrow{\alpha} P$; we proceed with induction on a derivation of this transition. By Lemma 9.2 there are two cases: either the transition under consideration stems directly from G, i.e., there exists a G' such that $G \xrightarrow{\alpha} G'$ and $P = G'[\mu X.E/X]$, or X is exposed in G, $\mu X.E \xrightarrow{\alpha} P$ and every derivation of $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation.

In the first case, Lemma 9.1 implies $G[\mu X.F/X] \xrightarrow{\alpha} G'[\mu X.F/X]$ and $P = G'[\mu X.E/X] \mathcal{R}_{E,F} G'[\mu X.F/X]$, so, since $\leftrightarrow_b^\Delta$ is reflexive, also $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta G'[\mu X.F/X]$.

In the second case, since the considered derivation of the transition $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation, and the last rule applied in this subderivation must be rule 2, it follows that the considered derivation of $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P$ as a proper subderivation. So by the induction hypothesis there exists a Q such that $E[\mu X.F/X] \xrightarrow{\alpha} Q$ and $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q$. Furthermore, since $E \leftrightarrow_{rb}^\Delta F$, whence $E[\mu X.F/X] \leftrightarrow_{rb}^\Delta F[\mu X.F/X]$, it follows that there exists an R such that $F[\mu X.F/X] \xrightarrow{\alpha} R$ and $Q \leftrightarrow_b^\Delta R$. It follows from $F[\mu X.F/X] \xrightarrow{\alpha} R$ that $\mu X.F \xrightarrow{\alpha} R$. Since X is exposed in G, Lemma 9.1 yields $G[\mu X.F/X] \xrightarrow{\alpha} R$. From $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q$ and $Q \leftrightarrow_b^\Delta R$ it follows that $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta R$. □

As an immediate corollary to Lemma 19 we get that if $E \leftrightarrow_{rb}^\Delta F$, then $\mathcal{R}_{E,F}$ satisfies the first condition of rooted divergence-preserving branching bisimulations up to $\leftrightarrow_b^\Delta$.

Corollary 20. *$\mathcal{R}_{E,F}$ satisfies condition (U1) of Definition 17.*

With a little more work, Lemma 19 can also be used to derive that $\mathcal{R}_{E,F}$ satisfies the second condition of rooted divergence-preserving branching bisimulations up to $\leftrightarrow_b^\Delta$. To this end, we first prove the following lemma.

Lemma 21. *Let P and Q be closed process expressions. If $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q$ and $P \xrightarrow{\alpha} P'$, then there exist Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q''$ and $P' \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q'$.*

Proof. Suppose that $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q$ and $P \xrightarrow{\alpha} P'$. Then there exists an R such that $P \mathcal{R}_{E,F} R \leftrightarrow_b^\Delta Q$, and according to the definition of $\mathcal{R}_{E,F}$ there exists an X-closed process expression G such that either $P = G[\mu X.E/X]$ and $R = G[\mu X.F/X]$ or $P = G[\mu X.F/X]$ and $R = G[\mu X.E/X]$. There is clearly no loss of generality in assuming that $P = G[\mu X.E/X]$ and $R = G[\mu X.F/X]$. By Lemma 19, there exists an R' such that $R \xrightarrow{\alpha} R'$ and $P' \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta R'$. Hence, since $R \leftrightarrow_b^\Delta Q$, there exist Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $R \leftrightarrow_b^\Delta Q''$ and $R' \leftrightarrow_b^\Delta Q'$. It follows that $P \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q''$ and $P' \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q'$, so the proof of the lemma is complete. □

Applying Lemma 21 with induction on the length of a transition sequence that gives rise to $P \twoheadrightarrow P'' \xrightarrow{\alpha} P'$, it is straightforward to establish the following corollary.

Corollary 22. *$\mathcal{R}_{E,F}$ satisfies condition (U2) of Definition 17.*

Proof. Let $P_n, \dots, P_0, P', Q \in \mathcal{P}$, such that $P_n \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q$, $P_{i+1} \xrightarrow{\tau} P_i$ for all $0 \le i < n$, and $P_0 \xrightarrow{(\alpha)} P'$; we prove with induction on n that there exists Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$ $P_0 \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q''$, and $P' \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q'$. Note that, since $\leftrightarrow_b^\Delta$ is reflexive, it then follows that $\mathcal{R}_{E,F}$ satisfies (U2) of Definition 17.

If $n = 0$, then we distinguish two cases: If $\alpha = \tau$ and $P_0 = P'$, then we can take $Q'' = Q' = Q$. If $\alpha \neq \tau$ or $P_0 \neq P'$, then $P_0 \xrightarrow{\alpha} P'$, and the result follows from Lemma 21.

Suppose that $n > 0$. Then $P_n \xrightarrow{\tau} P_{n-1}$, so by Lemma 21 there exist Q_n and Q_{n-1} such that $Q \twoheadrightarrow Q_n \xrightarrow{(\tau)} Q_{n-1}$, $P_n \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q_n$, and $P_{n-1} \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q_{n-1}$. Furthermore, by the induction hypothesis, there exist Q' and Q'' such that $Q_{n-1} \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P_0 \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q''$, and $P' \mathcal{R}_{E,F} \,;\, \leftrightarrow_b^\Delta Q'$. Clearly, we then also have that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$. □

It remains to establish that $\mathcal{R}_{E,F}$ satisfies the third condition of rooted divergence-preserving branching bisimulations up to $\leftrightarrow_b^\Delta$.

Lemma 23. *Let G and H be X-closed process expressions such that $G \leftrightarrow_b^\Delta H$. If there exists an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $G[\mu X.E/X]{=}P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there also exists an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $H[\mu X.F/X] = Q_0$, $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$, and $P_{\sigma(\ell)}\ \mathcal{R}_{E,F}\ ;\leftrightarrow_b^\Delta Q_\ell$ for all $\ell \in \omega$.*

Proof. Suppose that there exists an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $G[\mu X.E/X] = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. By Corollary 11 there is an infinite sequence of X-closed process expressions $(G_k)_{k\in\omega}$ such that $P_k = G_k[\mu X.E/X]$ and either $G_k \xrightarrow{\tau} G_{k+1}$ or $E \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$. We shall define simultaneously, with induction on ℓ, an infinite sequence of X-closed process expressions $(H_\ell)_{\ell\in\omega}$ with $H_0 = H$ and $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$, and a mapping $\sigma : \omega \to \omega$, such that $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$. This will suffice, because, for all $\ell \in \omega$, defining Q_ℓ as $H_\ell[\mu X.F/X]$ we obtain $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_{\sigma(\ell)} = G_{\sigma(\ell)}[\mu X.E/X]\ \mathcal{R}_{E,F}\ G_{\sigma(\ell)}[\mu X.F/X] \leftrightarrow_b^\Delta H_\ell[\mu X.F/X] = Q_\ell$.

Suppose, by way of induction hypothesis, that H_ℓ and $\sigma(\ell)$ have been defined already, such that $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$. By Corollary 11 there are two cases:

1. $G_{\sigma(\ell)+k} \xrightarrow{\tau} G_{\sigma(\ell)+k+1}$ for all $k \in \omega$. Then, since $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$, by Corollary 15.3 there exists an X-closed process expression H' such that $H_\ell \xrightarrow{\tau} H'$ and $G_{\sigma(\ell)+k} \leftrightarrow_b^\Delta H'$ for some $k \in \omega$. We define $H_{\ell+1} = H'$ and $\sigma(\ell+1) = \sigma(\ell)+k$. Now $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$ by Lemma 9.1 and $G_{\sigma(\ell+1)} \leftrightarrow_b^\Delta H_{\ell+1}$.

2. There is a $k \in \omega$ such that $G_{\sigma(\ell)+i} \xrightarrow{\tau} G_{\sigma(\ell)+i+1}$ for all $i < k$, X is exposed in $G_{\sigma(\ell)+k}$ and $E \xrightarrow{\tau} G_{\sigma(\ell)+k+1}$. Then, since $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$, by Corollary 15.1 and with induction on i there exists a sequence $H'_0, \ldots, H'_m$ and a mapping $\rho : \{0,\ldots,m\} \to \{0,\ldots,k\}$ with $\rho(m){=}k$ such that $H_\ell = H'_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} H'_m$ and $G_{\sigma(\ell)+\rho(j)} \leftrightarrow_b^\Delta H'_j$. Using Corollary 15.2, we may furthermore assume that X is exposed in H'_m.

   If $m > 0$, then we define $H_{\ell+1} = H'_1$ and $\sigma(\ell+1) = \sigma(\ell)+\rho(1)$. Now $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$ by Lemma 9.1 and $G_{\sigma(\ell+1)} \leftrightarrow_b^\Delta H_{\ell+1}$.

   So it remains to consider the case that $m = 0$. Since $E \leftrightarrow_{rb}^\Delta F$, there exists, by Corollary 16, an X-closed process expression F' such that $F \xrightarrow{\tau} F'$ and $G_{\sigma(\ell)+k+1} \leftrightarrow_b^\Delta F'$. We now define $H_{\ell+1} = F'$ and $\sigma(\ell+1) = \sigma(\ell)+k+1$. We then have that $G_{\sigma(\ell+1)} = G_{\sigma(\ell)+k+1} \leftrightarrow_b^\Delta H_{\ell+1}$, and $F[\mu X.F/X] \xrightarrow{\tau} H_{j+1}[\mu X.F/X]$ by Lemma 9.1. So $\mu X.F \xrightarrow{\tau} H_{j+1}[\mu X.F/X]$ by rule 2, and Lemma 9.1 yields $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$, using that X is exposed in H_ℓ. ☐

From Lemma 23 with $G = H$ we immediately get the following corollary.

Corollary 24. *$\mathcal{R}_{E,F}$ satisfies condition (U3) of Definition 17.*

The relation $\mathcal{R}_{E,F}$ is symmetric by definition and we have now also proved that it satisfies conditions (U1), (U2) and (U3), so we have established the following result.

Corollary 25. *$\mathcal{R}_{E,F}$ is a rooted divergence-preserving branching bisimulation up to $\leftrightarrow_b^\Delta$.*

3.3 The main results

We can now establish that $\leftrightarrow_{rb}^\Delta$ is compatible with $\alpha.\_$, $\mu X.\_$ and $+$.

Proposition 26. *If $E \leftrightarrow_{rb}^\Delta F$, then $\alpha.E \leftrightarrow_{rb}^\Delta \alpha.F$ for all $\alpha \in \mathcal{A}_\tau$, $E + H \leftrightarrow_{rb}^\Delta F + H$ and $H + E \leftrightarrow_{rb}^\Delta H + F$ for all process expressions H, and $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$.*

Proof. To prove that $\leftrightarrow_{rb}^{\Delta}$ is compatible with $\alpha.$ and $+$ is straightforward. (First, establish the property for closed terms, and then use that substitution distributes over $\alpha.$ and $+$.)

It remains to prove that $\leftrightarrow_{rb}^{\Delta}$ is compatible with $\mu X.\_$, i.e., that $E \leftrightarrow_{rb}^{\Delta} F$ implies $\mu X.E \leftrightarrow_{rb}^{\Delta} \mu X.F$. Note that in the special case that E and F are X-closed this immediately follows from Corollary 25 and Proposition 18. Now, for the general case, let E and F be process expressions and suppose that $E \leftrightarrow_{rb}^{\Delta} F$. Let $X, \vec{Y}$ be a sequence of variables that at least includes the variables with a free occurrence in E or F, and such that X does not occur in $\vec{Y}$. Then, according to the definition of $\leftrightarrow_{rb}^{\Delta}$ on process expressions with free variables (Definition 7), we have that, for every closed process expression P and for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{Y}$, $E[P,\vec{P}/X,\vec{Y}] \leftrightarrow_{rb}^{\Delta} F[P,\vec{P}/X,\vec{Y}]$. So, clearly, also $E[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^{\Delta} F[\vec{P}/\vec{Y}]$, and since $E[\vec{P}/\vec{Y}]$ and $F[\vec{P}/\vec{Y}]$ are X-closed, it follows that $\mu X.E[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^{\Delta} \mu X.F[\vec{P}/\vec{Y}]$. Since X is not among the $\vec{Y}$, we may conclude that $(\mu X.E)[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^{\Delta} (\mu X.F)[\vec{P}/\vec{Y}]$ for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{Y}$, and hence $\mu X.E \leftrightarrow_{rb}^{\Delta} \mu X.F$. □

We have now obtained our main result that $\leftrightarrow_{rb}^{\Delta}$ is a congruence. In fact, it is the coarsest contained in $\leftrightarrow_{b}^{\Delta}$.

Theorem 27. *The relation $\leftrightarrow_{rb}^{\Delta}$ is the coarsest congruence contained in $\leftrightarrow_{b}^{\Delta}$.*

Proof. By Propositions 6 and 26, the relation $\leftrightarrow_{rb}^{\Delta}$ is a congruence. To prove that it is coarsest, it suffices to prove that for every relation $\mathcal{R} \subseteq \leftrightarrow_{b}^{\Delta}$ that is compatible with $+$ we have that $\mathcal{R} \subseteq \leftrightarrow_{rb}^{\Delta}$. Let P and Q be closed process expressions, and suppose that $P \mathrel{\mathcal{R}} Q$.

Since by Proposition 12 the set of closed process expressions reachable from P and Q is finite and $\mathcal{A}$ is non-empty, there exists a natural number $n \in \omega$ such that for all R reachable from P or Q it holds that $R \not\leftrightarrow_{b}^{\Delta} a^n$. This implies that for all R reachable from P or Q it holds that $R \not\leftrightarrow_{b}^{\Delta} P + a^{n+1}$ and $R \not\leftrightarrow_{b}^{\Delta} Q + a^{n+1}$.

Since $\mathcal{R}$ is compatible with $+$, we have that $P + a^{n+1} \mathrel{\mathcal{R}} Q + a^{n+1}$, and hence $P + a^{n+1} \leftrightarrow_{b}^{\Delta} Q + a^{n+1}$. To prove (R1), suppose that $P \xrightarrow{\alpha} P'$. Then $P + a^{n+1} \xrightarrow{\alpha} P'$, so by Lemma 2 there exist closed process expressions Q' and Q'' such that $Q + a^{n+1} \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P + a^{n+1} \leftrightarrow_{b}^{\Delta} Q''$ and $P' \leftrightarrow_{b}^{\Delta} Q'$. Since $a \neq \tau$, we have that $Q'' = Q + a^{n+1}$, for otherwise Q'' is reachable from Q and $Q'' \leftrightarrow_{b}^{\Delta} P + a^{n+1}$. Moreover, $Q'' \xrightarrow{\alpha} Q'$, for otherwise $P' \leftrightarrow_{b}^{\Delta} Q' = Q'' = Q + a^{n+1}$. Condition (R2) follows by symmetry. □

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