The Logical Validation of Mathematical Diagrammatic Proofs

Christina L. Jenkin
Knowledge System Group
School of Computer Science and Engineering
University of New South Wales, Sydney 2052, Australia
E-mail: tinaj@cse.unsw.edu.au, tina@alum.rpi.edu

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Abstract

Diagrams have been used for problem solving for thousands of years but have only recently had a resurgence into mainstream science with applications in cognitive science, artificial intelligence, computer science, physics, mathematics, and other disciplines. Diagrammatic reasoning is "the understanding of concepts and ideas by the use of diagrams and imagery, as opposed to linguistic or algebraic representations"[GCN95, on back cover]. This paper aims to introduce the reader to diagrammatic reasoning, specifically in the area of diagrammatic proofs, logically validate the soundness of the construction steps in a diagrammatic proof, as well as help develop a theoretical basis for computing directly with diagrammatic representations. In doing this, the work of Jamnik (in [Jam99]) and of Foo, et al. (in [FNP99]) will be extended. This will be done through an analysis of diagrammatic proofs of geometric theorems and a study of some problematic proofs in this area. In addition, a proof showing the equivalence of the two solutions to the problem of generalization presented in [FNP99] and a link between traditional theories of computation, such as fixed points, invariants, and continuations, with diagrammatic proofs is shown. In essence, this paper intends to help advance the understanding of what is involved in diagrammatic proofs, why they work, and why they sometimes do not work as well as show that diagrams alone can be regarded as legitimate (or even desirable) proofs in the area of geometric theorems. Hopefully, this will help to open new opportunities for study and development in the justification and in later work on the automation of diagrammatic proofs.

1 Introduction

> *The use of diagrams as reasoning aids has a long history, but the serious investigation of what is involved in such reasoning is recent.*[FNP99]

"Draw a diagram." Every student has heard a teacher at some point use this phrase. Whether in mathematics, physics, chemistry, or engineering, teachers encourage students to use diagrams as a way to help them understand and solve a problem. Jamnik writes that, "diagrams... guide human mathematical thought, and enable a mathematician to understand instantly the problem represented by the diagram, *how* to go about solving the problem, and *why* the solution is correct."[Jam99] Because of these traits, diagrams have been used to solve problems as far back as Ancient Greece.[1]

On the back cover of [GCN95], diagrammatic reasoning is defined to be the "understanding of concepts and ideas by the use of diagrams and imagery, as opposed to linguistic or algebraic representations." It is the study of the use of diagrams in (and, more recently, *as*) formal reasoning aids and proofs, and the field is growing rapidly. "[In] the last twenty years, researchers from various fields, such as cognitive science, artificial intelligence, computer science, physics, and mathematics returned to the use of diagrams and tried to re-establish a formal role of diagrams in proofs."[Jam99]

For an excellent collection of recent investigations into diagrammatic reasoning, see [GCN95]. It contains a history, the underlying theory, and numerous examples of diagrammatic reasoning. Glasgow, et al. have compiled a good illustration of the current effort to understand the logical and computational characteristics of diagrammatic reasoning as well as its powers and limitations.

It is almost unanimously agreed upon that information from diagrams seems to be easily understood by humans. These facts, relationships, etc. that humans can process and understand from a diagram with little effort are called "free rides" in [FNP99], who attributes their discussion to [Shi96] and [Gur98] and refers the reader there for further details. Although they are of great interest in this field, still little is understood about "free rides" because what actually makes diagrams *easier* to understand is hard to explain and analyze. The role of diagrams in problems that are essentially spatial seems quite obvious, but, as we will show, diagrammatic reasoning also has many benefits in the area of non-spatial reasoning.

[1] Allwein estimates even earlier when he states in [All98] that "[diagrammatic] reasoning ... has been with humans at least since the first written forms of communication. It probably began with the first person to draw a map to explain to another person how to get from here to there."

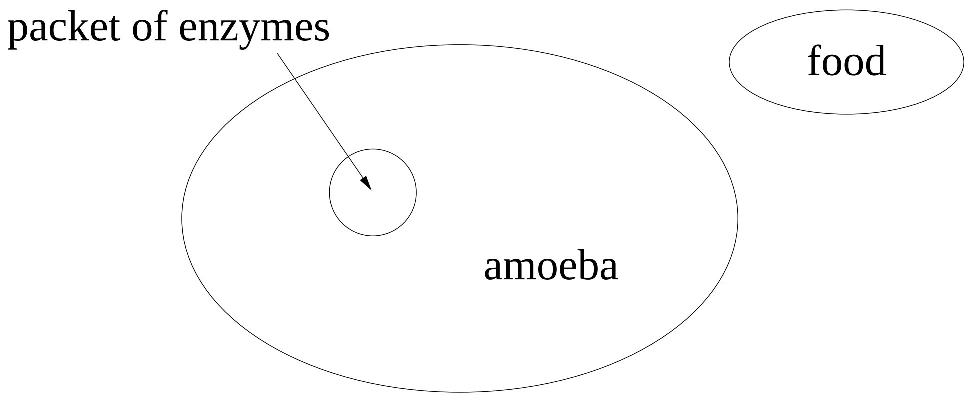

Figure 1: Simulating Phagocytosis and Exocytosis

1.1 Spatial Reasoning

Diagrammatic reasoning is widely used in the area of spatial reasoning, especially in geometry problems. Geometric problems will be revisited later, in Section 1.3. Spatial reasoning problems warrant themselves to the use of diagrammatic reasoning because of the strategies by which the diagrams can be manipulated, the simple detection of visual information, and the inferences which can be easily drawn.

A good example of a spatial reasoning problem comes from a course offered a couple of years ago in Dresden. In that course, we students were asked to code relations and transitions of a spatial reasoning problem involving an amoeba, a food packet, and a packet of enzymes, as in Figure 1, so that the computer could solve it. For the sake of brevity, we will not go into the details of the problem, which can be found in [CCR92]. The challenge of this assignment was to interpret the common-sense spatial reasoning that we already know from everyday life regarding the transition states and relations between the objects involved and put it into a logical language that the computer could understand. Essentially, our minds are doing diagrammatic reasoning, but the challenge was translating it for a computer.

Examples like this demonstrate the standard attitude of the traditional logician towards the role of diagrams in proofs. Tennant expresses it best when he wrote in [Ten86],

> [The diagram] is only a heuristic to prompt certain trains of inference; ... it is dispensable as a proof-theoretic device; indeed, ... it has no proper place in the proof as such. For the proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array.[2]

[2] Barwise and Etchemendy reply to this comment in [BE95] by saying that "... this

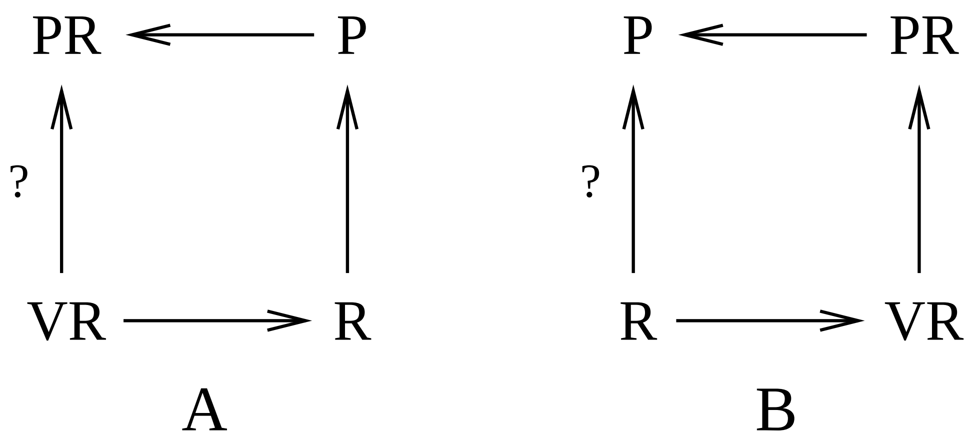

Figure 2: A) Mapping from a Visual to a Non-visual Problem. VR is a visual representation from which it is desired to compute a predicate PR. R is a non-visual representation of VR such that P and, hence, PR can quickly be computed by the symbolic architecture. B) Mapping from a Non-visual to a Visual Problem. R is a non-visual representation from which it is desired to compute a predicate P. VR is a visual representation of R such that PR and, hence, P can quickly be computed by the visual architecture. *Taken from [GCN95, page xxiv].*

Indeed, most logicians believe that diagrams are simply not rigorous enough to constitute a proof on their own. For this reason, even in the case of spatial reasoning, logicians tend to map the problem into some logical language, solve the problem, then use the reverse mapping back to the original diagram, as in Figure 2A. While there is nothing wrong with this,[3] why should we turn an essentially visual problem that we humans can see, interpret, and understand in seconds into a difficult logical representation which can only be understood by a few people after a time studying it? Why not use visual representations to solve visual problems? It will be seen that, in some situations, manipulations to a diagram itself can be a sound form of reasoning, without the need for a mapping to symbolic logic.

Where did diagrams get such a bad reputation anyway? Barwise and Etchemendy say that the suspicion comes from the founders of geometry because of certain proofs that were believed to have been misguided by poor diagrams. Because of this, the founders believed the same thing as Tennant's quote above: that diagrams are dispensable and should be eliminated from

dogma is misguided. We believe that many of the problems people have putting their knowledge of logic to work, whether in machines or in their own lives, stems from the logocentricity that has pervaded its study for the past hundred years."

[3]We will actually use this same technique, but in reverse, a little later to demonstrate non-spatial problems.

any proof. The text should "stand on its own two feet." However, Barwise and Etchemendy argue that this reasoning is irrelevant.

> If we threw out every form of reasoning that could be misapplied by the careless reasoner, we would have nothing left. Mathematical induction, for example, would certainly have to go. No, the correct response is not to throw out methods of proof that have been misapplied in the past, but rather to give a careful analysis of such methods with the aim of understanding exactly when they are valid and when they are not.[BE95]

For this reason, Section 3 is dedicated to the analysis of some misleading proofs that are guided by poor diagrams.

1.2 Non-spatial Reasoning

What about problems that are not spatial? Can diagrams be used for non-spatial reasoning problems? A resounding yes. The key is a mapping such as in Figure 2B, where there is a non-visual representation, R, that is mapped into a visual representation, VR. If this visual representation can be solved, then the reverse mapping can be used to find the solution to the original, non-visual problem.

This may seem overly complex; however, many such mappings are frequently used everyday. Some examples include maps, blueprints, bar charts, line graphs, trees, Venn Diagrams, Euler's Circles, and timelines. For example, in order to prove that a kangaroo is an Australian animal, we can reason using the tree or the Euler's circles in Figure 3. First we map our non-visual understanding to a visual diagram (draw the tree or circles) then use the diagram to reason about our problem (to notice where the kangaroo node or circle lie in the diagram). By doing this before mapping back to our non-visual problem (mapping the node or circle back to the concept of a kangaroo), we can conclude that a kangaroo is, indeed, an Australian animal.

When we can successfully map the non-visual representation to the tree, Venn diagram, etc., we are able to reason with only the diagram, regardless of its interpretation. For example, regardless of what was instantiated on the nodes of the tree in Figure 3, we can reason about the nodes and branches. However, we must be careful about this because, as stated in [WLZ95], the interpretation of graphical constants cannot be arbitrary because of facts that emerge after constructed into a diagram. For example, if we represent the fact that Andrew (A) is Bill's (B) father by an Euler Circle showing that B is included in A, then we can infer that Bill's son, Charles (C), is also Andrew's son. This is because C is included in B, which is included in A, so we can infer that C is included in A. For this reason, we have to take care when constructing diagrams to ensure that we have a good

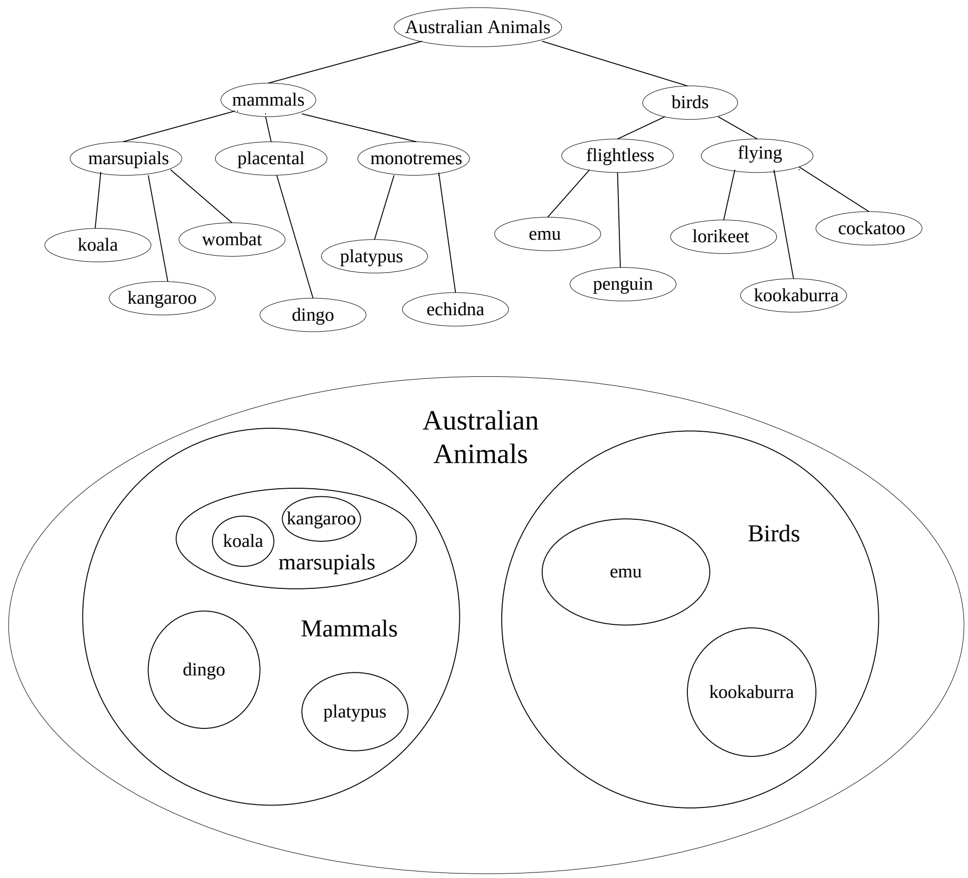

Figure 3: Examples of diagrams used in non-spatial reasoning include trees and Euler's Circles.

representation such that no extraneous facts will emerge. Unfortunately, there are currently no guidelines for choosing such a representation, which is why it is an interest for future work (see Section 6.1.2).

Now, some may ask a question similar to the one that was asked in the previous section: why turn a non-spatial problem into something spatial? One answer is simply because people use visual representations in order to understand complex things. There is something about visualizations that makes them easier for humans to understand and even show comprehension.[4] Difficult concepts are usually mapped into some type of diagram or picture. An example of the way humans use diagrams for comprehension is in mathematical problems. For example, try the following example from geometry:[5]

> Given the following information:
>
> - Two transversals intersect two parallel lines and intersect with each other at a point x between two parallel lines.
> - One of the transversals bisects the segment of the other that is between the two parallel lines.
>
> Prove that the two triangles formed by the transversals are congruent.

The first thing that most people will do is draw a diagram like the one in Figure 4. Indeed, most people require the aid of a picture to solve this sort of problem; they find it impossible to complete without it.

Another reason why it is beneficial to consider diagrammatic representations is because of possible computational gains that come from different representations. There are good discussions regarding this point in [LS95] and [Slo95]. A large part of the study of computer science definitely involves the data structure used to represent what is being reasoned about. Computational efficiency depends partially on how good this representation is. By finding a "better" representation, the computational efficiency and ease at which a problem is solved will increase.

These are a couple reasons why it is beneficial for us to consider diagrams seriously as not only reasoning *aids*, but also as rigorous enough to be reasoning *tools*. In fact, Shin has already thoroughly analyzed Venn diagrams as a formal system. In [Shi91], she showed that the usual rules of inference are sound with respect to the logical consequence relation of that formal system, and, in [Shi94], she presented a formal system based on Venn

[4] Qin and Simon (in [QS95]) actually found close links between the quality and accuracy of diagrams drawn by people and their understanding of the concepts in Einstein's 1905 paper on special relativity, which was published with no diagrams.

[5] This example was also shown in [LS95].

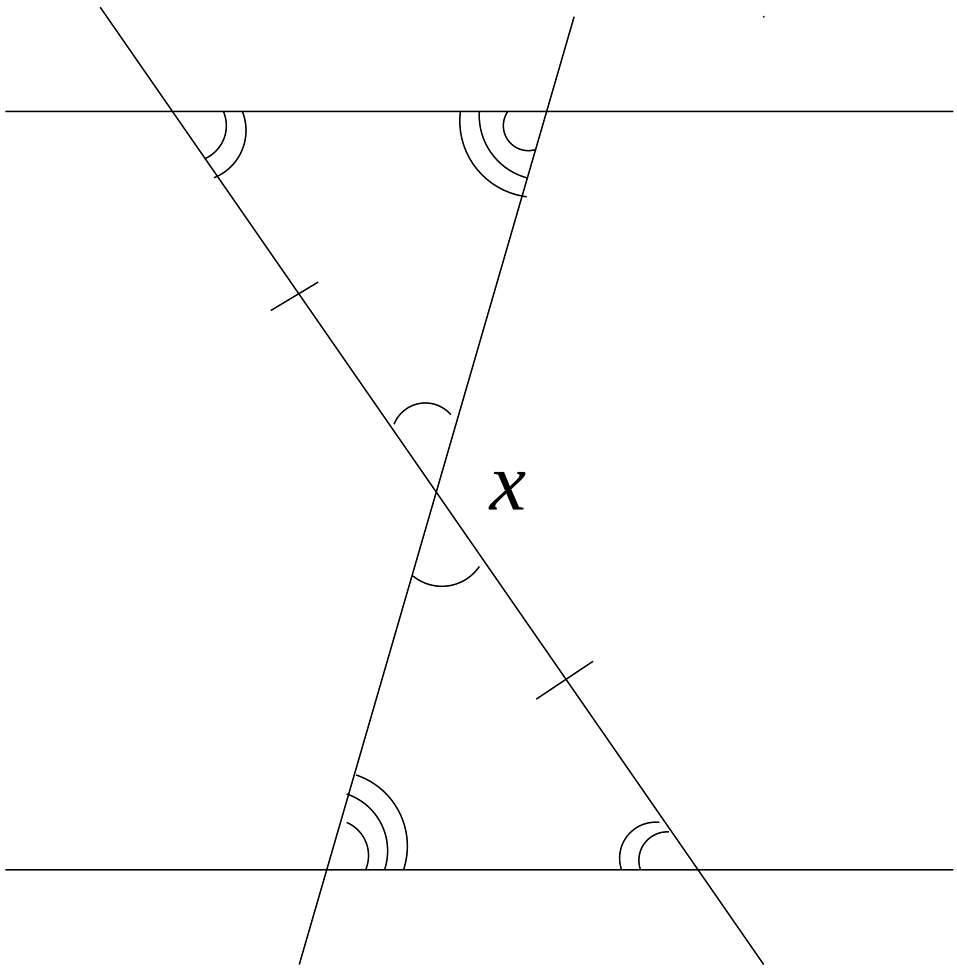

Figure 4: Diagram used as visual aid in geometry example.

diagrams that is sound and complete for some specific form of syllogistic reasoning. This is an amazing step in the right direction as it shows us that some diagrams have already been taken seriously and proven to be as sound as other forms of reasoning.

1.3 Geometric Problems and Proofs

The discussion on the use of diagrams in geometric problems has been saved for last because this is the area where diagrams are most naturally used. "Geometry has been used by many researchers as the domain in which to study reasoning with diagrams for an obvious reason; It is the only domain in which a diagram itself (or more precisely, the spatial relations among the diagrammatic elements - lines, points, angles, etc.) is the subject of study."[Iwa95] Iwasaki states it best as she continues, in the same article, by saying:

> Geometry diagrams have characteristics that are not shared by diagrams in any other domains. Most importantly, geometry diagrams stand for themselves. In other words, they are not abstractions of the real world or anything else that is the real object of interest. In almost every other domain, diagrams represent something other than themselves that one is trying to study. As a consequence, in any other domain one must have sufficient information about what is represented by the diagram in order to understand or to reason with it. In contrast, a geometry diagram usually includes all the information that one needs to solve the problem, and one only needs knowledge of geometry to understand and reason with the diagram. For this reason, geometry can be seen as the ideal domain in which diagrammatic reasoning process can be studied without having to worry about all the background knowledge of the domain, most of which is not represented in the diagram at all.

For those reasons, this study is concentrated on the area of diagrammatic reasoning in geometry.[6]

Furthermore, a very interesting area of diagrammatic reasoning is, indeed, in the area of proofs. "In the domain of mathematical proofs where diagrams have had a long history, we have an opportunity to investigate in detail and in a controlled setting the various perceptual devices and cognitive processes that facilitate diagrammatically based arguments."[FNP99] We plan to take advantage of this opportunity. Because geometry is essentially spatial, we plan to show that, just as in the time of Euclid, a diagram may be a sufficient, and even desirable, proof.

Historically, geometry theorem proving on computers seriously began in the fifties with the work of Gelernter, Hanson, and Loveland in [GHL63]. Gelernter created in the first AI model of a geometry proof solver, and it worked by using a diagram to control the search for a proof. Basically, diagrams were only used as a heuristic to reject backward paths when they became implausible in the diagram. We refer the interested reader to [Gel63] and [GHL63] for more detail.[7] Furthermore, in Chapter 2 of [Jam99], Jamnik

[6]However, it must also be noted that the same traits listed above that make geometry "the ideal domain" for diagrammatic reasoning, also make it unrepresentative of diagrammatic reasoning in general. In other areas, the interaction between the process of reasoning with the visual information in the diagram and that of reasoning with non-visual knowledge of the domain is essential to understand the whole problem solving process.

[7]The reader may also be interested in reading the work of Chou, et al. in [Cho88] and [CGZ94] regarding (non-diagrammatic) machine proofs in geometry. While they are irrelevant to this paper, they are still an interesting implementation of automated proofs of geometric theorems.

surveys several other diagrammatic systems which have been implemented (including Gelernter's). She states that "[all] of the described diagrammatic reasoning systems use diagrams in the search for an essentially algebraic proof of a theorem." While all of this work is historically interesting, it is irrelevant to this paper because diagrams were used only as a heuristic to cut the search space. The ultimate goal of this paper is not to develop a "better" mathematical theorem prover; it is to use mathematical proofs as a domain in which to study and justify diagrammatic reasoning. It will be shown in this paper that, in some domains, diagrams can be sound proofs by themselves, without the need for symbolic logic.

Because of time limitations and for the sake of thoroughness, this study is further concentrated on diagrammatic proofs in mathematics,[8] specifically proofs that are presented in [Nel93]. We are very fortunate that Nelson has compiled such a comprehensive collection of diagrammatic proofs.

1.4 Motivation and Aims

Diagrams have been used in theorems and proofs of geometry since the time of Aristotle and Euclid. It has even been said that pictorial representations of information predate textual representations by at least six thousand years. However, starting with the invention of formal axiomatic logic in the sense of Frege, Russell, and Hilbert, diagrams have been denied a formal role in theorem proving. A goal of this paper is to investigate whether and how diagrams can be used in formal proofs by analyzing them in a similar manner as in [Jam99] when she said that her "motivation is not to discover diagrammatic proofs, but to study them in order to understand them better and be able to formalise them." It will be shown that, in the domain of geometry, each step of a diagrammatic proof can be logically validated by traditional computational theories.

Although diagrammatic representations may need to be augmented in the future by other representations to completely represent many problem domains, a sharply focused exploration of the exclusive use of diagrammatic representations is necessary to bring our understanding of them to the same level as other, better understood representation methods. Only when equality is achieved in our understanding of all representation methods can we make informed judgements concerning their respective uses.

However, until a better understanding is achieved and diagrammatic proofs are considered on par with symbolic reasoning proofs, we must justify every step of a proof. Indeed, Foo, et al. believe that explanations into the effectiveness of diagrammatic proofs of mathematical theorems have an obligation "...to give an account – using standard mathematics, metamathematics, logic or computation theory – of why that mode of reasoning

[8]More specifically, mathematical formulas in which we can appeal to a feature of a diagram that conveys the truth of a theorem in two-dimensional space.

is sound."[FNP99] We will do just this in the area of mathematical diagrammatic proofs.

Furthermore, this paper will concentrate on two kinds of mathematical proofs, called Categories 1 and 3 by [Jam99], in order to further the work of Jamnik and possibly find a foundation for later work on automating diagrammatic proofs of those categories. In fact, a hope is that the work in this paper helps tackle the problem of finding the underlying theory regarding diagrammatic proofs of geometric theorems in order to pave the way to discovering the principles for the mechanical generation and verification of diagrammatic proofs.

This will be done by, firstly, giving an introduction and analysis of mathematical and geometric diagrammatic proofs. Then, an examination into some of problematic, misleading diagrams and their proofs will be given. Until we overcome the suspicion surrounding the use of diagrams in proofs, we cannot begin the study of diagrammatic proofs. Included also are explanations into the effectiveness of diagrammatic proofs of mathematical theorems using "machinery" from symbolic proofs to support them, thus showing that diagrammatic proofs are as rigorous as "formal" algebraic proofs. In addition, two solutions to the problem of generalization of Category 1 proofs were presented in [FNP99]. A proof that these two solutions are equivalent is also included. Finally, a link between traditional theories of computation that involve concepts like invariants, continuations, and fixed points, and diagrammatic reasoning is also presented. As an example, the diagrammatic proof that the sum of the geometric series $1/4 + 1/16 + 1/64 + \cdots$ equals $1/3$ is used.

In essence, the aim of this paper is to logically validate the soundness of construction steps in certain diagrammatic proofs as well as to give a good introduction to diagrammatic reasoning and to help develop a theoretical basis for computing directly with diagrammatic representations. In doing this, we intend help advance the understanding of what is involved in diagrammatic proofs, why they work, and why they sometimes do not work. We will also show that diagrams alone can be regarded as legitimate (or even desirable) proofs in the area of mathematical theorems. Hopefully, this will help to open new opportunities for study and development in the underlying theory and in later work on the automation of diagrammatic proofs.

1.5 Layout of Paper

This paper is laid out in seven sections.

Section 1 This section, the introduction. Includes a general overview of diagrammatic reasoning and diagrammatic proofs, motivation and aims, and layout of paper.

Section 2 Introduces Jamnik's taxonomy for categorizing proofs and provides examples of common diagrammatic proofs of mathematical theorems and an analysis of each proof. Definitions and concepts that will be used throughout this paper will also be introduced.

Section 3 As stated in Section 1.1, there are some incorrect "proofs" that are misled by poor diagrams. This section contains an analysis of some misleading proofs and their corresponding diagrams in order to show when they are valid and when they are not.

Section 4 In [FNP99], two solutions to the problem of generalization were presented. This section contains a proof that these two solutions are actually equivalent.

Section 5 This section focuses on a specific Category 3 proof. It formalizes the geometric intuition we humans use when looking at the proof, justifies every step in the proof by linking it with traditional computational theories, and develops a meaning and representation for the ellipsis in the proof.

Section 6 Future work. If we had another six months (or more) to work, this section includes the things that would be explored.

Section 7 A summary and discussion regarding what was presented and problems that were solved in this paper.

2 Taxonomy and Analysis of Examples

As stated in the previous section, this paper will concentrate on diagrammatic proofs of geometric theorems in order to advance the understanding in this area and to find the underlying foundations of possible automation in the future. Although a thorough justification of each proof is not given in this section, the aim is to introduce the reader to diagrammatic reasoning with a good set of examples and a brief analysis of each.

In Chapter 3 of [Jam99], a taxonomy for distinguishing three different types of diagrammatic proofs was established. Jamnik presented proofs from [Nel93] and categorized them according to specific aspects of the proof. In this section, her taxonomy will be presented, as it will be used throughout the paper. Then, examples from each category will be given. As stated in Section 1, since this study concentrates on theorems of Categories 1 and 3 and since Category 2 proofs are so thoroughly covered by Jamnik, there will be less detail for the examples of Category 2. All proofs presented here can also be found in [Nel93] and many are also discussed in [Jam99]. Although these proofs have appeared in numerous places before, to our knowledge, this is the first time such an investigation of them has been compiled.

2.1 Definition

Jamnik defines three categories of diagrammatic proofs as follows, reproduced from [Jam99]:

Category 1 Non-inductive theorems. Usually, there is only one representative diagram for all instances of the theorem. There is no need for induction to prove the general case: proofs are not schematic. Simple geometric manipulations of a diagram prove the individual case. Abstraction is required to show that this proof will hold for all [parameters]. Theorems are of continuous space.

Category 2 Inductive theorems with a parameter. A diagram is representative of a particular instance of a theorem. Proofs are schematic: they require induction for the general diagram of magnitude n (a concrete diagram cannot be drawn for this instance). And alternative method can sometimes be used to capture the generality of the proof. Theorems are of discrete space.

Category 3 Theorems whose proofs are inherently inductive: for each individual concrete case of the diagram they need an inductive step to prove the theorem. Every particular instance of a theorem, when represented as a diagram requires the use of abstractions to represent infinity. Theorems are of continuous space.

2.2 Examples

Information about each individual example will be given. Every example will be set up roughly the same, with differences only between the various categories. The following will be given for the examples:

Explanation A brief explanation of the diagrammatic proof. Ideally, no explanation should be necessary, but, for the sake of thoroughness, a brief clarification is given.

Construction A brief description of the construction of the proof (more necessary for Categories 2 and 3).[9]

Hypotheses A set of formulas that become the implicit hypotheses of the proof. In essence, these are the theorems that are required to validate every step of the transformation from one diagram into another (most necessary for Category 1 proofs in order to generalize the proof using the Theorem of Constants, see Section 4). To our knowledge, besides a brief mention in [FNP99], no other paper has so thoroughly discussed

[9] As you will see, the construction for a Category 2 proof is more like a *deconstruction* or a decomposition.

the necessary formulas required to prove theorems of Category 1. As was mentioned in Section 1, until diagrammatic proofs are considered on par with algebraic, symbolic proofs, these hypotheses are needed to justify every step of the proof. In addition, as will be further discussed in Section 3, using diagrams as well as symbolic reasoning in a proof provides a validity check to help prevent problematic proofs.

Invariant After each step of the transformation (for Category 1 proofs) or each construction step (for Category 3 proofs), the invariant is the aspect of the proof that does not change. Invariants are necessary in order to complete many of the proofs in Categories 1 and 3.

Actions The actions that are performed on a diagram in order to transform it to the solution to the theorem. After each action, no invariant should have been altered (for Categories 1 and 3). For Category 1 proofs, a diagram representing the left-hand side of the equation is usually transformed using these actions into a diagram representing the right-hand side of the equation. For Category 3 proofs, a complete picture of the theorem is only represented with an abstraction, like ellipsis (...), because it requires infinitely many construction steps for completion. A construction step is a single run-through of the set of actions. The actions that will be used in this section are:

- **slide** - Slides a specific piece of a diagram within a specific area.
- **arrange** - Similar to *slide*, but the piece can be placed wherever necessary.
- **split** - A figure in the diagram is split into at least two pieces.
- **adjust dimension** - The dimensions (length, width, height, radius, etc.) of a figure are adjusted. Usually, the dimensions are adjusted in such a way as to maintain a constant area.
- **make square** - This is the same action as *adjust dimension*, but *make square* is more intuitive and descriptive of what is specifically being done to the dimensions.
- **scale** - Produces a scaled version of some specific part of the diagram.

2.3 Category 1 Proofs

Most Category 1 proofs use the area and length of a geometric figure to represent a mathematical expression. Manipulation and generalization are then used to transform one diagrammatic representation into another (usually, to transform the diagrammatic representation of one side of an equation to the diagrammatic representation of the other side). Generalization of the diagrams is implied but can also be proven. See Section 4 for more detail.

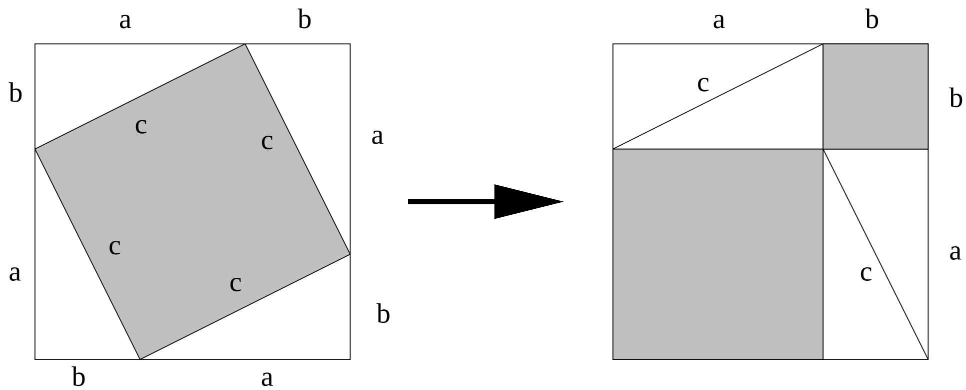

Figure 5: Pythagorean Theorem

2.3.1 Pythagorean Theorem *(Figure 5)*

Explanation The *Pythagorean Theorem* states that the sum of the squares of the legs of a right triangle equals the square of the hypotenuse, i.e. $a^2 + b^2 = c^2$, where a and b are the legs of the right triangle, and c is its hypotenuse. In [Nel93], six different diagrammatic proofs of this theorem are given. Figure 5 is one of these proofs, and Figure 7 is another. Nelson attributes this solution to the unknown author of the *Chou pei suan ching*, circa BC 200.

The key to this proof is that the area of the square stays the same, regardless of how we rearrange the four right triangles within it. Since the area of the triangles also do not change when we move them around, we can infer that the shaded area of the right-hand square must be the same as the shaded area of the left-hand square. The shaded area of the left-hand square is c^2, and the shaded area of the right-hand square is $a^2 + b^2$. Therefore, we know that $a^2 + b^2 = c^2$, where a, b, and c are the two legs and hypotenuse of our right triangle, respectively.

Construction Take any right triangle. Call one leg a, the other leg b, and the hypotenuse c. Arrange the triangle, along with three identical right triangles, around the edges of a square with sides of length $a + b$, as in the left-hand side of Figure 5. The hypotenuses of the four triangles will form a square of area c^2 contained within the larger square.

The transformation from the left-hand side of Figure 5 to the right-hand side consists of sliding the four triangles like tiles within the large square to form two rectangles of size $a \times b$ in the upper-left-hand and lower-right-hand corners of the square as in Figure 6.[10] The area not occupied by the triangles must be the same as in the previous example. The unoccupied area consists

[10]This method of transformation is thanks to [HL98].

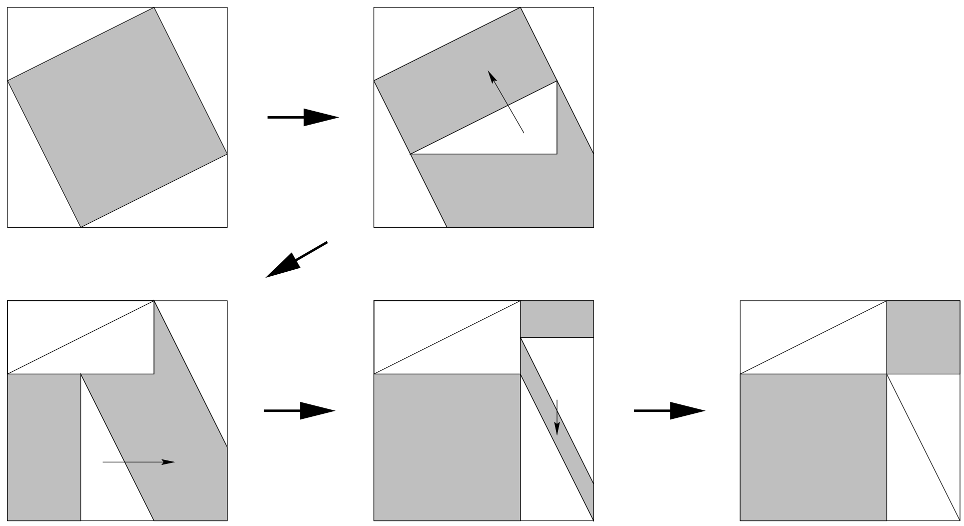

Figure 6: Transformation for diagrammatic proof of Pythagorean Theorem.

of two squares: one of size a^2 and the other of size b^2. Therefore, the total unoccupied (shaded) area is $a^2 + b^2$.

Finally, we have shown that, given a triangle of dimensions a, b, and c (*leg, leg, hypotenuse,* respectively), $a^2 + b^2 = c^2$, which is precisely the Pythagorean Theorem.

Hypotheses Some of the hypotheses required for this proof include triangle congruence properties, theorems, and definitions, the fact that the sum of the angles of a triangle is 180°, and the area of a square is the length of its side squared. Furthermore, we also must know that shapes and areas are invariant under rotations and translations and that manipulations of figures follow algebraic laws. This last fact makes sense with respect to what we are doing with the diagrams. Since the visual diagram is, in essence, just another representation of a mathematical expression, each manipulation done to it is, in essence, the same as performing some algebraic laws on that expression. So, dividing the area of a square in half is the same as dividing the expression x^2 by 2.

Invariant Total area stays the same.

Actions See Figure 6.

1. slide
2. slide
3. slide

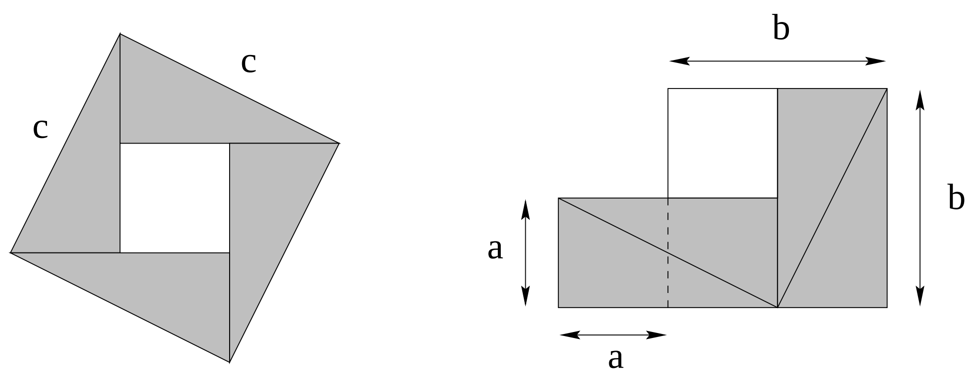

Figure 7: Pythagorean Theorem

2.3.2 Another Proof of the Pythagorean Theorem *(Figure 7)*

Explanation This is another proof of the Pythagorean Theorem, which was also shown in the previous section. Figure 7 is taken from page 4 of [Nel93], who attributes the proof to Bhāskara in the 12th century.

The key to this proof, just like the previous version of the proof, is that the total area contained within the square and triangles remains the same, no matter how they are arranged. The area of the square on the left is c^2. When the pieces are rearranged, the same area is equal to a^2+b^2. Therefore, since the area does not change, we know that $a^2 + b^2 = c^2$, when we are given a triangle of size a, b, c (*leg, leg, hypotenuse*, respectively).

Construction Take any triangle with legs of length a and b and hypotenuse of length c. Without loss of generalization, let us say that a is the shorter leg of the triangle and b is the longer one. Form a square with sides of length c by arranging four identical triangles as in the left-hand side of Figure 7. The area of this square is c^2. Now move two of the triangles to form the figure on the right side of Figure 7. To figure out the area of this figure, we can break it up into two squares. The larger square has an area of b^2 and the smaller square has an area of a^2. Therefore, the total area of the figure is $a^2 + b^2$. Finally, since the area of a figure does not change simply by moving it around, the area of the figure on the left must be the same as the area of the figure on the right, i.e. we have that $a^2 + b^2 = c^2$.

Hypotheses Same as the hypotheses in the previous section.

Invariant Total area stays the same.

Actions

1. arrange

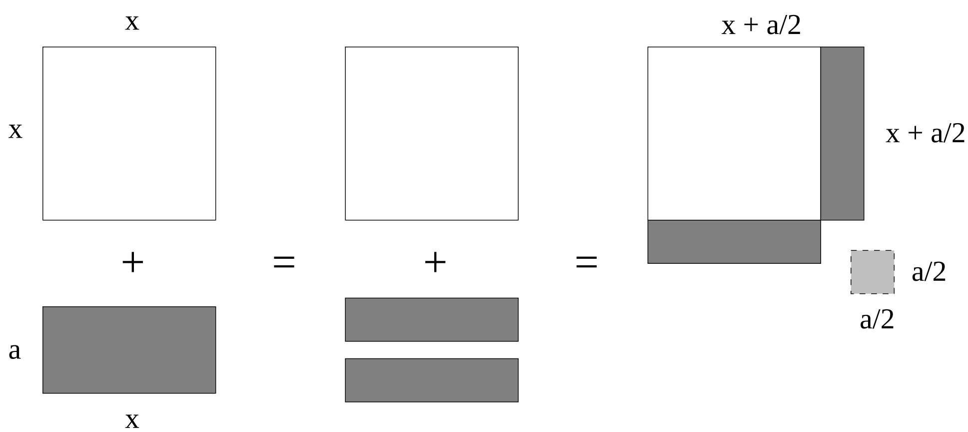

Figure 8: Completing the Square: $x^2 + ax = (x + a/2)^2 - (a/2)^2$

2.3.3 Completing the Square *(Figure 8)*

Explanation We want to prove that the theorem for completing the square, namely, $x^2 + ax = (x + a/2)^2 - (a/2)^2$, is true. This proof is also on page 19 of [Nel93].

Represent x^2 as a square of side length x and ax as a rectangle with sides of length a and x. The sum of the area of these two figures is the sum $x^2 + ax$, which is the left-hand side of the theorem. By splitting the rectangle in half and arranging the two halves on the sides of the square, it is easy to see that a square with sides of length $x + a/2$ is almost formed, except for a small square with sides of $a/2$.

Construction Take any x and a. Construct a square with sides of length x and a rectangle with sides of length a and x to represent x^2 and ax, respectively. Split the rectangle into two identical rectangles each with sides of length x and $a/2$. Arrange these smaller rectangles onto the bottom and right-hand sides of the square. If one adds a square with sides of length $a/2$, a square with sides of length $x + a/2$ will be left. Therefore, the area of the original square, x^2, plus the area of the rectangle, ax, plus the area of the smaller square with sides $a/2$, is equal to the area of a square with sides of length $x + a/2$. This means that $x^2 + ax + (a/2)^2 = (x + a/2)^2$, which is the same as the original theorem.

Hypotheses Most hypotheses for this proof are similar to the ones mentioned for the Pythagorean Theorem in Section 2.3.1. These include the invariance of shapes and areas under rotations and translations, the area of a square is the length of the side squared, the area of a rectangle is *length* $\times$ *width*, and that manipulations of figures follow algebraic rules.

Invariant Total area stays the same.

Actions

1. split
2. arrange

2.3.4 Sum of Squares *(Figure 9)*

Explanation In Figure 9 is Diophantus of Alexandria's "Sum of Squares" Identity, from page 22 of [Nel93], as solved by Nelson himself. It states that $(a^2+b^2)(c^2+d^2) = (ad+bc)^2 + (bd-ac)^2$. Basically, the proof relies on a lot of dimension changing and rearranging of pieces, while maintaining the overall area.

Construction Take any a, b, c, and d. Form a rectangle with sides of length a^2+b^2 and c^2+d^2. This rectangle can be divided into four rectangles of size a^2c^2, b^2c^2, a^2d^2, and b^2d^2. Because of the algebraic property that $x^2y^2 = (xy)^2$, we can change the dimension of each of these rectangles to make them each squares with areas of $(ac)^2$, $(bc)^2$, $(ad)^2$, $(bd)^2$, respectively.

If the squares of size $(ac)^2$ and $(bd)^2$ are put together, they can be split into three pieces: two rectangles with dimensions of $ac \times bd$ and one square with sides of length $bd-ac$.[11] The two rectangles with dimensions of $ac \times bd$ (area $= abcd$) can be changed into rectangles with dimensions of $ad \times bc$, since this does not alter the total area. Finally, these rectangles can be arranged alongside the squares of size $(bc)^2$ and $(ad)^2$ to get a square with sides of length $ad+bc$.

Consequently, we have transformed a square of area $(a^2+b^2)(c^2+d^2)$ into two squares of area $(ad+bc)^2$ and $(bd-ac)^2$ by using only operations that preserve area. Therefore, the area is the same, hence they are equal, and the theorem is proven.

Hypotheses Again, like in the previous three sections, the hypotheses for this proof include properties of squares and rectangles, the invariance of shapes and areas under rotations and translations, the addition and subtraction of areas, and algebraic identities.

Invariant Total area stays the same.

[11] This can also be seen by simple algebra on the areas of the rectangles: $2 \cdot abcd + (bd-ac)^2 = 2 \cdot abcd + ((bd)^2 - 2 \cdot abcd + (ac)^2) = (ac)^2 + (bd)^2$

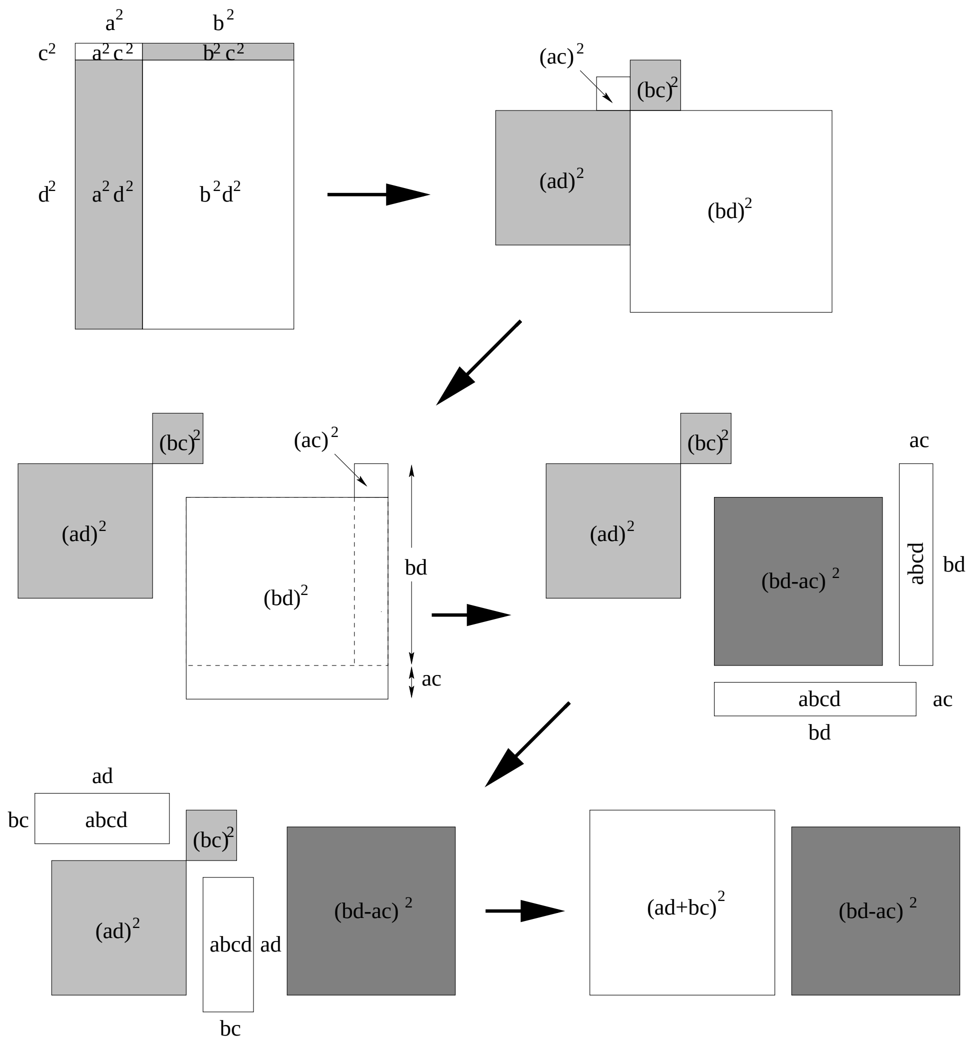

Figure 9: Diophantus of Alexandria's "Sum of Squares" Identity: $(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (bd - ac)^2$

Actions

1. make square
2. arrange
3. split
4. adjust dimension
5. arrange

Note Some may have noted that this proof is a bit more complex and slightly less intuitive (contains fewer "free rides") than others that have been presented so far. This proof, in fact, has been debated recently regarding whether it constitutes a diagrammatic proof or not because it is not as "transparent" or intuitive as, for example, the proof of the Pythagorean Theorem in Section 2.3.1. The fact is that there is currently no delimitation as to when a proof is "more inclined" to be proven diagrammatically or symbolically. We believe that there is actually a continuum of proofs: some which yield much better diagrammatic proofs, some which yield nicer symbolic proofs, and some that lie in the "grey" area between.

2.4 Category 2 Proofs

With Category 2 proofs, most often dots or squares are used to represent individual numbers, as area and length were used in Category 1 proofs to represent expressions.

Diagrammatic proofs of Category 2 are thoroughly discussed by Jamnik. Because of this, and because this research concentrates of proofs of Categories 1 and 3, this section will not contain as much detail as the others. For a complete discussion of Category 2 proofs, refer to [Jam99]. A discussion on the justification and generalization of diagrammatic proofs of this category is given in Section 2.4.4.

2.4.1 Sum of Odd Natural Numbers *(Figure 10)*

Explanation This theorem states that the sum of the first n odd natural numbers is equal to n^2, i.e. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. It is on page 71 of [Nel93] and page 41 of [Jam99] (it is also used throughout Jamnik's thesis as an example for the performance of DIAMOND, her semi-automatic theorem prover). Nelson attributes this solution to Nicomachus of Gerasa (circa AD 100).

The reason that this theorem is a typical example of a diagrammatic proof is because it is so self-explanatory. It is easy to see that one can start with a single point, and each L that is added is the next consecutive odd

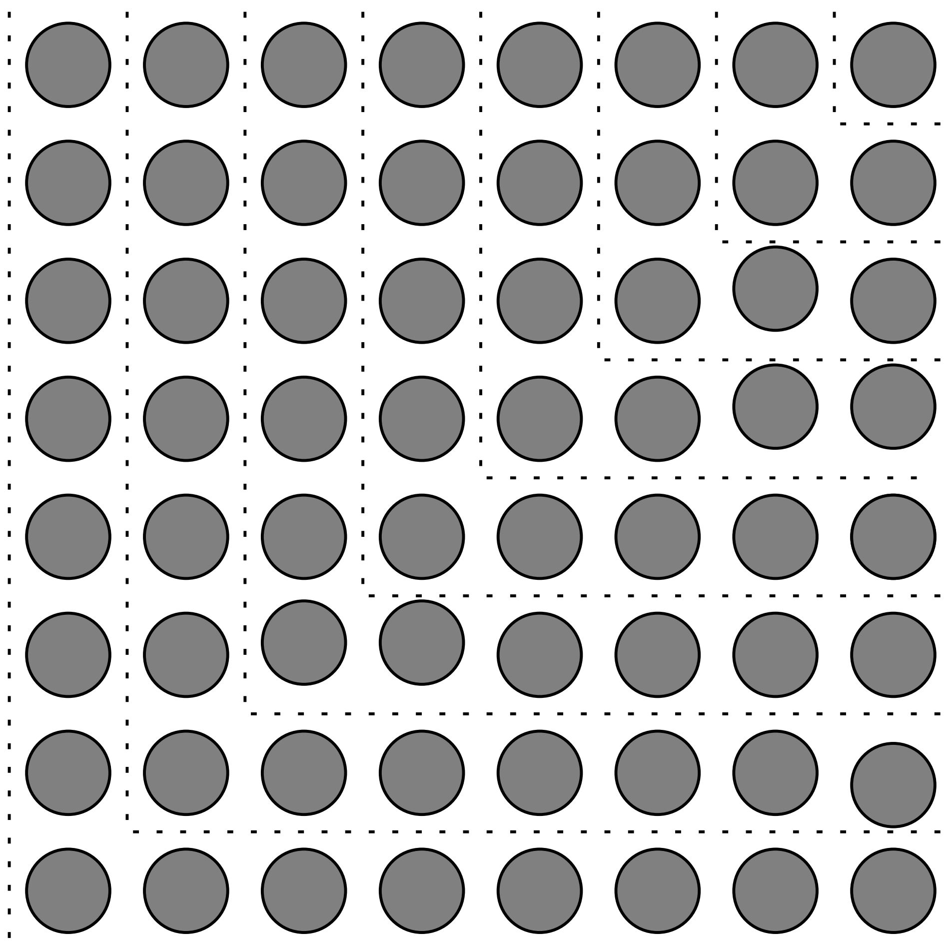

Figure 10: Sum of Odd Natural Numbers: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

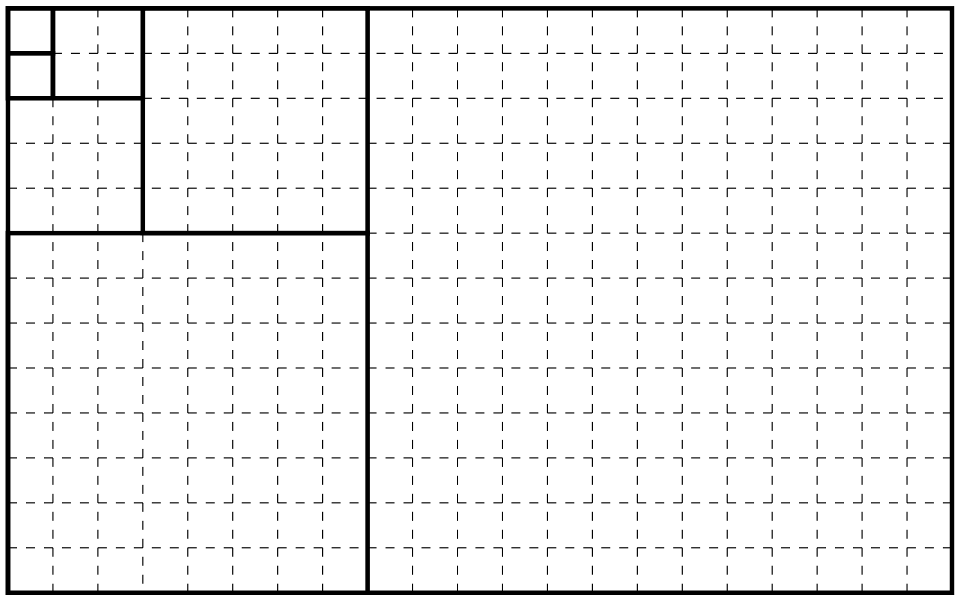

Figure 11: Sum of Squares of Fibonacci Numbers:
$F_1 = F_2 = 1$; $F_{n+2} = F_{n+1} + F_n \Rightarrow F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$

natural number. So, after n odd numbers are added, a square with sides of length n is created. This, of course, has an value of n^2.

Construction Take a square of area n^2. Cut it into L-shaped pieces (an L-shaped piece consists of two adjoining sides of the square). Since each L contains two consecutive sides of the square, if the square has the side length of k, the corresponding L will be of size $2k - 1$ (the length k from each size, minus the one shared vertex point). This is an odd number. Therefore, since each L-shaped piece represents an odd number and since they are consecutive odd numbers, it is easy to see that the original square of area n^2 is the sum of the first n odd natural numbers.

2.4.2 Sum of Squares of Fibonacci Numbers *(Figure 11)*

Explanation Fibonacci numbers are well known throughout mathematics. The definition used here is from [Nel93]. A Fibonacci number is defined as the sum of the two previous Fibonacci numbers, where the first two Fibonacci numbers are 1. In other words, $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. This theorem (also seen on page 83 of [Nel93]) is for the sum of the squares of Fibonacci numbers. It states that the sum of the squares of the first n Fibonacci numbers is equal to the product of the nth and $(n+1)$th Fibonacci numbers. In symbols, this is $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

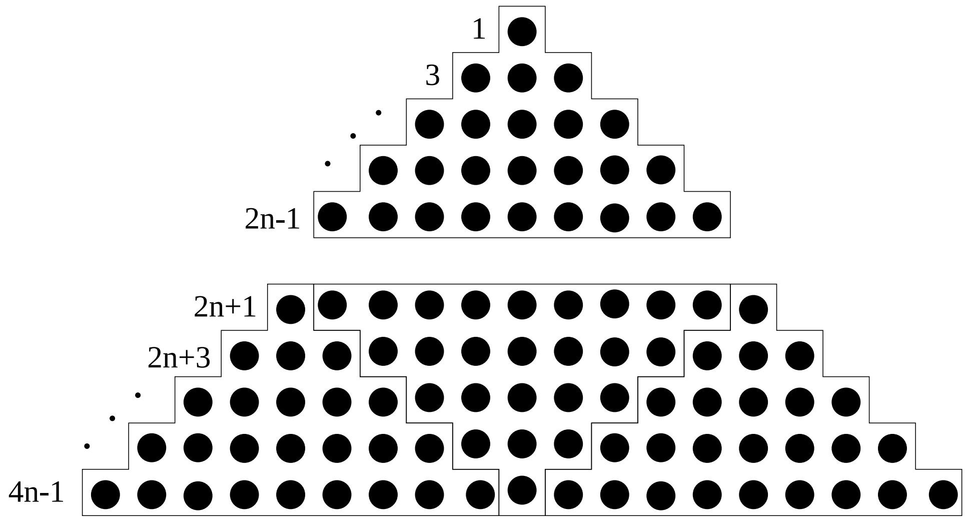

Figure 12: $\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(4n-1)}$

The reason that this proof works is because of the very nature of Fibonacci numbers. When the next square of a Fibonacci number is added to the diagram, it is added to the side of the rectangle adjacent to the two Fibonacci numbers previous to it (the two numbers which were added to get that number). Therefore, a rectangle with dimension $F_n \times F_{n+1}$ always remains.

Construction Take a rectangle of dimension $F_n \times F_{n+1}$. Remove a square of area F_n^2 from a corner of it. This leaves a rectangle of dimension $F_{n-1} \times F_n$. Continue by removing a square of area F_{n-1}^2 from a corner of this rectangle. This leaves a rectangle with dimensions $F_{n-2} \times F_{n-1}$. Repeat this step of removing a square with an area equal to the short side of the current rectangle squared until you are left with a single square. This represents $F_1 = 1$. By observation, one can see that the areas which were removed are consecutive squares of Fibonacci numbers.

2.4.3 Sequence of Odd Integers *(Figure 12)*

Explanation Nelson attributes this property of the sequence of odd integers to Galileo in 1615. Despite the fact that it is so interesting, this proof does not appear in much of the literature, besides [Nel93].

The theorem states that the sum of the first n odd numbers divided by the sum of the next n odd numbers is $1/3$. In mathematical notation, that is

$$\frac{1+3+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(4n-1)} = \frac{1}{3}.$$

It is proven by representing the sum of the numerator by a triangle and showing that a similar representation of the denominator can be broken into three copies of the triangle formed by the numerator.

Construction For a specific n, represent the sum of the first n odd numbers by a triangle of height n. Similarly, represent the sum of the next n odd numbers by a trapezoid with the top row of length $2n + 1$ and the bottom row of length $4n - 1$. Finally, break the bottom trapezoid into three equal sections as in Figure 12. Each of these sections is identical to the triangle representing the numerator. Therefore, the denominator is exactly three times larger than the numerator, making the fraction equal to 1/3.

2.4.4 Discussion

Since we will not return to proofs of this category again in this paper, we should take this opportunity to discuss the way Jamnik overcomes the problem of representing the general *(uninstantiated)* versions of Category 2 theorems. She accomplishes this by instantiating the diagrams to particular values, then uses induction to prove the theorem. The generality of the proof is proven using schematic proofs. Jamnik sketches the basic idea behind schematic proofs in Chapter 4 of her thesis, from which the following is taken:

> A schematic proof is a program with some parameters. By instantiation of these parameters the program generates ground examples of a particular proof. ... Thus, a diagrammatic schematic proof is a program which applies geometric operations to diagrams when given some value of the parameter. In this way, we eliminate the need for general diagrams, and instead use a general number of applications of geometric operations.

In this way, Jamnik solves the problem plaguing the automation of Category 2 theorems and is able to create a semi-automatic theorem prover based on diagrams, DIAMOND, which solves proofs of this category. See [Jam99] for more detail.

2.5 Category 3 Proofs

The main distinction of Category 3 proofs is that they require abstractions to represent infinity (note the use of ellipsis (...) in Figures 13A and 14A). The ellipsis imply that the construction steps can be completed to infinity. Because of this, a bit more information, like fixed points and continuations, are used to help describe, understand, and solve the proofs. However, these will not be discussed here; rather, they are thoroughly discussed in Section 5 along with the complete formalization and justification of each step of the proof in Section 2.5.2. This section is only to introduce the reader to examples of proofs in this category.

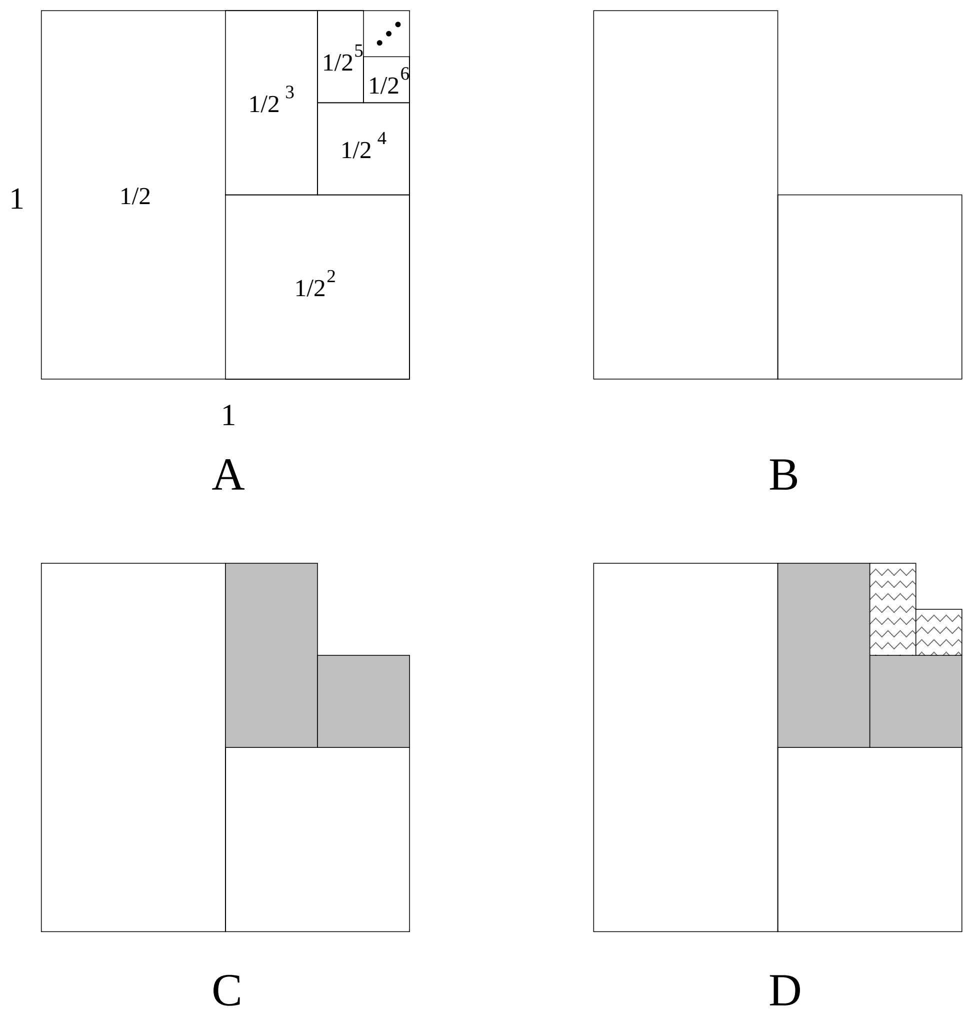

Figure 13: Geometric Sum: $\sum_{i=1}^{\infty} 1/2^i = 1/2 + 1/4 + 1/8 + \cdots = 1$

2.5.1 Geometric Sum *(Figure 13)*

Explanation The proof of the Geometric Sum $1/2+1/4+1/8+\cdots = 1$ is given on page 118 of [Nel93] as well as in [Jam99] and [FNP99]. Nelson also gives a general version of the Geometric Sum, whose solution he attributes to Warren Page. The full proof is shown in Figure 13A.

Take a unit square. Cut it down the middle and remove half. Cut the remaining half down the middle and, again, remove one half. Continue repeating this process indefinitely. It is easy to see that, each time, you are removing $1/2^i$ from the square as i goes to infinity. This represents decomposing the geometric sum.

Construction When constructing this proof, the key idea is to start with an L-shaped region, let us call it L_1, as in Figure 13B. This corresponds to the sum $1/2 + 1/2^2$. Proceeding, scale L_1 to get L_2, and glue L_2 onto L_1, as in diagram C. This corresponds to the sum $(1/2+1/2^2) + (1/2^3 + 1/2^4)$. Continue the next step of the construction by scaling L_2 to get L_3, which corresponds to the sum $1/2^5 + 1/2^6$. Likewise, glue L_3 to the diagram of L_1 and L_2, to get the diagram in Figure 13D.

With each addition of an L-shaped region, L_i, a new pair of summands, $1/2^{(2i-1)} + 1/2^{2i}$, is added to the diagram. For the sake of simplicity, call the summands corresponding to L_i S_i. In other words, $S_i = 1/2^{(2i-1)} + 1/2^{2i}$. Using this, one can say that diagram B corresponds to the sum S_1, diagram C corresponds to the sum S_1+S_2, and diagram D corresponds to the sum $S_1 + S_2 + S_3$. Therefore, we have established a one-to-one relationship between the area of each L_i and the sum of each S_i.

The construction can be continued by scaling and translating each L_i in the diagram, which corresponds to summing the S_i's, as i goes to infinity. The L_i's will sum to the total square, which has area of 1. Therefore, the corresponding S_i's will also sum to 1.

Hypotheses Hypotheses for this diagrammatic proof include facts that we have seen in Section 2.3, like that shapes and areas are invariant under rotations and translations, that manipulations follow algebraic laws, and properties concerning the areas of squares and rectangles. We also need the fact that scaling does not change proportions.

Invariant For any number of construction steps, sum will always be ≤ 1.

Actions

1. scale
2. arrange

2.5.2 Sum of Geometric Series *(Figure 14)*

Explanation This proof is very similar to the Geometric Sum in Section 2.5.1.[12] Basically, it states that the sum $1/4+1/16+1/64+1/256+\cdots$ equals $1/3$. It is presented in [Nel93] on page 121 as well as a solution to the general version of it, attributed to Sunday A. Ajose. The proof is in Figure 14A.

Take an L-shaped region with area $3/4$, and shade the center square, as in Figure 14B. It is trivial to see that the shaded region is one-third of the total region. Now, if this L is scaled to one-half its size and glued to the first

[12]This proof is also analyzed in great detail in Section 5.

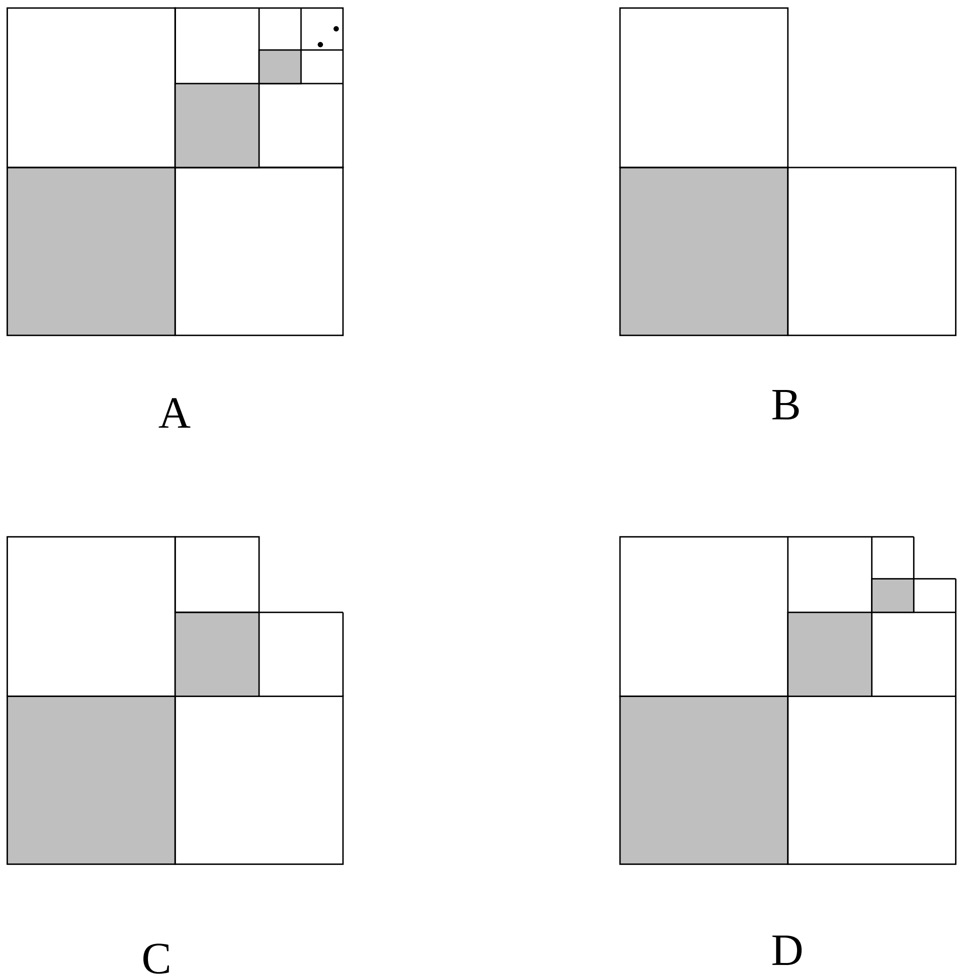

Figure 14: Sum of Geometric Series: $\sum_{i=1}^{\infty} 1/4^i = 1/4 + 1/16 + 1/64 + \cdots = 1/3$

L, it is easy to see in diagram C that the shaded region is still one-third of the total region. Next, scale the smaller L to one-half its size and glue the new piece onto the diagram, as in diagram D. The shaded region is still one-third of the total. If this process is continued indefinitely, we have shown in Section 2.5.1 that the L-shaped regions will eventually converge to the unit square, i.e. to one. Therefore, since every construction step maintains the invariant that the shaded region is one-third of the total region, the area of the shaded region will converge to 1/3.

Construction The construction of this proof is exactly like the construction of the proof of the Geometric Sum in the previous section, except that the center square of the L-shaped region is shaded, as in Figure 14A. Also, each L_n represents $1/4^n$ (the area of the shaded region) instead of a sum of the entire area, as in the previous theorem. Therefore, at each construction step, we only add the area of the shaded region to the sum. Then, we use the fact that the L's sum to one and the invariant that, after each construction, the shaded area is still one-third of the total area to conclude that the shaded area goes to 1/3.

Hypotheses Facts involving

- invariance of shapes and areas under rotations and translations
- adding one area to another
- scaling does not change proportions

are included in the hypotheses of this diagrammatic proof.

Invariant Shaded area is one-third of total area.[13]

Actions

1. scale
2. arrange

[13]Besides the proofs of Category 1 which required invariants for choosing the correct actions in order to complete the proof, this is the only proof so far that has really used an invariant to help solve the problem. It is interesting to note that, without the observation of this invariant, this proof could not be completed.

2.6 Summary

This section gave the reader an introduction to diagrammatic proofs of geometric theorems by presenting Jamnik's taxonomy for distinguishing three different types of diagrammatic proofs and providing an analysis of some examples of diagrammatic proofs from each category. Analyzing and categorizing diagrams is definitely a first step on the way to understanding them. We have seen in this section how diagrams are representative of mathematical expressions and how the actions on them are representative of operations on an expression. Furthermore, we have seen how standard mathematics can justify the soundness of reasoning diagrammatically. For this reason, for some theorems, diagrams can be used as proofs in the same manner as proofs written in algebraic or symbolic language.

3 Problematic Proofs

Here is a riddle to get the blood flowing back to your brain:[14]

> Three men were dining one evening. The total of their meal was \$30, so each man gave the waiter a \$10 note. When the waiter went to pay the bill, he realized that a mistake had been made, and the meal had only cost \$25. On his way back to the men's table to return the last \$5, it occurred to him that \$5 would be difficult to split between three men. So he pocketed \$2, figuring that the men would not be expecting any money in return and would be happy if any of it came back, and gave each of the men \$1. Therefore, each man had paid \$9 for the meal. $9 \times 3 = 27$. $27 + 2$ *(in the waiter's pocket)* $= 29$, but the men originally paid \$30. Where did the last dollar go?

The answer to this riddle lies in the deceiving way that it is presented. Up until the statement $27 + 2 = 29$, the riddle is very clear and straightforward. While that statement is true in itself, putting it into this riddle leads the reader astray. If, instead of that statement, the riddler had written "$9 \times 3 = 27$, but the bill was only \$25. Where did the extra \$2 go?", the riddle would be trivial enough for any student who is observant and knows simple arithmetic to solve. However, because of one misleading statement, it becomes a riddle that is not obvious to most people upon first inspection.

Diagrams can mislead in the same manner. In Section 1.1, there is a discussion regarding the bad reputation and suspicion surrounding the use of diagrams because of certain misleading diagrams that lead to incorrect proofs. Barwise and Etchemendy were also quoted when they said that misapplied diagrams should be studied so that we have a thorough understanding of when they are appropriate and valid and when they are not.

[14] A similar version of this riddle appears in [Nor44].

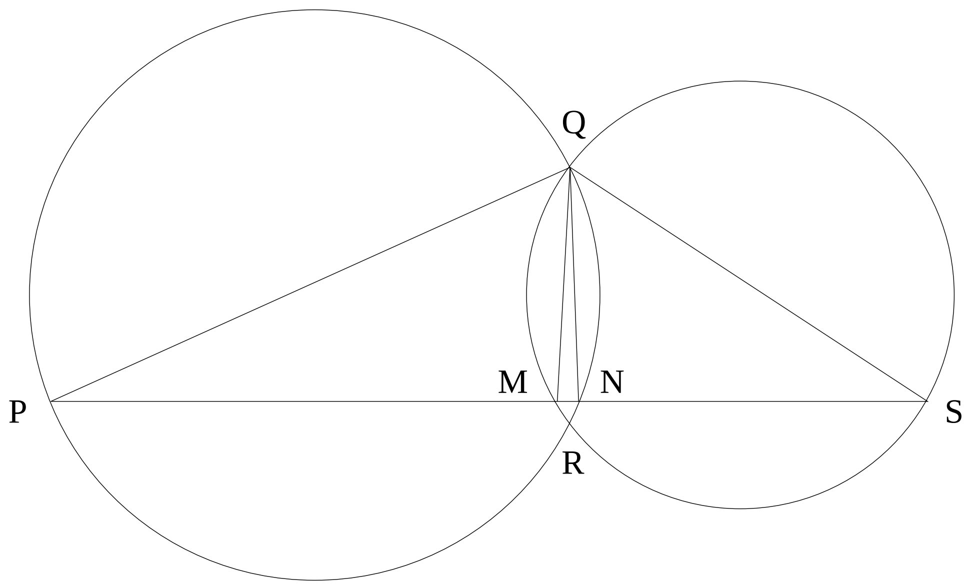

Figure 15: Diagram for proof that there are two perpendicular lines from a point to a line

The aim of this section is to do just that. Diagrammatic reasoning, like any form of reasoning, can be misapplied and can lead to incorrect conclusions. The key is to use different methods of reasoning as verification for a proof. We will analyze a few misleading proofs and their corresponding diagrams, show why they are not valid, as well as show when diagrams are necessary in order to invalidate other misleading forms of reasoning.

We are very fortunate to have a wonderful collection of various misleading proofs in geometry in Chapter 6 of [Nor44], appropriately titled *Geometrical Fallacies*. All of the "proofs" below are from this collection.

3.1 Diagram Problems

People do not normally have compasses, rulers, and protractors with them at all times (or, at least, *normal* people do not), so, most often, diagrams are drawn roughly. This is why improperly drawn diagrams are the number one problem with misleading proofs.

This section contains four examples of incorrect proofs which are suggested by misleading, improperly drawn diagrams.

To prove that there are two perpendiculars from a point to a line Let any two circles intersect at points Q and R, as in Figure 15. Draw diameters PQ and QS, and draw the line PS. Label the points where PS intersects with the circles M and N, respectively. Furthermore, draw lines

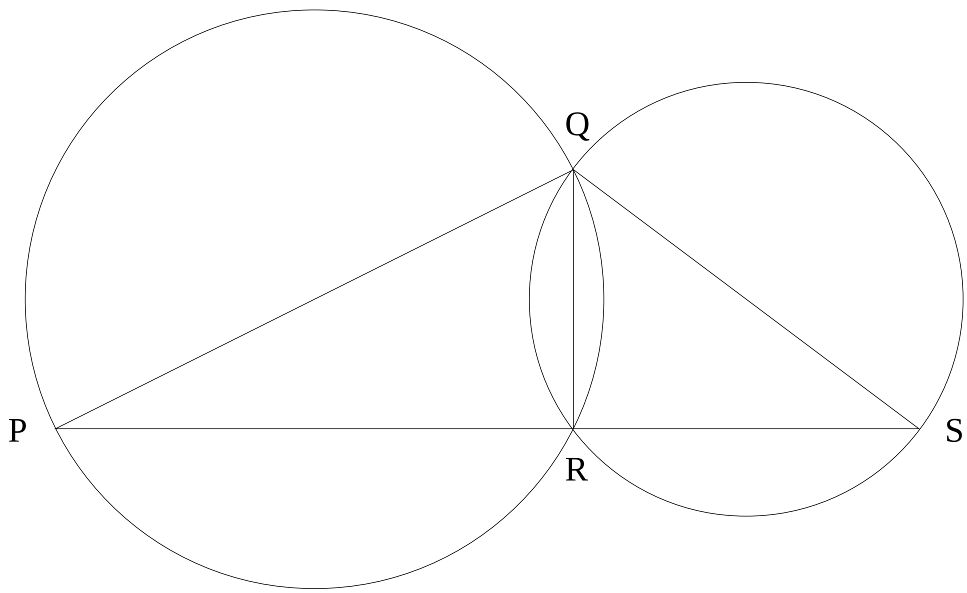

Figure 16: Properly drawn diagram disproving that there are two perpendicular lines from a point to a line

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MQ and NQ. Therefore, $\angle PMQ$ and $\angle SNQ$ must be right angles *(an angle inscribed in a semi-circle is a right angle)*. Hence, MQ and NQ are both perpendicular to PS through Q.

Obviously, this proof cannot be valid, and we are reluctant to believe that there are two perpendicular lines to a line through a point not on the line. The problem with this proof lies in the incorrectly drawn diagram in Figure 15. In a correctly drawn diagram, Figure 16, one can see that the line PS intersects the circles at precisely point R, not at two distinct points M and N. To prove that this is, indeed, the case, one only needs to see that $\angle PRQ$ and $\angle SRQ$ are right angles *(an angle inscribed in a semi-circle is a right angle)*, so their sum must be a straight angle. Therefore, PRS must be a straight line, and, since there is only one straight line between P and S, it must go through R.

To prove that a right angle is equal to an obtuse angle Let $ABCD$ be any rectangle. Draw a line BE outside the rectangle and equal in length to AD, as in Figure 17. Draw the line DE, and construct the perpendicular bisectors of AB and DE, where G and F are their midpoints, respectively. Since AB and DE are not parallel by construction, their perpendicular bisectors cannot be parallel and will, therefore, intersect at a point P. Construct AP, BP, DP, and EP. By construction, we know that $AD = BE$.

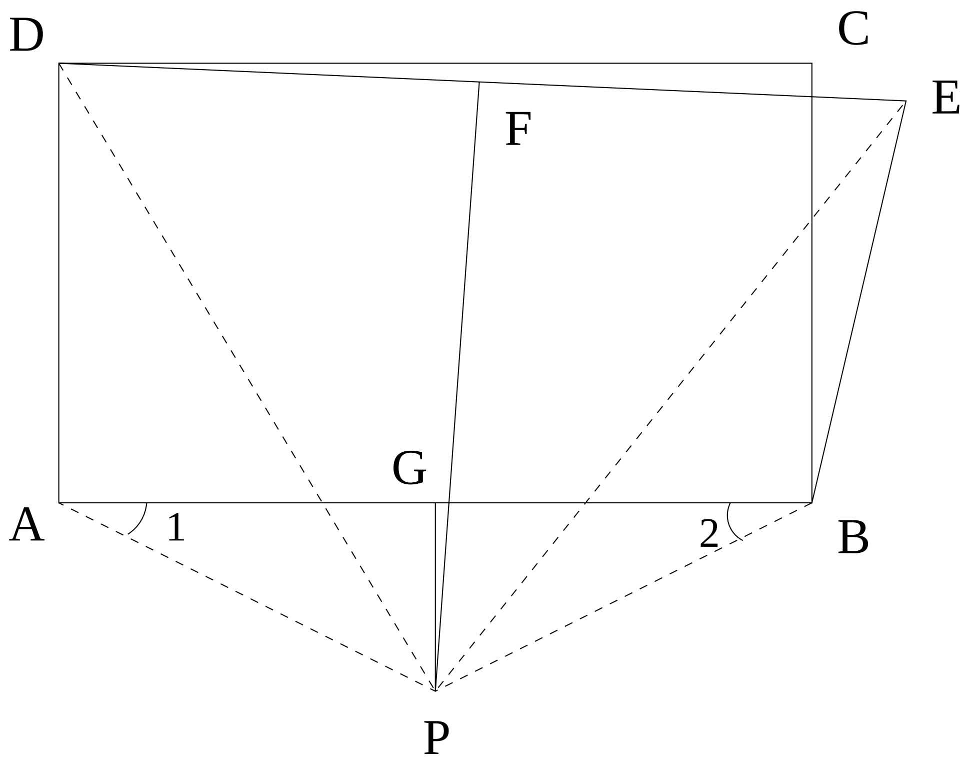

Figure 17: Diagram for proof that a right angle is equal to an obtuse angle

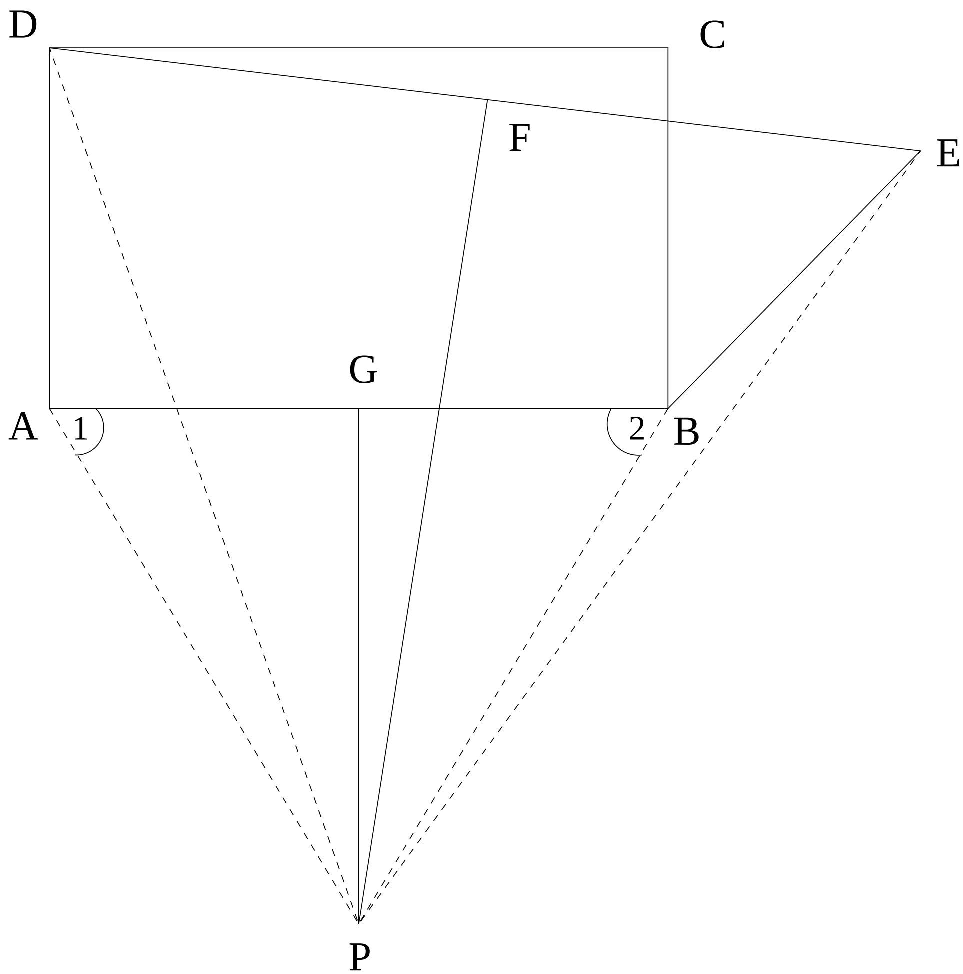

Figure 18: Properly drawn diagram disproving that a right angle is equal to an obtuse angle

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Furthermore, because any point in the perpendicular bisector of a line is equidistant from the ends of the line, $AP = BP$ and $EP = DP$. Therefore, $\triangle APD$ and $\triangle BPE$ are congruent *(if three sides of a triangle are equal respectively to three sides of another triangle, then they are congruent)*, and, hence, $\angle DAP = \angle EBP$. However, we also know that $\angle 1 = \angle 2$ *(angles opposite the equal sides of an isosceles triangle are equal)*. Subtracting this from the equality $\angle DAP = \angle EBP$ gives us that $\angle DAG = \angle EBG$. Since $\angle DAG$ is a right angle *(given)* and $\angle EBG$ is an obtuse angle *(by construction)*, we have proven that a right angle is equal to an obtuse angle.

Again, we are naturally sceptical of this proof because right angles and obtuse angles should not be equal, by definition. So, we have to look closely

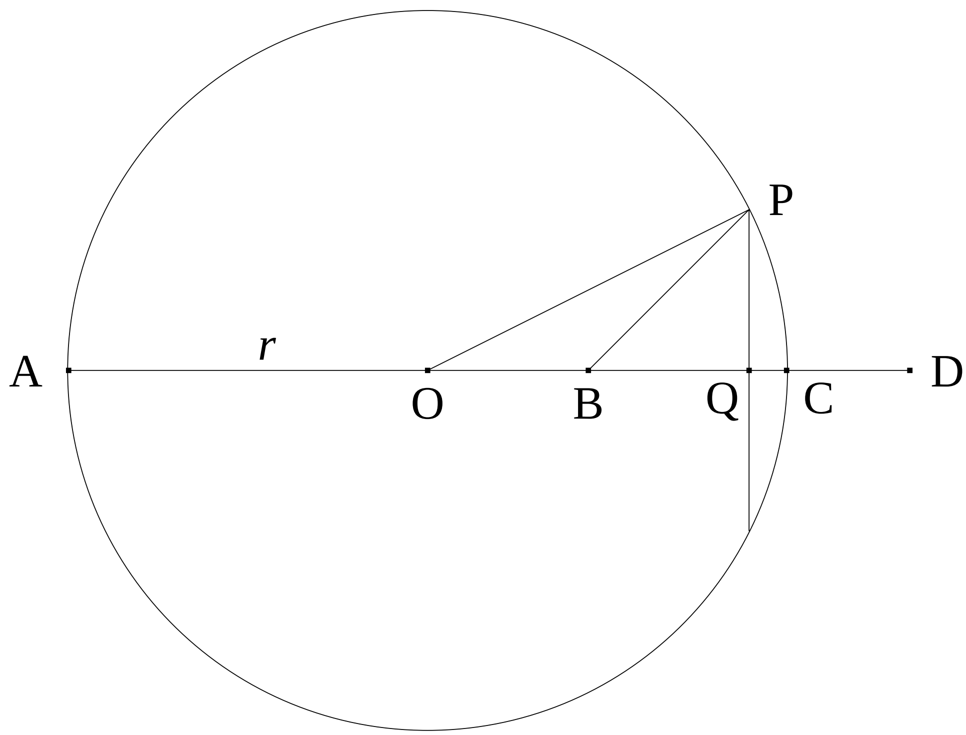

Figure 19: Diagram for proof that every point inside a circle must lie on the circumference of the circle

at our "proof". And, again, as in the previous example, the diagram is to blame for misleading us.

Figure 18 is the properly drawn diagram. In this picture, it is easily seen that the line EP lies outside of the rectangle $ABCD$. Although our proof regarding the equality of $\angle DAP$ and $\angle EBP$ is still valid, we can now see that $\angle 2$ is no longer part of $\angle EBP$.

To prove that every point inside a circle must lie on the circumference of the circle Let B be any point within any circle O, and draw a diameter AC through B, as in Figure 19. Find point D on AC extended (produced) so that AB/BC equals AD/DC. Draw the perpendicular bisector of BD, PQ, where Q is the midpoint of BD and P is where it intersects with the circle. Draw OP and BP.

If r is the radius of the circle, then

$$\begin{aligned} AB &= r + OB, \\ BC &= r - OB, \\ AD &= OD + r, \\ DC &= OD - r. \end{aligned}$$

Using these, the proportion $AB/BC = AD/DC$ can be rewritten and simplified as follows:

$$\begin{aligned}
\frac{r+OB}{r-OB} &= \frac{OD+r}{OD-r} \\
(r+OB)(OD-r) &= (r-OB)(OD+r) \\
r\cdot OD - r^2 + OB\cdot OD - r\cdot OB &= r\cdot OD + r^2 - r\cdot OB - OB\cdot OD \\
2\cdot OB\cdot OD &= 2\cdot r^2 \\
OB\cdot OD &= r^2
\end{aligned}$$

Furthermore, from the picture, we know that

$$OB = OQ - BQ \tag{1}$$

and that

$$OD = OQ + QD. \tag{2}$$

Since Q bisects BD, $BQ = QD$. We can, therefore, rewrite equation 2 as

$$OD = OQ + BQ. \tag{3}$$

By multiplying equations 1 and 3, we get

$$OB\cdot OD = (OQ)^2 - (BQ)^2,$$

but, since $OB\cdot OD = r^2$, we now have

$$r^2 = (OQ)^2 - (BQ)^2. \tag{4}$$

Next, we can apply the Pythagorean Theorem (see Section 2.3.1 on page 14 for definition) to $\triangle OQP$ and $\triangle BQP$ to get the following

$$\begin{aligned}
(OP)^2 &= (OQ)^2 + (QP)^2 \\
(BP)^2 &= (BQ)^2 + (QP)^2,
\end{aligned}$$

respectively. If we subtract the second equation from the first one, we get

$$(OP)^2 - (BP)^2 = (OQ)^2 - (BQ)^2.$$

Since $OP = r$, the above can be rewritten as

$$r^2 - (BP)^2 = (OQ)^2 - (BQ)^2.$$

Finally, replace the right-hand side of the above equation with its value given in equation 4 to get

$$\begin{aligned}
r^2 - (BP)^2 &= r^2 \\
(BP)^2 &= 0 \\
BP &= 0
\end{aligned}$$

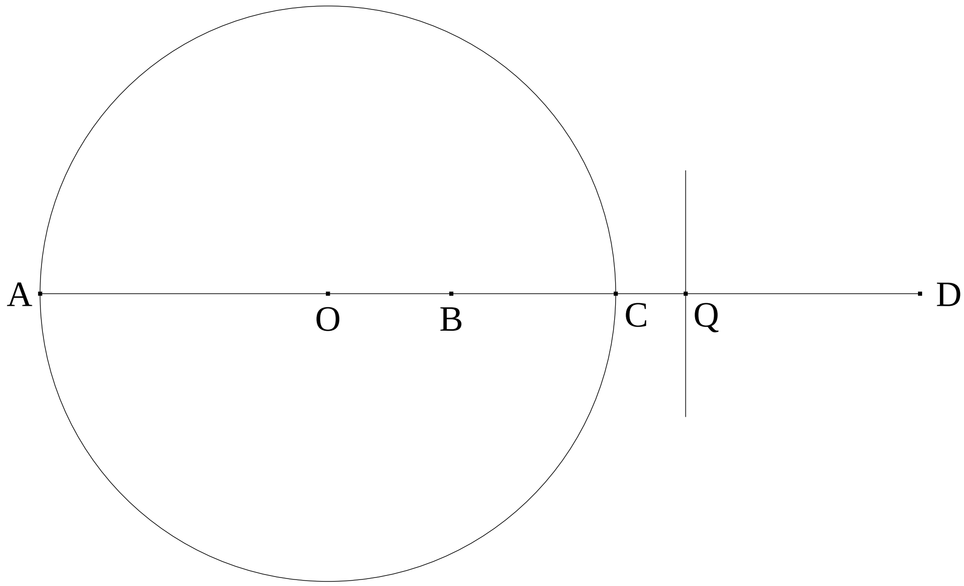

Figure 20: Properly drawn diagram disproving that every point inside a circle must lie on the circumference of that circle

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However, if $BP = 0$, then B and P must be the same point. Therefore, since P lies on the circumference of the circle O, B, originally given as an internal point of O, must also lie on the circumference.

There is a lot of algebra in this example. So, upon discovering the ridiculous theorem that one is able to "prove", the first instinct is to look at the algebra. However, the problem in this example, like the previous two, lies in the misleading diagram. Let us examine the original proportion $AB/BC = AD/DC$. Since B in an internal point and D is an external point, AD must be greater than AB. Therefore, DC must be greater than BC. It then follows that Q, the midpoint of BD must fall outside the circle, as in Figure 20. This means that the perpendicular bisector of BD does not intersect the circle at all. In other words, P does not exist, so our proof is faulty as soon as we use P.

3.1.1 For Fun

After reading all of the previous misleading proofs and why they are wrong, look at the following fallacy. See if you can tell where the problem lies.

To prove that any triangle is isosceles Let $\triangle ABC$ be any triangle. Construct the bisector of $\angle C$ and the perpendicular bisector of AB. Call their intersection G. Now construct the lines AG and BG as well as the lines DG and FG that run through G and are perpendicular to AC and BC, respectively, as in Figure 21A. We know that $\angle 1$ and $\angle 2$ are equal because CG bisects $\angle C$ and we know that $\angle 3 = \angle 4$ because they are both right angles (since DG and FG are perpendicular to AC and BC, respectively). Therefore, $\triangle CGD$ and $\triangle CGF$ are congruent because of the above information and because CG is common for both triangles *(if two angles and a side of one triangle are equal respectively to two angles and a side of another triangle, then the triangles are congruent)*. It follows that $DG = FG$ *(corresponding parts of congruent triangles are congruent)*.

Continuing, since G lies on the perpendicular bisector of AB, $AG = BG$ *(any point on the perpendicular bisector of a line is equidistant from the ends of the line)*. Furthermore, because $\angle 5$ and $\angle 6$ are right angles, $AG = BG$, and $DG = FG$ (we proved in the previous paragraph), $\triangle ADG$ is congruent to $\triangle BGF$ *(if the hypotenuse and another side of a right triangle are equal to the hypotenuse and another side of a second right triangle, then the triangles are congruent)*.

From these two pairs of congruent triangles, we know that $DC = FC$ and $AD = BF$. Adding these together proves that $AC = BC$, which is precisely the definition of an isosceles triangle.

Surely, this cannot be right. The sceptic immediately asks how we know G falls inside $\triangle ABC$. It very well might not, so let us explore the other options.

What if G bisects AB, i.e. it coincides with E, as in Figure 21B? Fortunately (or unfortunately), we can use the same proof for this diagram. We can prove that $\triangle CGD$ and $\triangle CGF$ are congruent as well as $\triangle ADG$ and $\triangle BGF$. Therefore, $DC = FC$ and $AD = BF$, so $AC = BC$, and $\triangle ABC$ is isosceles.

A similar thing can be said if G falls outside $\triangle ABC$, but close enough so that D and F still fall on AC and BC, respectively, as in Figure 21C.

Still, what if G falls so far outside the triangle that D and F fall on the extensions of AC and BC, as in Figure 21D? We can still prove that both $\triangle CGD$ and $\triangle CGF$ as well as $\triangle ADG$ and $\triangle BGF$ are congruent, so $DC = FC$ and $AD = BF$. However, this time $AC = DC - AD$ and $BC = FC - BF$. Regardless, we still see that $AC = BC$, so $\triangle ABC$ is isosceles.

Finally, one can think of one last case. What if the bisector of $\angle C$ and the bisector of AB are parallel, so they do not meet? In this case, let us call the point P where the bisector of $\angle C$ meets AB, as in Figure 21E. However, it is proven that $\triangle APC$ and $\triangle BPC$ are congruent because $\angle 1 = \angle 2$, $\angle 7 = \angle 8$ (they are both right angles), and because CP is common in both triangles. Therefore, yet again we can see that $AC = BC$, and the triangle is isosceles.

It appears that we have exhausted all possibilities, yet we have always proven that $\triangle ABC$ is isosceles. Is this possible? Do we really have to accept such a seemingly absurd notion?

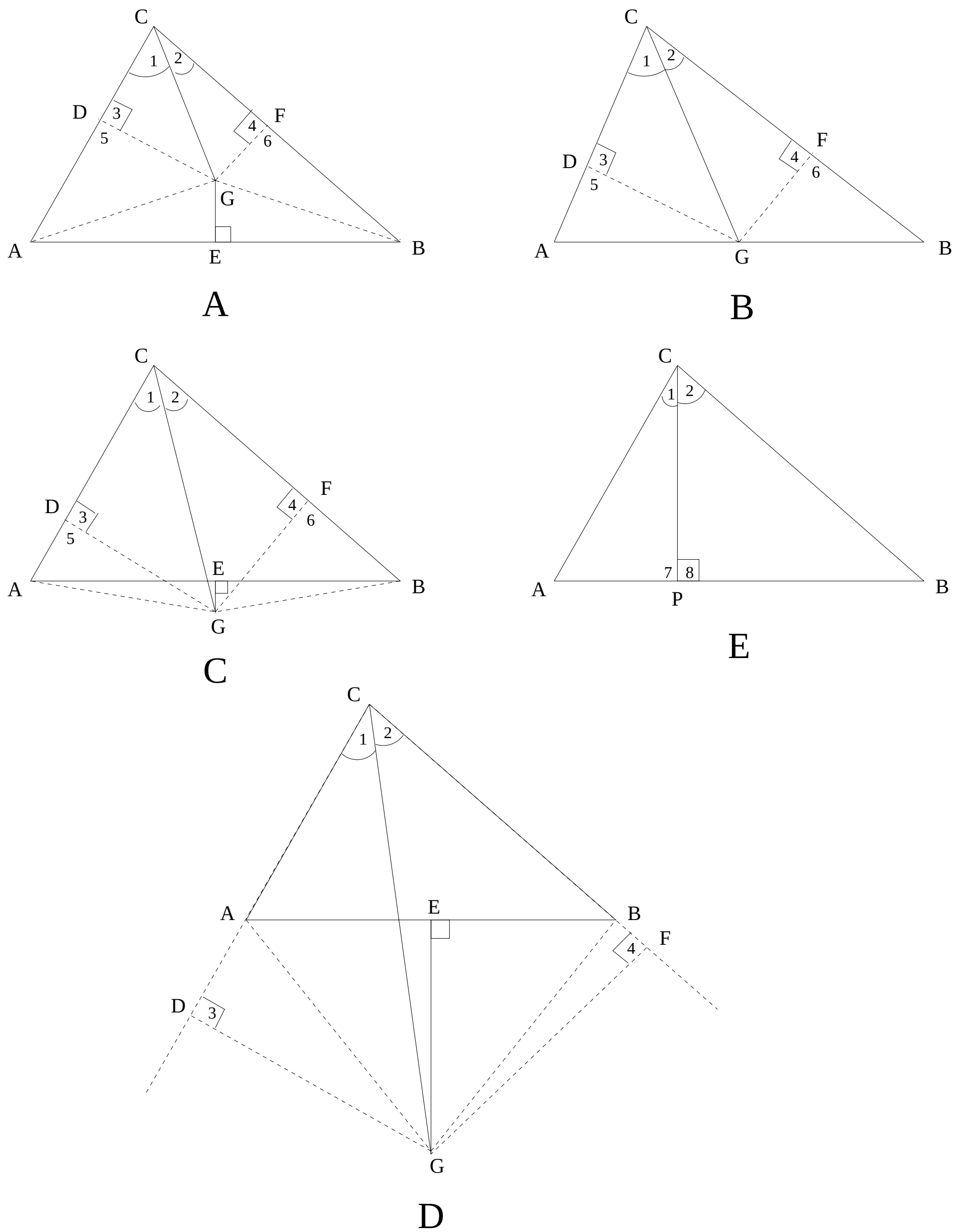

Figure 21: Diagrams for proof that all triangles are isosceles

Of course, no, we do not because this problem is in the *Problematic Proofs* section of this paper. The question is, can you see where the problem is? The answer is in Section 3.3.

3.1.2 Discussion

Although this section contained many normally trivial proofs that were lead astray by improperly drawn diagrams, this will not be a problem when diagrammatic proofs are automated. The problems in this section were, for the most part, simple human errors that could occur in any form of reasoning and would not occur in a computer. Furthermore, on the side of the human reasoner, it is comforting to know that, in such cases, algebra and logic can be used to show that the diagrams are, indeed, misleading, so that we can validate or invalidate a proof using other representation methods.

3.2 Helpful Diagrams

We know that, just as diagrams can be misleading, any form of reasoning can be misleading in the same way. In the previous section, we saw misleading diagrams that were shown, using algebra, to be invalid. In this section, we will show that, when logic and algebra are misleading, we can similarly use diagrams to help show that it is incorrect. Two "proofs" where diagrams help show that the logic is problematic follow.

To prove that two unequal lines are equal Let $\triangle ABC$ be any triangle and draw a line PQ parallel to AB, as in Figure 22. We then know that $\triangle ABC$ and $\triangle PQC$ are similar *(if a line is drawn parallel to one side of a triangle and intersecting the other two sides, it cuts off a triangle similar to the given one)*. Consequently,

$$\frac{AB}{PQ} = \frac{AC}{PC} \tag{5}$$

(corresponding sides of two similar triangles are proportional). This can be rewritten as

$$AB \cdot PC = AC \cdot PQ. \tag{6}$$

Continuing, multiply both sides by $AB - PQ$ to get

$$(AB)^2 \cdot PC - AB \cdot PC \cdot PQ = AB \cdot AC \cdot PQ - (PQ)^2 \cdot AC. \tag{7}$$

Add $AB \cdot PC \cdot PQ$ and subtract $AB \cdot AC \cdot PQ$ from both sides to get

$$(AB)^2 \cdot PC - AB \cdot AC \cdot PQ = AB \cdot PC \cdot PQ - (PQ)^2 \cdot AC. \tag{8}$$

Then, factorize

$$AB(AB \cdot PC - AC \cdot PQ) = PQ(AB \cdot PC - PQ \cdot AC) \tag{9}$$

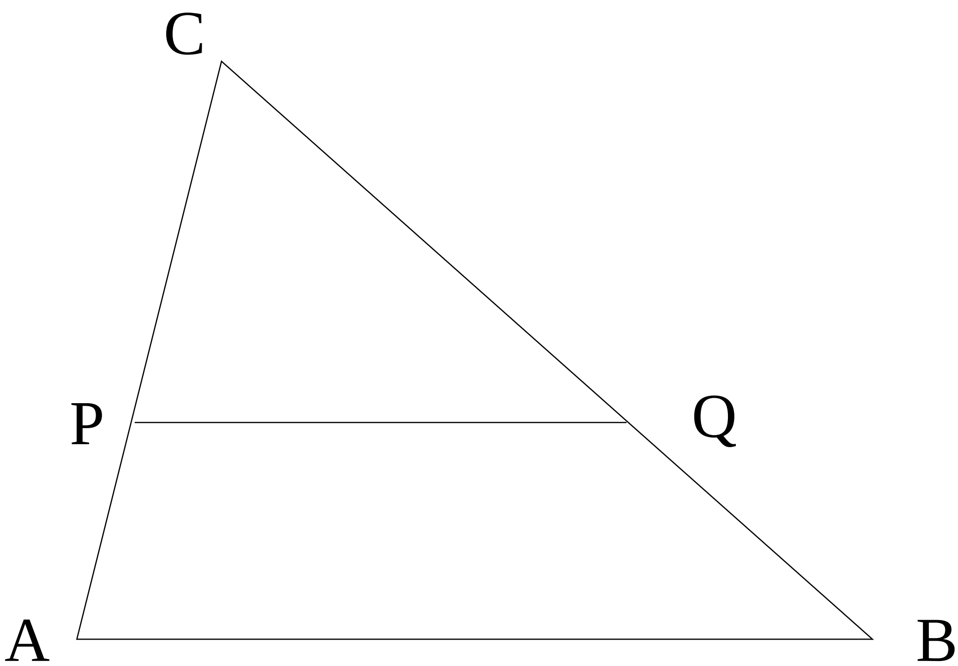

Figure 22: Diagram for proof that two unequal lines are equal

and divide both sides by $AB \cdot PC - AC \cdot PQ$ to have, finally, that

$$AB = PQ. \tag{10}$$

As convincing as the proof may be, our intuition (and eyes) tell us that this result is incorrect. We can see plainly in Figure 22 that AB is *not* the same length as PQ. Furthermore, Figure 22 is too simple to contain the problem, so it must be in the algebra itself. Looking at the picture is what informs us that there is some mistake in the above "proof."

The problem lies in step 10 when we divide by $AB \cdot PC - AC \cdot PQ$. As seen in step 6, $AB \cdot PC = AC \cdot PQ$. Therefore, in step 10 we are dividing by zero. As we have seen, such simple things like this, when overlooked, can lead to very curious results.

To prove that the sum of the angles of a spherical triangle is $180°$
Let $\triangle ABC$ be any spherical triangle, and choose any point P within $\triangle ABC$, as in Figure 23. Pass great circles through P and A, B, and C, respectively, dividing $\triangle ABC$ into three smaller spherical triangles. If we call the sum of the angles of any spherical triangle $x°$, then the sum of the angles of the three smaller spherical triangles is $3x°$. Included in this sum is the sum of the angles around point P, which is $360°$. So, the sum of the angles of

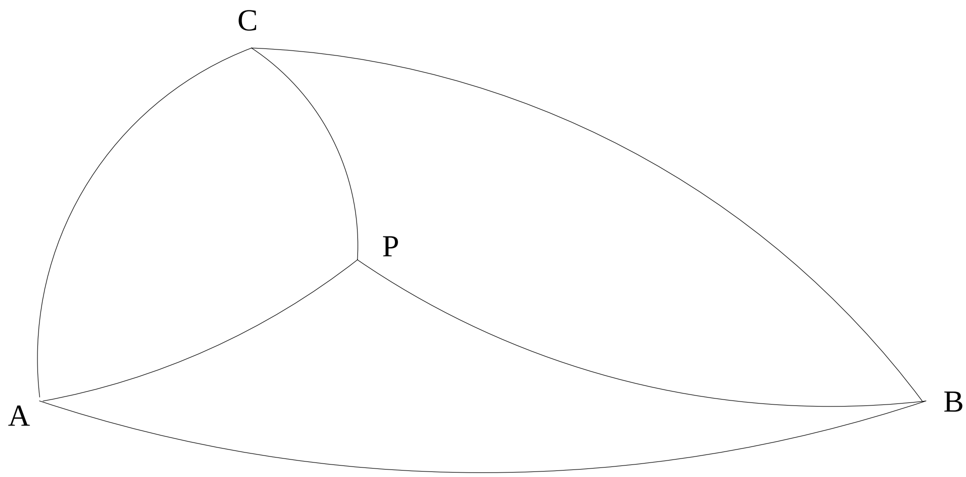

Figure 23: Diagram for proof that the sum of the angles of a spherical triangle is 180°

$\triangle ABC$ is equal to the sum of the angles of all three triangles minus the sum of the angles around P, i.e. $x = 3x - 360$. Therefore, $x = 180$.

However, a quick look again at Figure 23 should set off a mental flag in the mind of the prover. If one would draw straight lines joining A, B, and C, as in Figure 24, she would see that the angles of the included triangle are each smaller than the angles of the spherical triangle. Since we know that the sum of the angles of a straight-sided triangle is precisely 180°, the sum of the angles of the spherical triangle could not be 180°. Upon seeing this, we know that there is some sort of problem.

This is probably the easiest example from which to find the problem. There are two major problems with the reasoning in this "proof." The first one lies in the fact that the sum of the angles of a spherical triangle can be anywhere between, but not including, 180° and 540°. So, we cannot assume that the sum of the angles of *any* spherical triangle is some specific x. In addition, our statement that the sum of the angles of $\triangle ABC$ is equal to $3x - 360$ is also false. This statement implies that the angles of spherical triangles add in the same way as in straight-line triangles, but this is not the case.

3.2.1 Discussion

Errors can occur in proofs for a number of reasons, including reasoning problems, false assumptions, and algebraic problems. However, in many cases, a diagram can help a person see that their logic is incorrect because it does not agree with the diagram. Indeed, just as algebra helped "diagnose" the

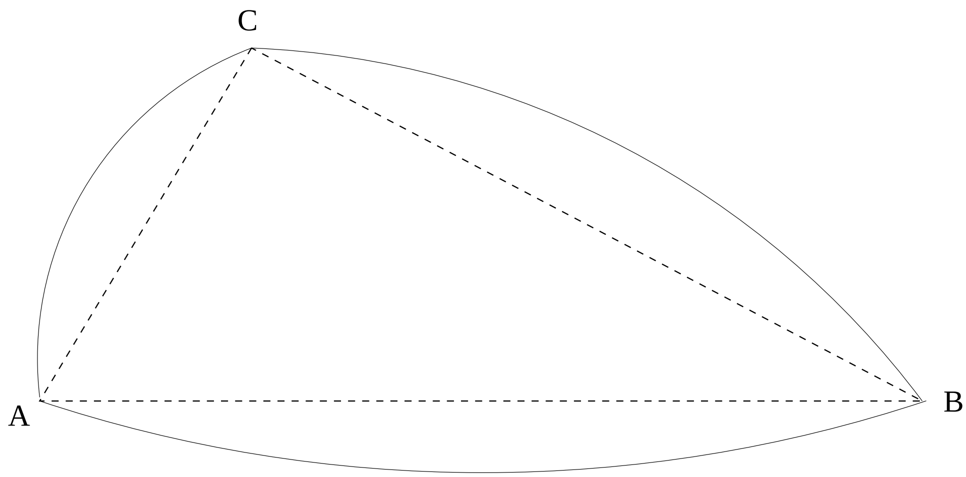

Figure 24: Diagram showing that the sum of the angles of a spherical triangle cannot be 180°

problems plaguing the diagrammatic proofs of the previous section, diagrams can help find the problems in symbolic proofs as well. In fact, diagrams are already used as a consistency check in areas such as model theory. If, for example, a diagrammatic model or counter-example can be found, it is considered a valid "counter-proof" for that theory. Therefore, we can see that, in cases like that and in proofs such as the onces above, a diagram is essential to the proof. For this reason, disregarding diagrams, as suggested by Tennant in Section 1.1, would leave proofs such as the ones above without a "validity check."

3.3 Answer to Question from Section 3.1.1

In Section 3.1.1, the "proof" that all triangles are isosceles was presented, and we asked the reader to try to find what was wrong with the proof. The answer is below.

There are two major deficiencies with this "proof". The first one is similar to the examples in Section 3.1 because the diagrams are drawn incorrectly. However, this is not the main problem with this proof; it just leads to the other problem, which is with the reasoning involved: the proof does not explore all cases. This is also a major problem with beginning math, computer science, and logic students. Whenever something is broken up into cases, we must **ensure** that we have exhausted all possibilities.

The final possibility, or counter-example, for this proof is in Figure 25. Indeed, if the diagram had been drawn properly, we would see that this is the only possibility (except the mirror possibility that D falls inside the

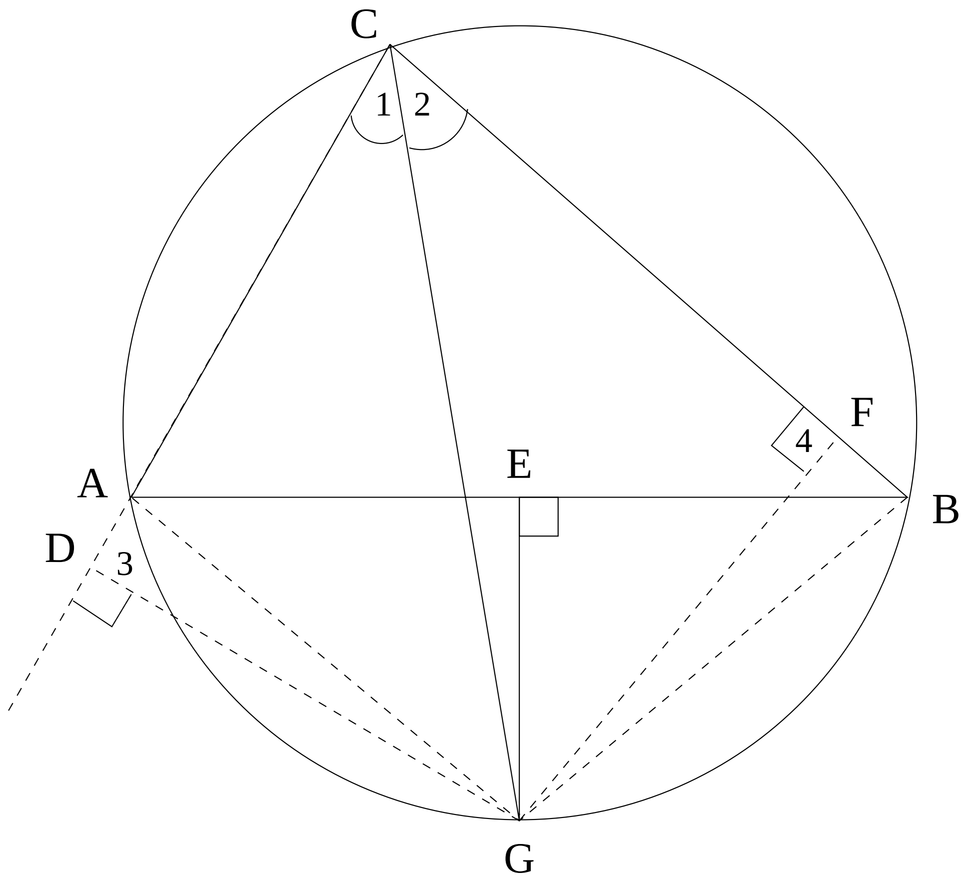

Figure 25: Correctly drawn diagram contradicting proof that all triangles are isosceles

triangle and F falls outside the triangle, which is essentially the same).[15] This figure shows that, if F falls inside the triangle and D falls outside the triangle, then, indeed, $CD = CF$ and $DA = FB$, just as we had proven before. However, in this case, $CB = CF + FB$ and $CA = CD - DA$. Therefore, $CD \neq CA$, and $\triangle ABC$ is not isosceles.

3.4 Summary

The eye can be deceiving. Just as riddles cause one's mind to be deceived, misleading diagrams can lead normally simple proofs astray. However, since diagrams are not the only forms of reasoning that are misleading and prone to errors, we cannot disregard any form of reasoning that can be misleading, or else nothing would be left. In essence, every form of reasoning can be

[15] See [Nor44] for a full proof as to why this is the only possibility.

misapplied and can lead to incorrect conclusions.[16] So, instead of tossing it aside, we have to study it to find when it is valid and when it is not. Furthermore, because every form of reasoning can be misleading, it is beneficial to use more than one method of reasoning in order to guarantee correspondence between them and to use one method to justify the other. Northrop states that "[formal] deduction in geometry is to some extent a combination of seeing and reasoning, for in the proof of any theorem the logical processes of the mind are guided by and checked against what the eye sees in the figure." In other words, diagrams and logic should work together. Therefore, in order to ensure that a diagram or the corresponding algebra is valid, we must verify that multiple forms of reasoning agree.

4 Generalization of Category 1 Proofs

In Section 2.1 on page 12, the definition for diagrammatic proofs of Category 1, as developed by Jamnik, was given. This section concentrates on proofs of that category.

As humans, we sometimes assume the generalization of a proof, which can lead to problems such as in the previous section. Balbiani and Fariñas del Cerro state in [BdC99] that "...logicians have rarely considered that a formal proof could be based on diagrammatic expressions. The truth of the matter is, fallacious arguments come from an improper use of diagrams which are mostly considered as the instances rather than the general form of a concept." For this reason, an analysis of problematic proofs was given in the previous section. We continue addressing this comment by formalizing and justifying the generalization of Category 1 proofs.

As well as the hypotheses given in Section 2, Category 1 proofs require generalization to show that the proof is not specific to the diagram shown. "The diagram is in fact acting as a kind of representative sampler of an infinite class of diagrams."[HL98] Generalization is the last step of any Category 1 proof, so the justification of such a proof must include the generalization step.

A common example of a proof of Category 1, the diagrammatic proof of the Pythagorean Theorem, was analyzed in Section 2.3.1. Since this proof will be used as an example throughout this section, it is reprinted in Figure 26.

Most Category 1 geometric diagrammatic proofs contain a diagram A that is transformed, via diagrammatic operations T, to the diagram B. The transformation, T, consists of a single run-through of the set of actions described in Section 2.2. In this section, we look at proving the generalization of the proof, i.e. showing that the diagram A and the transformation T are

[16] One only needs to observe an introductory course on any form of reasoning to see students misapplying it and coming up with tremendous results.

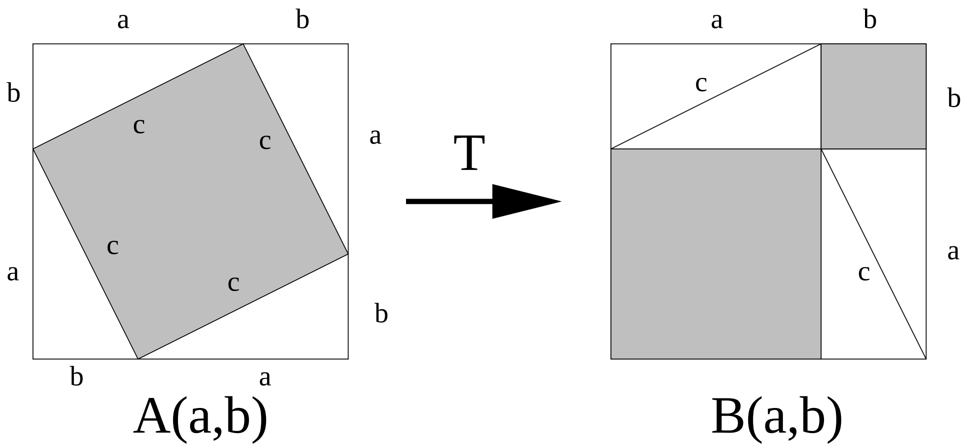

Figure 26: Pythagorean Theorem: $a^2 + b^2 = c^2$

general and can be applied to similar diagrams (they are not special cases), in order to justify that the proof is sound. There are two ways of doing this, as outlined by Foo, et al. in [FNP99]. One relies on the notion of continuity, while the other corresponds to a powerful meta-theorem in logic: the Theorem on Constants. We want to show that these two solutions to the problem of generalization are equivalent.

First, some definitions that are needed during this section will be given. Then, a description of both solutions to the problem of generalization is presented. Finally, we will prove that they are equivalent. As the definitions and descriptions of the arguments are given, we will use the diagrammatic proof of the Pythagorean Theorem in Figure 26 as an example. The references to the specific instance of the proof are in **bold print**. See Section 4.5 for further examples.

4.1 Definitions

For each diagrammatic proof $A \xrightarrow{T} B$ (that is, for each diagram A which can be transformed using T to a diagram B), there are parts of the diagram which can be changed and things which cannot be changed (when we say "can" and "cannot" be changed, we actually mean aspects of the diagram which we want changed and aspects of the diagram that we do not want changed). For example, if we are proving something about the area of a square, then we can fluctuate the length of the sides but do not want to alter the fact that they are of the same length, or else we will loose the aspect of squareness. So, define $A(\overline{x})$ to be the diagram A, where $\overline{x}$ is the n-tuple consisting of all of the alterable aspects of A.

For example, in the proof of the Pythagorean Theorem in Figure 26, define the left diagram to be $A(a, b)$ and, similarly, the

diagram on the right to be $B(a,b)$ because the values of both a and b can be altered in this diagram.

Furthermore, let us define two perturbation functions, P and R, which are required for the continuity argument. The perturbation function P alters a diagram in such a way to show that no absolute magnitude is used in the diagram, while the perturbation function R alters the diagram in such a way to show that the relative ratio of certain parts of the diagram is not material. Each function can only alter the x_i's within the n-tuple $\overline{x}$. However, we must note that, when we use $P(A(\overline{x}))$ and $P(B(\overline{x}))$ (or $R(A(\overline{x}))$ and $R(B(\overline{x}))$) in the same equation, then we assume that P (or R) alters the same aspects of both graphs, i.e. the same x_i in $\overline{x}$.

For the Pythagorean Theorem, in order to show that no absolute magnitude is used in the diagram, P must alter a and b by $\pm\epsilon$. In order to show that the relative ratio of certain parts of the diagram is not material, R must show a scaled counterpart of the same diagrams, i.e. $A(ra, rb)$ and $B(ra, rb)$.

4.2 Continuity Argument

The continuity argument for the generalization of a proof says that the transformation T is continuous if all $P(B(\overline{x}))$ and $R(B(\overline{x}))$ converge to $B(\overline{x})$ whenever all $P(A(\overline{x}))$and $R(A(\overline{x}))$ converge to $A(\overline{x})$. $P(A(\overline{x}))$ converges to $A(\overline{x})$ means that, when x_i of $\overline{x}$ in $A(\overline{x})$ is altered by some ϵ to get $P(A(\overline{x}))$, then $P(A(\overline{x}))$ must go to $A(\overline{x})$ as ϵ goes to zero. Similarly, $R(A(\overline{x}))$ converges to $A(\overline{x})$ means that, when x_j of $\overline{x}$ in $A(\overline{x})$ is altered by some r to get $R(A(\overline{x}))$, then $R(A(\overline{x}))$ must go to $A(\overline{x})$ as r goes to one.

Therefore, since T is continuous, it is independent of which $P(A(\overline{x}))$ or $R(A(\overline{x}))$ it is applied to. In other words, if the diagram $A(\overline{x})$ can be transformed using T to a diagram $B(\overline{x})$, then any perturbation $P(A(\overline{x}))$ or $R(A(\overline{x}))$ can be transformed using T to the corresponding perturbed version of $B(\overline{x})$ which alters the same x_i of $\overline{x}$. So, if T is continuous, then we know that

$$\begin{aligned} & A(\overline{x}) \xrightarrow{T} B(\overline{x}) \quad \Rightarrow \quad P(A(\overline{x})) \xrightarrow{T} P(B(\overline{x})) \\ \text{AND} \quad & A(\overline{x}) \xrightarrow{T} B(\overline{x}) \quad \Rightarrow \quad R(A(\overline{x})) \xrightarrow{T} R(B(\overline{x})). \end{aligned}$$

In Figure 27 is an example of the first half of the continuity argument, as applied to the Pythagorean Theorem. If the original diagram A is altered by various degrees of ϵ, we get the A_i. As ϵ goes to zero, the A_i go to A. As the A_i go to A, the corresponding B_i also go to B. The second half of the argument, as applied to this theorem, is in Figure 28. Similarly, as the original diagram A is altered by various degrees of r, then we get the A_j in the figure. As r goes to one, the A_j go to A and the corresponding B_j go to B.

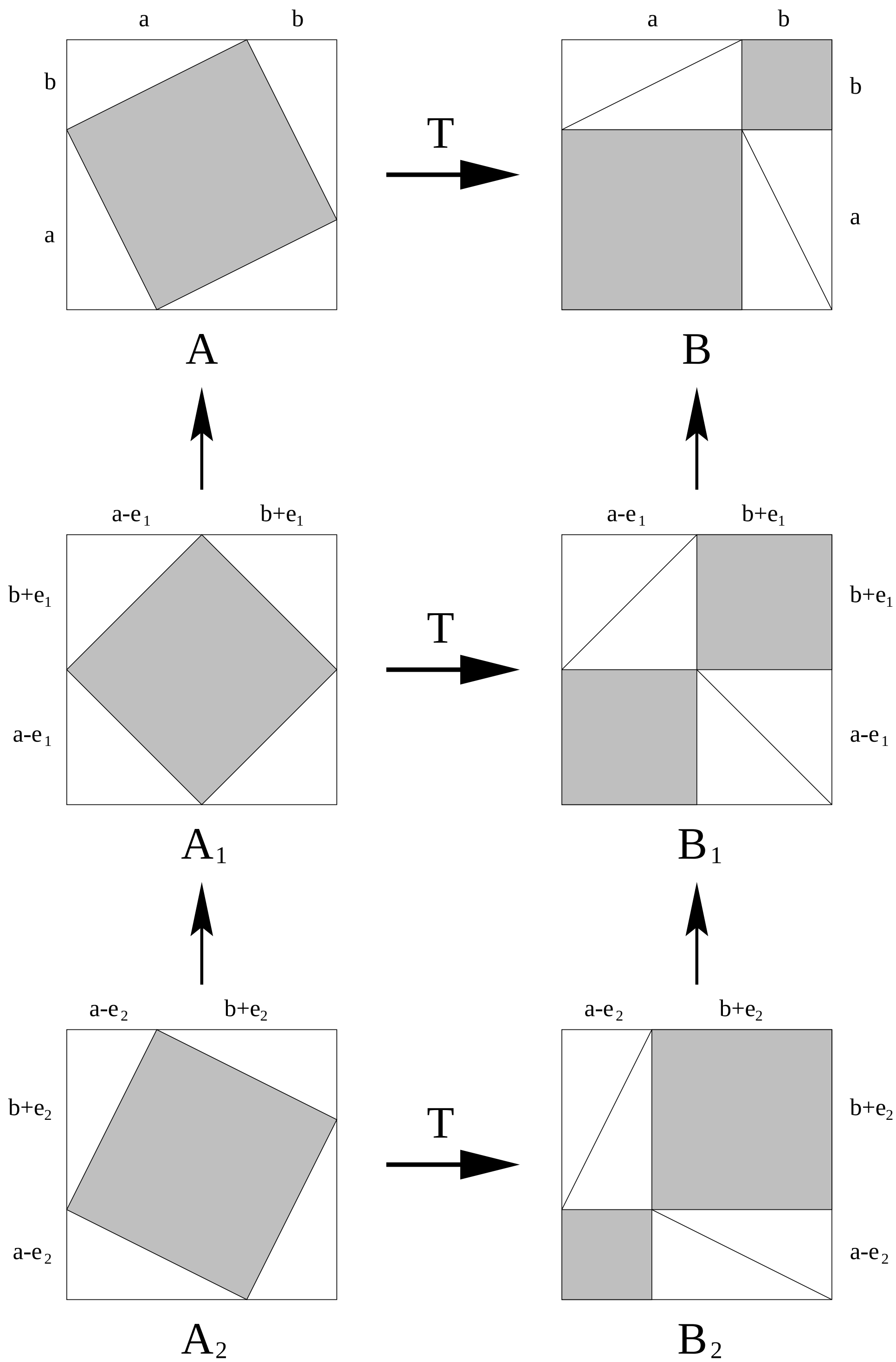

Figure 27: Continuity Argument: as ϵ goes to zero, $B(a-\epsilon, b+\epsilon)$ goes to $B(a,b)$ when $A(a-\epsilon, b+\epsilon)$ goes to $A(a,b)$.

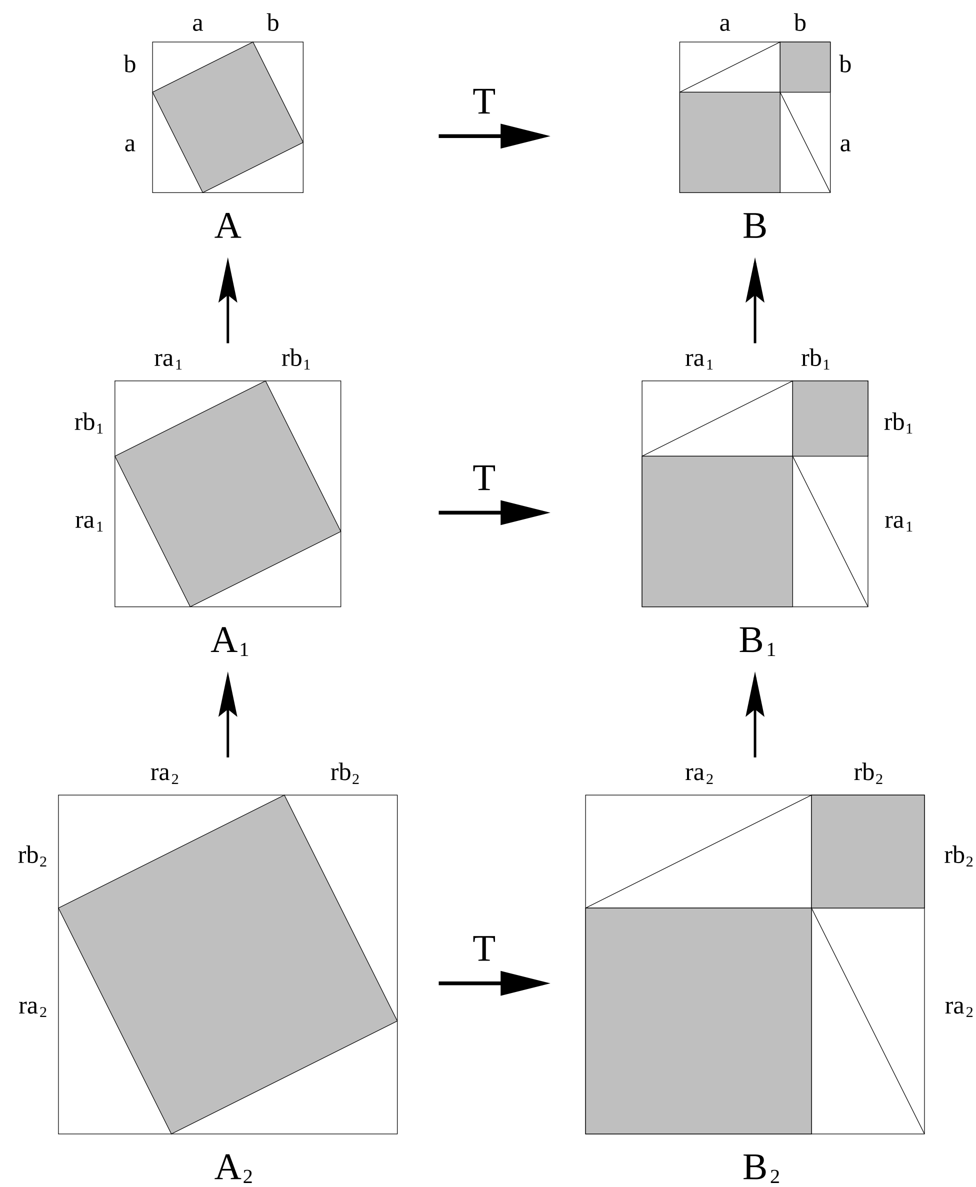

Figure 28: Continuity Argument: as r goes to one, $B(ra, rb)$ goes to $B(a, b)$ when $A(ra, rb)$ goes to $A(a, b)$.

4.3 Theorem on Constants Argument

Although part of the natural development of logic, the Theorem on Constants (TOC) has been traditionally attributed to Shoenfield, and its proof is on page 33 of his book [Sho67]. One version, taken from [FNP99], can be stated as follows. Suppose Γ is a set of formulas and $\alpha(x)$ is a formula with a free variable x, and the constant d does not appear in any formula in Γ. Further, suppose $\Gamma \vdash \alpha[x/d]$, where the notation $\alpha[x/d]$ signifies the substitution of d for x in $\alpha(x)$. Then we may infer $\Gamma \vdash \forall x.\alpha(x)$.

In our case, the diagrammatic version of the TOC for the generalization of a diagram is that, if Γ is a set of formulas that support the transformation of a diagram $A[\overline{x}/\overline{d}]$, via T, to $B[\overline{x}/\overline{d}]$, and if no d_i in $\overline{d}$ is specifically mentioned in Γ,[17] then we can say that Γ supports the transformation of A for any set $\overline{x}$ of alterable aspects of the diagram. In other words,

$$\Gamma \vdash (A[\overline{x}/\overline{d}] \stackrel{T}{\to} B[\overline{x}/\overline{d}]) \Rightarrow \Gamma \vdash \forall \overline{x}.(A(\overline{x}) \stackrel{T}{\to} B(\overline{x})).$$

We can see that Γ is the set of hypotheses described in Section 2.2 on page 12.

What does it mean that Γ *supports* a transformation T? There is a set of formulas Γ, which become the implicit hypotheses of the proof. Each step in the transformation T is based on one or more of these general hypotheses. For example, the transformation step done by sliding a triangle from one place to another is supported by the fact that shapes and areas are invariant (rigid) under rotations and translations. So, if that fact is in Γ, then Γ supports sliding. Another example is: the transformation step of breaking a rectangle of dimension $l \times w$ into two identical right triangles with legs of length l and w is supported by the following four facts and formulas:

1. areas stay constant under divisions

2. the area of a rectangle is length $\times$ width

3. the area of a right triangle is $1/2 \times$ (*product of legs*)

4. $\forall l, w . l \times w = 2 \times (1/2 \times (l \times w))$

Again, if all four of these facts are contained in Γ (as well as algebraic definitions for the product of two numbers), then the transformation step of breaking a rectangle of dimension $l \times w$ into two identical right triangles with legs of length l and w is supported by Γ. However, it is very important to note that l and w, as well as the length and width of the rectangle and the

[17] There is one exception to this rule: a specific x of $A(\overline{x})$ may be mentioned if and only if all of the cases are also explicitly mentioned. For example, if we know that x_j is an integer, we can have one formula if x_j is even and one formula if it is odd. As long as all cases are covered, it is permitted.

length of the legs of a right triangle, are universally quantified in Γ. Except for the last item, universal quantification is implied in the above facts.

If each step of a transformation is supported by Γ then the entire transformation is supported by Γ. In other words, if we start with a diagram A and apply T to get B such that the non-alterable aspects and relationships in B are derivable from those in A via hypotheses Γ, then Γ supports the transformation T.

Returning to the Pythagorean Theorem example, we can see in Figure 6 on page 15 that there are three actions in the transformation from A to B:

1. **slide**
2. **slide**
3. **slide**

As stated in Section 2.3.1, *slide* is supported by the hypothesis stating the invariance of shapes and areas under rotations and translations because slide is basically just a translation. Other hypotheses are also used to support the actions above and to support our conclusion. These include triangle congruence properties, the area of a square is the length of the side squared, the sum of the angles of a triangle is $180°$, etc.

Written formally, we are given a Γ that contains the facts above. We are also given a diagram $A(a', b')$, where a' and b' are the constants assigned to variables a and b, as seen on the left-hand side of Figure 26 (in other words, a and b in the diagram are general variables, for which we have instantiated the constants a' and b', respectively, in order to get the figure shown). Then, we apply T, which consists of the three actions above to get $B(a', b')$. Since neither a nor b is mentioned in Γ, the relationships in B (such as the fact that the shaded area is equal to the sum $a^2 + b^2$ and must be the same as the shaded area of A) are derivable from those in A via the general hypotheses in Γ. Therefore, Γ supports $B(a', b')$. Then, by the TOC, for any values of a and b, Γ supports the transformation of $A(a, b)$. In other words, the diagram in Figure 26 is generalizable using the TOC for any values a and b.

4.4 Proof

In order to prove that the continuity argument and the TOC argument are the same, two things must be shown:

1. if a proof can be generalized using the continuity argument, it can be generalized using TOC argument, and

2. if a proof can be generalized using the TOC argument, it can be generalized using the continuity argument.

For the sake of simplicity, throughout this proof, as the original diagram $A(\overline{x})$ is altered by various degrees of ϵ, call the perturbations A_i in such a way that, as ϵ goes to zero, A_i goes to A. Similarly, label the perturbations of B such that $A_i \xrightarrow{T} B_i$. Furthermore, as the original diagram is scaled by various degrees of r, call the perturbations A_j in such a way that, as r goes to one, A_j goes to A, and label the scaled versions of B such that $A_j \xrightarrow{T} B_j$.

Part 1 The continuity argument basically argues for the generalization of a proof with respect to choosing dimensions and scaling. Because the diagrammatic proof $A \xrightarrow{T} B$ can be generalized using the continuity argument, we know that the A_i's and A_j's both converge to A, the B_i's and B_j's both converge to B, and that the A_i's and A_j's together span the space of possible perturbations of A. In other words, there is a continuum of possible values for every x_i in $\overline{x}$ so that the theorem still holds. **For example, in the Pythagorean Theorem example, a and b can be any value from zero to infinity. Without loss of generality, we can say that a and b can be any value within the continuous interval $[0, 1]$.**[18] This means that, for any choice $\overline{d}$ within that continuum, we know, by the continuity argument, that $A(\overline{d}) \xrightarrow{T} B(\overline{d})$. Furthermore, assume that we have a set of hypotheses, Γ, that support this transformation.

Now, choose another value within the interval of possible values for $\overline{x}$ and call it $\overline{d'}$. Because $\overline{d'}$ comes from the interval formed by the continuity argument, it is a perturbed version of $\overline{d}$. Just as in Figures 27 and 28, as $\overline{d'}$ is altered by small increments of ϵ and/or small increments of r, it will eventually converge to $\overline{d}$. Therefore, since $\Gamma \vdash (A(\overline{d}) \xrightarrow{T} B(\overline{d}))$, then we also know that $A(\overline{d'}) \xrightarrow{T} B(\overline{d'})$ and that Γ supports this transformation. Since $\overline{d'}$ is a general value within the possible range, Γ supports the transformation $A(\overline{x}) \xrightarrow{T} B(\overline{x})$ for all $\overline{x}$. This is, indeed, the diagrammatic version of the TOC.

Part 2 Since $A(\overline{x})$ and T can be generalized using the TOC, we know that Γ supports the transformation of $A(\overline{x})$ whenever it supports the transformation of $A(\overline{d})$. So, Γ must support the transformation of all $A(\overline{d_n})$ whenever it supports the transformation of $A(\overline{d})$, where d_n is such that, as n goes to zero, d_n goes to d.[19] Each $A(\overline{d_i})$ will be called A_i, each $B(\overline{d_i})$ will be B_i, each $A(\overline{d})$ will be A, and each $B(\overline{d})$ will be B.

[18] Another example is: if we are trying to find possible values for x in the equation $x^2 \leq x$, then only values of x within the interval $[0, 1]$ can be chosen in order for the equation to hold true.

[19] The constants $\overline{d_n}$ may be either $\overline{d} + n \cdot \epsilon$ or $(n + 1) \cdot \overline{d}$ in order to accomodate the perturbations caused by epsilon or scaling, respectively.

We can now see that, whenever Γ supports the transformation $A \xrightarrow{T} B$, then it also supports all $A_i \xrightarrow{T} B_i$. Since each A_i and B_i is just a permutation of A and B, respectively, (because only the constants within $\overline{x}$ are being altered), and since A_i and B_i will converge to A and B, respectively, as d_i goes to d, we have precisely the continuity argument.

4.5 Further Examples

Here are a couple more examples of Diagrammatic Proofs.

4.5.1 Completing the Square

Look again at Figure 8 on page 17. The leftmost diagram[20] can be called $A(x,a)$ because the values of x and a can be altered in this diagram. Similarly, the rightmost diagram is called $B(x,a)$. We want to transform $A(x,a)$ to $B(x,a)$ using T. In this case, T consists of

1. split
2. arrange

Split is supported by the facts that manipulations of figures follow algebraic rules and that split does not alter the area involved, which is the invariant for this proof. *Arrange* is supported by the invariance of shapes and areas under rotations and translations. Other hypotheses required include: the area of a square is the length of the side squared and the area of a rectangle is *length* $\times$ *width*. Neither x nor a is mentioned in any of these hypotheses.

Consequently, this proof can be generalized using the Theorem on Constants. Since we have just proven that the TOC and continuity arguments are equivalent, this proof should also be generalizable by the continuity argument. This means that, as ϵ goes to zero, B_i will go to B because A_i goes to A and, as r goes to one, B_j will go to B because A_j goes to A. Indeed, as x and a are altered by various degrees of ϵ and r, we can "see" that the same T is applicable and that the proof still holds.[21] Therefore, it holds for all values of the universally quantified variables, x and a.

4.5.2 Sum of Squares

In Figure 9 on page 19, the first diagram can be called $A(a,b,c,d)$ because we want to be able to change the values of all four variables. Similarly, the last diagram, containing both squares, is called $B(a,b,c,d)$. The other four diagrams are intermediary steps in the transformation T. In this example, T has five steps. They are:

[20] The leftmost diagram consists of both the square AND the rectangle.

[21] For the sake of conciseness, a demonstration of this will not be given, but we ask the reader to "see" it him- or herself.

1. make square
2. arrange
3. split
4. change dimension
5. arrange

Again, each step is supported by at least one hypothesis. *Make square*, *change dimension*, and *split* are each supported by the facts that manipulations of figures follow algebraic rules and that split does not alter the area involved, which is the invariant for this proof. Identical to the proof in Section 4.5.1, *arrange* is supported by the invariance of shapes and areas under rotations and translations. Other hypotheses include the areas of squares and rectangles, as mentioned above. Again, neither a, b, c, nor d is mentioned in any of these hypotheses.

Similar to the previous example, this proof can be generalized using the TOC and, consequently, also using the continuity argument. Again, the reader should be able to visualize that, as ϵ goes to zero and r goes to one, A_i and A_j go to A and B_i and B_j go to B. Furthermore, since all of the hypotheses are general, the steps of T are applicable for all values of a, b, c, and d. Therefore, this is a valid proof of the above theorem.

4.6 Summary

For Category 1 proofs, "[generalization] is required in the end to show that the theorem holds for all values of universally quantified variables." [Jam99] There are two solutions to the problem of generalization of Category 1 proofs, as presented in [FNP99]. It was shown that they are equivalent. Since generalization is the last step of a Category 1 proof, the formalization and justification of this step is necessary for the logical verification of these proofs. This, as well as the hypotheses in Section 2, prove that diagrammatic proofs of Category 1 are sound in their reasoning and should be regarded with the same respect as the corresponding algebraic proofs.

5 Linking Traditional Computational Theories with Diagrammatic Proofs

Category 3 proofs require abstractions. "Conducting proofs and using abstractions in diagrams is problematic ... since it is difficult to keep track of these abstractions while manipulating the diagram during the proof procedure." [Jam99] For this reason, Jamnik excludes all theorems of Category 3 from the problem domain of her semi-automated theorem prover, DIAMOND. The aim of this research is to justify the soundness of diagrammatic proofs of mathematical theorems using traditional, symbolic reasoning.

In this section, we show the formalization of a Category 3 proof and a justification of each step of it. We do this by linking the cognitive intuition of diagrammatic reasoning to traditional theories of computation, like theories involving *fixed points*, *invariants*, and *continuations*. Most important is the link between continuations and the abstractions used in Category 3 proofs because this link helps us deal with the problem of ellipsis. This ability to link diagrammatic reasoning with such a well-established field as computational logic greatly progresses its study and understanding.

This section consists of an example of this link. In essence, we justify every diagrammatic step of a proof and develop a meaning for and a way to represent the ellipsis in the diagram.

5.1 Review of Example Diagrammatic Proof

Let us look again at the diagrammatic proof that the sum of the geometric series $1/4 + 1/16 + 1/64 + \cdots$ equals $1/3$, which is discussed in [Nel93], [FNP99], [Jam99], as well as in Section 2.5.2 of this paper. For convenience, Figure 14 is reproduced as Figure 29 on the following page. Diagram A in this figure is the full diagrammatic proof of this theorem.

When constructing this proof, the key idea is to start with an L-shaped region, let us call it L_1, where the center section (one-third of the area of L_1) is shaded, as in Figure 29D_1. This corresponds to the number $1/4$.

Proceeding, scale L_1 to get L_2, and glue L_2 onto L_1, as in diagram D_2. The shaded area of L_2 corresponds to the number $1/16$ and the shaded area of diagram D_2 corresponds to the sum $1/4 + 1/16$. Similarly, continue the next step of the construction by scaling L_2 to get L_3, which corresponds to the number $1/64$. Likewise, glue L_3 to the diagram of L_1 and L_2, to get Figure 29D_3.

With each addition of an L-shaped region, L_i, a new summand, $1/4^i$, is added to the total. For the sake of simplicity, call the summand corresponding to the shaded area of each L_i, S_i. In other words, $S_i = 1/4^i$. Using this, the shaded area of diagram D_1 corresponds to S_1, the shaded area of diagram D_2 corresponds to the sum $S_1 + S_2$, and the shaded area of diagram D_3 corresponds to the sum $S_1 + S_2 + S_3$. Therefore, we have established a one-to-one relationship between the shaded area of each L_i and each S_i. We must also make the crucial observation that, at each step of the construction, the shaded area is exactly one-third of the total area, i.e. the sum of the total area of the first n L_i's is equal to three times the sum of the first n S_i's.

The construction is continued by scaling and translating each L_i in the diagram, whose shaded areas correspond to summing the S_i's, as i goes to

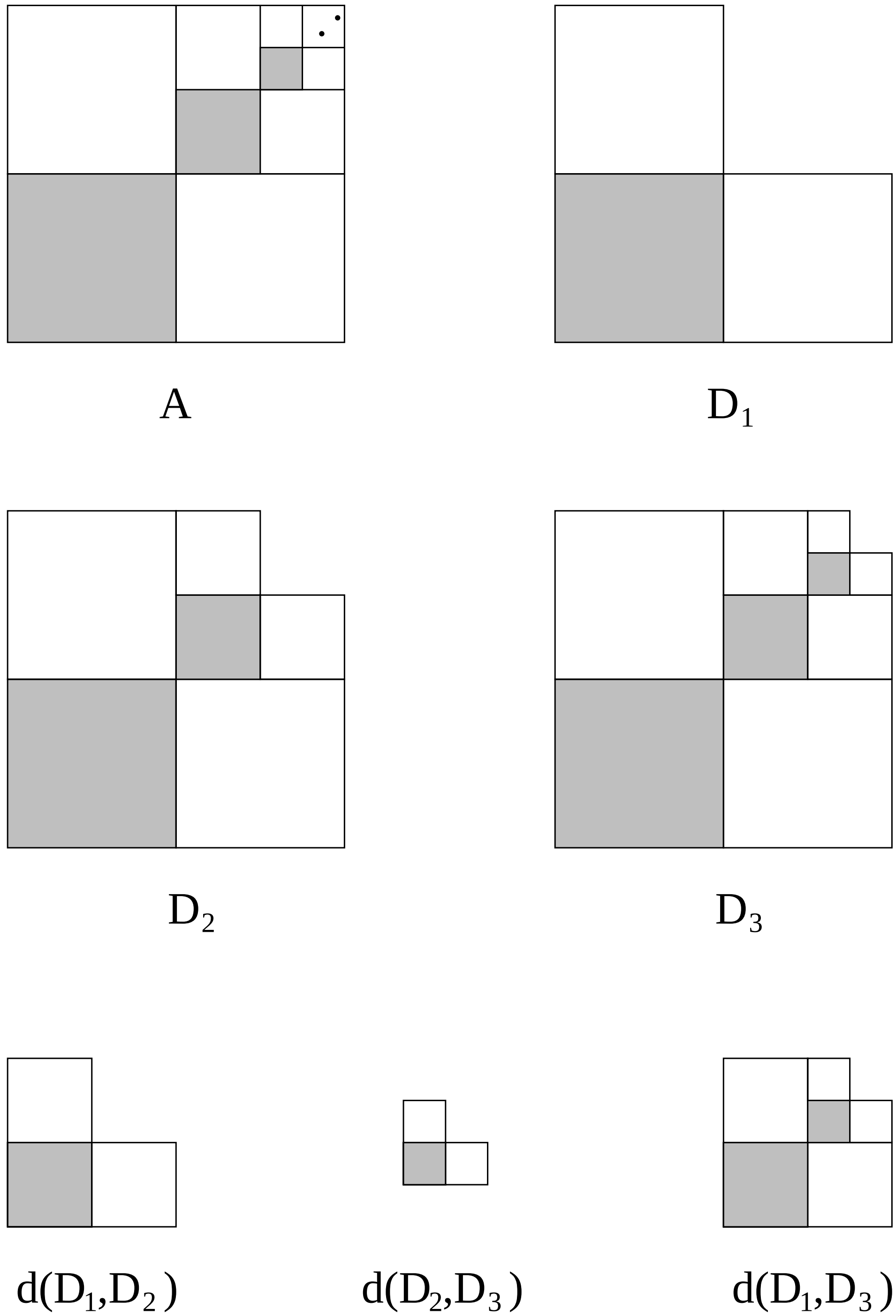

Figure 29: Construction steps in the proof of $\sum_{i=1}^{\infty} 1/4^i = 1/3$ and difference metric, d.

infinity. The L_i's will sum to the total square, which has area of 1.[22] Thus, the shaded area goes to 1/3 because it is still one-third of the total area after each construction step.

It is interesting to note that the algebraic ("traditional") proof involves induction on the S_i's, while the diagrammatic proof involves induction on the L_i's. It is because of the obvious one-to-one relationship between them that reasoning with diagrams as opposed to traditional algebraic or logical methods can be shown to hold. We also use this relationship to help justify the soundness of the diagrammatic reasoning involved in the proof.

As discussed in [FNP99], the most important thing to note is that the completed square is the limit of the sequence of constructions, each of which is a *monotonic* and *Markovian* addition to its predecessor; monotonic because each construction step adds new information that is distinct from the old information, which is not changed, and Markovian because only the most recent piece of information is used to construct the next one.

5.2 Formalizing Geometric Intuition

The above is very intuitive and is easy for a human to "see" the proof. However, can we prove that it is, in fact, a legitimate proof? Is this simple diagrammatic proof rigorous enough? In order to prove that it is a valid proof, we will formalize the geometric intuition that was used above and justify each step of the proof.

First, we must notice that each diagram D_i, defined by

$$D_i = \bigcup_{j=1}^{i} L_j,$$

represents a member of a sequence. The layout of the proof goes as follows: we start by proving that this sequence converges to a fixed point, then show that an invariant that holds for each diagram in the sequence will also hold at its fixed point. Finally, we will show the meaning of the ellipsis $(\dots)$ to finish the formalization of the proof.

All of the definitions and theorems used below are taken from [Cai94], except where noted.

5.2.1 Transformation Functions

In this section, we define the transformation, F, that glues a scaled and translated L-shaped region onto the previous diagram at the appropriate place in order to form the next diagram in our sequence.

[22] The fact that the L-shaped regions sum to 1 is proven diagrammatically in [Nel93], [FNP99], [Jam99], and in Section 2.5.1 of this paper.

Formalization Continued Place the initial diagram of the sequence, D_1, which consists of only L_1, in the first quadrant of the co-ordinate axis such that its bottom-left corner is at the origin and its X- and Y- edges have a length of 1. The second L-shaped region, L_2, is a scaled version of L_1. Its bottom-left corner is at the point $(1/2, 1/2)$, and its X- and Y- edges each have a length of $1/2$. Similarly, the third region, L_3, is a scaled version of L_2, with its bottom-left corner at $(1/4, 1/4)$, and with X- and Y- edges of length $1/4$. The following transformation, T, taken from [FNP99, notation slightly altered], transforms each L_k to L_{k+1}:

$$L_{k+1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} (L_k - d_k) + \begin{bmatrix} 1/2^k \\ 1/2^k \end{bmatrix} + d_k, \tag{11}$$

where d_k is the vector to the bottom-left corner of L_k. Basically, this transformation moves L_k to the origin, scales its X- and Y- edges by $1/2$, then glues it to its "proper" place on the diagram.

Therefore, diagram D_2 in Figure 29 can be denoted as $L_1 \cup L_2$, diagram D_3 is $L_1 \cup L_2 \cup L_3$, and a general diagram D_n is $L_1 \cup L_2 \cup \cdots \cup L_n$. Using this ,we can write a formal transformation expressing the progression from one diagram to the next by a transformation F as follows (taken from [FNP99, notation slightly altered]):

$$F(L_1 \cup \cdots \cup L_k) = (L_1 \cup \cdots \cup L_k) \cup T(L_k). \tag{12}$$

For example, using this, the transformation of D_2 to D_3 is:

$$F(D_2) = F(L_1 \cup L_2) = L_1 \cup L_2 \cup T(L_2) = L_1 \cup L_2 \cup L_3 = D_3.$$

5.2.2 Sequence Converges to a Fixed Point

In this section, we will show that the sequence of diagrams, D_n, converges to a fixed point. A *fixed point*, as defined in [Sto77], of an expression F is any X such that $F(X) = X$. We will show that D_n converges to a fixed point by defining a *difference function* and proving that it is a *metric*. Then, we will show that the transformation F is a *contraction map*, and, hence, the sequence D_n is a *Cauchy sequence*. This leads us to the *Banach Fixed Point (BFP) Theorem* which, finally, proves that the sequence *converges* to a unique fixed point. We finish by proving that this fixed point is, indeed, the unit square, which has an area of 1.

Formalization Continued The difference between two diagrams is just the L-shaped piece(s) that was (were) added from one to get the other. For example, the difference between diagrams D_1 and D_2 is precisely the region L_2, the difference between diagrams D_2 and D_3 is L_3, and the difference between diagrams D_1 and D_3 is the region $L_2 \cup L_3$, as in Figure 29. We can define the a *difference function*, d, as follows:

$$d(D_m, D_n) = \mu \left(\bigcup_{i=min\{m,n\}+1}^{max\{m,n\}} L_i \right),$$

were μ is a Riemann *measure* in two-dimensional Euclidean space, $\mathbf{E}^2$.[23] In other words, it is the area of the one or more L-shaped regions that make up the difference between two diagrams.[24] Let us now see if this is a metric over the set containing the collection of possible diagrams; call it $\mathcal{U}$.

According to [Cai94], d is a *metric* if it satisfies the following five conditions:[25]

1. $d(X, Y) \geq 0, \forall X, Y \in \mathcal{U}$

   In other words, for every pair of diagrams, the difference between them cannot be an empty diagram, i.e. must have an area greater than or equal to zero. Since the difference between two diagrams is the area of the union of one or more L-shaped regions, and since an L-shaped region will always have an area larger than zero (it can be very small, though), this condition is satisfied. Formally, because we are only considering positive areas, the union of such areas can never have a negative value.

2. $d(X, X) = 0, \forall X \in \mathcal{U}$

   This condition is obvious because there is no difference in area between a diagram and itself. Formally, $d(D_k, D_k) = \mu(\cup_{i=k+1}^{k} L_i)$. Since $\cup_{i=k+1}^{k} L_i$ is empty, its area is zero. Hence, this condition is satisfied.

3. $d(X, Y) = d(Y, X), \forall X, Y \in \mathcal{U}$

   Since areas cannot be negative, d is symmetric. In other words, the difference diagrams in Figure 29 could also be written as $d(D_2, D_1)$, $d(D_3, D_2)$, and $d(D_3, D_1)$, respectively. Formally, because of the *max* and *min* in the definition of d, the relative values of m and n do not matter.

4. $d(X, Z) \leq d(X, Y) + d(Y, Z), \forall X, Y, Z \in \mathcal{U}$

   By the very nature of the function d, we, in fact, know that $d(X, Z)$ equals $d(X, Y) + d(Y, Z)$. For example, look again at $d(D_1, D_2)$, $d(D_2, D_3)$, and $d(D_1, D_3)$ in Figure 29. Formally, take any diagrams D_X, D_Y, and

[23] See [Sag74] or [Wei] for more information on Riemann measures and integrals.

[24] Because the L_i's are disjoint, the right-hand side of the above equation can equivalently be written as $\sum_{i=min\{m,n\}+1}^{max\{m,n\}} \mu(L_i)$.

[25] Actually, Cain calls a function that satisfies the first four conditions a *pseudometric*, and a *metric* is defined as a pseudometric which also satisfies condition 5.

D_Z. Because of condition 3, we can assume that $X \leq Y \leq Z$ without loss of generality. This condition now says that:

$$\begin{aligned} d(D_X, D_Z) &\leq d(D_X, D_Y) + d(D_Y, D_Z) \\ \mu\left(\bigcup_{i=X+1}^{Z} L_i\right) &\leq \mu\left(\bigcup_{i=X+1}^{Y} L_i\right) + \mu\left(\bigcup_{i=Y+1}^{Z} L_i\right) \\ \mu\left(\bigcup_{i=X+1}^{Z} L_i\right) &\leq \mu\left(\bigcup_{i=X+1}^{Y} L_i \cup \bigcup_{i=Y+1}^{Z} L_i\right) \end{aligned}$$

Because the L_i's are disjoint, we can rewrite the right-hand side of the last line as

$$\mu\left(\bigcup_{i=X+1}^{Z} L_i\right),$$

which is the same as the left-hand side. Thus, the two sides are equivalent, and this condition holds.

5. $d(X, Y) = 0$ only if $X = Y$

   It is trivial to see that the only way for d to be zero is if $m = n$. Formally, we will prove it by contradiction. Assume that there exist two diagrams, D_X and D_Y, such that $D_X \neq D_Y$, but $d(D_X, D_Y) = 0$. Without loss of generality, because condition 3 is satisfied, assume that $X \leq Y$. By the definition of d, we have $d(D_X, D_Y) = \mu(\bigcup_{X+1}^{Y} L_i) = 0$. The only way that this holds is if the union from $X + 1$ to Y is empty. Therefore, $X + 1$ must be greater than Y. However, since we assumed that $X \leq Y$, the only solution is that $X = Y$. This contradicts our assumption that $D_X \neq D_Y$ by the unique name assumption. Therefore, such diagrams, D_X and D_Y, cannot exist, and the condition holds.

Therefore, d is a metric, and $(\mathcal{U}, d)$ is a *metric space.*

Continuing, the transformation $F : \mathcal{U} \to \mathcal{U}$ over the metric space $(\mathcal{U}, d)$ is a *contraction map* if there is a real number $k < 1$ such that

$$d(F(X), F(Y)) \leq k \cdot d(X, Y), \forall X, Y \in \mathcal{U}.$$

Since each L_{k+1} is half of the size of L_k, their areas must differ by a factor of 1/4. Because the difference between two diagrams is the area of one or more L-shaped regions, we know that this equation holds for any $k \geq 1/4$. Let us look at an example, using diagrams D_1 and D_2:

$$\begin{aligned} d(F(D_1), F(D_2)) &\leq k \cdot d(D_1, D_2) \\ d(D_2, D_3) &\leq k \cdot \mu(L_2) \\ \mu(L_3) &\leq k \cdot \mu(L_2) \end{aligned}$$

Since the area of L_3 is $3/64$ and the area of L_2 is $3/16$, we can see that, for $k \geq 1/4$, the last equation holds. Therefore, F is a contraction map.

We now know that the sequence D_n, defined by $D_n = F(D_{n-1})$, where $D_1 = L_1$, is a *Cauchy sequence* because of Proposition 8.24 from [Cai94] below:

> Let $f : X \to X$ be a contraction map from a pseudometric space (X, d) into itself. Let $x_0 \in X$, and for each $n \in \mathcal{Z}_+$, define $x_n = f(x_{n-1})$. Then the sequence (x_n) is a Cauchy sequence.

Therefore, by the definition of Cauchy sequence, for each $\delta > 0$, there is an integer N such that $d(D_i, D_j) < \delta$ whenever $i > N$ and $j > N$. In other words, the difference between the successive D_n's is approaching zero as n goes to infinity.

Furthermore, by Proposition 8.5 in [Cai94], we know that, since D_n is a Cauchy sequence over the metric space $(\mathcal{U}, d)$, the set $\{D_n : n \in \mathcal{Z}_+\}$ is bounded. Since F is monotonic and D_n is bound, we also know that D_n converges because every bounded monotonic sequence *converges.*[Wei]

Finally, because of the Cauchy Criterion for Convergence (Theorem 62.1) in [Sag74], we know that $(\mathcal{U}, d)$ is a complete metric space. The *Cauchy Criterion for Convergence* states that a sequence in $\mathbf{E}^n$ converges to a limit in $\mathbf{E}^n$ if and only if it is a Cauchy sequence, and a *complete metric space* is one where every Cauchy sequence in the space converges to an element of that space.[Sag74] Therefore, since $(\mathcal{U}, d)$ is based in $\mathbf{E}^2$, any Cauchy sequence in that metric space will converge to a limit in that space. Thus, $(\mathcal{U}, d)$ is complete.

Using this, we would like to apply the *BFP Theorem* in order to prove that D_n converges to a *unique* fixed point. The BFP states that if $(\mathcal{U}, d)$ is a complete metric space and $F : \mathcal{U} \to \mathcal{U}$ is a contraction map, then there is exactly one point $Z \in \mathcal{U}$ such that $F(Z) = Z$, which is a fixed point. Moreover, if X_0 is any point in $\mathcal{U}$, and $X_n = F(X_{n-1})$ for all $n \in \mathcal{Z}_+$, then the sequence (X_n) converges to Z. Simply stated, "[if] T is a contraction defined on a complete metric space X, then T has a unique fixed point."[Sim63]

However, due to the fact that $\mathcal{U}$ contains diagrams that are not in the sequence D_n, we cannot apply this theorem directly. Instead, we can apply a weaker theorem which follows directly from the standard proof of the BFP theorem. For the sake of brevity, that proof will not be included here. In the weaker version of the BFP, F is a contraction map on a subspace $(\mathcal{U}', d)$, where only members of the sequence D_n are in $\mathcal{U}'$. In other words, the new subspace is generated by applying F to subsequent diagrams of a sequence, starting from an initial diagram D_0. This sequence will still converge to a unique fixed point.

Now we know that there is a unique *fixed point* Z to which the sequence D_n converges, but how do we know what that fixed point is? We can obviously see in Figure 29 that the L-shaped pieces converge to the square, so the sequence D_n should converge to the fixed point 1. But, how do we show that?

Pick any point within the square. Eventually, there will be an L-shaped piece, L_w, which contains that point. This means that any diagram D_v, where $v \geq w$, will contain L_w and, hence, also contain that point. Therefore, the L-shaped pieces converge to the square, and the sequence D_n converges to 1.

5.2.3 Invariant

We have shown that the L-shaped regions converge to the square, which has a total area of 1. However, this proof contains another piece of inference that must be accounted for: the observation that the shaded area is always one-third of the total area. We will represent this using an *invariant*. An invariant, as defined in [Sho67], is something (in this case, a specific aspect of the diagram) that does not change after each permutation, transformation, or construction step. Another good definition of invariant comes from [Wei] and follows:

> A quantity which remains unchanged under certain classes of transformations. Invariants are extremely useful for classifying mathematical objects because they usually reflect intrinsic properties of the object of study.

It can be argued that every diagrammatic proof has an invariant.

Basically, we will base the use of an invariant closely around the idea of a loop invariant from programming languages, which captures the relationship between the variables on each circuit of the loop.[McG80] In other words, the truth of a loop invariant at the entry point of a loop guarantees its truth at the exit, denoted $\phi[L] \to \phi$, where ϕ is the invariant and L is the loop. In our case, as was briefly mentioned in Section 2, if the invariant is true before a construction step (the application of F), then it must also be true after.

In this section, we introduce the concept of a invariant into our proof in order to prove that, at the fixed point determined in the last section, the total area of the shaded region is 1/3.

Formalization Continued We introduce an invariant ϕ, which, in this case, stands for the statement "shaded area = 1/3 total area". Furthermore, we would like to form the following rule:

$$\phi[L_1 \cup \cdots \cup L_k] \to \phi[F(L_1 \cup \cdots \cup L_k)]. \tag{13}$$

This rule states that, if the invariant ϕ is true for a certain diagram D_k, then it is also true for the next diagram in the sequence, D_{k+1}. This rule is a formal statement of what we "see" in the diagrammatic proof. Now we must prove that it is true.

Let us look at the transformation F in equation 12. All that this equation does, essentially, is add a new L-shaped region to the diagram using the transformation T. So, if rule 13 holds for T, then it is true for F.

Transformation T scales and transposes an L-shaped region. As discussed in Sections 2 and 4, translating does not alter the area of the region, which is the invariant, ϕ, in this problem. Furthermore, scaling the region by half does not change the relationship between the shaded and unshaded regions, either. This is because, if X is one-third of Y, then $X/2$ must be one-third of $Y/2$. Another way to look at it is that, since each L_k is composed of two plain squares and one shaded square, we can say that $L_k = B_k \cup C_k$, where B_k represents the two unshaded squares and C_k represents the shaded square. Using this, we can rewrite the definition of T in equation 11 as follows:

$$\begin{aligned} L_{k+1} \quad = \quad & \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} (B_k - d_k) + \begin{bmatrix} 1/2^k \\ 1/2^k \end{bmatrix} + d_k \quad \cup \\ & \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} (C_k - d_k) + \begin{bmatrix} 1/2^k \\ 1/2^k \end{bmatrix} + d_k. \end{aligned}$$

Since we are performing the same transformation (scaling and translating) on both parts of the diagram (B_k and C_k), the invariant ϕ will hold. This is because translating does not alter ϕ and scaling each piece by half also does not change their relation to each other. Therefore, the transformation T does not alter the invariant ϕ; thus, we know that rule 13 holds for the transformation F.

Consequently, we can use rule 13 as our induction step in order to prove this theorem. In other words, we know from Figure $29D_1$ that the shaded area is one-third of the total area (i.e. ϕ holds), and, given rule 13, we know that, if ϕ holds for a diagram D_k, then it holds for diagram D_{k+1}. Therefore, by induction, we know that ϕ holds for all diagrams in the sequence D_n. Since we also know that D_n converges to 1, ϕ must also hold at that fixed point. Hence, the area of shaded regions converges to 1/3, and the theorem is proven.

5.2.4 Continuation

To continue in the formalization of this proof, we would like to define a semantics for the ellipsis in Figure 29A. We use *continuations* for this purpose.

A continuation, as explained in [McG80], describes the state change that occurs as a result of applying another construction step of our proof. Continuations, typically used in programming language theory and computational theory, are a formal way of expressing each step of a diagrammatic proof. Basically, a continuation allows one "...the option of deciding whether or not to go on to [the next step of the proof]."[Sto77] In this example, a continuation is used to describe the entire proof in a step-by-step manner, therefore dealing with the problem of ellipsis (...) in proofs. In other words, the continuations denote a "lazy evaluator" for the finite diagram constructed so far.

Formalization Continued The following, taken from [McG80], is a good discussion on the use of continuations:

> We have already seen that the value $\mathcal{C}[\Gamma]$ of a command is a function which takes as parameter an environment ρ. We now revise this idea and arrange that it takes two parameters ρ and θ and write this in the form of a curried function $\mathcal{C}[\Gamma]\rho\theta$. The extra parameter θ is the *command continuation*; it describes the state change that occurs as a result of executing the remainder of the program, i.e. the commands that would be executed if Γ terminated naturally.

In our case, the environment, ρ, involves the dimension and location of the latest L-shaped region added to the diagram. So, L_1 (Figure 29D_1) becomes $L_1\rho\theta$, where θ is the continuation.

To deal with diagrams D_2 and D_3, we must observe that "executing $[\Gamma_1;\Gamma_2]$ with continuation θ involves executing Γ_1 with the continuation consisting of executing Γ_2 with the original continuation, θ."[Sto77, notation slightly altered] This gives us

$$(L_1 \cup F[L_1])\rho\theta = L_1\rho\{F[L_1]\rho\theta\} \tag{14}$$

for diagram D_2 and

$$(L_1 \cup F[L_1;L_2])\rho\theta = L_1\rho\{F[L_1]\rho\{F[L_2]\rho\theta\}\} \tag{15}$$

for diagram D_3.

The real usefulness of continuations, however, can be seen in the description of diagram A, which contains ellipsis. Diagram A, as well as all possible diagrams associated with the theorem being proven in Figure 29, can be described using the following general expression:[26]

$$(L_1 \cup F[L_1;L_2;L_3;\ldots])\rho\theta.$$

[26]If we define some L_0 such that $F(L_0) = L_1$, then we can simplify this expression as well as equations 14 and 15 to be $F[L_0;L_1;L_2;\ldots]\rho\theta$, $F[L_0;L_1]\rho\theta$, and $F[L_0;L_1;L_2]\rho\theta$, respectively.

Diagram A can be written, with a slight abuse of notation, as

$$(L_1 \cup F[L_1; L_2; L_3; \ldots])\rho\theta = L_1\rho\{F[L_1]\rho\{F[L_2]\rho\{F[L_3]\rho\{F[L_4; \ldots]\rho\theta\}\}\}\}.$$

In addition, equations 14, 15, and similar expressions are equivalent to the above expression in that diagram D_1 is

$$(L_1 \cup F[L_1; L_2; L_3; \ldots])\rho\theta = L_1\rho\theta,$$

while diagrams D_2 and D_3 are

$$(L_1 \cup F[L_1; L_2; L_3; \ldots])\rho\theta = L_1\rho\{F[L_1]\rho\theta\}$$

and

$$(L_1 \cup F[L_1; L_2; L_3; \ldots])\rho\theta = L_1\rho\{F[L_1]\rho\{F[L_2]\rho\theta\}\},$$

respectively. Moreover, any diagram, D_k, can be written

$$(L_1 \cup F[L_1; L_2; \ldots])\rho\theta = L_1\rho\{F[L_1]\rho\{F[L_2]\rho\{\cdots\{F[L_{k-2}]\rho\{F[L_{k-1}]\rho\theta\}\}\cdots\}\}\}.$$

Now, in every diagram, the abstraction is hidden in the continuation. In this way, we can continue to produce subsequent diagrams using a single expression, and can successfully hide the abstractions.[27]

Returning to the proof, using rule 13 and the continuation above, we can formalize this final rule, using induction:

$$\{\phi[L_1 \cup \cdots \cup L_k] \to \phi[F(L_1 \cup \cdots \cup L_k)]\} \to \phi[(L_1 \cup F[L_1; L_2; \ldots; L_{k-1}])\rho\theta] \tag{16}$$

In other words, if ϕ holds for diagram D_k implies that it holds for D_{k+1}, then it holds for every diagram. Finally, we have one rule that proves this theorem.

Rule 16 is a formal statement that invariants across constructions are inductive "free rides". It captures the essence of what we humans can "see" very easily in Figure 29.

5.3 Summary

Our cognitive understanding of a proof is sometimes very difficult to formalize. Nevertheless, until diagrammatic proofs are believed to be rigorous enough to stand alone, we must formalize them and justify each step of the proof with traditional computational theory. It is exciting, though, that we are able to use the well-established ideas of *invariants*, *continuations*, and *fixed points* in the study of diagrammatic proofs. Finding a relationship between a well-studied field and a relatively new field is a huge step forward and beneficial to all fields. This link between the two will help the understanding of both areas as well as provide a large progression in the area of diagrammatic reasoning.

[27] It is interesting to note that the same expression $(L_1 \cup F[L_1; L_2; L_3; \ldots])\rho\theta$ can also be used to describe the proof of the geometric sum from Section 2.5.1 (the diagram is Figure 13).

6 Future Work

During the course of this research, certain theoretical and practical topics have surfaced and begged to be studied. Unfortunately, due to the relatively short duration of this project (six months), a lot of exploration in other areas of diagrammatic reasoning had to be left on the table.

Some further work that would be interesting to explore, if given more time, include the following:

- **Theoretical Work** - topics of theoretical interest include the gaining of a better understanding of diagrammatic reasoning (*Section 6.1.1*), a study regarding how to choose between different diagrammatic representations, when there is more than one, which includes an exploration into the guidelines for representation (*Section 6.1.2*), the formal proof that spatial reasoning tools, such as trees, are sound (*Section 6.1.3*), and computational efficiency gains that might be achieved through the incorporation of diagrammatic reasoning (*Section 6.1.4*).

- **Practical Work** - practical work that could be pursued, if given more time, include the automation of diagrammatic proofs of Categories 1 and 3 (*Section 6.2.1*) and the ability of agents to understand, process, accept, and communicate diagrammatically, therefore eliminating an agent's need for textual representations and improving their understanding of the natural world (*Section 6.2.2*).

6.1 Theoretical Work

Although many theoretical issues were dealt with in this research, the underlying theory regarding diagrammatic reasoning is far from complete. There are many questions still unanswered regarding this subject, and, given more time, the following are a few that would definitely be explored.

6.1.1 Better Understanding of Diagrammatic Reasoning

Upon attaining more time to work in this area, the main focus of a continued study would definitely be to acquire a better understanding of diagrammatic reasoning: its strengths, its weaknesses, how it works, and where it can be best applied. A rigorous study of diagrammatic reasoning must be completed in order to combine it with other methods of reasoning and representation. This, in turn, will help provide a scientific base for diagrammatic information that can be stored in and reasoned with by a computer, therefore improving the human-computer interaction.

6.1.2 Which Representation is Best?

All of the proofs in this paper (especially the examples in Section 2) assumed that the person attempting to prove a theorem was able to construct a corresponding diagram that not only successfully represented all of the information in the theorem, but also did not use irrelevant (unnecessary) information or constraints on the diagram. In addition, certain operations which were applied to the diagram in order to arrive at the desired proof had to be applicable on the specified diagram. However, as of yet, no guidelines as to the "best" way to represent a problem have been written. In addition, when there is more than one choice of representation of a certain theorem, the "best" representation may not be obvious. Plus, the "best" representation with respect to how much *work* is required or with respect to how *convincing* the proof is may not be the same representation in some cases.

In our best estimation, a further study into this area would lead into the area of cognitive science and psychology, as we are now dealing with the way people perceive information. Still, it would be a very interesting study and would have extremely beneficial results, especially in the area of interfaces.

6.1.3 Soundness of Spatial Reasoning "Tools"

As stated in Section 1.2, in order to make complex concepts easier for us humans to understand, visual aids such as trees, maps, blueprints, charts, graphs, Euler's Circles, and Venn diagrams are used to represent non-spatial ideas. Furthermore, Shin proved in [Shi91] and [Shi94] that Venn diagrams are sound and complete for some forms of reasoning. An interesting project would be to complete such a proof of (at least) soundness for other visual "aids" such as trees. This task should not be very difficult, in our best guess.

6.1.4 Computational Efficiency Gains

Anderson believes that "the main thrust of diagrammatic reasoning research to date (from an artificial intelligence perspective) has been a search for computational efficiency gains through representations, and related inference mechanisms, that analogously model a problem domain."[And99] Indeed, incorporating the power of diagrammatic reasoning along with traditional forms of reasoning will allow us to diminish the shortcomings of each. If we are able to combine their strengths in order to overcome their weaknesses, the result would be an improvement of computational efficiency.

It would be interesting to examine how agreement and continuity between groups of related diagrams can be supported for computational purposes. Anderson states that, "[this] approach has been successful in producing diagrammatic solutions to problems in a number of contexts including heuristic development, spatial configuration, inductive learning, case-based reasoning, proof support, inference from cartograms, scheduling, and DNA sequencing."

6.2 Practical Work

The aim of most practical work in the area of diagrammatic reasoning is to endow a machine with capabilities similar to a human's for reasoning directly with diagrammatic information. This would be beneficial to human-computer interfacing as well as to the way agents interact with a real-world environment.

Although this paper concentrates mostly on theoretical aspects of diagrammatic proofs, if one had more time to concentrate on this project, one might enjoy working on some practical applications of this study. Any of the studies from the diverse collection of various AI applications of diagrammatic reasoning in [GCN95] would be interesting, although future practical work which stem directly from the work in this paper would be best. This would include the automation of diagrammatic proofs of Categories 1 and 3 and the diagrammatic communication with agents.

6.2.1 Automation

As stated in Section 1, one aim of this paper is to extend the work of Jamnik, who was able to automate diagrammatic proofs of Category 2. Indeed, in [Jam99], Jamnik presented her semi-automated diagrammatic theorem prover, DIAMOND. Naturally, after studying the underlying theory of proofs of Categories 1 and 3, the next step is to automate them in a similar way. Alternately, DIAMOND could be further studied and possibly extended to include proofs of these categories. It would be a very interesting exercise in both code development and, most importantly, in the understanding of diagrammatic proofs. It would also be a first in the field since, to our knowledge, besides Jamnik's DIAMOND, no one else has developed a solely diagrammatic theorem prover, and, with the addition of the automation of Categories 1 and 3, it would include all categories.

6.2.2 Diagrammatic Communications with Agents

The current work of Anderson revolves around "developing an agent with full diagrammatic reasoning capabilities on par with human beings."[And00] He continues by saying that "[if] a picture is worth a thousand words, we would like to obviate the need for this text by being able to communicate with an agent directly via pictures or diagrams. An agent should be able to accept and understand such diagrammatic input from us and our environment as well as being able to produce such diagrams in its attempt to communicate diagrammatically representable notions to us." Indeed, the ability to avert the need for inference to reveal what is in the textual representations is very appealing for agents in the natural world. Agents who can understand, process, accept, and communicate diagrammatically will have a tremendous advantage over agents who cannot. We believe, as does Anderson, that this is the way of the future, and, therefore, it is a definite possibility for future investigations.

6.3 Summary

During any study of limited duration, some things that look interesting must be left for the next researcher. Although this is an advantage for the "next researcher", who will be able to benefit from the work accomplished, it is hard to leave an area which is so intriguing untouched. Especially in a field so relatively new, one can see many paths that are not yet travelled but must, unfortunately, walk away from them. This section listed some of the areas that, if given another year or more to work, one could pursue. These include more theoretical work like that of this paper as well as practical work to apply what has been discovered.

7 Conclusion and Discussion

Diagrammatic reasoning is representing and reasoning about concepts and ideas with diagrams instead of mathematical and logical symbols. In the future, though, in order to completely cover all problem domains and to ensure that proofs are not led astray, diagrammatic representations may need to be combined with the strengths of other representations. However, because the study of diagrammatic reasoning is so new, a rigorous investigation is required to increase our understanding of it and bring it up to the same level as that of the other, better understood representation methods. Once an equal, thorough understanding of all representation methods is achieved, we can make informed judgements regarding their respective uses. Nevertheless, until then, we must logically validate the soundness of every step of a diagrammatic proof. This paper is meant to be a part of achieving such a meticulous understanding and formalization. Various problems were presented, solved, and justified in this paper.

Section 2 introduced the reader to examples of diagrammatic proofs from [Nel93] and presented an analysis of each proof. Without a complete study of what constitutes a diagrammatic proof and how it works, we cannot begin the study of diagrammatic reasoning. We were also able to see how symbolic logic hypotheses can be used to justify a diagrammatic argument. Furthermore, we must balance this with a study of diagrammatic proofs that do not work, as in Section 3. Here, we were able to analyze some misleading diagrams and why they do not work. On the other side of that, an analysis of misleading symbolic logic that was shown to be incorrect by diagrams was also given, proving that neither the diagrammatic nor symbolic representation should be thrown out, but must work together. In [Sim95], Simon states that, "thanks to Descartes and Dedekind and others, we can see the logical identity (knowledge equivalence) of symbolic and diagrammatic representations of a given problem ... or that we will be able to draw the same inference from both." While some problems warrant their solution to a certain type of representation and reasoning, most lie in a "grey" area and can be reasoned with using various representations. Knowing that these representations can be informationally equivalent without necessarily being computationally equivalent allows us the opportunity to utilize the strengths of each representation on a *per problem* basis.

Category 1 diagrammatic proofs require generalization to show that the theorem holds for all values of universally quantified variables. Foo, et al. showed in their IJCAI'99 paper that there are two ways to prove this. In Section 4, it was proven that these two solutions are actually equivalent. This means that the generalization required at the end of a proof can be accomplished with either the Theorem on Constants or by the continuity argument.

Diagrammatic proofs of Category 3 have another problem: they require abstractions such as ellipsis (...) to represent infinite completion, and there are no instances of a theorem. Interestingly enough, although these features do not make Category 3 proofs more complex for the cognitive understanding of the proof, they do make the formalization and justification of them more difficult as they are higher-order problems. Nevertheless, in Section 5, we formalized and justified every step of a specific Category 3 diagrammatic proof using well-established ideas from traditional computational theories, such as fixed points, invariants, and continuations. Continuations, in particular, allowed us to define a semantics for the ellipsis in the diagram of the proof. The establishment of such a link between diagrammatic reasoning and traditional computational theories is beneficial for the understanding and development of both areas.

The future of this research was discussed in Section 6. There are so many advancements in this area that it would be impossible to list all of the ways that this work could be continued in the future. Still, beyond the primary goal of completely understanding diagrammatic reasoning and its underlying processes, in general, research in diagrammatic reasoning has two main aims, as best stated by Simon:

> The first is to deepen our understanding of ourselves and the ways in which we think. That deeper understanding is already beginning to enhance our sophistication and effectiveness in using visual displays, in books and on computer screens, to communicate and teach. The second goal is to provide an essential scientific base for constructing representations of diagrammatic information that can be stored and processed by computers, en-

> abling computers to achieve some of the computational efficiencies in their thinking that diagrams now provide to human beings. [Sim95]

Diagrammatic reasoning has a long history, but has just recently had a rebirth and is exploding into new life. This is because, much of what is explicit to humans in pictorial representations ("free rides") is, at best, implicit in textual ones, requiring inference to reveal it. This loses the "essence" and speed of a pictorial representation. Imagine what might be gained by the restoration of this explicitness through the practice of computing directly with pictorial representations. Because of all of the obvious benefits of diagrammatic reasoning, work in this area is blossoming and taking great strides in the right direction to prove that computing directly with diagrams is a realistic goal for the not-so-distant future.

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