Unique Solutions of Guarded Recursive Equations

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25 August 2025

Recursive specifications

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$$\begin{aligned} X &= a.X + b.Y \\ Y &= c.X + d.Y \end{aligned}$$

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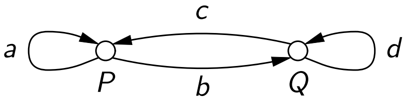

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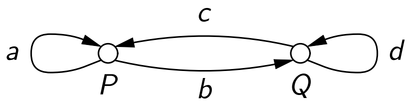

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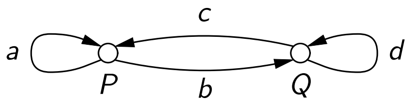

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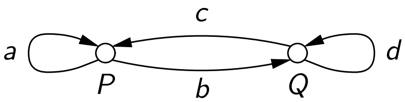

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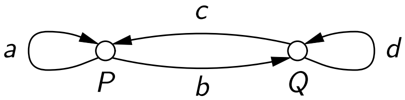

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Goal: For a general class of languages,
find a general class of recursive specifications
and show that they have unique solutions
up to strong bisimilarity, or some other equivalence.

Syntax

We postulate an infinite collection $Var \ni X, Y, Z, \ldots$ of *variables*.

A *signature* Σ is a collection of operator symbols f,
each with an *arity* n.

The set of *terms* or *expressions* $\mathbb{T}(\Sigma)$ is defined inductively:

- variables are terms,
- if $f \in \Sigma$ has arity n and $E_1, \ldots, E_n \in \mathbb{T}(\Sigma)$ then $f(E_1, \ldots, E_n) \in \mathbb{T}(\Sigma)$,

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A term is *closed* if it has no *free* variables.
Closed terms denote *processes* in process algebra.

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The meaning of a process is a state in a labelled transition system. For a language given by a signature Σ, the set $\mathrm{T}(\Sigma)$ of closed terms or processes form the states.

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$$a.x \xrightarrow{a} x \qquad \begin{array}{l} X = a.X + b.Y \\ Y = c.X + d.Y \end{array} \qquad \frac{\langle \mathcal{S}_X | \mathcal{S} \rangle \xrightarrow{a} z}{\langle X | \mathcal{S} \rangle \xrightarrow{a} z} \qquad \frac{a.\langle X | \mathcal{S} \rangle + b.\langle Y | \mathcal{S} \rangle \xrightarrow{a} z}{\langle X | \mathcal{S} \rangle \xrightarrow{a} z}$$

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$$\frac{x \xrightarrow{a} x'}{en_R(x) \xrightarrow{a} en_R(x')} \qquad \frac{x \stackrel{a}{\not\longrightarrow}}{en_R(x) \xrightarrow{a} en_R(x)}\ (a \in R)$$

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$$\frac{x \xrightarrow{a} x'}{x \triangleright y \xrightarrow{a} x' \triangleright y}\ (a \neq sw) \qquad \frac{x \xrightarrow{sw} x'}{x \triangleright y \xrightarrow{sw} y \triangleright x'}$$

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The meaning of negative premises

$$a.x \xrightarrow{a} x \qquad \frac{\langle \mathcal{S}_X | \mathcal{S} \rangle \xrightarrow{a} z}{\langle X | \mathcal{S} \rangle \xrightarrow{a} z}$$

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Let $\mathcal{S}$ be the recursive specification $X = b.0 \parallel_b en_b(X)$.

Under the assumption $\langle X | \mathcal{S} \rangle \stackrel{b}{\not\longrightarrow}$ one proves $\langle X | \mathcal{S} \rangle \xrightarrow{b} 0 \parallel_b en_b(\langle X | \mathcal{S} \rangle)$.
However, there is no outright proof of any transition $\langle X | \mathcal{S} \rangle \xrightarrow{b} Q$.

The meaning of negative premises

The well founded semantics assigns to every set $\mathcal{T}$ of transition rules a 3-valued labelled transition system (CT, PT) with $CT \subseteq PT$, such that

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CT^- is the set of negative literals that do not deny any $\beta \in PT^+$ $\quad \beta \in CT^+_\lambda$ iff $\mathcal{T} \vdash \frac{CT^-}{\beta}$.

Here $P \stackrel{c}{\not\longrightarrow}$ denies $P \stackrel{c}{\longrightarrow} Q$.

In fact, it assign the unique 3-valued LTS with these properties.

Main theorem

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guarded recursive specifications have unique solutions
up to strong bisimulation equivalence / ready simulation equivalence.

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Example: $X = a.X$ is a guarded recursive specification
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Example: $X = a.X$ is a guarded recursive specification
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If P and Q and processes that satisfy

$$P =_B a.P \qquad \text{and} \qquad Q =_B a.Q$$

then they are both solutions of this rec. spec. up to $=_B$.

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The theorem now says that $P =_B Q$.

Bisimulation equivalence

A *bisimulation* $\mathscr{B}$ is a binary relation on processes such that,

- if $P \mathrel{\mathscr{B}} Q$ and $P \xrightarrow{a} P'$, then $\exists Q'$ with $Q \xrightarrow{a} Q'$ and $P' \mathrel{\mathscr{B}} Q'$,
- if $P \mathrel{\mathscr{B}} Q$ and $Q \xrightarrow{a} Q'$, then $\exists P'$ with $P \xrightarrow{a} P'$ and $P' \mathrel{\mathscr{B}} Q'$.

Write $P \equiv_B Q$ if $P \mathrel{\mathscr{B}} Q$ for some bisimulation $\mathscr{B}$.

Ready simulation equivalence

A *ready simulation* $\mathscr{B}$ is a binary relation on processes such that,

- if $P \mathrel{\mathscr{B}} Q$ and $P \xrightarrow{a} P'$, then $\exists Q'$ with $Q \xrightarrow{a} Q'$ and $P' \mathrel{\mathscr{B}} Q'$,
- if $P \mathrel{\mathscr{B}} Q$ and $Q \xrightarrow{a}$, then $P \xrightarrow{a}$

Write $P \sqsubseteq_{RS} Q$ if $P \mathrel{\mathscr{B}} Q$ for some ready simulation $\mathscr{B}$.

Write $P \equiv_{RS} Q$ if $P \sqsubseteq_{RS} Q$ and $Q \sqsubseteq_{RS} P$.

simulation equivalence

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Write $P \sqsubseteq_S Q$ if $P \mathrel{\mathscr{B}} Q$ for some simulation $\mathscr{B}$.

Write $P \equiv_S Q$ if $P \sqsubseteq_S Q$ and $Q \sqsubseteq_S P$.

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  The above defines the ntyft/ntxft format
- No *lookahead*.

A language in ready simulation format

$$a.x \xrightarrow{a} x \qquad \frac{\langle \mathcal{S}_X | \mathcal{S} \rangle \xrightarrow{a} z}{\langle X | \mathcal{S} \rangle \xrightarrow{a} z}$$

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$$\frac{x \xrightarrow{a} x'}{x \rhd y \xrightarrow{a} x' \rhd y}\ (a \neq sw) \qquad \frac{x \xrightarrow{sw} x'}{x \rhd y \xrightarrow{sw} y \rhd x'}$$

$$\frac{en_b(x \rhd y) \xrightarrow{a} x'}{h(x,y,z) \xrightarrow{a} h(z,x',y)}$$

$\mathbf{0}$ $\qquad a \in Act \qquad R, S \subseteq Act$

Guards, and guarded occurrences of variables

An occurrence of a variable x in a term E is *guarded* if it occurs in a subterm $f(E_1, \ldots, E_n)$ of E, such that, for some $1 \leq i \leq n$,

- the occurrence lays in E_i,
- and f guards its i^{th} argument.

f *guards* its i^{th} argument if for each transition rule

$$\frac{H}{f(x_1, \ldots, x_n) \xrightarrow{a} E}$$

there is no premise $G \xrightarrow{b} y$ or $G \stackrel{b}{\not\longrightarrow}$ in H,
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The variable x does not.

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Example: the variable y occurs guarded in the term $en_b(x \triangleright y)$.
The variable x does not.
Example: the variable z occurs guarded in the term $h(x, y, z)$.
The variable x does not. The occurrence of y is guarded.

Guarded recursive specifications

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A recursive specification $\mathcal{S}$ is *manifestly guarded* if $\forall X, Y \in V_{\mathcal{S}}$ all occurrences of the variable Y in the term $\mathcal{S}_X$ are guarded.

It is *guarded* if it can be converted into a manifestly guarded recursive specification by repeated substitution of expressions $\mathcal{S}_Y$ for variables $Y \in V_{\mathcal{S}}$ occurring in expressions $\mathcal{S}_Z$ for $Z \in V_{\mathcal{S}}$.

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Example: The recursive specification $\{X = h(a.b.sw.X, X, X)\}$ is manifestly guarded.

Example: $\{X = Y \triangleright X,\ Y = (a.b.sw.X) \triangleright Y\}$ is guarded.

Main theorem

For any language with a SOS in the ready simulation format,
guarded recursive specifications have unique solutions
up to strong bisimulation equivalence / ready simulation equivalence.

Main theorem

For any language with a SOS in the positive ready simulation format, guarded recursive specifications have unique solutions up to simulation equivalence.

The recursive specification principle (RSP)

Guarded recursive specifications have at most one solution.

$$\frac{\vec{P} \equiv \mathcal{S}[\vec{P}] \qquad \vec{Q} \equiv \mathcal{S}[\vec{Q}]}{\vec{P} \equiv \vec{Q}}$$

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Let $\mathcal{S}$ be

$$\begin{aligned} X &= a.X + b.Y \\ Y &= c.X + d.Y \end{aligned}$$

Let $\vec{P} = (P_X, P_Y)$ be a vector (pair) of two processes, such that

$$\begin{aligned} P_X &\equiv a.P_X + b.P_Y \\ P_Y &\equiv c.P_X + d.P_Y \end{aligned}$$

Then I write $\vec{P} \equiv \mathcal{S}[\vec{P}]$. In this case $\vec{P}$ is a *solution* of $\mathcal{S}$.

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$$\frac{\vec{P} \equiv \mathcal{S}[\vec{P}] \qquad \vec{Q} \equiv \mathcal{S}[\vec{Q}]}{\vec{P} \equiv \vec{Q}} \qquad \frac{\vec{P} \sqsubseteq \mathcal{S}[\vec{P}] \qquad \vec{Q} \sqsubseteq \mathcal{S}[\vec{Q}]}{\vec{P} \quad ?? \quad \vec{Q}}.$$

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$$\begin{aligned} P_X &\sqsubseteq a.P_X + b.P_Y \\ P_Y &\sqsubseteq c.P_X + d.P_Y \end{aligned}$$

Then I write $\vec{P} \sqsubseteq \mathcal{S}[\vec{P}]$. In this case $\vec{P}$ is a *presolution* of $\mathcal{S}$.

The recursive specification principle (RSP)

Guarded recursive specifications have at most one solution.

$$\frac{\vec{P} \equiv \mathcal{S}[\vec{P}] \qquad \vec{Q} \equiv \mathcal{S}[\vec{Q}]}{\vec{P} \equiv \vec{Q}} \qquad \frac{\vec{P} \sqsupseteq \mathcal{S}[\vec{P}] \qquad \vec{Q} \sqsupseteq \mathcal{S}[\vec{Q}]}{\vec{P} \quad ?? \quad \vec{Q}}.$$

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If $\equiv$ is the kernel of $\sqsubseteq$ (meaning $P \equiv Q \Leftrightarrow P \sqsubseteq Q \wedge Q \sqsubseteq P$) then RSP for $\sqsubseteq$ implies RSP for $\equiv$.

RSP for the simulation preorder

Let $\mathcal{S} := \{X = a.X + b.0\}$.
Take $P := \langle Y | \{Y = a.Y\} \rangle$ and $Q := \langle Z | \{Z = a.Z \; + a.b.0\} \rangle$.

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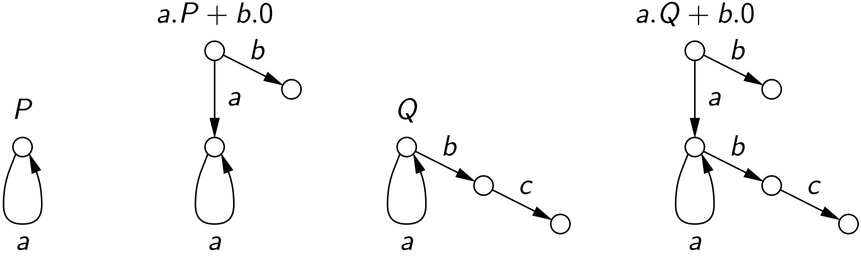

Now $P \sqsubseteq_S \mathcal{S}[P] = a.P + b.0$ and $\mathcal{S}[Q] = a.Q + b.0 \sqsubseteq_S Q$.
Hence, by RSP, $P \sqsubseteq_S Q$.

The recursive definition principle (RDP)

Guarded recursive specifications have at least one solution.

$$\langle V_{\mathcal{S}} | \mathcal{S} \rangle \equiv \mathcal{S}[\langle V_{\mathcal{S}} | \mathcal{S} \rangle]$$

Here $\langle V_{\mathcal{S}} | \mathcal{S} \rangle$ is the vector with components $\langle X | \mathcal{S} \rangle$ for each bound variable X of $\mathcal{S}$. It is the default solution of $\mathcal{S}$.

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In combination with RDP, RSP can be restated as:

$$\frac{\vec{P} \equiv \mathcal{S}[\vec{P}]}{\vec{P} \equiv \langle V_{\mathcal{S}} | \mathcal{S} \rangle} \qquad \frac{\vec{P} \sqsubseteq \mathcal{S}[\vec{P}]}{\vec{P} \sqsubseteq \langle V_{\mathcal{S}} | \mathcal{S} \rangle} \qquad \frac{\mathcal{S}[\vec{Q}] \sqsubseteq \vec{Q}}{\langle V_{\mathcal{S}} | \mathcal{S} \rangle \sqsubseteq \vec{Q}}\ . \tag{1}$$

Lookahead

$$\frac{x \xrightarrow{a} y \quad y \xrightarrow{b} z}{f(x) \xrightarrow{b} z}$$

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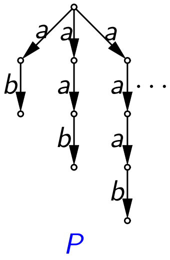
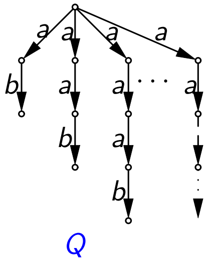
This is why I have no unique-solution results for systems with lookahead.

Linear time semantics





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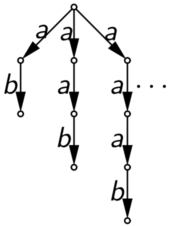
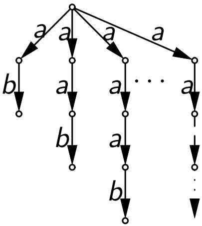


P



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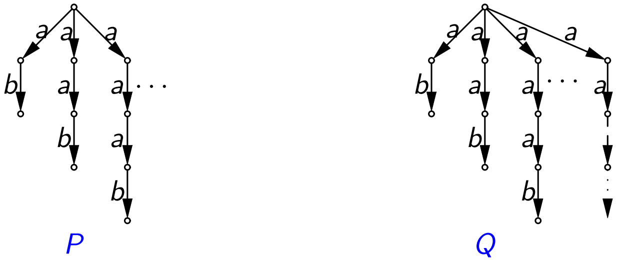

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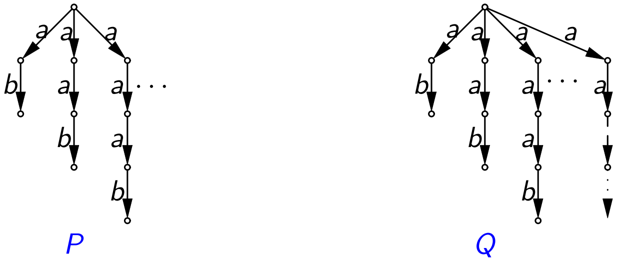

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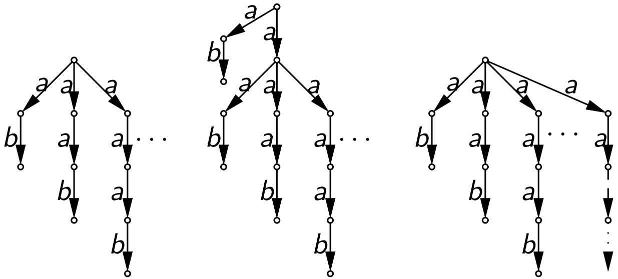

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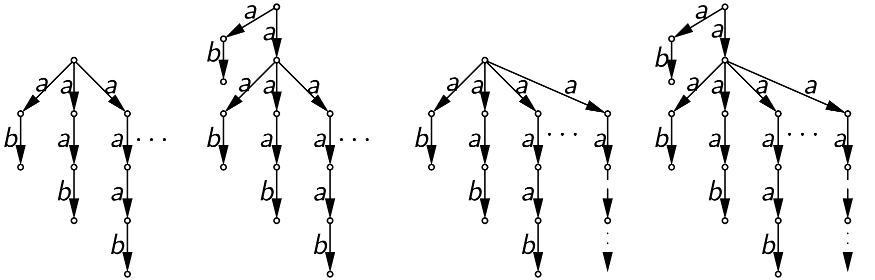

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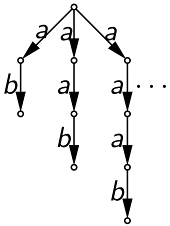

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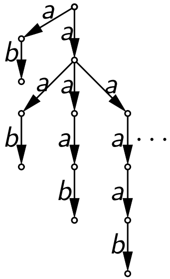

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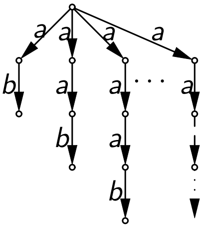

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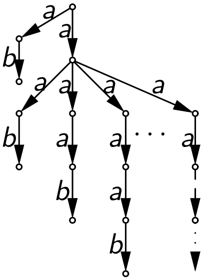

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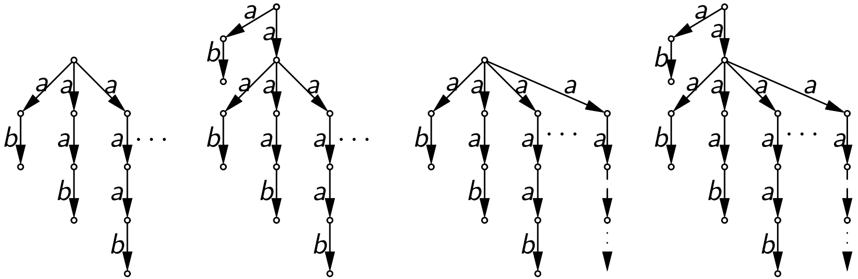

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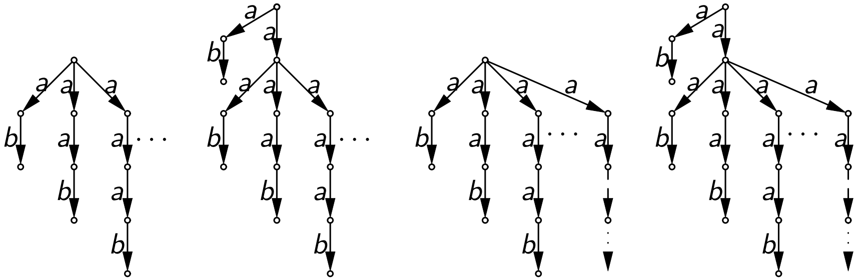

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For such semantics, guarded recursive specifications do not have unique solutions.

Full congruence for guarded recursion

$$\mathcal{S} \equiv \mathcal{S}' \Rightarrow \langle V_{\mathcal{S}} | \mathcal{S} \rangle \equiv \langle V_{\mathcal{S}} | \mathcal{S}' \rangle$$

i.e.

if $\mathcal{S}_Y \equiv \mathcal{S}'_Y$ for all $Y \in V_{\mathcal{S}}$ then $\langle X | \mathcal{S} \rangle \equiv \langle X | \mathcal{S}' \rangle$ for all $X \in \mathcal{S}$.

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This holds if either $\mathcal{S}$ or $\mathcal{S}'$ is guarded.

Ground-complete axiomatisations

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The same works for any GSOS language!

Future work

It would be great to extend this work to branching bisimilarity and/or other weak equivalences.