Asymptotic behaviour of some families of orthonormal polynomials and an associated Hilbert space

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Abstract

We characterise the asymptotic behaviour of families of orthonormal polynomials whose recursion coefficients satisfy certain conditions, satisfied for example by the Hermite polynomials and, more generally, by families with recursion coefficients of the form $c(n+1)^p$ for $0 < p < 1$. We then use this result to show that, in a Hilbert space associated with a family of orthonormal polynomials whose recursion coefficients satisfy such conditions, every two sinusoids of unequal positive frequencies are mutually orthogonal.[1]

[1]This paper is dedicated to my wife Sharon Younghi Choi; without her love and patience this work would have never seen the daylight.

All equalities and asymptotic estimates in this paper have been validated using *Mathematica*™ software package (Wolfram Research, Inc.). Only the basic functionality of Mathematica has been used. These Mathematica calculations are available online at `http://www.cse.unsw.edu.au/~ignjat/diff/christoffel.zip`. The files can be viewed using either *Mathematica* software package or *Wolfram CDF Player* which is available for free at `http://demonstrations.wolfram.com/download-cdf-player.html`.

1 Introduction

Let $\gamma_n > 0$ be the recursion coefficients which correspond to a symmetric positive definite family of ***orthonormal*** polynomials $P_n(\omega)$, i.e., such that

$$\gamma_n P_{n+1}(\omega) = \omega\, P_n(\omega) - \gamma_{n-1}\, P_{n-1}(\omega), \tag{1.1}$$

and let s_n be the first and d_n the second order forward finite differences of these recursion coefficients:

$$s_n = \gamma_{n+1} - \gamma_n; \qquad\qquad d_n = s_{n+1} - s_n.$$

We will consider families of orthonormal polynomials such that the corresponding recursion coefficients γ_n satisfy the following conditions[1].

($\mathcal{C}_1$) There exist n_0, m_0 such that $\gamma_{n+m} > \gamma_n$ holds for all $n \geq n_0$ and all $m \geq m_0$.

A sequence γ_n which satisfies condition ($\mathcal{C}_1$) will be called an *almost increasing sequence*; an *almost decreasing sequence* is defined in an analogous way. Clearly, every increasing sequence is also an almost increasing sequence with $n_0 = 0$ and $m_0 = 1$.

($\mathcal{C}_2$) $\gamma_n \to \infty$; ($\mathcal{C}_3$) $s_n \to 0$; ($\mathcal{C}_4$) $\gamma_n d_n \to 0$;

($\mathcal{C}_5$) $\displaystyle\sum_{n=0}^{\infty} \frac{|s_n|}{\gamma_n^2} < \infty$; ($\mathcal{C}_6$) $\displaystyle\sum_{n=0}^{\infty} \frac{|d_n|}{\gamma_n} < \infty$; ($\mathcal{C}_7$) $\displaystyle\sum_{j=0}^{\infty} \frac{1}{\gamma_j}$ diverges;

($\mathcal{C}_8$) for some $\kappa > 1$, the sum $\displaystyle\sum_{j=0}^{\infty} \frac{1}{\gamma_j^k}$ converges.

****Propositions marked with a star are proved in the Appendix.***

Lemma 1.* *Conditions ($\mathcal{C}_1$)-($\mathcal{C}_8$) are satisfied by the Hermite polynomials and, more generally, by families with recursion coefficients defined by $\gamma_n = c(n+1)^p$ for any $0 < p < 1$ and $c > 0$.* □

Lemma 2.* *Sequence $\gamma_n = (n+1)^p + p(-1)^n(n+1)^{p-1}$ also satisfies conditions ($\mathcal{C}_1$)-($\mathcal{C}_8$) whenever $0 < p < 1/2$; condition ($\mathcal{C}_1$) is satisfied with $n_0 = 0$ and $m_0 = 2$.* □

The goal of this paper is to prove the following theorem, obtain its two corollaries below as well as its harmonic analysis consequence, Theorem 23.

Theorem 3. *Assume that the recursion coefficients $\gamma_n > 0$ which correspond to a symmetric positive definite family of orthonormal polynomials $P_n(\omega)$ satisfy conditions ($\mathcal{C}_1$)- ($\mathcal{C}_8$); then*

$$0 < \lim_{n\to\infty} \frac{P_{2n}^2(\omega) + P_{2n+1}^2(\omega)}{\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}} < \infty.$$

□

Corollary 4.* *If γ_n satisfy the conditions of Theorem 3, then*

$$0 < \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P_k^2(\omega)}{\sum_{k=0}^{n} \frac{1}{\gamma_k}} = \lim_{n\to\infty} \frac{P_{2n}^2(\omega) + P_{2n+1}^2(\omega)}{\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}} < \infty.$$

□

Corollary 5.* *If $\gamma_n = c(n+1)^p$ for some $c > 0$ and some p such that $0 < p < 1$, then*

$$0 < \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P_k^2(\omega)}{(n+1)^{1-p}} < \infty.$$

□

Corollary 5 partly proves our conjecture from [1].

[1] In all likelihood, these conditions can be weakened and some, such as ($\mathcal{C}_7$) dropped, but, for the sake of simplicity, we stick with the present choice.

2 A Representation of Orthogonal Polynomials

In order to investigate the asymptotic behaviour of the sum $P^2_{2n}(\omega)+P^2_{2n+1}(\omega)$ as $n\to\infty$, we first note that, if we define complex valued functions

$$E_n(\omega)=i^{2n}P_{2n}(\omega)+i^{2n+1}P_{2n+1}(\omega)=(-1)^n(P_{2n}(\omega)+i\,P_{2n+1}(\omega)),$$

then $P^2_{2n}(\omega)+P^2_{2n+1}(\omega)=|E_n(\omega)|^2$. We now look for a recurrence which $E_n(\omega)$ satisfy.

Using the three term recurrence (1.1) we get

$$i^{2n}P_{2n}(\omega)=\frac{i\,\omega}{\gamma_{2n-1}}\,i^{2n-1}P_{2n-1}(\omega)+\frac{\gamma_{2n-2}}{\gamma_{2n-1}}\,i^{2n-2}P_{2n-2}(\omega),$$

and, by eliminating $P_{2n}(\omega)$ from the recurrence expression for $P_{2n+1}(\omega)$, we get

$$i^{2n+1}P_{2n+1}(\omega)=-\frac{\omega^2-\gamma^2_{2n-1}}{\gamma_{2n-1}\gamma_{2n}}\,i^{2n-1}P_{2n-1}(\omega)+\frac{i\,\omega\,\gamma_{2n-2}}{\gamma_{2n}\gamma_{2n-1}}\,i^{2n-2}P_{2n-2}(\omega).$$

If we add these two equations together, after some simplifications we obtain

$$E_n(\omega)=\left(\frac{\gamma_{2n-2}}{\gamma_{2n-1}}+i\,\frac{\omega\,\gamma_{2n-2}}{\gamma_{2n-1}\gamma_{2n}}\right)i^{2n-2}P_{2n-2}(\omega)+\left(-\frac{\omega^2-\gamma^2_{2n-1}}{\gamma_{2n-1}\gamma_{2n}}+i\,\frac{\omega}{\gamma_{2n-1}}\right)i^{2n-1}P_{2n-1}(\omega).$$

Since

$$i^{2n-2}P_{2n-2}(\omega)=\frac{E_{n-1}(\omega)+\overline{E_{n-1}(\omega)}}{2};\qquad i^{2n-1}P_{2n-1}(\omega)=\frac{E_{n-1}(\omega)-\overline{E_{n-1}(\omega)}}{2},\tag{2.1}$$

after a corresponding substitution and some simplifications we obtain

$$\begin{aligned}E_n(\omega)=&\left(-\frac{\omega^2-\gamma^2_{2n-1}}{2\gamma_{2n-1}\gamma_{2n}}+\frac{\gamma_{2n-2}}{2\gamma_{2n-1}}+i\left(\frac{\omega}{2\gamma_{2n-1}}+\frac{\omega\,\gamma_{2n-2}}{2\gamma_{2n-1}\gamma_{2n}}\right)\right)E_{n-1}(\omega)+\\&\left(\frac{\omega^2-\gamma^2_{2n-1}}{2\gamma_{2n-1}\gamma_{2n}}+\frac{\gamma_{2n-2}}{2\gamma_{2n-1}}+i\left(-\frac{\omega}{2\gamma_{2n-1}}+\frac{\omega\,\gamma_{2n-2}}{2\gamma_{2n-1}\gamma_{2n}}\right)\right)\overline{E_{n-1}(\omega)}.\end{aligned}\tag{2.2}$$

Thus, while polynomials $P_n(\omega)$ satisfy a three term recurrence, (2.2) is just a two term recurrence for $E_n(\omega)$.

Since the families of orthonormal polynomials considered in this paper are symmetric, we will restrict our attention to $\omega>0$; in all of our propositions the case when $\omega=0$ can easily be handled separately. Moreover, we will assume that $\omega>0$ is fixed and, to make our formulas more readable, we will usually suppress ω in our notation; thus, for example, we will write E_n instead of $E_n(\omega)$.

To get a more compact form of equality (2.2) we define for all $n\geq 1$,

$$\mathrm{cs}(n)=-\frac{\omega^2-\gamma^2_{2n-1}}{2\gamma_{2n-1}\gamma_{2n}}+\frac{\gamma_{2n-2}}{2\gamma_{2n-1}};\qquad \mathrm{sn}(n)=\frac{\omega}{2\gamma_{2n-1}}+\frac{\omega\,\gamma_{2n-2}}{2\gamma_{2n-1}\gamma_{2n}};\qquad \zeta_n=\mathrm{cs}(n)+i\,\mathrm{sn}(n);\tag{2.3}$$

$$\widetilde{\mathrm{cs}}(n)=\frac{\omega^2-\gamma^2_{2n-1}}{2\gamma_{2n-1}\gamma_{2n}}+\frac{\gamma_{2n-2}}{2\gamma_{2n-1}};\qquad \widetilde{\mathrm{sn}}(n)=-\frac{\omega}{2\gamma_{2n-1}}+\frac{\omega\,\gamma_{2n-2}}{2\gamma_{2n-1}\gamma_{2n}};\qquad \widetilde{\zeta}_n=\widetilde{\mathrm{cs}}(n)+i\,\widetilde{\mathrm{sn}}(n);\tag{2.4}$$

Equation (2.2) now becomes

$$E_n=\zeta_nE_{n-1}+\widetilde{\zeta}_n\overline{E_{n-1}}.\tag{2.5}$$

We also define

$$\alpha_n=|\zeta_n|;\qquad \widetilde{\alpha}_n=|\widetilde{\zeta}_n|;\qquad \theta_n=\arg(\zeta_n);\qquad \widetilde{\theta}_n=\arg(\widetilde{\zeta}_n).$$

Thus,

$$\mathrm{cs}(n)=\alpha_n\cos\theta_n;\qquad \mathrm{sn}(n)=\alpha_n\sin\theta_n;$$
$$\widetilde{\mathrm{cs}}(n)=\widetilde{\alpha}_n\cos\widetilde{\theta}_n;\qquad \widetilde{\mathrm{sn}}(n)=\widetilde{\alpha}_n\sin\widetilde{\theta}_n.$$

Let $\Phi_{-1} = 0$ and for all $n \geq 0$ let Φ_n be the least number larger than Φ_{n-1} such that

$$\Phi_n \equiv \arg E_n \mod 2\pi.$$

Thus, for $n \geq 0$, Φ_n is a sequence of positive reals, monotonically increasing in n; in signal processing terminology, Φ_n is the *unwound phase of* E_n. We now define

$$\Delta_n = \Phi_n - \Phi_{n-1} > 0. \tag{2.6}$$

Finally, we let $\mathrm{A}(n) = |E_n|$; using (2.1) we have

$$P_{2n}(\omega) = (-1)^n A(n) \cos \Phi_n; \qquad P_{2n+1}(\omega) = (-1)^n A(n) \sin \Phi_n.$$

Taking the complex conjugate of both sides of equation (2.5) we obtain

$$\overline{E_n} = \overline{\zeta_n}\ \overline{E_{n-1}} + \overline{\widetilde{\zeta}_n} E_{n-1}. \tag{2.7}$$

Multiplying the corresponding sides of (2.5) and (2.7) we get

$$|E_n|^2 = (|\zeta_n|^2 + |\widetilde{\zeta}_n|^2)|E_{n-1}|^2 + 2\,\mathfrak{Re}(\zeta_n \overline{\widetilde{\zeta}_n} E_{n-1}^2),$$

i.e.,

$$\mathrm{A}(n)^2 = \mathrm{A}(n-1)^2 \left(\alpha_n^2 + \widetilde{\alpha}_n^2 + 2\alpha_n \widetilde{\alpha}_n \cos(2\Phi_{n-1} + \theta_n - \widetilde{\theta}_n)\right). \tag{2.8}$$

Let us define

$$\mu(0) = \mathrm{A}(0); \quad \mu(n) = \sqrt{{\alpha_n}^2 + \widetilde{\alpha}_n^2 + 2\alpha_n \widetilde{\alpha}_n \cos(2\Phi_{n-1} + \theta_n - \widetilde{\theta}_n)}, \quad (n > 0); \tag{2.9}$$

then (2.8) is equivalent to

$$\mathrm{A}(n) = A(n-1)\mu(n). \tag{2.10}$$

Consequently,

$$\mathrm{A}(n) = \prod_{j=0}^{n} \mu(n)$$

and

$$\frac{P_{2n}(\omega)^2 + P_{2n+1}(\omega)^2}{\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}} = \frac{\prod_{j=0}^{n} \mu(j)^2}{\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}}.$$

Taking the logarithm of both sides and letting

$$\mathcal{S}_n = 2\sum_{j=0}^{n} \ln \mu(j) - \ln\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right),$$

we conclude that, in order to prove Theorem 3, it is enough to prove that $\mathcal{S}_n$ converges to a finite limit as $n \to \infty$.

Let us define

$$\lambda_0 = \frac{1}{\frac{1}{\gamma_0} + \frac{1}{\gamma_1}}; \qquad \lambda_n = \frac{\frac{1}{\gamma_{2n-2}} + \frac{1}{\gamma_{2n-1}}}{\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}}, \quad (n \geq 1).$$

We can now represent $-\ln\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)$ as a telescopic sum,

$$-\ln\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right) = \sum_{j=1}^{n+1} \ln \lambda_{j-1}$$

and obtain

$$\mathcal{S}_n = 2\ln \mu(0) + \ln \lambda_n + \sum_{j=1}^{n} \left(2\ln \mu(j) + \ln \lambda_{j-1}\right). \tag{2.11}$$

Before proceeding with the proof of convergence of $\mathcal{S}_n$, we must first prove some elementary properties of the basic sequences $\alpha_n, \widetilde{\alpha}_n, \theta_n, \widetilde{\theta}_n$ and Δ_n.

3 Properties of the Basic Sequences

Let us define

$$\epsilon_n = \gamma_{2n-2}\gamma_{2n} - \gamma_{2n-1}^2;$$

it is easy to verify that

$$\epsilon_n = \gamma_{2n-1}d_{n-2} - s_{2n-2}s_{2n-1}. \tag{3.1}$$

The following Lemma can be verified by straightforward computations.

Lemma 6.

$$(a)\ \ \mathrm{cs}(n) = \left(1 - \frac{s_{2n-1}}{\gamma_{2n}}\right)\left(1 - \frac{\omega^2 - \epsilon_n}{2\,\gamma_{2n-1}^2}\right); \qquad (b)\ \ \widetilde{\mathrm{cs}}(n) = \frac{\omega^2 + \epsilon_n}{2\,\gamma_{2n-1}\gamma_{2n}}; \tag{3.2}$$

$$(a)\ \ \mathrm{sn}(n) = \frac{\omega}{\gamma_{2n}}\left(1 + \frac{d_{2n-2}}{2\,\gamma_{2n-1}}\right); \qquad (b)\ \ \widetilde{\mathrm{sn}}(n) = \frac{\omega\,(d_{2n-2} - 2s_{2n-1})}{2\,\gamma_{2n-1}\gamma_{2n}}; \tag{3.3}$$

$$(a)\ \ \theta_n = \arg\left(1 - \frac{\omega^2 - \epsilon_n}{2\gamma_{2n-1}^2} + i\,\frac{\omega}{\gamma_{2n-1}}\left(1 + \frac{d_{2n-2}}{2\gamma_{2n-1}}\right)\right); \quad (b)\ \ \widetilde{\theta}_n = \arg\left(1 + \frac{\epsilon_n}{\omega^2} + i\,\frac{d_{2n-2} - 2s_{2n-1}}{\omega}\right); \tag{3.4}$$

$$(a)\ \ \alpha_n^2 = \widetilde{\alpha}_n^2 + 1 - \frac{2s_{2n-1} - d_{2n-2}}{\gamma_{2n}}; \qquad (b)\ \ \widetilde{\alpha}_n^2 = \frac{\omega^4}{4\gamma_{2n-1}^2\gamma_{2n}^2}\left(\left(1 + \frac{\epsilon_n}{\omega^2}\right)^2 + \left(\frac{2s_{2n-1} - d_{2n-2}}{\omega}\right)^2\right); \tag{3.5}$$

$$\mathrm{sn}(n)^2 = \widetilde{\alpha}_n^2 + \frac{\omega^2}{\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{4\,\gamma_{2n}^2}\left(1 + \frac{\epsilon_n}{\omega^2}\right)^2 - \frac{2s_{2n-1} - d_{2n-2}}{\gamma_{2n}}\right). \tag{3.6}$$

□

We now use conditions ($\mathcal{C}_2$), ($\mathcal{C}_3$) and ($\mathcal{C}_4$) as well as equality (3.1) to conclude that

$$\frac{\omega}{\gamma_{2n-1}}, s_{2n-1}, d_{2n-2}, \epsilon_n \to 0. \tag{3.7}$$

These facts together with (3.5) and (3.6) imply that

$$\alpha_n \to 1; \qquad \widetilde{\alpha}_n \to 0; \qquad \theta_n \to 0; \qquad \widetilde{\theta}_n \to 0. \tag{3.8}$$

We now return to equation (2.7) which for $n > 0$ can be written as

$$A(n)\mathrm{e}^{-\,i\,\Phi_n} = A(n-1)\left(\alpha_n \mathrm{e}^{-\,i(\Phi_{n-1}+\theta_n)} + \widetilde{\alpha}_n \mathrm{e}^{i(\Phi_{n-1}-\widetilde{\theta}_n)}\right). \tag{3.9}$$

Multiplying both sides by $\mathrm{e}^{i(\Phi_{n-1}+\theta_n)}/A(n-1)$ we get

$$\frac{A(n)}{A(n-1)}\,\mathrm{e}^{i(\theta_n - \Delta_n)} = \alpha_n + \widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}. \tag{3.10}$$

This together with (2.10) implies that

$$\mu(n) = \left|\alpha_n + \widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right|, \tag{3.11}$$

and that for an integer k such that $|k| \leq 1$,

$$\theta_n - \Delta_n = \arg\!\left(\alpha_n + \widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right) + 2k\pi. \tag{3.12}$$

Note that by (3.8)

$$\left|\arg\!\left(\alpha_n + \widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right)\right| \leq \arcsin\frac{\widetilde{\alpha}_n}{\alpha_n} < \frac{\pi}{2}; \tag{3.13}$$

see Figure 3.1.

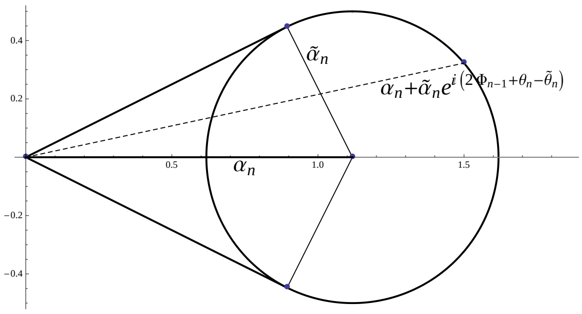

Figure 3.1:

Since (3.3) implies $\mathrm{sn}(n) > 0$, we get $\theta_n > 0$; on the other hand (3.6) and (3.7) imply that eventually $\mathrm{sn}(n) > \widetilde{\alpha}_n$. This, together with (3.13) implies

$$\theta_n = \arcsin\frac{\mathrm{sn}(n)}{\alpha_n} > \arcsin\frac{\widetilde{\alpha}_n}{\alpha_n} \geq \left|\arg\left(\alpha_n + \widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right)\right|.$$

Consequently, for all sufficiently large n,

$$\theta_n > \left|\arg\left(\alpha_n + \widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right)\right|. \tag{3.14}$$

However, since $0 < \theta_n \leq \pi$ and $0 \leq \Delta_n < 2\pi$, (3.12) and (3.14) imply

$$\theta_n - \Delta_n = \arg\left(\alpha_n + \widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right). \tag{3.15}$$

This, together with (3.13), also implies

$$\theta_n - \arcsin\left(\frac{\widetilde{\alpha}_n}{\alpha_n}\right) \leq \Delta_n \leq \theta_n + \arcsin\left(\frac{\widetilde{\alpha}_n}{\alpha_n}\right). \tag{3.16}$$

4 A Few More Calculations

Equations (3.11) and (3.15) express $\mu(n)$ and Δ_n via Φ_{n-1}. For a reason which will be clear later[1], we need to represent $\mu(n-1)$ and Δ_{n-1} also via Φ_{n-1}, rather than Φ_{n-2}. To this end, we use equations (2.5) and (2.7) to express E_{n-1} in terms of E_n and $\overline{E_n}$, obtaining

$$E_{n-1} = \frac{\overline{\zeta_n} E_n - \widetilde{\zeta}_n \overline{E_n}}{|\zeta_n|^2 - |\widetilde{\zeta}_n|^2}. \tag{4.1}$$

Note that (4.1) implies

$$\frac{E_{n-1}}{E_n} = \frac{\overline{\zeta_n} - \widetilde{\zeta}_n \frac{\overline{E_n}}{E_n}}{|\zeta_n|^2 - |\widetilde{\zeta}_n|^2},$$

which, together with (2.6), in turn yields

$$\frac{A_{n-1}}{A_n}\mathrm{e}^{-i\,\Delta_n} = \frac{\overline{\zeta_n} - \widetilde{\zeta}_n \mathrm{e}^{-i\,2\Phi_n}}{|\zeta_n|^2 - |\widetilde{\zeta}_n|^2},$$

i.e.,

$$\frac{\mathrm{e}^{-i\,\Delta_n}}{\mu(n)} = \frac{\alpha_n \mathrm{e}^{-i\,\theta_n} - \widetilde{\alpha}_n \mathrm{e}^{-i(2\Phi_n-\widetilde{\theta}_n)}}{\alpha_n^2 - \widetilde{\alpha}_n^2}. \tag{4.2}$$

Multiplying both sides by $\mathrm{e}^{i\,\theta_n}$ we get

$$\frac{\mathrm{e}^{i(\theta_n-\Delta_n)}}{\mu(n)} = \frac{\alpha_n - \widetilde{\alpha}_n \mathrm{e}^{-i(2\Phi_n-\theta_n-\widetilde{\theta}_n)}}{\alpha_n^2 - \widetilde{\alpha}_n^2}.$$

[1] Point Φ_{n-1} can be seen as a sampling point for an integrand, contained in the interval $[\Phi_{n-1} - \Delta_{n-1}, \Phi_{n-1} + \Delta_n]$, figuring in a Riemann sum for a corresponding integral; see the comment after (5.55).

This implies

$$\mu(n)=\frac{\alpha_n^2-\widetilde{\alpha}_n^2}{\left|\alpha_n-\widetilde{\alpha}_n \mathrm{e}^{-\,i(2\Phi_n-\theta_n-\widetilde{\theta}_n)}\right|}, \tag{4.3}$$

and, using the same reasoning as in the derivation of (3.15),

$$\theta_n-\Delta_n=\arg\left(\alpha_n-\widetilde{\alpha}_n \mathrm{e}^{-\,i(2\Phi_n-\theta_n-\widetilde{\theta}_n)}\right). \tag{4.4}$$

Finally, substituting n with $n-1$ in (4.3) and (4.4) we get

$$\mu(n-1)=\frac{\alpha_{n-1}^2-\widetilde{\alpha}_{n-1}^2}{\left|\alpha_{n-1}-\widetilde{\alpha}_{n-1} \mathrm{e}^{-\,i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\right|}; \tag{4.5}$$

$$\theta_{n-1}-\Delta_{n-1}=\arg\left(\alpha_{n-1}-\widetilde{\alpha}_{n-1} \mathrm{e}^{-\,i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\right). \tag{4.6}$$

The following Lemma is proved by direct calculations and series expansions; it follows directly from the basic equalities summarised in the Appendix (equations 8.6-8.13).

Lemma 7.

$$\theta_n=\frac{\omega}{\gamma_{2n-1}}+\mathrm{O}\left(\frac{1}{\gamma_{2n-1}^3}\right); \tag{4.7}$$

$$\arcsin\left(\frac{\widetilde{\alpha}_n}{\alpha_n}\right)=\frac{\omega^2}{2\gamma_{2n-1}^2}\left(1+\mathrm{O}\left(\frac{1}{\gamma_{2n-1}^4}+\frac{s_{2n-1}}{\gamma_{2n-1}^3}+\gamma_{2n-1}d_{2n-2}+s_{2n-1}^2\right)\right). \tag{4.8}$$

□

From (3.16), (4.7) and (4.8) we get that for all sufficiently large n,

$$\frac{\omega}{\gamma_{2n-1}}-\frac{\omega^2}{\gamma_{2n-1}^2}<\Delta_n<\frac{\omega}{\gamma_{2n-1}}+\frac{\omega^2}{\gamma_{2n-1}^2}. \tag{4.9}$$

Using condition $(\mathcal{C}_7)$ we obtain

$$\Phi_n=\Phi_0+\sum_{k=1}^{n}\Delta_k\sim\sum_{k=0}^{n}\frac{\omega}{\gamma_{2k-1}}\to\infty. \tag{4.10}$$

5 Representing $\mathcal{S}_n$ as a Riemann Sum

A part of our strategy is to represent $\mathcal{S}_n$ as a Riemann sum. Using (3.11) and (4.5) we get

$$\begin{aligned}\ln\mu(n)+\ln\mu(n-1)=\ln(\alpha_{n-1}^2-\widetilde{\alpha}_{n-1}^2)+\ln\left|\alpha_n+\widetilde{\alpha}_n \mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right|-\\ \ln\left|\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\right|.\end{aligned} \tag{5.1}$$

Similarly, from (3.12) and (4.6) we also get that

$$\Delta_{n-1}+\Delta_n=\theta_{n-1}+\theta_n-\arg\big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{-\,i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\big)-\arg\big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\big). \tag{5.2}$$

Note that (see Figure 3.1)

$$-\frac{\pi}{2}<\arg\big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{-\,i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\big)<\frac{\pi}{2},$$

and

$$-\frac{\pi}{2}<\arg\big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\big)<\frac{\pi}{2}.$$

Since for all $z\in\mathbb{C}$ which are outside the branch cut $(-\infty,0]$ of the logarithm function we have

$$\ln|z|=\frac{1}{2}\big(\ln\overline{z}+\ln z\big);\qquad\qquad \arg z=\frac{i}{2}\big(\ln\overline{z}-\ln z\big),$$

equations (5.1) and (5.2) can be transformed into

$$\ln\mu(n)+\ln\mu(n-1)=\ln(\alpha_{n-1}^2-\widetilde{\alpha}_{n-1}^2)+\frac{1}{2}\ln\Big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{-\,i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\Big)+\frac{1}{2}\ln\Big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\Big)-$$
$$\frac{1}{2}\ln\Big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\Big)-\frac{1}{2}\ln\Big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{-\,i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\Big);\tag{5.3}$$

$$\Delta_{n-1}+\Delta_n=\theta_{n-1}+\theta_n-\frac{i}{2}\ln\big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\big)+\frac{i}{2}\ln\big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{-\,i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\big)-$$
$$\frac{i}{2}\ln\big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{-\,i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\big)+\frac{i}{2}\ln\big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\big).\tag{5.4}$$

Thus, quite remarkably and very fortunately for our proof, expressions $\ln\mu(n)+\ln\mu(n-1)$ as well as $\Delta_{n-1}+\Delta_n$ are both obtained via the same logarithms, $\ln\left(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}\right)$ and $\ln\left(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}\right)$. Let us define for all $n\geq 1$,

$$F_n(z)=2\ln\lambda_{n-1}+2\ln(\alpha_{n-1}^2-\widetilde{\alpha}_{n-1}^2)+\ln\left(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{-\,i(\theta_n-\widetilde{\theta}_n)}z^{-1}\right)-\ln\left(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{i(\theta_{n-1}+\widetilde{\theta}_{n-1})}z^{-1}\right)+$$
$$\ln\Big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{i(\theta_n-\widetilde{\theta}_n)}z\Big)-\ln\Big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{-\,i(\theta_{n-1}+\widetilde{\theta}_{n-1})}z\Big);\tag{5.5}$$

$$G_n(z)=2(\theta_n+\theta_{n-1})-i\ln\left(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{-\,i(\theta_n-\widetilde{\theta}_n)}z^{-1}\right)+i\ln\left(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{i(\theta_{n-1}+\widetilde{\theta}_{n-1})}z^{-1}\right)+$$
$$i\ln\Big(\alpha_n+\widetilde{\alpha}_n\mathrm{e}^{i(\theta_n-\widetilde{\theta}_n)}z\Big)-i\ln\Big(\alpha_{n-1}-\widetilde{\alpha}_{n-1}\mathrm{e}^{-\,i(\theta_{n-1}+\widetilde{\theta}_{n-1})}z\Big);\tag{5.6}$$

If we let

$$H_n(z)=\frac{F_n(z)}{G_n(z)};$$

then (5.3) and (5.4) imply that for all $n\geq 2$,

$$H_n(\mathrm{e}^{i\,2\Phi_{n-1}})(\Delta_{n-1}+\Delta_n)=\ln\mu(n)+\ln\mu(n-1)+\ln\lambda_{n-1}.\tag{5.7}$$

Thus,

$$\begin{aligned}\sum_{j=2}^{n}H_j(\mathrm{e}^{i\,2\Phi_{j-1}})(\Delta_{j-1}+\Delta_j)&=\sum_{j=2}^{n}(\ln\mu(j)+\ln\mu(j-1)+\ln\lambda_{j-1})\\&=\sum_{j=1}^{n}(2\ln\mu(j)+\ln\lambda_{j-1})-\ln\mu(n)-\ln\mu(1)-\ln\lambda_0.\end{aligned}$$

Using (2.11) we now get

$$\mathcal{S}_n=\sum_{j=2}^{n}H_j(\mathrm{e}^{i\,2\Phi_{j-1}})(\Delta_{j-1}+\Delta_j)+\ln\lambda_n+\ln\mu(n)+\ln\lambda_0+2\ln\mu(0)+\ln\mu(1).$$

It is easy to see that $\lambda_n\to 1$ and $\mu(n)\to 1$. Thus, to prove that $\mathcal{S}_n$ is convergent, it is enough to show that the sum

$$\mathcal{S}_n^*=\sum_{j=2}^{n}H_j(\mathrm{e}^{i\,2\Phi_{j-1}})(\Delta_{j-1}+\Delta_j)\tag{5.8}$$

converges to a finite limit.

We now note that the sum

$$\sum_{j=2}^{n}H_j(\mathrm{e}^{i\,2\Phi_{j-1}})(\Delta_{j-1}+\Delta_j)$$

resembles a Riemann sum, with a partition of the interval of integration $[2\Phi_1 - \Delta_1, 2\Phi_{n-1} + \Delta_n]$ into subintervals $[2\Phi_{j-1} - \Delta_{j-1}, 2\Phi_{j-1} + \Delta_j]$ and integrand evaluated at sampling points $2\Phi_{j-1}$, except that $H_j(\mathrm{e}^{i\,t})$ is a sequence of functions, rather than a single function. However, since for all j functions $H_j(\mathrm{e}^{i\,x})$ are 2π periodic, we can expand them into their Fourier series and, as we will see, this will reduce $\mathcal{S}_n$ to Riemann sums of some damped complex exponentials. However, before proceeding with such a strategy, we first reduce functions $H_n(\mathrm{e}^{i\,t})$ to functions $h_n(\mathrm{e}^{i\,t})$ which do not contain any finite differences, plus some remainders whose sum is absolutely convergent. This is not just a simplification but it is crucial for obtaining almost monotonic Fourier coefficients, which we will need in the course of our argument.

Let us define

$$\begin{aligned} f(x,z) = {} & \ln\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)z^{-1}\right) - \ln\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)z^{-1}\right) \\ & - \ln\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)z\right) + \ln\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)z\right); \end{aligned} \tag{5.9}$$

$$\begin{aligned} g(x,z) = {} & -i\ln\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)z^{-1}\right) + i\ln\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)z^{-1}\right) + \\ & i\ln\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)z\right) - i\ln\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)z\right) + \\ & 2\,i\ln\left(1+\frac{x^4}{4}\right) - 2\,i\ln\left(1+\frac{x^4}{4}-\frac{2x^2}{2}+2\,i\,x\left(1-\frac{x^2}{2}\right)\right); \end{aligned} \tag{5.10}$$

$$h(x,z) = \frac{f(x,z)}{g(x,z)}. \tag{5.11}$$

Lemma 8.*

$$\begin{aligned} F_n(\mathrm{e}^{i\,t}) = {} & f\left(\frac{\omega}{\gamma_{2n-1}}, \mathrm{e}^{i\,t}\right) - \frac{d_{2n-3}\left(1-\frac{2\omega}{\gamma_{2n-1}}\sin t\right) + d_{2n-2}\left(1-\cos t-\frac{2\omega}{\gamma_{2n-1}}\sin t\right) - d_{2n-4}\left(\cos t+\frac{2\omega}{\gamma_{2n-1}}\sin t\right)}{\gamma_{2n-1}} - \\ & \frac{4\omega}{\gamma_{2n-1}^2}s_{2n-1}\sin t + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3}+\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \end{aligned} \tag{5.12}$$

$$\begin{aligned} G_n(\mathrm{e}^{i\,t}) = {} & g\left(\frac{\omega}{\gamma_{2n-1}}, \mathrm{e}^{i\,t}\right) + \frac{2\omega\,(2(1-\cos t)s_{2n-1}+\cos t\; d_{2n-4}-(1-\cos t)d_{2n-3}-(2-\cos t)d_{2n-2})}{\gamma_{2n-1}^2} - \\ & \frac{(d_{2n-4}+d_{2n-2})\sin t}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3}+\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right) \end{aligned} \tag{5.13}$$

□

Lemma 9.*

$$\begin{aligned} H_n(\mathrm{e}^{i\,t}) = {} & h\left(m, \frac{\omega}{\gamma_{2n-1}}, \mathrm{e}^{i\,t}\right) + \frac{d_{2n-4}\cos t - d_{2n-3} - (1-\cos t)d_{2n-2}}{4\omega} + \frac{\omega\; s_{2n-1}^2}{\gamma_{2n-1}} - \frac{s_{2n-1}}{\gamma_{2n-1}}\sin t \\ & + \mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-2}|}{\gamma_{2n-1}}+\frac{|s_{2n-1}|}{\gamma_{2n-1}^2}\right). \end{aligned} \tag{5.14}$$

□.

Substituting (5.14) in (5.8), since $(\Delta_{j-1}+\Delta_j) = \mathrm{O}\left(\frac{1}{\gamma_{2j-1}}\right)$, we obtain

$$\mathcal{S}_n^* = \sum_{j=2}^{n} h\left(\frac{\omega}{\gamma_{2j-1}}, \mathrm{e}^{i\,2\Phi_{j-1}}\right)(\Delta_{j-1}+\Delta_j) + \mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-3}|}{\gamma_{2n-1}}+\frac{|s_{2n-1}|}{\gamma_{2n-1}^2}\right). \tag{5.15}$$

Conditions $(\mathcal{C}_5)$ and $(\mathcal{C}_6)$ now imply that it is sufficient to prove that the sum

$$\mathcal{S}' = \sum_{j=2}^{\infty} h\left(\frac{\omega}{\gamma_{2j-1}}, \mathrm{e}^{i\,2\Phi_{j-1}}\right)(\Delta_{j-1}+\Delta_j) \tag{5.16}$$

is convergent.

We now deal with the fact that in the above sum for every sampling point Φ_{j-1} the value of the parameter ω/γ_{2j-1} is different. To this end, we now treat x as a fixed parameter and expand the real valued 2π-periodic function $h(x, \mathrm{e}^{i\,t})$ into Fourier series with respect to variable t:

$$h(x, \mathrm{e}^{i\,t}) = \sum_{m=-\infty}^{\infty} c_m(x) \mathrm{e}^{i\,m\,t},$$

with $\{c_m(x)\}_{m\in\mathbb{Z}}$ given by

$$c_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x, \mathrm{e}^{i\,t})\, \mathrm{e}^{-\,i\,mt} \mathrm{d}t. \tag{5.17}$$

Elementary transformations of (5.9) and (5.10) yield the following Lemma.

Lemma 10.*

$$f\big(x, \mathrm{e}^{i\,t}\big) = \ln\left(1 + \frac{2x^2\left(1-\frac{x^2}{2}\right)\cos t}{1+\frac{x^4}{2} - x^2\left(1-\frac{x^2}{2}\right)\cos t - x^3 \sin t}\right); \tag{5.18}$$

$$g\big(x, \mathrm{e}^{i\,t}\big) = 2\arctan\frac{2x\left(1-\frac{x^2}{2}\right)\left(1-\frac{x}{2}\sin t\right)}{1-2x^2+x^3\sin t}. \tag{5.19}$$

Thus,

$$h\big(x, \mathrm{e}^{i\,t}\big) = \frac{\ln\left(1 + \frac{2x^2\left(1-\frac{x^2}{2}\right)\cos t}{1+\frac{x^4}{2} - x^2\left(1-\frac{x^2}{2}\right)\cos t - x^3 \sin t}\right)}{2\arctan\frac{2x\left(1-\frac{x^2}{2}\right)\left(1-\frac{x}{2}\sin t\right)}{1-2x^2+x^3\sin t}}. \tag{5.20}$$

□.

Using (5.18), (5.19) and (5.20), it is easy to verify that $f\big(x, \mathrm{e}^{i\,t}\big)$ and $g\big(x, \mathrm{e}^{i\,t}\big)$ have the following properties for all $0 < x < 1/4$ and all $-\pi \le t \le \pi$:

$$f\Big(x, \mathrm{e}^{i\left(\frac{\pi}{2}-t\right)}\Big) = -f\Big(x, \mathrm{e}^{i\left(\frac{\pi}{2}+t\right)}\Big); \qquad f\Big(x, \mathrm{e}^{i\left(-\frac{\pi}{2}-t\right)}\Big) = -f\Big(x, \mathrm{e}^{i\left(-\frac{\pi}{2}+t\right)}\Big). \tag{5.21}$$

$$g\Big(x, \mathrm{e}^{i\left(\frac{\pi}{2}-t\right)}\Big) = g\Big(x, \mathrm{e}^{i\left(\frac{\pi}{2}+t\right)}\Big); \qquad g\Big(x, \mathrm{e}^{i\left(-\frac{\pi}{2}-t\right)}\Big) = g\Big(x, \mathrm{e}^{i\left(-\frac{\pi}{2}+t\right)}\Big). \tag{5.22}$$

Thus, $h\big(x, \mathrm{e}^{i\,t}\big)$ satisfies

$$h\Big(x, \mathrm{e}^{i\left(\frac{\pi}{2}-t\right)}\Big) = -h\Big(x, \mathrm{e}^{i\left(\frac{\pi}{2}+t\right)}\Big); \qquad h\Big(x, \mathrm{e}^{i\left(-\frac{\pi}{2}-t\right)}\Big) = -h\Big(x, \mathrm{e}^{i\left(-\frac{\pi}{2}+t\right)}\Big). \tag{5.23}$$

Representing the Fourier coefficients $c_m(x)$ in the form

$$\begin{aligned} c_m(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\,i\,mt} h(x, \mathrm{e}^{i\,t}) \mathrm{d}t \qquad (5.24)\\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\,i\,m\left(-\frac{\pi}{2}+t\right)} h\Big(x, \mathrm{e}^{i\left(-\frac{\pi}{2}+t\right)}\Big)\,\mathrm{d}t + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\,i\,m\left(\frac{\pi}{2}+t\right)} h\Big(x, \mathrm{e}^{i\left(\frac{\pi}{2}+t\right)}\Big)\,\mathrm{d}t \end{aligned}$$

and using

$$\cos\left(k\left(\frac{\pi}{2}-t\right)\right) = (-1)^k \cos\left(k\left(\frac{\pi}{2}+t\right)\right); \quad \cos\left(k\left(-\frac{\pi}{2}-t\right)\right) = (-1)^k \cos\left(k\left(-\frac{\pi}{2}+t\right)\right); \tag{5.25}$$

$$\sin\left(k\left(\frac{\pi}{2}-t\right)\right) = (-1)^{k+1} \sin\left(k\left(\frac{\pi}{2}+t\right)\right); \quad \sin\left(k\left(-\frac{\pi}{2}-t\right)\right) = (-1)^{k+1} \sin\left(k\left(-\frac{\pi}{2}+t\right)\right), \tag{5.26}$$

it easy to prove the following Lemma.

Lemma 11. *For all real x such that $|x| < 1/4$,*
(i) $c_0(x) = 0$;
(ii) for all $m \neq 0$ the coefficients $c_m(x)$ are purely imaginary for all even m and real for all odd m. □

Note that for all real x such that $|x| < 1/4$, function $h(x, \mathrm{e}^{i\,t})$ is real valued and thus $c_{-m}(x) = \overline{c_m(x)}$ for all m.

Lemma 12. *For every m, $|c_m(x)|$ decrease monotonically in x in a sufficiently small neighbourhood of* 0.

Proof. To prove the above Lemma[1] we consider function

$$c_m(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-imt} h(z, \mathrm{e}^{it}) \mathrm{d}t$$

on the compact set $U = \{(z,t) : |z| \le 1/2,\ |t| \le \pi\} \subset \mathbb{Z}\times\mathbb{R}$. Clearly, for every fixed t function $\mathrm{e}^{-imt}h(z,\mathrm{e}^{it})$ is analytic on the disc $|z| \le 1/2$ in the complex plane. Thus, for every closed contour $C \subset U$, Fubini's and Cauchy's theorems imply

$$\oint_C \int_{-\pi}^{\pi} \mathrm{e}^{-imt} h(z,\mathrm{e}^{it}) \mathrm{d}t\mathrm{d}z = \int_{-\pi}^{\pi}\oint_C \mathrm{e}^{-imt} h(z,\mathrm{e}^{it}) \mathrm{d}z\mathrm{d}t = 0.$$

Consequently, by Morera's theorem, function $c_m(z)$ is analytic on the disc $|z| \le 1/2$. Note that for real x, $|x| < 1/4$, $c_m(x)$ and $c'_m(x)$ are real, and if there were no neighbourhood of 0 in which $c_m(x)$ is monotonic, $c'_m(x)$ would change its sign infinitely many times in every neighbourhood of 0 and thus also have infinitely many zeros in the set $\{z : |z| \le 1/2\}$, which is impossible because $c'_m(z)$ is also analytic on that set. ☐

We now want to establish the asymptotic behaviour of $c_m(x)$ as $x \to 0$. Note that

$$c_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x,\mathrm{e}^{it})\,\mathrm{e}^{-imt}\mathrm{d}t = \frac{1}{2\pi}\oint_{|z|=1} z^{-m} h(x,z)\frac{\mathrm{d}z}{i\,z} = \frac{1}{2\pi\, i}\oint_{|z|=1} z^{-m-1}h(x,z)\mathrm{d}z.$$

We assume that x is a fixed parameter such that $0 < x < 1/4$ and look for the singularities of $z^{-m-1}h(x,z)$. Clearly, $p_0 = 0$ is a pole of this function for $m \ge 0$. For z such that $|z| \le 1$ we can combine the first two logarithms in $f(x,z)$ and $g(x,z)$ (given by (5.9) and (5.10)) and then multiply both the numerator and the denominator by z, thus eliminating z^{-1} and obtaining that for all $|z| \le 1$ functions $f(x,z)$ and $g(x,z)$ satisfy

$$\begin{aligned} f(x,z) = \ln\frac{\left(1+\frac{x^4}{4}\right)z + \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)}{\left(1+\frac{x^4}{4}\right)z - \frac{x^2}{2}\left(1-\frac{x^2}{2} + i\,x\right)} &- \ln\left(1+\frac{x^4}{4} - \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)z\right) \\ &+ \ln\left(1+\frac{x^4}{4} + \frac{x^2}{2}\left(1-\frac{x^2}{2} + i\,x\right)z\right); \end{aligned} \tag{5.27}$$

$$\begin{aligned} g(x,z) = -\,i\ln\frac{\left(1+\frac{x^4}{4}\right)z + \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)}{\left(1+\frac{x^4}{4}\right)z - \frac{x^2}{2}\left(1-\frac{x^2}{2} + i\,x\right)} + i\ln\left(1+\frac{x^4}{4} - \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)z\right) - \\ i\ln\left(1+\frac{x^4}{4} + \frac{x^2}{2}\left(1-\frac{x^2}{2} + i\,x\right)z\right) + 2\,i\ln\left(1+\frac{x^4}{4}\right) - 2\,i\ln\left(1+\frac{x^4}{4} - \frac{2x^2}{2} + 2\,i\,x\left(1-\frac{x^2}{2}\right)\right). \end{aligned} \tag{5.28}$$

In order to keep the cuts which we will have to make in the complex plane as simple as possible, we do not combine the remaining logarithms. Considering the logarithms appearing in $f(x,z)$ and $g(x,z)$, we obtain for the numerator of the fraction inside the first logarithm

$$\left(1+\frac{x^4}{4}\right)z + \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right) = 0 \quad \text{if and only if} \quad z = w_1 = \frac{x^2}{2\left(1+\frac{x^4}{4}\right)}\left(-1+\frac{x^2}{2} + i\,x\right). \tag{5.29}$$

Setting the denominator to zero produces

$$\left(1+\frac{x^4}{4}\right)z - \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right) = 0 \quad \text{if and only if} \quad z = w_2 = \frac{x^2}{2\left(1+\frac{x^4}{4}\right)}\left(1-\frac{x^2}{2} + i\,x\right). \tag{5.30}$$

Considering now the values of the fraction in side the first logarithm for any $z = a + i\,b$ such that $z \ne w_1$ and $z \ne w_2$, a direct calculation shows that

$$\mathfrak{Im}\left(\frac{\left(1+\frac{x^4}{4}\right)(a+i\,b) + \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)}{\left(1+\frac{x^4}{4}\right)(a+i\,b) - \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)}\right) = 0 \qquad \text{if an only if} \qquad b = \mathfrak{Im}(w_1) = \mathfrak{Im}(w_2). \tag{5.31}$$

For such a value of b we also get that

$$\mathfrak{Re}\left(\frac{\left(1+\frac{x^4}{4}\right)(a+i\,b) + \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)}{\left(1+\frac{x^4}{4}\right)(a+i\,b) - \frac{x^2}{2}\left(1-\frac{x^2}{2} - i\,x\right)}\right) < 0 \qquad \text{if an only if} \qquad \mathfrak{Re}(w_1) < a < \mathfrak{Re}(w_2). \tag{5.32}$$

[1]This must be a consequence of some well known and general fact, but the author could not find a reference to cite.

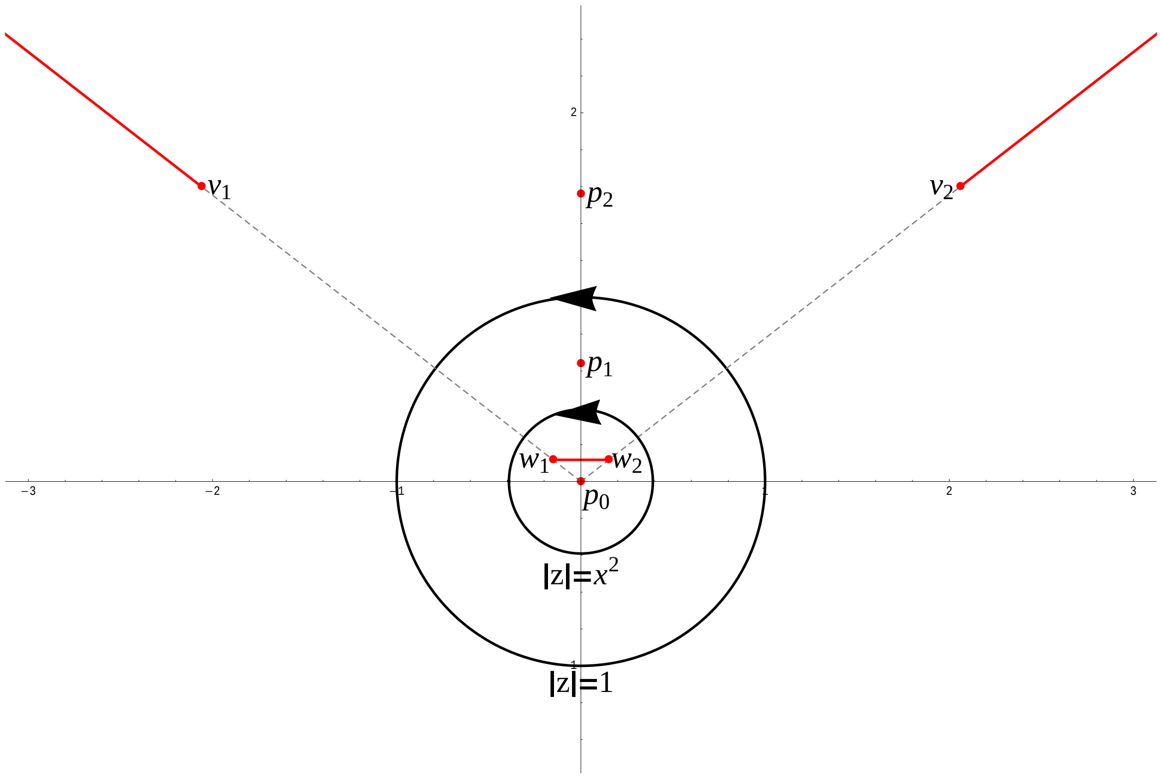

Figure 5.1: Integration contours with cuts and poles

Thus, we make a cut in the complex plane which is a segment of a line with end points w_1 and w_2, see Figure 5.1. Note that

$$|w_1| = |w_2| = \frac{x^2}{2\sqrt{1+\frac{x^4}{4}}} < \frac{x^2}{2}. \tag{5.33}$$

Consequently, the entire cut is contained in a disc $\{z \,:\, |z| \le x^2\}$. Such a feature is one of the benefits of pairing $\mu(n)$ with $\mu(n-1)$ and Δ_n with Δ_{n-1}.

For the remaining two logarithms we obtain

$$1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)z=0 \qquad \text{if and only if} \qquad z=v_1=1-\frac{2}{x^2}+i\,\frac{2}{x}; \tag{5.34}$$

$$1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)z=0 \qquad \text{if and only if} \qquad z=v_2=-1+\frac{2}{x^2}+i\,\frac{2}{x}. \tag{5.35}$$

Representing again z as $z=a+i\,b$, we obtain that

$$\mathfrak{Im}\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)(a+i\,b)\right)=0 \qquad \text{if and only if} \qquad b=-\frac{a\,x}{1-\frac{x^2}{2}}, \tag{5.36}$$

and for such a, b we have

$$\mathfrak{Re}\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)(a+i\,b)\right)<0 \qquad \text{if and only if} \qquad a<\mathfrak{Re}(v_1). \tag{5.37}$$

For $a=\mathfrak{Re}(v_1)$ we get that $b=\mathfrak{Im}(v_1)$. Thus, we make a cut in the complex plane which is a part of a line passing through v_1 and the origin, and is the half line starting at v_1 and not containing the origin; see Figure 5.1.

Similarly, for the last logarithm we have

$$\mathfrak{Im}\left(\ln\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)(a+i\,b)\right)\right)=0 \qquad \text{if and only if} \qquad b=\frac{a\,x}{1-\frac{x^2}{2}}. \tag{5.38}$$

For such a, b and for $x>0$ we have

$$\mathfrak{Re}\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)(a+i\,b)\right)<0 \qquad \text{if and only if} \qquad a>\mathfrak{Re}(v_2). \tag{5.39}$$

For $a = \mathfrak{Re}(v_2)$ we get that $b = \mathfrak{Im}(v_2)$. Thus, we make a cut in the complex plane which is a part of a line passing through the origin and v_2 and is a half line starting at v_2, also pointing away from the origin.

The last possible remaining singularity can occur only when $g(x, z) = 0$. It is easy to see that for x and z such that $0 < x < 1/4$ and $|z| \leq 1$ all logarithms in $g(x, z)$ except the first one can be combined into a single one, thus obtaining after some simplification that for $0 < x < 1/4$ and $|z| \leq 1$,

$$g(x,z) = -\,i\ln\frac{\left(1+\frac{x^4}{4}\right)z + \frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)}{\left(1+\frac{x^4}{4}\right)z - \frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)} - i\ln\frac{\left(1-x\left(\frac{x}{2}-i\right)\right)\left(1-x\left(\frac{x}{2}(1+z)-i\right)\right)}{\left(1-x\left(\frac{x}{2}+i\right)\right)\left(1-x\left(\frac{x}{2}(1-z)+i\right)\right)}. \tag{5.40}$$

Let us set

$$\tilde{g}(x,z) = -\,i\ln\frac{\left(1+\frac{x^4}{4}\right)z + \frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)\left(1-x\left(\frac{x}{2}-i\right)\right)\left(1-x\left(\frac{x}{2}(1+z)-i\right)\right)}{\left(1+\frac{x^4}{4}\right)z - \frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)\left(1-x\left(\frac{x}{2}+i\right)\right)\left(1-x\left(\frac{x}{2}(1-z)+i\right)\right)}; \tag{5.41}$$

then $g(x, z) = \tilde{g}(x, z) + 2k\pi$ for some k such that $|k| \leq 1$. If $g(x, z) = 0$ for some z, then $\tilde{g}(x, z) + 2k\pi = 0$. However, since $|\tilde{g}(x, z)| \leq \pi$ we get that $k = 0$ and $\tilde{g}(x, z) = 0$. On the other hand, if $\tilde{g}(x, z) = 0$, then $g(x, z) = 2k\pi$. Since the absolute value of the imaginary part of the first logarithm in $g(x, z)$ is smaller or equal to π and the absolute value of the imaginary part of the second logarithm is strictly smaller than π whenever $0 < x < 1/4$ and $|z| \leq 1$, we obtain that for such values of x and z we again have $k = 0$ and $g(x, z) = 0$. Thus, for $0 < x < 1/4$ and $|z| \leq 1$ $g(x, z) = 0$ if and only if $\tilde{g}(x, z) = 0$, i.e., if and only if

$$\frac{\left(1+\frac{x^4}{4}\right)z + \frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)}{\left(1+\frac{x^4}{4}\right)z - \frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)}\,\frac{\left(1-x\left(\frac{x}{2}-i\right)\right)\left(1-x\left(\frac{x}{2}(1+z)-i\right)\right)}{\left(1-x\left(\frac{x}{2}+i\right)\right)\left(1-x\left(\frac{x}{2}(1-z)+i\right)\right)} = 1. \tag{5.42}$$

The solutions to this equation are

$$p_1(x) = i\,\frac{x}{2\left(1+\sqrt{1-\frac{x^2}{4}}\right)}; \qquad\qquad p_2(x) = i\,\frac{2\left(1+\sqrt{1-\frac{x^2}{4}}\right)}{x}; \tag{5.43}$$

If x satisfies $0 < x < 1/4$, then pole p_1 lies inside the disc $\{z : |z| \leq 1\}$ but outside the disc $\{z : |z| \leq x^2\}$. Thus, we have

$$\oint_{|z|=1} z^{-m-1}h(x,z)\mathrm{d}z = \oint_{|z|=x^2} z^{-m-1}h(x,z)\mathrm{d}z + 2\pi\,i\,\mathrm{Res}\left(z^{-m-1}h(x,z);\,p_1\right). \tag{5.44}$$

Equations (5.17) and (5.44) yield

$$c_m(x) = \frac{1}{2\pi\,i}\oint_{|z|=x^2} z^{-m-1}h(x,z)\mathrm{d}z + \mathrm{Res}\left(z^{-m-1}h(x,z);\,p_1\right) \tag{5.45}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^{-2m}\mathrm{e}^{-\,i\,mt}h(x,x^2\mathrm{e}^{-\,i\,mt})\mathrm{d}t + \mathrm{Res}\left(z^{-m-1}h(x,z);\,p_1(x)\right). \tag{5.46}$$

Let us set $g'_z(x, z) = \partial g(x, z)/\partial z$; then direct calculations show that

$$g'_z(x,p_1(x)) = -8\,i\left(1-\frac{x^2}{2}\right)\sqrt{1-\frac{x^2}{4}}\left(1+\sqrt{1-\frac{x^2}{4}}\right); \tag{5.47}$$

$$f(x,p_1(x)) = \ln\left(1-8x^2\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{2}\right)^2 + 4\,i\,x\sqrt{1-\frac{x^2}{4}}\left(1-\frac{x^2}{2}\right)\left(1-2x^2-\frac{x^4}{2}\right)\right), \tag{5.48}$$

and we obtain that for all x such that $0 < x < 1/4$,

$$\mathrm{Res}\left(z^{-m-1}h(x,z);\,p_1(x)\right) = p_1(x)^{-m-1}\,\frac{f(x,p_1(x))}{g'_z(x,p_1(x))}.$$

One can verify that the real part of the logarithm in (5.48) is zero; thus, after some simplification, we obtain

$$\mathrm{Res}\left(z^{-m-1}h(x,z);\,p_1(x)\right) = \frac{i^{-m-1}x^{-m-1}}{2^{-m+2}\left(1+\sqrt{1-\frac{x^2}{4}}\right)^{-m}}\;\frac{\arctan\frac{4x\sqrt{1-\frac{x^2}{4}}\left(1-\frac{x^2}{2}\right)\left(1-2x^2+\frac{x^4}{2}\right)}{1-8x^2\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{2}\right)^2}}{\left(1-\frac{x^2}{2}\right)\sqrt{1-\frac{x^2}{4}}}$$

$$= i^{-m-1}\left(\frac{x}{4}\right)^{-m} + \mathrm{O}\left(x^{-m+2}\right).$$

Since $h(x, x^2 e^{-imt})$ is continuous for $(x,t) \in [0,1/2] \times [-\pi,\pi]$, for all $m \le -1$ we have

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} x^{-2m} e^{-imt} h(x, x^2 e^{-imt}) dt \right| < \frac{1}{2\pi} \int_{-\pi}^{\pi} x^{-2m} \left| h(x, x^2 e^{-imt}) \right| dt = O\left(x^{-2m}\right).$$

Consequently, we obtain that for $m \ge 1$

$$c_{-m}(x) = i^{m-1} \left(\frac{x}{4}\right)^m + O\left(x^{m+2}\right); \qquad c_m(x) = \overline{c_{-m}(x)} = (-i)^{m-1} \left(\frac{x}{4}\right)^m + O\left(x^{m+2}\right). \tag{5.49}$$

Let us set

$$c_m^n = c_m \left(\frac{\omega}{\gamma_{2n-1}}\right); \tag{5.50}$$

note that Lemma 11(i) implies that $c_0^n = 0$ for all n.

To summarise, having in mind (5.16), we obtain the following Lemma.

Lemma 13. *There exist $c_m^j \in \mathbb{C}$ such that*

$$h\left(\frac{\omega}{\gamma_{2j-1}}, e^{i\,2\Phi_{j-1}}\right) = \sum_{m=-\infty}^{\infty} c_m^j e^{i\,2m\Phi_{j-1}} \tag{5.51}$$

and which satisfy the following properties:

(i) $c_0^j = 0$ *and* $c_{-m}^j = \overline{c_m^j}$ *for all* $j \ge 2$ *and all* $m > 0$ *;*

(ii) for all j, if $m > 0$ is even, then $\mathfrak{Re}\,(c_m^j) = 0$ and for all sufficiently large j, $\mathfrak{Im}\,(c_m^j)$ are of the same sign;

(iii) for all j, if $m > 0$ is odd, then $\mathfrak{Im}\,(c_m^j) = 0$ and for all sufficiently large j, $\mathfrak{Re}\,(c_m^j)$ are of the same sign;

(iv) $|c_m^j|$ is monotonic in γ_{2j-1} for all sufficiently large j and is thus an almost decreasing sequence;

(v) $|c_m^j| = O\left(\frac{1}{\gamma_{2j-1}}\right)^m$ *for all sufficiently large j.*

We now return to the proof of convergence of the sum $\mathcal{S}_n'$ given by (5.16). Let j_0, m_0 be such that $\frac{\omega}{4\gamma_{j_0}} < 1$ and that, according to condition $(\mathcal{C}_1)$, for every $j > j_0$ and every $m \ge m_0$, $\gamma_{j+m} > \gamma_j$. Let also $j_1 = j_0 + m_0$. We will prove the convergence of the sum

$$\sum_{j=j_1}^{\infty} h\left(\frac{\omega}{\gamma_{2j-1}}, e^{i\,2\Phi_{j-1}}\right) (\Delta_{j-1} + \Delta_j). \tag{5.52}$$

By $(\mathcal{C}_8)$ there exists an an integer $\kappa \ge 2$ such that $\sum_{n=0}^{\infty} \gamma_n^{-\kappa}$ converges. Exchanging the order of summations, clearly it is enough to prove convergence of $\mathcal{S}_n^l$ and $\mathcal{S}_n^h$, where

$$\mathcal{S}_n^l = \sum_{j=j_1}^{n} \sum_{m=-\kappa+2}^{\kappa-2} c_m^j e^{i\,2m\Phi_{j-1}} (\Delta_{j-1} + \Delta_j); \tag{5.53}$$

$$\mathcal{S}_n^h = \sum_{j=j_1}^{n} \sum_{|m| \ge \kappa-1} c_m^j e^{i\,2m\Phi_{j-1}} (\Delta_{j-1} + \Delta_j). \tag{5.54}$$

We can now show that $\mathcal{S}_n^h$ converges. Using Lemma 13 *(v)*, (4.9) and definition (5.54) we get

$$\mathcal{S}_n^h = \sum_{j=j_1}^{n} \sum_{|m| \ge \kappa-1}^{\infty} |c_m^j| (\Delta_{j-1} + \Delta_j) = O\left(\sum_{j=j_1}^{n} \sum_{m=\kappa}^{\infty} \left(\frac{\omega}{4\gamma_{2j-1}}\right)^m\right).$$

Using condition $(\mathcal{C}_8)$,

$$\begin{aligned}
\sum_{j=j_1}^{n} \sum_{m=\kappa}^{\infty} \left(\frac{\omega}{4\gamma_{2j-1}}\right)^m &= \sum_{j=j_1}^{n} \sum_{m=0}^{\infty} \left(\frac{\omega}{4\gamma_{2j-1}}\right)^{m+\kappa} \le \sum_{j=j_1}^{n} \sum_{m=0}^{\infty} \left(\frac{\omega}{4\gamma_{2j_0-1}}\right)^m \left(\frac{\omega}{4\gamma_{2j-1}}\right)^{\kappa} \\
&= \sum_{m=0}^{\infty} \left(\frac{\omega}{4\gamma_{2j_0-1}}\right)^m \sum_{j=j_1}^{n} \left(\frac{\omega}{4\gamma_{2j-1}}\right)^{\kappa} = \frac{\sum_{j=j_1}^{n} \left(\frac{\omega}{4\gamma_{2j-1}}\right)^{\kappa}}{1 - \frac{\omega}{4\gamma_{2j_0-1}}} < \infty,
\end{aligned}$$

which implies that $\mathcal{S}_n^h$ converges.

Finally, to show that $\mathcal{S}_n^l$ converges, it is enough to show that for every m such that $1 \le m \le \kappa - 2$, the sum

$$\sigma_{m,n} = \sum_{j=2}^{n} (c_m^j \mathrm{e}^{i\,2m\Phi_{j-1}} + c_{-m}^j \mathrm{e}^{-\,i\,2m\Phi_{j-1}})(\Delta_{j-1} + \Delta_j)$$

converges as $n \to \infty$. Since by Lemma 13 *(i)*

$$\sigma_{m,n} = \sum_{j=2}^{n} 2\,\mathfrak{Re}(c_m^j \mathrm{e}^{i\,2m\Phi_{j-1}})(\Delta_{j-1} + \Delta_j),$$

using Lemma 13 *(ii)* and *(iii)* and (5.49), we get

$$\sigma_{m,n} = \begin{cases} \sum_{j=2}^{n} 2(-1)^{\frac{m}{2}} |c_m^j| \sin(2m\ \Phi_{j-1})(\Delta_{j-1} + \Delta_j) & \text{if } m \text{ is even;} \\ \\ \sum_{j=2}^{n} 2(-1)^{\frac{m-1}{2}} |c_m^j| \cos(2m\ \Phi_{j-1})(\Delta_{j-1} + \Delta_j) & \text{if } m \text{ is odd} \end{cases} \tag{5.55}$$

If we now define

$$\tau_m(x) = \begin{cases} 0, & \text{if } x < 2\Phi_1 - \Delta_1; \\ 2(-1)^{\frac{m}{2}} |c_m^j| \sin mx, & \text{if } m \text{ is even and } \ 2\Phi_{j-1} - \Delta_{j-1} \le x < 2\Phi_{j-1} + \Delta_j; \\ 2(-1)^{\frac{m-1}{2}} |c_m^j| \cos mx, & \text{if } m \text{ is odd and } \ 2\Phi_{j-1} - \Delta_{j-1} \le x < 2\Phi_{j-1} + \Delta_j, \end{cases} \tag{5.56}$$

then clearly the sum $\sigma_m(n)$ is the Riemann sum for the integral

$$I_m(n) = \int_{2\Phi_1 - \Delta_1}^{2\Phi_{n-1} + \Delta_n} \tau_m(x) dx, \tag{5.57}$$

with a partition of the interval $[2\Phi_1 - \Delta_1, 2\Phi_{n-1} + \Delta_n]$ into segments of the form $[2\Phi_{j-1} - \Delta_{j-1}, 2\Phi_{j-1} + \Delta_j]$ and with points $2\Phi_{j-1}$ as the sampling points for the integrand. However, in order to facilitate some estimates which we will have to make later[2], instead we define two functions $\tau_m^e(x)$ and $\tau_m^o(x)$; for $\tau_m^e(x)$ we consider partitions of the interval $[\Phi_1, \Phi_{2n+1}]$ into segments of the form $[\Phi_{2j-1}, \Phi_{2j+1}]$ and sampling points $\Phi_{2j} \in [\Phi_{2j-1}, \Phi_{2j+1}]$, while for function $\tau_m^o(x)$ we consider partitions of the interval $[\Phi_0, \Phi_{2n}]$ into segments of the form $[\Phi_{2j-2}, \Phi_{2j}]$ and sampling points $\Phi_{2j-1} \in [\Phi_{2j-2}, \Phi_{2j}]$. Thus, by splitting each sum $\sigma_{m,n}$ into two sums it follows that it is enough to prove the convergence of the following two sums:

$$\sigma_{m,n}^e = \begin{cases} \sum_{j=1}^{n} (-1)^{\frac{m}{2}} |c_m^{2j}| \sin(2m\ \Phi_{2j})(2\Delta_{2j} + 2\Delta_{2j+1}) & \text{if } m \text{ is even;} \\ \\ \sum_{j=2}^{n} (-1)^{\frac{m-1}{2}} |c_m^{2j}| \cos(2m\ \Phi_{2j})(2\Delta_{2j} + 2\Delta_{2j+1}) & \text{if } m \text{ is odd} \end{cases} \tag{5.58}$$

$$\sigma_{m,n}^o = \begin{cases} \sum_{j=2}^{n} (-1)^{\frac{m}{2}} |c_m^{2j-1}| \sin(2m\ \Phi_{2j-1})(2\Delta_{2j-1} + 2\Delta_{2j}) & \text{if } m \text{ is even;} \\ \\ \sum_{j=2}^{n} (-1)^{\frac{m-1}{2}} |c_m^{2j-1}| \cos(2m\ \Phi_{2j-1})(2\Delta_{2j-1} + 2\Delta_{2j}) & \text{if } m \text{ is odd} \end{cases} \tag{5.59}$$

We define

$$\tau_m^e(x) = \begin{cases} 0, & \text{if } x < 2\Phi_1; \\ 2(-1)^{\frac{m}{2}} |c_m^{2j}| \sin mx, & \text{if } m \text{ is even and } \ 2\Phi_{2j-1} \le x < 2\Phi_{2j+1}; \\ 2(-1)^{\frac{m-1}{2}} |c_m^{2j}| \cos mx, & \text{if } m \text{ is odd and } \ 2\Phi_{2j-1} \le x < 2\Phi_{2j+1}. \end{cases} \tag{5.60}$$

$$\tau_m^o(x) = \begin{cases} 0, & \text{if } x < 2\Phi_0; \\ 2(-1)^{\frac{m}{2}} |c_m^{2j+1}| \sin mx, & \text{if } m \text{ is even and } \ 2\Phi_{2j} \le x < 2\Phi_{2j+2}; \\ 2(-1)^{\frac{m-1}{2}} |c_m^{2j+1}| \cos mx, & \text{if } m \text{ is odd and } \ 2\Phi_{2j} \le x < 2\Phi_{2j+2}, \end{cases} \tag{5.61}$$

as well as definite integrals

$$I_m^e(n) = \int_{2\Phi_1}^{2\Phi_{2n+1}} \tau_m^e(x) dx; \qquad I_m^o(n) = \int_{2\Phi_0}^{2\Phi_{2n}} \tau_m^o(x) dx. \tag{5.62}$$

[2]This is to avoid having to take a square root of the righthand sides of equations (6.4) and (6.7), and thus avoid dealing with the necessary branch cuts for these square roots.

Clearly, $\sigma^e_{m,n}$ and $\sigma^o_{m,n}$ are the Riemann sums for $I^e_m(n)$ and $I^o_m(n)$, respectively. Thus, to show that $\sigma^e_{m,n}$ and $\sigma^o_{m,n}$ converge as $n \to \infty$, it is enough to show that both integrals converge, as well as that the errors of approximating these integrals by the corresponding Riemann sums also converge.

Let us fix m and assume that k_0, j_0 are such that for all k, j satisfying $k \geq k_0$ and $j \geq j_0$ we have $\gamma_{2j+k} > \gamma_{2j}$; then by Lemma 13 *(iv)*, $|c^{j+k}_m| < |c^j_m|$. Since $\Phi_j \to \infty$, integral $I^e_m(n)$ can be represented as a sum of integrals of $\tau^e_m(x)$ between the consecutive zeroes of $\tau^e_m(x)$, which are points of the form $l\pi/m$, for l an integer. These integrals are alternating in sign. Since $\Delta_j = \Phi_j - \Phi_{j-1} \to 0$, eventually the number of points Φ_j in any interval of length π/m will be greater that k_0. Thus, eventually for all sufficiently large x, $|\tau^e_m(x + \pi/m)| < |\tau^e_m(x)|$, and so the absolute values of integrals of $\tau^e_m(x)$ between the consecutive zeroes of $\tau^e_m(x)$ will be monotonically decreasing. Consequently,

$$\lim_{n\to\infty} I^e_m(n) = \int_{2\Phi_1}^{\infty} \tau^e_m(x)dx < \infty, \tag{5.63}$$

and the same applies to $I^o_m(n)$.

Finally, to conclude that σ^n_m converges as $n \to \infty$, it is enough to show that the sum of errors of approximation of $I^e_m(n)$ by the Riemann sum $\sigma^e_{m,n}$ and of $I^o_m(n)$ by the Riemann sum $\sigma^o_{m,n}$, i.e.,

$$\mathcal{E}_m(n) = \sum_{j=2}^{n} c^{j-1}_m \left(\int_{\Phi_{j-2}}^{2\Phi_j} e^{i\,mx}dx - e^{i\,m\,2\Phi_{j-1}}(2\Delta_{j-1} + 2\Delta_j) \right) \tag{5.64}$$

also converges as $n \to \infty$. We achieve that by applying the same technique several times: we represent $\mathcal{E}_m(n)$ as a Riemann sum for integrals of some damped complex exponentials and then show that these integrals converge and that the Fourier coefficients of the new error terms are decreased by a factor of at least $1/\gamma^2_{2n-1}$. Thus, after a few iterations, the resulting error terms will be $\mathrm{O}\left(1/\gamma_{2n-1}\right)^{\kappa}$ and thus absolutely convergent.

6 Estimating $\mathcal{E}_m(n)$

A direct calculation shows that

$$\int_{2\Phi_{j-2}}^{2\Phi_j} e^{i\,mx}dx - e^{i\,m\,2\Phi_{j-1}}(2\Delta_{j-1} + 2\Delta_j) = e^{i\,m\,2\Phi_{j-1}} \left(\frac{i\left(e^{-2\,i\,m\Delta_{j-1}} - e^{-2\,i\,m\Delta_j}\right)}{2m(\Delta_{j-1} + \Delta_j)} - 1 \right) (2\Delta_{j-1} + 2\Delta_j). \tag{6.1}$$

From (3.10) and (2.10) we obtain

$$\mu(n)\, e^{-\,i\,\Delta_n} = e^{-\,i\,\theta_n} \left(\alpha_n + \widetilde{\alpha}_n e^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)} \right). \tag{6.2}$$

Taking the complex conjugate of both sides we get

$$\mu(n)\, e^{i\,\Delta_n} = e^{i\,\theta_n} \left(\alpha_n + \widetilde{\alpha}_n e^{-\,i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)} \right). \tag{6.3}$$

By dividing each side of (6.3) by the corresponding side of (6.2) we get

$$e^{i\,2\Delta_n} = e^{i\,2\theta_n} \frac{\alpha_n + \widetilde{\alpha}_n e^{-\,i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}}{\alpha_n + \widetilde{\alpha}_n e^{i(2\Phi_{n-1}+\theta_n-\widetilde{\theta}_n)}}. \tag{6.4}$$

Similarly, substituting n by $n-1$ in (4.2) we get

$$\frac{e^{-\,i\,\Delta_{n-1}}}{\mu(n-1)} = \frac{\alpha_{n-1} e^{-\,i\,\theta_{n-1}} - \widetilde{\alpha}_{n-1} e^{-\,i(2\Phi_{n-1}-\widetilde{\theta}_{n-1})}}{\alpha^2_{n-1} - \widetilde{\alpha}^2_{n-1}}. \tag{6.5}$$

Taking the complex conjugates of both sides of (6.5) produces

$$\frac{e^{i\,\Delta_{n-1}}}{\mu(n-1)} = \frac{\alpha_{n-1} e^{i\,\theta_{n-1}} - \widetilde{\alpha}_{n-1} e^{i(2\Phi_{n-1}-\widetilde{\theta}_{n-1})}}{\alpha^2_{n-1} - \widetilde{\alpha}^2_{n-1}}. \tag{6.6}$$

Dividing both sides of (6.5) by the corresponding sides of (6.6) yields

$$e^{-\,i\,2\Delta_{n-1}} = e^{-\,i\,2\theta_{n-1}} \frac{\alpha_{n-1} - \widetilde{\alpha}_{n-1} e^{-\,i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}}{\alpha_{n-1} - \widetilde{\alpha}_{n-1} e^{i(2\Phi_{n-1}-\theta_{n-1}-\widetilde{\theta}_{n-1})}}. \tag{6.7}$$

Combining (6.4) with (6.7) we get

$$\left(e^{-i\,2m\Delta_{n-1}} - e^{i\,2m\Delta_n}\right) = \left(e^{-i\,2\theta_{n-1}}\frac{\alpha_{n-1}-\widetilde{\alpha}_{n-1}e^{-i\,2\Phi_{n-1}}e^{i(\theta_{n-1}+\widetilde{\theta}_{n-1})}}{\alpha_{n-1}-\widetilde{\alpha}_{n-1}e^{i\,2\Phi_{n-1}}e^{-i(\theta_{n-1}+\widetilde{\theta}_{n-1})}}\right)^m - \left(e^{i\,2\theta_n}\frac{\alpha_n+\widetilde{\alpha}_n e^{-i\,2\Phi_{n-1}}e^{-i(\theta_n-\widetilde{\theta}_n)}}{\alpha_n+\widetilde{\alpha}_n e^{i\,2\Phi_{n-1}}e^{i(\theta_n-\widetilde{\theta}_n)}}\right)^m$$

Let us define

$$L_n(m,z) = \left(e^{-i\,2\theta_{n-1}}\frac{\alpha_{n-1}-\widetilde{\alpha}_{n-1}e^{i(\theta_{n-1}+\widetilde{\theta}_{n-1})}z^{-1}}{\alpha_{n-1}-\widetilde{\alpha}_{n-1}e^{-i(\theta_{n-1}+\widetilde{\theta}_{n-1})}z}\right)^m - \left(e^{i\,2\theta_n}\frac{\alpha_n+\widetilde{\alpha}_n e^{-i(\theta_n-\widetilde{\theta}_n)}z^{-1}}{\alpha_n+\widetilde{\alpha}_n e^{i(\theta_n-\widetilde{\theta}_n)}z}\right)^m. \tag{6.8}$$

Using (5.4) we get that

$$e^{i\,m2\Phi_{n-1}}\left(\frac{i\left(e^{-i\,2m\Delta_{n-1}} - e^{i\,2m\Delta_n}\right)}{2m(\Delta_{n-1}+\Delta_n)} - 1\right) = e^{i\,m2\Phi_{n-1}}\left(\frac{i\,L_n(m, e^{i\,2\Phi_{n-1}})}{2m(\Delta_{n-1}+\Delta_n)} - 1\right). \tag{6.9}$$

Thus, using (5.6),

$$\mathcal{E}_m(n) = \sum_{j=1}^{n} c_n^m e^{i\,m2\Phi_{n-1}}\left(\frac{i\,L_n(m, e^{i\,2\Phi_{n-1}})}{m\,G_n\left(e^{i\,2\Phi_{n-1}}\right)} - 1\right)(2\Delta_{n-1}+2\Delta_n). \tag{6.10}$$

In order to obtain Fourier series coefficients which are monotonic in γ_n, we again need to eliminate all finite differences from $L_n(m, e^{i\,2\Phi_{n-1}})/G_n(e^{i\,2\Phi_{n-1}})$. We define

$$l(m,x,z) = \left(\frac{1-\frac{x^2}{2} - i\,x - \frac{x^2}{2}z^{-1}}{1-\frac{x^2}{2} + i\,x - \frac{x^2}{2}z}\right)^m - \left(\frac{1-\frac{x^2}{2} + i\,x + \frac{x^2}{2}z^{-1}}{1-\frac{x^2}{2} - i\,x + \frac{x^2}{2}z}\right)^m. \tag{6.11}$$

Lemma 14.*

$$L_n(m, e^{i\,t}) = l\left(m, \frac{\omega}{\gamma_{2n-1}}, e^{i\,t}\right) + \frac{i\,m(d_{2n-4}+d_{2n-2})\sin t}{\gamma_{2n-1}} - \frac{4\,i\,m\,\omega(1-\cos t)s_{2n-1}}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-2}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right). \tag{6.12}$$

□

Lemma 15.*

$$\frac{L_n(m, e^{i\,t})}{G_n\left(e^{i\,2\Phi_{n-1}}\right)} = \frac{l\left(m, \frac{\omega}{\gamma_{2n-1}}, e^{i\,t}\right)}{g\left(\frac{\omega}{\gamma_{2n-1}}, e^{i\,t}\right)} + \frac{i\,m(d_{2n-4}+d_{2n-2})^2\sin^2 t}{16\omega^2} + \mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-2}|}{\gamma_{2n-1}} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^2} + \frac{s_{2n-1}^2}{\gamma_{2n-1}}\right). \tag{6.13}$$

□

Since $\Delta_{n-1}+\Delta_n = \mathrm{O}\left(1/\gamma_{2n-1}\right)$, Lemma 15 together with conditions $(\mathcal{C}_5)$ and $(\mathcal{C}_6)$ imply that it suffices to show that for every m the sum

$$\mathcal{E}_m^*(n) = \sum_{j=1}^{n} c_m^j e^{i\,2m\Phi_{j-1}}\left(\frac{i\,l\left(m, \frac{\omega}{\gamma_{2j-1}}, e^{i\,2\Phi_{j-1}}\right)}{m\,g\left(\frac{\omega}{\gamma_{2j-1}}, e^{i\,2\Phi_{j-1}}\right)} - 1\right)(2\Delta_{j-1}+2\Delta_j)$$

converges as $n \to \infty$.

We now consider m and x fixed parameters and expand into Fourier series with respect to variable t functions

$$\varepsilon_m(x,t) = e^{i\,mt}\left(\frac{i\,l(m,x,e^{i\,t})}{m\,g(x,e^{i\,t})} - 1\right). \tag{6.14}$$

Thus, with

$$b_k^m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon_m(x,t)e^{-i\,kt}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{i\,l(m,x,e^{i\,t})}{m\,g(x,e^{i\,t})} - 1\right)e^{i(m-k)t}dt, \tag{6.15}$$

we have

$$\varepsilon_m(x,t) = \sum_{k=-\infty}^{\infty} b_k^m(x)\mathrm{e}^{i\,kt} \tag{6.16}$$

One can directly verify that for all x and all t such that $0 < x < 1/4$ and $-\pi \leq t \leq \pi$,

$$\left|\frac{1-\frac{x^2}{2}-i\,x-\frac{x^2}{2}\mathrm{e}^{-\,i\,t}}{1-\frac{x^2}{2}+i\,x-\frac{x^2}{2}\mathrm{e}^{i\,t}}\right| = \left|\frac{1-\frac{x^2}{2}+i\,x+\frac{x^2}{2}\mathrm{e}^{-\,i\,t}}{1-\frac{x^2}{2}-i\,x+\frac{x^2}{2}\mathrm{e}^{i\,t}}\right| = 1. \tag{6.17}$$

This is easily seen to imply that $l(-m,x,\mathrm{e}^{i\,t}) = \overline{l(m,x,\mathrm{e}^{i\,t})}$ which in turn implies the following Lemma.

Lemma 16. *For all m,n and all k, $b_k^{m,n} = \overline{b_{-k}^{-m,n}}$.* □

It is straightforward to verify the following Lemma.

Lemma 17. *Let*

$$\mathrm{rl}(x,t) = 1 - \frac{2x^2\left(1-\frac{x}{2}\sin t\right)^2}{1+\frac{x^4}{2}-x^2\left(1-\frac{x^2}{2}\right)\cos t - x^3\sin t}; \quad \mathrm{il}(n,t) = -\frac{2x\left(1-\frac{x}{2}\sin t\right)\left(1-x^2\cos^2\frac{t}{2}\right)}{1+\frac{x^4}{2}-x^2\left(1-\frac{x^2}{2}\right)\cos t - x^3\sin t}; \tag{6.18}$$

$$\mathrm{rr}(n,t) = 1 - \frac{2x^2\left(1-\frac{x}{2}\sin t\right)^2}{1+\frac{x^4}{2}-x^2\left(1-\frac{x^2}{2}\right)\cos t - x^3\sin t}; \quad \mathrm{ir}(n,t) = \frac{2x\left(1-x^2\sin^2\frac{t}{2}\right)\left(1-\frac{x}{2}\sin t\right)}{1+\frac{x^4}{2}-x^2\left(1-\frac{x^2}{2}\right)\cos t - x^3\sin t}. \tag{6.19}$$

Then

$$l\left(m,x,\mathrm{e}^{i\,t}\right) = \left(\mathrm{rl}\,(x,t)+i\,\mathrm{il}\,(x,t)\right)^m - \left(\mathrm{rr}\,(x,t)+i\,\mathrm{ir}\,(x,t)\right)^m. \tag{6.20}$$

□

Let

$$\Psi(x,m,t) = \frac{i\,\left(\left(\mathrm{rl}(x,t)+i\,\mathrm{il}(x,t)\right)^m - \left(\mathrm{rr}(x,t)+i\,\mathrm{ir}(x,t)\right)^m\right)}{2m\,\mathrm{g}(x,t)}; \tag{6.21}$$

then

$$b_k^{m,n} = b_k^m\left(\frac{\omega}{\gamma_{2n-1}}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi\left(\frac{\omega}{\gamma_{2n-1}},m,t\right)\mathrm{e}^{i(m-k)t}dt. \tag{6.22}$$

It is straightforward to verify directly the following Lemma.

Lemma 18.

$$\mathrm{rl}\left(n,\frac{\pi}{2}-t\right) = \mathrm{rr}\left(n,\frac{\pi}{2}+t\right); \qquad \mathrm{rl}\left(n,-\frac{\pi}{2}-t\right) = \mathrm{rr}\left(n,-\frac{\pi}{2}+t\right); \tag{6.23}$$

$$\mathrm{il}\left(n,\frac{\pi}{2}-t\right) = -\mathrm{ir}\left(n,\frac{\pi}{2}+t\right); \qquad \mathrm{il}\left(n,-\frac{\pi}{2}-t\right) = -\mathrm{ir}\left(n,-\frac{\pi}{2}+t\right). \tag{6.24}$$

□

If we represent $b_k^{m,n}$ in the form

$$b_k^{m,n} = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Psi\left(\frac{\omega}{\gamma_{2n-1}},m,-\frac{\pi}{2}+t\right)\mathrm{e}^{i(m-k)\left(-\frac{\pi}{2}+t\right)}dt + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Psi\left(\frac{\omega}{\gamma_{2n-1}},m,\frac{\pi}{2}+t\right)\mathrm{e}^{i(m-k)\left(\frac{\pi}{2}+t\right)}dt,$$

then Lemma 18, equations (5.22) as well as trigonometric equalities (5.25) and (5.26) imply that, depending on the parity of $m-k$, either real or imaginary part of $b_k^{m,n}$ is zero, thus obtaining the following Lemma.

Lemma 19. *The coefficients $b_k^{m,n}$ are real if m and k are of the same parity and purely imaginary otherwise.* □

The coefficients $b_k^m(x)$ can be proven to be monotonic in x in a sufficiently small neighbourhood of 0 in the same way as this was done for $c_m(x)$. Thus, since γ_{2n-1} are almost increasing, we get the following Lemma.

Lemma 20. *The absolute values $|b_k^{m,n}|$ of the coefficients $b_k^{m,n}$ form an almost decreasing sequence.*

A direct computation shows that

$$\lim_{v\to 0}\Psi(v,n,m,t)=0; \qquad \lim_{v\to 0}\frac{\partial}{\partial v}\Psi(v,n,m,t)=0; \qquad \lim_{v\to 0}\frac{\partial^2}{\partial v^2}\Psi(v,n,m,t)=-\frac{4m^2}{3}. \tag{6.25}$$

Considering the second order Taylor series of $\Psi(x,m,t)$ at $x=0$, we obtain that for some $M>0$ inequality $|\Psi(x,m,t)|<M|x|^2$ holds for all t. This implies that for all m,n,

$$b_k^{m,n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi\left(\frac{\omega}{\gamma_{2n-1}},m,t\right)\mathrm{e}^{i(m-k)t}dt=\mathrm{O}\left(\frac{1}{\gamma_{2n-1}^2}\right). \tag{6.26}$$

We now consider

$$\frac{l(m,x,z)}{g(x,z)}=\frac{\left(\frac{1-\frac{x^2}{2}-i\,x-\frac{x^2}{2}z^{-1}}{1-\frac{x^2}{2}+i\,x-\frac{x^2}{2}z}\right)^m-\left(\frac{1-\frac{x^2}{2}+i\,x+\frac{x^2}{2}z^{-1}}{1-\frac{x^2}{2}-i\,x+\frac{x^2}{2}z}\right)^m}{-i\left(\ln\frac{\left(1+\frac{x^4}{4}\right)z+\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)}{\left(1+\frac{x^4}{4}\right)z-\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)}+\ln\frac{\left(1-x\left(\frac{x}{2}-i\right)\right)\left(1-x\left(\frac{x}{2}(1+z)-i\right)\right)}{\left(1-x\left(\frac{x}{2}+i\right)\right)\left(1-x\left(\frac{x}{2}(1-z)+i\right)\right)}\right)}. \tag{6.27}$$

We have already found the singularities and branch cuts for $g(x,z)$; we also saw that if $g(x,z)=0$ and $x<1/4$ and $|z|<1$ then $z=p_1(x)$. However, a direct substitution shows that $l(m,x,p_1(x))=0$. Thus $z=p_1(x)$ is a removable singularity of $l(m,x,z)/g(x,z)$ and we get that integration over the unit circle can be replaced by integration over a circle of radius x^2, thus obtaining

$$b_k^m(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{i\,l\left(m,x,x^2\mathrm{e}^{i\,t}\right)}{m\,g(x,x^2\mathrm{e}^{i\,t})}-1\right)\left(x^2\mathrm{e}^{i\,t}\right)^{m-k}dt. \tag{6.28}$$

If $k>0$ then this implies $b_{-k}^m(x)=\mathrm{O}\left(x^{2(k+m)}\right)$. Using Lemma 16 we obtain $b_k^m(x)=\mathrm{O}\left(x^{2(|k|-|m|)}\right)$ and thus $b_k^{m,j}=\mathrm{O}\left(\frac{\omega}{\gamma_{2j-1}}\right)^{2(|k|-|m|)}$. Using Lemma 11$(v)$ we obtain

$$c_m^j b_k^{m,j}=\mathrm{O}\left(\frac{\omega}{\gamma_{2j-1}}\right)^{2|k|-|m|} \tag{6.29}$$

The sum $\mathcal{E}_m^*(n)$ can now be represented as

$$\mathcal{E}_m^*(n)=\sum_{j=1}^{n}\sum_{|k|\le\frac{\kappa+|m|}{2}-2}c_m^j b_k^{m,j}\mathrm{e}^{i\,kt}(2\Delta_{j-1}+2\Delta_j)+\sum_{j=1}^{n}\sum_{|k|>\frac{\kappa+|m|}{2}-1}c_m^j b_k^{m,j}\mathrm{e}^{i\,kt}(2\Delta_{j-1}+2\Delta_j). \tag{6.30}$$

For $|k|\ge 1/2(\kappa+|m|)-1$ equation (6.29) implies $c_m^j b_k^{m,j}=\mathrm{O}\left(1/\gamma_{2j-1}\right)^{\kappa-1}$; thus, the second sum converges for the same reason as the sum (5.54). It is now enough to show that for every k such that $|k|\le\frac{\kappa+|m|}{2}-2$, the sum

$$\mathcal{E}_m^*(n)=\sum_{j=1}^{n}c_m^j b_k^{m,j}\mathrm{e}^{i\,k2\Phi_{j-1}}(2\Delta_{j-1}+2\Delta_j) \tag{6.31}$$

converges. However, since $b_k^{m,j}=\mathrm{O}\left(\omega/\gamma_{2n-1}\right)^2$ for all m,k,j, we have that

$$c_m^j b_k^{m,j}=\mathrm{O}\left(\frac{\omega}{\gamma_{2j-1}}\right)^{m+2}. \tag{6.32}$$

Thus, we can repeat the above argument on the sums

$$\sum_{j=1}^{n}c_m^j b_k^{m,j}\mathrm{e}^{i\,k2\Phi_{j-1}}(2\Delta_{j-1}+2\Delta_j) \tag{6.33}$$

until the coefficients become on the order of $(\omega/\gamma_{2n-1})^{\kappa-1}$ and thus the corresponding sums absolutely convergent. This concludes our proof of Theorem 3.

7 A Hilbert space associated with orthonormal polynomials

We now present an application of Corollary 4; in fact, this application was author's sole motivation for the present work.

Assuming that the families of orthogonal polynomials we consider satisfy our conditions ($\mathcal{C}_1$)-($\mathcal{C}_8$), we define a corresponding family of linear differential operators $\mathcal{K}_t^n$ by

$$\mathcal{K}_t^n = (-i)^n P_n\left(i\,\frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{7.1}$$

Such operators have real coefficients and satisfy

$$\mathcal{K}_t^n[\mathrm{e}^{i\,\omega t}] = i^n P_n(\omega)\,\mathrm{e}^{i\,\omega t}. \tag{7.2}$$

Denoting by D_t differentiation with respect to variable t, it is easy to see that such operators satisfy the recurrence

$$\gamma_n\,\mathcal{K}_t^{n+1} = D_t \circ \mathcal{K}_t^n + \gamma_{n-1}\,\mathcal{K}_t^{n-1}, \tag{7.3}$$

with the same coefficients $\gamma_n > 0$ as in (1.1). We will make use of the Christoffel-Darboux equality for orthogonal polynomials,

$$(\omega - \sigma)\sum_{k=0}^{n} P_k(\omega)P_k(\sigma) = \gamma_n(P_{n+1}(\omega)P_n(\sigma) - P_{n+1}(\sigma)P_n(\omega)). \tag{7.4}$$

Differential operators $\mathcal{K}_t^m$ which correspond to polynomials $P_n(\omega)$ have a corresponding property: for all $n \in \mathbb{N}$ and $f, g \in C^\infty$,

$$D_t\left[\sum_{m=0}^{n}\mathcal{K}_t^m[f]\,\mathcal{K}_t^m[g]\right] = \gamma_n\,(\mathcal{K}_t^{n+1}[f]\,\mathcal{K}_t^n[g] + \mathcal{K}_t^n[f]\,\mathcal{K}_t^{n+1}[g]). \tag{7.5}$$

More details about operators $\mathcal{K}^n$ can be found in [1].

In the remaining part of this paper all functions $f : \mathbb{R} \to \mathbb{R}$ which we consider are assumed to be analytic on $\mathbb{R}$.

Definition 1. *We denote by $\mathcal{L}$ the vector space of functions $f(t)$ such that the sequence of corresponding functions $\{\nu_n^f(t)\}_{n\in\mathbb{N}}$ defined by*

$$\nu_n^f(t) = \frac{\sum_{k=0}^{n}\mathcal{K}^k[f](t)^2}{\sum_{k=0}^{n}\frac{1}{\gamma_k}} \tag{7.6}$$

converges uniformly on every finite interval $I \subset \mathbb{R}$.

Lemma 21. *Let $f, g \in \mathcal{L}$; if we define a corresponding sequence of functions $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ by*

$$\sigma_n^{fg}(t) = \frac{\sum_{k=0}^{n}\mathcal{K}^k[f](t)\mathcal{K}^k[g](t)}{\sum_{k=0}^{n}\frac{1}{\gamma_k}}, \tag{7.7}$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function. In particular, $\{\nu_n^f(t)\}_{n\in\mathbb{N}}$ also converges to a constant function.

Proof. Since $\nu_n^f(t)$ and $\nu_n^g(t)$ given by (7.6) converge uniformly on every finite interval, the same holds for the sequence $\sigma_n^{fg}(t)$. Consequently, it is enough to show that for all t, $\lim_{n\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_n^{fg}(t) = 0$.

Let

$$S_k(t) = \mathcal{K}^k[f](t)^2 + \mathcal{K}^{k+1}[f](t)^2 + \mathcal{K}^k[g](t)^2 + \mathcal{K}^{k+1}[g](t)^2.$$

It is easy to see that convergence of the sequences $\nu_n^f(t)$, $\nu_n^g(t)$ imply that

$$\frac{\sum_{k=0}^{n} S_k(t)}{\sum_{k=0}^{n}\frac{1}{\gamma_k}}$$

also converges everywhere to some $\alpha(t)$. We now show that if t is such that $\alpha(t) > 0$, then there are infinitely many k such that $S_k(t) < 2\alpha(t)/\gamma_k$. Assume the opposite, and let K be such that $S_k(t) \geq 2\alpha(t)/\gamma_k$ for all $k \geq K$. Then we would have that for all $n \geq K$,

$$\frac{\sum_{k=K}^{n} S_k(t)}{\sum_{k=K}^{n}\frac{1}{\gamma_k}} \geq 2\alpha(t). \tag{7.8}$$

Thus, since

$$\frac{\sum_{k=0}^{n} S_k(t)}{\sum_{k=0}^{n}\frac{1}{\gamma_k}} = \frac{\sum_{k=0}^{K-1} S_k(t)}{\sum_{k=0}^{n}\frac{1}{\gamma_k}} + \frac{\sum_{k=K}^{n} S_k(t)}{\sum_{k=K}^{n}\frac{1}{\gamma_k}}\left(1 - \frac{\sum_{k=0}^{K-1}\frac{1}{\gamma_k}}{\sum_{k=0}^{n}\frac{1}{\gamma_k}}\right)$$

we would have

$$\frac{\sum_{k=0}^{n} S_k(t)}{\sum_{k=0}^{n}\frac{1}{\gamma_k}} > \alpha(t)$$

for all sufficiently large n, which contradicts the definition of $\alpha(t)$. Consequently, for infinitely many n

$$\frac{S_n}{2} < \frac{\alpha(t)}{\gamma_n}.$$

On the other hand,

$$|\mathcal{K}^{n+1}[f](t)\,\mathcal{K}^{n}[g](t)| + |\mathcal{K}^{n}[f](t)\,\mathcal{K}^{n+1}[g](t)| < \frac{S_n}{2},$$

and the last two inequalities imply

$$|\mathcal{K}^{n+1}[f](t)\,\mathcal{K}^{n}[g](t)| + |\mathcal{K}^{n}[f](t)\,\mathcal{K}^{n+1}[g](t)| < \frac{\alpha(t)}{\gamma_n}.$$

Using (7.5), from this we obtain

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma_n^{fg}(t)\right| &\leq \frac{\gamma_n}{\sum_{k=0}^{n}\frac{1}{\gamma_k}}\left(|\mathcal{K}^{n+1}[f](t)\,\mathcal{K}^{n}[g](t)| + |\mathcal{K}^{n}[f](t)\,\mathcal{K}^{n+1}[g](t)|\right)\\ &\leq \frac{\alpha(t)}{\sum_{k=0}^{n}\frac{1}{\gamma_k}}\end{aligned}$$

Since the above inequality holds for infinitely many n and the denominator diverges, $\liminf_{n\to\infty}\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma_n^{fg}(t)\right| = 0$ and since $\lim_{n\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_n^{fg}(t)$ exists, it must be equal to zero. ☐

Corollary 22. *Let $\mathcal{L}_0$ be the vector space consisting of functions $f(t)$ such that $\lim_{n\to\infty}\nu_n^f(t) = 0$; then in the quotient space $\mathcal{L}_2 = \mathcal{L}/\mathcal{L}_0$ we can introduce a scalar product by the following formula whose right hand side is independent of t:*

$$\langle f, g\rangle = \lim_{n\to\infty}\frac{\sum_{k=0}^{n}\mathcal{K}^k[f](t)\,\mathcal{K}^k[g](t)}{\sum_{k=0}^{n}\frac{1}{\gamma_k}}. \tag{7.9}$$

☐

Theorem 23. *If the recursion coefficients satisfy conditions ($\mathcal{C}_1$)-($\mathcal{C}_8$), then in the associated space $\mathcal{L}_2$ every sine wave has a finite positive norm and any two sine waves with unequal positive frequencies are mutually orthogonal.*

Proof. Let $\omega \neq 0$; since

$$\sum_{k=0}^{2n-1}(\mathcal{K}^k[\sin\omega t])^2 = \sum_{k=0}^{n-1}P_{2k}^2(\omega)\sin^2\omega t + \sum_{k=0}^{n-1}P_{2k+1}^2(\omega)\cos^2\omega t < \sum_{k=0}^{2n-1}P_k^2(\omega)$$

and since by our Corollary 4

$$0 < \lim_{n\to\infty}\frac{\sum_{j=0}^{n}P_j(\omega)^2}{\sum_{j=0}^{n}\frac{1}{\gamma_j}} < \infty; \tag{7.10}$$

we get that the sequence of functions

$$\frac{\sum_{k=0}^{n}(\mathcal{K}^k[\sin\omega t])^2}{\sum_{k=0}^{n}\frac{1}{\gamma_n}}$$

converges uniformly on every finite interval. Thus, by our Lemma 21, the limit of the sequence is independent of t. By substituting first $t = 0$ and then also $t = \pi/(2\omega)$ we obtain that

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{n}P_{2j}(\omega)^2}{\sum_{j=0}^{n}\frac{1}{\gamma_j}} = \lim_{n\to\infty}\frac{\sum_{j=0}^{n}P_{2j+1}(\omega)^2}{\sum_{j=0}^{n}\frac{1}{\gamma_j}}, \tag{7.11}$$

which implies that

$$\|\sin\omega t\|^2 = \lim_{n\to\infty}\frac{\sum_{k=0}^{n}(\mathcal{K}^k[\sin\omega t])^2}{\sum_{k=0}^{n}\frac{1}{\gamma_n}} = \frac{1}{2}\lim_{n\to\infty}\frac{\sum_{k=0}^{n}P_j(\omega)^2}{\sum_{k=0}^{n}\frac{1}{\gamma_n}}.$$

Thus, $\sin\omega t\in\mathcal{L}_2$. Let $\omega\neq 0,\ \sigma\neq 0\,,\ \omega\neq\sigma$. Note that

$$\sum_{k=0}^{2n}\mathcal{K}^k[\sin\omega t]\mathcal{K}^k[\sin\sigma t] = \sum_{k=0}^{n}P_{2k}(\omega)P_{2k}(\sigma)\sin\omega t\sin\sigma t + \sum_{k=0}^{n-1}P_{2k+1}(\omega)P_{2k+1}(\sigma)\cos\omega t\cos\sigma t. \tag{7.12}$$

It is easy to see that this implies

$$2\sum_{k=0}^{2n}\mathcal{K}^k[\sin\omega t]\mathcal{K}^k[\sin\sigma t] = \left(\sum_{k=0}^{n-1}P_{2k+1}(\omega)P_{2k+1}(\sigma) - \sum_{k=0}^{n}P_{2k}(\omega)P_{2k}(\sigma)\right)\cos(\omega+\sigma)t + \sum_{k=0}^{2n}P_k(\omega)P_k(\sigma)\cos(\omega-\sigma)t. \tag{7.13}$$

By Theorem 3 we have

$$\lim_{n\to\infty}\frac{P_{2n}^2(\omega)+P_{2n+1}^2(\omega)}{\frac{1}{\gamma_{2n}}+\frac{1}{\gamma_{2n+1}}} = L(\omega) > 0;$$

thus, for sufficiently large n,

$$P_{2n}(\omega) < \sqrt{2\left(\frac{1}{\gamma_{2n}}+\frac{1}{\gamma_{2n+1}}\right)L(\omega)};\qquad P_{2n+1}(\omega) < \sqrt{2\left(\frac{1}{\gamma_{2n}}+\frac{1}{\gamma_{2n+1}}\right)L(\omega)},$$

and we obtain

$$|\gamma_{2n}(P_{2n+1}(\omega)P_{2n}(\sigma) - P_{2n+1}(\sigma)P_{2n}(\omega))| < 4\gamma_{2n}\left(\frac{1}{\gamma_{2n}}+\frac{1}{\gamma_{2n+1}}\right)\sqrt{L(\omega)L(\sigma)} \to 8\sqrt{L(\omega)L(\sigma)}. \tag{7.14}$$

By (7.4) we get that

$$(\omega-\sigma)\sum_{k=0}^{2n}P_k(\omega)P_k(\sigma) = \gamma_{2n}(P_{2n+1}(\omega)P_{2n}(\sigma) - P_{2n+1}(\sigma)P_{2n}(\omega)). \tag{7.15}$$

From (7.15) and(7.14) we obtain

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{2n}P_j(\omega)P_j(\sigma)}{\sum_{j=0}^{n}\frac{1}{\gamma_j}} = 0. \tag{7.16}$$

Thus, by (7.13)

$$\lim_{n\to\infty}\frac{2\sum_{k=0}^{2n}\mathcal{K}^k[\sin\omega t]\mathcal{K}^k[\sin\sigma t]}{\sum_{k=0}^{n}\frac{1}{\gamma_n}} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}P_{2k+1}(\omega)P_{2k+1}(\sigma) - \sum_{k=0}^{n}P_{2k}(\omega)P_{2k}(\sigma)}{\sum_{k=0}^{n}\frac{1}{\gamma_n}}\cos(\omega+\sigma)t \tag{7.17}$$

Since by our Lemma 21 the value of

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{2n}\mathcal{K}^k[\sin\omega t]\mathcal{K}^k[\sin\sigma t]}{\sum_{k=0}^{n}\frac{1}{\gamma_n}}$$

is independent of t, we get that

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{n}P_{2j+1}(\omega)P_{2j+1}(\sigma)}{\sum_{j=0}^{n}\frac{1}{\gamma_j}} = \lim_{n\to\infty}\frac{\sum_{j=0}^{n}P_{2j}(\omega)P_{2j}(\sigma)}{\sum_{j=0}^{n}\frac{1}{\gamma_j}}, \tag{7.18}$$

and

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{2n}\mathcal{K}^k[\sin\omega t]\mathcal{K}^k[\sin\sigma t]}{\sum_{k=0}^{n}\frac{1}{\gamma_n}} = 0$$

□

It would be interesting to study inclusion relations between our spaces and the classical spaces of almost periodic functions, as well as such inclusions between our spaces for different families of orthogonal polynomials depending on the relationship of the asymptotic behaviour of the corresponding recursion coefficients γ_n.

8 Appendix

We here provide a summary of basic calculations as well as proofs of technical lemmas.

Lemma: 1. Conditions ($\mathcal{C}_1$)-($\mathcal{C}_8$) are satisfied by the Hermite polynomials and, more generally, by families with recursion coefficients defined by $\gamma_n = c(n+1)^p$ for any $0 < p < 1$ and $c > 0$.

Proof. Follows from the fact that, in this case, for all finite differences $\Delta^k(n)$ we have $\Delta^k(n) = \mathrm{O}\left(n^{p-k}\right)$. Thus, in particular, $s_n = \mathrm{O}\left(n^{p-1}\right)$ and $d_n = \mathrm{O}\left(n^{n-2}\right)$ and we obtain

$$s_n = \mathrm{O}\left(n^{p-1}\right) \to 0; \tag{8.1}$$

$$\gamma_n d_n = \mathrm{O}\left(n^{p}\right)\mathrm{O}\left(n^{p-2}\right) = \mathrm{O}\left(n^{2p-2}\right) \to 0; \tag{8.2}$$

$$\sum_{n=0}^{\infty} \frac{s_n}{\gamma_n^2} = \sum_{n=0}^{\infty} \mathrm{O}\left(n^{p-1}\right)\mathrm{O}\left(n^{-2p}\right) = \mathrm{O}\left(\sum_{n=0}^{\infty} n^{-p-1}\right) < \infty \tag{8.3}$$

$$\sum_{n=0}^{\infty} d_n = \sum_{n=0}^{\infty} O(n^{p-2}) = O\left(\sum_{n=0}^{\infty} n^{p-2}\right) < \infty \tag{8.4}$$

Note that (8.3) holds just in case $p > 0$, while the rest of the conditions hold just in case $p < 1$. Also, (8.4) is stronger than condition ($\mathcal{C}_6$). □

Lemma: 2. Conditions ($\mathcal{C}_1$)-($\mathcal{C}_8$) are satisfied by almost increasing sequence $\gamma_n = (n+1)^p + p(-1)^n(n+1)^{p-1}$ for $0 < p < 1/2$; condition ($\mathcal{C}_1$) is satisfied with $n_0 = 0$ and $m_0 = 2$.

Corollary: 4. If γ_n satisfy conditions $(\mathcal{C}_1) - (\mathcal{C}_8)$, then also $\lim\limits_{N\to\infty} \dfrac{\sum_{n=0}^{N} P_n^2(\omega)}{\sum_{n=0}^{N} \frac{1}{\gamma_n}} = L$.

Proof. Let N be such that $\left|\dfrac{P_{2n-1}^2(\omega) + P_{2n}^2(\omega)}{\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}} - L\right| < \varepsilon$ for all $n \geq N$; then it is easy to verify that for every $M > N$ also

$$\left|\frac{\sum_{n=N}^{M}\left(P_{2n}^2(\omega) + P_{2n+1}^2(\omega)\right)}{\sum_{n=N}^{M}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)} - L\right| \leq \frac{\sum_{n=N}^{M}\left|P_{2n}^2(\omega) + P_{2n+1}^2(\omega) - L\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)\right|}{\sum_{n=N}^{M}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)}$$
$$\leq \frac{\sum_{n=N}^{M}\varepsilon\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)}{\sum_{n=N}^{M}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)} = \varepsilon$$

and consequently also

$$\left|\frac{\sum_{n=0}^{2M} P_n^2(\omega)}{\sum_{n=0}^{2M}\frac{1}{\gamma_n}} - L\right| \leq \left|\frac{\sum_{n=0}^{2N-2} P_n^2(\omega)}{\sum_{n=0}^{2M}\frac{1}{\gamma_n}}\right| + \left|\frac{\sum_{n=N}^{M}\left(P_{2n}^2(\omega) + P_{2n+1}^2(\omega)\right)}{\sum_{n=N}^{M}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)}\,\frac{\sum_{n=N}^{M}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)}{\sum_{n=0}^{M}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)} - L\right|$$
$$= \left|\frac{\sum_{n=0}^{2N-2} P_n^2(\omega)}{\sum_{n=0}^{2M}\frac{1}{\gamma_n}}\right| + \left|(L+\varepsilon)\,\left(1 - \frac{\sum_{n=0}^{N}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)}{\sum_{n=0}^{M}\left(\frac{1}{\gamma_{2n}} + \frac{1}{\gamma_{2n+1}}\right)}\right) - L\right|$$

which, using the fact that $\sum_{n=0}^{2M}\frac{1}{\gamma_n}$ diverges, can be made arbitrarily small. □

Corollary: 5. If $\gamma_n = (n+1)^p$ for some p such that $0 < p < 1$, then $0 < \lim\limits_{N\to\infty} \dfrac{\sum_{n=0}^{N} P_n^2(\omega)}{(N+1)^{1-p}} < \infty$.

Proof. Follows from the previous Lemma and the fact that in this case

$$\sum_{k=0}^{n}\frac{1}{\gamma_k} = \mathrm{O}\left(\sum_{k=0}^{n}(k+1)^{-p}\right) = \mathrm{O}\left(\int_1^{n+1} x^{-p}dx\right) = \mathrm{O}\left(\frac{(n+1)^{1-p} - 1}{1-p}\right). \tag{8.5}$$

□

8.1 Basic Asymptotic Equalities

Let us set $\mathrm{r}(n) = \frac{\gamma_n}{\gamma_{n-1}}$; then we obtain

$$\mathrm{r}(2n) = 1 + \frac{s_{2n-1}}{\gamma_{2n-1}}; \qquad \frac{1}{\mathrm{r}(2n)} = 1 - \frac{s_{2n-1}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right)$$

$$\mathrm{r}(2n-1) = 1 + \frac{s_{2n-1} - d_{2n-2}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \quad \mathrm{r}(2n-2) = 1 + \frac{s_{2n-1} - d_{2n-2} - d_{2n-3}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right).$$

These asymptotic representations, together with (2.3) and (2.4) yield

$$\mathrm{cs}(n) = 1 - \frac{\omega^2}{2\gamma_{2n-1}^2} - \frac{2s_{2n-1} - d_{2n-2}}{2\gamma_{2n-1}} + \frac{\omega^2 s_{2n-1}}{2\gamma_{2n-1}^3} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.6}$$

$$\mathrm{sn}(n) = \frac{\omega}{\gamma_{2n-1}}\left(1 - \frac{2s_{2n-1} - d_{2n-2}}{2\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right)\right); \tag{8.7}$$

$$\widetilde{\mathrm{cs}}(n) = \frac{\omega^2}{2\gamma_{2n-1}^2} + \frac{d_{2n-2}}{2\gamma_{2n-1}} - \frac{\omega^2 s_{2n-1}}{2\gamma_{2n-1}^3} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.8}$$

$$\widetilde{\mathrm{sn}}(n) = -\frac{\omega(2s_{2n-1} - d_{2n-2})}{2\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^3}\right); \tag{8.9}$$

$$\mathrm{cs}(n-1) = 1 - \frac{\omega^2}{2\gamma_{2n-1}^2} - \frac{2s_{2n-1} - 2d_{2n-2} - 2d_{2n-3} - d_{2n-4}}{2\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.10}$$

$$\mathrm{sn}(n-1) = \frac{\omega}{\gamma_{2n-1}} + \frac{\omega(2s_{2n-1} - 2d_{2n-2} + d_{2n-4}}{2\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^3}\right); \tag{8.11}$$

$$\widetilde{\mathrm{cs}}(n-1) = \frac{\omega^2}{2\gamma_{2n-1}^2} + \frac{d_{2n-4}}{2\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.12}$$

$$\widetilde{\mathrm{sn}}(n-1) = -\frac{\omega(2s_{2n-1} - 2d_{2n-2} - 2d_{2n-3} - d_{2n-4})}{2\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^3}\right). \tag{8.13}$$

These formulas yield the following representations:

$$\alpha_n^2 = 1 + \frac{\omega^4}{4\gamma_{2n-1}^4} - \frac{2s_{2n-1} - d_{2n-2}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.14}$$

$$\alpha_{n-1}^2 = 1 + \frac{\omega^4}{4\gamma_{2n-1}^4} - \frac{2s_{2n-1} - 2d_{2n-2} - 2d_{2n-3} - d_{2n-4}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.15}$$

$$\widetilde{\alpha}_n^2 = \frac{\omega^2}{4\gamma_{2n-1}^2}\left(\frac{\omega^2}{\gamma_{2n-1}^2} + \frac{d_{2n-2}^2}{\omega^2} + \frac{2d_{2n-2}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right)\right); \tag{8.16}$$

$$\widetilde{\alpha}_{n-1}^2 = \frac{\omega^2}{4\gamma_{2n-1}^2}\left(\frac{\omega^2}{\gamma_{2n-1}^2} + \frac{d_{2n-2}^2}{\omega^2} + \frac{2d_{2n-2}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right)\right); \tag{8.17}$$

$$\ln \lambda_{n-1} = \frac{4s_{2n-1} - 4d_{2n-2} - 3d_{2n-3} - d_{2n-4}}{2\gamma_{2n-1}} + \mathrm{O}\left(\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right). \tag{8.18}$$

Equalities (8.6)-(8.18) will be referred to as *the basic equalities.*

Lemma: 8.

$$F_n(\mathrm{e}^{i\,t}) = f\left(\frac{\omega}{\gamma_{2n-1}}, \mathrm{e}^{i\,t}\right) - \frac{d_{2n-3}\left(1 - \frac{2\omega}{\gamma_{2n-1}}\sin t\right) + d_{2n-2}\left(1 - \cos t - \frac{2\omega}{\gamma_{2n-1}}\sin t\right) - d_{2n-4}\left(\cos t + \frac{2\omega}{\gamma_{2n-1}}\sin t\right)}{\gamma_{2n-1}} - \frac{4\omega}{\gamma_{2n-1}^2} s_{2n-1}\sin t + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.19}$$

$$G_n(\mathrm{e}^{i\,t}) = g\left(\frac{\omega}{\gamma_{2n-1}}, \mathrm{e}^{i\,t}\right) + \frac{2\omega\,(2(1-\cos t)s_{2n-1} + \cos t\; d_{2n-4} - (1-\cos t)d_{2n-3} - (2-\cos t)d_{2n-2})}{\gamma_{2n-1}^2} - \frac{(d_{2n-4} + d_{2n-2})\sin t}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right) \tag{8.20}$$

□

Proof. We introduce the following auxiliary functions.

$$\mathrm{CN}(n) = \mathrm{cs}(n)\widetilde{\mathrm{cs}}(n) + \mathrm{sn}(n)\widetilde{\mathrm{sn}}(n); \qquad \mathrm{CM}(n) = \mathrm{cs}(n-1)\widetilde{\mathrm{cs}}(n-1) - \mathrm{sn}(n-1)\widetilde{\mathrm{sn}}(n-1); \tag{8.21}$$

$$\mathrm{SN}(n) = \mathrm{sn}(n)\widetilde{\mathrm{cs}}(n) - \mathrm{cs}(n)\widetilde{\mathrm{sn}}(n); \qquad \mathrm{SM}(n) = \mathrm{sn}(n-1)\widetilde{\mathrm{cs}}(n-1) + \mathrm{cs}(n-1)\widetilde{\mathrm{sn}}(n-1); \tag{8.22}$$

$$\mathrm{MN}(n) = \mathrm{cs}(n-1)\mathrm{cs}(n) - \mathrm{sn}(n-1)\mathrm{sn}(n); \qquad \mathrm{MM}(n) = \mathrm{cs}(n)\mathrm{sn}(n-1) + \mathrm{cs}(n-1)\mathrm{sn}(n); \tag{8.23}$$

$$\mathrm{HH}(n,t) = \alpha_n^2 + (\mathrm{CN}(n) - i\,\mathrm{SN}(n))\mathrm{e}^{-i\,t} \qquad \mathrm{LL}(n) = \alpha_{n-1}^2 + (\mathrm{CM}(n) + i\,\mathrm{SM}(n))\mathrm{e}^{-i\,t}; \tag{8.24}$$

$$\mathrm{KK}(n) = \alpha_n^2 + (\mathrm{CN}(n) + i\,\mathrm{SN}(n))\mathrm{e}^{i\,t}; \qquad \mathrm{RR}(n) = \alpha_{n-1}^2 - (\mathrm{CM}(n) - i\,\mathrm{SM}(n))\mathrm{e}^{i\,t} \tag{8.25}$$

then a straightforward calculation shows that for $G(z)$ defined via (5.6) we have for all $-\pi \le t \le \pi$,

$$G_n(\mathrm{e}^{i\,t}) = i\left(\ln(\alpha_n^2\alpha_{n-1}^2) - 2\ln(\mathrm{MN}(n) + i\,\mathrm{MM}(n)) - \ln \mathrm{HH}(n, 2\Phi_{n-1}) + \ln \mathrm{LL}(n, 2\Phi_{n-1}) - \ln \mathrm{RR}(n, 2\Phi_{n-1}) + \ln \mathrm{KK}(n, 2\Phi_{n-1})\right) \tag{8.26}$$

Similarly, for $F_n(z)$ defined by (5.5) we have for all $-\pi \le t \le \pi$,

$$F_n(\mathrm{e}^{i\,t}) = 2\ln(\alpha_{n-1}^2 - \widetilde{\alpha}_{n-1}^2) + \ln\frac{\alpha_n^2}{\alpha_{n-1}^2} + 2\ln\lambda_{n-1} + \ln \mathrm{HH}(n, 2\Phi_{n-1}) - \ln \mathrm{LL}(n, 2\Phi_{n-1}) + \ln \mathrm{KK}(n, 2\Phi_{n-1}) - \ln \mathrm{RR}(\mathrm{n}, 2\Phi_{\mathrm{n-1}}) \tag{8.27}$$

The basic equalities now yield the following asymptotic representations:

$$\ln(\alpha_{n-1}^2 - \widetilde{\alpha}_{n-1}^2) = \frac{2d_{2n-2} + 2d_{2n-3} + d_{2n-4} - 2s_{2n-1}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right);$$

$$\ln\frac{\alpha_n^2}{\alpha_{n-1}^2} = -\frac{d_{2n-2} + 2d_{2n-3} + d_{2n-4}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-1}|}{\gamma_{2n-1}^5} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right);$$

$$\ln(\alpha_n^2\alpha_{n-1}^2) = 2\ln\left(1 + \frac{\omega^4}{4\gamma_{2n-1}^4}\right) + \frac{3d_{2n-2} + 2d_{2n-3} + d_{2n-4} - 4s_{2n-1}}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right);$$

$$\mathrm{MN}(n) + i\,\mathrm{MM}(n) = 1 - \frac{2\omega^2}{\gamma_{2n-1}^2} + \frac{\omega^4}{4\gamma_{2n-1}^4} + \frac{2\,i\,\omega}{\gamma_{2n-1}}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2}\right) + \frac{3d_{2n-2} + 2d_{2n-3} + d_{2n-4} - 4s_{2n-1}}{2\gamma_{2n-1}} - \frac{2\,i\,\omega\, s_{2n-1}}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-4}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.28}$$

$$\mathrm{HH}(n) = 1 + \frac{\omega^4}{4\gamma_{2n-1}^4} + \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} - i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{-i\,t} + \frac{(2 + \cos t - i\sin t)d_{2n-2} - 4s_{2n-1}}{2\gamma_{2n-1}} - \frac{(i\cos t + \sin t)\omega\, s_{2n-1}}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.29}$$

$$\mathrm{LL}(n) = 1 + \frac{\omega^4}{4\gamma_{2n-1}^4} - \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} + i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{-i\,t} + \frac{4(d_{2n-2} + d_{2n-3} - s_{2n-1}) + d_{2n-4}(2 - \cos t + i\sin t)}{2\gamma_{2n-1}} + \frac{\omega\, s_{2n-1}(i\cos t + \sin t)}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-4}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.30}$$

$$\mathrm{RR}(n) = 1 + \frac{\omega^4}{4\gamma_{2n-1}^4} - \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} - i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{i\,t} + \frac{4(d_{2n-2} + d_{2n-3} - s_{2n-1}) + d_{2n-4}(2 - \cos t + i\sin t)}{2\gamma_{2n-1}} + \frac{\omega\, s_{2n-1}(-\,i\cos t + \sin t)}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-4}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.31}$$

$$\mathrm{KK}(n) = 1 + \frac{\omega^4}{4\gamma_{2n-1}^4} + \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} + i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{i\,t} + \frac{(2+\cos t + i\sin t)d_{2n-2} - 4s_{2n-1}}{2\gamma_{2n-1}} + \frac{(i\cos t - \sin t)\omega\, s_{2n-1}}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.32}$$

The above equalities yield

$$\ln(\mathrm{MN}(n) + i\,\mathrm{MM}(n)) = \ln\left(1 - \frac{2\omega^2}{\gamma_{2n-1}^2} + \frac{\omega^4}{4\gamma_{2n-1}^4} + \frac{2\,i\,\omega}{\gamma_{2n-1}}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2}\right)\right) + \frac{3d_{2n-2} + 2d_{2n-3} + d_{2n-4} - 4s_{2n-1}}{2\gamma_{2n-1}} - \frac{2\,i\,\omega\, s_{2n-1}}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-4}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.33}$$

(8.34)

$$\ln \mathrm{HH}(n) = \ln\left(1 + \frac{\omega^4}{4\gamma_{2n-1}^4} + \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} - i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{-\,i\,t}\right) + \frac{(2+\cos t - i\sin t)d_{2n-2} - 4s_{2n-1}}{2\gamma_{2n-1}} - \frac{(i\cos t + \sin t)\omega\, s_{2n-1}}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.35}$$

(8.36)

$$\ln \mathrm{LL}(n) = \ln\left(1 + \frac{\omega^4}{4\gamma_{2n-1}^4} - \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} + i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{-\,i\,t}\right) + \frac{\omega\, s_{2n-1}(i\cos t + \sin t)}{\gamma_{2n-1}^2} + \frac{4(d_{2n-2} + d_{2n-3} - s_{2n-1}) + d_{2n-4}(2 - \cos t + i\sin t)}{2\gamma_{2n-1}} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-4}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.37}$$

(8.38)

$$\ln \mathrm{RR}(n) = \ln\left(1 + \frac{\omega^4}{4\gamma_{2n-1}^4} - \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} - i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{i\,t}\right) + \frac{\omega\, s_{2n-1}(-\,i\cos t + \sin t)}{\gamma_{2n-1}^2} + \frac{4(d_{2n-2} + d_{2n-3} - s_{2n-1}) + d_{2n-4}(2 - \cos t + i\sin t)}{2\gamma_{2n-1}} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-4}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.39}$$

(8.40)

$$\ln \mathrm{KK}(n) = \ln\left(1 + \frac{\omega^4}{4\gamma_{2n-1}^4} + \frac{\omega^2}{2\gamma_{2n-1}^2}\left(1 - \frac{\omega^2}{2\gamma_{2n-1}^2} + i\,\frac{\omega}{\gamma_{2n-1}}\right)\mathrm{e}^{i\,t}\right) + \frac{(2+\cos t + i\sin t)d_{2n-2} - 4s_{2n-1}}{2\gamma_{2n-1}} + \frac{(i\cos t - \sin t)\omega\, s_{2n-1}}{\gamma_{2n-1}^2} + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.41}$$

These equalities imply (5.12) and (5.13). □

Lemma: 9.

$$H_n(\mathrm{e}^{i\,t}) = h\left(m, \frac{\omega}{\gamma_{2n-1}}, \mathrm{e}^{i\,t}\right) + \frac{d_{2n-4}\cos t - d_{2n-3} - (1-\cos t)d_{2n-2}}{4\omega} + \frac{\omega\, s_{2n-1}^2}{\gamma_{2n-1}} - \frac{s_{2n-1}}{\gamma_{2n-1}}\sin t + \mathrm{O}\left(\frac{|d_{2n-4}| + |d_{2n-3}| + |d_{2n-2}|}{\gamma_{2n-1}} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^2}\right).$$

Proof. Equality (5.13) yields

$$\frac{1}{G_n(\mathrm{e}^{i\,t})} = \frac{1}{g\left(\frac{\omega}{\gamma_{2n-1}}, \mathrm{e}^{i\,t}\right)} + \frac{(1 - 4\cos t - \cos 2t)d_{2n-4} + 4(1-\cos t)d_{2n-3} + (9 - 4\cos t - \cos 2t)d_{2n-2}}{32\omega} - \frac{(1-\cos t)s_{2n-1}}{4\omega} + \mathrm{O}\left(\frac{|d_{2n-2}| + |d_{2n-3}| + |d_{2n-4}|}{\gamma_{2n-1}} + \frac{|s_{2n-1}|}{\gamma_{2n-1}} + s_{2n-1}^2\right). \tag{8.42}$$

Multiplying with $F_n(\mathrm{e}^{i\,t})$ given by (5.12) after some simplification we obtain (5.14). □

Lemma: 10.

$$f\big(x,\mathrm{e}^{i\,t}\big)=\ln\left(1+\frac{2x^2\left(1-\frac{x^2}{2}\right)\cos t}{1+\frac{x^4}{2}-x^2\left(1-\frac{x^2}{2}\right)\cos t-x^3\sin t}\right);$$

$$g\big(x,\mathrm{e}^{i\,t}\big)=2\arctan\frac{2x\left(1-\frac{x^2}{2}\right)\left(1-\frac{x}{2}\sin t\right)}{1-2x^2+x^3\sin t}.$$

Thus,

$$h\big(x,\mathrm{e}^{i\,t}\big)=\frac{\ln\left(1+\frac{2x^2\left(1-\frac{x^2}{2}\right)\cos t}{1+\frac{x^4}{2}-x^2\left(1-\frac{x^2}{2}\right)\cos t-x^3\sin t}\right)}{2\arctan\frac{2x\left(1-\frac{x^2}{2}\right)\left(1-\frac{x}{2}\sin t\right)}{1-2x^2+x^3\sin t}}.$$

□.

Proof. It is easy to see that the logarithms in equation (8.26) can be combined and this yields

$$f\big(x,\mathrm{e}^{i\,t}\big)=\ln\frac{\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)\mathrm{e}^{i\,t}\right)\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)\mathrm{e}^{-\,i\,t}\right)}{\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)\mathrm{e}^{i\,t}\right)\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)\mathrm{e}^{-\,i\,t}\right)}. \tag{8.43}$$

Another simplification yields

$$f\big(x,\mathrm{e}^{i\,t}\big)=\ln\left(1+\frac{2x^2\left(1-\frac{x^2}{2}\right)\cos t}{1+\frac{x^4}{2}-x^2\left(1-\frac{x^2}{2}\right)\cos t-x^3\sin t}\right). \tag{8.44}$$

Similarly, combining the logarithms in $g\big(x,\mathrm{e}^{i\,t}\big)$ we obtain

$$g\big(x,\mathrm{e}^{i\,t}\big)=-\,i\ln\frac{\left(1-x\left(\frac{x}{2}-i\right)\right)\left(1-x\left((1+\mathrm{e}^{i\,t})\frac{x}{2}-i\right)\right)\left(1+\frac{x^4}{4}+\frac{x^2}{2}\left(1-\frac{x^2}{2}-i\,x\right)\mathrm{e}^{-\,i\,t}\right)}{\left(1-x\left(\frac{x}{2}+i\right)\right)\left(1-x\left((1-\mathrm{e}^{i\,t})\frac{x}{2}+i\right)\right)\left(1+\frac{x^4}{4}-\frac{x^2}{2}\left(1-\frac{x^2}{2}+i\,x\right)\mathrm{e}^{-\,i\,t}\right)}. \tag{8.45}$$

One then verifies that the real part of the logarithm in (8.45) is zero and that

$$g\big(x,\mathrm{e}^{i\,t}\big)=\arg\left(\left(1-2x^2+x^3\sin t+2\,i\,x\left(1-\frac{x^2}{2}\right)\left(1-2x^2\left(1-\frac{x}{2}\sin t\right)\right)\right)^2\right), \tag{8.46}$$

which yields (5.19). Finally, combining (5.18) and (5.19) one obtains (5.20). □

Lemma: 14.

$$L_n(m,\mathrm{e}^{i\,t})=l\bigg(m,\frac{\omega}{\gamma_{2n-1}},\mathrm{e}^{i\,t}\bigg)+\frac{i\,m(d_{2n-4}+d_{2n-2})\sin t}{\gamma_{2n-1}}-\frac{4\,i\,m\,\omega(1-\cos t)s_{2n-1}}{\gamma_{2n-1}^2}+$$
$$\mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-2}|}{\gamma_{2n-1}^2}+\frac{|s_{2n-1}|}{\gamma_{2n-1}^3}+\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right).$$

□

Proof. We define two additional auxiliary functions:

$$\mathrm{RN}(n)=\frac{(\mathrm{cs}(n)+i\,\mathrm{sn}(n))^2}{\alpha_n^2};\qquad\qquad \mathrm{RM}(n)=\frac{(\mathrm{cs}(n-1)-i\,\mathrm{sn}(n-1))^2}{\alpha_{n-1}^2};$$

Then a direct calculation shows that for $L_n(m,z)$ defined via (6.8) we have

$$L_n\left(m,\mathrm{e}^{i\,t}\right)=\left(\frac{\mathrm{RM}(n)\ \mathrm{LL}(n,t)}{\mathrm{RR}(n,t)}\right)^m-\left(\frac{\mathrm{RN}(n)\ \mathrm{HH}(n,t)}{\mathrm{KK}(n,t)}\right)^m \tag{8.47}$$

The basic equalities yield

$$\mathrm{RN}(n)=1-\frac{\frac{2\omega^2}{\gamma_{2n-1}^2}-i\,\frac{2\omega}{\gamma_{2n-1}}\left(1-\frac{\omega^2}{2\gamma_{2n-1}^2}\right)}{1+\frac{\omega^4}{4\gamma_{2n-1}^4}}+\mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3}+\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.48}$$

$$\mathrm{RM}(n)=1-\frac{\frac{2\omega^2}{\gamma_{2n-1}^2}+i\,\frac{2\omega}{\gamma_{2n-1}}\left(1-\frac{\omega^2}{2\gamma_{2n-1}^2}\right)}{1+\frac{\omega^4}{4\gamma_{2n-1}^4}}-4\,i\,\frac{\omega\ s_{2n-1}}{\gamma_{2n-1}^2}+2\,i\,\frac{\omega(d_{2n-3}+2d_{2n-2})}{\gamma_{2n-1}^2}+\mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3}+\frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right). \tag{8.49}$$

Combining the above with equations (8.29)-(8.32) we obtain

$$\frac{\mathrm{RM}(n)\ \mathrm{LL}(n,t)}{\mathrm{RR}(n,t)} = \frac{1-\frac{\omega}{\gamma_{2n-1}}\left(i+\frac{\omega}{2\gamma_{2n-1}}\left(1+\mathrm{e}^{-i\,t}\right)\right)}{1+\frac{\omega}{\gamma_{2n-1}}\left(i+\frac{\omega}{2\gamma_{2n-1}}\left(1+\mathrm{e}^{i\,t}\right)\right)} - \frac{2\,i\,\omega\,(2-\cos t)s_{2n-1}}{\gamma_{2n-1}^2} + \frac{i\,d_{2n-4}\sin t}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-2}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right); \tag{8.50}$$

$$\frac{\mathrm{RN}(n)\ \mathrm{HH}(n,t)}{\mathrm{KK}(n,t)} = \frac{1+\frac{\omega}{\gamma_{2n-1}}\left(i-\frac{\omega}{2\gamma_{2n-1}}\left(1-\mathrm{e}^{-i\,t}\right)\right)}{1-\frac{\omega}{\gamma_{2n-1}}\left(i+\frac{\omega}{2\gamma_{2n-1}}\left(1-\mathrm{e}^{i\,t}\right)\right)} - \frac{2\,i\,\omega\,s_{2n-1}\cos t}{\gamma_{2n-1}^2} + \frac{d_{2n-2}\sin t}{\gamma_{2n-1}}\left(\frac{2\omega}{\gamma_{2n-1}}-i\right) + \mathrm{O}\left(\frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right). \tag{8.51}$$

Finally, combining these two equations we get

$$L_n\left(m,\mathrm{e}^{i\,t}\right) = \left(\frac{1-\frac{\omega}{\gamma_{2n-1}}\left(i+\frac{\omega}{2\gamma_{2n-1}}\left(1+\mathrm{e}^{-i\,t}\right)\right)}{1+\frac{\omega}{\gamma_{2n-1}}\left(i+\frac{\omega}{2\gamma_{2n-1}}\left(1+\mathrm{e}^{i\,t}\right)\right)}\right)^m - \left(\frac{1+\frac{\omega}{\gamma_{2n-1}}\left(i-\frac{\omega}{2\gamma_{2n-1}}\left(1-\mathrm{e}^{-i\,t}\right)\right)}{1-\frac{\omega}{\gamma_{2n-1}}\left(i+\frac{\omega}{2\gamma_{2n-1}}\left(1-\mathrm{e}^{i\,t}\right)\right)}\right)^m - \frac{8\,i\,m\,\omega\,s_{2n-1}\ \sin^2\left(\frac{t}{2}\right)}{\gamma_{2n-1}^2} + \frac{i\ m(d_{2n-4}+d_{2n-2})\sin t}{\gamma_{2n-1}} + \mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-2}|}{\gamma_{2n-1}^2} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^3} + \frac{s_{2n-1}^2}{\gamma_{2n-1}^2}\right), \tag{8.52}$$

which proves (4.1). □

Lemma: 15.

$$\frac{L_n(m,\mathrm{e}^{i\,t})}{G_n\left(\mathrm{e}^{i\,2\Phi_{n-1}}\right)} = \frac{l\left(m,\frac{\omega}{\gamma_{2n-1}},\mathrm{e}^{i\,t}\right)}{g\left(\frac{\omega}{\gamma_{2n-1}},\mathrm{e}^{i\,t}\right)} + \frac{i\,m(d_{2n-4}+d_{2n-2})^2\sin^2 t}{16\omega^2} + \mathrm{O}\left(\frac{|d_{2n-4}|+|d_{2n-3}|+|d_{2n-2}|}{\gamma_{2n-1}} + \frac{|s_{2n-1}|}{\gamma_{2n-1}^2} + \frac{s_{2n-1}^2}{\gamma_{2n-1}}\right). \tag{8.53}$$

Proof. Follows directly from Lemma 14 and (8.42). □

Bibliography

[1] A. Ignjatovic, Chromatic Derivatives, chromatic expansions and associated function spaces, East Journal on Approximations, Vol. 15, No. 2, 2009, pp. 263-302.