EPDL: A Logic for Causal Reasoning

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Abstract

This paper is twofold. First, we presentes an extended system $EPDL$ of propositional dynamic logic by allowing a proposition as a modality in order to represent and specify indirect effects of actions and causal propagation. An axiomatic deductive system is given which is sound and complete with respect to the corresponding semantics. The resultant system provides a unified treatment of direct and indirect effects of actions. Second, we reduce the $EPDL$ into a mutlimodal logic by deleting the component of action in order to obtain an axiomatized logical system for causal propagation. A characterization theorem of the logic is given. Properties of causal reasoning with the logic are discussed.

1 Introduction

Dynamic logic is one of the formalisms for specifying and reasoning about action and change that has been proposed explicitly or implicitly by several authors, such as [Harel 1979] [Rosenschein 1981] [Kautz 1982] [Giacomo and Lenzerini 1995] [Prendinger and Schurz 1996] [Castilho et al 1999], in the last twenty years. There are some features of dynamic logic which distinguish it from the other formalisms of action.

First, dynamic logic was originally developed for modelling programs. From the computer science perspective, any program can be viewed as an action and any action can be implemented by program. Therefore dynamic logic should be a natural formalism for reasoning about action. [1]

Second, dynamic logic can express and specify compound actions [2], which are necessary to represent complex plans ([Rosenschein 1981] [Kautz 1982]), robot controllers ([Levesque *et al.* 1997]), automatic systems and some special kinds of complex actions(c.f.Example 6). Dynamic logic expresses such actions more naturally than other action formalisms (see [Levesque *et al.* 1997]).

Third, dynamic logic is a logical system with sound and complete axiomatization and well-developed Kripke semantics. Both its proof and model theory have reached a high degree of sophistication through the development of theoretical computer science. Some features, such as decidability and the finite model property of propositional dynamic logic (PDL), and techniques such as bisimulation and filtration, are well understood.

However, this does not mean that the existing dynamic logic is adequate for modeling action and change.

In dynamic logic, the causal relation between an action and its effects is expressed by action or program modality. Let α be a action and A a property. "α **causes** A to be true" can be expressed by the formula $[\alpha]A$. For instance, $[Shoot]\neg alive$ means that the action $Shoot$ causes a turkey to be dead. In other words, $\neg alive$ is the effect of action $Shoot$. In many cases, effects of an action can be further propagated through causal relations between affairs. For instance, if a turkey is shot down, its death will cause the turkey to be unable to walk: $\neg alive$ **causes** $\neg walking$. However, this can not be formally expressed in traditional dynamic logic. Obviously it can not simply written as either $\neg alive \rightarrow \neg walking$ or $[\neg alive]\neg walking$. The

[1] See the preface of [Harel 1979].

[2] Compound actions are actions generated from primitive actions by the program connectives $;, \cup, ?, *$.

first expression is unsuitable because it is equivalent to $walking \rightarrow alive$ but "$\neg alive$ **causes** $\neg walking$" does not mean "$walking$ **causes** $alive$". The second one is wrong because it is not allowed in the syntax of traditional dynamic logic. One idea is why not just extend the traditional dynamic logic in the way of allowing a proposition as a modality, say $[\varphi]$ for proposition φ. Apparently, there are many similarities between *causation triggered by action* and *causation triggered by proposition*. For instance, "*switch_on* **causes** *light*" can be explained by both of the statements:

(1) Switching on the circuit causes the light to be on;

(2) The switch is on causes the light to be on.

In the first statement, *switch_on* is viewed as an action, so *light* is a direct effect of the action. In the second one, it is treated as a proposition, so *light* is an indirect effect of some action, say *Toggle_switch*, through the causal propagation of the proposition *switch_on*. Therefore if we allow a proposition as a modality[3], we will have a way to unify the treatment of the causation triggered by actions and by propositions, and likewise the treatment of direct and indirect effects of actions.

In this paper, we first present an extended system $EPDL$ of propositional dynamic logic by allowing a proposition as a modality to encompass causation between propositions and indirect effects of actions. The extension is as conservative as possible to preserve desirable attributes of PDL, in particular its inferential style, soundness and completeness of the axiom system, decidability and the finite model property. The system enables a unified treatment of direct and indirect effects of action. Secondly, but perhaps more interestingly, we reduce the $EPDL$ into a multiple modal logic by deleting action component in order to obtain a logic for causal propagation[4]. The resultant system has a ready axiomatic deductive system and semantics of dynamic logic. We present a characterization theorem of this logic and discuss its properties in causal reasoning.

2 Extended Propositional Dynamic Logic

In this section, we extend the PDL syntax by introducing proposition modalities, axioms and intended semantics. The resulting logic is called EPDL.

[3] A similar approach has been used by [Groeneveld 1995] to deal with changes in knowledge. Besides the motivation, however, the semantics and deduction system are totally different from ours.

[4] We can not do the same with PDL because otherwise the resultant system will collapse to the classical propositional logic.

2.1 Language of $EPDL$

The alphabet of the language $\mathcal{L}_{EPDL}$ of the extended propositional dynamic logic consists of a set **Flu** of countable fluent symbols and a set $\mathbf{Act}_P$ of countable primitive action symbols.

Proposition ($\varphi \in \mathbf{Pro}$), formula ($A \in \mathbf{Fma}$) and action ($\alpha \in \mathbf{Act}$) are defined by the following BNF rules:

$\varphi ::= f \mid \neg\varphi \mid \varphi_1 \to \varphi_2$
$A ::= f \mid \neg A \mid A_1 \to A_2 \mid [\alpha]A \mid [\varphi]A$
$\alpha ::= a \mid \alpha_1; \alpha_2 \mid \alpha_1 \cup \alpha_2 \mid \alpha^* \mid A?$
where $f \in \mathbf{Flu}$ and $a \in \mathbf{Act}_P$.

The intended meaning for $[\alpha]A$ is "α (always) causes A if α is feasible". For example, $[Turn\_off]\neg light$ says that "turning off the switch causes the light to be off". The dual operator $\langle\alpha\rangle A$, defined as usual, reads as "α is feasible and possibly (or may) cause(s) A to be true". For instance, $\langle Spin\rangle\neg loaded$ says that "spinning a gun may cause it to be unloaded". Note that $\langle\alpha\rangle\top$ means "α is feasible or executable".

The definition of $\top$(**true**), $\bot$(**false**), $\vee$, $\wedge$, $\leftrightarrow$ are as usual. A literal is a fluent or its negation. The set of all the literals in $\mathcal{L}_{EPDL}$ is denoted by $\mathbf{Flu}_L$.

The formula $[\varphi]A$, called a *propositional causation*, represents the cause-effect relationship between the proposition φ and the formula A, read as "φ **causes** A". For example, $[short\text{-}circuit]damaged$ says that "being short-circuit causes the circuit to be damaged".

Notice the difference between $[\varphi]$ and $[\varphi?]$. $\varphi?$ is an action, called a *test action*, which can be compounded with other actions but φ can not be compounded with actions.

We introduce the following two notations for the future use:
$\langle[\alpha]\rangle A =_{def} \langle\alpha\rangle\top \wedge [\alpha]A$, meaning "$\alpha$ must cause A";
$\prec \alpha \succ A =_{def} \langle\alpha\rangle\top \to \langle\alpha\rangle A$, meaning "if α is feasible, α may cause A".
Note that these two modal operators are dual, i.e. $\prec \alpha \succ A = \neg\langle[\alpha]\rangle\neg A$.

2.2 Semantics

The semantics of EPDL is the standard PDL semantics plus the interpretation of propositional causation.

As usual, a model for $\mathcal{L}_{EPDL}$ is a structure $M = (W, \mathcal{R}, V)$, where $\mathcal{R} = \{R_\alpha : \alpha \in \mathbf{Act}\} \cup \{R_\varphi : \varphi \in \mathbf{Pro}\}$. The components of W, $\{R_\alpha : \alpha \in$

Act} and V are as same as the ones of PDL model. For any propositional formula φ, R_φ is also a binary relation on W.

The satisfaction relation $M \models_s A$ is defined as follows:

$M \models_w f$ iff $f \in V(f)$, for any $f \in \mathbf{Flu}$;

$M \models_w A_1 \rightarrow A_2$ iff $M \models_w A_1$ implies $M \models_w A_2$

$M \models_w [\gamma]A$ iff for all $w' \in S$, $wR_\gamma w'$ implies $M \models_{w'} A$, where $\gamma \in \mathbf{Act} \cup \mathbf{Pro}$.

This means that the modalities of both action and proposition are normal modal connectives. As in any modal logic, $A \in \mathbf{Fma}$ is valid in M, written as $M \models A$, if $M \models_w A$ for all $w \in W$. "$\models A$" means "A is valid in any model".

It is easy to see that if $\models_{PL}$ denotes the satisfiablity relation of the classical propositional logic, then for any $\varphi \in \mathbf{Fma}_P$,

$$\models_{PL} \varphi \text{ if and only if } \models \varphi.$$

The conditions for *standard models* of $\mathcal{L}_{EPDL}$ consist of the ones for program connectives:

$R_{\alpha;\beta} = R_\alpha \circ R_\beta$

$R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$

$R_{\alpha^*} = R_\alpha^*$

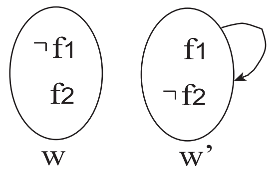
$R_{A?} = \{(w,w) : M \models_w A\}$

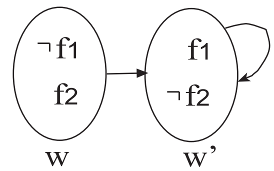
and the extra conditions for the propositional causation:

1. If $M \models_w \varphi$, then $(w,w) \in R_\varphi$.
2. If $\models \varphi_1 \rightarrow \varphi_2$, then $R_{\varphi_1} \subseteq R_{\varphi_2}$.

The first condition for the propositional causation says that $[\varphi]$ is a kind of weakening of $[\varphi?]$. In fact, for any proposition φ, we have $\models [\varphi]A \rightarrow [\varphi?]A$ because $R_{\varphi?} \subseteq R_\varphi$. The following example shows that they are different.

Example 1 *Let $\mathcal{L}$ be an language of EPDL in which $\mathbf{Flu} = \{f_1,\ f_2\}$ and $\mathbf{Act}_P = \{\}$. Let $M = (W, \mathcal{R}, V)$ be a standard model of $\mathcal{L}$, where $W = \{w, w'\}$, $R_{f_1} = \{(w,w'),\ (w',w')\}$, $V(f_1) = \{w'\}$, $V(f_2) = \{w\}$ and the others could be anything consistent with the conditions of standard models (see the following figure). Then we have $M \models_w [f_1?]f_2$, but $M \not\models_w [f_1]f_2$.*





Accessibility relation for $[f_1?]$. *Accessibility relation for* $[f_1]$.

From this example we can see that to verify $M \models_w [\varphi?]A$, we only need to consider the truth-values of φ and A in the current world w whereas to verify $M \models_w [\varphi]A$, we not only need to check whether A follows φ if φ is true in the world w but also have to consider the case when an action changes the world to other relevant worlds. In other words, the test action only concerns the current state of the system while a causal relation is sensitive to both the history and the future evolution of the system because it is action-relevant.

2.3 Deductive system

The axiom system for $EPDL$ consists of the following axiom schemes and inference rules:

(1). Axiom schemes:

- all tautologies of propositional calculus.
- all axioms for compound programs:
  $Comp : [\alpha;\beta]A \leftrightarrow [\alpha][\beta]A$
  $Alt : [\alpha \cup \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$
  $Test : [A?]B \leftrightarrow (A \rightarrow B)$
  $Mix : [\alpha*]A \rightarrow A \wedge [\alpha][\alpha*]A$
  $Ind : [\alpha^*](A \rightarrow [\alpha]A) \rightarrow (A \rightarrow [\alpha^*]A)$
- EK axiom: $[\gamma](A \rightarrow B) \rightarrow ([\gamma]A \rightarrow [\gamma]B)$
- CW axiom: $[\varphi]A \rightarrow [\varphi?]A$

(2). Inference rules:

- MP: From A and $A \rightarrow B$, infer B.
- EN: From A, infer $[\gamma]A$.
- LC: From $\varphi_1 \rightarrow \varphi_2$, infer $[\varphi_2]A \rightarrow [\varphi_1]A$.

where φ, $\varphi_1, \varphi_2 \in \mathbf{Pro}$, $A \in \mathbf{Fma}$, $\alpha \in \mathbf{Act}$ and $\gamma \in \mathbf{Pro} \cup \mathbf{Act}$.

Provability in $EPDL$ is denoted by $\vdash$. A formula A is called to be *provable* from a set Γ of formulas, denoted by $\Gamma \vdash A$, if there exists $A_0, \cdots, A_n \in \Gamma$ such that $\vdash A_0 \rightarrow (\cdots \rightarrow (A_{n-1} \rightarrow A) \cdots)$. Γ is *consistent* in $EPDL$ if $\Gamma \nvdash \bot$.

Comparing with the axiomatic system of PDL, we find that the K axiom and the inference rule necessitation N of PDL have been extended into new forms EK and EN, respectively, so that they are not only applicable for actions but also for proposition. The EK specifies propagation of causations under logical implication. The inference rule EN says that tautologies can be caused by anything.

The axiom CW and the inference rule LC are new added specially for propositional causation. The axiom CW is corresponding to the semantic condition 1 of propositional causation. It reflects the standard way to find and judge a causal relation. Since it is equivalent to $[\varphi]A \rightarrow (\varphi \rightarrow A)$, CW also specifies the relationship between propositional causality and logical implication[5]. Notice that this rule coincides with the conditional logic axiom $\varphi > \psi \rightarrow (\varphi \rightarrow \psi)$ if we view the propositional causation $[\varphi]\psi$ as a conditional assertion (See [Nute 1984] [Gardenfors 1988]). Note that if we add an extra axiom "$[\varphi?]A \rightarrow [\varphi]A$" into the deductive system, $EPDL$ will collapse to PDL.

The inference rules LC comes from the conditional logic (see [Gardenfors 1988] p.149). It specifies the relationship between logical relevance and causal relevance, which says that if φ_2 is a logical consequence of φ_1, then φ_2 **causes** A must imply that φ_1 **causes** A.[6]

We would like remark that our axiomatic system is by no means enough for specifying the causality. We do not aim to proposal an ideal causal theory to satisfy all purpose. In fact, we try to introduce as few axioms as possible in order to make the resultant system simpler, less controversial and meanwhile satisfy the requirement for dealing with the fundamental problems in reasoning about action and change. It is greatly encouraged to add new axioms or modify the extant axioms of EPDL in order to satisfy some special requirement.

The following Lemmas can be derived straightforward from the deductive system of $EDPL$.

Lemma 1

[5] CW also shows that the propositional causality we consider here is deterministic because probabilistic causality does not implies material implication.

[6] This rule could be controversy because it face the same problem about "false antecedent" as the material implication do. For instance, LC implies that $\vdash [\varphi]\psi \rightarrow [\perp]\psi$, which says that if something causes ψ, then false can cause it as well. A remedy for this problem is substituting LC by an axiom of conditional logic: $[\varphi]\psi \wedge \neg[\varphi]\neg\chi \rightarrow [\varphi \wedge \chi]\psi$ (see [Nute 1984]). But this will invoke much more complicated semantics.

1. $\vdash [\varphi]A \to (\varphi \to A)$

2. *If* $\vdash A \leftrightarrow B$, *then* $\vdash [\varphi]A \leftrightarrow [\varphi]B$

3. *If* $\vdash \varphi_1 \leftrightarrow \varphi_2$, *then* $\vdash [\varphi_1]A \leftrightarrow [\varphi_2]A$

Note that we have not the following tautologies:
$\vdash (\psi_1 \to \psi_2) \to ([\varphi]\psi_1 \to [\varphi]\psi_2)$,
$\vdash (\varphi_1 \to \varphi_2) \to ([\varphi_2]\psi \to [\varphi_1]\psi)$,
because the former implies that $\psi_1, \psi_2 \vdash [\varphi]\psi_1 \to [\varphi]\psi_2$, which says that "if two things happened at the same time, they must be caused by the same reason". And the later implies that $\varphi_1, \varphi_2 \vdash [\varphi_2]\psi \to [\varphi_1]\psi$, which says that "if two things happened at the same time, they will cause same results".

2.4 Soundness and completeness

The soundness of axioms and inference rules of $EPDL$ is easy to check, so we have

Theorem 1 (Soundness) *If* $\vdash A$, *then* A *is valid in all standard models of* $\mathcal{L}_{EPDL}$.

For the completeness of the deductive system for $EPDL$, we can follow Fischer and Ladner's proof for PDL step by step while adding the proof for extended components. We will only present the part for the new introduced components and omit all the other steps (refer to [Goldblatt 1987] for details).

Let Γ be a set of formulas. The Fisher-Ladner closure of Γ, denoted by $FL(\Gamma)$, is the smallest set Δ of formulas satisfying the following conditions:

$\Gamma \subseteq \Delta$;
Δ is closed under subformulas;
$[\varphi]A \in \Delta$ implies $\varphi \in \Delta$;
$[\alpha;\beta]A \in \Delta$ implies $[\alpha][\beta]A \in \Delta$;
$[\alpha \cup \beta]A \in \Delta$ implies $[\alpha]A \in \Delta$ and $[\beta]A \in \Delta$;
$[\alpha*]A \in \Delta$ implies $[\alpha][\alpha*]A \in \Delta$;
$[A?]B \in \Delta$ implies $A \in \Delta$.

It is easy to see that Fisher and Ladner's Lemma remains true, that is, Γ is finite implies that $FL(\Gamma)$ is finite. We shall call Γ to be *FL-closed* if Γ is finite and $FL(\Gamma) = \Gamma$.

A *canonical* model of $\mathcal{L}_{EPDL}$ based on the above deductive system is a model $M^C = (S^C, \{R^C_\alpha : \alpha \in \mathbf{Act}\} \cup \{R^C_\varphi : \varphi \in \mathbf{Fma}_P\}, V^C)$ where

- $W^C = \{$maximal consistent sets of formulas of $\mathcal{L}_{EPDL}\}$;
- $R^C_\gamma = \{(w, w') : \forall [\gamma] A \in w (A \in w')\}$,
  where $\gamma \in \mathbf{Fma}_P \cup \mathbf{Act}$;
- $V^C(f) = \{w \in W^C : f \in w\}$.

It is not hard to prove that $M^C \models_w A$ iff $A \in w$ for any $w \in W^C$. Now for a given canonical model M^C and a FL-closed set Γ of formulas, we can define a Γ-filtration M^Γ of M^C in the usual way except for letting
$R^\Gamma_\varphi = \{(|u|, |v|) : \exists u' \in |u| \, \exists v' \in |v| \, (u' R^C_\varphi v')\}$.

Following the same steps of the proof of Filtration Lemma in PDL (see Filtration Lemma 10.8, page 115 in [Goldblatt 1987]), we can prove that M^Γ is a Γ-filtration of M^C. Therefore, For any $A \in \Gamma$,

$$M^C \models_w A \text{ iff } M^\Gamma \models_{|w|} A$$

Although M^C may not be standard, M^Γ is, that is

Lemma 2 *M^Γ is a standard model.*

Theorem 2 (Completeness) *If A is valid in all standard models of $\mathcal{L}_{EPDL}$, then $\vdash A$.*

Similar to the PDL, we have the following decidability of EPDL.

Proposition 1 *Validity in $EPDL$ is decidable in deterministic exponential time.*

The following proposition is interesting because a primitive action symbol could act as a free variable of action. However, we can not quantify over action variables in dynamic logic. This might be the weakness of dynamic logic comparing to situation calculus in expressing action effects.

Proposition 2 *If $A \vdash B(a)$ where a is a primitive action symbol in $\mathcal{L}_{EPDL}$ occurring in $B(a)$ but not occurring in A, then $A \vdash B(\alpha)$ for any action α in which there is no the occurrence of a, where $B(\alpha)$ is the result of substituting α for all occurrences of a in $B(a)$.*

3 Reasoning on Action Descriptions

One purpose of action logic is to provide a formal language to describe effects of actions. We can distinguish two kinds of such descriptions: one is called *action description* which specifies the generic effects of actions and causal relationships in a dynamic system; the other is *query*, which is used for expressing a prediction of the effects of a certain action running under a certain situation or an explanation of observed phenomena. This categorization is similar to the one in action languages [Gelfond and Lifschitz 1998], where action description languages are used to express action descriptions and action query languages for describing queries. An important difference with our approach is that we use the same language to describe both kinds of information. Of cause, they have different inference mechanisms. In this section, we shall develope some special inference mechanism for reasoning about action effects by the extended dynamic logic.

3.1 Action description

An action description of a dynamic system is a set of formulas which specifies effects of actions, causal relations, domain constraints and qualifications of action execution.

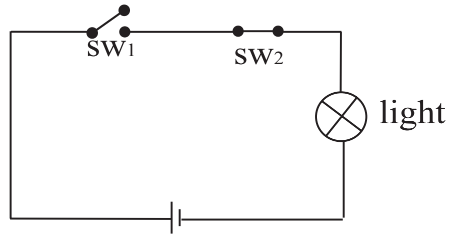
Example 2 *Consider the Yale Shooting Problem in [Hanks and McDermott 1987]. Let* $\mathbf{Flu} = \{alive, loaded, walking\}$ *and* $\mathbf{Act}_P = \{Load, Shoot, Wait\}$. *Then this problem can be described by the following action description:*

$$\Sigma = \left\{ \begin{array}{l} \neg loaded \to [Load]loaded \\ loaded \to [Shoot]\neg alive \\ loaded \to [Shoot]\neg loaded \\ [\neg alive]\neg walking \\ \langle Load\rangle\top, \langle Wait\rangle\top, \langle Shoot\rangle\top \end{array} \right\}$$

The first three sentences describe the direct effects of action *Load* and *Shoot*. The fourth one specifies the causal relation between *alive* and *working*. The last three express the qualification of execution of three actions, meaning they are all executable in any case. Note that we have not listed the unaffected information, or *frame axioms,* into the action description. In other words, this action description is not a complete specification of the problem. A treatment of frame axioms was presented in a sequent paper.

Example 3 *Consider the circuit of the following figure. Let* Σ *be the action description of this circuit:*

$$\Sigma = \left\{ \begin{array}{l} \neg sw_i \rightarrow [Toggle_i]sw_i \\ sw_i \rightarrow [Toggle_i]\neg sw_i \\ [sw_1 \wedge sw_2]light \\ [\neg sw_1 \vee \neg sw_2]\neg light \\ \langle Toggle_i \rangle \top \\ i = 1, 2 \end{array} \right\}$$



The first two expressions describe the direct effects of action $Toggle_1$ and $Toggle_2$. The middle two expressions specify the causal relation among the fluents in the circuit. "$\langle Toggle_i \rangle \top$" means the action $Toggle_i$ is always executable.

3.2 Reasoning on Action Descriptions

We can easily see that a formula in an action description is different from an ordinary formula. For instance, the sentence "$loaded \rightarrow [Shoot]\neg alive$" in an action description means that whenever $loaded$ is true, $Shoot$ must cause $\neg alive$. In situation calculus language, it can be expressed as $\forall s(loaded(s) \rightarrow \neg alive(do(Shoot, s)))$ (see [Reiter 1991]). A similar expression in dynamic logic by introducing a special action "**any**", meaning "any action", is $[\mathbf{any}](loaded \rightarrow [Shoot]\neg alive)$. With help of $[\mathbf{any}]$, a sentence in an action description can be easily differentiated from an ordinary formula. In [Goranko and Passy 1992], they showed that $[\mathbf{any}]$ is exactly an S_5-modality. Therefore, if we want to introduce **any** formally as [Prendinger and Schurz 1996] and [Castilho et al 1999][7] did, we must extend our system further by adding all the axioms of S_5 to the modal operator **any** and an extra axiom "$[\mathbf{any}]A \rightarrow [\alpha]A$", to say that **any** is a universal action. Instead of doing this, however, we prefer a simpler way to deal with action description by treating it as extra axioms in reasoning with action, like the way in situation calculus (action description as domain axioms, see [Reiter 1991]).

3.2.1 Σ-Provability

Let Σ be an action description. A formula A is a Σ-*theorem*, written by $\vdash^{\Sigma} A$, if it belongs to the least set of formulas which contains all the theorems of $EPDL$, all the elements of Σ, and is closed under MP and EN.

[7] [Castilho et al 1999] treated $[\mathbf{any}]$ as S_4-modality.

As usual, for any $\Gamma \subseteq \mathbf{Fma}$, a sentence A is Σ-*provable* from Γ, written by $\Gamma \vdash^{\Sigma} A$, if there exist $A_1, \cdots, A_n \in \Gamma$ such that

$$\vdash^{\Sigma} A_1 \to (\cdots (A_n \to A) \cdots).$$

Note that we do not require that Σ-provability is closed under LC. In fact, it is provable in the following sense[8]:

If $\vdash \varphi_1 \to \varphi_2$, then $\vdash^{\Sigma} [\varphi_2]A \to [\varphi_1]A$.

See the following Lemma 3 [3].

Example 4 *Let* Σ *be the action description in Example* 2*. We prove that* $\{\neg loaded\} \vdash^{\Sigma} [load; shoot]\neg alive$.

Proof. Since $loaded \to [shoot]\neg alive \in \Sigma$, by (EN), $[load](loaded \to [shoot]\neg alive)$ is a Σ-theorem. Thus $[load]loaded \to [load][shoot]\neg alive$ and then $[load]loaded \to [load; shoot]\neg alive$ are Σ-theorem. By $\neg loaded \to [load]loaded \in \Sigma$, we have $\neg loaded \to [load]loaded$ is a Σ-theorem. Thus we obtain $\neg loaded \to [load; shoot]\neg alive$ is Σ-theorem. So $\vdash^{\Sigma} \neg loaded \to [load; shoot]\neg alive$, i.e., $\neg loaded \vdash^{\Sigma} [load; shoot]\neg alive$, as desired. □

In order to make the prove clear, we write the above procedure of deduction in the following form:

$\neg loaded$

(1)* $\vdash^{\Sigma} loaded \to [Shoot]\neg alive$ (AD)

(2)* $\vdash^{\Sigma} [Load](loaded \to [Shoot]\neg alive)$ (1 and EN)

(3)* $\vdash^{\Sigma} [Load]loaded \to [Load][Shoot]\neg alive$ (2 and EK)

(4)* $\vdash^{\Sigma} [Load]loaded \to [Load; Shoot]\neg alive$ (3 and $Comp$)

(5)* $\vdash^{\Sigma} \neg loaded \to [Load]loaded$ (AD)

(6)* $\vdash^{\Sigma} \neg loaded \to [Load; Shoot]\neg alive$ (4 and 5)

(7). $\vdash^{\Sigma} [Load; Shoot]\neg alive$ (Γ and 6)

where AD indicates "Action Description in Σ"; $Comp$ is an axiom of PDL; EN and EK are the inference rules of EPDL; Γ represent the promises. "*" means the formula is a Σ -theorem, so can use inference rules EN.

The reader is invited to verify the following inference relations (with standard PDL program-like abbreviations):

[8]Note that the following inference is not true:
if $\vdash^{\Sigma} \varphi_1 \to \varphi_2$, then $\vdash^{\Sigma} [\varphi_2]A \to [\varphi_1]A$.
As we mentionedd before, LC reflects the causal propogation through logical necessarity ($\vdash \varphi_1 \to \varphi_2$) rather than accidental truth link.

1. $\neg loaded \vdash^{\Sigma} \langle [Load; Shoot] \rangle \neg alive$

2. $\vdash^{\Sigma} \langle [Load; Wait;$ **if** $\neg loaded?$ **do** $Load$ **endif**; $Shoot] \rangle \neg alive$

We remark that we can neither prove nor refute $\neg loaded \vdash^{\Sigma} \langle [Load; Wait; Shoot] \rangle \neg alive$ now because it requires frame axioms. The frame problem will be dealt with elsewhere.

Example 5 *Consider the action description in Example 3. We can easily prove the following inference relations:*
1. $\vdash^{\Sigma} (sw_1 \wedge sw_2) \leftrightarrow light$
2. $sw_1 \vdash^{\Sigma} [Toggle_1] \neg light$
3. $\vdash^{\Sigma} [\neg sw_1] \neg light$

The first expression shows that the causal laws "$[sw_1 \wedge sw_2] light$" and "$[\neg sw_1 \vee \neg sw_2] \neg light$" imply the domain constraint "$(sw_1 \wedge sw_2) \leftrightarrow light$". So, for example, if we know switch 1 is closed but the light is off, we can infer that switch 2 must be open, i.e. $sw_1 \wedge \neg light \vdash^{\Sigma} \neg sw_2$. The second inference relation reflects the propagation of effects of action $Toggle_1$ through the causal law $[\neg sw_1 \vee \neg sw_2] \neg light$. The third one is a derived causal law. It comes from the causal law $[\neg sw_1 \vee \neg sw_2] \neg light$ and the logical relation $\vdash \neg sw_1 \rightarrow (\neg sw_1 \vee \neg sw_2)$ by applying LC rule.

Example 6 *Consider an action description* Σ *as follows:*

$$\Sigma = \left\{ \begin{array}{l} thirsty \rightarrow \langle Drink\_a\_mouthful\_of\_water \rangle \neg thirsty \\ thirsty \rightarrow \langle Drink\_a\_mouthful\_of\_water \rangle thirsty \end{array} \right\}$$

which says that drinking a mouthful of water may quench a thirst or not. Then we can prove that

$\vdash^{\Sigma} \langle [$**while** $thirsty$ **do** $Drink\_a\_mouthful\_of\_water] \rangle \neg thirsty$

which says that drinking enough water can certainly quench a thirst.

Similar examples can be also found in fuzzy logic, such as "bald head paradox"(pulling out a hair does not cause bald) and "smoking paradox" (smoking one cigarette can not cause lung cancer). Note that this kind of aggregative effects of actions is not easily expressed in the other formalisms of action.

The following lemma will be useful in discussion the properties of propositional causation (see section 4).

Lemma 3

1. *If* $\vdash A$, *then* $\vdash^{\Sigma} A$.
2. *If* $A \in \Sigma$, *then* $\vdash^{\Sigma} [\gamma]A$.
3. *If* $\vdash \varphi \rightarrow \psi$, *then* $\vdash^{\Sigma} [\psi]A \rightarrow [\varphi]A$.
4. *If* $\vdash^{\Sigma} [\gamma]\varphi \wedge [\varphi]A$, *then* $\vdash^{\Sigma} [\gamma]A$.
5. *If* $\Sigma_1 \vdash C$ *for each* $C \in \Sigma_2$, *then* $\vdash^{\Sigma_2} A$ *implies* $\vdash^{\Sigma_1} A$.

where $A \in \mathbf{Fma}$, $\gamma \in \mathbf{Fma}_P \cup \mathbf{Act}$ *and* φ, $\psi \in \mathbf{Fma}_P$.

3.2.2 Semantics of Σ-provability

A standard model M is called a Σ-*model* if $M \models B$ for any $B \in \Sigma$. A is Σ-*valid*, written by $\models^{\Sigma} A$, if it is valid in every Σ-model. The intended semantics for Σ-provability is A is Σ-provable if and only if A is valid in every Σ-model. The following lemmas sketch the proof of soundness and completeness of Σ-provability.

An action description Σ is called to be *uniformly consistent* if $\not\vdash^{\Sigma} \bot$. In the other words, $\bot$ could never be a Σ-theorem. A set $\Gamma \subseteq \mathbf{Fma}$ is said to be Σ-*consistent* if $\Gamma \not\vdash^{\Sigma} \bot$. Γ is *maximal* Σ-*consistent* if

1. it is Σ-consistent,
2. $\Sigma \subseteq \Gamma$, and
3. for any $A \in \mathbf{Fma}$, either $A \in \Gamma$ or $\neg A \in \Gamma$.

Lemma 4 $\Gamma \not\vdash^{\Sigma} A$ *iff* $\Gamma \cup \{\neg A\}$ *is* Σ-*consistent.*

Similar to the ordinary provability of $EPDL$, we can easily to prove that

Lemma 5 *If* Γ *is maximal* Σ-*consistent, then* $\Gamma \vdash^{\Sigma} A$ *implies* $A \in \Gamma$*;*

A similar proof of Lindenbaum Lemma can lead to the following lemma.

Lemma 6 *Every* Σ-*consistent set can be extended to a maximal* Σ-*consistent set.*

Then we have

Theorem 3 (soundness and completeness of Σ-provability)

$$\vdash^{\Sigma} A \text{ if and only if } \models^{\Sigma} A.$$

Similar to the decidability of $EPDL$, we have

Corollary 1 *Σ-provability is decidable.*

Note that Σ-provability is not strong complete as usual. That is to say that $\Gamma \models^{\Sigma} A$ need not imply $\Gamma \vdash^{\Sigma} A$. But it is true if Γ is a finite set.

4 A Logic for Causal Propagation

The philosophical consideration of causality has been around at least since the time of David Hume (1739). The recent investigations of causality with logical approach started from Lewis (1973). AI researchers found it's importance from the early 80's (c.f. [Shoham 1990][Sosa and Tooley 1993]).

[Lewis 1973] offered an account of causality based on his conditional logic. Although his formalism captures some basic properties of causal reasoning, not all the rules for counterfactuals are suitable for causality. For instance, the identity law: $\varphi \rightarrow \varphi$ is sound for counterfactuals, but not sound for causality (see [Shoham 1990]).

An interesting observation is if we delete all the components of action in $EPDL$, then it will degenerate into a mutimodal logic with propositions as modality, which is very similar to conditional logic ([Nute 1984] [Friedman, et al 1996]). Precisely, we delete all the components about action from the language, semantics and deductive system of $EPDL$. The reminder system, denoted by $EPDL^{-}$, will be a logic on causal relation $[\varphi]\psi$. This offers a logic for causal reasoning or more precisely for causal propagation. Since causal reasoning has been ubiquitous in everyday life and so confused with other kind of reasoning, it seems impossible to have a causal theory which satisfies philosophers, physicists, mathematicians and computer scientists. What we are wondering here is what is the characteristics of $EPDL^{-}$ and how much it satisfies our intuitive appeal about causal reasoning?

4.1 Basic properties of causal propagation

One task of a causal theory is to provide an account of causal laws. A *causal law* is a causal relation which truth does not depend on which action would be taken. For instance, $[\neg alive]\neg walking$ is generally viewed as a causal law because its truth does not depend on any actions. We represent a causal law with the form $[\varphi]\psi$ in action description[9]. So any causal formula $[\varphi]\psi$ in action description is a causal law.

Given an action description Σ, which could include some causal laws, if $\vdash^{\Sigma} [\varphi]\psi$, we call it *derived causal law.* The most important question that arises in causal theory is what causal laws can be derived from other causal laws. For instance, given causal laws "*raining* **causes** *wet*" and "*wet* **causes** *slippery* " in action description Σ, can we derive that "*raining* **causes** *slippery*", i.e.,$\vdash^{\Sigma} [raining]slippery$.

The following properties of causal reasoning in $EPDL^{-}$ which follows from Lemma 3 are intuitively desired:

1. If $\vdash^{\Sigma} [\varphi]\psi$ and $\vdash \psi \to \chi$, then $\vdash^{\Sigma} [\varphi]\chi$ (*Right Weakening).*
2. If $\vdash^{\Sigma} [\chi]\psi$ and $\vdash \varphi \to \chi$, then $\vdash^{\Sigma} [\varphi]\psi$ (*Left Strengthening*).
3. If $\vdash^{\Sigma} [\varphi]\psi$ and $\vdash^{\Sigma} [\psi]\chi$, then $\vdash^{\Sigma} [\varphi]\chi$ (*Transitivity*).

The first two specify causal propagation through logical relevancy. The third one reflects transitivity of causation.

Another property, which is not very intuitive but still reasonable, is:

4. If $\vdash^{\Sigma} [\varphi]\varphi$ and $\vdash^{\Sigma} \varphi \to \psi$, then $\vdash^{\Sigma} [\varphi]\psi$. (*Self Causation)*

In words, if φ can be self-caused, then whenever ψ follows φ, φ can cause ψ. This gives a sufficient condition to obtain a causal law. Generally, it is very hard to conceive a self-caused proposition. However, if we are restricted to explaining action effects by using only propositional causation and domain constraint, one way to do it is viewing the direct effects of actions as self-caused propositions (see [McCain and Turner 1997]).

4.2 Characteristic theorem of $EPDL^{-}$

We have presented some basic properties of causal laws in our causal theory. So if we have $[raining]wet$ and $[wet]slippery$ in Σ, by *transitivity* we have $\vdash^{\Sigma} [raining]slippery$, which is desired. However, do we also have $\vdash^{\Sigma} [\neg slippery]\neg raining$, which is not desired. This phenomenon is called the direction of causation, the essential difference between causation and implication. Precisely, $\neg\psi \to \neg\varphi$ always follows $\varphi \to \psi$, however, $[\neg\psi]\neg\varphi$

[9] We do not consider conditional causal law: $\varphi \to [\psi]\chi$ here.

does not necessarily follow $[\varphi]\psi$ no matter that $[\varphi]\psi$ is a causal law or ordinary causal relation. One crucial criterion to the success of a causal logic is whether it can represent the direction of causation while preserve the contrapostion of logical implication. To show our causal theory satisfies such a criterion, we have to prove, for instance, that $\vdash^\Sigma \neg slippery \rightarrow \neg raining$ is provable but $\vdash^\Sigma [\neg slippery]\neg raining$ is not provable with causal laws $\Sigma = \{[raining]slippery\}$ or $\Sigma = \{[raining]wet, [wet]slippery\}$. This is generally requires to provide a full characterization of our framework for causal reasoning.

To focus on this problem, consider action descriptions Σ which consists of only causal laws. We denote

$$D(\Sigma) = \{\varphi \rightarrow \psi : [\varphi]\psi \in \Sigma\},$$

which includes all the domain constraints implied by the causal laws in Σ.

Definition 1 *Let Σ be a set of causal laws. Σ^* is called to be the completion of Σ if it can be generated by the following rules:*

1. *If $[\varphi]\psi \in \Sigma$, then $[\varphi]\psi \in \Sigma^*$;*
2. *If $\vdash^\Sigma \psi$, then $[\varphi]\psi \in \Sigma^*$;*
3. *If $[\varphi]\psi \in \Sigma^*$, then $[\chi](\varphi \rightarrow \psi) \in \Sigma^*$.*
4. *If $[\varphi]\psi \in \Sigma^*$ and $\vdash \chi \rightarrow \varphi$, then $[\chi]\psi.\in \Sigma^*$;*
5. *If $[\varphi]\psi \in \Sigma^*$ and $\vdash \psi \rightarrow \chi$, then $[\varphi]\chi \in \Sigma^*$;*
6. *If $[\varphi]\psi_1$, $[\varphi]\psi_2 \in \Sigma^*$, then $[\varphi](\psi_1 \wedge \psi_2) \in \Sigma^*$.*

The following lemma says that Σ^* is a conservative extension of Σ under Σ-provability.

Lemma 7 *If $[\varphi]\psi \in \Sigma^*$, then $\vdash^\Sigma [\varphi]\psi$.*

The following technical lemma says that for propositional query, there is no difference between causal laws and domain constraints.

Lemma 8 $\vdash^\Sigma \varphi$ *if and only if* $D(\Sigma) \vdash \varphi$.

Lemma 9 $[\varphi]\psi \in \Sigma^*$ *iff* $D(\Sigma) \vdash \psi$ *or there exists* $[\chi]\lambda \in \Sigma$ *such that* $\varphi \vdash \chi$ *and* $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$.

The following theorem characterizes causal propagation.

Theorem 4 *$\vdash^{\Sigma} [\varphi]\psi$ if and only if $[\varphi]\psi \in \Sigma^*$.*

By combining Lemma 9 and Theorem 4, we have the following

Corollary 2 *$\not\vdash^{\Sigma} [\varphi]\psi$ iff*

1. $D(\Sigma) \not\vdash \psi$ and $\varphi \not\vdash \chi$ for each $[\chi]\lambda \in \Sigma$, or

2. $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \not\vdash \psi$.

For example, if $\Sigma = \{[raining]wet,\ [wet]slippery\}$, then we have:

1. $\vdash^{\Sigma} [raining]slippery$ because $[raining]wet \in \Sigma$ and $\{wet,\ wet \rightarrow slippery\} \vdash slippery$.

2. $\vdash^{\Sigma} \neg slippery \rightarrow \neg raining$ because $D(\Sigma) \vdash \neg slippery \rightarrow \neg raining$.

3. $\not\vdash^{\Sigma} [\neg slippery]\neg raining$ because there are no causal laws $[\chi]\lambda \in \Sigma$ such that $\neg slippery \vdash \chi$.

Example 7 *Consider the circuit in Example 3. The causal relation in this circuit can be described by the following causal laws:*

$$\Sigma^1 = \left\{ \begin{array}{l} [sw_1 \wedge sw_2]light \\ [\neg sw_1 \vee \neg sw_2]\neg light \end{array} \right\}$$

According to the corollary 2, it is much easy to know that $\not\vdash^{\Sigma^1} [light](sw_1 \wedge sw_2)$ and $\not\vdash^{\Sigma^1} [sw_1 \wedge \neg light]\neg sw_2$ because there is no any causal law $[\chi]\lambda \in \Sigma^1$ such that $light \vdash \chi$ or $sw_1 \wedge \neg light \vdash \chi$.

To make the characterization theorem clearer, let's consider another corollary of Theorem 4.

Corollary 3 *Let Σ be a set of causal laws and $\mathcal{D} \subseteq \mathbf{Fma}_P$. Then $\vdash^{\Sigma \cup \mathcal{D}} [\varphi]\psi$ iff*

1. $D(\Sigma) \cup \mathcal{D} \vdash \psi$ or

2. $\bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \neq \phi$ and $D(\Sigma) \cup \mathcal{D} \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$.

Suppose that $\mathcal{D}$ is a set of domain constraints. $D(\Sigma)$ are the domain constraints implied by Σ. So the domain constraints in effect is $D(\Sigma) \cup \mathcal{D}$. The Corollary 3 says that φ **causes** ψ under Σ if and only if ψ is implies either by domain constraints or by domain constraints plus effects of φ through propagation of causal laws.

Note that Theorem 4 tells us what can be derived from a set of causal laws and Corollary 2 shows us what can not be derived from it. Therefore we have had a full characterization of causal reasoning in $EPDL^-$. With help of this result, we can offer a solution to the ramification problem based on $EPDL$ (see a sequent paper for details).

4.3 More properties of causal propagation

There have been many proposals in different formalisms for causal propagation ([Lin 1995] [McCain and Turner 1997] [Giunchiglia, et al 1997] [Thielscher 1997] and etc.). Few of them are axiomatized. Among them, McCain and Turner's causal theory [McCain and Turner 1997] offered a modal expression of causation. In [Turner 1999], a causal law was written by the modal formula "$\varphi \to \mathbf{C}\psi$", where $\mathbf{C}$ is a S_5-modality. Besides S_5 axioms, however, $\mathbf{C}$ also relies on some fix-point condition which is not axiomatized. Another important difference between our approach and the others is that we permit the generation of new causal laws from a given set of causal laws whereas most of other formalisms only consider causal propagation in semantic level. Nevertheless, there are lots of properties are comparable among these systems.

[Schwind 1999] presented a comparative tableaux showing the properties of causal propagation satisfied by the main formalisms of causal reasoning. We are not going to advertise our system by showing which properties it has had. Notwithstanding it seems a shortcut to differentiate our formalism from the others by extending this tableaux with $EPDL^-$.

The main properties which were considered in [Schwind 1999] include *Monotonicity, Transitivity, Contraposition, Conjunction, Reflexivity, Conjunctive Antecedents, Disjunctive Antecedents, Right Weakening, Left Logical Equivalence.*

Besides the properties we have shown in Section 4.1, the following two are also easy to be verified:

5 If $\vdash^{\Sigma} [\varphi]\psi \wedge [\varphi]\chi$, then $\vdash^{\Sigma} [\varphi](\psi \wedge \chi)$ (*Conjunction)*

6 If $\vdash^{\Sigma} [\varphi]\psi$ and $\vdash \varphi \leftrightarrow \chi$, then $\vdash^{\Sigma} [\chi]\psi$ (*Left Logical Equivalence*).

Therefore we know that *Transitivity, Conjunction, Right Weakening and Left Logical Equivalence* are true in $EPDL^-$. With Corollary 2, we can easily falsify *Contraposition* and *Reflexivity.* Now we discuss the other rules.

- Monotonicity

Our approach to causality is definitely monotonic in both of the following senses:

a). If $\vdash^{\Sigma} [\varphi]\psi$, then $\vdash^{\Sigma\cup\Delta} [\varphi]\psi$.

b). If $\vdash^{\Sigma} [\varphi]\psi$, then $\vdash^{\Sigma} [\varphi \wedge \chi]\psi$.

We are keen on the monotonic approach not only because it is simple but also it is essential to our methodology on reasoning about action and causality. If $rain \wedge unberella$ does not cause wet, we think that $rain$ is not necessarily to cause wet. It is true that we can't have a complete list of the qualifications upon which $rain$ **causes** wet, but this is another problem which we call it the *qualification problem of effect propagation.*

- Conjunctive Antecedents

Conjunctive Antecedents means if $\varphi \wedge \psi$ **causes** χ, then either φ **causes** χ or ψ **causes** χ. It is not necessarily true in $EPDL^-$. For instance, we have $\vdash^{\Sigma} [sw_1 \wedge sw_2]light$ but we have neither $\vdash^{\Sigma} [sw_1]light$ nor $\vdash^{\Sigma} [sw_2]light$.

- Disjunctive Antecedents

Disjunctive Antecedents means if φ **causes** χ and ψ **causes** χ, then $\varphi \wedge \psi$ **causes** χ. It is also not necessarily true in $EPDL^-$ because it independent with the extant axioms of $EPDL^-$. *Disjunctive Antecedents* is one of the most controversial rules in conditional logic (See [Nute 1984]). Although in causal logic it is not as bad as in conditional logic, it is a question that if it is intuitive appeal. We did not consider it as an axiom, but it can be an interesting topic for the future research.

According to Schwind's tableaux, $EPDL^-$ most closes to McCain and Turner's causal theory. The only difference is $EPDL^-$ has not *Disjunctive Antecedents* but their system has. In fact, $EPDL^-$ is slightly weaker than their's. The following proposition show the relation between them.

Proposition 3 *Let Σ be a set of causal laws. Let $\Sigma^I = \{\psi : I \vdash \varphi$ for $[\varphi]\psi \in \Sigma\}$, where I is an interpretation of* **Flu**. *If I is a model of Σ^I, then I is a model of $(\Sigma^*)^I$.*

If we take McCain and Turner's causal theory as the semantics of $EPDL^-$, this proposition shows that $EPDL^-$ is sound with this semantics. However, it is not complete with this semantics in the sense that if I is the unique model of both Σ^I and $(\Sigma \cup \{[\varphi]\psi\})^I$, then $[\varphi]\psi \in \Sigma^*$.

We would like to see if McCain and Turner's causal theory can be axiomatized by extending $EPDL^-$ axiomatic system.

We remark that not every property of $EPDL^-$ is desired. As we footnoted before, the inference rule LC has the same problem of material implication, that is, false can cause anything. More precisely, if $\vdash^\Sigma [\varphi]\psi$, then $\vdash^\Sigma [\bot]\psi$. A variant of this example looks even worse: if $\vdash^\Sigma [\varphi]\psi$, then $\vdash^\Sigma [\varphi \wedge \neg\psi]\psi$, which says, for instance, if we know *raining* causes *wet*, then *raining* and *dry* still cause *wet*. Although we do not think this is a serious problem to our approach, we would like to see any solution to this problem.

5 Conclusion and Discussion

We have presented an extended system $EPDL$ of propositional dynamic logic to represent and specify indirect effects of actions and propagation of causality. The extended propositional dynamic logic provides a unified formalism for reasoning about (direct and indirect) effects of actions. The extension is so slight that only one axiom and one inference rule were added[10]. It has been shown that the resultant system captures most basic properties of causal reasoning. Such simplicity may reflect the essential similarity between direct and indirect effects of actions.

As one of the formalisms for reasoning about action and causality, $EPDL$ is quite different from the other formalisms, such as the situation calculus [Reiter 1991], the action language [Gelfond and Lifschitz 1998] [Giunchiglia, et al 1997] and fluent calculus, as it does not have a built-in solution to the frame problem or the ramification problem. In fact, dynamic logic is a pure monotonic logic which only offers built-in expression of compound actions and causal relations. It can serve as a representational language and reasoning platform for constructing a monotonic or nonmonotonic system to deal with various problems in reasoning about actions (c.f. [Giacomo and Lenzerini 1995] and [Prendinger and Schurz 1996]). We will present our solutions to these problem based on $EPDL$ in the sequent papers.

We have presented a causal logic $EPDL^-$ which is a reduction of $EPDL$ by deleting the component of action. It is an axiomatized logical system for causal propagation. We presented a characterisation theorem of the system and discussed its propertied in causal reasoning. Regarding to the similarity between direct and indirect effects of actions, the semantic of dynamic logic for causality seems intuitively appealing.

[10] More precisely, there are another axiom and inference rule of PDL were revised.

$EPDL^-$ can be viewed as a conditional logic. This means we can endow a conditional logic the semantics of dynamic logic. It is interesting that whether we can give $EPDL^-$ a semantics of conditional logic.

We believe that dynamic logic as an axiomatic logical system of action has a very promising future for applications in planning, cognitive robot and intelligent agent. It is hoped to extend EPDL further with the approach in [Peleg 1987] [Chen and Giacomo 1999] and *etc.* in order to express concurrent actions. However, a big challenge for reasoning about action with dynamic logic is whether first-order dynamic logic can be also extended, without lost of completeness of its deductive system and embedability of first-order infinitary logic, to the case where primitive actions could be more general than just deterministic valuations.

Appendix:

Proofs of Theorems

Theorem 1 (Soundness)*If* $\vdash A$ *, then* A *is valid in all standard models of* $\mathcal{L}_{EPDL}$.

Proof. We only need to verify the extra axioms and inference rules.

For (EK), suppose that $M \models_w [\varphi](\psi_1 \to \psi_2)$ and $M \models_w [\varphi]\psi_1$, so for any w' with $wR_\varphi w'$, $M \models_{w'} \psi_1 \to \psi_2$ and $M \models_{w'} \psi_1$. Thus $M \models_{w'} \psi_2$. That means $M \models_w [\varphi]\psi_2$.

For (CW), given any standard model M, suppose that $M \models_w [\varphi]A$ and $M \models_w \varphi$. According to the semantics, $(w, w) \in R_\varphi$. Thus $M \models_w A$. That means (CW) is true in M.

For the inference rule (EN), suppose that A is true in any world of an standard model M. Then for any state w, if $wR_\varphi w'$ then $M \models_{w'} A$. That means $M \models_w [\varphi]A$.

For the inference rule (LC), assume that $\varphi_1 \to \varphi_2$ is true in any standard model M. By the semantics, $R_{\varphi_1} \subseteq R_{\varphi_2}$. Therefore $M \models_w [\varphi_2]A$ implies $M \models_w [\varphi_1]A$. □

Lemma 2 M^Γ *is a standard model.*

Proof. We only consider the extended conditions for $EPDL$.

For the condition 1, suppose that $M^\Gamma \models_{|w|} \varphi$ for $\varphi \in \Gamma$. Then $M^C \models_w \varphi$, or, $\varphi \in w$. If $[\varphi]A \in w$, then $\varphi \to A \in w$ by the axiom (CW), hence $A \in w$.

That means $\{A : [\varphi]A \in w\} \subseteq w$. According to the construction of canonical model, $(w, w) \in R^C_\varphi$. Therefore, $(|w|, |w|) \in R^\Gamma_\varphi$ by the construction of Γ-filtration.

For the condition 2, suppose that $\models \varphi_1 \to \varphi_2$. Since φ_1 and φ_2 are classical propositional formula, so according to the completeness of classical propositional logic, $\varphi_1 \to \varphi_2$ is a tautology, so it is a tautology of $EPDL$. Let $(|u|, |v|) \in R^\Gamma_{\varphi_1}$. Then there exist $u' \in |u|$ and $v' \in |v|$ such that $u' R^C_{\varphi_1} v'$. That is $\{A : [\varphi_1]A \in u'\} \subseteq v'$. Assume that $[\varphi_2]A \in u'$. For u' is closed under the rule (LC), $\varphi_1 \to \varphi_2 \in u'$ implies $[\varphi_2]A \to [\varphi_1]A \in u'$. Hence $[\varphi_1]A \in u'$, so $A \in v'$. That means $\{A : [\varphi_2]A \in u'\} \subseteq v'$, or $u' R^C_{\varphi_2} v'$. Therefore $(|u|, |v|) \in R^\Gamma_{\varphi_2}$, as desired. □

Theorem 2 (Completeness) *If A is valid in all standard models of $\mathcal{L}_{EPDL}$, then $\vdash A$.*

Proof. Suppose that $\nvdash A$, then for any canonical model M^C of $\mathcal{L}_{EPDL}$, there is a state $w \in W^C$ such that $\neg A \in W^C$. Thus $M^C \models_w \neg A$.

Now let $\Gamma = FL(\{\neg A\})$ and M^Γ be a Γ-filtration of M^C. Then $M^\Gamma \models_{|w|} \neg A$. This contradicts to the promise of the theorem. □

Proposition 2 *If $A \vdash B(a)$ where a is a primitive action symbol in $\mathcal{L}_{EPDL}$ occurring in $B(a)$ but not occurring in A, then $A \vdash B(\alpha)$ for any action α in which there is no the occurrence of a, where $B(\alpha)$ is the result of substituting α for all occurrences of a in $B(a)$.*

Proof. It is sufficient to prove it in semantics according to the soundness and completeness theorems of $EPDL$. Suppose $M = (W, \{R_\alpha : \alpha \in \mathbf{Act}\} \cup \{R_\varphi : \varphi \in \mathbf{Fma}_P\}, V)$ is a standard model of $\mathcal{L}_{EPDL}$. Let M' is the model which is as same as M except that let $R_a = R_\alpha$. It is easy to see that M' is a standard model of $\mathcal{L}_{EPDL}$ because a does not occur in α. For any $w \in W$, if $M \models_w A$, then $M' \models_w A$ for there is no any occurrence of a in A. It follows by $A \models B(a)$ that $M' \models_w B(a)$. By $R_a = R_\alpha$, we have $M' \models B(\alpha)$. Since there is no any occurrence of a in $B(\alpha)$, we obtain $M \models B(\alpha)$. □

Example 6 Prove that

$\vdash^\Sigma \langle[\mathbf{while}\ thirsty\ \mathbf{do}\ Drink\_a\_mouthful\_of\_water]\rangle\neg thirsty$

Proof. Let D_m_w denote $Drink\_a\_mouthful\_of\_water$. It is easy to see that $\vdash [\mathbf{while}\ thirsty\ \mathbf{do}\ D\_m\_w]\neg thirsty$. Now we prove that $\vdash^\Sigma \langle\mathbf{while}\ thirsty\ \mathbf{do}\ D\_m\_w\rangle\neg thirsty$.

(1)* $\vdash^{\Sigma}$ $thirsty \to \langle D\_m\_w\rangle \neg thirsty$ (AD)
(2)* $\vdash^{\Sigma}$ $thirsty \to (thirsty \wedge \langle D\_m\_w\rangle \neg thirsty)$ (1)
(3)* $\vdash^{\Sigma}$ $[(thirsty?; D\_m\_w)*](thirsty \to (thirsty \wedge \langle D\_m\_w\rangle \neg thirsty))$ (2 and EN)
(4)* $\vdash^{\Sigma}$ $\langle (thirsty?; D\_m\_w)*\rangle thirsty \to \langle (thirsty?; D\_m\_w)*\rangle (thirsty \wedge \langle D\_m\_w\rangle \neg thirsty))$ (3)
(5)* $\vdash^{\Sigma}$ $[(thirsty?; D\_m\_w)*]\neg thirsty \vee \langle (thirsty?; D\_m\_w)*\rangle (thirsty \wedge \langle D\_m\_w\rangle \neg thirsty))$ (4)
(6)* $\vdash^{\Sigma}$ $\neg thirsty \vee \langle (thirsty?; D\_m\_w)*\rangle (thirsty \wedge \langle D\_m\_w\rangle \neg thirsty))$ (5 and Mix)
(7)* $\vdash^{\Sigma}$ $\neg thirsty \to \langle (thirsty?; D\_m\_w)*\rangle \neg thirsty$ (Mix)
(8)* $\vdash^{\Sigma}$ $\langle (thirsty?; D\_m\_w)*\rangle \langle D\_m\_w\rangle \neg thirsty \to \langle (thirsty?; D\_m\_w)*\rangle \neg thirsty$ (Mix)
(9)* $\vdash^{\Sigma}$ $\langle (thirsty?; D\_m\_w)*\rangle \neg thirsty$ (7 and 8)
(10)* $\vdash^{\Sigma}$ $\langle (thirsty?; D\_m\_w)*\rangle (\neg thirsty \wedge \neg thirsty)$ (9)
(11)* $\vdash^{\Sigma}$ $\langle (thirsty?; D\_m\_w)*; \neg thirsty?\rangle \neg thirsty$ (10 and $Test$)
(12)* $\vdash^{\Sigma}$ $\langle \textbf{while } thirsty \textbf{ do } D\_m\_w\rangle \neg thirsty$ (11) □

Lemma 3

1. $\vdash A$ *implies* $\vdash^{\Sigma} A$.
2. $A \in \Sigma$ *implies* $\vdash^{\Sigma} [\gamma]A$.
3. *If* $\vdash \varphi \to \psi$, *then* $\vdash^{\Sigma} [\psi]A \to [\varphi]A$.
4. *If* $\vdash^{\Sigma} [\gamma]\varphi \wedge [\varphi]A$, *then* $\vdash^{\Sigma} [\gamma]A$.
5. *If* $\Sigma_1 \vdash C$ *for each* $C \in \Sigma_2$, *then* $\vdash^{\Sigma_2} A$ *implies* $\vdash^{\Sigma_1} A$.

Proof. (1) and (2) are straightforward.

For (3), suppose that $\vdash \varphi \to \psi$. By (LC) we have $\vdash [\psi]A \to [\varphi]A$, so $\vdash^{\Sigma} [\psi]A \to [\varphi]A$.

For (4), suppose that $\vdash^{\Sigma} [\gamma]\varphi \wedge [\varphi]A$. Then by (CW) we have $\vdash^{\Sigma} \varphi \to A$. So $\vdash^{\Sigma} [\gamma](\varphi \to A)$, or, $\vdash^{\Sigma} [\gamma]\varphi \to [\alpha]A$. Therefore $\vdash^{\Sigma} [\gamma]A$.

For (5), since $\Sigma_1 \vdash C$ implies $\vdash^{\Sigma_1} C$, any Σ_2-theorem is Σ_1-theorem. □

Lemma 4 $\Gamma \not\vdash^{\Sigma} A$ *iff* $\Gamma \cup \{\neg A\}$ *is* Σ*-consistent.*

Proof. Suppose that $\Gamma \not\vdash^{\Sigma} A$. If $\Gamma \cup \{\neg A\}$ is not Σ-consistent, that is, $\Gamma \cup \{\neg A\} \vdash^{\Sigma} \bot$, then $\Gamma \vdash^{\Sigma} A$, a contradiction. Conversely, if $\Gamma \cup \{\neg A\}$ is Σ-consistent but $\Gamma \vdash^{\Sigma} A$, then there exist $A_1, \cdots A_n \in \Gamma$ such that $\vdash^{\Sigma} A_1 \to (\cdots (A_n \to A) \cdots)$. Then $\vdash^{\Sigma} A_1 \to (\cdots (A_n \to (\neg A \to \bot)) \cdots)$. That means that $\Gamma \cup \{\neg A\}$ is not Σ-consistent, a contradiction. □

Theorem 3 (soundness and completeness for Σ-provability)

$$\vdash^{\Sigma} A \textit{ if and only if } \models^{\Sigma} A.$$

Proof. For soundness, suppose that $\vdash^{\Sigma} A$. For any Σ-model M, according to the completeness of $EPDL$, all the theorems of $EPDL$ belong to $\{B : M \models B\}$. It is not hard to verify that $\{B : M \models B\}$ is closed under MP and EN. Therefore $\{B : M \models B\}$ contains all the Σ-theorems, specially, $A \in \{B : M \models B\}$, or $M \models^{\Sigma} A$.

For completeness, suppose that $\models^{\Sigma} A$ but $\not\vdash^{\Sigma} A$. Then $\{\neg A\}$ is Σ-consistent and can be extended to a maximal Σ-consistent set. Then we construct a canonical Σ-model M^C of $\mathcal{L}_{EPDL}$ such that

$W^C = \{w : w \text{ is maximal } \Sigma\text{-consistent}\}$

$R^C_{\gamma} = \{(w, w') : \forall [\gamma] A \in w (A \in w')\}$, where $\gamma \in \mathbf{Fma}_P \cup \mathbf{Act}$

$V^C(f) = \{w \in W^C : f \in w\}$

It is easy to see that M^C exists and we can verify that $M^C \models_w B$ iff $B \in w$ for all $w \in W^C$ by using the lemmas in Section 3.2.2. Since for any $w \in W^C$, $\Sigma \subseteq w$, every element of Σ is valid in M^C, but A is not valid in M^C, hence[11], it is not valid in the $FL(\Sigma \cup \{A\})$-filtration of M^C, which contradicts to the condition of the theorem. Therefore $\vdash^{\Sigma} A$, as desired. □

Lemma 7 If $[\varphi]\psi \in \Sigma^*$, then $\vdash^{\Sigma} [\varphi]\psi$.

Proof. Straightforward from the construction (Definition 1) and Lemma 3.

Lemma 8 $\vdash^{\Sigma} \varphi$ if and only if $D(\Sigma) \vdash \varphi$.

Proof. We only prove the non-trivial direction. Suppose that $\vdash^{\Sigma} \varphi$. If $D(\Sigma) \not\vdash \varphi$, then $D(\Sigma) \cup \{\neg\varphi\}$ is consistent. Thus we can construct a standard model $M = (\{w\}, \{R_{\psi} : \psi \in \mathbf{Fma}_P\}, V)$ such that

- $R_{\psi} = \{(w, w) : M \models_w \psi\}$ and

[11] A very careful check of every step in the proof for Filtration Lemma has to been made because we have changed the definition of canonical model.

- $M \models_w D(\Sigma) \cup \{\neg\varphi\}$.

It is easy to see that M is a Σ-model and $M \models \{\neg\varphi\}$, which contradicts $\vdash^{\Sigma} \varphi$. $\Box$

Lemma 9 *$[\varphi]\psi \in \Sigma^*$ iff $D(\Sigma) \vdash \psi$ or there exists $[\chi]\lambda \in \Sigma$ such that $\varphi \vdash \chi$ and $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$.*

Proof. Let
$P_1(\varphi) =$"$D(\Sigma) \vdash \varphi$";
$P_2(\varphi) =$"$\exists[\chi]\lambda \in \Sigma(\varphi \vdash \chi)$";
$P_3(\varphi, \psi) =$"$D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$";
$P(\varphi, \psi) = P_1(\psi) \vee (P_2(\varphi) \wedge P_3(\varphi, \psi))$.
Then this lemma says that "$[\varphi]\psi \in \Sigma^*$ iff $P(\varphi, \psi)$".

"$\Rightarrow$" We prove by induction on the rules in Definition 1 that if $[\varphi]\psi \in \Sigma^*$, then $P(\varphi, \psi)$.

Rule 1. If $[\varphi]\psi \in \Sigma$, then $P_2(\varphi) \wedge P_3(\varphi, \psi)$ is obviously true. So is $P(\varphi, \psi)$.

Rule 2. If $\vdash^{\Sigma} \psi$, then by Lemma 8 $P_1(\psi)$ holds. So do $P(\varphi, \psi)$.

Rule 3. Suppose that $[\varphi]\psi \in \Sigma$. By hypothesis of induction, $[\varphi]\psi$ satisfies $P(\varphi, \psi)$. If $P_1(\psi)$, then $P_1(\varphi \to \psi)$; otherwise, $P_2(\varphi) \wedge P_3(\varphi, \psi)$, that is, $\exists[\chi]\lambda \in \Sigma(\varphi \vdash \chi)$ and $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$. Then there exist $\chi_1, \cdots, \chi_n$ such that for each j $(1 \leq j \leq n)$, $\varphi \vdash \chi_j$ and there are $\lambda_1^j, \cdots, \lambda_m^j$ such that for any k $(i \leq k \leq m_j)$, $[\chi_j]\lambda_k^j \in \Sigma$ and $D(\Sigma) \cup \{\lambda_k^j : 1 \leq j \leq n \text{ and } 1 \leq k \leq m\} \vdash \psi$. Since $\varphi \vdash \chi_j$ and $[\chi_j]\lambda_k^j \in \Sigma$, we obtain that $\{\varphi\} \cup D(\Sigma) \vdash \lambda_k^j$ for each j and k because $\chi_j \to \lambda_k^j \in D(\Sigma)$. Thus $\{\varphi\} \cup D(\Sigma) \vdash \psi$, or, $D(\Sigma) \vdash \varphi \to \psi$. That is $P_1(\varphi \to \psi)$, so $P(\chi, \varphi \to \psi)$.

Rule 4. Suppose that $\vdash \chi \to \varphi$ and $[\varphi]\psi \in \Sigma$. By hypothesis of induction, $[\varphi]\psi$ satisfies $P(\varphi, \psi)$. Then if $P_1(\psi)$, we have $P(\chi, \psi)$; otherwise, $P_2(\varphi) \wedge P_3(\varphi, \psi)$ holds. Since $P_2(\varphi)$, or there exists $[\chi_1]\lambda \in \Sigma$ such that $\varphi \vdash \chi_1$, we have $\chi \vdash \chi_1$ because $\chi \vdash \varphi$. So $P_2(\chi)$ holds. To show $P_3(\chi, \psi)$, since $\chi \vdash \varphi$ implies that $\bigcup_{\varphi \vdash \chi'} \{\lambda : [\chi']\lambda \in \Sigma\} \subseteq \bigcup_{\chi \vdash \chi'} \{\lambda : [\chi']\lambda \in \Sigma\}$, we yield that $P_3(\varphi, \psi)$ implies $P_3(\chi, \psi)$. Therefore $P(\chi, \psi)$.

Rule 5. Suppose that $\vdash \psi \to \chi$ and $[\varphi]\psi \in \Sigma$. By hypothesis of induction, $[\varphi]\psi$ satisfies $P(\varphi, \psi)$. Then if $P_1(\psi)$, we have $P_1(\chi)$; otherwise, $P_2(\varphi) \wedge P_3(\varphi, \psi)$. In this case, we only need to prove $P_3(\varphi, \chi)$. Since $\psi \vdash \chi$,

$D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$ implies $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \chi$. So $P_3(\varphi, \psi)$ implies $P_3(\varphi, \chi)$. Therefore $P(\varphi, \chi)$.

Rule 6. Similar to the rule 5.

"$\Leftarrow$" If $D(\Sigma) \vdash \psi$, then by Lemma 8 $\vdash^{\Sigma} \psi$. It follows from rule 2 that $[\varphi]\psi \in \Sigma^*$.

If both $P_2(\varphi)$ and $P_3(\varphi, \psi)$ hold, or, $\exists[\chi]\lambda \in \Sigma(\varphi \vdash \chi)$ and $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$, then there exist $\chi_1, \cdots, \chi_n$ such that for each j $(1 \leq j \leq n)$, $\varphi \vdash \chi_j$ and there are $\lambda_1^j, \cdots, \lambda_m^j$ such that for any k $(i \leq k \leq m_j)$, $[\chi_j]\lambda_k^j \in \Sigma$ and $D(\Sigma) \cup \{\bigwedge_{j=1}^{n} \bigwedge_{k=1}^{m_j} \lambda_k^j\} \vdash \psi$. Since $[\chi_j]\lambda_k^j \in \Sigma$ for any k $(i \leq k \leq m_j)$, $[\chi_j](\bigwedge_{k=1}^{m_j} \lambda_k^j) \in \Sigma^*$. Again, by $\varphi \vdash \chi_j$, it follows that $[\varphi](\bigwedge_{k=1}^{m_j} \lambda_k^j) \in \Sigma^*$. Then we have $[\varphi](\bigwedge_{j=1}^{n} \bigwedge_{k=1}^{m_j} \lambda_k^j) \in \Sigma^*$. On the other hand, it is always true that $[\varphi] \bigwedge_{[\chi]\lambda \in \Sigma} (\chi \to \lambda) \in \Sigma^*$, that is, $[\varphi](\bigwedge D(\Sigma)) \in \Sigma^*$. So $[\varphi](\bigwedge D(\Sigma) \wedge (\bigwedge_{j=1}^{n} \bigwedge_{k=1}^{m_j} \lambda_k^j)) \in \Sigma^*$. By $D(\Sigma) \cup \{\bigwedge_{j=1}^{n} \bigwedge_{k=1}^{m_j} \lambda_k^j\} \vdash \psi$ and rule 5 we conclude that $[\varphi]\psi \in \Sigma^*$. □

Theorem 4 $\vdash^{\Sigma} [\varphi]\psi$ if and only if $[\varphi]\psi \in \Sigma^*$.

Proof. The direction of "if" is straightforward from Lemma 7.

For the other direction, if $\vdash^{\Sigma} \psi$, then by Lemma 8, $D(\Sigma) \vdash \psi$. So by Lemma 9, $[\varphi]\psi \in \Sigma^*$. This means that we can assume that $\nvdash^{\Sigma} \psi$, or, $D(\Sigma) \nvdash \psi$.

Now suppose that $[\varphi]\psi \notin \Sigma^*$. We prove that $\nvdash^{\Sigma} [\varphi]\psi$.

By Lemma 9 and the above assumption, $\forall[\chi]\lambda \in \Sigma(\varphi \nvdash \chi)$, or $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \nvdash \psi$.

Case 1: $\forall[\chi]\lambda \in \Sigma(\varphi \nvdash \chi)$. Since $\nvdash^{\Sigma} \psi$, there exists a Σ-model M and a worlds w_0 of M such that $M \models_{w_0} \neg\psi$.

Case 2: $\exists[\chi]\lambda \in \Sigma$ $(\varphi \vdash \chi)$ and $D(\Sigma) \cup \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \nvdash \psi$. By Lemma 8, $\bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \nvdash^{\Sigma} \psi$. So $\bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \cup \{\neg\psi\}$ is Σ-consistent. Thus there exists a Σ-model M and a world w_0 such that $M \models_{w_0} \bigcup_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \cup \{\neg\psi\}$.

For both cases, we call the model M^- is the reduction of M if it is as same as M except for $R^-_\chi = R_{\chi ?}$ for any $\chi \in \mathbf{Fma}_P$. In the other words, R^-_χ is generated by deleting all the non-reflexive elements from R_χ. It is easy to see that $M \models_w [\chi]\lambda$ implies $M^- \models_w [\chi]\lambda$ because $R^-_\chi \subseteq R_\chi$ for any world w of M and any $\chi \in \mathbf{Fma}_P$. Then M is a Σ-model implies that M^- is a Σ-model.

Let M^+ is just as same as M^- except for $R^+_\chi = R^-_\chi \cup \{(w_0, w_0)\}$ whenever $\varphi \vdash \chi$. Then

(a). M^+ is a standard model. In fact, if $M^+ \models_w \chi$, then $(w, w) \in R_{\chi ?} = R^-_\chi \subseteq R^+_\chi$. Suppose that $\models \chi_1 \to \chi_2$. If $\varphi \vdash \chi_1$, then $\varphi \vdash \chi_2$. It follows that $R^-_{\chi_1} \subseteq R^-_{\chi_2}$ iff $R^-_{\chi_1} \cup \{(w_0, w_0)\} \subseteq R^-_{\chi_2} \cup \{(w_0, w_0)\}$ iff $R^+_{\chi_1} \subseteq R^+_{\chi_2}$. If $\varphi \not\vdash \chi_1$ but $\varphi \vdash \chi_2$, $R^-_{\chi_1} \subseteq R^-_{\chi_2}$ implies $R^-_{\chi_1} \subseteq R^-_{\chi_2} \cup \{(w_0, w_0)\}$, or $R^+_{\chi_1} \subseteq R^+_{\chi_2}$. If both $\varphi \not\vdash \chi_1$ and $\varphi \not\vdash \chi_2$, $R^+_{\chi_1} \subseteq R^+_{\chi_2}$ holds obviously.

(b). M^+ is a Σ-model. In fact, for any $[\chi]\lambda \in \Sigma$, according to the construction of M^+, if $\varphi \not\vdash \chi$ or if $\varphi \vdash \chi$ and $w \neq w_0$, then $M^+ \models_w [\chi]\lambda$ iff $M^- \models_w [\chi]\lambda$. If $\varphi \vdash \chi$ and $w = w_0$, according to the speciality of w_0 in case 2, $M \models_{w_0} \lambda$. So $M^+ \models_w [\chi]\lambda$. Therefore M^+ is a Σ-model.

Finally, $M^+ \models_{w_0} \neg[\varphi]\psi$ because $(w_0, w_0) \in R_\varphi$ but $M^+ \models_{w_0} \neg\psi$. □

Proposition 3 *Let Σ be a set of causal laws. Let $\Sigma^I = \{\psi : I \vdash \varphi$ for $[\varphi]\psi \in \Sigma\}$, where I is an interpretation of* **Flu**. *If I is a model of Σ^I, then I is a model of $(\Sigma^*)^I$.*

Proof. Assume that $\psi \in (\Sigma^*)^I$. Then there exists φ such that $I \vdash \varphi$ and $[\varphi]\psi \in \Sigma^*$. According to Lemma 9,

(1). $D(\Sigma) \vdash \psi$ or

(2). $\bigcup\limits_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \neq \phi$ and $D(\Sigma) \cup \bigcup\limits_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \vdash \psi$.

If (1) is true, we have $I \vdash \psi$ because $I \vdash \bigwedge D(\Sigma)$. If (2) is true, since $\bigcup\limits_{\varphi \vdash \chi} \{\lambda : [\chi]\lambda \in \Sigma\} \subseteq \Sigma^I$, we obtain $I \vdash \psi$. □

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