Phylogeny, Genealogy, and the Linnaean Hierarchy: Formal Proofs

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Abstract

Phylogenetic terms (monophyly, polyphyly, and paraphyly) were first used in the context of a phylogenetic tree. However, the only possible source for a phylogeny is a genealogy. This paper presents formal definitions for phylogenetic terms in a genealogical context and shows that their properties match their intuitive meanings. Moreover, by presenting the definitions in a genealogical context, a firm connection between genealogy and phylogeny is established. To support the correctness of the definitions, results will show that they satisfy the appropriate properties in the context of a phylogenetic tree.

Ancestors in a phylogenetic tree are viewed as theoretical entities since no means exist for proving ancestral relationships. As such, groups of terminal species are often considered. This will impact on phylogenetic concepts. Results will be presented showing that monophyly and polyphyly have reasonable interpretations in this context while the notion of paraphyly becomes degenerate. The vigorous debate about whether biological taxa should be monophyletic will also be addressed. Results will be presented showing why the monophyletic condition will make a Linnaean classification entirely monotypic.
Keywords: phylogeny, monophyly, polyphyly, Linnaean hierarchy, knowledge representation

1 Introduction

A graph has for a long time been widely recognised as a more accurate model in which to study phylogenetics [8]. Yet, phylogenetic concepts are defined relative to phylogenetic trees; a more constrained model. The aim here is to redefine phylogenetic concepts within a graph model. This is important because the meaning and intuition behind phylogenetic concepts may not carry over into this more general model. Proofs will we presented showing the relationship between monophyletic, polyphyletic, and paraphyletic groups. Superimposing a Linnaean classification scheme on top of the graph model, several theorems will show that a totally monophyletic Linnaean classification is trivial; every rank is monotypic. The exact reasons for this will be examined. From this, a weakened notion of a monophyletic classification will be presented which can be accommodated within a Linnaean classification. Significant steps will also be taken showing how a phylogenetic tree can be generated from a graph.

Hennig [8] argues that a genealogical network is better than a hierarchy for modelling genetic inheritance. Figure 1.1 is taken from Figure 4 in Hennig's *Phylogenetic Systematics*. It differs slightly from the original in not having male and female individuals. What the figure shows is how individuals are related to each other through family ties. It also shows how an individual can pass on some or all of it's genetic legacy to zero or more progeny. Moreover it depicts how one group *A* splits into two *separate* groups - *B* and *C*. This can be summarised as a phylogenetic tree where a parent group *A* gives rise to two leaves *B* and *C*. However, if we begin with only a genealogical network, convenient labels such as *A*, *B*, and *C* will be missing. Perhaps more importantly, the very suggestive wedge which cleaves the graph will be missing. How are such labels and wedges to be defined from a genealogical network? Also, how can a genealogical network be summarised as a phylogenetic tree? Such issues will be the study of this paper.

2 The Genealogical Network and Descent Groups

A directed acyclic graph (DAG) is a structure that models biological reproduction well. With a tree, where each individual has at most one parent, parthenogenesis is modelled by linking a parent to each offspring. A DAG allows an individual to have multiple parents. This allows sexual reproduction to be modelled. To a lesser extent it also models fungal anastomosis and endosymbiosis. These biological processes are very different to sexual reproduction. However, in terms of mapping the transfer of genetic material (from source to destination), the DAG is an adequate representation. Both hyphal anastomosis (fungi) and endosymbiosis involve the addition of genetic material to generate a 'new' organism. Perhaps the one form of 'reproduction' that the DAG cannot model is plasmid exchange. This is due to the acyclic constraint on a DAG. However, one possibility is to interpret the plasmid exchange process as producing new individuals; a form of reproduction. A DAG consists of two components: a set of individuals and a parent relation.

Definition 1 (Genealogical Network) *A genealogical network G is a pair (X, p) where X is a finite set and p is a binary relation on X subject to the restriction that p is acyclic, i.e., there does not exist a sequence $x_1, x_2, \ldots, x_n$ of elements of X such that:*

1. $n \geq 2$

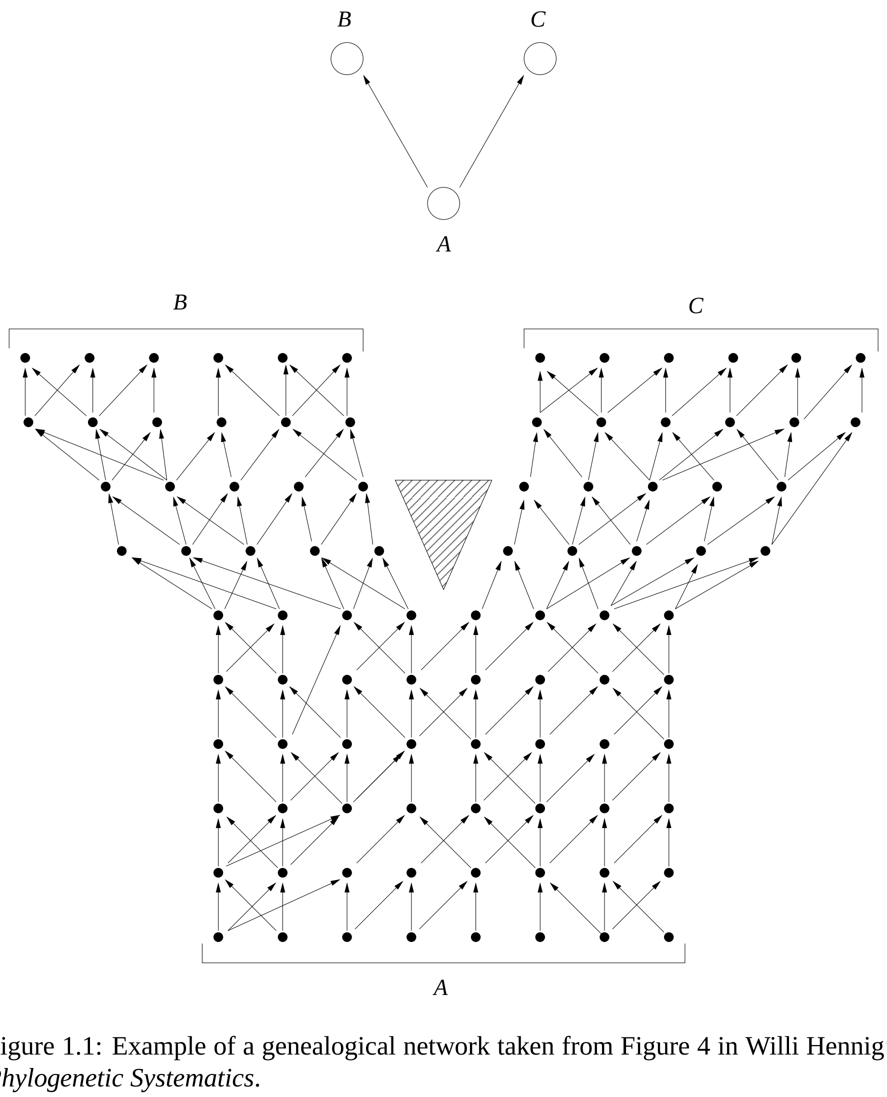

Figure 1.1: Example of a genealogical network taken from Figure 4 in Willi Hennig's *Phylogenetic Systematics*.

2. $x_1 = x_n$

3. $(x_i, x_{i+1}) \in p$ *for every* $i, 1 \leq i \leq n-1$.

The genealogical network defines a class of structures. One particular structure will actually represent a complete history of life on earth; showing the precise genetic heritage of every living organism. This structure is of course inaccessible. However, it is possible to derive properties that are satisfied by *all* genealogical networks. Such properties will then be satisfied by the network that does represent the genetic history for life.

Given a genealogical network $G = (X, p)$, call X the *population* of G and p the *parent of relation over* X *in* G. The set X represents things that have lived or are living and elements of X are called *individuals*. Elements (x_1, x_2) of p can be read as "x_1 is a parent of x_2 or "x_1 donates genetic material to x_2". This definition does not imply that an organism is an unstructured point (or that mathematics is restricted to modelling an organism as a point), only that for the *purposes* of mapping genetic heritage it is *adequate* to view an organism as a point.

The parent relation can be generalised to an *ancestor of* relation. An ancestor is an individual which donates genetic material through a sequence of descendants to an individual.

Definition 2 (Ancestor) *Consider a genealogical network* $G = (X, p)$ *and an individual* $x \in X$. *An individual* $a \in X$ *is an* ancestor *of* x *if and only if there exists* $x_1, x_2, \ldots x_n \in X$ *such that*

1. $n \geq 1$

2. $x_1 = a$

3. $x_n = x$

4. if $n > 1$*, then* $(x_i, x_{i+1}) \in p$ *for every* i, $1 \leq i \leq n-1$

The last condition on the sequence makes every individual an ancestor of itself. While this may seem unintuitive, it makes more concise and legible a number of definitions and results.

A number of the preliminary results presented in this section have exact analogues in the study of Graph Theory in mathematics. However, these observations are presented here so that the paper is self–contained.

One basic property is that the ancestor relation form a partial order. This means that the concept of ancestor defines lines of descent in a genealogical network.

Observation 1 (Ancestor Relation A Partial Order) *Consider a genealogical network* $G = (X, p)$. *The ancestor relation is a partial order over* X*, i.e.,*

1. for every $x \in X$, x *is an ancestor of* x

2. for every $x, y \in X$*, if* x *is an ancestor of* y *and* y *is an ancestor of* x*, then* $x = y$

3. for every $x, y, z \in X$*, if* x *is an ancestor of* y *and* y *is an ancestor of* z*, then* x *is an ancestor of* z

A common criterion used for defining the concepts of monophyly, polyphyly, and paraphyly is *descendant closure* – a fixed–point idea that any descendant of a group member is contained in the group. For Hennig [8], Nelson [11], Farris [5], this is central to defining monophyly. Nelson and Farris carry this further and use this idea for defining paraphyly and polyphyly as well. Sets of individuals from a genealogical network that satisfy this criterion (here called *descent groups*) are already of interest. The remainder of this section is dedicated to the properties of descent groups. In the jargon of mathematics, a *descent group* is a collection of individuals that is closed under the parent relation.

Definition 3 (Descent Group) *Given a genealogical network $G = (X, p)$, a* descent group *D in G is a set such that:*

1. *$D \subseteq X$*
2. *for every $x \in X$, if there exists an $a \in D$ such that a is an ancestor of x, then $x \in D$*

The set of descent groups in a genealogical network are naturally structured into a bounded lattice. At one end of the lattice, the set of individuals X is a descent group. At the other end, the empty set is also a descent group.

Observation 2 (Limiting Descent Groups) *Given a genealogical network $G = (X, p)$,*

1. *$\emptyset$ is a descent group*
2. *X is a descent group*

The individuals from two descent groups taken together constitute another descent group. Similarly, the individuals common to both descent groups also constitute a descent group. The symbol $\cap$ denotes set intersection while $\cup$ denotes set union. Given sets A and B, $A \cap B$ is the set of elements which are common to both A and B. Similarly, an element is contained in $A \cup B$ exactly when it is contained in either A or B.

Observation 3 (Descent Group Closure Under Intersection and Union) *Consider a genealogical network $G = (X, p)$. If D_1 and D_2 be two descent groups in G, then*

1. *$D_1 \cap D_2$ is a descent group in G.*
2. *$D_1 \cup D_2$ is a descent group in G.*

Using set union and set intersection as the basis for defining the “meet” and “join” structures descent groups into a lattice. through set union and set intersection.

Definition 4 (Descent Group Ordering) *Consider a genealogical network $G = (X, p)$. For every pair D_1, D_2 of descent groups in G, say that $D_1 \leq D_2$ if and only if $D_1 \subseteq D_2$.*

Observation 4 (Descent Group Partial Ordering) *Consider a genealogical network $G = (X, p)$. Let $descent(G)$ denote the set of all descent groups in G. Then, the binary relation $\leq$ is a partial order over $descent(G)$.*

Definition 5 (Meet and Join) *Consider a genealogical network $G = (X, p)$. For any two descent groups D_1, D_2 in G, define the* meet *and the* join *as follows:*

1. *the* meet *of D_1 and D_2 is $D_1 \cap D_2$*
2. *the* join *of D_1 and D_2 is $D_1 \cup D_2$*

Observation 5 (Descent Groups Form a Bounded Lattice) *Consider a genealogical network $G = (X, p)$. Let $descent(G)$ denote the set of all descent groups in G. Then, $(descent(G), \leq)$ is a bounded lattice.*

Intuitively, a descent group should contain progenitors that are the source of all genetic material to members of the descent group. A progenitor in a descent group does not have any parents in the descent group.

Definition 6 (Progenitor) *Let D be a descent group in a genealogical network $G = (X, p)$. A* progenitor *x of D satisfies:*

1. *$x \in D$*
2. *for every $y \in X$, if $(y, x) \in p$, then $y \notin D$*

The progenitors of a descent group can be gathered into a set.

Definition 7 (Set of Progenitors) *Let D be a descent group in a genealogical network $G = (X, p)$. The* set of progenitors *$P(D)$ of D is the set $P(D) = \{x \in D \mid x$ is a progenitor in $D\}$.*

An individual in a descent group fall into two categories: a progenitor or the descendant of a progenitor.

Observation 6 (Progenitor as Founding Ancestor) *Let $P(D)$ be the progenitor set for a descent group D in genealogical network $G = (X, p)$. For every $x \in D$, either*

1. *$x \in P(D)$, or*
2. *there exists a $y \in P(D)$ such that $y \neq x$ and y is an ancestor of x*

Given a set of individuals, the descendants of any individual in the set can be generated and collected via a closure operation.

Definition 8 (Set Closure) *Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. Define the closure of A by:*

$$cl(A) = \{x \in X | \text{ for some } a \in A \text{ is an ancestor of } x\}$$

The function cl defines a closure operator. This will be central to a number results concerned with descent group construction.

Observation 7 (cl a Closure Operator) *Consider a genealogical network $G = (X, p)$. The function $cl : 2^X \rightarrow 2^X$ is a closure operator, i.e.,*

1. *for every $A \subseteq X$, $A \subseteq cl(A)$*
2. *for every $A, B \subseteq X$, if $A \subseteq B$, then $cl(A) \subseteq cl(B)$*

3. *for every $A \subseteq X$, $cl(cl(A)) = cl(A)$*

The function cl distributes over set union but not intersection.

Observation 8 (Distributivity of cl) *Consider a genealogical network $G = (X, p)$. For every $A, B \subseteq X$*

1. $cl(A \cup B) = cl(A) \cup cl(B)$
2. $cl(A \cap B) \subseteq cl(A) \cap cl(B)$

It does not follow that $cl(A) \cap cl(B) \subseteq cl(A \cap B)$ for arbitrary subsets A and B of X. Consider $A = \{a\}$ and $B = \{b\}$ such that both a and b are parents of y. Then y is in the intersection of $cl(A)$ and $cl(B)$ but $A \cap B = \emptyset$, so $y \notin cl(A \cap B)$.

The closure of a set of individuals is a descent group.

Observation 9 (Descent Group Generator) *Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. The set $cl(A)$ is a descent group in G.*

Unsurprisingly, the closure function does not add anything to a descent group.

Observation 10 (Closure Generates Descent Group) *Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. If A is a descent group in G, then $cl(A) = A$.*

When applying the closure operator, it may the case that some elements can be removed without affecting the outcome. However, in the case where the removal of any element results in a smaller descent group, the parent set is said to be a *minimal generating set*.

Definition 9 (Minimal Generating Set) *Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. Say that A is a* minimal generating set *if and only if for every $A^{'} \subset A$, $cl(A^{'}) \subset cl(A)$.*

In applying the closure operator on a parent set, some parents are unnecessary. This occurs when a parent is the descendant of another parent. The lack of such unnecessary elements is equivalent to the notion of a minimal generating set.

Observation 11 (Witness To Minimality) *Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. If A is not a minimal generating set, then for some $a, b \in A$, $a \neq b$ and b is an ancestor of a.*

The progenitors of a descent group constitute a minimal generating set.

Observation 12 (Progenitor Set is Minimal) *Consider a genealogical network $G = (X, p)$. For every descent group D in G, $P(D)$ is a minimal generating set.*

For minimal generating sets, the progenitor function is the inverse of the closure operator.

Observation 13 (Minimal Generating Set and Progenitor Set) *Consider a genealogical network $G = (X, p)$. For every subset A of X, if A is a minimal generating set, then $P(cl(A)) = A$.*

For descent groups, the closure operator is the inverse of the progenitor function.

Observation 14 (Progenitors Generate Descent Group) *Consider a genealogical network $G = (X, p)$. For every descent group D in G, $cl(P(D)) = D$.*

A descent group is reconstructed by gathering all descendants of the progenitors.

Observation 15 (Progenitors Cover Descent Group Exactly) *Consider a genealogical network $G = (X, p)$. For every descent group D in G, $\bigcup_{x \in P(D)} cl(\{x\}) = D$.*

The progenitors of a descent group identify the descent group exactly. This greatly simplifies issues when reasoning about descent groups – only the progenitors need to be considered.

Observation 16 (Progenitors Identify Descent Group) *Consider a genealogical network $G = (X, p)$. For every pair of descent groups D_1 and D_2 in G, $D_1 = D_2$ if and only if $P(D_1) = P(D_2)$.*

Any non–empty descent group has at least one progenitor.

Observation 17 (Non-empty Descent Group Implies Progenitor) *Consider a genealogical network $G = (X, p)$. For every descent group D in G, $P(D) = \emptyset$ if and only if $D = \emptyset$.*

Since progenitors uniquely identify a descent group, the progenitors from the conglomeration of two descent groups can be determined exactly.

Observation 18 (Progenitors for Descent Group Union) *Consider a genealogical network $G = (X, p)$ and two descent groups D_1 and D_2 in G. Then,*

$$\begin{aligned} P(D_1 \cup D_2) \quad = \quad & (P(D_1) \setminus (D_2 \cap P(D_1))) \cup \\ & (P(D_2) \setminus (D_1 \cap P(D_2))) \cup \\ & (P(D_1) \cap P(D_2)) \end{aligned}$$

A useful corollary to this is the special case where a single new individual is incorporated into a descent group. This will be critical when construction descent groups by adding one individual at a time.

When a descent group is enlarged by adding an individual there are one of two possible outcomes. In the first case, when the individual is already in the descent group, the descent group remains unchanged. In the second case (as shown in Figure 2.1) the individual obliterates some progenitors of the descent group and the newcomer is a progenitor. The progenitors of the enlarged group and the original group are related as follows.

Corollary 1 (New Individual and Progenitors) *Let D be a descent group in a genealogical network $G = (X, p)$. For any $x \in X$,*

$$P(D \cup cl(\{x\})) = \begin{cases} P(D) & \text{if } x \in D \\ \{x\} \cup (P(D) \setminus (cl(\{x\} \cap P(D))) & \text{otherwise} \end{cases}$$

An often used concept in phylogenetics is the idea of a "most recent common ancestor". This is defined as any common ancestor that is *not* the ancestor of any other common ancestor.

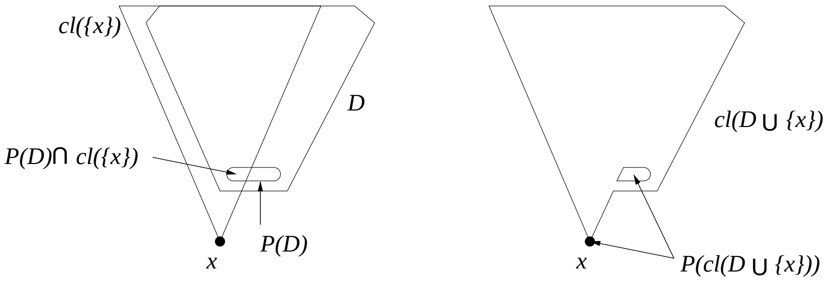

Figure 2.1: Adding a new individual to a descent group obliterates some old progenitors.

Definition 10 (Most Recent Common Ancestors) *Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. Say that $x \in X$ is a* most recent common ancestor *of A when*

1. *for every $a \in A$, x is an ancestor of a*
2. *for every $y \in X$, if $x \neq y$ and for every $a \in A$, y is an ancestor of a, then x is not an ancestor of y.*

Define the function $MRCA : 2^X \to 2^X$ by

$$MRCA(A) = \{x \in X \mid x \textit{ is a most recent common ancestor of } A\}$$

The concept of most recent common ancestor collapses to that of ancestor in some cases. One case occurs when considering two individuals where one is the ancestor of the other.

Observation 19 (An Ancestor Is Most Recent) *Consider a genealogical network $G = (X, p)$. For every $x_1, x_2 \in X$, if x_1 is an ancestor of x_2, then $MRCA(\{x_1, x_2\}) = \{x_1\}$.*

The presence of a common ancestor guarantees the presence of a most recent common ancestor.

Observation 20 (Some Common Ancestors are Most Recent) *Consider a genealogical network $G = (X, p)$. For every $S \subseteq X$, $y_1 \in X$, if for every $x \in S$, y_1 is an ancestor of x, then $MRCA(S) \neq \emptyset$.*

3 Monophyly, Polyphyly, and Paraphyly

The terms ‘monophyly’, ‘polyphyly’, and ‘paraphyly’ have been the subject of debate [8, 1, 5, 10]. Not only are the meanings of these terms disputed but the terms themselves. Some [10] have argued that Hennig’s concept of ‘monophyly’ be termed ‘holophyly’. However, the results contained in this paper will provide reasons for the retention of ‘monophyly’; ‘polyphyly’ can be shown to be a plural of ‘monophyly’.

Criteria used for distinguishing monophyly, polyphyly, and paraphyly include:

1. whether a group contains all descendants (descent closure)
2. the number of ancestors that give rise to a group

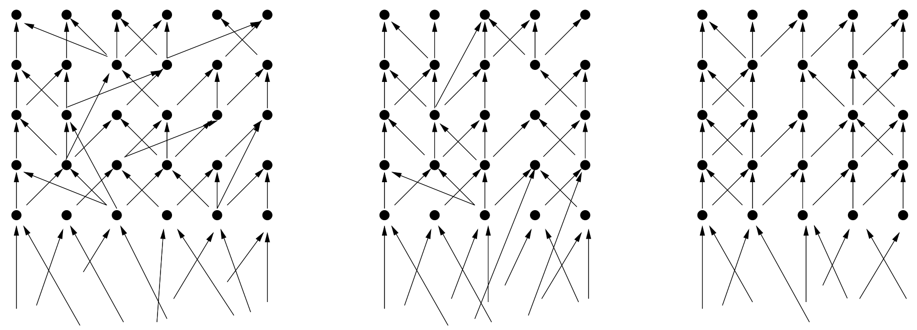

Figure 3.1: Figure from Hennig's *Phylogenetic Systematics* depicted 3 separate monophyletic lineages
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3. whether the most recent common ancestor of the group has a phenetically similar descendant in the group [1]

The meanings of these phylogenetic terms as defined by Farris [5] will be adopted. Farris argues that these terms should be conditional on a *specified phylogeny* and not on character traits – even though a phylogeny is ultimately inferred from characters.

Farris [5] distinguishes between monophyletic and polyphyletic groups based a closure criterion on most recent common ancestors. A monophyletic group contains all most recent common ancestors. A polyphyletic group does not. This intuition does not translate directly into a framework with a genealogical network which models individual organisms. The context of Farris' definition is that of species. Given a genealogical network, the only applicable definition of species that can be defined is one based on reproductive isolation.

Further intuition into how the concepts of monophyly and polyphyly can be defined is obtained from the work of Hennig [8]. In Figure 3.1, three monophyletic lineages are depicted. Collectively, they constitute a polyphyletic group. Notice that each monophyletic lineage has more than one progenitor; supporting our decision to allow a descent group to have multiple progenitors. Considering any two individuals in separate lineages, it can be seen that their most recent common ancestor is not present in either group. This intuitive idea can be used to formalise the notion of disconnection between two descent groups.

Definition 11 (Disconnected Descent Groups) *Let D_1 and D_2 be descent groups in a genealogical network $G = (X, p)$. D_1 and D_2 are* disconnected *when for every* $x_1 \in D_1, x_2 \in D_2$

1. $MRCA(\{x_1, x_2\}) \cap D_1 = \emptyset$*, and*

2. $MRCA(\{x_1, x_2\}) \cap D_2 = \emptyset$.

The following gives a much simpler way of interpreting disconnectedness in descent groups; they don't intersect. The result shows an equivalence between these two ideas; disconnectedness could have been defined by the non–intersection of descent groups.

Observation 21 (Disconnected Descent Groups Don't Intersect) *Consider a genealogical network $G = (X, p)$ and descent groups D_1, D_2 in G. D_1 and D_2 are disconnected if and only if $D_1 \cap D_2 = \emptyset$.*

Consider a situation where two descent groups are disconnected. Making one group smaller will not establish a connection.

Observation 22 (The Smaller the More Disconnected) *Consider a genealogical network $G = (X, p)$ and three descent groups D_0, D_1, D_2 in G. If D_1 and D_2 are disconnected and $D_0 \subseteq D_1$, then D_0 and D_2 are disconnected.*

A partition of a descent group into two disconnected sub–groups also generates a partition of the progenitors.

Observation 23 (Progenitors and Disconnected Descent Groups) *Consider a genealogical network $G = (X, p)$ and two descent groups D_1 and D_2 in G. If D_1 and D_2 are disconnected and $D = D_1 \cup D_2$, then*

1. $P(D) = P(D_1) \cup P(D_2)$
2. $P(D_1) \cap P(D_2) = \emptyset$

Disconnection is preserved under set union. If a descent group is disconnected with two other descent groups, it is also disconnected with the union of two other descent groups.

Observation 24 (Disconnection Preserved Under Union) *Consider a genealogical network $G = (X, p)$ and three descent groups D_0, D_1, D_2 in G. If D_0, D_1 and D_2 are pairwise disconnected, then $(D_0 \cup D_1)$ and D_2 are disconnected.*

This result generalises to the situation when there are a finite number of other descent groups.

Corollary 2 (Disconnection Preserved Under General Union) *Consider a genealogical network $G = (X, p)$ and descent groups $D_0, D_1, ..., D_k$ in G. If $D_0, D_1, \ldots D_k$ are pairwise disconnected, then $\bigcup_{0 \leq i < k} D_i$ and D_k are disconnected.*

A monophyletic group is a descent group that cannot be partitioned into two disconnected descent groups.

Definition 12 (Monophyletic Group) *A descent group D in a genealogical network $G = (X, p)$ is* monophyletic *if and only if there does not exist descent groups D_1 and D_2 such that,*

1. $D_1 \neq \emptyset$
2. $D_2 \neq \emptyset$
3. $D_1 \cup D_2 = D$
4. D_1 *and* D_2 *are disconnected*

A monophyletic descent group will be more simply referred to as a monophyletic group.

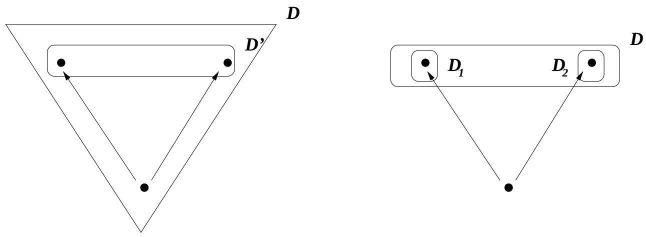

Figure 3.2: Examples showing that the monophyletic property is neither preserved under descent group subsets nor descent group union.

Descent groups consisting of a single individual are monophyletic by the above definition. The monophyletic condition requires a partition with two non-empty sub-descent groups; implying that a non–monophyletic group must contain at least two elements.

Farris does not comment on whether a polyphyletic group should be closed under descent (i.e., contain all descendants). The decision taken here is to define a polyphyletic group a descent group. This, as will subsequently be seen, will make a polyphyletic group the concretion of monophyletic groups. A descent group is *polyphyletic* exactly when it is not monophyletic.

Definition 13 (Polyphyletic Group) *A descent group D in a genealogical network $G = (X, p)$ is* polyphyletic *if and only if D is not monophyletic. The phrase* polyphyletic group *will be shorthand for* polyphyletic descent group.

The partitioning idea generates some trivial monophyletic groups – the empty set and individuals without descendants.

Observation 25 (Limiting Monophyletic Descent Groups) *Consider a genealogical network $G = (X, p)$. Then*

1. *the empty set $\emptyset$ is a monophyletic group in G*
2. *for every $x \in X$, if for every $y \in X$, $(x, y) \notin p$, then $\{x\}$ is a monophyletic group*

Figure 3.2 shows that the monophyletic condition is not preserved under the subset relation. On the left, D is clearly a monophyletic group but the subset D' of D is a polyphyletic descent group. The figure on the right shows that the union of two monophyletic groups is not necessarily monophyletic. Both D_1 and D_2 are monophyletic, but their union D is polyphyletic.

Figure 3.3 shows that the monophyletic condition is violated under intersection. In the diagram, both D_1 and D_2 are non–closed monophyletic groups. Their intersection D is a polyphyletic group.

Figure 3.4 shows two intersecting monophyletic descent groups D_1 and D_2. However, it is neither the case that $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$. Such a property would be favourable if we wish to structure monophyletic groups into a tree.

Since progenitors identify a descent group, the concept of polyphyly is expressible in terms of progenitors. The following recasts the polyphyletic definition in terms of progenitors and whether they have shared descendants. In a monophyletic group, any partition of the progenitors in connected be two progenitors which share a descendant; preventing any split into non–intersecting descent groups.

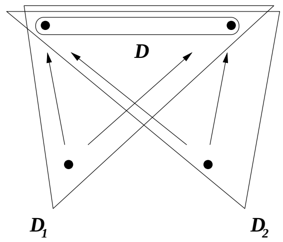

Figure 3.3: Example showing that the monophyletic property is not preserved under non–closed descent group intersection.

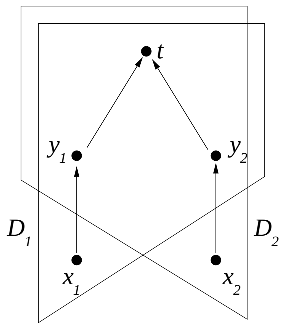

Figure 3.4: Example showing that the presence two intersecting monophyletic descent groups does not imply that one is a subset of the other.

Observation 26 (Monophyly and Progenitors) *Consider a descent group D in a genealogical network $G = (X, p)$. D is polyphyletic if and only if there exists a partition of the progenitors of D into two subsets X_1 and X_2 such that:*

1. $X_1 \cup X_2 = P(D)$
2. $X_1 \cap X_2 = \emptyset$
3. *for every $x \in D$, $x_1 \in X_1$ and $x_2 \in X_2$, it is not the case that both x_1 and x_2 are an ancestor of x*

By Hennig [8], the progenitors of a monophyletic group should constitute a single biological species. The only possible formulation of a biological species in a genealogical network is that of reproductive connectedness.

This above result gives a historical notion of reproductive connectedness. Consider some ancestral population A. A would be said to contain (at least) two separate reproductive populations if A can be partitioned into A_1 and A_2 such that the descendants or A_1 are distinct from those of A_2, i.e., $cl(A)$ is polyphyletic. Conversely, A can be thought to consist of a single reproductive population if every partition of A does not create two reproductively isolated populations; all partitions have a shared descendant.

A descent group with a single progenitor is always monophyletic.

Corollary 3 (Single Progenitor Generates Monophyletic Descent Group) *Consider a genealogical network $G = (X, p)$. For every $x \in X$, $cl(\{x\})$ is a monophyletic group.*

A notion of polyphyletic degree can be defined in terms of the number of pieces a descent group can be chopped into. Each piece is not arbitrary, it must be a descent group.

Definition 14 (Polyphyletic Degree) *A descent group D in a genealogical network $G = (X, p)$ is* polyphyletic of degree k *for $k \geq 1$ if and only if there exists descent groups $D_1, D_2, \ldots, D_k$ such that:*

1. $\bigcup_{1 \leq i \leq k} D_i = D$
2. $D_i \neq \emptyset$ *for every i, $1 \leq i \leq k$*
3. D_I *and* D_J *are disconnected for every I, J, $1 \leq I < J \leq k$*

The sequence of descent groups $D_1, \ldots, D_k$ is called a witness *to D being polyphyletic of degree k.*

As a measure, any non–empty descent group is polyphyletic of degree 1; a minimal value.

Observation 27 (Minimal Polyphyletic Degree) *Consider a descent group D in a genealogical network $G = (X, p)$. If $D \neq \emptyset$, then D is polyphyletic of degree 1.*

Combining two disconnected descent groups results in the addition of the polyphyletic degree of the constituents.

Observation 28 (Polyphyletic Degree and Descent Group Union) *Consider a genealogical network $G = (X, p)$ and descent groups D_1, D_2 in G. Suppose D_1 and D_2 are polyphyletic of degree k_1 and k_2 respectively. If $D_1 \cap D_2 = \emptyset$, then $D_1 \cup D_2$ is polyphyletic of degree $k_1 + k_2$.*

Each piece in the carving of a descent group contains at least one progenitor.

Observation 29 (Partition of Progenitors) *Consider a descent group D in a genealogical network $G = (X, p)$. If $D_1, \ldots, D_k$ is a witness to D being polyphyletic of degree k, then for every k, $1 \leq i \leq k$, there exists an $x_i \in P(D)$ such that $x_i \in D_i$.*

For polyphyletic degree to be a reasonable measure, a descent group should be polyphyletic for all degrees less than or equal to some boundary value k and not polyphyletic for degrees above k. The following shows that polyphyletic degree is implied for smaller values.

Observation 30 (Polyphyletic Degree Preserved Downwards) *Consider a descent group D in a genealogical network $G = (X, p)$. If D is polyphyletic of degree k, then for every l, $1 \leq l < k$, D is polyphyletic of degree l.*

The above result shows that polyphyletic degree forms a measure on a descent group. It establishes a border so that a descent group is polyphyletic to all degrees below the border but not polyphyletic to any degree above the border. This prompts the following definition of *polyphyletic of maximal degree*.

Definition 15 (Maximal Polyphyletic Degree) *Consider a descent group D in a genealogical network $G = (X, p)$. Say that D is* polyphyletic of maximal degree k *if and only if D is polyphyletic of degree k and D is not polyphyletic of degree $k + 1$.*

Non–empty monophyletic groups are exactly those with a maximal polyphyletic degree of 1.

Observation 31 (Monophyly and Maximal Polyphyletic Degree) *Consider a descent group D in a genealogical network $G = (X, p)$. D is polyphyletic of maximal degree 1 if and only if $D \neq \emptyset$ and D is monophyletic.*

Splits of a descent group into pieces that witness the maximal degree are almost exactly the same; they are permutations of each other.

Observation 32 (Maximal Polyphyletic Degree Witnesses are Permutations) *Consider a genealogical network $G = (X, p)$ and a descent group D in G. Suppose D is polyphyletic of maximal degree k. If $D_1, \ldots, D_k$ and $D_1^{'}, \ldots, D_k^{'}$ are witnesses to D being polyphyletic of degree k, then there exists a permutation ϕ from $\{D_1, \ldots D_k\}$ to $\{D_1^{'}, \ldots, D_k^{'}\}$ such that $D_i = \phi(D_i)$ for every i, $1 \leq i \leq k$.*

Any witness to the maximal polyphyletic degree of a descent group consists of monophyletic pieces.

Observation 33 (Polyphyletic Border) *Consider a genealogical network $G = (X, p)$ and a descent group $D \neq \emptyset$ in G which is polyphyletic of degree k. Suppose that $D_1, \ldots D_k$ is a witness to D being polyphyletic of degree k. D is* not *polyphyletic of degree $k + 1$ if and only if for every i, $1 \leq i \leq k$, D_i is monophyletic.*

The maximal polyphyletic degree of a descent group can be decreased by adding a new progenitor that glues together disconnected pieces. A piece of the polyphyletic group is glued to the new progenitor if it contains a descendant of the new progenitor. When a new progenitor connects to every piece of a polyphyletic group, the enlarged group is monophyletic.

Observation 34 (Preserving Monophyly) *Consider a genealogical network $G = (X, p)$ and a descent group D in G. For every $x \in X$, and witness $D_1, \ldots D_k$ to D being polyphyletic of maximal degree k, if $D_i \cap cl(\{x\}) \neq \emptyset$ for every i, $1 \leq i \leq k$, then $cl(\{x\} \cup \bigcup_{1 \leq i \leq k} D_i$ is a monophyletic group.*

The number of pieces connected by a new progenitor determines exactly the amount by which the maximal polyphyletic degree is reduced.

Observation 35 (Reducing Polyphyletic Degree) *Consider a genealogical network $G = (X, p)$ and a descent group D in G which is polyphyletic of maximal degree k. For every $x \in X$ and witness $D_1, \ldots, D_M, \ldots, D_k$ to D being polyphyletic of degree k, if*

1. *$D_i \cap cl(\{x\}) = \emptyset$ for every i, $1 \leq i < M$*
2. *$D_i \cap cl(\{x\}) \neq \emptyset$ for every i, $M \leq i \leq k$,*

then $cl(D \cup \{x\})$ is polyphyletic of maximal degree M.

In a Linnaean classification, higher ranks can be viewed as being more inclusive. For instance, living organisms which are classed as *Crustacea* and a strict subset of those that are classed as *Animalia*. In trying to marry the monophyletic criterion onto a Linnaean classification, it is necessary to establish the circumstances in which larger monophyletic groups can be derived from smaller ones. The following shows how a new progenitor can be grafted onto a monophyletic group.

Observation 36 (Enlarging Monophyletic Descent Groups) *Consider a genealogical network $G = (X, p)$ and a descent group D in G. For every $x \in X$, if D is a monophyletic group and $D \cap cl(\{x\}) \neq \emptyset$, then $cl(D \cup \{x\})$ is a monophyletic group.*

Generalising the previous result, the following shows that the combination of two overlapping monophyletic groups is monophyletic.

Observation 37 (Monophyletic Union) *Consider a genealogical network $G = (X, p)$ and two non–empty monophyletic groups D_1 and D_2 in G. $D_1 \cup D_2$ is monophyletic if and only if $D_1 \cap D_2 \neq \emptyset$.*

This result has tremendous repercussions for the creation of a monophyletic Linnaean hierarchy. Consider a particular family f in a Linnaean hierarchy that contains exactly two genera g_1 and g_2. The two genera are disjoint since any organism cannot be simultaneously assigned to *both* g_1 *and* g_2. Now, if the two genera are monophyletic groups, then f *cannot* be monophyletic since the combination of g_1 and g_2 in f are witness to f being polyphyletic. The following result shows that this reasoning holds even if f contains more than two families. It shows how a monophyletic group can be constructed from multiple monophyletic pieces. The construction is incremental and each successive piece must overlap with the current construction.

Observation 38 (Monophyly and General Union) *Consider a genealogical network $G = (X, p)$ and descent groups $D_1, \ldots D_k$ in G. Suppose for every i, $1 \leq i \leq k$, D_i is non–empty and monophyletic. Then, $\bigcup_{1 \leq i \leq k} D_i$ is monophyletic if and only if there exists a permutation ϕ of $\{1, \ldots, k\}$ such that for every j, $1 \leq j < k$, $\bigcup_{1 \leq i \leq j} D_{\phi(i)} \cap D_{\phi(j+1)} \neq \emptyset$.*

Paraphyletic sets are generated from excising pieces from a monophyletic group. The excised pieces are themselves monophyletic descent groups; collected together they constitute a descent group. A paraphyletic set is a monophyletic group with an excised sub–descent group.

Definition 16 (Paraphyly) *Consider a genealogical network $G = (X, p)$. A non–empty subset E of X is said to be a* paraphyletic group *in G if and only if there exists descent groups D and $D^{'}$ in G such that*

1. $D^{'} \subseteq D$
2. $D \cap D^{'} \neq \emptyset$
3. $E = D \setminus D^{'}$
4. *D is monophyletic*

When a pair of descent groups D and $D^{'}$ satisfy the above conditions for a paraphyletic group E, the pair $(D, D^{'})$ is called a witness *to paraphyletic group E. Moreover, D and $D^{'}$ are called the* inclusion group *and* exclusion group *respectively.*

The first two conditions make the definition sensible. Firstly, the individuals removed from a monophyletic group must come from the descent group. Secondly, there must actually be something removed; the removal process is not trivial. Quite clearly, a strongly paraphyletic group is also a paraphyletic group.

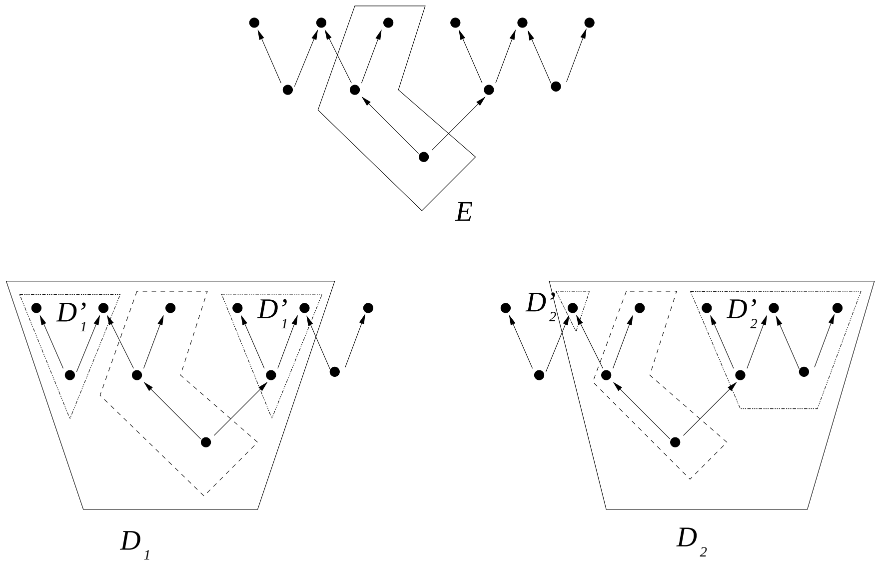

Figure 3.5: Example showing that paraphyletic group E can have multiple pairs of witnesses that define E.

Intuitively, the three different types of phylogenetic groups (monophyletic group, polyphyletic group, and paraphyletic group) should be mutually exclusive. The following result shows exactly this. Previously, descent groups were classed as either monophyletic or polyphyletic. Given the definition above, it can be shown that a paraphyletic group is not a descent group and thus distinct from monophyletic groups and polyphyletic groups.

Observation 39 (Paraphyletic Group) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Then E is* not *a descent group.*

In general, a paraphyletic group can have several witnesses. An example can be seen in Figure 3.5.

Witnesses to a paraphyletic group satisfy a number constraints. Any ancestor in the inclusion group of an element in the paraphyletic group cannot be in the exclusion group. The progenitors of an inclusion group satisfy several constraints. Firstly, at least one progenitor in the inclusion group is in the paraphyletic group. For all witnesses, those progenitors of the inclusion group that are in the paraphyletic group are the same. Smaller inclusion groups imply smaller exclusion groups.

Observation 40 (Paraphyletic Witness Constraints) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Suppose that $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E. Then,*

1. *for every $x \in E, y \in D_1$, if y is an ancestor of x, then $y \in E$*
2. *there exists an $x \in P(D_1)$ such that $x \in E$*
3. *$P(D_1) \cap E = P(D_2) \cap E$*
4. *for every $x_1 \in D_1, x_2 \in D_2$, if $x_2 \notin D_1$ and x_2 is an ancestor of x_1, then $x_1 \in D_1^{'}$.*

5. *if* $D_1 \subseteq D_2$*, then* $D_1^{'} \subseteq D_2^{'}$

The witnesses to a paraphyletic group are gathered together as follows:

Definition 17 (Paraphyletic Witness Set) *Consider a genealogical network* $G = (X, p)$ *and a paraphyletic group* E *in* G*. The* witness set *of* E*, denoted by* $[E]$*, is defined by*

$$[E] = \{(D, D^{'}) \mid (D, D^{'}) \textit{ is a witness to } E\}$$

A paraphyletic group is contained in the inclusion group of all witnesses. Similarly, no element of a paraphyletic group is contained in any exclusion group.

Observation 41 (Witness Set Constraints) *Consider a genealogical network* $G = (X, p)$ *and a paraphyletic group* E *in* G*. Then,*

1. $E \subseteq \bigcap_{(D,D')\in[E]} D$
2. $E \cap \bigcup_{(D,D')\in[E]} D^{'} = \emptyset$

Given that there are multiple witnesses to a paraphyletic group in general, it is interesting to consider whether any witness is in some way *canonical*. One idea is that smaller witnesses are more canonical. Here smallness is defined in terms of the descent groups that are used to construct the paraphyletic; both the monophyletic group that determines what might be in the paraphyletic group and the excised descent group that determines what is not in the paraphyletic group.

Definition 18 (Smaller Paraphyletic Witness) *Consider a genealogical network* $G = (X, p)$ *and a paraphyletic group* E *in* G*. Let* $(D_1, D_1^{'})$ *and* $(D_2, D_2^{'})$ *be witnesses to* E*.*

Say that $(D_1, D_1^{'})$ *is* smaller than $(D_2, D_2^{'})$ *(or conversely say that* $(D_2, D_2^{'})$ *is* larger than $(D_1, D_1^{'})$*) exactly when* $D_1 \subseteq D_2$ *and* $D_1^{'} \subseteq D_2^{'}$*.*

Say that $(D_1, D_1^{'})$ *is* minimal *exactly when for every witness* (D, D') *to* E*, if* $(D, D^{'})$ *is smaller than* $(D_1, D_1^{'})$*, then* $(D_1, D_1^{'})$ *is smaller than* $(D, D^{'})$*.*

Given witnesses to a paraphyletic group, larger and (sometimes) smaller witnesses can be constructed.

Observation 42 (Paraphyletic Witness Structure) *Consider a genealogical network* $G = (X, p)$ *and a paraphyletic group* E *in* G*. Suppose that* $(D_1, D_1^{'})$ *and* $(D_2, D_2^{'})$ *are witnesses to* E*. Then,*

1. $(D_1 \cup D_2, D_1^{'} \cup D_2^{'}) \in [E]$ *and is larger than* $(D_1, D_1^{'})$
2. $(D_1, D_1^{'} \cup D_2^{'}) \in [E]$ *and is larger than* $(D_1, D_1^{'})$
3. *if* $D_1 \cap D_2$ *is monophyletic, then* $(D_1 \cap D_2, D_1^{'}) \in [E]$ *and is smaller than* $(D_1, D_1^{'})$
4. *if* $D_1 \cap D_2$ *is monophyletic, then* $(D_1 \cap D_2, D_1^{'} \cup D_2^{'}) \in [E]$

Another way to determine a canonical witness to a paraphyletic group is to measure the complexity of the excised descent group in a witness to the paraphyletic group. The smaller the degree to which the excised descent group is polyphyletic, the more canonical the witness.

Definition 19 (Paraphyletic Degree) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Say that E is* paraphyletic of degree k *if there exists a witness $(D, D^{'})$ to E such that $D^{'}$ is polyphyletic of degree k.*

Moreover, say that E is paraphyletic of maximal degree k *exactly when E is paraphyletic of degree k and* not *paraphyletic of degree $k + 1$.*

As might be expected, several properties relating to polyphyletic degree carry over to the notion of paraphyletic degree. The smallest paraphyletic degree is 1.

Observation 43 (Smallest Paraphyletic Degree) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Then E is paraphyletic of degree 1.*

Paraphyletic degree is persevered downwards. If a paraphyletic group is paraphyletic to degree k, it is also paraphyletic to all degrees less than k.

Observation 44 (Lower Paraphyletic Degrees Preserved) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. If E is paraphyletic of degree k, then for every l, $1 \leq l < k$, E is paraphyletic of degree l.*

Unfortunately, the notions of paraphyletic degree and smallness do not coincide when trying to isolate a canonical witness to a paraphyletic set. To show this, a few preliminary definitions are necessary.

The previously defined concept of "progenitor" can be extended to cover sets of individuals in general. A progenitor of a set is any individual that does not have any parent in that set. This extended definition does not alter the notion of "progenitor" for descent groups as the definition is exactly the same as applied to descent groups.

Definition 20 (General Progenitor) *Consider a genealogical network $G = (X, p)$ and a subset Y of X. A* progenitor *of Y is an individual $x \in X$ such that:*

1. $x \in Y$
2. *for every $z \in X$ such that $(z, x) \in p$, $z \notin Y$*

Moreover, the progenitor set *of Y, $P(Y)$ is defined as*

$$P(Y) = \{y \in Y \mid y \text{ is a progenitor of } Y\}$$

A weaker notion of paraphyletic witness can be defined by dropping the monophyletic restriction on the first component of a witness.

Definition 21 (A Weak Paraphyletic Witness) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. A* weak witness *to E is a pair of descent groups $(D, D^{'})$ such that:*

1. $D^{'} \subseteq D$
2. $D \cap D^{'} \neq \emptyset$
3. $E = D \setminus D^{'}$

Based on smallness, a canonical weak paraphyletic witness can be defined for a paraphyletic group.

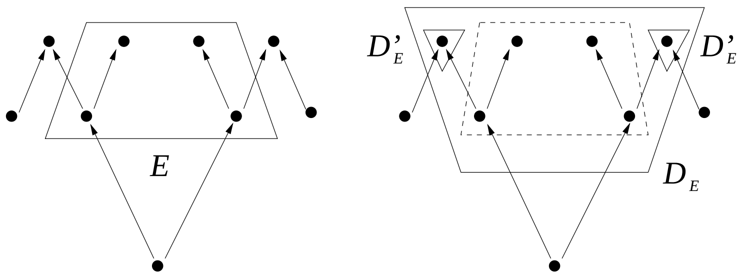

Figure 3.6: Example showing that the canonical witness to a paraphyletic set E is not necessarily a strong witness.

Definition 22 (Canonical Weak Paraphyletic Witness) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Define the* weak canonical witness *of E as the pair $(D_E, D_E^{'})$ where*

$$\begin{aligned} D_E &= cl(P(E)) \\ D_E^{'} &= D_E \setminus E \end{aligned}$$

The progenitors of the canonical weak witness are exactly those of the paraphyletic group.

Observation 45 (Progenitors of the Canonical Weak Witness) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. If $(D_E, D_E^{'})$ is the canonical weak witness of E, then $P(D_E) = P(E)$.*

The canonical weak witness is contained in all witnesses to a paraphyletic group.

Observation 46 (Canonical Weak Witnesses Contained in Witnesses) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Let $(D_E, D_E^{'})$ be the canonical weak witness of E and $(D, D^{'})$ an arbitrary witness to E. Then $D_E \subseteq D$ and $D_E^{'} \subseteq D^{'}$.*

As the name insinuates, the canonical weak witness is a weak witness.

Observation 47 (Canonical Weak Witness A Weak Witness) *Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Then the weak canonical witness of E, $(D_E, D_E^{'})$ is a weak witness of E.*

The weak canonical witness to a paraphyletic group is not necessarily a witness. An example can be seen in Figure 3.6. The reason for this is that $D_E = cl(P(E))$ is possible polyphyletic.

Smaller witnesses do not necessarily have smaller paraphyletic degrees. Moreover, the canonical weak witness can have a paraphyletic degree which is *larger* than other witnesses. Both of these statements are exemplified by Figure 3.7. The paraphyletic group E is shown with two witnesses $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$. Note that $(D_1, D_1^{'})$ is the canonical weak witness which, in this case, is also a witness that is paraphyletic of degree 2. The larger witness $(D_2, D_2^{'})$ is paraphyletic of degree 1.

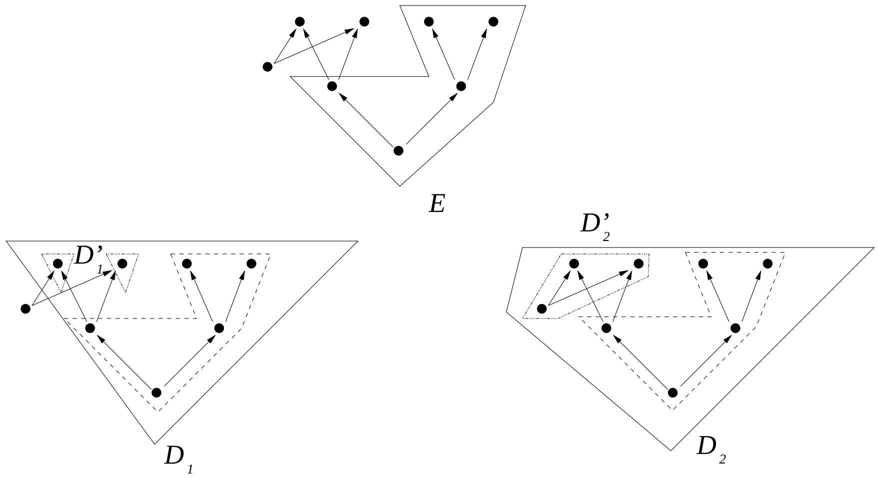

Figure 3.7: Example showing witnesses to a paraphyletic group such that (i)a smaller witness can have a larger paraphyletic degree, and (ii) the canonical weak witness has a paraphyletic degree that is larger than other witnesses.

4 From Graph to Tree

Accepting the correctness of a graph model of genetic inheritance and the usefulness of phylogenetic trees, the question of how to transform a genealogical network to a phylogenetic tree becomes crucial. Since a genealogical network is more general than a phylogenetic tree, a tree should represent a summary of a graph; a tree will necessarily correspond on a number of graphs.

Published phylogenetic trees are typically leaf–labelled; internal nodes are tacit. There are good reasons, as we shall see, for doing this.

A *terminal* is an individual at the 'bottom' of the genealogical network – an individual with no descendants;

Definition 23 (Terminal and Terminal Group) *Consider a genealogical network $G = (X, p)$. A* terminal *in G is an individual $x \in X$ that has no descendants, i.e., for every $y \in X$, $(x, y) \notin p$.*

A terminal group *T in G is a subset X such that for every $t \in T$, t is a terminal in G.*

From an arbitrary collection of individuals Y, elements that are terminals are filtered and collected as the terminal set of Y.

Definition 24 (Terminal Set) *Consider a genealogical network $G = (X, p)$ and Y a subset of X. Define the* terminal set *of Y, $Term(Y)$, to be $Term(Y) = \{t \in Y \mid t \text{ is a terminal in } G\}$.*

The presence of a progenitor exactly ensures the presence of a terminal in a descent group.

Observation 48 (Presence of Terminals) *Consider a genealogical network $G = (X, p)$. For every $A \subseteq X$, $Term(cl(A)) = \emptyset$ if and only if $A = \emptyset$.*

More terminals are generated higher up a genealogical network. The terminal descendants of an individual are always fewer than those of an ancestor of that individual.

Observation 49 (Terminal Sets and Ancestors) *Consider a genealogical network $G = (X, p)$ and two individuals x and y in X. If x is an ancestor of y, then*

$$Term(cl(\{y\})) \subseteq Term(cl(\{x\}))$$

The terminal sets of descent groups preserve set theoretic structure. The $Term$ function distributes over the set union and set intersection of descent groups. Moreover, the subset relation between descent groups is preserved.

Observation 50 (Terminal Set Properties) *Consider a genealogical network $G = (X, p)$ and descent groups D_1 and D_2 in G. Then,*

1. $Term(D_1 \cup D_2) = Term(D_1) \cup Term(D_2)$
2. $Term(D_1 \cap D_2) = Term(D_1) \cap Term(D_2)$
3. *if $D_1 \subseteq D_2$, then $Term(D_1) \cap Term(D_2)$*

Many descent groups may have the same terminals. Descent groups can be grouped into equivalence classes based on their terminal set.

Definition 25 (Descent Groups for a Terminal Group) *Consider a genealogical network $G = (X, p)$ and a terminal group T in G. Define the* class of descent groups for *T, $[T]$, to be $[T] = \{D \mid D$ is a descent group in G and $Term(D) = T\}$.*

For any terminal group T, there is at least one descent group with terminals that match T exactly. This is, of course, T itself. T constitutes a descent group since no member of T has any descendants.

Observation 51 (Class of Descent Groups Non-empty) *Consider a genealogical network $G = (X, p)$ and a terminal group T in G. The class of descent groups for T is non-empty because $T \in [T]$.*

A terminal group is as polyphyletic as a group can be. Since each terminal has no descendants, there is no connection between it and other parts of the descent group. For a terminal group, each terminal represents a disconnected monophyletic group.

Observation 52 (Non-trivial Terminal Group Polyphyletic) *Consider a genealogical network $G = (X, p)$ and a terminal group T in G. If $T \neq \emptyset$, T is a polyphyletic group of maximal degree $\mid T \mid$.*

Different descent groups in $[T]$ have different maximal polyphyletic degrees. Larger descent groups have a smaller maximal polyphyletic degree.

Observation 53 (Subsets and Polyphyletic Degree in $[T]$) *Consider a genealogical network $G = (X, p)$, a terminal group T in G, and two descent groups D_1 and $D_2 \in [T]$. If $D_1 \subseteq D_2$ and D_2 is polyphyletic of degree m, then D_1 is polyphyletic of degree m.*

Since larger descent groups in $[T]$ tend to have smaller polyphyletic degrees, it is of interest to see how a descent group in $[T]$ can be enlarged. This enlarged group should also be in $[T]$. When a new individual is added, this happens exactly when the terminal descendants of the new individual are contained in T.

Observation 54 (Adding an Individual and Remaining in $[T]$) *Consider a genealogical network $G = (X, p)$, a terminal group T in G, and a descent group $D \in [T]$. For every $x \in X$, $Term(cl(\{x\})) \subseteq T$ if and only if $cl(D \cup \{x\}) \in [T]$.*

Results can also be established about the polyphyletic degree of every descent group in $[T]$. When every terminal in T is derived from a separate lineage, the descent groups in $[T]$ have the largest possible polyphyletic degree. Two terminals are from separate lineages if their most recent common ancestor has descendants outside of T.

Observation 55 (Separate Lineages) *Consider a genealogical network $G = (X, p)$ and a terminal group T in G. If*

1. *for every $t_1, t_2 \in T$, if $t_1 \neq t_2$ and for every $y \in MRCA(\{t_1, t_2\}), Term(cl(\{y\})) \not\subseteq T$, and*
2. *$D \in [T]$,*

then D is polyphyletic of degree $\mid T \mid$.

Exact conditions can be established for determining the presence of a monophyletic group in $[T]$. This occurs when every partition of T is bridged by some ancestor. This ancestor has descendants in each component of the partition and it's terminal descendants are all within T.

Corollary 4 (Monophyletic Descent Group in $[T]$) *Consider a genealogical network $G = (X, p)$ and a terminal group T in G. All descent groups in $[T]$ are polyphyletic if and only if for some T_1 and T_2*

1. *$T_1 \neq \emptyset$,*
2. *$T_2 \neq \emptyset$,*
3. *$T_1 \cup T_2 = T$,*
4. *$T_1 \cap T_2 = \emptyset$, and*
5. *for every $t_1 \in T_1, t_2 \in T_2$, and for every $y \in MRCA(\{t_1, t_2\}), Term(cl(\{y\})) \not\subseteq T$.*

There is a descent group in $[T]$ which larger than all other descent groups in $[T]$. Moreover, if $[T]$ contains a monophyletic group, then this largest element of $[T]$ is also monophyletic. This canonical element of $[T]$ will be called the *ancestor set* of T.

Observation 56 (Maximal Monophyletic Descent Group in $[T]$) *Consider a genealogical network $G = (X, p)$ and a terminal group T in G. If $[T]$ contains a monophyletic group, then the set*

$$D_{max} = \{x \in X \mid Term(cl(\{x\})) \subseteq T\}$$

is a monophyletic group such that for every monophyletic group D in $[T]$, $D \subseteq D_{max}$.

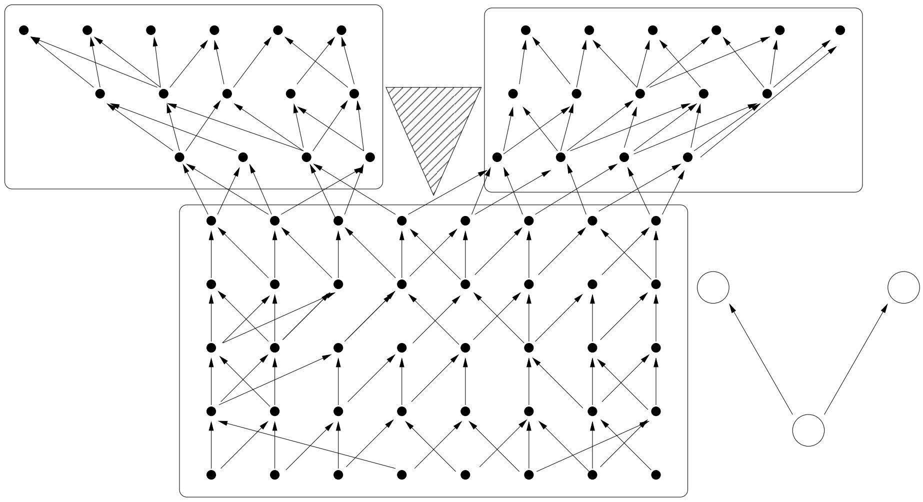

Figure 4.1: A figure adapted from Willi Hennig's *Phylogenetic Systematics* showing how species might be constructed from a genealogical network.

Hennig provides some graphical intuition into the meaning of species as shown in Figure 4.1. The exact placement of the border seems to be dependant on an alignment of generations. However, such alignments are not always present; as shown in Figure 1.1.

Slightly contrasting, the approach here is to construct a maximal descent group based on terminal groups. Figure 4.2 shows two maximal descent groups based on two terminal groups T_1 and T_2.

The ancestor set of a terminal group contains any ancestor of some terminal in the terminal group.

Definition 26 (Ancestor Set) *Consider a genealogical network $G = (X, p)$ and a terminal group T in G. Define the* ancestor set *$A(T)$ of T to be*

$$A(T) = \{x \in X \mid Term(cl(\{x\})) \subseteq T\}$$

The function that generates ancestor sets satisfies a number of structural properties across terminal groups. Structure is preserved exactly across subset and set intersection. Also, two terminal sets have common elements exactly when their corresponding ancestor sets have common elements. However, structure is not exactly preserved across set union. This result is absolutely vital. As will be subsequently shown, a monophyletic Linnaean classification is totally monotypic. The result will provide a means for weakening the monophyletic definition that will allow, if so desired, a Linnaean classification to be monophyletic.

Observation 57 (Ancestor Set Relations) *Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G.*

1. $T_1 \subseteq T_2$ *if and only if* $A(T_1) \subseteq A(T_2)$
2. $T_1 \cap T_2 = \emptyset$ *if and only if* $A(T_1) \cap A(T_2) = \emptyset$
3. $A(T_1 \cap T_2) = A(T_1) \cap A(T_2)$

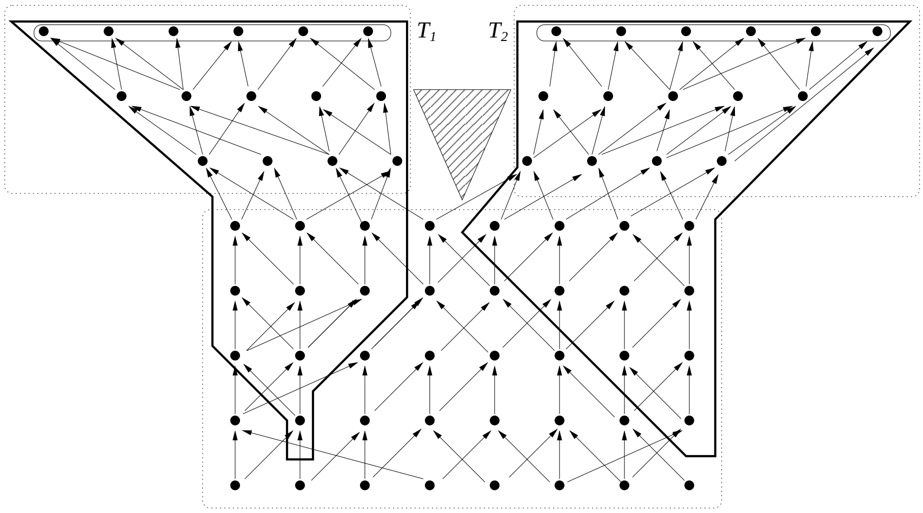

Figure 4.2: A figure adapted from Willi Hennig's *Phylogenetic Systematics*. The dotted lines indicate Hennig's species while the heavier lines demarcate maximal descent groups based on terminal groups T_1 and T_2.

4. $A(T_1 \cup T_2) \supseteq A(T_1) \cup A(T_2)$

It is not generally true that union of two ancestor sets is a subset of the ancestor set of the union. Figure 4.2 show an example. The ancestor set of $T_1 \cup T_2$ would consist of more elements that the ancestor set of T_1 and T_2 combined.

The ancestor set satisfies a number of properties that make it a canonical element of $[T]$. Firstly, it is an element of $[T]$. It is also a descent group with a maximal polyphyletic degree that is the smallest of all descent groups in $[T]$. Finally, it is the largest descent group in $[T]$.

Observation 58 (Ancestor Set Properties) *Consider a genealogical network $G = (X, p)$ and a terminal set T in G.*

1. $A(T) \in [T]$
2. $A(T)$ *is a descent group*
3. *for every descent group $D \in [T]$, $D \subseteq A(T)$*
4. *if $A(T)$ is polyphyletic of maximal degree k, then for every descent group $D \in [T]$, D is polyphyletic of degree k.*

Consider how the function that generates ancestor sets does not distribute over set union. One half of the result does follow, i.e., $A(T_1 \cup T_2) \supseteq A(T_1) \cup A(T_2)$. However, it is possible that $A(T_1 \cup T_2)$ contains elements not present in either $A(T_1)$ or $A(T_2)$. A graphic illustration of this can be seen in Figure 4.2. In the figure, heavy lines demarcate $A(T_1)$ and $A(T_2)$. $A(T_1 \cup T_2)$ contains all individual in the picture. This 'gap' between $A(T_1)$ and $A(T_2)$ contain individuals that all satisfy a very specific property, viz., they only have terminal descendants in $T_1 \cup T_2$ and at least one descendant in T_1 and one in T_2. These individuals bridge T_1 and T_2.

Observation 59 (Union Gap) *Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G. For every $x \in A(T_1 \cup T_2)$, if $x \notin A(T_1) \cup A(T_2)$, then $Term(cl(\{x\})) \cap T_1 \neq \emptyset$ and $Term(cl(\{x\})) \cap T_2 \neq \emptyset$.*

A single terminal has a monophyletic ancestor set.

Observation 60 (Single Term Generates Monophyletic Descent Group) *Consider a genealogical network $G = (X, p)$ and a terminal $t \in X$. Then, $A(\{t\})$ is monophyletic.*

A Linnaean hierarchy aggregates taxa at a rank into a taxon at the next higher rank. If a Linnaean classification only applies to terminals, it is of interest to see how the monophyly property of ancestor sets behaves when terminal sets are aggregated. Basically, the monophyly property is *lost* when there are no bridging elements between two terminal groups, viz., the function that generates ancestor sets distributes over set union.

Observation 61 (Union is Monophyletic) *Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G. Suppose $A(T_1)$ is monophyletic and $A(T_2)$ is monophyletic.*

1. *if $A(T_1 \cup T_2)$ is polyphyletic then $A(T_1 \cup T_2) = A(T_1) \cup A(T_2)$.*
2. *if $T_1 \neq \emptyset$, $T_2 \neq \emptyset$, $T_1 \cap T_2 = \emptyset$, and $A(T_1 \cup T_2) \subseteq A(T_1) \cup A(T_2)$, then $A(T_1 \cup T_2)$ is polyphyletic.*

The monophyly property of ancestor sets is preserved when larger terminal groups are aggregated.

Observation 62 (Ancestor Set Monophyletic Monotonicity) *Consider a genealogical network $G = (X, p)$ and terminal groups T_1, T_2, and T_3 in G. If*

1. *$T_i \neq \emptyset$, for $i = 1, 2,$ and 3*
2. *$T_1 \cap T_i = \emptyset$, for $i = 2$ and 3*
3. *$T_2 \subseteq T_3$*
4. *$A(T_i)$ is monophyletic for $i = 1, 2,$ and 3*
5. *$A(T_1 \cup T_2)$ is monophyletic,*

then $A(T_1 \cup T_3)$ is monophyletic.

5 A Linnaean Classification

Since the work of Linnaeus in the mid 18^{th} century, biological organisms have been classified by placing them in a balanced taxonomic tree. In *Origin of Species*, Darwin [3] argued that biological classification should, and to some extent does, reflect the recency of common ancestry. Under the Linnaean taxonomy, this means that organisms which share a recent common ancestor are grouped closely while organisms which are distantly related are far apart.

Can a Linnaean classification be entirely monophyletic? The idea that a Linnaean classification cannot be entirely monophyletic has a long history [12, 4]. It has been proposed [7, 2] that the reason for this is that a Linnaean system cannot accommodate both extant and fossil species.

The complex methods used by taxonomists to build a classification (in the sense of [10]) are not addressed here – only the structure of a Linnaean classification. This structure is a hierarchical series of ranks that contain classes where "lower classes nested within higher classes". A set theoretic formulation of this structure was provided by Gregg [6]. This inciteful study also presented the logical properties of such a classification scheme. Through a series of theorems, Gregg shows that the consequences of the definition match the intuition that surrounds a classification system. The definition consists of two parallel systems. The taxonomic system consists of taxonomic groups arranged in a hierarchy or tree. Mirroring this is the category system which places taxonomic categories in a hierarchy. For this paper, it is the taxonomic system which is of greater importance. The aim is to determine whether it is possible to create a taxonomic system that consists entirely of monophyletic groups. Even if possible, what properties would such a system possess.

Gregg (and [9]) first defines taxa and *subsequently* attaches a rank to a taxon. By doing so, Gregg encounters the problem of monotypic taxa; a taxon can only be placed at a single rank. Rather than altering the definitions, arguments were presented against monotypic taxa. The approach taken here is to alter the definitions. In fact, the problem is easily resolved by changing the order of construction: first define ranks and *then* define the relationships between ranks. Taxa of different ranks can then refer to exactly the same set of organisms. A full definition of a Linnaean hierarchy will not be presented here; only the mandatory ranks. Interleaving of non–mandatory ranks is not problematic but is beyond the scope of this paper. As will become apparent, the Linnaean hierarchy presented here will have no problem with monotypic categories.

A mandatory Linnaean rank assigns every organisms to exactly one taxon. Moreover, it is assumed that a taxon must contain at least one organism. In what follows, the adjective 'mandatory' will be left implicit for conciseness; this paper will only consider the mandatory ranks.

Definition 27 (Linnaean Rank) *Consider a finite set Y. A* Linnaean rank $\mathbf{R}$ over Y, *is a set of sets $\{G_1, \ldots, G_k\}$ that partitions Y, i.e.,*

1. $G_i \neq \emptyset$ *for every* i, $1 \leq i \leq k$
2. $G_i \cap G_j = \emptyset$ *for every* i, j, $1 \leq i < j \leq k$
3. $\bigcup_{1 \leq i \leq k} G_i = Y$

A higher Linnaean rank forms a coarser partition of organisms. Viewed in another way, a higher rank aggregates taxa from lower ranks. Thus a taxon at a higher rank contains at least one taxon at a lower rank. Also, taxa at a higher rank cannot split a taxon at a lower rank; a taxon at a higher rank either contains, entirely, a taxon at a lower rank or does not intersect it – all or nothing.

Definition 28 (Linnaean Rank Hierarchy) *Consider a finite set Y and two Linnaean ranks $\mathbf{R}_1$ and $\mathbf{R}_2$ over Y. Say that $\mathbf{R}_1$ is* above $\mathbf{R}_2$ *(or alternately $\mathbf{R}_2$ is* below $\mathbf{R}_1$*) when*

1. *for every $G \in \mathbf{R}_1$, there exists a $G^{'} \in \mathbf{R}_2$ such that $G^{'} \subseteq G$.*
2. *for every $G \in \mathbf{R}_1$ and $G^{'} \in \mathbf{R}_2$, either $G \cap G^{'} = \emptyset$ or $G^{'} \subseteq G$.*

From the definitions of a Linnaean rank and the notion of higher, some intuitive results immediately follow. Firstly, a taxon at a higher ranks contains at least one taxon at a lower rank. Also, a taxon at a higher rank is an aggregation of a number of taxa at a lower rank. Finally, a taxon at a higher rank contains exactly those taxa at a lower rank that intersect with it.

Observation 63 (Content of Higher Ranks) *Consider a finite set Y and two Linnaean ranks $\mathbf{R}_1$ and $\mathbf{R}_2$ over Y. Suppose that $\mathbf{R}_1$ is* above *$\mathbf{R}_2$. Then,*

1. *for every $G' \in \mathbf{R}_2$, there exists a unique $G \in \mathbf{R}_1$ such that $G^{'} \subseteq G$*
2. *for every $G \in \mathbf{R}_1$, $G = \bigcup_{G' \in \mathbf{R}} G^{'}$ for some non-empty $\mathbf{R} \subseteq \mathbf{R}_2$*
3. *for every $G \in \mathbf{R}_1$, $G = \bigcup_{G' \in \mathbf{R}_2 \text{ and } G' \subseteq G} G^{'}$.*

The concept of 'above' is a transitive relation. This means that a sequence of Linnaean ranks, each consecutive pair satisfying the 'above' relation, forms a linear chain.

Observation 64 ('Above' Transitive) *Consider a finite set Y and Linnaean ranks $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ over Y. If $\mathbf{R}_1$ is* above *$\mathbf{R}_2$ and $\mathbf{R}_2$ is above $\mathbf{R}_3$, then $\mathbf{R}_1$ is above $\mathbf{R}_3$.*

A Linnaean classification is simply a sequence of Linnaean ranks where each rank is above the next rank in the sequence.

Definition 29 (Extensive Linnaean Classification) *Consider a finite set Y. An* extensive Linnaean classification L over Y *is a sequence of Linnaean ranks $(\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y such that $\mathbf{R}_i$ is above $\mathbf{R}_{i+1}$ for every i, $1 \leq i \leq k-1$.*

Since the publication of Darwin's *On the Origin of Species* [3], many biologists have argued over how biological classification should reflect evolution. One particular interpretation would insist on all biological taxa being monophyletic.

Definition 30 (Strong Monophyletic Extensive Linnaean Classification) *Consider a genealogical network $G = (X, p)$ and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. Say that L is* strongly monophyletic *when for every i, $1 \leq i \leq n$ and every $G \in \mathbf{R}_i$, G is a monophyletic group.*

A slight weakening of the "strong monophyletic" condition on an extensive Linnaean classification will allow the groups in the lowest rank to be an arbitrary partition. This is worth considering since many have expressed the opinion that "species" need not be monophyletic.

Definition 31 (Weak Monophyletic Extensive Linnaean Classification) *Consider a genealogical network $G = (X, p)$ and an extensive Linnaean classification $L = (\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n)$ over X. Say that L is* weak monophyletic *when for every i, $1 \leq i \leq n-1$ and every $G \in \mathbf{R}_i$, G is a monophyletic group.*

In a monophyletic Linnaean classifications, the removal of the lowest rank leaves a strong monophyletic Linnaean classification.

Observation 65 (Slicing the Last Rank) *Consider a genealogical network $G = (X, p)$ and an extensive Linnaean classification $L = (\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n)$ over X. Let $L^{'} = (\mathbf{R}_1, \ldots, \mathbf{R}_{n-1})$. Then*

1. $L^{'}$ *is an extensive Linnaean classification over* X

2. *if* L *is strongly monophyletic, then* $L^{'}$ *is strongly monophyletic*

3. *if* L *is weakly monophyletic, then* $L^{'}$ *is strongly monophyletic*

The Linnaean hierarchy becomes less useful with more monotypic taxa. The following shows that in a strongly monophyletic extensive Linnaean classification, *all* ranks are exactly the same. The only interest in such a classification is in the highest rank.

Theorem 1 (Monophyletic Linnaean Incompatibility) *Consider a genealogical network* $G = (X, p)$ *and a Linnaean classification* $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ *over* X*. If* L *is strongly monophyletic, then*

1. $\mathbf{R}_i = \mathbf{R}_n$ *for every* i*,* $1 \leq i \leq n$

2. X *is polyphyletic of maximal degree* $| \mathbf{R}_n |$

Consider the case that X is monophyletic - where life has a single origin. This makes a strongly monophyletic extensive Linnaean classification even more degenerate. In this case, every rank is monotypic.

Corollary 5 (Monophyletic Linnaean Incompatibility) *Consider a genealogical network* $G = (X, p)$ *and a Linnaean classification* $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ *over* X*. If* L *is strongly monophyletic and* X *is a monophyletic group in* G*, then* $\mathbf{R}_i = \{X\}$ *for every* i*,* $1 \leq i \leq n$*.*

It may be argued that the lowest Linnaean rank (containing species say) need not be monophyletic. However, this only delays the collapse of the classification by 1 rank. In an extensive Linnaean classification, removing the lowest rank still leaves an extensive Linnaean classification.

Corollary 6 (Monophyletic Linnaean Incompatibility) *Consider a genealogical network* $G = (X, p)$ *and a Linnaean classification* $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ *over* X*. If* L *is weakly monophyletic, then*

1. $\mathbf{R}_i = \mathbf{R}_{n-1}$ *for every* i*,* $1 \leq i \leq n - 1$

2. X *is polyphyletic of maximal degree* $| \mathbf{R}_{n-1} |$

A monophyletic origin to life once again leaves all ranks, except the bottom rank, monotypic.

Corollary 7 (Monophyletic Linnaean Incompatibility) *Consider a genealogical network* $G = (X, p)$ *and a Linnaean classification* $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ *over* X*. If* L *is weakly monophyletic and* X *is a monophyletic group in* G*, then* $\mathbf{R}_i = \{X\}$ *for every* i*,* $1 \leq i \leq n - 1$*.*

An example will help to provide insight into the reasons for these results. Consider two monophyletic families f_1 and f_2. The dictates of an extensive Linnaean classification require o to contain a number of families. Suppose o contain exactly f_1 and f_2. However, the only way that o can be monophyletic is if o contains something to connect f_1 and f_2; otherwise f_1 and f_2 will testify to o being polyphyletic. The only way out of this impasse is if o contains a single family. This will apply to all ranks.

A Linnaean rank by Definition 27 is a container for individuals. Is this reasonable? Is a Linnaean family a container for genera or a container for all individuals that belong to species as part of genera in the family? Either answer to this question makes no real difference to a Linnaean classification and only the mildest mark on the above results. Suppose that individuals are only considered at the lowest rank and that a rank is only a container for elements of the rank below it. This more *intensive* perspective on a Linnaean classification can be defined as follows:

Definition 32 (Intensive Linnaean Classification) *Consider a finite set Y. An* intensive Linnaean classification $L^{'}$ over Y *is a sequence of Linnaean ranks $(\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ such that*

1. *$\mathbf{R}^{'}_n$ is a Linnaean rank over Y*
2. *$\mathbf{R}^{'}_i$ is a Linnaean rank over $\mathbf{R}^{'}_{i+1}$, for every i, $1 \leq i < n$*

The tremendous similarity between an intensive and an extensive Linnaean classification is shown by the fact that they are clearly inter–translatable. In the first part of the translation, the intention of an extensive taxon is defined. At the lowest rank, the intention and extension are the same. For an extensive group at a higher rank, the intensive counterpart is simply the collection of intentions of those extensive taxa at the next lower rank.

Definition 33 (Intention of an Extensive Group) *Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Consider an arbitrary group $G_i \in \mathbf{R}_i$. Define the* intention $I(G_i)$ *of G_i recursively as follows:*

$$I(G_i) = \begin{cases} G_i & \text{if } i = n \\ \{I(G_{i+1}) \mid G_{i+1} \subseteq G_i \text{ and } G_{i+1} \in \mathbf{R}_{i+1}\} & \text{otherwise} \end{cases}$$

An intensive Linnaean classification can be defined from an extensive one by translating each extensive taxon and placing the translation at the same rank in the intensive classification.

Definition 34 (Extensive to Intensive) *Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Define the* intensive counterpart *of L to be $\psi(L) = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ where*

$$\mathbf{R}^{'}_i = \{I(G_i) \mid G_i \in \mathbf{R}_i\}$$

The phrase *intensive counterpart* is applicable to two contexts: (i) the counterpart to an extensive group, and (ii) the counterpart to an extensive classification. This ambiguity is deliberate since the phrase has essentially the same meaning; only the context is different.

The properties of the function that generates intensive groups matches exactly that define an intensive Linnaean classification.

Observation 66 (Properties of I) *Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Then,*

1. *$I(G_i) \neq \emptyset$ for every i, $1 \leq i \leq n$ and $G_i \in \mathbf{R}_i$*

2. $I(G) \cap I(G^{'}) = \emptyset$ *or* $I(G) = I(G^{'})$ *for every* i, $1 \leq i \leq n$ *and* $G, G^{'} \in \mathbf{R}_i$

3. $\bigcup_{G \in \mathbf{R}_{i-1}} I(G) = \{I(G_i) \mid G_i \in \mathbf{R}_i\}$, *for every* i, $1 < i \leq n$.

Overall, the translation is sensible. An extensive classification is translated into an intensive classification.

Observation 67 (ψ Makes An Intensive Classification) *Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Then the intensive counterpart of L is an intensive Linnaean classification over Y.*

Translating in the opposite direction, the organisms contained in an intensive taxon can be gathered recursively. As noted earlier, at the lowest rank, intention and extension are the same. For an intensive taxon at a higher rank, the extension function merely gathers the intentions of each taxon contained in the intensive taxon.

Definition 35 (Extension of an Intensive Group) *Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ over Y. Consider an arbitrary group $G^{'}_I \in \mathbf{R}^{'}_I$. Define the* extension $E(G^{'}_I)$ *of $G^{'}_I$ recursively.*

$$E(G^{'}_I) = \begin{cases} G^{'}_I & \textit{if } I = n \\ \bigcup_{G^{'}_{I+1} \in G^{'}_I} E(G^{'}_{I+1}) & \textit{otherwise} \end{cases}$$

An intensive Linnaean classification has an extensive counterpart. Each intensive taxon is translated separately and placed at the same corresponding rank.

Definition 36 (Intensive To Extensive) *Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ over Y. Define the* extensive counterpart *of $L^{'}$ to be $\psi^{'}(L^{'}) = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ where*

1. $\mathbf{R}_n = \mathbf{R}^{'}_n$

2. $\mathbf{R}_i = \{E(G^{'}) \mid G^{'} \in \mathbf{R}^{'}_i\}$

The extension function satisfies a number of properties that show the translation satisfies the properties of an extensive Linnaean classification.

Observation 68 (Extension Properties) *Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ over Y.*

1. $E(G^{'}) \neq \emptyset$ *for every* i, $1 \leq i \leq n$ *and* $G^{'} \in \mathbf{R}^{'}_i$

2. $E(G^{'}) = E(H^{'})$ *if and only if* $G^{'} = H^{'}$ *for every* i, $1 \leq i \leq n$ *and* $G^{'}, H^{'} \in \mathbf{R}^{'}_i$

3. *either* $E(G^{'}) = E(H^{'})$ *or* $E(G^{'}) \cap E(H^{'}) = \emptyset$ *for every* i, $1 \leq i \leq n$ *and* $G^{'}, H^{'} \in \mathbf{R}^{'}_i$

4. $\bigcup_{G^{'} \in \mathbf{R}^{'}_i} E(G^{'}) = Y$ *for every* i, $1 \leq i \leq n$

The translation from an intensive Linnaean classification is sensible. The structure resulting from the translation is an extensive Linnaean classification.

Observation 69 ($\psi^{'}$ Makes an Extensive Classification) *Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ over Y. The* extensive counterpart *of $L^{'}$, $\psi^{'}(L^{'}) = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ is an extensive Linnaean classification over Y.*

To show that the two translation functions are inverses of each other, two structural results need to be shown. Firstly, extension function composed with the intention function results in no change to an extensive taxon.

Observation 70 (Extension Preserved By I) *Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Let $\phi(L) = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ be the intensive counterpart of L. For an arbitrary i, $1 \leq i \leq n$, $G_i \in \mathbf{R}_i$,*

$$E(I(G_i)) = G_i$$

The intention function composed with the extension function on an intensive taxon results in no change to the intensive taxon.

Observation 71 (Intension Preserved By E) *Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ over Y. Let $\phi^{'}(L^{'}) = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ be the extensive counterpart of $L^{'}$. For an arbitrary i, $1 \leq i \leq n$, $G^{'}_i \in \mathbf{R}^{'}_i$, $I(E(G^{'}_i)) = G^{'}_i$.*

An extensive Linnaean classification is unchanged after performing two rounds of translation.

Corollary 8 (Extensive Circle) *Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Then $\psi^{'}(\psi(L)) = L$*

An intensive Linnaean classification is unchanged after performing two rounds of translation.

Corollary 9 (Intensive Circle) *Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}^{'}_1, \ldots, \mathbf{R}^{'}_n)$ over Y. Then $\psi(\psi^{'}(L^{'})) = L^{'}$*

It may be argued that the genealogical network is an invalid model because systematic biology work with phylogenetic trees. However, this is not the case because a phylogenetic tree is a *specialisation* of a genealogical network. A phylogenetic tree is simply a genealogical network with the *added* assumption that each point in a genealogical network has *at most* one parent. Given the monotonic meta–logic used, this means that all results proven about a genealogical network apply to a phylogenetic tree.

Published phylogenetic tree (and the computer algorithms used to generate them) feature only labelled leaves. Only rarely are internal nodes to the tree labelled; no common ancestors are labelled. This suggests that a weaker notion of monophyly may be adopted. For instance, rather than applying a classification scheme to all individuals in a genealogical network, apply it to only the end points (i.e. terminals) in the genealogical network. Also, the concept of monophyly can be adapted to a set of terminals. The following definition facilitates this.

Definition 37 (Terminal Group Basis) *Consider a genealogical network $G = (X, p)$ and a terminal set T in G. Say that T is* allowably monophyletic *exactly when $A(T)$ is monophyletic.*

Observation 56 shows how the concept of *allowably monophyletic* is a weakened version of monophyletic.

Technically, the problem of trying to create a monophyletic Linnaean classification is the fact that if D_1 and D_2 are non–empty monophyletic groups, then $D_1 \cup D_2$ is polyphyletic. However, this does not apply to the notion of *allowably monophyletic*. If T_1 and T_2 are non–empty allowably monophyletic, then it is *not* necessarily the case that $T_1 \cup T_2$ is not allowably monophyletic. In fact, Corollary 4 gives the exact conditions under which $T_1 \cup T_2$ is allowably monophyletic.

Definition 38 (Allowably Monophyletic Linnaean Classification) *Consider a genealogical network $G = (X, p)$ and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over $Term(X)$. Say that L is* allowably monophyletic *when for every i, $1 \leq i \leq n$ and every $G \in \mathbf{R}_i$, G is allowably monophyletic.*

Observation 72 (Allowable Conglomerations) *Consider a genealogical network $G = (X, p)$ and terminal groups $T_1, \ldots, T_k$. Suppose that*

1. *$T_i \neq \emptyset$ for every i, $1 \leq i \leq k$*
2. *T_i is allowably monophyletic, for every i, $1 \leq i \leq k$*
3. *$T_i \cap T_j = \emptyset$ for every i, j, $1 \leq i < j \leq k$.*

If for every i, $1 \leq i < k$, $T_i \cup T_{i+1}$ is allowably monophyletic, then $\bigcup_{1 \leq i \leq k} T_i$ is allowably monophyletic.

6 Consequences of a Genealogical Tree

Ever since Darwin [3], phylogenetic trees have been depicted as branching trees. It is therefore important that our definitions are investigated with respect to a tree model. Fortunately, a genealogical network is a more general structure than a phylogenetic tree. A network can be converted into a tree by placing an extra assumption: that an individual has at most one parent.

Definition 39 (Genealogical Tree) *Consider a genealogical network $G = (X, p)$. Say that G is a* genealogical tree *when for every $x, y, z \in X$, if $(y, x) \in p$ and $(z, x) \in p$, then $y = z$.*

In a genealogical tree, all ancestors of an individual occur in a single line of descent.

Observation 73 (Single Ancestor Path) *Consider a genealogical tree $G = (X, p)$ and an individual $x \in X$. For any $x_1, x_2 \in X$, if both x_1 and x_2 are ancestors of x, then x_1 is an ancestor of x_2 or x_2 is an ancestor of x_1.*

Progenitors in 'separate' parts of a tree generate distinct descendants.

Observation 74 (Disjoint Descent Groups in a Tree) *Consider a genealogical tree $G = (X, p)$ and subsets X_1 and X_2 of X. Suppose that X_1 and X_2 are minimal generating sets. Moreover, suppose that for every $x_1 \in X_1$ and $x_2 \in X_2$ that x_1 is not an ancestor of x_2 and x_2 is not an ancestor of x_1. Then, $cl(X_1) \cap cl(X_2) = \emptyset$.*

Having defined a genealogical tree, the definitions for monophyly, paraphyly, and polyphyly can now be examined in the context in which they were originally devised. A monophyletic group can be shown to consist exactly of one ancestral individual and all descendants of that individual.

Observation 75 (Monophyletic Group) *Consider a genealogical tree $G = (X, p)$ and a descent group D in G. D is monophyletic if and only if $\mid P(D) \mid = 1$.*

A descent group that is polyphyletic of maximal degree k consists of exactly k distinct progenitors and all descendants of those progenitors.

Observation 76 (Polyphyletic Group) *Consider a genealogical tree $G = (X, p)$ and a descent group D in G. D is polyphyletic of maximal degree k if and only if $\mid P(D) \mid = k$.*

In a genealogical network, a paraphyletic group potentially has many witnesses. In a tree, there is only one witness to a paraphyletic group.

Observation 77 (Paraphyletic Set in a Family Tree) *Consider a genealogical tree $G = (X, p)$ and a paraphyletic group E in G. If $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E, then $D_1 = D_2$ and $D_1^{'} = D_2^{'}$.*

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A Proofs for Section 2

Observation 1 [Ancestor Relation A Partial Order] Consider a genealogical network $G = (X, p)$. The ancestor relation is a partial order over X, i.e.,

1. for every $x \in X$, x is an ancestor of x
2. for every $x, y \in X$, if x is an ancestor of y and y is an ancestor of x, then $x = y$
3. for every $x, y, z \in X$, if x is an ancestor of y and y is an ancestor of z, then x is an ancestor of z

Proof
Consider a genealogical network $G = (X, p)$. For reflexivity, consider an arbitrary $x \in X$. The singleton sequence x trivially satisfies the first three conditions for x to be an ancestor of x. The fourth condition is trivially satisfied since the length of the sequence is 1. Thus, for every $x \in X$, x is an ancestor of x.

For antisymmetry, consider arbitrary elements $x, y \in X$. Suppose that a is an ancestor of b and b is an ancestor of a. By Definition 2, there exists sequences $x_1, \ldots, x_n$ and $x_n, \ldots, x_{n+m-1}$ such that $x_1 = a$ and $x_n = b$ and $x_{n+m} = a$ with $n, m \geq 1$. Since p is acyclic, it must be the case that $n = m = 1$. Therefore, since $x_n = b$ and $x_{n+m-1} = a$, it obtains that $a = b$. Thus, for every $x, y \in X$, if x is an ancestor of y and y is an ancestor of x, then $x = y$.

For transitivity, consider arbitrary elements $x, y, z \in X$. Suppose that x is an ancestor of y and y is an ancestor of z. By Definition 2, there exists sequences $x_1, x_2, \ldots x_n$ and $x_n, \ldots x_{n+m-1}$ such that:

1. $n, m \geq 1$
2. $x_1 = x$, $x_n = y$, and $x_{n+m-1} = z$
3. for every i, $1 \leq i \leq x_{n+m-2}$, $(x_i, x_{i+1}) \in P$

Therefore, the sequence $x_1, x_2, \ldots, x_n, \ldots x_{n+m-1}$ is a witness to the fact that x is an ancestor of z. Therefore, for every $x, y, z \in X$, if x is an ancestor of y and y is an ancestor of z, then x is an ancestor of z Thus, the ancestor relation is transitive and a partial order. $\square$

Observation 2 [Limiting Descent Groups] Given a genealogical network $G = (X, p)$,

1. $\emptyset$ is a descent group
2. X is a descent group

Proof
Consider a genealogical network $G = (X, p)$ and the empty set $\emptyset$. Clearly, $\emptyset \subseteq X$. Moreover, the constraint that descendants of a descent group elements must be included in the descent group is trivially satisfied since $\emptyset$ contains no possible ancestors.

Consider the set X. Clearly, $X \subseteq X$. Moreover, X contains all descendants since descendants can only come from X. Thus X is a descent group. $\square$

Observation 3 [Descent Group Closure Under Intersection and Union] Consider a genealogical network $G = (X, p)$. If D_1 and D_2 be two descent groups in G, then

1. $D_1 \cap D_2$ is a descent group in G.

2. $D_1 \cup D_2$ is a descent group in G.

Proof
Let D_1, D_2 be descent groups in genealogical network $G = (X, p)$.

1. Clearly $D_1 \cap D_2 \subseteq X$ since D_1 and D_2 are already subsets of X. Consider an $x \in X$ and $a \in D_1 \cap D_2$. Suppose that a is an ancestor of x. Since $a \in D_1$, by the definition of a descent group (Definition 3), $x \in D_1$. Similarly, $x \in D_2$. Thus, $x \in D_1 \cap D_2$. Therefore, $D_1 \cap D_2$ is a descent group in G.

2. Let D_1, D_2 be descent groups in genealogical network $G = (X, p)$. Clearly $D_1 \cup D_2 \subseteq X$ since D_1 and D_2 are already subsets of X. Consider an $x \in X$ and $a \in D_1 \cup D_2$. Suppose that a is an ancestor of x. If $a \in D_1$, then $x \in D_1$ (and $x \in D_1 \cup D_2$) since D_1 is a descent group. If $a \notin D_1$, then $a \in D_2$. Since D_2 is a descent group, $x \in D_2$ and consequently $x \in D_1 \cup D_2$. Thus, $D_1 \cup D_2$ is a descent group in G.

□

Observation 4 [Descent Group Partial Ordering] Consider a genealogical network $G = (X, p)$. Let $descent(G)$ denote the set of all descent groups in G. Then, the binary relation $\leq$ is a partial order over $descent(G)$.

Proof
Consider a genealogical network $G = (X, p)$. Let $descent(G)$ denote the set of all descent groups in G. Since the $\subseteq$ relation is reflexive, transitive, and antisymmetric it follows that $\leq$ is a partial order over $descent(G)$. □

Observation 5 [Descent Groups Form a Bounded Lattice] Consider a genealogical network $G = (X, p)$. Let $descent(G)$ denote the set of all descent groups in G. Then $(descent(G), \leq)$ is a bounded lattice.

Proof
Consider a genealogical network $G = (X, p)$. Let $descent(G)$ denote the set of all descent groups in G.

Consider $(descent(G), \leq)$. Let D be any element of $descent(G)$. By Observation 2, $\emptyset$ and X are elements of $descent(G)$. Moreover, $\emptyset \subseteq D \subseteq X$. Thus $\emptyset \leq D \leq X$ and $descent(G)$ is bounded below by $\emptyset$ and above by X.

Let D_1 and D_2 be arbitrary elements of $descent(G)$. By Definition 5, the join of D_1 and D_2 is defined by $D_1 \cup D_2$. By Observation 3, $D_1 \cup D_2 \in descent(G)$. Also, $D_1 \subseteq D_1 \cup D_2$ and $D_2 \subseteq D_1 \cup D_2$. Thus, $D_1 \leq D_1 \cup D_2$ and $D_2 \leq D_1 \cup D_2$. Let D be any descent group in $descent(G)$. Suppose that $D_1 \leq D$ and $D_2 \leq D$. By Definition 4, $D_1 \subseteq D$ and $D_2 \subseteq D$. Thus $D_1 \cup D_2 \subseteq D$, i.e., the $D_1 \cup D_2 \leq D$. Therefore, $D_1 \cup D_2$ is the least upper bound of D_1 and D_2. Similarly, by Definition 5, the meet of D_1 and D_2 is defined by $D_1 \cap D_2$. By Observation 3, $D_1 \cap D_2 \in descent(G)$. Also, $D_1 \supseteq D_1 \cap D_2$ and $D_2 \supseteq D_1 \cap D_2$. Thus, $D_1 \cap D_2 \leq D_1$ and $D_1 \cap D_2 \leq D_2$. Let D be any descent group in $descent(G)$. Suppose that $D \leq D_1$ and $D \leq D_2$. By Definition 4, $D \subseteq D_1$ and $D \subseteq D_2$. Thus $D \subseteq D_1 \cap D_2$, i.e., the $D \leq D_1 \cap D_2$. Therefore,

$D_1 \cap D_2$ is the greatest lower bound of D_1 and D_2. Therefore, $(descent(G), \leq)$ is a bounded lattice. $\square$

Observation 6 [Progenitor as Founding Ancestor] Let $P(D)$ be the progenitor set for a descent group D in genealogical network $G = (X, p)$. For every $x \in D$, either

1. $x \in P(D)$, or
2. there exists a $y \in P(D)$ such that $y \neq x$ and y is an ancestor of x

Proof
Consider the progenitor set $P(D)$ of a descent group D in a genealogical network $G = (X, p)$. Let x be an arbitrary member of D. Further suppose that for every progenitor $y \in P(D)$, y is not an ancestor of x. Construct a sequence $x_1, x_2, \ldots, x_n$ ($n \geq 1$) such that:

1. $x = x_n$
2. $(x_i, x_{i+1}) \in p$
3. there does not exist a $z \in D$ such that $(z, x_1) \in p$

Such a chain can be constructed. If x is a progenitor, then the sequence consists of simply x itself. Otherwise, there exists a $z \in D$, such that $(z, x) \in p$. The process repeats for z. It cannot continue indefinitely since X is finite and p is acyclic. If $n = 1$, then x is progenitor. Otherwise, x_1 is a progenitor and x_1 is an ancestor of x. Moreover x_1 is distinct from x since G is acyclic. $\square$

Observation 7 [cl a Closure Operator] Consider a genealogical network $G = (X, p)$. The function $cl : 2^X \to 2^X$ is a closure operator, i.e.,

1. for every $A \subseteq X$, $A \subseteq cl(A)$
2. for every $A, B \subseteq X$, if $A \subseteq B$, then $cl(A) \subseteq cl(B)$
3. for every $A \subseteq X$, $cl(cl(A)) = cl(A)$

Proof
Consider a genealogical network $G = (X, p)$. To show that cl is monotonic, note that for any individual $x \in X$, x is an ancestor of x. By the definition of cl (Definition 8), if $A \subseteq X$ and $a \in A$, $a \in cl(A)$. Thus, for any $A \subseteq X$, $A \subseteq cl(A)$.

Consider two subsets A, B of X. Suppose that $A \subseteq B$. Let $x \in cl(A)$. Then for some $a \in A$, a is an ancestor of x. Since $A \subseteq B$, $a \in B$. Thus $x \in cl(B)$ and therefore $cl(A) \subseteq cl(B)$.

Consider a subset A of X. By monotonicity, $cl(A) \subseteq cl(cl(A))$. Let $x \in cl(cl(A))$. Then for some $a \in cl(A)$, a is an ancestor of x. Since $a \in cl(A)$, for some $b \in A$, b is an ancestor of a. By transitivity of the ancestor relationship, b is an ancestor of x and $b \in A$. Thus $x \in cl(A)$ and $cl(cl(A)) \subseteq cl(A)$. Therefore $cl(cl(A)) = cl(A)$.

Hence, cl is a closure operator. $\square$

Observation 8 [Distributivity of cl] Consider a genealogical network $G = (X, p)$. For every $A, B \subseteq X$

1. $cl(A \cup B) = cl(A) \cup cl(B)$
2. $cl(A \cap B) \subseteq cl(A) \cap cl(B)$

Proof
Consider a genealogical network $G = (X, p)$. Let $A, B \subseteq X$ be arbitrary.

Let $x \in cl(A \cup B)$ be arbitrary. Then for some $a \in A \cup B$, a is an ancestor of x. If $a \in A$, then $x \in cl(A)$ and $x \in cl(A) \cup cl(B)$. If $a \notin A$, then $a \in B$. Moreover, $x \in cl(B)$ and $x \in cl(A) \cup cl(B)$.

Let $x \in cl(A) \cup cl(B)$ be arbitrary. If $x \in cl(A)$, then by the monotonicity of cl (Observation 7), $x \in cl(A \cup B)$. If $x \notin cl(A)$, then $x \in cl(B)$. Once again, by the monotonicity of cl (Observation 7), $x \in cl(A \cup B)$.

Let $x \in cl(A \cap B)$ be arbitrary. Then for some $a \in A \cap B$, a is an ancestor of x. This implies that $a \in A$ and $a \in B$. Hence $x \in cl(A)$ and $x \in cl(B)$ which implies that $x \in cl(A) \cap cl(B)$. $\square$

Observation 9 [Descent Group Generator] Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. The set $cl(A)$ is a descent group in G.
Proof
Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. By the definition of cl (Definition 8), clearly $cl(A) \subseteq X$. Let $x \in X$ and suppose there exists a $a \in cl(A)$ such that a is an ancestor of x. Since $a \in cl(A)$, for some $b \in A$, b is an ancestor of a. Thus b is an ancestor of x. By definition of $cl(A)$ (Definition 8), $x \in cl(A)$. Therefore $cl(A)$ is a descent group. $\square$

Observation 10 [Closure Generates Descent Group] Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. If A is a descent group in G, then $cl(A) = A$.

Proof
Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$ and suppose that A is a descent group. By Observation 7, cl is a closure operator. This implies that $A \subseteq cl(A)$. Consider an arbitrary $x \in cl(A)$. Then for some $a \in A$, a is an ancestor of x. Since A is a descent group, $x \in A$. Thus $A \supseteq cl(A)$ and $A = cl(A)$. $\square$

Observation 11 [Witness To Minimality] Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$. If A is not a minimal generating set, then for some $a, b \in A$, $a \neq b$ and b is an ancestor of a.

Proof
Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$ and suppose A is *not* a minimal generating set. Then for some $A^{'} \subset A$, $cl(A^{'}) \supseteq cl(A)$. Since cl is a closure operator (Observation 7), $cl(A^{'}) \subseteq cl(A)$. Thus, $cl(A^{'}) = cl(A)$. Let a be any witness to the fact that $A^{'} \subset A$, i.e., $a \in A$ and $a \notin A^{'}$. Since $a \in cl(A)$ and $cl(A) = cl(A^{'})$, for some $b \in A^{'}$, b is an ancestor of a. Clearly $a \neq b$ since $a \notin A^{'}$ and $b \in A^{'}$. Since $A^{'} \subset A$, $b \in A$. Thus, for some $a, b \in A$, $a \neq b$ and b is an ancestor of a. $\square$

Observation 12 [Progenitor Set is Minimal] Consider a genealogical network $G = (X, p)$. For every descent group D in G, $P(D)$ is a minimal generating set.

Proof
Consider a genealogical network $G = (X, p)$. Consider an arbitrary descent group D in G. Let a, b be arbitrary elements in $P(D)$. Suppose that $a \neq b$. Further, suppose that b is an ancestor of a. Consider a path from b to a, $x_1, x_2, \ldots, x_n$ where $x_1 = b$ and $x_n = a$. Since $a \neq b$, $n \geq 2$. Consider (x_{n-1}, x_n) (this pair exists since $n \geq 2$). By the definition of a path, this pair is an element of p. Also, $x_{n-1} \in D$ since D is a descent group. By the definition of a progenitor (Definition 6), $x_{n-1} \notin D$. This is a contradiction. Thus b is not an ancestor of a. By the converse of Observation 11, this implies that $P(D)$ is a minimal generating set. □

Observation 13 [Minimal Generating Set and Progenitor Set] Consider a genealogical network $G = (X, p)$. For every subset A of X, if A is a minimal generating set, then $P(cl(A)) = A$.

Proof
Consider a genealogical network $G = (X, p)$. Let A be an arbitrary subset of X. Suppose that A is a minimal generating set.

Let x be an arbitrary element of $P(cl(A))$. By the definition of a progenitor (Definition 6), $x \in cl(A)$. Since $x \in cl(A)$, by Definition 8, for some $a \in A$, a is an ancestor of x. Moreover, since cl is a closure operator by Observation 7, $a \in cl(A)$. For a proof by contradiction, suppose that $a \neq x$. Consider any path from a to x, $x_1, x_2, \ldots, x_n$ where $x_1 = a$, $x_n = x$, and $n \geq 2$. Consider the pair (x_{n-1}, x_n). This pair exists since $n \geq 2$. By the definition of a path the pair $(x_{n-1}, x_n) \in p$. Since $cl(A)$ is a descent group $x_{n-1} \in cl(A)$. By the definition of a progenitor (Definition 6) for x $(= x_n)$, $x_{n-1} \notin cl(A)$. This is a contradiction. Thus $a = x$ and $x \in A$. Hence, $P(cl(A)) \subseteq A$.

Consider an arbitrary $a \in A$. Since cl is a closure operator (by Observation 7), $a \in cl(A)$. For a proof by contradiction, suppose that $a \notin P(cl(A))$. If a is not a progenitor in $cl(A)$, there exists a progenitor (by Observation 6) $y \in P(cl(A))$ such that y is an ancestor of a. Clearly $a \neq y$ since $a \notin P(cl(A))$ and $y \in P(cl(A))$. There are two cases to consider: $y \in A$ or $y \notin A$. In the case that $y \in A$, since $a \neq y$, by the converse of Observation 11, A is not a minimal generating set with a and y as witnesses. This contradiction will give that $a \in P(cl(A))$. Now, consider the second case, $y \notin A$. Since $y \in cl(A)$, there exists a $z \in A$ such that z is an ancestor of y. Consider the concatenation of a path from z to y and a path from y to a, $x_1, x_2, \ldots, x_n$ where $x_1 = z$, $x_n = a$. Since $z \in A$ and $y \notin a$, the length of a path from z to y must be at least 2. Thus $n \geq 2$. Since G is acyclic, it follows that $z \neq a$. By the converse of Observation 11, A is not a minimal generating set with a and z as witnesses. This contradiction will give that $a \in P(cl(A))$. Hence, $A \subseteq P(cl(A))$.

Therefore, $P(cl(A)) = A$. □

Observation 14 [Progenitors Generate Descent Group] Consider a genealogical network $G = (X, p)$. For every descent group D in G, $cl(P(D)) = D$.

Proof
Consider a genealogical network $G = (X, p)$. Let D be an arbitrary descent group in

G.

First, show that $D \subseteq cl(P(D))$. Let x be an arbitrary element in D. By Observation 6, either $x \in P(D)$ or there exists a $y \in P(D)$ such that y is an ancestor of x. In the first case $x \in P(D)$, since cl is a closure operator (Observation 7), $x \in cl(P(D))$. In the second case, there exists a $y \in P(D)$ such that y is an ancestor of x. By the definition of cl (Definition 8), $x \in cl(P(D))$. Thus $D \subseteq cl(P(D))$.

Next, show that $cl(P(D)) \subseteq D$. Let x be an arbitrary element of $cl(P(D))$. By the definition of cl (Definition 8), for some $y \in P(D)$, y is an ancestor of x. Certainly by the definition of a progenitor (Definition 6), $y \in D$. Moreover, since D is a descent group and y is an ancestor of x, $x \in D$. Thus $cl(P(D)) \subseteq D$.

Therefore, $cl(P(D)) = D$. $\square$

Observation 15 [Progenitors Cover Descent Group Exactly] Consider a genealogical network $G = (X, p)$. For every descent group D in G, $\bigcup_{x \in P(D)} cl(\{x\}) = D$.

Proof
Consider a genealogical network $G = (X, p)$. Let D be an arbitrary descent group in G.

Firstly, let y be an arbitrary element in D. Then for some progenitor z in $P(D)$, z is an ancestor of y; in the case that y is a progenitor of D, $y = z$. In either case, $y \in cl(\{z\})$ since z is an ancestor of y. Thus $y \in \bigcup_{x \in P(D)} cl(\{x\})$ and $D \subseteq \bigcup_{x \in P(D)} cl(\{x\})$.

Let y be an arbitrary element of $\bigcup_{x \in P(D)} cl(\{x\})$. Then for some $z \in P(D)$, $y \in cl(\{z\})$. Since z is a progenitor of D, $z \in D$ and z is an ancestor of y. Thus, since D is a descent group, $y \in D$. Hence $\bigcup_{x \in P(D)} cl(\{x\}) \subseteq D$.

Therefore, $\bigcup_{x \in P(D)} cl(\{x\}) = D$. $\square$

Observation 16 [Progenitors Identify Descent Group] Consider a genealogical network $G = (X, p)$. For every pair of descent groups D_1 and D_2 in G, $D_1 = D_2$ if and only if $P(D_1) = P(D_2)$.

Proof
Consider a genealogical network $G = (X, p)$. Let D_1 and D_2 be arbitrary descent groups in G.

Suppose $D_1 = D_2$. Clearly, by the definition of the progenitor set (Definition 6) P is a function from 2^X to 2^X. Thus, if $D_1 = D_2$, then $P(D_1) = P(D_2)$.

Suppose that $P(D_1) = P(D_2)$. Since the progenitor set contains all founding ancestors, by Observation 14, $D_1 = cl(P(D_1))$ and $D_2 = cl(P(D_2))$. Since $P(D_1) = P(D_2)$, we obtain $D_1 = cl(P(D_2)) = D_2$.

Therefore, $D_1 = D_2$ if and only if $P(D_1) = P(D_2)$. $\square$

Observation 17 [Non-empty Descent Group Implies Progenitor] Consider a genealogical network $G = (X, p)$. For every descent group D in G, $P(D) = \emptyset$ if and only if $D = \emptyset$.

Proof
Consider a genealogical network $G = (X, p)$ and an arbitrary descent group D in G.

If $D = \emptyset$, then $P(D) = \emptyset$ since by the definition of a progenitor (Definition 6), any element of $P(D)$ must come from D.

Suppose that $P(D) = \emptyset$. By the definition of cl (Definition 8), $cl(P(D)) = \emptyset$ since $P(D)$ are the ancestors of any element in $cl(P(D))$. By the fact that progenitors generate a descent group (Observation 14), $cl(P(D)) = D$. Thus $D = \emptyset$. $\Box$

Observation 18 [Progenitors for Descent Group Union] Consider a genealogical network $G = (X, p)$ and two descent groups D_1 and D_2 in G. Then,

$$\begin{aligned} P(D_1 \cup D_2) \quad = \quad & (P(D_1) \setminus (D_2 \cap P(D_1))) \cup \\ & (P(D_2) \setminus (D_1 \cap P(D_2))) \cup \\ & (P(D_1) \cap P(D_2)) \end{aligned}$$

Proof
Consider a genealogical network $G = (X, p)$ and two descent groups D_1 and D_2 in G.

Let y be an arbitrary element of $P(D_1) \setminus (D_2 \cap P(D_1))$. Then $y \in P(D_1)$ and $y \notin (D_2 \cap P(D_1))$. Which is equivalent to $y \in P(D_1)$ and $y \notin D_2$. Since $y \in D_1$, $y \in D_1 \cup D_2$ and is a candidate for being an element of $P(D_1 \cup D_2)$. Let x be an arbitrary element of X. Suppose $(x, y) \in p$. Since y is a progenitor of D_1, this implies $x \notin D_1$. Also, we have that $y \notin D_2$ which implies $x \notin D_2$ because otherwise the presence of x in D_2 implies $y \in D_2$ since D_2 is a descent group. Thus $x \notin D_1 \cup D_2$ and $y \in P(D_1 \cup D_2)$.

A totally symmetric argument will show that if $y \in (P(D_2) \setminus (D_1 \cap P(D_2)))$ then $y \in P(D_1 \cup D_2)$.

Consider the case where $y \in P(D_1) \cap P(D_2)$. Then $y \in D_1 \cup D_2$. Consider an arbitrary $x \in X$ and suppose that $(x, y) \in p$. Since $y \in P(D_1)$, $x \notin D_1$. Symmetrically, since $y \in P(D_2)$, $x \notin D_2$. Thus $x \notin D_1 \cup D_2$ and $y \in P(D_1 \cup D_2)$.

Hence $((P(D_1) \setminus (D_2 \cap P(D_1))) \cup (P(D_2) \setminus (D_1 \cap P(D_2))) \cup (P(D_1) \cap P(D_2)) \subseteq P(D_1 \cup D_2)$.

Now, let y be an arbitrary element of $P(D_1 \cup D_2)$. Then, by the definition of progenitor (Definition 6),

1. $y \in D_1 \cup D_2$
2. for every $x \in X$, if $(x, y) \in p$ then $x \notin D_1 \cup D_2$

Since $y \in D_1 \cup D_2$, without loss of generality, suppose $y \in D_1$. Consider an arbitrary $x \in X$ and suppose that $(x, y) \in p$. Then, $x \notin D_1 \cup D_2$. By monotonicity, $x \notin D_1$. Hence $y \in P(D_1)$. There are now two cases: $y \notin D_2$ and

$y \in D_2$. Consider the case where $y \notin D_2$. Then $y \in P(D_1) \setminus (D_2 \cap P(D_1))$. Consider, the second case, $y \in D_2$. Then, any parent of y is not in $D_1 \cup D_2$; implying that $y \in P(D_2)$. Hence $y \in P(D_1) \cap P(D_2)$. Thus, $P(D_1 \cup D_2) \subseteq ((P(D_1) \setminus (D_2 \cap P(D_1))) \cup (P(D_2) \setminus (D_1 \cap P(D_2))) \cup (P(D_1) \cap P(D_2)))$.

Therefore $P(D_1 \cup D_2) = ((P(D_1) \setminus (D_2 \cap P(D_1))) \cup (P(D_2) \setminus (D_1 \cap P(D_2))) \cup (P(D_1) \cap P(D_2)))$. $\square$

Corollary 1 [New Individual and Progenitors] Let D be a descent group in a genealogical network $G = (X, p)$. For any $x \in X$,

$$P(D \cup cl(\{x\})) = \begin{cases} P(D) & \text{if } x \in D \\ \{x\} \cup (P(D) \setminus (cl(\{x\} \cap P(D))) & \text{otherwise} \end{cases}$$

Proof
Let D be a descent group in a genealogical network $G = (X, p)$. Moreover, let x be an arbitrary element of X.

Since the singleton set $\{x\}$ is clearly a minimal generating set by Definition 9, $P(cl(\{x\})) = \{x\}$ by Observation 13. Now, by Observation 18

$$P(D \cup cl(\{x\})) = (P(D) \setminus (cl(\{x\}) \cap P(D))) \cup (\{x\} \setminus (D \cap \{x\})) \cup (P(D) \cap \{x\})$$

Consider the case, where $x \in D$, then

$$\begin{aligned} P(D \cup cl(\{x\})) &= (P(D) \setminus (cl(\{x\}) \cap P(D))) \cup (P(D) \cap \{x\}) \\ &\quad \text{since } \{x\} \setminus (D \cap \{x\}) = \emptyset \text{ when } x \in D \\ &= (P(D) \setminus (\{x\} \cap P(D)) \cup (P(D) \cap \{x\}) \\ &\quad \text{since } x \in D\text{, no proper descendant of } x \text{ can be a progenitor of } D \\ &\quad \text{and } P(D) \cap cl(\{x\}) = P(D) \cap \{x\} \\ &= P(D) \text{ since } P(D) \subseteq (P(D) \cap \{x\}) \end{aligned}$$

Now, consider the case where $x \notin D$, then

$$\begin{aligned} P(D \cup cl(\{x\})) &= (P(D) \setminus (cl(\{x\}) \cap P(D))) \cup \{x\} \cup (P(D) \cap \{x\}) \\ &\quad \text{since } x \notin D\text{, } \{x\} \setminus (D \cap \{x\}) = \{x\} \\ &= (P(D) \setminus (cl(\{x\}) \cap P(D))) \cup \{x\} \\ &\quad \text{since } x \notin D\text{, } x \text{ cannot be progenitor of } D \text{ and } P(D) \cap \{x\} = \emptyset \end{aligned}$$

Therefore,

$$P(D \cup cl(\{x\})) = \begin{cases} P(D) & \text{if } x \in D \\ \{x\} \cup (P(D) \setminus (cl(\{x\}) \cap P(D))) & \text{otherwise} \end{cases}$$

$\square$

Observation 19 [An Ancestor Is Most Recent] Consider a genealogical network $G = (X, p)$. For every $x_1, x_2 \in X$, if x_1 is an ancestor of x_2, then $MRCA(\{x_1, x_2\}) = \{x_1\}$.

Proof
Given a genealogical network $G = (X, p)$, let $x_1, x_2 \in X$ be two individuals. Suppose that x_1 is an ancestor of x_2. Then, x_1 is clearly an ancestor of x_1 and x_2. Let y be an ancestor of x_1 and x_2 with $x_1 \neq y$. Then x_1 is not an ancestor of y since G is acyclic. Thus, by Definition 10, x_1 is a most recent common ancestor of $\{x_1, x_2\}$.

Suppose there exists $z \in MRCA(\{x_1, x_2\})$ such that $z \neq x_1$. Then for every $y \in X$, if y is an ancestor of x_1 and x_2, then z is not an ancestor of y. This will hold for $y = x_1$. Since x_1 is an ancestor for x_1 and x_2, we obtain that z is not an ancestor of x_1. This is a contradiction. Thus $MRCA(\{x_1, x_2\}) = \{x_1\}$. □

Observation 20 [Some Common Ancestors are Most Recent] Consider a genealogical network $G = (X, p)$. For every $S \subseteq X$, $y_1 \in X$, if for every $x \in S$, y_1 is an ancestor of x, then $MRCA(S) \neq \emptyset$.

Proof
Consider a genealogical network $G = (X, p)$. Let S be an arbitrary subset of X and y_1 be arbitrary element of X. Suppose that, for every $s \in S$, y_1 is an ancestor of s. For a proof by contradiction, suppose that $MRCA(S) = \emptyset$. $MRCA(S) = \emptyset$ by Definition 10 implies that for any $x \in X$,

1. there exists an $s \in S$ such that x is not an ancestor of s, or

2. for some $y \in X$, y is an ancestor of every $s \in S$ and x is an ancestor of y

Starting with y_1 - which is an ancestor of every $s \in S$ - generate an infinite sequence $y_1, \ldots, y_i, y_{i+1}, \ldots$ of distinct elements of X each of which is an ancestor of every $s \in S$. Suppose that y_i is an ancestor of every $s \in S$. Then, there exists a y_{i+1} distinct from y_i which is an ancestor of every $s \in S$ and y_i is an ancestor of y_{i+1}. In this sequence, there cannot exist a pair of elements y_I and y_J such that $I \neq J$ and $y_I = y_J$. If this were the case, then the sequence $y_I, \ldots y_J$ would bear witness, via the transitivity of the ancestor relationship, that G is a cyclic graph. Hence, each pair of elements in the sequence $y_1, \ldots, y_i, y_{i+1}, \ldots$ are distinct. This implies that X is infinite. This contradicts our assumption that X is finite. Therefore $MRCA(S) \neq \emptyset$. □

B Proofs For Section 3

Observation 21 [Disconnected Descent Groups Don't Intersect] Consider a genealogical network $G = (X, p)$ and descent groups D_1, D_2 in G. D_1 and D_2 are disconnected if and only if $D_1 \cap D_2 = \emptyset$.

Proof
Consider a genealogical network $G = (X, p)$ and descent groups D_1, D_2 in G.

($\Rightarrow$) Suppose that D_1 and D_2 are disconnected. For a proof by contradiction, suppose that $D_1 \cap D_2 \neq \emptyset$. Let $x \in D_1 \cap D_2$. Then $x \in D_1$ and $x \in D_2$. Since x is an ancestor of x, by Observation 19, $MCRA(\{x, x\}) = \{x\}$. Thus $MRCA(\{x, x\}) \cap D_1 = \{x\}$. Contradicting the fact that D_1 and D_2 are disconnected. Therefore, $D_1 \cap D_2 = \emptyset$.

($\Leftarrow$) Suppose that $D_1 \cap D_2 = \emptyset$. Clearly, if either $D_1 = \emptyset$ or $D_2 = \emptyset$, then D_1 and D_2 are disconnected. Let x_1 and x_2 be arbitrary elements of D_1 and D_2 respectively. Let y be an arbitrary element of $MRCA(\{x_1, x_2\})$. Then y cannot be an element of D_1 or D_2. For a proof by contraction, assume the contrary. Without loss of generality, assume that $y \in D_1$. Then y is an ancestor of x_2. Since D_1 is a descent group, this implies that $x_2 \in D_1$. This is a contradiction since we have that $D_1 \cap D_2 = \emptyset$.

Therefore D_1 and D_2 are disconnected if and only if $D_1 \cap D_2 = \emptyset$. $\square$

Observation 22 [The Smaller the More Disconnected] Consider a genealogical network $G = (X, p)$ and three descent groups D_0, D_1, D_2 in G. If D_1 and D_2 are disconnected and $D_0 \subseteq D_1$, then D_0 and D_2 are disconnected.

Proof
Consider a genealogical network $G = (X, p)$ and three descent groups D_0, D_1, D_2 in G. Suppose that D_1 and D_2 are disconnected and $D_0 \subseteq D_1$. Since D_1 and D_2 are disconnected, for every $x_1 \in D_1, x_2 \in D_2$, $MRCA(\{x_1, x_2\}) \cap D_1 = \emptyset$ and $MRCA(\{x_1, x_2\}) \cap D_2 = \emptyset$. Moreover, since $D_0 \subset D_1$, this implies that for every $x_0 \in D_0, x_2 \in D_2$, that $MRCA(\{x_0, x_2\}) \cap D_0 = \emptyset$ and $MRCA(\{x_0, x_2\}) \cap D_2 = \emptyset$. Thus, D_0 and D_2 are disconnected descent groups. $\square$

Observation 23 [Progenitors and Disconnected Descent Groups] Consider a genealogical network $G = (X, p)$ and two descent groups D_1 and D_2 in G. If D_1 and D_2 are disconnected and $D = D_1 \cup D_2$, then

1. $P(D) = P(D_1) \cup P(D_2)$
2. $P(D_1) \cap P(D_2) = \emptyset$

Proof
Consider a genealogical network $G = (X, p)$ and two descent groups D_1 and D_2 in G. Suppose that D_1 and D_2 are disconnected and let $D = D_1 \cup D_2$.

Let x be an arbitrary element of $P(D)$. Without loss of generality, suppose that $x \in D_1$. Let y be an arbitrary element of X and suppose that $(y, x) \in p$. Since $x \in P(D)$, by the definition of a progenitor (Definition 6), $y \notin D$. By the definition of D, $y \notin D_1$. Hence $x \in P(D_1)$ and $P(D) \subseteq P(D_1) \cup P(D_2)$.

Let x be an arbitrary element of $P(D_1) \cup P(D_2)$. Without loss of generality, suppose $x \in P(D_1)$. Since $D = D_1 \cup D_2$, $x \in D$. Let y be an arbitrary element of X and suppose that $(y, x) \in p$. Since x is a progenitor in D_1, this implies that $y \notin D_1$. There is a chance that $y \in D_2$. However, if we assume this, then $x \in D_2$ since D_2 is a descent group. However, since D_1 and D_2 are disconnected, and disconnected descent groups do not intersect (Observation 21), $x \notin D_2$. This contradiction means that $y \notin D_2$. Then $y \notin D$ and $x \in P(D)$. Hence $P(D_1) \cup P(D_2) \subseteq P(D)$.

Therefore $P(D) = P(D_1) \cup P(D_2)$.

By Definition 7, $P(D_1) \subseteq D_1$ and $P(D_2) \subseteq D_2$. Since D_1 and D_2 are disconnected, by Observation 21, $D_1 \cap D_2 = \emptyset$, $P(D_1) \cap P(D_2) = \emptyset$. $\square$

Observation 24 [Disconnection Preserved Under Union] Consider a genealogical network $G = (X, p)$ and three descent groups D_0, D_1, D_2 in G. If D_0, D_1 and D_2 are pairwise disconnected, then $(D_0 \cup D_1)$ and D_2 are disconnected.

Proof
Consider a genealogical network $G = (X, p)$ and three descent groups D_0, D_1, D_2 in G. Suppose D_0, D_1 and D_2 are pairwise disconnected. Consider $(D_0 \cup D_1)$ and D_2. Given the symmetry in the result between D_0 and D_1, w.l.o.g, consider an arbitrary $x \in D_0$ and $x_2 \in D_2$. Straightaway, $MRCA(\{x, x_2\}) \cap D_2 = \emptyset$ since D_0 and D_2 are disconnected descent groups. By the same reason, $MRCA(\{x, x_2\}) \cap D_0 = \emptyset$.

Now, break the scenario into two cases: $x \in D_1$ and $x \notin D_1$. Firstly, consider $x \in D_1$. Since D_1 and D_2 are disconnected, $MRCA(\{x, x_2\}) \cap D_1 = \emptyset$. Thus, $MRCA(\{x, x_2\}) \cap (D_0 \cup D_1) = \emptyset$. Now consider the second case where $x \notin D_1$. Suppose there exists a $y \in D_1$ such that $y \in MRCA(\{x, x_2\})$. This implies that y is an ancestor of x. By Observation 19, it obtains that $MRCA(\{x, y\}) = \{y\}$. This contradicts the assumption that D_0 and D_1 are disconnected. Thus $MRCA(\{x, x_2\}) \cap D_1 = \emptyset$ and $MRCA(\{x, x_2\}) \cap (D_0 \cup D_1) = \emptyset$.

Therefore, $(D_0 \cup D_1)$ and D_2 are disconnected descent groups. □

Corollary 2 [Disconnection Preserved Under General Union] Consider a genealogical network $G = (X, p)$ and descent groups $D_0, D_1, ..., D_k$ in G. If $D_0, D_1, \ldots D_k$ are pairwise disconnected, then $\bigcup_{0 \leq i < k} D_i$ and D_k are disconnected.

Proof
Consider a genealogical network $G = (X, p)$ and descent groups $D_0, D_1, ..., D_k$ in G. Suppose $D_0, D_1, \ldots D_k$ are pairwise disconnected. Then, by Observation 21, $D_k \cap D_i = \emptyset$ for every i, $0 \leq i < k$. Then $D_k \cap \bigcup_{0 \leq i < k} D_i = \emptyset$. Once again, by Observation 21, $\bigcup_{0 \leq i < k} D_i$ and D_k are disconnected. □

Observation 25 [Limiting Monophyletic Descent Groups] Consider a genealogical network $G = (X, p)$. Then

1. the empty set $\emptyset$ is a monophyletic group in G
2. for every $x \in X$, if for every $y \in X$, $(x, y) \notin p$, then $\{x\}$ is a monophyletic group

Proof
Consider a genealogical network $G = (X, p)$.

1. By Observation 2, $\emptyset$ is a descent group in G. Clearly, $\emptyset$ cannot be partitioned into two non–empty descent groups. Thus, $\emptyset$ is monophyletic.
2. Consider an $x \in X$ such that for every $y \in X$, $(x, y) \notin p$. Since x has no descendants, $\{x\}$ is a descent group in G. Moreover, any partition of $\{x\}$ into two non–empty sub–descent groups, would require that the size of $\{x\}$ be at least two. Thus, $\{x\}$ is monophyletic.

□

Observation 26 [Monophyly and Progenitors] Consider a descent group D in a genealogical network $G = (X, p)$. D is polyphyletic if and only if there exists a partition of the progenitors of D into two subsets X_1 and X_2 such that:

1. $X_1 \cup X_2 = P(D)$
2. $X_1 \cap X_2 = \emptyset$
3. for every $x \in D$, $x_1 \in X_1$ and $x_2 \in X_2$, it is not the case that both x_1 and x_2 are an ancestor of x

Proof
Let D be a descent group in a genealogical network $G = (X, p)$.

$(\Leftarrow)$ Suppose that there exists a partition of the progenitors of D into two subsets X_1 and X_2 such that:

1. $X_1 \neq \emptyset$
2. $X_2 \neq \emptyset$
3. $X_1 \cup X_2 = P(D)$
4. $X_1 \cap X_2 = \emptyset$
5. for every $x \in D$, $x_1 \in X_1$ and $x_2 \in X_2$, it is not the case that both x_1 and x_2 are an ancestor of x

This partition of the progenitors can be used to construct a partition of D into two non–intersecting descent groups. By Observation 15,

$$D = \bigcup_{x \in P(D)} cl(\{x\})$$

Since $X_1 \cup X_2 = P(D)$,

$$D = \bigcup_{x \in X_1} cl(\{x\}) \cup \bigcup_{x \in X_2} cl(\{x\})$$

Let $D_1 = \bigcup_{x \in X_1} cl(\{x\})$ and $D_2 = \bigcup_{x \in X_2} cl(\{x\})$. Since both X_1 and X_2 are non-empty, D_1 and D_2 are non-empty. By Observation 9, for any $x \in P(D)$, $cl(\{x\})$ is a descent group. This combined by a finite application of Observation 3 (the set union of two descent groups is a descent group), give that both D_1 and D_2 are descent groups. To show that $D_1 \cap D_2 = \emptyset$, consider an arbitrary $x \in D$. For a proof by contradiction, suppose that $x \in D_1$ and $x \in D_2$. Then by Observation 6, there exist progenitors $y_1 \in P(D_1)$ and $y_2 \in P(D_2)$ such that both y_1 and y_2 are progenitors of x. This contradicts the last assumption listed above. Therefore, $D_1 \cap D_2 = \emptyset$. Since D can be partitioned into two non-empty non-intersecting sub-descent groups, D is polyphyletic by Definition 13.

$(\Rightarrow)$ Suppose that D is polyphyletic. Then by Definition 13, there exists descent groups D_1 and D_2 such that:

1. $D_1 \neq \emptyset$
2. $D_2 \neq \emptyset$
3. $D_1 \cup D_2 = D$
4. $D_1 \cap D_2 = \emptyset$

In a similar vein to the above argument, this partition of D into two two descent groups generates a partition of the progenitors of D. Let $X_1 = P(D_1)$ and $X_2 = P(D_2)$. By Observation 23, $X_1 \cup X_2 = P(D)$ and $X_1 \cap X_2 = \emptyset$. Moreover, since neither D_1 nor D_2 are empty, by Observation 17, $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. Consider arbitrary $x \in D$, $x_1 \in X_1$ and $x_2 \in X_2$. To show that it is not the case that both x_1 and x_2 are ancestors of x, assume the contrary. Then $x \in D_1$ since D_1 is a descent group and $x_1 \in X_1$. Similarly $x \in D_2$. This implies that $D_1 \cap D_2 \neq \emptyset$. This contradiction gives that: there exists a partition of the progenitors of D into two subsets X_1 and X_2 such that:

1. $X_1 \neq \emptyset$
2. $X_2 \neq \emptyset$
3. $X_1 \cup X_2 = P(D)$
4. $X_1 \cap X_2 = \emptyset$
5. for every $x \in D$, $x_1 \in X_1$ and $x_2 \in X_2$, it is not the case that both x_1 and x_2 are an ancestor of x

$\square$

Corollary 3 [Single Progenitor Generates Monophyletic Descent Group] Consider a genealogical network $G = (X, p)$. For every $x \in X$, $cl(\{x\})$ is a monophyletic group.

Proof
Consider a genealogical network $G = (X, p)$. Let x be an arbitrary element of X. By Observation 13, the progenitors of the closure of a set exactly match the set, i.e., $P(cl(\{x\})) = \{x\}$. Since there is only one progenitor, no non-trivial partition exists of $P(cl(\{x\}))$. By Observation 26, this implies that $cl(\{x\})$ is monophyletic. $\square$

Observation 27 [Minimal Polyphyletic Degree] Consider a descent group D in a genealogical network $G = (X, p)$. If $D \neq \emptyset$, then D is polyphyletic of degree 1.

Proof
Consider a descent group D in a genealogical network $G = (X, p)$. Suppose $D \neq \emptyset$. Then D itself forms a single set partition of D. Moreover, since D is non–empty it obtains that D is polyphyletic of degree 1. $\square$

Observation 28 [Polyphyletic Degree and Descent Group Union] Consider a genealogical network $G = (X, p)$ and descent groups D_1, D_2 in G. Suppose D_1 and D_2 are polyphyletic of degree k_1 and k_2 respectively. If $D_1 \cap D_2 = \emptyset$, then $D_1 \cup D_2$ is polyphyletic of degree $k_1 + k_2$.

Proof
Consider a genealogical network $G = (X, p)$ and descent groups D_1, D_2 in G. Suppose D_1 and D_2 are polyphyletic of degree k_1 and k_2 respectively. Let $D_1^{'}, \ldots D_{k_1}^{'}$ be witness to D_1 being polyphyletic of degree k_1. Also, let $D_{k_1+1}^{'}, \ldots, D_{k_1+k_2}^{'}$ be witness to D_2 being polyphyletic of degree k_2. Then $D_1^{'}, \ldots, D_{k_1+k_2}^{'}$ shows that $D_1 \cup D_2$ is polyphyletic of degree k_1+k_2. Firstly, $\bigcup_{1 \leq i \leq k_1+k_2} D_i^{'} = D_1 \cup D_2$ since $D_1^{'}, \ldots D_{k_1}^{'}$ partition D_1 and $D_{k_1+1}^{'}, \ldots, D_{k_1+k_2}^{'}$ partition D_2. Also, each $D_i^{'}$ is non–empty from the original partitions. Consider arbitrary I, J, $1 \leq I < J \leq k_1 + k_2$. If $J \leq k_1$, then $D_I^{'} \cap D_J^{'} = \emptyset$ since $D_1^{'}, \ldots D_{k_1}^{'}$ are witness to D_1 being polyphyletic of degree k_1. Similarly if $I > k_1$, then $D_I^{'} \cap D_J^{'} = \emptyset$ since $D_{k_1+1}^{'}, \ldots, D_{k_1+k_2}^{'}$ witness that D_2 is polyphyletic of degree k_2. For the remaining case, $I \leq k_1$ and $J > k_1$. Then, $D_I^{'} \subseteq D_1$ and $D_J^{'} \subseteq D_2$. Since $D_1 \cap D_2 = \emptyset$, $D_I^{'} \cap D_J^{'} = \emptyset$. Hence $D_1 \cup D_2$ is polyphyletic of degree $k_1 + k_2$. □

Observation 29 [Partition of Progenitors] Consider a descent group D in a genealogical network $G = (X, p)$. If $D_1, \ldots, D_k$ is a witness to D being polyphyletic of degree k, then for every k, $1 \leq i \leq k$, there exists an $x_i \in P(D)$ such that $x_i \in D_i$.

Proof
Consider a descent group D in a genealogical network $G = (X, p)$. Let $D_1, \ldots, D_k$ be a witness to D being polyphyletic of degree k. Consider an arbitrary D_i where $1 \leq i \leq k$. For a proof by contradiction, suppose that $D_i \cap P(D) = \emptyset$. Since $D_i \neq \emptyset$, let $x \in D_i$ be arbitrary. Since x is not a progenitor of D, there exists a progenitor y of D such that y is an ancestor of x (Observation 6). Since the witness partitions D, $y \in D_j$ for some $j \neq i$ and $1 \leq j \leq k$. Since D_j is a descent group, $x \in D_j$. This contradicts that $D_i \cap D_j = \emptyset$ (Definition 14). Thus $D_i \cap P(D) \neq \emptyset$. □

Observation 30 [Polyphyletic Degree Preserved Downwards] Consider a descent group D in a genealogical network $G = (X, p)$. If D is polyphyletic of degree k, then for every l, $1 \leq l < k$, D is polyphyletic of degree l.

Proof
Consider a descent group D in a genealogical network $G = (X, p)$. Suppose that D is polyphyletic of degree k. By the definition of polyphyletic degree (Definition 14), there exist descent groups $D_1, \ldots, D_k$ such that $\bigcup_{1 \leq i \leq k} D_i = D$ and for every $1 \leq I < J \leq k$, D_I and D_J are disconnected. Consider the sequence $D_1, \ldots, D_{l-1}, \bigcup_{l \leq i \leq k} D_i$. Clearly $D_1, \ldots, D_{k-1}$ are descent groups that are pairwise disconnected. Also, $\bigcup_{1 \leq i \leq l-1} D_i \cup \bigcup_{l \leq i \leq k} D_i = D$. Now, consider $\bigcup_{l \leq i \leq k} D_i$. By Observation 3, $\bigcup_{l \leq i \leq k} D_i$ is a descent group. Moreover, by Observation 24 (descent groups are closed under set union), for every j, $1 \leq j \leq l-1$, D_i and $\bigcup_{l \leq i \leq k} D_i$ are disconnected. Therefore, D is polyphyletic of degree l. □

Observation 31 [Monophyly and Maximal Polyphyletic Degree] Consider a descent group D in a genealogical network $G = (X, p)$. D is polyphyletic of maximal degree 1 if and only if $D \neq \emptyset$ and D is monophyletic.

Proof
Consider a descent group D in a genealogical network $G = (X, p)$.

Suppose that D is polyphyletic of maximal degree 1. Then D is polyphyletic of degree 1 which means, by Definition 31, that for some non-empty descent group D_1, $D_1 = D$. Thus, $D \neq \emptyset$. Also, D is not polyphyletic of degree 2, which clearly means that D is not polyphyletic, i.e., D is monophyletic.

Suppose that $D \neq \emptyset$ and D is monophyletic. Then, D by itself constitutes a partition (trivial) of D by one set. Moreover, $D \neq \emptyset$. Thus, D is polyphyletic of degree 1. To show that D is not polyphyletic of degree 2, assume the contrary. Then for some non–empty descent groups D_1 and D_2, $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = D$. Thus, D is polyphyletic; contradicting that D is monophyletic. Thus, D is not polyphyletic of degree 2. Hence, D is polyphyletic of maximal degree 1. $\square$

Observation 32 [Maximal Polyphyletic Degree Witnesses are Permutations] Consider a genealogical network $G = (X, p)$ and a descent group D in G. Suppose D is polyphyletic of maximal degree k. If $D_1, \ldots, D_k$ and $D_1^{'}, \ldots, D_k^{'}$ are witnesses to D being polyphyletic of degree k, then there exists a permutation ϕ from $\{D_1, \ldots D_k\}$ to $\{D_1^{'}, \ldots, D_k^{'}\}$ such that $D_i = \phi(D_i^{'})$ for every i, $1 \leq i \leq k$.

Proof
Consider a genealogical network $G = (X, p)$ and a descent group D in G. Suppose D is polyphyletic of degree k and not polyphyletic of degree $k+1$. Let $D_1, \ldots, D_k$ and $D_1^{'}, \ldots, D_k^{'}$ be witnesses to D being polyphyletic of degree k, i.e,

1. for every i, $1 \leq i \leq k$, $D_i \neq \emptyset$ and $D_i^{'} \neq \emptyset$,
2. for every i, j, $1 \leq i < j \leq k$, $D_i \cap D_j = \emptyset$ and $D_i^{'} \cap D_j^{'} = \emptyset$, and
3. $\bigcup_{1 \leq i \leq k} D_i = D$ and $\bigcup_{1 \leq i \leq k} D_i^{'} = D$,

To show a permutation ϕ exists between $\{D_1, \ldots D_k\}$ and $\{D_1^{'}, \ldots, D_k^{'}\}$ such that $D_i = \phi(D_i)$ for every i, $1 \leq i \leq k$, it is sufficient to show that for every D_i and $D_j^{'}$, if $1 \leq i, j \leq k$ and $D_i \cap D_j^{'} \neq \emptyset$, then $D_i = D_j^{'}$. For a proof by contradiction, suppose for some D_I and $D_J^{'}$ where $1 \leq I, J \leq k$ that $D_I \cap D_J^{'} \neq \emptyset$ and $D_I \neq D_J^{'}$. Let

1. $D_I^{-} = D_I \setminus D_J^{'}$
2. $D_I^{=} = D_I \cap D_J^{'}$
3. $D_I^{+} = D_J^{'} \setminus D_I$

Partially overlapping pieces in two witnesses attest that a piece can be more finely chopped. The proof shows how this chopping can be done. Since $D_I \neq D_J^{'}$, $D_I^{-} \neq \emptyset$ or $D_I^{+} \neq \emptyset$. Also, $D_I^{=} \neq \emptyset$. Due to the symmetry of the situation, assume that $D_I^{-} \neq \emptyset$. Now, by construction, $D_I^{-} \cap D_I^{=} = \emptyset$. What remains is to show that D_I^{-} is descent group contained in D. Since $D_I^{-} \subseteq D_I$, $D_I^{-} \subseteq D$. Let x be an arbitrary individual in X and suppose that for some $a \in D_I^{-}$, a is an ancestor of x. Since D_I is a descent group, $x \in D_I$. Suppose that $x \in D_I^{=}$. Now since $D_1^{'}, \ldots, D_k^{'}$ partition D, for some $D_L^{'}$, $1 \leq L \leq k$, $a \in D_L^{'}$. Moreover, by construction $D_J^{'} \neq D_L^{'}$. Since $D_L^{'}$ is a descent group, $x \in D_L^{'}$. This gives that $D_J^{'} \cap D_L^{'} \neq \emptyset$. Thus $x \notin D_I^{=}$ and $x \in D_I^{-}$. Thus D_I^{-} is a descent group. By a symmetric argument, D_I^{+} is a descent group.

In the sequence $D_1, \dots D_k$, replace D_I by D_I^- and $(D_I^= \cup D_I^+)$. Now, each set is a descent group. In the above argument, D_I^- was shown to be a descent group. Also by Observation 3, $D_I^=$ is a descent group and $(D_I^= \cup D_I^+)$ is also a descent group. This gives a partition of D into $k+1$ non-empty descent groups and implies that D is polyphyletic of degree $k+1$. This contradictions gives us that for every D_i and $D_j^{'}$, if $1 \leq i, j \leq k$ and $D_i \cap D_j^{'} \neq \emptyset$, then $D_i = D_j^{'}$. From this a function ϕ from $\{D_1, \dots, D_k\}$ to $\{D_1^{'}, \dots, D_k^{'}\}$ can be constructed where $\phi(D_i) = D_j^{'}$ where j is the smallest value such that $D_i \cap D_j^{'} \neq \emptyset$. This function is well defined since each D_i is non-empty, i.e., it contains an element x of D and $D_1^{'}, \dots, D_k^{'}$ partitions D. By choosing the smallest j such that $D_i \cap D_j^{'} \neq \emptyset$, each D_i is mapped to a unique $D_j^{'}$. Consider any $D_J^{'}$, $1 \leq J \leq K$. Since $D_J^{'} \neq \emptyset$ and $D_J^{'}$ is a part of the partition of D, it contains an element x of D. Also, since $D_1, \dots, D_k$ partitions D, there exists a D_I such that $x \in D_I$. Since $D_I \cap D_J^{'} \neq \emptyset$, $\phi(D_I) = D_J^{'}$. Thus, ϕ is an onto function. Suppose that $\phi(D_I) = \phi(D_J)$ for some I, J, $1 \leq I, J \leq k$. Let $\phi(D_I) = D_K^{'}$. Then for some $x \in D$, $x \in D_K^{'}$ and $x \in D_I$. Moreover, $x \in D_J$. Since $D_1, \dots, D_k$ is a partition of D, $D_I = D_J$. Therefore, ϕ is an injective function. Now since we have established that for every D_i and $D_j^{'}$, if $1 \leq i, j \leq k$ and $D_i \cap D_j^{'} \neq \emptyset$, then $D_i = D_j^{'}$. For every i, $1 \leq i \leq k$, $\phi(D_i) = D_i$. $\square$

Observation 33 [Polyphyletic Border] Consider a genealogical network $G = (X, p)$ and a descent group $D \neq \emptyset$ in G which is polyphyletic of degree k. Suppose that $D_1, \dots D_k$ is a witness to D being polyphyletic of degree k. D is *not* polyphyletic of degree $k+1$ if and only if for every i, $1 \leq i \leq k$, D_i is monophyletic.

Proof

Consider a genealogical network $G = (X, p)$ and a descent group $D \neq \emptyset$ in G which is polyphyletic of degree k. Suppose that $D_1, \dots D_k$ are descent groups that are witness to D being polyphyletic of degree k, i.e.,

1. $D_i \neq \emptyset$ for every i, $1 \leq i \leq k$
2. $D_i \cap D_j = \emptyset$ for every i, j, $1 \leq i < j \leq k$
3. $\bigcup_{1 \leq i \leq k} D_i = D$

($\Rightarrow$) Suppose that D is not polyphyletic of degree $k+1$. For a proof by contradiction, suppose that D_I is polyphyletic for some I, $1 \leq I \leq k$. By Definition 13, D_I can be partitioned into two non-empty descent groups $D_{I,1}$ and $D_{I,2}$. Then, the sequence

$$D_1, \dots, D_{I-1}, D_{I,1}, D_{I,2}, D_{I+1}, \dots, D_k$$

is a witness that D is polyphyletic of degree $k+1$ since $D_{I,1} \cup D_{I,2} = D_I$ and $D_{I,1} \cap D_{I,2} = \emptyset$. Moreover, for any $i \neq I$, $D_i \cap D_{I,1} = \emptyset$ and $D_i \cap D_{I,2} = \emptyset$ since $D_i \cap D_I = \emptyset$ and both $D_{I,1}$ and $D_{I,2}$ are subsets of D_I. This contradiction gives that D_i is monophyletic for every i, $1 \leq i \leq k$.

($\Leftarrow$) Suppose that D_i is monophyletic for every i, $1 \leq i \leq k$. For a proof by contradiction, suppose that D is polyphyletic of degree $k+1$. Then there exists descent groups $D_1^{'}, \dots, D_k^{'}, D_{k+1}^{'}$ such that

1. $D_i^{'} \neq \emptyset$ for every i, $1 \leq i \leq k+1$
2. $D_i^{'} \cap D_j^{'} = \emptyset$ for every i, j, $1 \leq i < j \leq k+1$
3. $\bigcup_{1 \leq i \leq k+1} D_i^{'} = D$

Now, consider the relationship between D_I and $D_J^{'}$ for arbitrary I, J, $1 \leq I \leq k$ and $1 \leq J \leq k+1$. Suppose that $D_I \cap D_J^{'} \neq \emptyset$. To show that this implies that $D_I \subseteq D_J^{'}$, assume the contrary, i.e, that $D_I \setminus D_J^{'} \neq \emptyset$. This non–empty intersection will generate a contradiction against the monophyly of D_i. This will come about through a non–trivial partition of the progenitors of D_i. Now, consider a partition of the $P(D_I)$ into X_1 and X_2 such that $X_1 = \{x \in P(D_I) \mid x \notin D_I \setminus D_J^{'}\}$ and $X_2 = P(D_1) \setminus X_1$. Now, since $D_I \setminus D_J^{'} \neq \emptyset$, there exists $y \in D_I \setminus D_J^{'}$. Then for some $x \in P(D_I)$, x is an ancestor of y. Now $x \notin D_I \cap D_J^{'}$, because otherwise $y \in D_I \cap D_J^{'}$ since $D_I \cap D_J^{'}$ is a descent group by Observation 3. Thus $X_1 \neq \emptyset$. Now, since $D_I \cap D_J^{'} \neq \emptyset$, there exists a $z \in D_I \cap D_J^{'}$. Then for some $w \in P(D_I)$, w is an ancestor of z. Then $w \in D_J^{'}$ because otherwise w will be an element of $D_K^{'}$ for some K, $1 \leq K \leq k+1$ and $J \neq K$ since $D_1^{'}, \ldots D_{k+1}^{'}$ partition D. Then since $D_K^{'}$ is a descent group, $z \in D_K^{'}$. Thus $D_J^{'} \cap D_K^{'} \neq \emptyset$ and $D_J^{'} \neq D_K^{'}$. This contradiction gives that $w \in D_J^{'}$ and $w \notin D_I \setminus D_J^{'}$. Thus $w \in X_2$ and $X_2 \neq \emptyset$. Now since D_I is monophyletic with X_1 and X_2 constituting a non–empty partition of $P(D_I)$, by Observation 26, there exists a $x_1 \in X_1$, $x_2 \in X_2$ and $y \in D_I$ such that both x_1 and x_2 are ancestors of y. Now, let $D_K^{'} \neq D_J^{'}$ contain x_1. Since $D_K^{'}$ and $D_J^{'}$ are descent groups, $y \in D_K^{'}$ and $y \in D_J^{'}$. This contradicts that $D_K^{'} \cap D_J^{'} = \emptyset$. Hence $D_I \subseteq D_J^{'}$.

Thus, for every i, j, $1 \leq i \leq k$ and $1 \leq j \leq k+1$,

1. $D_i \cap D_j^{'} = \emptyset$, or
2. $D_i \subseteq D_j^{'}$.

Now construct a mapping ϕ from $\{D_1, \ldots, D_k\}$ to $\{D_1^{'}, \ldots, D_{k+1}^{'}\}$. For each $D_I \in \{D_1, \ldots, D_k\}$, ϕ maps D_I to $D_J^{'}$ where J is the smallest value such that for some $x \in D_I$, $x \in D_J^{'}$. Such a mapping is well defined since $D_I \neq \emptyset$, $x \in D$ and $D_1^{'}, \ldots, D_{k+1}^{'}$ partition D. Now, since $\phi(D_I) \cap D_I \neq \emptyset$, $D_I \subseteq \phi(D_I)$.

$$\begin{aligned} D &= \bigcup_{1 \leq i \leq k} D_i \text{ since } D_1, \ldots, D_k \text{ partitions } D \\ &\subseteq \bigcup_{1 \leq i \leq k} \phi(D_i) \text{ since } D_i \subseteq \phi(D_i) \text{ for each } i \end{aligned}$$

Consider $\bigcup_{1 \leq i \leq k} \phi(D_i)$. This is the range of the function ϕ. Since the size of the domain of ϕ is k, the size of the range of ϕ is at most k. Then, since the codomain of ϕ has size $k+1$, there exists some $D_J^{'} \in \{D_1^{'}, \ldots, D_{k+1}^{'}\}$ which is not in the range of ϕ. Moreover, $D_J^{'}$ is non–empty. Thus,

$$D \subset \bigcup_{1 \leq i \leq k} \phi(D_i) \cup D_J^{'} \subseteq \bigcup_{1 \leq j \leq k+1} D_i^{'} = D \text{ since } D_1^{'}, \ldots, D_{k+1}^{'} \text{ partitions } D$$

This contradiction gives us that for some i, $1 \leq i \leq k$, D_i is polyphyletic.

Therefore, D is *not* polyphyletic of degree $k+1$ if and only if for every i, $1 \leq i \leq k$, D_i is monophyletic. $\square$

Observation 34 [Preserving Monophyly] Consider a genealogical network $G = (X, p)$ and a descent group D in G. For every $x \in X$, and witness $D_1, \ldots D_k$ to D being polyphyletic of maximal degree k, if $D_i \cap cl(\{x\}) \neq \emptyset$ for every i, $1 \leq i \leq k$, then $cl(\{x\} \cup \bigcup_{1 \leq i \leq k} D_i$ is a monophyletic group.

Proof
Consider a genealogical network $G = (X, p)$ and a descent group D in G. Let x be an arbitrary individual in X. Suppose that $D_1, \ldots D_k$ is a witness to D being polyphyletic of maximal degree k such that $D_i \cap cl(\{x\}) \neq \emptyset$ for every i, $1 \leq i \leq k$.

Certainly since cl generates descent groups, by Observation 9, $cl(\{x\})$ is a descent group. Moreover, since descent groups are closed under set union, repeated applications of Observation 3 gives that $cl(\{x\}) \cup \bigcup_{1 \leq i \leq k} D_i$ is a descent group.

Since $D_i \cap D_j = \emptyset$ for every i, j, $1 \leq i < j \leq k$, repeated applications of Corollary 23 will give that $P(D) = \bigcup_{1 \leq i \leq k} P(D_i)$. By adding $cl(\{x\})$ to D, Observation 1 gives that $P(cl(\{x\}) \cup D) = \{x\} \cup (P(D) \setminus (cl(\{x\}) \cap P(D)))$.

The proof proceeds by showing that a partition of the progenitors of $cl(\{x\}) \cup D$ corresponds to a partition of one of the monophyletic pieces of D. The monophyly of this piece is then inherited by $cl(\{x\}) \cup D$. Now consider an arbitrary partition of $P(cl(\{x\}) \cup D)$ into two non–empty sets X_1 and X_2 such that $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = P(cl(\{x\}) \cup D)$. Without loss of generality, suppose that $x \in X_1$. Consider an arbitrary $x_2 \in X_2$. Then $x_2 \neq x$ since $X_1 \cap X_2 = \emptyset$. This implies that $x_2 \in P(D)$ and x is not an ancestor of x_2. Since $P(D) = \bigcup_{1 \leq i \leq k} P(D_i)$, for some I, $1 \leq I \leq k$, $x_2 \in P(D_I)$. From X_1 and X_2, construct two sets Y_1 and Y_2 where

$$\begin{aligned} Y_1 &= (P(D_I) \cap cl(\{x\})) \cup (P(D_I) \cap X_1) \\ Y_2 &= P(D_I) \cap X_2 \end{aligned}$$

Then,

$$\begin{aligned} Y_1 \cup Y_2 &= (P(D_I) \cap cl(\{x\})) \cup (P(D_I) \cap X_1) \cup (P(D_I) \cap X_2) \\ &= (P(D_I) \cap cl(\{x\})) \cup (P(D_I) \cap (X_1 \cup X_2)) \\ &= (P(D_I) \cap cl(\{x\})) \cup \\ &\quad (P(D_I) \cap (\{x\} \cup (\bigcup_{1 \leq i \leq k} P(D_i) \setminus (\bigcup_{1 \leq i \leq k} P(D_i) \cap cl(\{x\}))) \\ &\quad \text{since } X_1 \text{ and } X_2 \text{ partition } P(cl(\{x\} \cup D) \\ &= (P(D_I) \cap cl(\{x\})) \cup \\ &\quad (P(D_I) \cap (\bigcup_{1 \leq i \leq k} P(D_i) \setminus (\bigcup_{1 \leq i \leq k} P(D_i) \cap cl(\{x\}))) \\ &= (P(D_I) \cap cl(\{x\})) \cup (P(D_I) \setminus (P(D_I) \cap cl(\{x\}))) \\ &= P(D_I) \end{aligned}$$

Also, $Y_1 \cap Y_2 = \emptyset$ since $X_1 \cap X_2 = \emptyset$ and any element of $(P(D_I) \cap cl(\{x\})$ is not an element of $P(cl(\{x\}) \cup D)$ and $Y_2 \subseteq P(cl\{x\} \cup D)$. Now, $Y_2 \neq \emptyset$ since $x_2 \in Y_2$. Suppose that $Y_1 \neq \emptyset$. Then, since D_I is monophyletic, by Observation 26, there exists $y_1 \in Y_1$, $y_2 \in Y_2$, $z \in D_I$ such that both y_1 and y_2 is an ancestor of z. Now, since $Y_2 \subseteq X_2$, $y_2 \in X_2$. If $y_1 \in P(D_I) \cap X_1$ then $y_1 \in X_1$. Otherwise $y_1 \in P(D_I) \cap cl(\{x\})$ and x will be an ancestor of y_1 with $x \in X_1$ being an ancestor of z. Either way, z is the descendant of an element an element in X_1 and an element of X_2.

However, it is possible that $Y_1 = \emptyset$. This happens when both $P(D_I) \cap cl(\{x\}) = \emptyset$ and $P(D_I) \cap X_1 = \emptyset$. When $P(D_I) \cap cl(\{x\}) = \emptyset$, $P(D_I) \subset P(cl(\{x\} \cup D$. Now since $P(D_I) \cap X_1 = \emptyset$ and also X_1 with X_2 partitioning $P(D_I) \subset P(cl(\{x\}) \cup D$, it obtains that $P(D_I) \subseteq X_2$. By the construction of Y_2, $Y_2 = P(D_I)$. By assumption, D_I and $cl(\{x\})$ have a non–empty intersection, i.e., there exists $z \in D_I \cap cl(\{x\})$. Then $x \in X_1$ is an ancestor of z. Now since D_I is a descent group and $z \in D_I$, by Observation 6 there exists a $y \in P(D_I)$ such that y is an ancestor of z. Now, since $Y_2 = P(D_I)$ and $Y_2 \subseteq X_2$, $y \in X_2$.

Therefore, by Observation 26, $cl(\{x\}) \cup \bigcup_{1 \leq i \leq k} D_i$ is a monophyletic group.
□

Observation 35 [Reducing Polyphyletic Degree] Consider a genealogical network $G = (X, p)$ and a descent group D in G which is polyphyletic of maximal degree k. For every $x \in X$ and witness $D_1, \ldots, D_M, \ldots, D_k$ to D being polyphyletic of degree k, if

1. $D_i \cap cl(\{x\}) = \emptyset$ for every i, $1 \leq i < M$
2. $D_i \cap cl(\{x\}) \neq \emptyset$ for every i, $M \leq i \leq k$,

then $cl(D \cup \{x\})$ is polyphyletic of maximal degree M.

Proof
Consider a genealogical network $G = (X, p)$ and a descent group D in G which is polyphyletic of maximal degree k. Suppose for some $x \in X$ there exists descent groups $D_1, \ldots, D_M, \ldots, D_k$ such that

1. $D_i \neq \emptyset$ for every i, $1 \leq i \leq k$
2. $D_i \cap D_j = \emptyset$ for every i, j, $1 \leq i < j \leq k$
3. $\bigcup_{1 \leq i \leq k} D_i = D$
4. $D_i \cap cl(\{x\}) = \emptyset$ for every i, $1 \leq i < M$
5. $D_i \cap cl(\{x\}) \neq \emptyset$ for every i, $M \leq i \leq k$

Consider the sequence $D_1, \ldots, D_{M-1}, (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)$. The aim is to show that this sequence is a witness to the fact that $cl(D \cup \{x\})$ is polyphyletic of degree M. Certainly $D_1, \ldots, D_{M-1}$ constitute a sequence of disjoint descent groups. As for $(cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)$, by Observation 9 $cl(\{x\})$ is a descent group and since descent groups are closed under set union repeated applications of Observation 3 will give that $(cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)$ is a descent group. Consider for an arbitrary J,

$1 \leq J \leq M-1$, $D_J \cap (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)$. Suppose this intersection is non–empty, i.e., for some $y \in X$, $y \in D_J$ and $y \in (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)$. Now, $y \notin cl(\{x\})$ by the assumption that $D_i \cap cl(\{x\}) = \emptyset$ for every i, $1 \leq i < M$. Then for some I, $M \leq I \leq k$, $y \in D_I$. This gives a non–empty intersection between D_I and D_J ($I \neq J$). This is a contradiction since $D_1, \ldots D_k$ is a partition of D. Hence, for every J, $1 \leq J \leq M-1$, $D_J \cap (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i) = \emptyset$.

Now, to show that $D_1, \ldots, D_{M-1}, (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)$ is a partition of $cl(D \cup \{x\})$ it is necessary to show that $\bigcup_{1 \leq j \leq M-1} D_j \cup (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i) = cl(D \cup \{x\})$.

$$
\begin{aligned}
cl(D \cup \{x\}) &= cl(D) \cup cl(\{x\}) \text{ since } cl \text{ distributes over set union (Observation 8)} \\
&= D \cup cl(\{x\}) \text{ since } D \text{ is a descent group by Observation 10, } cl(D) = D \\
&= D_1, \ldots, D_k \cup cl(\{x\}) \text{ since } \textstyle\bigcup_{1 \leq i \leq k} D_i = D \\
&= \bigcup_{1 \leq j \leq M-1} D_j \cup (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)
\end{aligned}
$$

Thus the sequence $D_1, \ldots, D_{M-1}, (cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_i)$ is a witness to $cl(D \cup \{x\})$ being polyphyletic of degree M. Now, since D is polyphyletic to degree k and not polyphyletic of degree $k+1$, by Observation 33, for every i, $1 \leq i \leq k$, D_i is monophyletic; especially for $i \geq M$. Now, since $cl\{x\} \cap D_i \neq \emptyset$, for every i, $M \leq i \leq k$, by Observation 34, $cl(\{x\}) \cup \bigcup_{M \leq i \leq k} D_I$ is a monophyletic group. Then, since $D_1, \ldots, D_{M-1}, cl\{x\} \cup \bigcup_{M \leq i \leq k} D_I$ witness that $cl(D \cup \{x\})$ is polyphyletic of degree M, it follows by Observation 33 that $cl(D \cup \{x\})$ is not polyphyletic of degree $M+1$. $\square$

Observation 36 [Enlarging Monophyletic Descent Groups] Consider a genealogical network $G = (X, p)$ and a descent group D in G. For every $x \in X$, if D is a monophyletic group and $D \cap cl(\{x\}) \neq \emptyset$, then $cl(D \cup \{x\})$ is a monophyletic group.

Proof
Consider a genealogical network $G = (X, p)$ and a descent group D in G. Let $x \in X$ be arbitrary. Suppose that D is monophyletic and $cl(\{x\}) \cap D \neq \emptyset$.

Consider the case where $x \in D$. Then by Corollary 1, $P(D \cup cl(\{x\})) = P(D)$. Then, by Observation 16, $D \cup cl(\{x\}) = D$. Hence, since D is monophyletic, $D \cup cl(\{x\})$ is monophyletic.

Consider the case where $x \notin D$. By Corollary 1, $P(D \cup cl(\{x\})) = \{x\} \cup (P(D) \setminus (cl(\{x\}) \cap P(D)))$. Let X_1 and X_2 be an arbitrary partition of $P(cl(D \cup \{x\}))$ such that $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = P(D \cup cl(\{x\}))$. From X_1 and X_2 construct the sets Y_1 and Y_2 where $Y_2 = X_2$ and $Y_1 = (X_1 \setminus \{x\}) \cup (P(D) \cap cl(\{x\}))$. Then Y_2 is non–empty (since $X_2 = Y_2$ is non–empty) and together Y_1 with Y_2 split $P(D)$. This is because

$$
\begin{aligned}
Y_1 \cup Y_2 &= (X_1 \setminus \{x\}) \cup (P(D) \cap cl(\{x\})) \cup X_2 \\
&= ((X_1 \cup X_2) \setminus \{x\}) \cup (P(D) \cap cl(\{x\})) \text{ since } x \notin X_2 \\
&= (\{x\} \cup (P(D) \setminus (cl(\{x\}) \cap P(D)))) \setminus \{x\}) \cup (P(D) \cap cl(\{x\})) \\
&\quad \text{since } X_1 \text{ and } X_2 \text{ partition } P(cl(D \cup \{x\})) \\
&= (P(D) \setminus (cl(\{x\}) \cap P(D))) \cup (P(D) \cap cl(\{x\}))
\end{aligned}
$$

$$= \quad P(D)$$

Now, it is possible that $Y_1 = \emptyset$. This happens exactly when $X_1 = \{x\}$ and $P(D) \cap cl(\{x\}) = \emptyset$. Which implies that $X_2 = P(D)$. Now, since $cl(\{x\}) \cap D \neq \emptyset$. Then there exists a $z \in cl(\{x\}) \cap D$ such that x (an element of X_1) is an ancestor of z and some $y \in P(D)$ (an element of X_2) such that y is an ancestor of z. Thus $D \cup cl(\{x\})$ is monophyletic.

Suppose $Y_1 \neq \emptyset$. Now since D is monophyletic, for some $y_1 \in Y_1$, $y_2 \in Y_2$, $z \in D$, both y_1 and y_2 are ancestors of z. Since $Y_2 = X_2$, $y_2 \in X_2$. There are two cases for y_1. Either $y_1 \in X_1$ or $y_1 \in (P(D) \cap cl(\{x\}))$. Consider the case where $y_1 \in (P(D) \cap cl(\{x\}))$. Then x is an ancestor of y_1. Either way, for some $x_1 \in X_1$, $y_2 \in X_2$, both x_1 and y_2 are ancestors of z. Thus, by Observation 26, $cl(D \cup \{x\})$ is monophyletic. $\square$

Observation 37 [Monophyletic Union] Consider a genealogical network $G = (X, p)$ and two non–empty monophyletic groups D_1 and D_2 in G. $D_1 \cup D_2$ is monophyletic if and only if $D_1 \cap D_2 \neq \emptyset$.

Proof
Consider a genealogical network $G = (X, p)$ and two non–empty monophyletic groups D_1 and D_2 in G.

If $D_1 \cap D_2 = \emptyset$, then D_1 and D_2 are witness to $D_1 \cup D_2$ being polyphyletic.

Suppose that $D_1 \cap D_2 \neq \emptyset$. To show that $D_1 \cup D_2$ is monophyletic, we gradually enlarge D_1 with the progenitors of D_2. At each stage the construction will be a monophyletic descent group. The last construction will be $D_1 \cup D_2$. Impose an index on the elements of $P(D_2) = \{x_1, \ldots, x_k\}$. Also, form sequences of sets $(D_i^{'})$ and (Y_i) where $D_0^{'} = D_1$ and $Y_0 = \emptyset$. Inductively define $D_i^{'}$ and Y_i as follows:

$$\begin{aligned} Y_{i+1} &= Y_i \cup \{x_J\} \text{where } J \text{ is the smallest value such that } x_J \in P(D_2) \setminus Y_i \text{ and} \\ & \quad D_i^{'} \cap cl(\{x_J\}) \neq \emptyset \\ D_{i+1}^{'} &= D_i^{'} \cup cl(Y_{i+1}) \end{aligned}$$

where $i + 1 \leq k$.

From the structure of the construction, it is possible that the sequence is not well–defined. Consider Y_1 and $D_1^{'}$. Since $D_1 \cap D_2 \neq \emptyset$, there exists a $x_J \in P(D_2)$ such that $D_1 \cap cl(\{x_J\}) \neq \emptyset$. Hence Y_1 is well defined and, consequently, $D_1^{'}$ is well defined. Moreover, Y_1 has 1 element. Suppose $D_i^{'}$ and Y_i are well defined and Y_i has i elements for every i, $1 \leq i < k$. Consider the partition of $P(D_2)$ into Y_i and $P(D_2) \setminus Y_i$. Y_i is non–empty since $1 \leq i$ and $P(D_2) \setminus Y_i$ is non–empty since $i < k$. Then, since D_2 is monophyletic, there exists a $z \in D_2$, $x_L \in Y_i$ and $x_M \in P(D_2) \setminus Y_i$ such that both x_L and x_M are ancestors of z. Since $Y_i \subseteq D_i^{'}$, $D_i^{'} \cap cl(\{x_M\}) \neq \emptyset$. Hence Y_{i+1} and $D_{i+1}^{'}$ are well defined. Moreover, Y_{i+1} has $i+1$ elements since $x_M \notin Y_i$ ($x_M \in P(D_2) \setminus Y_i$).

For every i, $0 \leq i \leq k$, $Y_i \subseteq P(D_2)$. $Y_0 = \emptyset$. Thus $Y_0 \subseteq P(D_2)$. Suppose $Y_I \subseteq P(D_2)$ for some I, $0 \leq I < k$. Then $Y_{I+1} = Y_I \cup \{x\}$ for some $x \in P(D_2)$. Thus $Y_{I+1} \subseteq P(D_2)$. Now, $P(D_2)$ has k elements. Then, since Y_k has k elements and $Y_k \subseteq P(D_2)$, $Y_k = P(D_2)$. Moreover, since $D_k^{'} = D_{k-1}^{'} \cup cl(Y_k) = D_1 \cup D_2$ – since by Observation 15 $D_2 = cl(P(D_2))$.

For every i, $0 \leq i \leq k$, $D_i^{'}$ is monophyletic. By definition $D_0^{'} = D_1$. Since D_1 is monophyletic $D_0^{'}$ is monophyletic. Suppose that $D_I^{'}$, for some I, $0 \leq I < k$. Then

$$\begin{aligned} D_{I+1}^{'} &= D_I^{'} \cup cl(Y_{I+1}) \\ &= D_I^{'} \cup cl(Y_I \cup (Y_{I+1} \setminus Y_I)) \text{ by construction} \\ &= D_I^{'} \cup cl(Y_I) \cup cl(Y_{I+1} \setminus Y_I) \text{ since } cl \text{ distributes over union (Observation 8)} \\ &= D_I^{'} \cup cl(\{x\}) \text{ for some } x \in P(D_2) \text{ (by construction)} \end{aligned}$$

Now by Observation 3, $cl(\{x\})$ is monophyletic. Also, since D_I is monophyletic and $D_I \cap cl(\{x\}) \neq \emptyset$, by Observation 36, D_{I+1} is monophyletic. By induction $D_i^{'}$ is monophyletic for every i, $1 \leq i \leq k$.

Now, since $D_k^{'} = D_1 \cup D_2$, $D_1 \cup D_2$ is monophyletic. $\square$

Observation 38 [Monophyly and General Union] Consider a genealogical network $G = (X, p)$ and descent groups $D_1, \ldots D_k$ in G. Suppose for every i, $1 \leq i \leq k$, D_i is non–empty and monophyletic. Then, $\bigcup_{1 \leq i \leq k} D_i$ is monophyletic if and only if there exists a permutation ϕ of $\{1, \ldots, k\}$ such that for every j, $1 \leq j < k$, $\bigcup_{1 \leq i \leq j} D_{\phi(i)} \cap D_{\phi(j+1)} \neq \emptyset$.

Proof

Consider a genealogical network $G = (X, p)$ and descent groups $D_1, \ldots D_k$ in G. Suppose for every i, $1 \leq i \leq k$, D_i is monophyletic.

($\Leftarrow$) Suppose there exists a permutation ϕ of $\{1, \ldots, k\}$ such that for every j, $1 \leq j < k$, $\bigcup_{1 \leq i \leq j} D_{\phi(i)} \cap D_{\phi(j+1)} \neq \emptyset$. Since ϕ is a permutation on $\{1, \ldots, k\}$, quite clearly $\bigcup_{1 \leq i \leq k} D_i = \bigcup_{1 \leq i \leq k} D_{\phi(i)}$. To prove that $\bigcup_{1 \leq i \leq k} D_{\phi(i)}$ is monophyletic, perform induction on the number of descent groups. Let $P(I)$ be the proposition that $\bigcup_{1 \leq i \leq I} D_{\phi(i)}$ is monophyletic. Certainly $P(1)$ is true since $D_{\phi(1)}$ is monophyletic. Suppose $P(J)$ is true for some J, $1 \leq J < k$. Consider $\bigcup_{1 \leq i \leq J+1} D_{\phi(i)}$. This set equals $\bigcup_{1 \leq i \leq J} D_{\phi(i)} \cup D_{\phi(J+1)}$. Now, since $P(J)$ is true, $\bigcup_{1 \leq i \leq J} D_{\phi(i)}$ is monophyletic. Also, $D_{\phi(J+1)}$ is monophyletic. By assumption, $\bigcup_{1 \leq i \leq j} D_{\phi(i)} \cap D_{\phi(j+1)} \neq \emptyset$ for every j, $1 \leq j \leq k$. Now applying Observation 37 gives that $\bigcup_{1 \leq i \leq J} D_{\phi(i)} \cup D_{\phi(J+1)}$ is monophyletic. Thus $P(J+1)$ is true. By, induction $P(I)$ is true for all I, $1 \leq I \leq k$. By $P(k)$, $\bigcup_{1 \leq i \leq k} D_i$ is monophyletic.

($\Rightarrow$) Suppose that $\bigcup_{1 \leq i \leq k} D_i$ is monophyletic. Construct a function ϕ from $\{1, \ldots, k\}$ to $\{1, \ldots, k\}$ inductively. Let $\phi(1) = 1$. Now, assume that ϕ is defined for every i, $1 \leq i \leq J$ for some $J \leq k - 1$. Consider the descent groups, $\bigcup_{1 \leq i \leq J} D_{\phi(i)}$ and $\bigcup_{D \in \{D_1, \ldots, D_k\} \setminus \{D_{\phi(i)} | 1 \leq i \leq J\}} D$. Neither descent group is empty, since $1 \leq J \leq k-1$. Moreover, the two descent groups intersect because otherwise, $\bigcup_{1 \leq i \leq k} D_i$ is polyphyletic. Hence there exists a $M \in \{1, \ldots, k\} \setminus \{\phi(i) \mid 1 \leq i \leq J\}$ such that $\bigcup_{1 \leq i \leq J} D_{\phi(i)} \cap D_M \neq \emptyset$. Let $\phi(J+1) = N$ where N is the smallest value in $\{1, \ldots, k\} \setminus \{\phi(i) \mid 1 \leq i \leq J\}$ such that $\bigcup_{1 \leq i \leq J} D_{\phi(i)} \cap D_N \neq \emptyset$. This inductive procedure generates a permutation on $\{1, \ldots, k\}$. Moreover, for every j, $1 \leq j < k$, $\bigcup_{1 \leq i \leq j} D_{\phi(i)} \cap D_{\phi(j+1)} \neq \emptyset$. Now, let $P(I)$ be the proposition that

$\bigcup_{1 \leq i < I} D_{\phi(i)} \cap D_N \neq \emptyset$ for some $D_N \in \{D_1, \ldots, D_k\} \setminus \{D_{\phi(i)} \mid 1 \leq i \leq I\}$. Consider $P(1)$ if $1 < k$. Now, $\bigcup_{1 \leq i \leq k} D_i = D_1 \cup \bigcup_{2 \leq i \leq k} D_i$. Since $\bigcup_{1 \leq i \leq k} D_i$ is monophyletic, $D_1 \cap \bigcup_{2 \leq i \leq k} D_i \neq \emptyset$ (otherwise D_1 and $\bigcup_{2 \leq i \leq k} D_i$ are witness to $\bigcup_{1 \leq i \leq k} D_i$ being polyphyletic – $\bigcup_{2 \leq i \leq k} D_i$ is a descent group by Observation 3). Since $D_1 \cap \bigcup_{2 \leq i \leq k} D_i \neq \emptyset$, for some J, $2 \leq J \leq k$, $D_1 \cap D_J \neq \emptyset$. Let $\phi(2) = N$ where N is the smallest value between 2 and k such that $D_1 \cap D_N \neq \emptyset$. Thus $P(1)$ is true. Suppose $P(J)$ is true for some J, $1 \leq J < k - 1$. Then $\phi(J+1)$ is constructed such that $\bigcup_{1 \leq i \leq J+1} D_{\phi(i)} \cap D_{\phi(J+2)} \neq \emptyset$. Thus $P(J+1)$ is true. Hence by induction, for every j, $1 \leq j < k$, $\bigcup_{1 \leq i \leq j} D_{\phi(i)} \cap D_{\phi(j+1)} \neq \emptyset$. $\square$

Observation 39 [Paraphyletic Set] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Then E is *not* a descent group.

Proof
Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Consider an arbitrary witness $(D, D^{'})$ to E. For a proof by contradiction, suppose E is a descent group. Since E is a paraphyletic group $E \neq \emptyset$. Moreover, by Observation 3, $D \cap D^{'}$ is a descent group. Notice that E and $D \cap D^{'}$ are witness to D being polyphyletic since both sets are non–empty descent groups. Also, $E \cup (D \cap D^{'}) = D \setminus (D \cap D^{'}) \cup (D \cap D^{'}) = D$. This is a contradiction since D is monophyletic. Thus E is not a descent group. $\square$

Observation 40 [Paraphyletic Witness Constraints] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Suppose that $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E. Then,

1. for every $x \in E, y \in D_1$, if y is an ancestor of x, then $y \in E$
2. there exists an $x \in P(D_1)$ such that $x \in E$
3. $P(D_1) \cap E = P(D_2) \cap E$
4. for every $x_1 \in D_1, x_2 \in D_2$, if $x_2 \notin D_1$ and x_2 is an ancestor of x_1, then $x_1 \in D_1^{'}$.
5. if $D_1 \subseteq D_2$, then $D_1^{'} \subseteq D_2^{'}$

Proof
Consider a genealogical network $G = (X, p)$ and a non–empty paraphyletic group E in G. Suppose that $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E.

1. Consider an arbitrary $x \in E$ and $y \in D_1$. Suppose that y is an ancestor of x. For a proof by contradiction, suppose $y \notin E$. Then $y \in D_1^{'}$. Since $D_1^{'}$ is a descent group, $x \in D_1^{'}$. Hence $x \notin E$. Thus contradiction gives that $y \in E$.
2. Since E is a paraphyletic group, $E \neq \emptyset$. For a proof by contradiction, suppose that $P(D_1) \cap E = \emptyset$. Then $P(D_1) \subseteq D_1^{'}$. Since $D_1^{'}$ is a descent group, it follows that $D_1 \subseteq D_1^{'}$ and thus $E = \emptyset$. This contradiction gives that there exists an $x \in P(D_1)$ such that $x \in E$.
3. Consider an arbitrary $x \in P(D_1) \cap E$. Since $x \in E$, $x \in D_2$ since $(D_2, D_2^{'})$ is a witness to E. Consider an arbitrary $y \in X$ such that $(y, x) \in p$. Then $y \notin E$ since $y \notin D_1$ and $(D_1, D^{'})$ is a witness to E. Clearly, if $y \notin D_2$, then

$x \in P(D_2)$. Consider whether it is possible that $y \in D_2$. Since $y \notin E$, it must be the case that y is removed and is an element of $D_2^{'}$. If this is the case, then $D_2^{'}$ must also contain x since $D_2^{'}$ is a descent group. Hence, $y \notin D_2$ and $x \in P(D_2)$. Moreover, $x \in P(D_2) \cap E$. By symmetry, $P(D_2) \cap E \subseteq P(D_1) \cap E$. Thus $P(D_1) \cap E = P(D_2) \cap E$.

4. Consider an arbitrary $x_1 \in D_1$ and $x_2 \in D_2$. Suppose that x_2 is an ancestor of x_1 and $x_2 \notin D_1$. Since $x_2 \notin D_1$, $x_2 \notin E$. Thus $x_2 \in D_2^{'}$. Since $D_2^{'}$ is a descent group and x_1 is a descendant of x_2, $x_1 \in D_2^{'}$. Thus $x_1 \notin E$. Hence, $x_1 \in D_1^{'}$.

5. Suppose that $D_1 \subseteq D_2$. Let $x \in D_1^{'}$ be arbitrary. Then, $x \in D_1$ and $x \notin E$. Since $D_1 \subseteq D_2$, $x \in D_2$. Moreover, $x \in D_2^{'}$ since $x \notin E$. Hence, $D_1^{'} \subseteq D_2^{'}$.

□

Observation 41 [Witness Set Constraints] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Then,

1. $E \subseteq \bigcap_{(D,D')\in[E]} D$
2. $E \cap \bigcup_{(D,D')\in[E]} D^{'} = \emptyset$

Proof
Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G.

1. Consider an arbitrary $(D, D^{'}) \in [E]$. Then $E = D \setminus D^{'}$. Hence $E \subseteq D$. Thus, $E \subseteq \bigcap_{(D,D')\in[E]} D$.

2. Consider an arbitrary $(D, D^{'}) \in [E]$. Then $E = D \setminus D^{'}$. Hence $E \cap D^{'} = \emptyset$. Thus, $E \cap \bigcup_{(D,D')\in[E]} D^{'} = \emptyset$.

□

Observation 42 [Paraphyletic Witness Structure] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Suppose that $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E. Then,

1. $(D_1 \cup D_2, D_1^{'} \cup D_2^{'}) \in [E]$ and is larger than $(D_1, D_1^{'})$
2. $(D_1, D_1^{'} \cup D_2^{'}) \in [E]$ and is larger than $(D_1, D_1^{'})$
3. if $D_1 \cap D_2$ is monophyletic, then $(D_1 \cap D_2, D_1^{'}) \in [E]$ and is smaller than $(D_1, D_1^{'})$
4. if $D_1 \cap D_2$ is monophyletic, then $(D_1 \cap D_2, D_1^{'} \cup D_2^{'}) \in [E]$

Proof
Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Suppose that $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E.

1. Let $D = D_1 \cup D_2$ and $D^{'} = D_1^{'} \cup D_2^{'}$. Consider an arbitrary element x of E. Since $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E, $x \in D_1 \cap D_2$ and $x \notin D_1^{'} \cup D_2^{'}$. Hence $x \in (D_1 \cup D_2) \setminus (D_1^{'} \cup D_2^{'})$. Thus $x \in D \setminus D^{'}$.

   Consider an arbitrary element x of $D \setminus D^{'}$. Then $x \in D_1 \cup D_2$ and $x \notin D_1^{'} \cup D_2^{'}$.

Without loss of generality, suppose that $x \in D_1$. Since $(D_1, D_1^{'})$ is a witness to E, it obtains that $x \in E$.

Since D_1 and D_2 are monophyletic groups with a non–empty intersection (since $E \neq \emptyset$), $D = D_1 \cup D_2$ is a monophyletic group by Observation 37. Also $D_1^{'} \cup D_2^{'}$ is a descent group by Observation 3. Hence $(D, D^{'})$ is a witness to E. Moreover, by construction $D_1 \subseteq D$ and $D_1^{'} \subseteq D^{'}$. Thus, $(D_1, D_1^{'})$ is smaller than $(D, D^{'})$.

2. Let $D = D_1$ and $D^{'} = D_1^{'} \cup D_2^{'}$. Consider an arbitrary $x \in E$. Then $x \in D_1$ and $x \notin D_1^{'} \cup D_2^{'}$. Hence, $x \in D \setminus D^{'}$.

   Consider an arbitrary $x \in D \setminus D^{'}$. Then $x \in D_1$ and $x \notin D_1^{'} \cup D_2^{'}$. Hence $x \notin D^{'}$ and $x \in E$. Moreover, since D_1 is monophyletic and descent groups are closed under set union (Observation 3), $(D, D^{'})$ is a witness to E. Clearly, by construction $D = D_1$ and $D^{'} \supseteq D_1^{'}$; implying that $(D, D^{'})$ is larger than $(D_1, D_1^{'})$.

3. Let $D = D_1 \cap D_2$ and $D^{'} = D_1^{'}$. Suppose $D_1 \cap D_2$ is monophyletic. Consider an arbitrary element x of E. Since $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E, $x \in D_1 \cap D_2$ and $x \notin D_1^{'}$. Thus $x \in D \setminus D^{'}$.

   Consider an arbitrary element x of $D \setminus D^{'}$. Then $x \in D_1 \cap D_2$ and $x \notin D_1^{'}$. Hence $x \in D_1$ and $x \in E$.

   Thus $E = D \setminus D^{'}$. Moreover, by Observation 3, D is a descent group and monophyletic by assumption. Also, $D^{'} = D_1^{'}$ is a descent group. Thus $(D, D^{'})$ is a witness to E. Moreover, by construction, $D \subseteq D_1$ and $D^{'} = D_1^{'}$. Hence, $(D, D^{'})$ is smaller than $(D_1, D_1^{'})$.

4. Let $D = D_1 \cap D_2$ and $D^{'} = D_1^{'} \cup D_2^{'}$. Suppose $D_1 \cap D_2$ is monophyletic. Consider an arbitrary element x of E. Since $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E, $x \in D_1 \cap D_2$ and $x \notin D_1^{'} \cup D_2^{'}$. Thus $x \in D \setminus D^{'}$.

   Consider an arbitrary $x \in D \setminus D^{'}$. Then $x \in D_1 \cap D_2$ and $x \notin D_1^{'} \cup D_2^{'}$. Thus $x \in D_1$ and $x \notin D_1^{'}$. This implies that $x \in E$ and $E = D \setminus D^{'}$. Moreover, since $D_1 \cap D_2$ by assumption is monophyletic, and $D^{'}$ is a descent group (Observation 3), $(D, D^{'})$ is a witness to E.

$\square$

Observation 43 [Smallest Paraphyletic Degree] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Then E is paraphyletic of degree 1.

Proof

Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Let $(D, D^{'})$ be a witness to E. Since $D \cap D^{'} \neq \emptyset$, $D^{'} \neq \emptyset$. Then, by Observation 27, $D^{'}$ is polyphyletic of degree 1. Hence E is paraphyletic of degree 1. $\square$

Observation 44 [Lower Paraphyletic Degrees Preserved] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. If E is paraphyletic of degree k, then for every l, $1 \leq l < k$, E is paraphyletic of degree l.

Proof
Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Suppose E is paraphyletic of degree k. Then, there exists a witness $(D, D^{'})$ of E such that $D^{'}$ is polyphyletic of degree k. Then by Observation 30, for every l, $1 \leq l < k$, $D^{'}$ is polyphyletic of degree l. Thus, for every l, $1 \leq l < k$, E is paraphyletic of degree l. □

Observation 45 [Progenitors of the Canonical Weak Witness] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. If $(D_E, D^{'}_E)$ is the canonical weak witness of E, then $P(D_E) = P(E)$.

Proof
Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Now, consider arbitrary elements $a, b \in P(E)$ such that $a \neq b$. Suppose it is possible that a is an ancestor of b. Let $x_1, x_2, \ldots, x_n$ be a path from a to b where $a = x_1$ and $b = x_n$. Since $a \neq b$, $n \geq 2$ and x_{n-1} is well defined. Since b is a progenitor of E, $x_{n-1} \notin E$. Consider an arbitrary witness $(D, D^{'})$ to E. Since $a \in E$ and D is a descent group, a, b, and x_{n-1} are elements of D. Since $x_{n-1} \notin E$, $x_{n-1} \in D^{'}$. Now, since $D^{'}$ is a descent group $b \in D^{'}$. Thus $b \notin E$ – contradicting that b is a progenitor of E. Thus a is not an ancestor of b. By Observation 11, this implies that $P(E)$ is a minimal generating set. Now, using Definition 9 we can obtain the progenitors of D_E, viz., $P(D_E) = P(E)$. □

Observation 46 [Canonical Weak Witnesses Contained in Witnesses] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Let $(D_E, D^{'}_E)$ be the canonical weak witness of E and $(D, D^{'})$ an arbitrary witness to E. Then $D_E \subseteq D$ and $D^{'}_E \subseteq D^{'}$.

Proof
Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Let $(D_E, D^{'}_E)$ be the canonical weak witness of E and $(D, D^{'})$ an arbitrary witness to E. Consider an arbitrary element $x \in D_E$. Then there exists a progenitor y of D_E such that y is an ancestor of x. Now $P(D_E) \subseteq E$ which implies that $P(D_E) \subseteq D$. Since D is a descent group and $y \in D$, $x \in D$. Thus $D_E \subseteq D$.

Now, consider an arbitrary element x of $D^{'}_E$. By construction, $x \in D_E$ and $x \notin E$. Since $x \in D_E$, $x \in D$. Since $x \notin E$, $x \in D^{'}$. Thus $D^{'}_E \subseteq D^{'}$. □

Observation 47 [Canonical Weak Witness A Weak Witness] Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Then the weak canonical witness of E, $(D_E, D^{'}_E)$ is a weak witness of E.

Proof
Consider a genealogical network $G = (X, p)$ and a paraphyletic group E in G. Let $(D_E, D^{'}_E)$ be the canonical weak witness of E. Firstly, consider D_E. Since cl generates descent groups (Observation 9), $D_E = cl(P(E))$ is a descent group.

Now, consider $D^{'}_E = D_E \setminus E$. Consider an arbitrary $x \in D^{'}_E$ and $y \in X$ such that x is an ancestor of y. Since $x \in D^{'}_E$, $x \in D_E$ and $x \notin E$. Since $x \in D_E$ and D_E is a descent group, $y \in D_E$. Also, let $(D, D^{'})$ be an arbitrary witness E. By

Observation 46, $D_E \subseteq D$ – which implies that $x, y \in D$. Now, since $x \notin E$, it must be the case that $x \in D^{'}$. Moreover, since $D^{'}$ is a descent group, $y \in D^{'}$. This would mean that $y \notin E$. Thus $D^{'}_E = D_E \setminus E$ would contain y. Hence $D^{'}_E$ is a descent group.

Clearly, by construction $D^{'}_E \subseteq D_E$. Similarly, by construction $D^{'}_E = D_E \setminus E$. Since $E \subseteq D_E$, it obtains that $D_E = D^{'}_E \cup E$. Also, since $E \cap D^{'}_E = \emptyset$, $E = D_E \setminus D^{'}_E$. Since E is a paraphyletic set, $D_E \cap D^{'}_E \neq \emptyset$ (otherwise E is a descent group). Hence $(D_E, D^{'}_E)$ is a weak witness to E.

□

C Proofs For Section 4

Observation 48 [Presence of Terminals] Consider a genealogical network $G = (X, p)$. For every $A \subseteq X$, $Term(cl(A)) = \emptyset$ if and only if $A = \emptyset$.

Proof
Consider a genealogical network $G = (X, p)$. Let $A \subseteq X$ be arbitrary.

Suppose that $A = \emptyset$. Then $cl(A) = \emptyset$. Since $Term(cl(A)) \subseteq cl(A)$, it obtains that $Term(cl(A)) = \emptyset$.

Suppose that $A \neq \emptyset$. For a proof by contradiction, suppose that $Term(cl(A)) = \emptyset$. By the definition of a terminal set (Definition 24), this means that for every $x \in cl(A)$, there exists a $y \in X$ such that $(x, y) \in p$ – every element of $cl(A)$ has a child. Now since $cl(A)$ is a descent group, this implies that for every $x \in cl(A)$, there exists a $y \in cl(A)$ such that $(x, y) \in p$. Let $n = \mid cl(A) \mid$. Since $A \neq \emptyset$, let x_1 be an arbitrary element of A. Since cl is a closure operator (Observation 7), $x_1 \in cl(A)$. Now form a sequence $x_1, x_2, \ldots x_{n+1}$ such that $(x_i, x_{i+1}) \in p$ for every i, $1 \leq i \leq n$. It is possible to construct such a sequence since every element of $cl(A)$ is the parent of another element in $cl(A)$. Now, if every element of this sequence is unique, then $\mid cl(A) \mid = n + 1$. This would contradict that $\mid cl(A) \mid = n$. Thus for $x_I = x_J$ for some I and J, $1 \leq I < J \leq n + 1$. This contradicts that G is acyclic. Thus, $Term(cl(A)) \neq \emptyset$. □

Observation 49 [Terminal Sets and Ancestors] Consider a genealogical network $G = (X, p)$ and two individuals x and y in X. If x is an ancestor of y, then

$$Term(cl(\{y\})) \subseteq Term(cl(\{x\}))$$

Proof
Consider a genealogical network $G = (X, p)$ and two individuals x and y in X. Suppose x is an ancestor of y. Let t be an arbitrary terminal in $Term(cl(\{y\}))$. Then, y is an ancestor of t (Definition 8). Hence, by the transitivity of the ancestor relationship (Observation 1), x is an ancestor of t and $t \in Term(cl(\{x\}))$. Therefore, $Term(cl(\{y\})) \subseteq Term(cl(\{x\}))$. □

Observation 50 [Terminal Set Properties] Consider a genealogical network $G = (X, p)$ and descent groups D_1 and D_2 in G. Then,

1. $Term(D_1 \cup D_2) = Term(D_1) \cup Term(D_2)$
2. $Term(D_1 \cap D_2) = Term(D_1) \cap Term(D_2)$
3. if $D_1 \subseteq D_2$, then $Term(D_1) \cap Term(D_2)$

Proof
Consider a genealogical network $G = (X, p)$ and descent groups D_1 and D_2 in G.

1. Let t be an arbitrary element of $Term(D_1 \cup D_2)$. This happens exactly when $t \in D_1 \cup D_2$ and for every $x \in X$, $(t, x) \notin p$. Which is equivalent to:

   (a) $t \in D_1$ and for every $x \in X$, $(t, x) \notin p$, *or*

   (b) $t \in D_2$ and for every $x \in X$, $(t, x) \notin p$

   Which is then equivalent to $t \in Term(D_1)$ or $t \in Term(D_2)$, i.e., $t \in Term(D_1) \cup Term(D_2)$.

2. Let t be an arbitrary element of $Term(D_1 \cap D_2)$. This happens exactly when $t \in D_1 \cap D_2$ and for every $x \in X$, $(t, x) \notin p$. Which is equivalent to:

   (a) $t \in D_1$ and for every $x \in X$, $(t, x) \notin p$, *and*

   (b) $t \in D_2$ and for every $x \in X$, $(t, x) \notin p$

   Which is exactly, $t \in Term(D_1)$ and $t \in Term(D_2)$, i.e., $t \in Term(D_1) \cap Term(D_2)$.

3. Suppose that $D_1 \subseteq D_2$. Let t be an arbitrary element of $Term(D_1)$. Then $t \in D_1$ and for every $x \in X$, $(t, x) \notin p$. Since $D_1 \subseteq D_2$, $t \in D_2$. Thus $t \in Term(D_2)$. Hence, $Term(D_1) \subseteq Term(D_2)$.

□

Observation 51 [Class of Descent Groups Non-empty] Consider a genealogical network $G = (X, p)$ and a terminal group T in G. The class of descent groups for T is non-empty because $T \in [T]$.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal group T in G. T is certainly a descent group since T is a subset of X and every element of T has no descendants. Moreover, the terminals in T are exactly T itself. □

Observation 52 [Non-trivial Terminal Group Polyphyletic] Consider a genealogical network $G = (X, p)$ and a terminal group T in G. If $T \neq \emptyset$, T is a polyphyletic group of maximal degree $\mid T \mid$.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal group T in G. Suppose $T \neq \emptyset$. Since T is finite, let $T = \{t_1, \ldots, t_k\}$ where $k \geq 1$ and $k = \mid T \mid$. Let $D_i = \{t_i\}$ for integer i ranging from 1 to k. Clearly each $D_i \neq \emptyset$ and $\bigcup_{1 \leq j \leq k} D_j = T$. Moreover, $D_i \cap D_j = \emptyset$ for every $1 \leq i < j \leq k$. Since each D_i consists of a single terminal, and t_i has no descendants, D_i is a descent group. Therefore T is polyphyletic of degree k.

With only k terminals, T can only be partitioned into k non–empty non–intersecting pieces. To show that T is not polyphyletic of degree $k+1$, assume the contrary. Suppose that T is polyphyletic of degree $k+1$. Then by Definition 14, there exists descent groups $D_1, \ldots D_{k+1}$ such that:

1. $D_i \neq \emptyset$, for every i, $1 \leq i \leq k+1$
2. $D_i \cap D_j = \emptyset$ for every i, j, $1 \leq i < j \leq k+1$
3. $\bigcup_{1 \leq i \leq k+1} D_i = T$

Since each D_i is non–empty, by Observation 48 and Observation 16, there exists a $t_i \in D_i$ such that $t_i \in T$. Collecting these witness terms gives that $\mid Term(T) \mid \geq k+1$. This contradiction gives that T is not polyphyletic of degree $k+1$. $\square$

Observation 53 [Subsets and Polyphyletic Degree in $[T]$] Consider a genealogical network $G = (X, p)$, a terminal group T in G, and two descent groups D_1 and $D_2 \in [T]$. If $D_1 \subseteq D_2$ and D_2 is polyphyletic of degree m, then D_1 is polyphyletic of degree m.

Proof
Consider a genealogical network $G = (X, p)$, a terminal group T in G, and two descent groups D_1 and $D_2 \in [T]$. Suppose that $D_1 \subseteq D_2$ and D_2 is polyphyletic of degree m. Since D_2 is polyphyletic of degree m, there are descent groups $D_1^{'}, \ldots D_m^{'}$ such that:

1. $D_i^{'} \neq \emptyset$ for every i, $1 \leq i \leq m$
2. $D_i^{'} \cap D_j^{'} = \emptyset$ for every i, j, $1 \leq i < j \leq m$
3. $\bigcup_{1 \leq i \leq m} D_i^{'} = D_2$

From this partition, construct a parallel sequence of m sets as follows:

$$D_i^{''} = \{x \in D_i^{'} \mid x \in D_1\}$$

for every i, $1 \leq i \leq m$. Each $D_i^{''}$ is just a restriction of $D_i^{'}$ to elements of D_1.

Firstly, each set $D_i^{''} \neq \emptyset$ for i, $1 \leq i \leq m$. Consider a fixed I, $1 \leq I \leq m$. Since $D_I^{'}$ is a non–empty descent group, by Observation 48 and Observation 17, there exists a terminal $t \in D_I^{'}$. Now, since $D_2 \in [T]$, this implies that $t \in T$. Then, since $D_1 \in [T]$, $t \in D_1$. Hence $t \in D_I^{''}$; implying that each set $D_i^{''} \neq \emptyset$ for every i, $1 \leq i \leq m$.

By construction, $D_i^{''} \subseteq D_i^{'}$ for every i, $1 \leq i \leq m$. Since $D_i^{'} \cap D_j^{'} = \emptyset$, for every i, j, $1 \leq i < j \leq m$, $D_i^{''} \cap D_j^{''} = \emptyset$, for every i, j, $1 \leq i < j \leq m$.

Let x be an arbitrary element of D_1. Since $D_1 \subseteq D_2$, $x \in D_2$. Since $D_1^{'}, \ldots, D_m^{'}$ is a partition of D_2, for some I, $1 \leq I \leq m$, $x \in D_I^{'}$. Moreover, by construction, $x \in D_I^{''}$. Hence, $\bigcup_{1 \leq i \leq m} D_I^{''} = D_1$.

Consider $D_I^{''}$ for some arbitrary I, $1 \leq I \leq m$. Let $x \in X$ and $a \in D_I^{''}$ be arbitrary. Suppose that a is an ancestor of x. Since $D_I^{''} \subseteq D_1$, $a \in D_1$. Moreover, since

D_1 is a descent group, $x \in D_1$. Also, since $D_I^{''} \subseteq D_I^{'}$ and $D_I^{'}$ is a descent group, $x \in D_I^{'}$. By construction, this implies that $x \in D_I^{''}$. Thus $D_i^{''}$ is a descent group.

Thus $D_1^{''}, \ldots, D_m^{''}$ are non–empty sets that partition D_1. Moreover each $D_i^{''}$, $1 \leq i \leq m$ are descent groups. Thus, D_1 is polyphyletic of degree m. $\square$

Observation 54 [Adding an Individual and Remaining in $[T]$**]** Consider a genealogical network $G = (X, p)$, a terminal group T in G, and a descent group $D \in [T]$. For every $x \in X$, $Term(cl(\{x\})) \subseteq T$ if and only if $cl(D \cup \{x\}) \in [T]$.

Proof
Consider a genealogical network $G = (X, p)$, a terminal group T in G, and a descent group $D \in [T]$. Let x be an arbitrary individual in X.

Certainly, if $x \in D$, then by Observation 10, $D = cl(D \cup \{x\})$. So $Term(cl(\{x\}) \subseteq T$ and $Term(cl(D \cup \{x\})) = T$.

Suppose that $x \notin D$.

($\Rightarrow$) Suppose that $Term(cl(\{x\})) \subseteq T$. Since cl is a closure operator (Observation 7) $cl(D \cup \{x\}) \supseteq D$ and thus $Term(cl(D \cup \{x\})) \supseteq T$. Let t be an arbitrary element of $Term(cl(D \cup \{x\}))$. By Corollary 1 and Observation 15, $cl(D \cup \{x\}) = \bigcup_{y \in \{x\} \cup (P(D) \setminus (cl\{x\} \cap P(D)))} cl(\{y\})$. So t must be a descendant of either x or a progenitor in D. Either way, $t \in T$ since $Term(cl(\{x\})) \subseteq T$.

($\Leftarrow$) Suppose that $Term(cl(D \cup \{x\}) = T$. Since cl is a closure operator (Observation 7), $cl(\{x\}) \subseteq cl(D \cup \{x\})$. Thus, $Term(\{x\}) \subseteq T$.

Hence, $Term(cl(\{x\})) \subseteq T$ if and only if $cl(D \cup \{x\}) \in [T]$. $\square$

Observation 55 [Separate Lineages] Consider a genealogical network $G = (X, p)$ and a terminal group T in G. If

1. for every $t_1, t_2 \in T$, if $t_1 \neq t_2$ and for every $y \in MRCA(\{t_1, t_2\})$, $Term(cl(\{y\})) \not\subseteq T$, and
2. $D \in [T]$,

then D is polyphyletic of degree $\mid T \mid$.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal group $T = \{t_1, \ldots t_k\}$ in G. Suppose that for every $t, t^{'} \in T$, if $t \neq t^{'}$ and for every $y \in MRCA(\{t, t^{'}\})$, $Term(cl(\{y\})) \not\subseteq T$. Also, suppose that $D \in [T]$. Consider an arbitrary $x \in P(D)$. It follows that x is the ancestor to only a single terminal in T. Suppose the contrary, there exists $t_I \in T$ and $t_J \in T$ such that $I \neq J$, $t_I \in cl(\{x\})$, and $t_J \in cl(\{x\})$. Then by Observation 20, there exists a $y \in MRCA(\{t_I, t_J\})$ such that x is an ancestor of y. Thus $Term(cl(\{y\})) \not\subseteq T$ and $Term(cl(\{x\})) \not\subseteq T$. This is a contradiction since $D \in [T]$. Thus there exists a unique $t \in T$ such that $Term(cl(\{x\})) = t$. Form sets $X_I = \{x \in P(D) \mid Term(cl(\{x\})) = t_I\}$. This gives a partition $X_1, \ldots, X_k$ of $P(D)$, i.e.,

1. for every i, $1 \leq i \leq k$, $X_i \neq \emptyset$
2. for every i, j, $1 \leq i < j \leq k$, $X_i \cap X_j = \emptyset$
3. $\bigcup_{1 \leq i \leq k} X_i = P(D)$

Note that since each terminal in T must have a progenitor in D, that each X_i is non–empty. From this partition, form descent groups $D_i = cl(X_i)$ for each i, $1 \leq i \leq k$. Since each X_i is non–empty, each D_i is non–empty. Also, since $\bigcup_{1 \leq i \leq k} X_i = P(D)$, $\bigcup_{1 \leq i \leq k} D_i = D$ by Observation 15. Moreover, for descent groups D_I, D_J where $I \neq J$, $D_I \cap D_J = \emptyset$ since $Term(D_I) = \{t_I\}$ and $Term(D_J) = \{t_J\}$. Therefore D is polyphyletic of degree k. □

Observation 78 (Sufficient Conditions For a Monophyletic Descent Group in $[T]$)
Consider a genealogical network $G = (X, p)$ and a terminal group T in G. If for some T_1 and T_2

1. $T_1 \neq \emptyset$,
2. $T_2 \neq \emptyset$,
3. $T_1 \cup T_2 = T$,
4. $T_1 \cap T_2 = \emptyset$, *and*
5. *for every $t_1 \in T_1, t_2 \in T_2$, and for every $y \in MRCA(\{t_1, t_2\}), Term(cl(\{y\})) \not\subseteq T$.*

then $[T]$ only contains polyphyletic groups.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal group T in G. Suppose there exists a non-trivial partition of T into two subsets, T_1 and T_2 that are both non-empty. Thus, $T_1 \cup T_2 = T$ and $T_1 \cap T_2 = \emptyset$. Moreover, assume that for every $t_1 \in T_1$, $t_2 \in T_2$, and for every $y \in MRCA(\{t_1, t_2\})$, $Term(cl(\{y\})) \not\subseteq T$. This condition will basically imply that T_1 and T_2 are the descendants of two separate groups of ancestors. These two ancestor groups cannot have a shared descendant.

For a proof by contradiction, suppose that there exists a monophyletic group D in $[T]$. Consider an arbitrary $x \in P(D)$. Suppose for some $t_1 \in T_1$ and $t_2 \in T_2$ that x is an ancestor to both t_1 and t_2. Then by Observation 20, for some $y \in D$, x is an ancestor of y and y is a most recent common ancestor of t_1 and t_2. Then we have that $Term(cl(\{y\})) \not\subseteq T$. This contradicts that $D \in [T]$. Thus for any $x \in P(D)$, the terminals in $cl(\{x\})$ are either contained in T_1 or T_2. Thus, there is a partition of $P(D)$ into X_1 and X_2 such that:

1. $X_1 \neq \emptyset$
2. $X_2 \neq \emptyset$
3. $X_1 \cup X_2 = P(D)$
4. $X_1 \cap X_2 = \emptyset$

5. for every $x \in X_1$, $Term(cl(\{x\})) \subseteq T_1$
6. for every $x \in X_2$, $Term(cl(\{x\})) \subseteq T_2$

Note that since neither T_1 nor T_2 are empty, neither X_1 nor X_2 are empty. Now, since D is monophyletic, by Observation 26, there exists $y \in D$, $x_1 \in X_1$, and $x_2 \in X_2$ such that both x_1 and x_2 are ancestors of y. Now, consider $Term(cl(\{y\}))$. Suppose there exists a $t \in T_1$ such that $t \in Term(cl(\{y\}))$. Then since x_2 is an ancestor of y and y is an ancestor of t, $t \in Term(cl\{y\}))$. This is a contradiction since $T_1 \cap T_2 = \emptyset$. Thus, for all $t \in Term(cl(\{y\}))$, $t \notin T_1$. By a symmetric argument we obtain that for all $t \in Term(cl(\{y\}))$, $t \notin T_2$. Since $D \in [T]$ it obtains that $Term(cl(\{y\})) = \emptyset$. This is a contradiction by Observation 48. Thus D is polyphyletic.

Therefore, all descent groups in $[T]$ are polyphyletic. $\square$

Observation 79 (Building a Monophyletic Descent Group in $[T]$) *Consider a genealogical network $G = (X, p)$ and a terminal group $T \neq \emptyset$ in G. If for every T_1 and T_2 such that*

1. $T_1 \neq \emptyset$,
2. $T_2 \neq \emptyset$,
3. $T_1 \cup T_2 = T$, *and*
4. $T_1 \cap T_2 = \emptyset$,

implies that for some $t_1 \in T_1$, $t_2 \in T_2$, and $x \in MRCA(\{t_1, t_2\})$, $Term(cl(\{x\})) \subseteq T$, then $[T]$ contains a monophyletic group.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal group $T \neq \emptyset$ in G. Suppose that for every T_1 and T_2, if

1. $T_1 \neq \emptyset$,
2. $T_2 \neq \emptyset$,
3. $T_1 \cup T_2 = T$, and
4. $T_1 \cap T_2 = \emptyset$

then, for some $t_1 \in T_1$, $t_2 \in T_2$, and $x \in MRCA(\{t_1, t_2\})$, $Term(cl(\{x\})) \subseteq T$.

Before proceeding with the formal proof, some notation will be useful. Firstly, impose an ordering on the individuals in X, i.e., let $X = \{x_1, \ldots, x_n\}$. Secondly, consider a polyphyletic group D in G which is polyphyletic of maximal degree k where $k \geq 2$. By Observation 32, partitions of D into k sub–descent groups are permutations of each other. Given an ordering of the individuals in X, it is possible to define a canonical partitioning of D into descent groups $D_1^{'}, D_2^{'}, \ldots D_k^{'}$. In this sequence $D_i^{'}$ is placed before $D_j^{'}$ if the smallest individual (based on the ordering in X) in $D_i^{'}$ comes before the smallest individual in $D_j^{'}$. This canonical partitioning of D can then be used to generate a canonical bi–partition of $Term(D)$, viz., $T_1 = Term(D_1^{'})$ and

$T_2 = Term(\bigcup_{2 \leq i \leq k} D_i^{'})$. Now, since $k \geq 2$, neither T_1 nor T_2 are empty. Suppose for some $t_1 \in T_1$, $t_2 \in T_2$, and $x \in MRCA(\{t_1, t_2\})$, $Term(cl(\{x\})) \subseteq T$. Call $x_m \in X$ the *canonical witness to the unification of* $D_1^{'}$ *and* $\bigcup_{2 \leq i \leq k} D_i^{'}$ if m is the smallest value such that some $t_1 \in T_1$, $t_2 \in T_2$, and $x_m \in MRCA(\{t_1, t_2\})$, $Term(cl(\{x_m\})) \subseteq T$ – by assumption such an individual is guaranteed to exist.

To construct a monophyletic group $D \in [T]$ consider the sequence D_i where $D_0 = T$ and

$$D_{i+1} = \begin{cases} D_i & \text{if } D_i \text{ is monophyletic} \\ D_i \cup cl(\{x_m\}) & \text{otherwise - where } D_1^{'}, D_2^{'}, \ldots, D_k^{'} \text{ is the canonical} \\ & \text{partition to } D_i \text{ being polyphyletic of maximal degree } k \\ & \text{and } x_m \text{ is the canonical witness to the unification of} \\ & D_1^{'} \text{ and } \bigcup_{2 \leq i \leq k} D_i^{'} \end{cases}$$

Now consider the terminal set in the D_i sequence. By Observation 51, $D_0 = T \in [T]$. Suppose $D_i \in [T]$ for some arbitrary $i \geq 0$. If D_i is monophyletic, $D_{i+1} = D_i$ which implies that $D_{i+1} \in T$. If D_i is not monophyletic, then $D_{i+1} = D_i \cup cl(\{x_m\})$ where $D_1^{'}, D_2^{'}, \ldots, D_k^{'}$ is the canonical partition to D_i being polyphyletic of maximal degree k and x_m is the canonical witness to the unification of $D_1^{'}$ and $\bigcup_{2 \leq i \leq k} D_i^{'}$. By construction of the canonical witness, $Term(cl(\{x_m\})) \subseteq T$. Thus, since $D_i \in [T]$, $cl(D_i \cup (\{x_m\})) \in [T]$ by Observation 54. Now, cl distributes over set union (Observation 8) and has no effect on descent groups (Observation 10), thus $cl(D_i \cup (\{x_m\})) = D_i \cup cl(\{x_m\})$. Hence $D_{i+1} \in [T]$. By induction, for every $i \geq 0$, $D_i \in [T]$.

Now consider the maximal degree to which each D_i is polyphyletic. Consider $D_0 = T$. By Observation 52, D_0 is polyphyletic of maximal degree $\mid T \mid$ and $\mid T \mid \leq max(\{\mid T \mid -0, 1\}$. Suppose that D_i is polyphyletic of maximal degree k and $k \leq max(\{\mid T \mid - i, 1\}$ for some arbitrary $i \geq 0$. If D_i is monophyletic, then $D_{i+1} = D_i$ and $D_{i+1} \neq \emptyset$ ($D_0 = T \neq \emptyset$ and the construction of D_{i+1} from D_i is monotonic), by Observation 31, $k = 1$ and $1 \leq max(\{\mid T \mid -(i+1), 1\}$. Suppose that D_i is polyphyletic. Then, $D_{i+1} = D_i \cup cl(\{x_m\})$ where $D_1^{'}, D_2^{'}, \ldots, D_k^{'}$ is the canonical partition to D_i being polyphyletic of maximal degree k and x_m is the canonical witness to the unification of $D_1^{'}$ and $\bigcup_{2 \leq i \leq k} D_i^{'}$. Since x_m is the canonical witness, there exists $t_1 \in Term(D_1^{'})$, $t_2 \in \bigcup_{2 \leq i \leq k} D_i^{'}$, and $x_m \in MRCA(\{t_1, t_2\})$. Thus $cl(\{x_m\}) \cap D_1^{'} \neq \emptyset$ since $t_1 \in D_1^{'}$ and $t_1 \in cl\{x_m\})$. Also, since $t_2 \in \bigcup_{2 \leq i \leq k} D_i^{'}$, for some $D_I^{'}$, $2 \leq I \leq k$, $t_2 \in D_I^{'}$. Moreover, $t_2 \in cl(\{x_m\})$. Given that $D_1^{'}, \ldots, D_k^{'}$ is the canonical partition of D and x_m intersects with at least two sets in the partition, by Observation 35, if $D_{i+1} = D_i \cup cl(\{x_m\})$ is polyphyletic of maximal degree $k^{'}$, then $k^{'} < k$. Now, by the inductive hypothesis, $k \leq max(\{\mid T \mid -i, 1\}$. Then $k^{'} < max(\{\mid T \mid -i, 1\}$ and $k^{'} \leq max(\{\mid T \mid -(i+1), 1\}$. Therefore, by induction, for every i, if D_i is polyphyletic of maximal degree k, then $k \leq max(\{\mid T \mid -i, 1\}$. Moreover, consider $D_{|T|+1}$. This descent group is polyphyletic of maximal degree 1. Thus, by Observation 31, $D_{|T|+1}$ is monophyletic.

Therefore, $[T]$ contains a monophyletic group. □

Corollary 4 [Monophyletic Descent Group in $[T]$] Consider a genealogical network $G = (X, p)$ and a terminal group T in G. All descent groups in $[T]$ are polyphyletic if and only if for some T_1 and T_2

1. $T_1 \neq \emptyset$,

2. $T_2 \neq \emptyset$,

3. $T_1 \cup T_2 = T$,

4. $T_1 \cap T_2 = \emptyset$, and

5. for every $t_1 \in T_1, t_2 \in T_2$, and for every $y \in MRCA(\{t_1, t_2\}), Term(cl(\{y\})) \not\subseteq T$.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal group T in G.

The "if" part is exactly the converse of Observation 79 – this result and proof appear in this appendix. The "only if" part is exactly Observation 78. Once again, this result is only contained in this appendix. □

Observation 56 [Maximal Monophyletic Descent Group in $[T]$] Consider a genealogical network $G = (X, p)$ and a terminal group T in G. If $[T]$ contains a monophyletic group, then the set

$$D_{max} = \{x \in X \mid Term(cl(\{x\})) \subseteq T\}$$

is a monophyletic group such that for every monophyletic group D in $[T]$, $D \subseteq D_{max}$.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal group T in G. Suppose $[T]$ contains a monophyletic group. Define the set D_{max} as follows:

$$D_{max} = \{x \in X \mid Term(cl(\{x\})) \subseteq T\}$$

Firstly $Term(D_{max}) = T$. Consider $Term(D_{max})$. For every $t \in T$, since t has no descendants, $Term(cl(\{t\})) = \{t\}$. Thus $t \in D_{max}$ and $T \subseteq D_{max}$. Also, for any terminal $t^{'} \notin T$, $Term(cl(\{t^{'}\})) = \{t^{'}\} \not\subseteq T$. Thus $t^{'} \notin T$. Hence, $Term(D_{max}) = T$.

Also, D_{max} is a descent group in G. Let x be an arbitrary individual in X. Let $a \in D_{max}$ be arbitrary and suppose that a is an ancestor of x. Since $a \in D_{max}$, $Term(cl(\{a\})) \subseteq T$. Now since a is an ancestor of x, the terminals of in the closure of x are smaller (Observation 49), i.e., $Term(cl(\{x\})) \subseteq Term(cl(\{a\}))$. Hence $Term(cl(\{x\})) \subseteq T$ and $x \in D_{max}$.

Finally, D_{max} is monophyletic. Consider an arbitrary partition of $P(D_{max})$ into two sets X_1 and X_2 such that neither X_1 nor X_2 are empty, $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = P(D_{max})$. This partition also generates a splitting of T into two sets - where $T_1 = Term(\bigcup_{x \in X_1} cl(\{x\}))$ and $T_2 = Term(\bigcup_{x \in X_2} cl(\{x\}))$. Now, neither T_1 nor T_2 are empty since neither X_1 nor X_2 are empty (Observation 48). Also $T_1 \cup T_2 = T$ since $D_{max} = \bigcup_{x \in P(D_{max})} cl(\{x\})$ by Observation 15. Now, suppose $T_1 \cap T_2 \neq \emptyset$, i.e., for some $t \in T$, $x_1 \in X_1$, and $x_2 \in X_2$, both x_1 and x_2 are ancestors of t. Then, by Observation 26, D_{max} is monophyletic. Consider the second case, $T_1 \cap T_2 = \emptyset$ – this is in fact an impossibility. Since T_1 and T_2 are a non–empty partition of T and $[T]$ contains a monophyletic group, by Corollary 4, for some $t_1 \in T_1$,

$t_2 \in T_2$, $y \in MRCA(\{t_1, t_2\})$, $Term(cl(\{y\})) \subseteq T$. Then, $y \in D_{max}$. Without loss of generality, suppose for some $x_1 \in X_1$, x_1 is an ancestor of y. Then x_1 is an ancestor of t_2 (Observation 1). Also, since $(\bigcup_{x \in X_1} cl(\{x\})$ is a descent group (Observation 3), $t_2 \in T_1$. This is a contradiction since $T_1 \cap T_2 = \emptyset$. Hence D_{max} is a monophyletic group.

Consider an arbitrary monophyletic group $D \in [T]$ and an arbitrary individual $x \in D$. For a contradiction, suppose there exists a $t^{'} \notin T$ and x is an ancestor of $t^{'}$. Since D is a descent group, this implies that $t^{'} \in D$. Hence $D \notin [T]$. Thus, $Term(cl(\{x\}) \subseteq T$ and $x \in D_{max}$. Hence $D \subseteq D_{max}$. $\square$

Observation 57 [Ancestor Set Relations] Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G.

1. $T_1 \subseteq T_2$ if and only if $A(T_1) \subseteq A(T_2)$
2. $T_1 \cap T_2 = \emptyset$ if and only if $A(T_1) \cap A(T_2) = \emptyset$
3. $A(T_1 \cap T_2) = A(T_1) \cap A(T_2)$
4. $A(T_1 \cup T_2) \supseteq A(T_1) \cup A(T_2)$

Proof
Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G.

1. Suppose that $T_1 \subseteq T_2$. Let x be an arbitrary element of $A(T_1)$. Then, $Term(cl(\{x\})) \subseteq T_1$. It follows that $Term(cl(\{x\})) \subseteq T_2$ since $T_1 \subseteq T_2$. Hence, $x \in A(T_2)$ and $A(T_1) \subseteq A(T_2)$.

   Suppose that $A(T_1) \subseteq A(T_2)$. Let t be an arbitrary element of T_1. Then $t \in A(T_1)$ since $cl(\{t\}) = \{t\}$. If follows that $t \in A(T_2)$, i.e., $\{t\} \subseteq T_2$. This implies that $t \in T_2$. Hence $T_1 \subseteq T_2$.

2. Suppose that $T_1 \cap T_2 = \emptyset$. For a proof by contradiction, suppose that $A(T_1) \cap A(T_2) \neq \emptyset$. Let x be an arbitrary element of $A(T_1) \cap A(T_2)$. Then $Term(cl(\{x\})) \subseteq T_1$ and $Term(cl(\{x\})) \subseteq T_2$. Since $Term(cl(\{x\}))$ is non–empty (Observation 48), there exists a term $y \in Term(cl(\{x\}))$ such that $y \in T_1 \cap T_2$. This is a contradiction. Thus, $A(T_1) \cap A(T_2) = \emptyset$.

   Suppose that $T_1 \cap T_2 \neq \emptyset$. Let t be an arbitrary element of $T_1 \cap T_2$. Then, since $cl(\{t\}) = t$, $t \in A(T_1)$ and $t \in A(T_2)$. Hence $A(T_1) \cap A(T_2) \neq \emptyset$.

3. Consider $A(T_1 \cap T_2)$. For an arbitrary $x \in X$, $x \in A(T_1 \cap T_2)$ if and only if $Term(cl(\{x\})) \subseteq T_1 \cap T_2$. This condition is equivalent to $Term(cl(\{x\})) \subseteq T_1$ and $Term(cl(\{x\})) \subseteq \cap T_2$ which is exactly the condition that $x \in A(T_1) \cap A(T_2)$. Thus $A(T_1 \cap T_2) = A(T_1) \cap A(T_2)$.

4. Let x be an arbitrary element of $A(T_1) \cup A(T_2)$. Then $Term(cl(\{x\})) \subseteq T_1$ or $Term(cl(\{x\})) \subseteq T_2$. This implies that $Term(cl(\{x\})) \subseteq T_1 \cup T_2$ and that $x \in A(T_1 \cup T_2)$. Hence, $A(T_1) \cup A(T_2) \subseteq A(T_1 \cup T_2)$.

□

Observation 58 [Ancestor Set Properties] Consider a genealogical network $G = (X, p)$ and a terminal set T in G.

1. $A(T) \in [T]$
2. $A(T)$ is a descent group
3. for every descent group $D \in [T]$, $D \subseteq A(T)$
4. if $A(T)$ is polyphyletic of maximal degree k, then for every descent group $D \in [T]$, D is polyphyletic of degree k.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal set T in G.

1. Consider $Term(A(T))$. For every $t \in T$, since t has no descendants, $Term(cl(\{t\})) = \{t\}$. Thus $t \in A(T)$ and $T \subseteq A(T)$. Also, for any terminal $t^{'} \notin T$, $Term(cl(\{t^{'}\})) = \{t^{'}\} \not\subseteq T$. Thus $t^{'} \notin T$. Hence, $Term(A(T)) = T$ and $A(T) \in [T]$.
2. Also, $A(T)$ is a descent group in G. Let x be an arbitrary individual in X. Suppose $a \in A(T)$ be arbitrary and suppose that a is an ancestor of x. Since $a \in A(T)$, $Term(cl(\{a\})) \subseteq T$. Now since a is an ancestor of x, the terminals of the closure of x are fewer (Observation 49), i.e., $Term(cl(\{x\})) \subseteq Term(cl(\{a\}))$. Hence $Term(cl(\{x\})) \subseteq T$ and $x \in A(T)$.
3. Consider an arbitrary descent group $D \in [T]$. Let x be an arbitrary element of D. Then $Term(cl(\{x\})) \subseteq T$. Otherwise, for some $t \notin T$, x is an ancestor of t and t would be an element of D since D is a descent group. This would imply that D cannot be a member of $[T]$. Thus, $Term(cl(\{x\})) \subseteq T$ and $x \in A(T)$. Hence, $D \subseteq A(T)$.
4. Suppose that $A(T)$ is polyphyletic of maximal degree k. Consider an arbitrary descent group $D \in [T]$ that is polyphyletic of maximal degree m. Then since $D \subseteq A(T)$, by Observation 53, D is polyphyletic of degree k.

□

Observation 59 [Union Gap] Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G. For every $x \in A(T_1 \cup T_2)$, if $x \notin A(T_1) \cup A(T_2)$, then $Term(cl(\{x\})) \cap T_1 \neq \emptyset$ and $Term(cl(\{x\})) \cap T_2 \neq \emptyset$.

Proof
Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G. Consider any $x \in A(T_1 \cup T_2)$ such that $x \notin A(T_1) \cup A(T_2)$. Then $Term(cl(\{x\})) \subseteq T_1 \cup T_2$. For a proof by contradiction, suppose that $Term(cl(\{x\})) \cap T_1 = \emptyset$. Then $Term(cl(\{x\})) \subseteq T_2$ and $x \in A(T_2)$. This contradiction gives that $Term(cl(\{x\})) \cap T_1 \neq \emptyset$. Similarly, $Term(cl(\{x\})) \cap T_2 \neq \emptyset$. □

Observation 60 [Single Term Generates Monophyletic Descent Group] Consider a genealogical network $G = (X, p)$ and a terminal $t \in X$. Then, $A(\{t\})$ is monophyletic.

Proof
Consider a genealogical network $G = (X, p)$ and a terminal $t \in X$. Then $A(\{t\})$ cannot be partitioned into two non–empty descent groups since $A(\{t\})$ has a single terminal t and a non–empty descent group has at least one terminal. Thus $A(\{t\})$ is monophyletic. $\square$

Observation 61 [Union is Monophyletic] Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G. Suppose $A(T_1)$ is monophyletic and $A(T_2)$ is monophyletic.

1. if $A(T_1 \cup T_2)$ is polyphyletic then $A(T_1 \cup T_2) = A(T_1) \cup A(T_2)$.
2. if $T_1 \neq \emptyset$, $T_2 \neq \emptyset$, $T_1 \cap T_2 = \emptyset$, and $A(T_1 \cup T_2) \subseteq A(T_1) \cup A(T_2)$, then $A(T_1 \cup T_2)$ is polyphyletic.

Proof
Consider a genealogical network $G = (X, p)$ and terminal groups T_1 and T_2 in G. Suppose $A(T_1)$ is monophyletic and $A(T_2)$ is monophyletic.

Suppose that $A(T_1 \cup T_2)$ is polyphyletic. Now, by Observation 57, $A(T_1 \cup T_2) \supseteq A(T_1) \cup A(T_2)$. Consider any $x \in A(T_1 \cup T_2)$. If $x \notin A(T_1) \cup A(T_2)$, then by Observation 59 $Term(cl(\{x\})) \cap T_1 \neq \emptyset$ and $Term(cl(\{x\})) \cap T_2 \neq \emptyset$. Consider the descent group $cl(\{x\}) \cup A(T_1) \cup A(T_2)$. Since both $cl(\{x\})$ and $A(T_1)$ are monophyletic and $cl(\{x\}) \cap A(T_1) \neq \emptyset$, by Observation 37, $cl(\{x\}) \cup A(T_1)$ is monophyletic. Once again, since $A(T_2)$ is monophyletic and $A(T_2) \cap (cl(\{x\}) \cup A(T_1)) \neq \emptyset$ ($A(T_2) \cap cl(\{x\}) \neq \emptyset$), by Observation 37, $cl(\{x\}) \cup A(T_1) \cup A(T_2)$ is monophyletic. Thus $[T_1 \cup T_2]$ contains a monophyletic descent group, which by Observation 56, $A(T_1 \cup T_2)$ is monophyletic. This contradiction gives that $x \in A(T_1) \cup A(T_2)$ and $A(T_1 \cup T_2) = A(T_1) \cup A(T_2)$.

Suppose that $T_1 \neq \emptyset$, $T_2 \neq \emptyset$, $T_1 \cap T_2 = \emptyset$, and $A(T_1 \cup T_2) \subseteq A(T_1) \cup A(T_2)$. By Observation 57, $A(T_1) \cup A(T_2) \subseteq A(T_1 \cup T_2)$. Thus $A(T_1 \cup T_2) = A(T_1) \cup A(T_2)$. Moreover, $A(T_1) \cap A(T_2) = \emptyset$. Otherwise, for some terminal t, $t \in T_1 \cap T_2$ - contradicting the assumption that $T_1 \cap T_2 = \emptyset$. Thus $A(T_1)$ and $A(T_2)$ are non–empty descent groups that partition $A(T_1 \cup T_2)$; witnesses to $A(T_1 \cup T_2)$ being polyphyletic. $\square$

Observation 62 [Ancestor Set Monophyletic Monotonicity] Consider a genealogical network $G = (X, p)$ and terminal groups T_1, T_2, and T_3 in G. If

1. $T_i \neq \emptyset$, for $i = 1, 2,$ and 3
2. $T_1 \cap T_i = \emptyset$, for $i = 2$ and 3
3. $T_2 \subseteq T_3$
4. $A(T_i)$ is monophyletic for $i = 1, 2,$ and 3
5. $A(T_1 \cup T_2)$ is monophyletic,

then $A(T_1 \cup T_3)$ is monophyletic.

Proof
Consider a genealogical network $G = (X, p)$ and terminal groups T_1, T_2, and T_3 in G. Suppose

1. $T_i \neq \emptyset$, for $i = 1, 2$, or 3
2. $T_1 \cap T_i = \emptyset$, for $i = 2$ or 3
3. $T_2 \subseteq T_3$
4. $A(T_1 \cup T_2)$ is monophyletic,

Since $T_1 \cup T_2$ is monophyletic, applying Observation 61 implies that $A(T_1 \cup T_2) \not\subseteq A(T_1) \cup A(T_2)$. Also, since in general $A(T_1) \cup A(T_2) \subseteq A(T_1 \cup T_2)$ (by Observation 57), it obtains that $A(T_1 \cup T_2) \supset A(T_1) \cup A(T_2)$. Thus, for some $x \in X$, $Term(cl(\{x\})) \subseteq T_1 \cup T_2$, $Term(cl(\{x\})) \cap T_1 \neq \emptyset$, and $Term(cl(\{x\})) \cap T_2 \neq \emptyset$. Now, since $T_2 \subseteq T_3$, it obtains that $Term(cl(\{x\})) \subseteq T_1 \cup T_3$ and $Term(cl(\{x\})) \cap T_3 \neq \emptyset$. Thus, for some $x \in X$, $x \notin A(T_1) \cup A(T_3)$ and $x \in A(T_1 \cup T_3)$ and $A(T_1) \cup A(T_3) \neq A(T_1 \cup T_3)$ and by Observation 61, $A(T_1 \cup T_3)$ is monophyletic. □

D Proofs For Section 5

Observation 63 [Content of Higher Ranks] Consider a finite set Y and two Linnaean ranks $\mathbf{R}_1$ and $\mathbf{R}_2$ over Y. Suppose that $\mathbf{R}_1$ is *above* $\mathbf{R}_2$. Then,

1. for every $G' \in \mathbf{R}_2$, there exists a unique $G \in \mathbf{R}_1$ such that $G^{'} \subseteq G$
2. for every $G \in \mathbf{R}_1$, $G = \bigcup_{G' \in \mathbf{R}} G^{'}$ for some non-empty $\mathbf{R} \subseteq \mathbf{R}_2$
3. for every $G \in \mathbf{R}_1$, $G = \bigcup_{G' \in \mathbf{R}_2 \text{ and } G' \subseteq G} G^{'}$.

Proof
Consider a finite set Y and two Linnaean ranks $\mathbf{R}_1$ and $\mathbf{R}_2$ over Y. Suppose that $\mathbf{R}_1$ is *above* $\mathbf{R}_2$.

Consider an arbitrary $G^{'} \in \mathbf{R}_2$. Then for some $y \in Y$, $y \in G^{'}$ and $y \notin H^{'}$ for any $H^{'} \in \mathbf{R}_2$ such that $G^{'} \neq H^{'}$. Since $\mathbf{R}_1$ is a Linnaean over rank over Y, for some $G \in \mathbf{R}_1$, $y \in G$. Thus $G^{'} \cap G \neq \emptyset$. Hence, since $\mathbf{R}_1$ is above $\mathbf{R}_2$, $G^{'} \subseteq G$. Moreover, for any $H \in \mathbf{R}_1$, if $H \neq G$, then by Definition 27, $H \cap G = \emptyset$ and $G^{'} \not\subseteq H$.

Consider an arbitrary $G \in \mathbf{R}_1$. By Definition 28, there exists a $G^{'} \in \mathbf{R}_2$ such that $G^{'} \subseteq G$. Consider, an arbitrary $H \in \mathbf{R}_2$ such that $G \cap H \neq \emptyset$. Then, by Definition 28, $H \subseteq G$. Thus, $G = \bigcup_{G' \in \{H \in \mathbf{R}_2 | H \cap G \neq \emptyset\}} G^{'}$. By Definition 28, $G = \bigcup_{G' \in \{H \in \mathbf{R}_2 | H \subseteq G\}} G^{'}$. □

Observation 64 ['Above' Transitive] Consider a finite set Y and Linnaean ranks $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ over Y. If $\mathbf{R}_1$ is *above* $\mathbf{R}_2$ and $\mathbf{R}_2$ is *above* $\mathbf{R}_3$, then $\mathbf{R}_1$ is *above* $\mathbf{R}_3$.

Proof

Consider a finite set Y and Linnaean ranks $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ over Y. Suppose $\mathbf{R}_1$ is *above* $\mathbf{R}_2$ and $\mathbf{R}_2$ is *above* $\mathbf{R}_3$. Consider an arbitrary $G \in \mathbf{R}_1$. Then since $\mathbf{R}_1$ is above $\mathbf{R}_2$, there exists a $G^{'} \in \mathbf{R}_2$ such that $G^{'} \subseteq G$. Since $\mathbf{R}_2$ is above $\mathbf{R}_3$, there exists a $G^{''} \in \mathbf{R}_3$ such that $G^{''} \subseteq G^{'}$. Hence $G^{''} \subseteq G$.

Consider an arbitrary $G \in \mathbf{R}_1$ and $G^{''} \in \mathbf{R}_3$. Suppose $G \cap G^{''} \neq \emptyset$. Let x be an arbitrary element of $G \cap G^{''}$. Then, for some $G^{'} \in \mathbf{R}_2$, $x \in G^{'}$. Since $x \in G^{'} \cap G^{''}$ and $\mathbf{R}_2$ is above $\mathbf{R}_3$, $G^{''} \subseteq G^{'}$. Also, since $\mathbf{R}_1$ is above $\mathbf{R}_2$ and $x \in G \cap G^{'}$, $G^{'} \subseteq G$. Hence, $G^{''} \subseteq G$.

Therefore $\mathbf{R}_1$ is above $\mathbf{R}_3$. □

Observation 65 [Slicing the Last Rank] Consider a genealogical network $G = (X, p)$ and an extensive Linnaean classification $L = (\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n)$ over X. Let $L^{'} = (\mathbf{R}_1, \ldots, \mathbf{R}_{n-1})$. Then

1. $L^{'}$ is an extensive Linnaean classification over X
2. if L is strongly monophyletic, then $L^{'}$ is strongly monophyletic
3. if L is weakly monophyletic, then $L^{'}$ is strongly monophyletic

Proof
Consider a genealogical network $G = (X, p)$ and an extensive Linnaean classification $L = (\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n)$ over X. Let $L^{'} = (\mathbf{R}_1, \ldots, \mathbf{R}_{n-1})$.

Since L is an extensive Linnaean classification over X, by Definition 29 $\mathbf{R}_i$ is a Linnaean rank for every i, $1 \leq i \leq n-1$. Moreover $\mathbf{R}_i$ is above $\mathbf{R}_{i+1}$ for every i, $1 \leq i < n-1$. Thus, $L^{'}$ is an extensive Linnaean classification over X.

If L is strongly monophyletic, then (Definition 30) for every $\mathbf{R}_i$ ($1 \leq i \leq n-1$) and $G \in \mathbf{R}_i$, G is monophyletic. Thus $L^{'}$ is strongly monophyletic.

If L is weakly monophyletic, then (Definition 31) for every $\mathbf{R}_i$ ($1 \leq i \leq n-1$) and $G \in \mathbf{R}_i$, G is monophyletic. Thus $L^{'}$ is strongly monophyletic. □

Theorem 1 [Monophyletic Linnaean Incompatibility] Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. If L is strongly monophyletic, then

1. $\mathbf{R}_i = \mathbf{R}_n$ for every i, $1 \leq i \leq n$
2. X is polyphyletic of maximal degree $\mid \mathbf{R}_n \mid$

Proof
Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. Suppose L is strongly monophyletic.

Consider an arbitrary $G \in \mathbf{R}_i$ where i can range from $1 \leq i \leq n-1$. Then, since $\mathbf{R}_i$ is above $\mathbf{R}_n$ (Observation 64), for some $G^{'} \in \mathbf{R}_n$, $G^{'} \subseteq G$. Now, by Observation 63, $G = \bigcup_{H \in \mathbf{R}} H$ for some $\mathbf{R} \subseteq \mathbf{R}_n$. For a contradiction, suppose that $G \neq G^{'}$, i.e. $\mathbf{R} \setminus \{G^{'}\} \neq \emptyset$. Then, let $H^{'} = \bigcup_{H \in \mathbf{R} \setminus \{G^{'}\}} H$. Now, $H^{'} \neq \emptyset$ since $\mathbf{R}_n$ is a partition of X into non–empty monophyletic groups. By Corollary 2, $G^{'}$ and $H^{'}$ are disconnected, i.e., $G^{'} \cap H^{'} = \emptyset$. Moreover, by Observation 3, $H^{'}$ is a descent group. Since $G = G^{'} \cup H^{'}$, this construction shows that G is polyphyletic. This contradiction gives that $G = G^{'}$. Thus, arbitrary elements of $\mathbf{R}_i$ equal single elements in $\mathbf{R}_n$. Since $\mathbf{R}_i$ and $\mathbf{R}_n$ are both partitions of X, $\mathbf{R}_i = \mathbf{R}_n$.

Since $\mathbf{R}_n$ is a partition of X into $\mid \mathbf{R}_n \mid$ non–empty descent groups, X is polyphyletic of degree $\mid \mathbf{R}_n \mid$. Moreover, since the descent groups in $\mathbf{R}_n$ are monophyletic, by Observation 33, X is polyphyletic of maximal degree $\mid \mathbf{R}_n \mid$. $\square$

Corollary 5 [Monophyletic Linnaean Incompatibility] Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. If L is strongly monophyletic and X is a monophyletic group in G, then $\mathbf{R}_i = \{X\}$ for every i, $1 \leq i \leq n$.

Proof
Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. Suppose L is strongly monophyletic and X is a monophyletic group in G. Consider $\mathbf{R}_n$. Since X is monophyletic, there cannot be a partition of X into two (or more) non–empty descent groups (Definition 13) with an empty intersection. Thus, $\mathbf{R}_n = \{X\}$. Then, by Theorem 1, $\mathbf{R}_i = \{X\}$ for every i, $1 \leq i \leq n$. $\square$

Corollary 6 [Monophyletic Linnaean Incompatibility] Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. If L is weakly monophyletic, then

1. $\mathbf{R}_i = \mathbf{R}_{n-1}$ for every i, $1 \leq i \leq n-1$
2. X is polyphyletic of maximal degree $\mid \mathbf{R}_{n-1} \mid$

Proof
Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. Suppose L is weakly monophyletic. Then clearly, $L^{'} = (\mathbf{R}_1, \ldots, \mathbf{R}_{n-1})$ is also a Linnaean classification over X (Observation 65). Moreover, $L^{'}$ is strongly monophyletic (Observation 65). The result follows by applying Theorem 1 to $L^{'}$. $\square$

Corollary 7 [Monophyletic Linnaean Incompatibility] Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. If L is weakly monophyletic and X is a monophyletic group in G, then $\mathbf{R}_i = \{X\}$ for every i, $1 \leq i \leq n-1$.

Proof
Consider a genealogical network $G = (X, p)$ and a Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over X. Suppose L is weakly monophyletic and X is a monophyletic group in G. Then clearly, $L^{'} = (\mathbf{R}_1, \ldots, \mathbf{R}_{n-1})$ is also a Linnaean classification over X (Observation 65). Moreover, $L^{'}$ is strongly monophyletic (Observation 65). The result follows by applying Corollary 5 to $L^{'}$. $\square$

Observation 66 [Properties of I] Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Then,

1. $I(G_i) \neq \emptyset$ for every i, $1 \leq i \leq n$ and $G_i \in \mathbf{R}_i$
2. $I(G) \cap I(G^{'}) = \emptyset$ or $I(G) = I(G^{'})$ for every i, $1 \leq i \leq n$ and $G, G^{'} \in \mathbf{R}_i$
3. $\bigcup_{G \in \mathbf{R}_{i-1}} I(G) = \{I(G_i) \mid G_i \in \mathbf{R}_i\}$, for every i, $1 < i \leq n$.

Proof
Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Let $P(k)$ be the proposition that $I(G) \neq \emptyset$ for every $G \in \mathbf{R}_k$. Let $Q(k)$ be the proposition that $I(G) \cap I(G^{'}) = \emptyset$ or $I(G) = I(G^{'})$ for every $G, G^{'} \in \mathbf{R}_k$. Let $R(k)$ be the proposition that $\bigcup_{G \in \mathbf{R}_{k-1}} I(G) = \{I(G_k) \mid G_k \in \mathbf{R}_k\}$. Propositions P and Q can be proven by induction on the 'height' of a Linnaean rank while R can be proved directly.

Consider $P(n)$. Consider an arbitrary $G \in \mathbf{R}_n$. Then by Definition 34, $I(G) = G$. Since $\mathbf{R}_n$ is a Linnaean rank over Y, $G \neq \emptyset$. Hence $P(n)$ is true. Consider $Q(n)$. Let $G, G^{'} \in \mathbf{R}_n$ be arbitrary. Since I is an identity function on members of $\mathbf{R}_n$ (Definition 34), $I(G) = G$ and $I(G^{'}) = G^{'}$. Now, since $\mathbf{R}_n$ is a Linnaean rank over Y, by Definition 27, either $G = G^{'}$ or $G \cap G^{'} = \emptyset$. Hence $I(G) = I(G^{'})$ or $I(G) \cap I(G^{'}) = \emptyset$. Thus, $Q(n)$ is true.

As inductive hypotheses, suppose that $P(i)$ and $Q(i)$ are true for some i, $2 < i \leq n$.

Consider $P(i-1)$. Since $P(i)$ is true, for every $G_i \in \mathbf{R}_i$, $I(G_i) \neq \emptyset$. Now, consider an arbitrary $G_{i-1} \in \mathbf{R}_{i-1}$. By Definition 34, $I(G_{i-1}) = \{I(G_i) \mid G_i \in \mathbf{R}_i \text{ and } G_i \subseteq G_{i-1}\}$. By Observation 63, there exists a $G_i \in \mathbf{R}_i$ such that $G_i \subseteq G_{i-1}$. And certainly, by the inductive hypothesis, $I(G_i) \neq \emptyset$ and thus $I(G_{i-1}) \neq \emptyset$. Thus, $P(i-1)$ is true. Therefore by induction, $I(G_i) \neq \emptyset$ for every i, $1 \leq i \leq n$ and $G_i \in \mathbf{R}_i$.

Consider $Q(i-1)$. Since $Q(i)$ is true, for every $G_i, G_i^{'} \in \mathbf{R}_i$, either $I(G_i) = I(G_i^{'})$ or $I(G_i) \cap I(G_i^{'}) = \emptyset$. Consider arbitrary $G_{i-1}, G_{i-1}^{'} \in \mathbf{R}_{i-1}$. Suppose that $I(G_{i-1}) \cap I(G_{i-1}^{'}) \neq \emptyset$. Let $X \in I(G_{i-1}) \cap I(G_{i-1}^{'})$. Thus by Definition 34, $X = I(G_i)$ for some $G_i \in \mathbf{R}_i$ and $G_i \subseteq G_{i-1}$. Also, $X = I(G_i^{'})$ for some $G_i^{'} \in \mathbf{R}_i$ and $G_i^{'} \subseteq G_{i-1}^{'}$. By the inductive hypothesis, this implies that $G_i^{'} = G_i$. By Observation 63, this implies that $G_{i-1} = G_{i-1}^{'}$. Thus, $Q(i-1)$ is true. There by induction, for every i, $1 \leq i \leq n$, $G_i, G_i^{'} \in \mathbf{R}_i$, either $I(G_i) = I(G_i^{'})$ or $I(G_i) \cap I(G_i^{'}) = \emptyset$.

Proposition $R(k)$ can be proved without directly (without induction) for any $1 < k \leq n$. Consider an arbitrary i, $1 < i \leq n$. By Definition 34,

$$\bigcup_{G \in \mathbf{R}_{i-1}} I(G) = \bigcup_{G \in \mathbf{R}_{i-1}} \{I(G_i) \mid G_i \subseteq G \text{ and } G_i \in \mathbf{R}_i\}$$

Let X be an arbitrary element of $\{I(G_i) \mid G_i \in \mathbf{R}_i\}$. Then $X = I(G_i)$ for some $G_i \in \mathbf{R}_i$. Since $\mathbf{R}_{i-1}$ is a Linnaean rank above $\mathbf{R}_i$, there exists a $G_{i-1} \in \mathbf{R}_{i-1}$ such that $G_i \subseteq G_{i-1}$ (Observation 63). Thus $X \in \bigcup_{G \in \mathbf{R}_{i-1}} \{I(G_i) \mid G_i \subseteq G \text{ and } G_i \in \mathbf{R}_i\}$. Consider an arbitrary $X \in \bigcup_{G \in \mathbf{R}_{i-1}} \{I(G_i) \mid G_i \subseteq G \text{ and } G_i \in \mathbf{R}_i\}$. Then for some $G \in \mathbf{R}_{i-1}, G_i \in \mathbf{R}_i$, $G_i \subseteq G$ and $X = I(G_i)$. Then certainly, $X \in \{I(G_i) \mid G_i \in \mathbf{R}_i\}$. Hence

$$\bigcup_{G \in \mathbf{R}_{i-1}} \{I(G_i) \mid G_i \subseteq G \text{ and } G_i \in \mathbf{R}_i\} = \{I(G_i) \mid G_i \in \mathbf{R}_i\}$$

and

$$\bigcup_{G \in \mathbf{R}_{i-1}} I(G) = \{I(G_i) \mid G_i \in \mathbf{R}_i\}$$

Thus $R(k)$ is true for all k, $1 < k \leq n$.
$\square$

Observation 67 [ψ Makes An Intensive Classification] Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Then the intensive counterpart of L is an intensive Linnaean classification over Y.

Proof
Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Let $\psi(L) = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ be the intensive counterpart to L.

By definition, $\mathbf{R}_n^{'} = \{I(G_n) \mid G_i \in \mathbf{R}_n\}$. Since I leaves extensive groups in the bottom rank unchanged, $\mathbf{R}_n^{'} = \{G_n \mid G_i \in \mathbf{R}_n\} = \mathbf{R}_n$. Then $\mathbf{R}_n^{'}$ is a Linnaean rank over Y since $\mathbf{R}_n$ is a Linnaean rank over Y.

Consider an arbitrary i, $1 \leq i < n$. By Definition 34, $\mathbf{R}_i^{'} = \{I(G_i) \mid G_i \in \mathbf{R}_i\}$. By Observation 66,

1. $I(G_i) \neq \emptyset$ for every $G_i \in \mathbf{R}_i$
2. $I(G) \cap I(G^{'}) = \emptyset$ or $I(G) = I(G^{'})$ for every $G, G^{'} \in \mathbf{R}_i$
3. $\bigcup_{G \in \mathbf{R}_i} I(G) = \{I(G_{i+1}) \mid G_{i+1} \in \mathbf{R}_{i+1}\}$

Now, since $\{I(G_{i+1}) \mid G_{i+1} \in \mathbf{R}_{i+1}\} = \mathbf{R}_{i+1}^{'}$ (by Definition 34, it obtains that $\mathbf{R}_i^{'}$ is a Linnaean rank over $\mathbf{R}_{i+1}^{'}$. Hence, $\psi(L)$ is an intensive Linnaean classification over Y. $\square$

Observation 68 [Extension Properties] Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ over Y.

1. $E(G^{'}) \neq \emptyset$ for every i, $1 \leq i \leq n$ and $G^{'} \in \mathbf{R}_i^{'}$
2. $E(G^{'}) = E(H^{'})$ if and only if $G^{'} = H^{'}$ for every i, $1 \leq i \leq n$ and $G^{'}, H^{'} \in \mathbf{R}_i^{'}$
3. either $E(G^{'}) = E(H^{'})$ or $E(G^{'}) \cap E(H^{'}) = \emptyset$ for every i, $1 \leq i \leq n$ and $G^{'}, H^{'} \in \mathbf{R}_i^{'}$
4. $\bigcup_{G^{'} \in \mathbf{R}_i^{'}} E(G^{'}) = Y$ for every i, $1 \leq i \leq n$

Proof
Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ over Y.

1. Let $P(k)$ be the proposition that $E(G^{'}) \neq \emptyset$ for every $G^{'} \in \mathbf{R}_k^{'}$. Consider, $P(n)$. Since $\mathbf{R}_n^{'}$ is a Linnaean rank over Y, by Definition 27, for every $G^{'} \in \mathbf{R}_n^{'}$, $G^{'} \neq \emptyset$. Moreover, for elements in the bottom rank $E(G^{'}) = G^{'}$. Thus $P(n)$ is true.

Suppose that $P(i)$ is true for some i, $1 < i \leq n$. Thus $E(H^{'}) \neq \emptyset$ for every $H^{'} \in \mathbf{R}^{'}_{i}$. Consider $P(i-1)$. Let $G^{'} \in \mathbf{R}^{'}_{i-1}$ be arbitrary. Then since $\mathbf{R}^{'}_{i-1}$ is a Linnaean rank over $\mathbf{R}^{'}_{i}$, $G^{'}$ is a non–empty subset of $\mathbf{R}^{'}_{i}$. By, the definition of extent, $E(G^{'}) = \bigcup_{H^{'} \in G^{'}} E(H^{'})$. By the inductive hypothesis, the extension of elements in $G^{'}$ is non–empty. Thus $E(G^{'}) \neq \emptyset$. Thus $P(i-1)$ is true. By induction, $P(k)$ is true for all k, $1 \leq i \leq n$.

2. Let $O(k)$ be the proposition that $E(G^{'}) = E(H^{'})$ if and only if $G^{'} = H^{'}$ for every $G^{'}, H^{'} \in \mathbf{R}^{'}_{k}$. Consider $O(n)$. Let $G^{'}, H^{'} \in \mathbf{R}^{'}_{n}$. Certainly, since E is a function, if $G^{'} = H^{'}$, then $E(G^{'}) = E(H^{'})$. Suppose that $E(G^{'}) = E(H^{'})$. By Definition 35, E leaves groups in the lowest rank unchanged. Hence $E(G^{'}) = G^{'}$ and $E(H^{'}) = H^{'}$. Thus, $G^{'} = H^{'}$ and $O(n)$ is true.

   Let $Q(k)$ be the proposition that $E(G^{'}) = E(H^{'})$ or $E(G^{'}) \cap E(H^{'}) = \emptyset$ for every $G^{'}, H^{'} \in \mathbf{R}^{'}_{k}$. Consider $Q(n)$. Let $G^{'}, H^{'} \in \mathbf{R}^{'}_{n}$ be arbitrary. Since $\mathbf{R}^{'}_{n}$ is a Linnaean rank over Y, by Definition 27, $G^{'} = H^{'}$ or $G^{'} \cap H^{'} = \emptyset$. Recall, that the extension function E leaves elements in the bottom rank of an intensive classification unchanged. So $E(G^{'}) = G^{'}$ and $E(H^{'}) = H^{'}$. Thus $E(G^{'}) = E(H^{'})$ or $E(G^{'}) \cap E(H^{'}) = \emptyset$. Thus $Q(n)$ is true.

   As inductive hypotheses, suppose that $O(i)$ and $Q(i)$ are true for some i, $1 < i \leq n$, viz.,

   (a) $E(H^{'}) = E(G^{'})$ if and only if $G^{'} = H^{'}$ for every $G^{'}, H^{'} \in \mathbf{R}^{'}_{i}$.

   (b) $E(G^{'}_{i}) = E(H^{'}_{i})$ or $E(G^{'}_{i}) \cap E(H^{'}_{i}) = \emptyset$ for every $G^{'}_{i}, H^{'}_{i} \in \mathbf{R}^{'}_{i}$.

   Consider $O(i-1)$ and arbitrary $G^{'}, H^{'} \in \mathbf{R}^{'}_{i-1}$. Since E is a function, if $G^{'} = H^{'}$, then $E(G^{'}) = E(H^{'})$. Suppose that $E(G^{'}) = E(H^{'})$. Since $\mathbf{R}^{'}_{i-1}$ is a Linnaean rank over $\mathbf{R}^{'}_{i}$, either $G^{'} = H^{'}$ or $G^{'} \cap H^{'} = \emptyset$. Suppose it is possible that $G^{'} \cap H^{'} = \emptyset$. Then since $G^{'} \neq \emptyset$ (Definition 27), there exists $X \in G^{'}$ and $X \notin H^{'}$.

   Now,

$$\begin{aligned} E(G^{'}) &= \bigcup_{G^{'}_{i} \in G^{'}} E(G^{'}_{i}) \\ E(H^{'}) &= \bigcup_{H^{'}_{i} \in H^{'}} E(H^{'}_{i}) \end{aligned}$$

   Since $O(i)$ is true and $Q(i)$ is true, $E(X) \cap E(H^{'}_{i}) = \emptyset$ for every $H^{'}_{i} \in H^{'}$. Also, since $E(X) \neq \emptyset$, this implies that $E(X) \not\subseteq E(H^{'})$. Since $E(X) \subseteq E(G^{'})$ it obtains that $E(G^{'}) \neq E(H^{'})$. This contradiction shows that $G^{'} = H^{'}$. Thus $O(i-1)$ is true. By induction, $O(k)$ is true for every k, $1 \leq k \leq n$.

   Consider $Q(i-1)$. Let $G^{'}, H^{'} \in \mathbf{R}^{'}_{i-1}$ be arbitrary. Since $\mathbf{R}^{'}_{i-1}$ is a Linnaean rank over $\mathbf{R}^{'}_{i}$. By Definition 27, either $G^{'} = H^{'}$ or $G^{'} \cap H^{'} = \emptyset$. In the case that $G^{'} = H^{'}$, since E is a function, $E(G^{'}) = E(H^{'})$. Consider the case where $G^{'} \neq H^{'}$, then $G^{'} \cap H^{'} = \emptyset$. Suppose that $E(G^{'}) \cap E(H^{'}) \neq \emptyset$. Then for some $G^{'}_{i} \in G^{'}$, $H^{'}_{i} \in H^{'}$, $E(G^{'}_{i}) \cap E(H^{'}_{i}) \neq \emptyset$. By the inductive hypothesis, this implies that $E(G_i^{'}) = E(H_i^{'})$. Since $O(i)$ is true, this implies that $G_i^{'} = H_i^{'}$. Thus, since $\mathbf{R}_{i-1}^{'}$ is a Linnaean rank over $\mathbf{R}_i^{'}$, it obtains that $G^{'} = H^{'}$. Thus $E(G^{'}) = E(H^{'})$. Hence $Q(i-1)$ is true. By induction, $Q(k)$ is true for every k, $1 \leq k \leq n$.

3. Let $R(k)$ be the proposition that $\bigcup_{G^{'} \in \mathbf{R}_k^{'}} E(G^{'}) = Y$. Consider $R(n)$. Since $\mathbf{R}_n^{'}$ is a Linnaean rank over Y, by Definition 27, $\bigcup_{G^{'} \in \mathbf{R}_n^{'}} G^{'} = Y$. Recall that the extension of sets in the lowest rank remains unaltered (Definition 35), i.e., $E(G^{'}) = G^{'}$ for every $G^{'} \in \mathbf{R}_n^{'}$. Thus $\bigcup_{G^{'} \in \mathbf{R}_n^{'}} E(G^{'}) = Y$. Hence, $R(n)$ is true.

   As the inductive hypothesis, suppose $R(i)$ is true for some i, $1 < i \leq n$, viz., $\bigcup_{G^{'} \in \mathbf{R}_i^{'}} E(G^{'}) = Y$. Consider $R(i-1)$.

$$\begin{aligned}
\bigcup_{G^{'} \in \mathbf{R}_{i-1}^{'}} E(G^{'}) &= \bigcup_{G^{'} \in \mathbf{R}_{i-1}^{'}} \left(\bigcup_{H^{'} \in G^{'}} E(H^{'}) \right) \text{ by Definition 35} \\
&= \bigcup_{H^{'} \in \mathbf{R}_i^{'}} E(H^{'}) \text{ since } \mathbf{R}_{i-1}^{'} \text{ is a Linnaean rank over } \mathbf{R}_i^{'} \\
&= Y \text{ by the inductive hypotheis, i.e., } R(i) \text{ is true}
\end{aligned}$$

   Thus $R(i-1)$ is true. By induction, $R(k)$ is true for every k, $1 \leq k \leq n$.

□

Observation 69 [$\psi^{'}$ Makes an Extensive Classification] Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ over Y. The *extensive counterpart* of $L^{'}$, $\psi^{'}(L^{'}) = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ is an extensive Linnaean classification over Y.

Proof
Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ over Y. Let $\psi^{'}(L^{'}) = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ be the extensive counterpart to $L^{'}$. By Observation 68, each $\mathbf{R}_i$ is a Linnaean rank over Y for every i, $1 \leq i \leq n$. What remains is to show that $\mathbf{R}_i$ is above $\mathbf{R}_{i+1}$ for every i, $1 \leq i < k$. So, consider an arbitrary j, $1 \leq j < k$. Let $G \in \mathbf{R}_j$ be arbitrary. By Definition 36, there exists a $G^{'} \in \mathbf{R}_j^{'}$ such that $E(G^{'}) = G$. Since $\mathbf{R}_j^{'}$ is a Linnaean rank over $\mathbf{R}_{j+1}^{'}$, there exists a $G_{j+1}^{'} \in \mathbf{R}_{j+1}^{'}$ such that $G_{j+1}^{'} \in G^{'}$. Now, the extent (Definition 35) of $G^{'}$ gives that $E(G^{'}) = \bigcup_{H^{'} \in G^{'}} E(H^{'})$. Thus, $E(G_{j+1}^{'}) \in \mathbf{R}_{j+1}$ and $E(G_{j+1}^{'}) \subseteq E(G^{'})$.

Consider an arbitrary $G_j \in \mathbf{R}_j$ and $G_{j+1} \in \mathbf{R}_{j+1}$. Then for some $G_j^{'} \in \mathbf{R}_j^{'}$, $E(G_j^{'}) = G_j$ and for some $G_{j+1}^{'} \in \mathbf{R}_{j+1}^{'}$, $E(G_{j+1}^{'}) = G_{j+1}$. Since $\mathbf{R}_j^{'}$ is a Linnaean rank over $\mathbf{R}_{j+1}^{'}$, either $G_{j+1}^{'} \in G_j^{'}$ or for some $H_j^{'} \in \mathbf{R}_j^{'}$, $H_j^{'} \cap G_j^{'} = \emptyset$ and $G_{j+1}^{'} \in H_j^{'}$. Consider the case where $G_{j+1}^{'} \in G_j^{'}$. Then, by Definition 35 $E(G_{j+1}^{'}) \subseteq E(G_j^{'})$, i.e., $G_{j+1} \subseteq G_j$. Consider the case where there exists a $H_j^{'} \in \mathbf{R}_j^{'}$, $H_j^{'} \cap G_j^{'} = \emptyset$ and $G_{j+1}^{'} \in H_j^{'}$. Then by Observation 68, $E(H_j^{'}) \cap E(G_j^{'}) = \emptyset$. Since $E(G_{j+1}^{'}) \subseteq E(H_j^{'})$, this implies that $E(G_{j+1}^{'}) \cap E(G_j^{'}) = \emptyset$. Thus $G_{j+1} \cap G_j = \emptyset$. Thus for an arbitrary j, $1 \leq j < n$, $\mathbf{R}_j$ is above $\mathbf{R}_{j+1}$. Thus $\psi^{'}(L^{'})$ is an extensive

Linnaean classification over Y. $\square$

Observation 70 [Extension Preserved By I] Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Let $\phi(L) = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ be the intensive counterpart of L. For an arbitrary i, $1 \leq i \leq n$, $G_i \in \mathbf{R}_i$,

$$E(I(G_i)) = G_i$$

Proof
Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Let $\phi(L) = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ be the intensive counterpart of L. Let $P(k)$ be the proposition that for every $G_k \in \mathbf{R}_k$, $E(I(G_k)) = G_k$.

Consider $P(n)$. Let $G_n \in \mathbf{R}_n$. Then $I(G_n) \in \mathbf{R}_n^{'}$ and $I(G_n) = G_n$. By Definition 35 the extension of G_n is simply G_n itself. Thus $E(I(G_n)) = G_n$ and $P(n)$ is true.

As the inductive hypothesis, suppose that $P(i)$ is true for some i, $1 < i \leq n$. Consider $P(i-1)$. Let G_{i-1} be an arbitrary element of $\mathbf{R}_{i-1}$. Then $I(G_{i-1}) = \{I(G_i) \mid G_i \in \mathbf{R}_i \text{ and } G_i \subseteq G_{i-1}\}$. Moreover $I(G_{i-1}) \in \mathbf{R}_{i-1}^{'}$. Then,

$$\begin{aligned}
E(I(G_{i-1})) &= \bigcup_{G_i^{'} \in I(G_{i-1})} E(G_i^{'}) \text{ by Definition 35} \\
&= \bigcup_{G_i \in \mathbf{R}_i \text{ and } G_i \subseteq G_{i-1}} E(I(G_i)) \text{ by Definition 34} \\
&= \bigcup_{G_i \in \mathbf{R}_i \text{ and } G_i \subseteq G_{i-1}} G_i \text{ by the inductive hypothesis} \\
&= G_{i-1} \text{ by Observation 63}
\end{aligned}$$

Thus, $P(i-1)$ is true. By induction $P(k)$ is true for all k, $1 \leq i \leq n$, i.e., for any $G_i \in \mathbf{R}_i$, $1 \leq i \leq n$, $E(I(G_i)) = G_i$. $\square$

Observation 71 [Intension Preserved By E] Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ over Y. Let $\phi^{'}(L^{'}) = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ be the extensive counterpart of $L^{'}$. For an arbitrary i, $1 \leq i \leq n$, $G_i^{'} \in \mathbf{R}_i^{'}$, $I(E(G_i^{'})) = G_i^{'}$.

Proof
Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ over Y. Let $\phi^{'}(L^{'}) = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ be the extensive counterpart of $L^{'}$.

Let $P(k)$ be the proposition that for every $G^{'} \in \mathbf{R}_k^{'}$, $I(E(G^{'})) = G^{'}$. Consider $P(n)$. Let $G^{'} \in \mathbf{R}_n^{'}$ be arbitrary. Then, by Definition 35, $E(G^{'}) = G^{'}$; $G^{'}$ resides in the bottom rank. Moreover, by Definition 34, $I(G^{'}) = G^{'}$ since $G^{'}$ lies in the bottom rank of $L^{'}$, $\mathbf{R}_n^{'}$. Thus, $I(E(G^{'})) = G^{'}$. Hence $P(n)$ is true.

As the inductive hypothesis, suppose that $P(i)$ is true for some i, $1 < i \leq n$, viz., $I(E(G^{'})) = G^{'}$ for every $G^{'} \in \mathbf{R}_i^{'}$. Consider $P(i-1)$. Let $G^{'}$ be an arbitrary element of $\mathbf{R}_{i-1}^{'}$. Then,

$$
\begin{aligned}
I(E(G^{'})) &= \{I(G_i) \mid G_i \subseteq E(G^{'}) \text{ and } G(i) \in \mathbf{R}_i\} \text{ Definition 34} \\
&= \{I(E(G_i^{'})) \mid E(G_i^{'}) \subseteq E(G^{'}) \text{ and } G_i^{'} \in \mathbf{R}_i^{'}\} \text{ Definition 34} \\
&= \{I(E(G_i^{'})) \mid G_i^{'} \in G^{'} \text{ and } G_i^{'} \in \mathbf{R}_i^{'}\} \text{ since } G_i^{'} \in G^{'} \text{ if and only if} \\
&\quad E(G_i^{'}) \subseteq E(G^{'})\text{; a straight consequence of Observation 68} \\
&= \{G_i^{'} \mid G_i^{'} \in G^{'} \text{ and } G_i^{'} \in \mathbf{R}_i^{'}\} \text{ by the inductive hypothesis} \\
&= G^{'}
\end{aligned}
$$

□

Corollary 8 [Extensive Circle] Consider a finite set Y and an extensive Linnaean classification $L = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over Y. Then $\psi^{'}(\psi(L)) = L$.
Proof
Consider an arbitrary group $G_i \in \mathbf{R}_i$. This becomes $I(G_i)$ in $\psi(L)$. Moreover, the group is placed in the i'th rank. Subsequently, this becomes $E(I(G_i))$ in $\psi^{'}(\psi(L))$. Once again, this is placed in the i'th rank. By Observation 71, $E(I(G_i)) = G_i$. Thus, $\psi^{'}(\psi(L)) = L$. □

Corollary 9 [Intensive Circle] Consider a finite set Y and an intensive Linnaean classification $L^{'} = (\mathbf{R}_1^{'}, \ldots, \mathbf{R}_n^{'})$ over Y. Then $\psi(\psi^{'}(L^{'})) = L^{'}$.
Proof
Consider an arbitrary group $G_i^{'} \in \mathbf{R}_i^{'}$. This becomes $E(G_i^{'})$ in $\psi^{'}(L)$. Moreover, the group is placed in the i'th rank. Subsequently, this becomes $I(E(G_i^{'}))$ in $\psi(\psi^{'}(L))$. Once again, this is placed in the i'th rank. By Observation 70, $I(E(G_i^{'})) = G_i^{'}$. Thus, $\psi(\psi^{'}(L^{'})) = L^{'}$. □

Observation 72 [Allowable Conglomerations] Consider a genealogical network $G = (X, p)$ and terminal groups $T_1, \ldots, T_k$. Suppose that

1. $T_i \neq \emptyset$ for every i, $1 \leq i \leq k$
2. T_i is allowably monophyletic, for every i, $1 \leq i \leq k$
3. $T_i \cap T_j = \emptyset$ for every i, j, $1 \leq i < j \leq k$.

If for every i, $1 \leq i < k$, $T_i \cup T_{i+1}$ is allowably monophyletic, then $\bigcup_{1 \leq i \leq k} T_i$ is allowably monophyletic.

Proof
Consider a genealogical network $G = (X, p)$ and terminal groups $T_1, \ldots, T_k$. Suppose that

1. $T_i \neq \emptyset$ for every i, $1 \leq i \leq k$
2. T_i is allowably monophyletic, for every i, $1 \leq i \leq k$
3. $T_i \cap T_j = \emptyset$ for every i, j, $1 \leq i < j \leq k$.

Suppose that for every i, $1 \leq i < k$, $T_i \cup T_{i+1}$ is allowably monophyletic. Proof proceeds by induction on the number of terminal groups. Let $P(I)$ be the proposition that $\bigcup_{1\leq i\leq I} T_i$ is allowably monophyletic. Certainly, $P(1)$ is true since T_1, by assumption, is allowably monophyletic. Suppose $P(j)$ is true for some j, $1 \leq j < k$. Consider $P(j+1)$. Since $P(j)$ is true, $\bigcup_{1\leq i\leq j} T_i$ is allowably monophyletic. Now, $T_{j+1} \cap \bigcup_{1\leq i\leq j} T_i = \emptyset$ since $T_{j+1} \cap T_i = \emptyset$ for every i, $1 \leq i \leq j$. Then, by Observation 62, $\bigcup_{1\leq i\leq j+1} T_i$ is allowably monophyletic. Thus, $P(I)$ is true for every I, $1 \leq I \leq k$. Then, by $P(k)$, $\bigcup_{1\leq i\leq k} T_i$ is allowably monophyletic. $\square$

E Proofs for Section 6

Observation 73 [Single Ancestor Path] Consider a genealogical tree $G = (X, p)$ and an individual $x \in X$. For any $x_1, x_2 \in X$, if both x_1 and x_2 are ancestors of x, then x_1 is an ancestor of x_2 or x_2 is an ancestor of x_1.

Proof
Consider a genealogical tree $G = (X, p)$ and an individual $x \in X$. Consider arbitrary $x_1, x_2 \in X$ such that both x_1 and x_2 are ancestors of x. Let $y_1, \ldots, y_n$ and $z_1, \ldots, z_m$ ($n, m \geq 1$) be paths from x_1 to x and x_2 to x respectively. Let $P(k)$ be the proposition that $y_{n-k} = z_{m-k}$. Consider $P(0)$. Since both sequences terminate at x, $y_n = z_m = x$. Hence $P(0)$ is true. Suppose that $P(i)$ is true for some i such that $n - i \geq 1$ and $m - i \geq 1$. Suppose that $n - (i+1) \geq 1$ and $m - (i+1) \geq 1$. Since $y_{n-i} = z_{m-i}$ and G is a genealogical tree, $y_{n-(i+1)} = z_{m-(i+1)}$. By, induction, either $y_1, \ldots, y_n$ is a subsequence of $z_1, \ldots, z_m$ or *vice versa*. Since $y_1 = x_1$ and $z_1 = x_2$, this implies that x_1 is an ancestor of x_2 or x_2 is an ancestor of x_1. $\square$

Observation 74 [Disjoint Descent Groups in a Tree] Consider a genealogical tree $G = (X, p)$ and subsets X_1 and X_2 of X. Suppose that X_1 and X_2 are minimal generating sets. Moreover, suppose that for every $x_1 \in X_1$ and $x_2 \in X_2$ that x_1 is not an ancestor of x_2 and x_2 is not an ancestor of x_1. Then, $cl(X_1) \cap cl(X_2) = \emptyset$.

Proof
Consider a genealogical tree $G = (X, p)$ and subsets X_1 and X_2 of X. Suppose that X_1 and X_2 are minimal generating sets. Moreover, suppose that for every $x_1 \in X_1$ and $x_2 \in X_2$ that x_1 is not an ancestor of x_2 and x_2 is not an ancestor of x_1. For a proof by contradiction, suppose that $cl(X_1) \cap cl(X_2) \neq \emptyset$. Let z be an arbitrary element of $cl(X_1) \cap cl(X_2)$. Then, by Observation 13, for some $x_1 \in X_1$ and $x_2 \in X_2$, both x_1 and x_2 are ancestors of z. $\square$

Observation 75 [Monophyletic Group] Consider a genealogical tree $G = (X, p)$ and a non–empty descent group D in G. D is monophyletic if and only if $\mid P(D) \mid = 1$.

Proof
Consider a genealogical tree $G = (X, p)$ and a non–empty descent group D in G.

($\Rightarrow$) Suppose that D is monophyletic. For a proof by contradiction, suppose $\mid P(D) \mid \neq 1$. Since D is non–empty, this discounts the possibility that $P(D)$ is empty. Thus, D has at least two progenitors. Suppose that x is a progenitor in $P(D)$. Consider

$D_1 = cl(\{x\})$ and $D_2 = cl(P(D) \setminus \{x\})$. By Observation 9, both D_1 and D_2 are descent groups. Now, $D_1 \cap D_2$ cannot intersect because otherwise x and some $y \in P(D) \setminus \{x\}$ share a common descendant z. By Observation 73, this implies that x is an ancestor of y or y is an ancestor of x. This is impossible since both x and y are progenitors of D. Thus D_1 and D_2 witness that D is polyphyletic. This contradicts our assumption that D is monophyletic. Hence, $\mid P(D) \mid= 1$.

($\Leftarrow$) This is just Corollary 3. $\square$

Observation 76 [Polyphyletic Group] Consider a genealogical tree $G = (X, p)$ and a non–empty descent group D in G. D is polyphyletic of maximal degree k if and only if $\mid P(D) \mid = k$.

Proof
Consider a genealogical tree $G = (X, p)$ and a non–empty descent group D in G.

($\Rightarrow$) Suppose that D is polyphyletic of maximal degree k. Then, it cannot be the case that $\mid P(D) \mid < k$ since $P(D)$ can be partitioned into k non–empty sets as witnessed by the fact that D is polyphyletic of degree k; each partition must contain at least one progenitor from D (Observation 29). Also, it is impossible for $\mid P(D) \mid > k$. Otherwise, let $P(D) = \{x_1, \ldots, x_m\}$ for some $m > k$. Let $D_i = cl(\{x_i\})$ for $1 \leq i \leq m$. For any $1 \leq I < J \leq m$, x_I is not an ancestor of x_J and x_J is not an ancestor of x_I since x_I and x_J are progenitors in D. Then by Observation 15 and Observation 74, $D_1, \ldots, D_m$ is a witness to D being polyphyletic of degree m. This is a contradiction since D is polyphyletic to maximal degree k. Thus $\mid P(D) \mid = k$.

($\Leftarrow$) Suppose that $\mid P(D) \mid = k$. Let $P(D) = \{x_1, \ldots, x_k\}$. Let $D_i = cl(\{x_i\})$ for every $i, 1 \leq i \leq k$. Then clearly $D_i \neq \emptyset$ and, by Observation 9, D_i is a descent group. By Observation 15, $\bigcup_{1 \leq i \leq k} D_i = D$. Also, for any $1 \leq I < J \leq m$, x_I is not an ancestor of x_J and x_J is not an ancestor of x_I since x_I and x_J are progenitors in D. Thus, by Observation 74, $D_i \cap D_j = \emptyset$ for every $i, j, 1 \leq i < j \leq k$. Thus D is polyphyletic of degree k. By Observation 29, D cannot be polyphyletic of degree m where $m > k$; each descent group in a polyphyletic witness must contain a progenitor. Thus, D is polyphyletic of maximal degree k. $\square$

Observation 80 (Paraphyletic Set in a Family Tree) *Consider a genealogical tree $G = (X, p)$ and a paraphyletic group E in G. If $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ are witnesses to E, then $D_1 = D_2$ and $D_1^{'} = D_2^{'}$.*

Proof
Consider a genealogical tree $G = (X, p)$ and a paraphyletic group E in G. Let $(D_1, D_1^{'})$ and $(D_2, D_2^{'})$ be witnesses to E. Firstly, prove that $D_1 = D_2$. Since $E \neq \emptyset$, $D_1 \cap D_2 \neq \emptyset$. Thus, by the converse of Observation 75, $D_1 = cl(\{x_1\})$ and $D_2 = cl(\{x_2\})$ for some $x_1, x_2 \in X$. Also, since $D_1 \cap D_2 \neq \emptyset$, by Observation 73 x_1 is an ancestor of x_2 or x_2 is an ancestor of x_1. Suppose x_1 is an ancestor of x_2. Suppose it is possible that $x_1 \neq x_2$. Then $x_1 \notin E$ and $x_1 \in D_1^{'}$. This is impossible because this implies that $D_1 = D_1^{'}$ which makes $D = \emptyset$. Thus $x_1 = x_2$ and $D_1 = D_2$. Since $D_1^{'} \subseteq D_1$ and $D_2^{'} \subseteq D_2$, this implies that $D_1^{'} = D_2^{'}$. $\square$