Sequential Payment Rules: Approximately Fair Budget Divisions via Simple Spending Dynamics

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In approval-based budget division, a budget needs to be distributed to some candidates based on the voters' approval ballots over these candidates. In the pursuit of simple, well-behaved, and approximately fair rules for this setting, we introduce the class of sequential payment rules, where each voter controls a part of the budget and repeatedly spends his share on his approved candidates to determine the final distribution. We show that all sequential payment rules satisfy a demanding population consistency notion and we identify two particularly appealing rules within this class called the maximum payment rule (MP) and the $\frac{1}{3}$-multiplicative sequential payment rule ($\frac{1}{3}$–MSP). More specifically, we prove that *(i)* MP is, apart from one other rule, the only monotonic sequential payment rule and gives a 2-approximation to a fairness notion called average fair share, and (ii) $\frac{1}{3}$–MSP gives a $\frac{3}{2}$-approximation to average fair share, which is optimal among sequential payment rules.

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1 Introduction

Suppose that a city council wants to support the local sport clubs by distributing a fixed amount of money between them. Moreover, in order to allocate the money in a satisfactory way, the council asks for the citizens' preferences over the sport clubs. However, given these preferences, how should the budget be distributed? This simple example describes a fundamental research problem in the realm of social choice theory, which we will refer to as budget division.[1] More specifically, the central problem of budget division is how to distribute a divisible resource among the candidates in a fair and structured way based on the voters' preferences over these candidates. Notably, this resource does not have to be money as budget division, e.g., also captures time-sharing problems (where we need to allocate time to various activities in a fixed time frame) or probabilistic voting (where we need to assign the probability to win the election to the candidates).

Perhaps due to these versatile applications, budget division has recently attracted significant attention [e.g., Airiau et al., 2019; Aziz et al., 2020; Brandl et al., 2021, 2022; Freeman et al., 2021; Brandt et al., 2024; Elkind et al., 2024; Ebadian et al., 2024] and it has been studied based on various

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[1] In the literature, budget division is referred to by a number of names such as randomized social choice [Bogomolnaia et al., 2005], fair mixing [Aziz et al., 2020], portioning [Airiau et al., 2019; Elkind et al., 2024], and (fractional) participatory budgeting [Fain et al., 2016; Aziz and Shah, 2021].

assumptions on the voters' preferences. For instance, Airiau et al. [2019] and Ebadian et al. [2024] study budget division under the assumption that the voters rank all candidates, Fain et al. [2016] suppose that the voters have cardinal utilities, and Freeman et al. [2021] initiated a line of work where voters report ideal resource distributions. By contrast, we will investigate budget division based on the assumption that the voters report approval ballots over the candidates, i.e., each voter only reports the set of candidates he approves of. This model has been first suggested by Bogomolnaia et al. [2005] and it is considered in numerous recent works [e.g., Aziz et al., 2020; Michorzewski et al., 2020; Brandl et al., 2021, 2022] because approval preferences achieve a favorable trade-off between the cognitive burden on the voters and the expressiveness of the ballot format.[2] The main study object of this paper thus are *distribution rules*, which map every profile of approval ballots to an allocation of the resource to the candidates.

A central objective for many budget division problems is to choose outcomes that are fair towards all voters: an appropriate amount of the budget should be allocated to the approved candidates of every group of voters. For instance, if each voter in our introductory example only approves of a single sport club, it seems desirable that every sport club gets a portion of the budget that is proportional to the number of voters approving it. Moreover, a multitude of axioms has been suggested to formalize fairness also in the context of general approval preferences [see, e.g., Bogomolnaia et al., 2005; Duddy, 2015; Aziz et al., 2020; Brandl et al., 2022], most of which are based on the idea that each voter controls a $\frac{1}{n}$ share of the total budget (where n is the number of voters). Two of the strongest fairness axioms based on this approach are *average fair share (AFS)* and *core*, both of which were suggested by Aziz et al. [2020]. Average fair share requires for every group of voters that approves a common candidate that the average utility of the voters in this group is at least as large as the utility they would obtain if the voters allocate their share to the commonly approved candidate. On the other hand, core applies the universal concept of core stability to budget division: there should not be a group of voters who can enforce an outcome that is better for each voter in the group by only using the share they deserve.

In the literature, there is only one distribution rule that is known to satisfy AFS and core—the Nash product rule (NASH) [Aziz et al., 2020]. This rule returns the distribution that maximizes the Nash social welfare, i.e., the product of the total share assigned to the approved candidates of each voter. However, while being fair, this rule is rather intricate and exhibits otherwise undesirable behavior. In particular, the distribution returned by NASH may contain irrational values [Airiau et al., 2019], which means that we can only approximate this rule in practice. Closely connected to this is the issue that, for many profiles, it is difficult even for experts to verify whether an allocation is the one selected by NASH. Furthermore, NASH fails *monotonicity* [Brandl et al., 2021]: if a candidate gains more support in the sense that an additional voter approves it, its share may decrease. For instance, in our introductory example, this may lead to paradoxical situations where it would be better for a sport club to be approved by less voters. Finally, we will show in this paper that NASH also behaves counter-intuitively with respect to *population consistency conditions*. Such conditions are well-studied in social choice theory [e.g., Smith, 1973; Young, 1975; Fishburn, 1978; Young and Levenglick, 1978; Brandl et al., 2016; Brandl and Peters, 2022] and roughly require that, when combining disjoint elections into one election, the outcome for the combined election should be consistent with the outcome for the subelections. In more detail, we will prove that, for NASH, it can happen that the share of a candidate in a combined election is less than its share in each subelection, even if the considered candidate obtains the maximal share in each subelection and

[2] We refer to the paper of Brams and Fishburn [2007] for a detailed discussion of the advantages of approval ballots and to the work of Fairstein et al. [2023] for an experimental evaluation of approval ballots in the context of participatory budgeting.

the distributions chosen for the subelections are structurally similar. For our sport club example, this means that the share of a sport club in a city-wide election may be less than the minimal share the sport club would obtain in district-level elections, even if the sport club obtains the largest share in each district and the distributions chosen for the districts are structurally similar.

Motivated by these observations, our central research question is whether there are simple combinatorial rules that exhibit more reasonable behavior than NASH while satisfying strong fairness guarantees. However, as it seems out of range to achieve full AFS or core with such rules, we will consider approximate variants of these axioms called α-AFS and α-core, which relax the constraints imposed by the original axioms by a $\frac{1}{\alpha}$ factor. Thus, 1-AFS and 1-core correspond to AFS and core, and the fairness guarantees of α-AFS or α-core become weaker as α increases. These approximate fairness notions allow us to quantify the fairness of every distribution rule by determining the minimal α for which it satisfies α-AFS or α-core, so we can derive a much more fine-grained picture about the fairness guarantees of distribution rules based on these axioms. Our formal goal is thus to identify distribution rules that satisfy monotonicity and/ or strong population consistency conditions while satisfying α-AFS or α-core for as small α as possible.

Contributions In an attempt to find simple and fair distribution rules, we first analyze several classical rules such as the conditional utilitarian rule [Duddy, 2015] and the fair utilitarian rule [Bogomolnaia et al., 2002; Brandl et al., 2021]. However, all of these rules give poor approximations to AFS and core and they typically violate even weak population consistency conditions (see Table 1 for details). Because of this, we introduce a new class of distribution rules called *sequential payment rules*. Intuitively, these rules work as follows: each voter is given a virtual budget of 1 and a payment willingness function specifies how much of his remaining budget a voter is willing to spend on his next candidate. Then, sequential payment rules repeatedly determine the candidate that maximizes the total payment willingness, let the voters send their corresponding payments to this candidate, and remove this candidate from consideration. Finally, the distribution chosen by a sequential payment rule is proportional to the budgets accumulated by the candidates.

As a first point on sequential payment rules, we show that all of them satisfy *ranked population consistency (RPC)*. This axiom formalizes a demanding population consistency condition which roughly requires that, if a rule chooses distributions that are structurally similar for two disjoint elections, then the share of every candidate in the joint election should be lower and upper bounded by its minimal and maximal share in the disjoint subelections. More specifically, for this axiom, we consider two distributions as similar if the orders obtained by sorting the candidates according to their shares are the same for both profiles. We believe this to be reasonable because, intuitively, there is relatively little conflict between distributions that agree on which candidates deserve a larger share than others. Since sequential payment rules satisfy RPC, they guarantee that the outcome for a joint election is closely related to the distributions selected for the subelections when these distributions are not conflicting and they thus afford a high degree of explainability in such situations.

As our second contribution, we analyze the class of sequential payment rules with respect to monotonicity and show that there are only two rules in this class that satisfy this axiom: the *maximum payment rule (*MP*)*, where voters are willing to spend their whole budget on their first approved candidate, and the *uncoordinated equal shares rule (*UES*)*, where the payment willingness of each voter is always the inverse of the number of his approved candidates. Moreover, while UES gives rather poor approximations to fairness, we show that MP satisfies 2-AFS and $\Theta(\log n)$-core. This shows that MP is also appealing from a fairness perspective, and thus answers our original question of whether there is a simple, fair, and well-behaved distribution rule in the affirmative.

	Monotonicity	Population consistency	α-AFS	α-core
NASH	✗	weak	1	1
CUT	✓	✗	$\Theta(n)$	$\Theta(n)$
FUT	✓	✗	$\Theta(n)$	$\Theta(n)$
UES	✓	strong	$\Theta(n)$	$\Theta(n)$
MP	✓	ranked	2	$\Theta(\log n)$
$\frac{1}{3}$–MSP	✗	ranked	$\frac{3}{2}$	$O(\log n)$

Table 1: Analysis of distribution rules with respect to monotonicity, population consistency, and their approximation ratios to AFS and core. Each row states for a distribution rule whether it satisfies monotonicity, the degree to which it satisfies population consistency, and its approximation ratio for AFS and core. The first four rows focus on established rules, while the fifth and sixth row discuss two rules designed in this paper. For population consistency, the symbol ✗ indicates that the rule fails all population consistency conditions considered in this paper, whereas "weak", "ranked", and "strong" indicate the exact population consistency condition that is satisfied (see Section 2.2 for details). Contributions of this paper are shaded gray.

We lastly turn to the problem of optimizing the approximation ratio to AFS within the class of sequential payment rules. To this end, we first characterize the AFS approximation ratio of γ-multiplicative sequential payment (γ–MSP) rules, for which the payment willingness for the i-th candidate is γ times the payment willingness of the $(i-1)$-th candidate. In particular, this result shows that $\frac{1}{3}$–MSP is a $\frac{3}{2}$-approximation for AFS, and we prove that no sequential payment rule satisfies α-AFS for a smaller α. Furthermore, by employing a more fine-grained analysis that takes the maximum ballot size of a profile into account, we derive a characterization of the $\frac{1}{3}$–MSP rule as the fairest sequential payment rule. A concise comparison of our two new rules (MP and $\frac{1}{3}$–MSP) and four classical rules with respect to our desiderata is given in Table 1.

Related Work The problem of budget division with dichotomous preferences was first considered by Bogomolnaia et al. [2005], who, inspired by randomized social choice, defined several distribution rules and studied them with respect to strategyproofness, efficiency, and fairness. In particular, Bogomolnaia et al. [2005] realized that there is a trade-off between these conditions because none of their rules simultaneously satisfies all three properties. This trade-off was further analyzed by Duddy [2015] and Brandl et al. [2021], the latter of whom proved that no strategyproof and efficient rule can satisfy minimal fairness requirements. In a similar vein, Michorzewski et al. [2020] showed that fair distribution rules only give poor guarantees on the utilitarian social welfare.

On a more positive note, Aziz et al. [2020] investigated distribution rules with respect to fairness notions and, in particular, showed that the Nash product rule satisfies AFS and core. Moreover, several other authors have analyzed further aspects of the Nash product rule [e.g., Guerdjikova and Nehring, 2014; Airiau et al., 2019; Brandl et al., 2022]. For instance, Brandl et al. [2022] show that NASH satisfies a strong participation incentive condition, and Guerdjikova and Nehring [2014] give a characterization of this rule based on a population consistency axiom and several other conditions. We moreover note that Brandl et al. [2022] present a dynamic process for computing the Nash product rule, which was first suggested by Cover [1984] in the context of portfolio selection in stock markets. More precisely, these authors demonstrate that, when the voters repeatedly update their spending such that it is proportional to the restriction of the current distribution to their approved alternatives, the distribution will converge against the one chosen by NASH.

Similar redistribution dynamics have recently attracted significant attention [e.g., Freeman et al., 2021; Brandl et al., 2022; Brandt et al., 2023; Haret et al., 2024], but these dynamics significantly differ from our work as we do not allow voters to reallocate spent money and mainly use spending dynamics to define rules instead of analyzing the dynamic itself.

Furthermore, our paper belongs to an extensive stream of research that studies fairness in budget division. In more detail, besides the paper by Aziz et al. [2020], fairness notions similar to AFS and core have been studied in budget division models with strict preferences [Airiau et al., 2019; Ebadian et al., 2024], cardinal utilities [Fain et al., 2016], and ideal distributions [Freeman et al., 2021; Caragiannis et al., 2024; Brandt et al., 2024; Elkind et al., 2024], as well as budget division models that impose additional structure on the outcomes [Bei et al., 2024; Lu et al., 2024]. For instance, Freeman et al. [2021] design a strategyproof and fair distribution rule under the assumption that the voters report ideal budget division. Moreover, analogous fairness concerns have attracted significant attention in settings that are closely related to budget division, such as committee voting [Faliszewski et al., 2017; Lackner and Skowron, 2023] and participatory budgeting [Aziz and Shah, 2021; Rey and Maly, 2023].

Lastly, we note that budget division is closely related to randomized social choice [Brandt, 2017], where the goal is to select a probability distribution over the candidates from which the final election winner will be chosen by chance. While the model of randomized social choice is mathematically identical to that of budget division, the goals, and thus also the considered axioms, are different. In particular, in randomized social choice, using a large amount of randomization is often met with criticism as this places a large degree of uncertainty on the final winner, whereas allocating the budget to many candidates seems uncontroversial in budget division.

2 Model and Axioms

For any positive integer $t \in \mathbb{N}$, we define $[t] = \{1, 2, \dots, t\}$. Let $N = \{1, \dots, n\}$ be a set of n voters and $C = \{x_1, \dots, x_m\}$ a set of m candidates. We assume that the voters have *dichotomous* preferences over the candidates, i.e., voters only distinguish between approved and disapproved candidates. Moreover, each voter $i \in N$ reports his preferences in the form of an *approval ballot* A_i, which is the non-empty set of his approved candidates. An *approval profile* $\mathcal{A} = (A_1, A_2, \dots, A_n)$ contains the approval ballots of all voters. Given an approval profile $\mathcal{A}$, we denote by $N_x(\mathcal{A}) = \{i \in N \colon x \in A_i\}$ the set of voters who approve of candidate x and we call $|N_x(\mathcal{A})|$ the *approval score* of candidate x. Finally, an *election instance* is a tuple $\mathcal{I} = (N, C, \mathcal{A})$ that specifies the set of voters N, the set of candidates C, and an approval profile $\mathcal{A}$ for the given sets of voters and candidates. Following the literature, we will typically equate election instances with the corresponding approval profiles as these contain all relevant information. Because we will study population consistency axioms in this paper, we define by $\mathcal{A} + \mathcal{A}'$ the approval profile that combines two voter-disjoint profiles $\mathcal{A}$ and $\mathcal{A}'$. More formally, given two instances $\mathcal{I} = (N, C, \mathcal{A})$ and $\mathcal{I}' = (N', C', \mathcal{A}')$ such that $C = C'$ and $N \cap N' = \emptyset$, $\mathcal{A} + \mathcal{A}'$ is the approval profile corresponding to the instance $\mathcal{I} + \mathcal{I}' = (N \cup N', C, (\mathcal{A}, \mathcal{A}'))$.

Given an approval profile, our goal is to distribute a divisible resource to the candidates. To this end, we will use distribution rules which determine for every election instance an allocation of the resource to the candidates. We will formalize such resource allocations by distributions that specify the share of the resource assigned to each candidate. More formally, a *distribution* p is a function of the type $C \to [0, 1]$ such that $\sum_{x \in C} p(x) = 1$, and we denote by $\Delta(C)$ the set of all distributions over C. Then, a *distribution rule* f is a function that maps each approval profile $\mathcal{A}$ to a distribution $p \in \Delta(C)$. For the ease of notation, we define by $f(\mathcal{A}, x)$ the share of the resource that the distribution rule f assigns to candidate x under the approval profile $\mathcal{A}$. For instance, $f(\mathcal{A}, x) = 0.5$ means that, for the profile $\mathcal{A}$, f assigns half of the resource to candidate x. Following the literature, we assume that the utility of a voter i for a distribution p is the total share assigned to his approved candidates, i.e., the utility function of voter i is given by $u_i(p) = \sum_{x \in A_i} p(x)$.

In the following three subsections, we will introduce the central axioms of this paper.

2.1 Monotonicity

As our first axiom, we will discuss monotonicity, which requires that the share of a candidate does not decrease if it gets additional supporters. Hence, this axiom can be seen as an incentive for candidates to gather as much support as possible.

Definition 2.1 (Monotonicity)**.** A distribution rule f is *monotonic* if $f(\mathcal{A}', x) \geq f(\mathcal{A}, x)$ for all approval profiles $\mathcal{A}$ and $\mathcal{A}'$, voters $i \in N$, and candidates $x \in C$ such that $A'_i = A_i \cup \{x\}$ and $A'_j = A_j$ for all $j \in N \setminus \{i\}$.

Analogous monotonicity notions are well-studied in numerous voting settings [e.g., Gibbard, 1977; Fishburn, 1982; Maskin, 1999; Sanver and Zwicker, 2012; Sánchez-Fernández and Fisteus, 2019], where it is typically possible to satisfy monotonicity with attractive voting rules. By contrast, in budget division with dichotomous preferences, monotonicity is known to be a rather restrictive concept as only few known rules satisfy this condition [Bogomolnaia et al., 2005; Brandl et al., 2021]. We nevertheless believe that this condition is a natural desideratum for budget divisions as it ensures that candidates are rewarded for obtaining more votes.

2.2 Population Consistency

We will next discuss several population consistency axioms, which aim to formalize that the chosen distribution for a joint election should be consistent with the distributions that are selected for its subelections. The standard notion of population consistency states that, if a rule chooses the same outcome for two disjoint elections, it should choose the same outcome also for the combined election. We will call this condition weak population consistency as it is the weakest population consistency axiom in this paper, and we define it as follows.

Definition 2.2 (Weak population consistency (WPC))**.** A distribution rule f satisfies *weak population consistency (WPC)* if $f(\mathcal{A} + \mathcal{A}') = f(\mathcal{A})$ for all voter-disjoint profiles $\mathcal{A}$ and $\mathcal{A}'$ with $f(\mathcal{A}) = f(\mathcal{A}')$.

Variants of this condition feature in numerous prominent results of social choice theory [e.g., Smith, 1973; Young, 1975; Fishburn, 1978; Young and Levenglick, 1978; Brandt, 2015; Lackner and Skowron, 2021; Brandl and Peters, 2022]. However, we believe that WPC is too weak in the context of budget division because the condition of choosing the exact same distribution in two profiles is too restrictive. For instance, assume a distribution rule f chooses for two profiles $\mathcal{A}$ and $\mathcal{A}'$ the distributions $p = f(\mathcal{A})$ and $q = f(\mathcal{A}')$ with $p(x) = 1$ and $q(x) = 0.99$. It then seems reasonable that candidate x also gets a share close to 1 in the combined election $\mathcal{A} + \mathcal{A}'$, but WPC does not impose any constraint on the distribution $f(\mathcal{A} + \mathcal{A}')$.

One possible solution for this is to demand that, for each alternative x, the share assigned to x in the joint profile $\mathcal{A} + \mathcal{A}'$ should be lower and upper bounded by its shares in $\mathcal{A}$ and $\mathcal{A}'$. We formalize this idea next.

Definition 2.3 (Strong population consistency (SPC))**.** A distribution rule f satisfies *strong population consistency (SPC)* if $\min\{f(\mathcal{A}, x), f(\mathcal{A}', x)\} \leq f(\mathcal{A} + \mathcal{A}', x) \leq \max\{f(\mathcal{A}, x), f(\mathcal{A}', x)\}$ for all candidates $x \in C$ and voter-disjoint profiles $\mathcal{A}$ and $\mathcal{A}'$.

Although we believe that SPC captures the idea of population consistency very well for budget division, this axiom turns out to be too prohibitive for dichotomous preferences. To see this, consider the profiles $\mathcal{A}$ and $\mathcal{A}'$ on the set of candidate $C = \{a, b, c\}$ such that all voters in $\mathcal{A}$ report $\{a, b\}$ and all voters in $\mathcal{A}'$ report $\{a, c\}$. For these profiles, it seems natural to choose the distributions p and q with $p(a) = p(b) = \frac{1}{2}$ and $q(a) = q(c) = \frac{1}{2}$, respectively. By contrast, in the profile $\mathcal{A} + \mathcal{A}'$, most rules will assign the entire budget to a since it is the only candidate that is approved by all voters, but this violates SPC.[3]

Because weak and strong population consistency are respectively too permissive and too restrictive, we will suggest a new intermediate population consistency notion which we call ranked population consistency. The idea of this notion is to only apply strong population consistency if the outcomes for $\mathcal{A}$ and $\mathcal{A}'$ are structurally similar. More specifically, we will only apply strong population consistency for profiles $\mathcal{A}$ and $\mathcal{A}'$ for which we obtain the same sequence of candidates when sorting the candidates in decreasing order of their assigned shares. To formalize this, we define the *distribution ranking* $\succsim^p$ of a distribution p as the binary relation over C given by $x \succsim^p y$ if and only if $p(x) \geq p(y)$ for all $x, y \in C$. Then, *ranked population consistency (RPC)* postulates strong population consistency only for profiles with identical distribution rankings.

Definition 2.4 (Ranked population consistency (RPC))**.** A distribution rule f satisfies *ranked population consistency (RPC)* if $\min\{f(\mathcal{A}, x), f(\mathcal{A}', x)\} \leq f(\mathcal{A} + \mathcal{A}', x) \leq \max\{f(\mathcal{A}, x), f(\mathcal{A}', x)\}$ for all candidates $x \in C$ and voter-disjoint profiles $\mathcal{A}$ and $\mathcal{A}'$ such that $\succsim^{f(\mathcal{A})} = \succsim^{f(\mathcal{A}')}$.

We note that SPC implies RPC and that RPC implies WPC, which gives a mathematical justification for our new axiom. Moreover, we believe that RPC is a desirable notion by itself because the precondition that $\succsim^{f(\mathcal{A})} = \succsim^{f(\mathcal{A}')}$ ensures that there is no severe conflict between the distributions $f(\mathcal{A})$ and $f(\mathcal{A}')$. In more detail, $\succsim^{f(\mathcal{A})} = \succsim^{f(\mathcal{A}')}$ means that, for all pairs of candidates, f agrees for $\mathcal{A}$ and $\mathcal{A}'$ on which of the candidates should obtain more budget, so it seems sensible that the distribution for the combined election $\mathcal{A} + \mathcal{A}'$ is closely related to the distributions for $\mathcal{A}$ and $\mathcal{A}'$. Finally, we note that it may seem even more intuitive to require for the definition of RPC that $\succsim^{f(\mathcal{A}+\mathcal{A}')} = \succsim^{f(\mathcal{A})}$ when $\succsim^{f(\mathcal{A})} = \succsim^{f(\mathcal{A}')}$. We decided against this since this condition is unrelated to weak and strong population consistency, but we observe that all of our results remain true for this alternative definition.

2.3 Fairness

A central concern in budget division problems is fairness: we should select outcomes that guarantee an appropriate amount of utility to all voters. In budget division, fairness is typically formalized based on the idea that each voter deserves to control a share of $\frac{1}{n}$ of the total budget. Maybe the most direct formalization of this idea is *decomposability* [Brandl et al., 2022], which requires that the selected distribution p can be decomposed into payments from individual voters such that each voter pays exactly $\frac{1}{n}$ to his approved candidates.

Definition 2.5 (Decomposability)**.** A distribution p is *decomposable* for an approval profile $\mathcal{A}$ if there exist individual distributions $\{p^i\}_{i \in N}$ such that (i) $p = \frac{1}{n} \sum_{i \in N} p^i$ and (ii) $p^i(x) = 0$ for all $i \in N, x \notin A_i$. A distribution rule f is decomposable if $f(\mathcal{A})$ is decomposable for every profile $\mathcal{A}$.

[3] When slightly modifying our setting, strong population consistency can be satisfied by reasonable rules. In more detail, when voters report strict preferences, it can be shown that proportional scoring rules [see, e.g., Barberà, 1979; Brandt, 2017] such as the harmonic rule by Boutilier et al. [2015] or the random dictatorship by Gibbard [1977] satisfy strong population consistency.

While decomposability formalizes a basic fairness notion, it fails to capture group synergies between voters. We thus view this condition as a minimal fairness requirement and only discuss decomposable rules in this work. By contrast, we will measure the fairness of distribution rules by studying stronger axioms in a quantitative manner. In particular, we will investigate the approximation ratio of distribution rules to average fair share (AFS) and core, two of the strongest fairness conditions in budget division. We will next discuss these two axioms, which were both suggested by Aziz et al. [2020] in the context of budget division.[4]

- *Average fair share (AFS)* lower bounds the average utility of every group of voters S that support a common candidate x by the average utility this group would enjoy if all voters in S allocate their $\frac{1}{n}$ share to x. More formally, a distribution p satisfies AFS for a profile $\mathcal{A}$ if $\frac{1}{|S|}\sum_{i\in S} u_i(p) \geq \frac{|S|}{n}$ for all groups of voters $S \subseteq N$ such that $\bigcap_{i\in S} A_i \neq \emptyset$.
- *Core* requires that there is no group of voters S that can reallocate their $\frac{1}{n}$ shares such that each voter in S weakly increases his utility and some voter in S strictly increases his utility. More formally, a distribution p satisfies core for a profile $\mathcal{A}$ if there is no distribution q and group of voters S such that $\frac{|S|}{n}u_i(q) \geq u_i(p)$ for all $i \in S$ and $\frac{|S|}{n}u_i(q) > u_i(p)$ for some $i \in S$.

We note that both of these axioms are very challenging as they give strong lower bounds on the utilities of the voters. To allow for a more fine-grained analysis, we will thus consider approximate versions of these axioms, called α-average fair share (α-AFS) and α-core, which relax the corresponding axioms by introducing a multiplicative approximation factor. In more detail, α-AFS weakens the lower bounds on the average utilities of groups of voters with a commonly approved candidate, whereas α-core rules out that there is a set of voters S such that the voters in S can obtain α times their original utility by reassigning their $\frac{1}{n}$ shares.

Definition 2.6 (α-AFS)**.** A distribution p satisfies *α-AFS* for an approval profile $\mathcal{A}$ and an approximate factor $\alpha \geq 1$ if $\alpha \cdot \frac{1}{|S|} \cdot \sum_{i\in S} u_i(p) \geq \frac{|S|}{n}$ for all $S \subseteq N$ with $\bigcap_{i\in S} A_i \neq \emptyset$.

Definition 2.7 (α-core)**.** A distribution p satisfies *α-core* for an approval profile $\mathcal{A}$ and an approximation factor $\alpha \geq 1$ if for all $S \subseteq N$, there is no distribution q such that $\frac{|S|}{n} \cdot u_i(q) \geq \alpha \cdot u_i(p)$ for all $i \in S$, and there is some voter $i \in S$ such that $\frac{|S|}{n} \cdot u_i(q) > \alpha \cdot u_i(p)$.

Moreover, we say that a distribution rule f satisfies α-AFS (resp., α-core) for an approximation factor $\alpha \geq 1$ if $f(\mathcal{A})$ satisfies α-AFS (resp., α-core) for every approval profile $\mathcal{A}$. These notions allow us to quantify the fairness of distribution rules by analyzing the minimal value of α for which they satisfy the given axioms. In particular, a smaller value of α corresponds to a higher degree of fairness and 1-AFS (resp., 1-core) is equivalent to the original notion of AFS (resp., core). We note that similar multiplicative approximations of fairness axioms have been considered before, albeit not in the context of budget division with dichotomous preferences [e.g., Peters and Skowron, 2020; Munagala et al., 2022; Bei et al., 2024; Ebadian et al., 2024].

Example 2.8 (Fairness axioms)**.** We will next discuss an example to further illustrate the fairness axioms in this paper. To this end, consider the following approval profile $\mathcal{A}$ with 6 voters and 4 candidates.

$$\mathcal{A}: \quad 1:\{a,b_1\} \quad 1:\{a,b_2\} \quad 1:\{a,b_3\} \quad 1:\{b_1\} \quad 1:\{b_2\} \quad 1:\{b_3\}$$

[4] We note that the general concept of the core was first suggested by Aumann [1961] in the context of cooperative games and that the core has recently been used as fairness notion in numerous social choice settings [e.g., Fain et al., 2016; Aziz et al., 2017; Ebadian et al., 2024].

Moreover, let p denote the distribution given by $p(b_1) = p(b_2) = p(b_3) = \frac{1}{3}$. First, we note that this distribution is decomposable for $\mathcal{A}$ as demonstrated by the following decomposition: the first and fourth voter pay their $\frac{1}{6}$ share to b_1, the second and fifth voter pay for b_2, and the third and sixth voter pay for b_3. However, the distribution fails both AFS and core because, intuitively, the voters approving a deserve half of the budget but each of them only has a utility of $\frac{1}{3}$. We thus compute the approximation ratio of p to core and AFS, and define to this end S as the group of voters approving a. By its definition, α-AFS requires that $\frac{\alpha}{3} = \alpha\frac{1}{|S|}\sum_{i\in S} u_i(p) \geq \frac{|S|}{|N|} = \frac{1}{2}$. Hence, p can only satisfy α-AFS for $\alpha \geq \frac{3}{2}$ and it turns out that it actually satisfies $\frac{3}{2}$-AFS by checking the constraints for all other candidates. On the other hand, for α-core, consider the distribution q given by $q(a) = 1$. For this distribution, it holds for all $i \in S$ that $\frac{1}{2} = \frac{|S|}{|N|}u_i(q) \geq u_i(p) = \frac{1}{3}$, so it follows that p can only satisfy α-core for $\alpha \geq \frac{3}{2}$. Moreover, it can again be checked that p satisfies $\frac{3}{2}$-core by considering all other groups of voters. Hence, while the distribution p may not fully satisfy AFS and core, it is still rather fair.

As the last point in this section, we will discuss the relation between α-AFS and α-core. To this end, we first recall that AFS and core are logically incomparable, i.e., there are instances where a distribution satisfies AFS but not core and vice versa [Aziz et al., 2020]. By contrast, we will next prove that the approximation ratios to core and AFS are mathematically related. We note that the implications given in the following proposition are asymptotically tight (see Remark 6 for details).

Proposition 2.9. *If a distribution p satisfies α-AFS for an approval profile $\mathcal{A}$, it satisfies $\alpha(1+\log(n))$-core. Conversely, if a distribution p satisfies α-core for an approval profile $\mathcal{A}$, it satisfies 2α-AFS.*

Proof. We will first show the implication from approximate AFS to approximate core and thus let p denote a distribution satisfying α-AFS for an approval profile $\mathcal{A}$. To show that p satisfies $\alpha(1+\log n)$-core, we need an auxiliary concept called *proportional fairness (PF)*. We thus say that a distribution p satisfies β-PF for an approval profile $\mathcal{A}$ if

$$\text{PF}(p) := \max_{q\in\Delta(C)} \frac{1}{n}\sum_{i\in N}\frac{u_i(q)}{u_i(p)} \leq \beta.$$

Ebadian et al. [2024] have shown that if a distribution satisfies β-PF for a profile $\mathcal{A}$, then the distribution also satisfies β-core. Thus, it suffices to show that p satisfies $\alpha(1+\log(n))$-PF. To this end, we note analogous to [Ebadian et al., 2024] that

$$\text{PF}(p) = \max_{q\in\Delta(C)} \frac{1}{n}\sum_{i\in N}\frac{u_i(q)}{u_i(p)} = \max_{x\in C}\frac{1}{n}\sum_{i\in N}\frac{u_i(x)}{u_i(p)} = \max_{x\in C}\frac{1}{n}\sum_{i\in N_x(\mathcal{A})}\frac{1}{u_i(p)}. \tag{1}$$

Consider now an arbitrary candidate x. Since p satisfies α-AFS, it holds that

$$\frac{1}{|N_x(\mathcal{A})|}\sum_{i\in N_x(\mathcal{A})} u_i(p) \geq \frac{1}{\alpha}\cdot\frac{|N_x(\mathcal{A})|}{n}.$$

Since the average utility of voters in $N_x(\mathcal{A})$ is at least $\frac{1}{\alpha}\cdot\frac{|N_x(\mathcal{A})|}{n}$, there must exist a voter $i_1 \in N_x(\mathcal{A})$ who gets utility $u_{i_1}(p) \geq \frac{1}{\alpha}\cdot\frac{|N_x(\mathcal{A})|}{n}$. Consider now the group of voters $N_x(\mathcal{A})\setminus\{i_1\}$. By applying the α-AFS bound to this group, we derive that the average utility of voters in $N_x(\mathcal{A})\setminus\{i_1\}$ is at least $\frac{1}{\alpha}\cdot\frac{|N_x(\mathcal{A})|-1}{n}$. Hence, there is a voter i_2 such that $u_{i_2}(p) \geq \frac{1}{\alpha}\cdot\frac{|N_x(\mathcal{A})|-1}{n}$. By applying this argument iteratively, we see that there exists an order of the voters in $N_x(\mathcal{A}) = \{i_1,\dots,i_{|N_x(\mathcal{A})|}\}$

such that $u_{i_t}(p) \geq \frac{1}{\alpha} \cdot \frac{|N_x(\mathcal{A})|-(t-1)}{n}$ for each $t \in [|N_x(\mathcal{A})|]$. Using this bound in conjunction with Equation (1), we infer that

$$\begin{aligned}
\mathrm{PF}(p) &= \max_{x\in C} \frac{1}{n} \sum_{i\in N_x(\mathcal{A})} \frac{1}{u_i(p)} \\
&= \max_{x\in C} \frac{1}{n} \sum_{t=1}^{|N_x(\mathcal{A})|} \frac{1}{u_{i_t}(p)} \\
&\leq \max_{x\in C} \frac{1}{n} \sum_{t=1}^{|N_x(\mathcal{A})|} \frac{\alpha n}{|N_x(\mathcal{A})| - (t-1)} \\
&\leq \alpha \cdot \max_{x\in C} (1 + \log(|N_x(\mathcal{A})|)) \\
&\leq \alpha(1+\log(n)).
\end{aligned}$$

Hence, p satisfies $\alpha(1+\log(n))$-PF, and therefore also $\alpha(1+\log(n))$-core.

We now show that if p satisfies α-core for an approval profile $\mathcal{A}$, then it also satisfies 2α-AFS. For this, assume that p is a distribution that satisfies α-core for an approval profile $\mathcal{A}$. Moreover, we consider an arbitrary group of voters S with $\bigcap_{i\in S} A_i \neq \emptyset$, let $x \in \bigcap_{i\in S} A_i$, and let q denote the distribution with $q(x) = 1$. Because p satisfies α-core and $x \in \bigcap_{i\in S} A_i$, there exists a voter $i_1 \in S$ such that $\alpha \cdot u_{i_1}(p) \geq \frac{|S|}{n} \cdot u_{i_1}(q) = \frac{|S|}{n}$. Next, by analyzing the group $S \setminus \{i_1\}$, we derive from α-core that there is a voter $i_2 \in S \setminus \{i_1\}$ such that $\alpha \cdot u_{i_2}(p) \geq \frac{|S|-1}{n}$ as the group of voters $S \setminus \{i_1\}$ can otherwise benefit by deviating to q. By repeatedly applying this argument, it follows that there is an order $i_1, \dots, i_{|S|}$ of the voters in S such that $\alpha \cdot u_{i_t}(p) \geq \frac{|S|-(t-1)}{n}$ for each $t \in [|S|]$. We hence conclude that

$$\sum_{i\in S} u_i(p) = \sum_{t=1}^{|S|} u_{i_t}(p) \geq \frac{1}{\alpha} \sum_{t=1}^{|S|} \frac{|S|-(t-1)}{n} = \frac{1}{\alpha} \sum_{t=1}^{|S|} \frac{t}{n} = \frac{1}{\alpha} \cdot \frac{|S| \cdot (|S|+1)}{2n}.$$

This means that $\frac{2\alpha}{|S|} \sum_{i\in S} u_i(p) \geq \frac{|S|+1}{n} \geq \frac{|S|}{n}$. Since this bound holds for every group $S \subseteq N$ with $\bigcap_{i\in S} A_i \neq \emptyset$, p satisfies 2α-AFS for $\mathcal{A}$. □

3 Analysis of Existing Rules

We will now analyze known decomposable rules, namely the Nash product rule (NASH), the conditional utilitarian rule (CUT), the fair utilitarian rule (FUT), and the uncoordinated equal shares rule (UES), with respect to monotonicity, our population consistency axioms, and their approximation ratios to AFS and core. We refer to the subsequent subsections for the definitions of these rules and to Table 1 for a summary of the results of our analysis. In particular, this table shows that only NASH satisfies AFS and core, but it violates monotonicity and only satisfies weak population consistency. Conversely, UES satisfies monotonicity and even strong population consistency, but is only a $\Theta(n)$-approximation ratio for AFS and core.[5] Finally, CUT and FUT are also only $\Theta(n)$-approximations to AFS and core and violate even weak population consistency. These results demonstrate the need for new rules in order to simulatenously satisfy fairness, monotonicity, and demanding population consistency notions.

[5]For the upper bounds on the approximation ratio to AFS and core, we note that for all profiles $\mathcal{A}$ and all decomposable distributions p, it holds that $u_i(p) \geq \frac{1}{n}$ for all $i \in N$ and that $\frac{1}{|S|} \sum_{i\in S} u_i(p) \geq \frac{1}{n}$ for all $S \subseteq N$. It thus follows immediately that every decomposable distribution rule satisfies n-core and n-AFS.

Nash product rule (NASH)

The Nash product rule (NASH) selects a distribution p maximizing the Nash welfare $\prod_{i\in N} u_i(p)$ [Fain et al., 2016; Aziz et al., 2020]. Although the solution to this convex optimization problem may be irrational, it can be approximated efficiently and its solution is guaranteed to satisfy both AFS and core. However, Brandl et al. [2021] showed that NASH fails monotonicity, and we will show next that NASH violates ranked population consistency. More specifically, we will demonstrate that even if a candidate receives the maximal share in two disjoint elections for which NASH returns distributions with identical distribution rankings, the candidates share can decrease when combining the two elections.

Proposition 3.1. NASH *satisfies weak population consistency but fails ranked population consistency.*

Proof. First, for showing that NASH satisfies weak population consistency, let $\mathcal{A}$ and $\mathcal{A}'$ denote two voter-disjoint approval profiles such that NASH returns for both profiles the same distribution p. Moreover, let N denote the set of voters corresponding to $\mathcal{A}$ and N' denote the set of voters corresponding to $\mathcal{A}'$. By the definition of NASH, it holds that p maximizes both $\prod_{i\in N} u_i(p)$ and $\prod_{i\in N'} u_i(p)$. This immediately implies that p also maximizes $\prod_{i\in N\cup N'} u_i(p)$. Hence, NASH will choose p also for $\mathcal{A}+\mathcal{A}'$. (We note that, strictly speaking, this argument requires consistent tie-breaking in case multiple distributions maximize the Nash social welfare for $\mathcal{A}$ or $\mathcal{A}'$, but it is easy to see that the argument still holds when imposing consistent tie-breaking).

Next, to see that NASH fails ranked population consistency, consider the following profiles $\mathcal{A}$ and $\mathcal{A}'$.

$$\begin{array}{llllll} \mathcal{A}: & 3\text{: } \{a\} & 2: \{b,c\} & 2: \{c\} & 3\text{: } \{a,b\} & \\ \mathcal{A}': & 1\text{: } \{a\} & 1: \{b\} & 1: \{c\} & 2: \{a,b\} & 4: \{a,c\}. \end{array}$$

By solving the corresponding convex programs, it can be shown that NASH chooses for $\mathcal{A}$ the lottery p given by $p(a)=0.6$, $p(b)=0$, and $p(c)=0.4$ and for $\mathcal{A}'$ the lottery q with $q(a)\approx 0.608$, $q(b)\approx 0.157$, and $q(c)\approx 0.235$. Hence, we have that $\succsim^p = \succsim^q$. However, for $\mathcal{A}+\mathcal{A}'$, NASH chooses the lottery r with $r(a)\approx 0.558$, $r(b)\approx 0.137$, and $r(c)\approx 0.305$. Thus, it holds that $p(a)>r(a)$ and $q(a)>r(a)$, which shows that ranked population consistency is violated even for the candidate that obtains the maximal share in two disjoint profiles. □

Remark 1. In the proof of Proposition 3.1 we show that the share of a candidate can decrease even if it has the maximal share in two profiles and the distribution rankings for these profiles agree. It can also be shown that the share of a candidate can significantly increase when combining two profiles for which NASH chooses distributions with the same distribution ranking, even if the considered candidate gets a share of 0 in either of the two profiles. To see this, let $\mathcal{A}$ denote the profile where 50 voters report $\{a\}$, 49 voters report $\{b,c\}$, 49 voters report $\{c\}$, and 50 voters report $\{a,b\}$. For this profile, NASH selects the distribution p with $p(a)=\frac{50}{99}$, $p(b)=0$, and $p(c)=\frac{49}{99}$. Next, let $\mathcal{A}'$ denote the profile where 1 voter report $\{a\}$, 1 voter report $\{c\}$ and 200 voters report $\{a,b\}$. NASH chooses for this profile the distribution q with $q(a)=\frac{201}{202}$, $q(b)=0$, and $q(c)=\frac{1}{202}$. It is straightforward to verify that the distribution rankings $\succsim^p$ and $\succsim^q$ coincide as $a\succ^x c\succ^x b$ for both $x\in\{p,q\}$. Finally, in the combined profile $\mathcal{A}+\mathcal{A}'$ with 400 voters, NASH selects the lottery r with $r(a)=\frac{51}{100}$, $r(b)=\frac{29}{100}$, and $r(c)=\frac{20}{100}$, thus demonstrating another severe violation of RPC.

Conditional Utilitarian Rule (CUT)

First introduced by Duddy [2015], the conditional utilitarian rule (CUT) gives every voter control over a share of $\frac{1}{n}$ and assumes that each voter uniformly distributes this share across the candidates in A_i with the highest approval scores. More formally, we denote by $\mathsf{CUT}(\mathcal{A}, i) = \arg\max_{x \in A_i} |N_x(\mathcal{A})|$ the subset of voter i's approved candidates with maximal approval score. Then, CUT returns the distribution p defined by $p(x) = \frac{1}{n} \sum_{i \in N} \frac{\mathbb{I}[x \in \mathsf{CUT}(\mathcal{A}, i)]}{|\mathsf{CUT}(\mathcal{A}, i)|}$ for all $x \in C$, where $\mathbb{I}[x \in \mathsf{CUT}(\mathcal{A}, i)]$ is the indicator function that is 1 if $x \in \mathsf{CUT}(\mathcal{A}, i)$ and 0 otherwise. CUT is known to satisfy strategyproofness and thus also monotonicity [Brandl et al., 2022]. However, as we show in the next two propositions, CUT fails even weak population consistency, and it is only a $\Theta(n)$-approximation to AFS and core.

Proposition 3.2. CUT *fails weak population consistency.*

Proof. We consider the following two profiles $\mathcal{A}$ and $\mathcal{A}'$, both of which contain 10 voters and 3 candidates $C = \{a, b, c\}$.

$$\begin{array}{lllll} \mathcal{A}: & 2: \{a\} & 4: \{a, c\} & 1: \{c\} & 3: \{b\} \\ \mathcal{A}': & 6: \{a\} & 2: \{b\} & 1: \{b, c\} & 1: \{c\} \end{array}$$

In both profiles, candidate a is approved by the most voters, so every voter who approves a sends his $\frac{1}{n}$ share to this candidate. Consequently, CUT assigns a share of $\frac{6}{10}$ to a in both $\mathcal{A}$ and $\mathcal{A}'$. Now, in $\mathcal{A}$, all of the remaining four voters approve only a single candidate, so we immediately get that CUT chooses the distribution p with $p(a) = \frac{6}{10}$, $p(b) = \frac{3}{10}$, and $p(c) = \frac{1}{10}$. On the other hand, for $\mathcal{A}'$, we note that three voters approve b but only two voters approve c, so the single voter approving $\{b, c\}$ sends his share to b. So, the outcome for $\mathcal{A}'$ is again the distribution p with $p(a) = \frac{6}{10}$, $p(b) = \frac{3}{10}$, and $p(c) = \frac{1}{10}$. Finally, for $\mathcal{A} + \mathcal{A}'$, CUT will first assign a share of $\frac{12}{20} = \frac{6}{10}$ to a because it is approved by the most voters. However, in $\mathcal{A} + \mathcal{A}'$, c is approved by 7 voters whereas b is only approved by 6 voters. Hence, the voter approving $\{b, c\}$ sends his share to c and c gets a share of $\frac{3}{20} > \frac{1}{10}$. This proves that CUT violates weak population consistency as it chooses the same distribution for $\mathcal{A}$ and $\mathcal{A}'$ but not for $\mathcal{A} + \mathcal{A}'$. □

Proposition 3.3. CUT *does not satisfy α-AFS or α-core for any $\alpha < \frac{n}{2} - 1$.*

Proof. Consider the following approval profile $\mathcal{A}$ for an even number of voters n, which are partitioned into two sets $S = \{1, \dots, \frac{n}{2} - 1\}$ and $S' = \{\frac{n}{2}, \dots, n-2\}$, and $\frac{n}{2}$ candidates $C = \{x^*, x_1, \dots, x_{n/2}\}$.

- Each voter $i \in S$ reports $A_i = \{x_i, x_{n/2}\}$.
- Each voter $i \in S'$ reports $A_i = \{x^*, x_1, \dots, x_{n/2-1}\}$.
- Voters n and $n-1$ report $A_{n-1} = A_n = \{x^*\}$.

Now, let p denote the distribution returned by CUT for this instance. Candidate x^* is approved by the most voters, so all these voters send their $\frac{1}{n}$ share to x^* and $p(x^*) = \frac{1}{2} + \frac{1}{n}$. Next, we consider the group of voters S. Each voter $i \in S$ will allocate his $\frac{1}{n}$ share to x_i since $|N_{x_i}(\mathcal{A})| > |N_{x_{n/2}}(\mathcal{A})|$ for all $i \in S$. Thus, $p(x_i) = \frac{1}{n}$ for all $i \in S$ and $p(x_{n/2}) = 0$. Together, we have that

$$\frac{1}{|S|} \sum_{i \in S} u_i(p) = \frac{1}{|S|} \sum_{i \in S} \sum_{x \in A_i} p(x) = \frac{1}{|S|} \cdot \frac{|S|}{n} = \frac{2}{n-2} \cdot \frac{|S|}{n}$$

Since $\bigcap_{i \in S} A_i = \{x_{n/2}\} \neq \emptyset$, this means that CUT does not satisfy α-AFS for any $\alpha < \frac{n}{2} - 1$.

Next, for core, consider the distribution q with $q(x_{n/2}) = 1$. Then, for all $i \in S$,

$$\frac{u_i(q)}{u_i(p)} \cdot \frac{|S|}{n} = \frac{1}{1/n} \cdot \frac{n/2 - 1}{n} = \frac{n}{2} - 1.$$

It follows that CUT cannot satisfy α-core for any $\alpha < \frac{n}{2} - 1$. □

Fair Utilitarian Rule (FUT)

First introduced by Bogomolnaia et al. [2002] and later rediscovered by Brandl et al. [2021], FUT dynamically constructs weights for each voter and returns a distribution which maximizes the resulting weighted utilitarian welfare. In more detail, the voters start with unit weights $\lambda_i = 1$ for all $i \in N$. Then, FUT identifies the set of candidates X_1 that maximize $\sum_{i \in N_x(\mathcal{A})} \lambda_i$ and sets $t = \sum_{i \in N_x(\mathcal{A})} \lambda_i$. Every voter i with $A_i \cap X_1 \neq \emptyset$ distributes their budget of $\frac{1}{n}$ uniformly among the candidates in $A_i \cap X_1$ and their λ_i is fixed. Then, the weights λ_i of all other voters increase at a common rate until a candidate obtains a score of t, i.e., until $\sum_{i \in N_x(\mathcal{A})} \lambda_i = t$ for some $x \in C \setminus X_1$. We then identify the set of candidates $X_2 \subseteq C \setminus X_1$ which each have a total weight of t, and every voter $i \in N$ who did not spend his budget yet and approves at least one candidate in X_2 uniformly distributes his $\frac{1}{n}$ share among the candidates $A_i \cap X_2$. FUT then again increases the weights of voters who did not spend their $\frac{1}{n}$ share and repeats this process until all voters have allocated their share to some candidates.

Just like CUT, FUT satisfies monotonicity [Brandl et al., 2021], but fails weak population consistency and only approximates AFS and core within a factor in $\Theta(n)$.

Proposition 3.4. FUT *fails weak population consistency.*

Proof. We consider the following two profiles $\mathcal{A}$ and $\mathcal{A}'$, both of which contain 14 voters and 3 candidates $C = \{a, b, c\}$.

$$\begin{array}{llllll} \mathcal{A}: & 3: \{a\} & 5: \{a,c\} & 1: \{c\} & 1: \{b,c\} & 4: \{b\} \\ \mathcal{A}': & 6: \{a\} & 2: \{a,b\} & 4: \{b\} & 2: \{c\} & \end{array}$$

For $\mathcal{A}$, FUT first assigns a share of $\frac{8}{14}$ to a as this candidate is approved by 8 voters. We then start increasing the weight of the remaining 6 voters and note that the total score of c will reach 8 when $\lambda_i = \frac{3}{2}$ and the total score of b is 8 when $\lambda_i = \frac{8}{5}$. Hence, FUT next assigns a share of $\frac{2}{14}$ to c and the remaining $\frac{4}{14}$ are assigned to b. This means that FUT chooses for $\mathcal{A}$ the distribution p with $p(a) = \frac{8}{14}$, $p(b) = \frac{4}{14}$, and $p(c) = \frac{2}{14}$. Next, for $\mathcal{A}'$, FUT again assigns a share of $\frac{8}{14}$ to a. Since all remaining voters only approve a single candidate and FUT is decomposable, we derive that FUT chooses for $\mathcal{A}'$ again the distribution p with $p(a) = \frac{8}{14}$, $p(b) = \frac{4}{14}$, and $p(c) = \frac{2}{14}$. Finally, in the joint profile $\mathcal{A} + \mathcal{A}'$, FUT again assigns a share of $\frac{8}{14}$ to a since it is approved by 16 of 28 voters. Next, we start increasing the weights and note that c reaches a total weight of 16 when $\lambda_i = \frac{11}{4}$ and b when $\lambda_i = \frac{14}{9}$. Hence, the 9 voters approving b now send their portion to this candidate, so b is assigned a share of $\frac{9}{28} > \frac{4}{14}$. This proves that FUT fails weak population consistency because it chooses the distribution p for both $\mathcal{A}$ and $\mathcal{A}'$ but not for $\mathcal{A} + \mathcal{A}'$. □

Proposition 3.5. FUT *does not satisfy α-AFS or α-core for any* $\alpha < \frac{n}{3} - 1$

Proof. Assume that n is divisible by 3 and consider the following approval profile $\mathcal{A}$ for the candidates $C = \{x^*, x_1, \dots, x_{n/3-1}, y\}$.

- Each voter $i \in \{1, \dots, \frac{n}{3} - 1\}$ reports $\{x^*, x_i\}$.
- Voters $\frac{n}{3}$ and $\frac{n}{3} + 1$ report $\{y\}$.
- Each voter $i \in \{\frac{n}{3} + 2, \dots, n\}$ reports $\{x_1, \dots, x_{n/3-1}, y\}$.

Observe that $|N_y(\mathcal{A})| = \frac{2n}{3} + 1 > |N_x(\mathcal{A})|$ for all $x \in C \setminus \{y\}$. Hence, FUT selects candidate y first and sets $p(y) = \frac{2}{3} + \frac{1}{n}$. Moreover, the weights of the voters in $N_y(\mathcal{A})$ are now fixed at 1. Next, the weights of all other agents uniformly increase until some candidate has score of $\frac{2n}{3}+1$. This occurs when $\lambda_i = 2$ for all $i \in N \setminus N_y(\mathcal{A})$ because $\sum_{i\in N_{x_i}(\mathcal{A})} \lambda_i = \frac{2n}{3} - 1 + \lambda_i$ for each $x_i \in \{x_1, \dots, x_{x/3-1}\}$ and $\sum_{i\in N_{x^*}(\mathcal{A})} \lambda_i = (\frac{n}{3}-1)\lambda_i$. Thus, each candidate x_i is assigned a share of $p(x_i) = \frac{1}{n}$ and x^* is not assigned any budget. Since all voter in $S = \{1, \dots, \frac{n}{3}-1\}$ approve x^*, we have that

$$\frac{1}{|S|}\sum_{i\in S} u_i(p) = \frac{1}{\frac{n}{3}-1}\cdot(\frac{n}{3}-1)\cdot\frac{1}{n} = \frac{1}{n}.$$

On the other hand, α-AFS requires that $\frac{1}{|S|}\sum_{i\in S} u_i(p) \geq \frac{1}{\alpha}\cdot\frac{|S|}{n} = \frac{1}{\alpha}(\frac{1}{3}-\frac{1}{n})$. Hence, we conclude that FUT fails α-AFS for every $\alpha < \frac{n}{3}-1$ when there are n voters.

For the α-core lower bound, we consider the distribution q with $q(x^*) = 1$. Then, we infer for every voter $i \in S$ that $u_i(p) = \frac{1}{n}$, but $\frac{|S|}{n}\cdot u_i(q) = \frac{1}{3}-\frac{1}{n}$. This shows that FUT fails α-AFS for every $\alpha < \frac{n}{3}-1$. □

Uncoordinated Equal Shares Rule (UES)

UES assumes that each voter spends his $\frac{1}{n}$ share uniformly among his approved alternatives, i.e., it chooses the distribution p given by $p(x) = \sum_{i\in N_x(\mathcal{A})}\frac{1}{n|A_i|}$ for all $x \in C$ [Michorzewski et al., 2020]. Perhaps surprisingly, it can be shown that UES satisfies both monotonicity and even strong population consistency. However, the approximation ratio of UES to AFS and core is in $\Theta(n)$.

Proposition 3.6. UES *does not satisfy α-AFS or α-core for any $\alpha < \frac{n}{2}$.*

Proof. Consider the following approval profile $\mathcal{A}$ for n voters and $1 + n(n-1)$ candidates. Each voter i approves candidate x^*, $n-1$ additional candidates, and the sets $A_i \setminus \{x^*\}$ are pairwise disjoint. More formally, when denoting the set of candidates C by $C = \{x^*\} \cup \{x^i_j \colon i \in [n], j \in [n-1]\}$, then the approval ballot of each voter i is given by $A_i = \{x^*\} \cup \{x^i_j \colon j \in [n-1]\}$. Now, let p denote the distribution returned by UES. It holds that $p(x^*) = \frac{1}{n}$ and that $p(x^j_i) = \frac{1}{n^2}$, which implies that $u_i(p) = \frac{2n-1}{n^2}$ for each voter $i \in N$. Consequently, we derive that $\frac{1}{n}\sum_{i\in N} u_i(p) = \frac{2n-1}{n^2}$. On the other hand, α-AFS requires that $\frac{1}{n}\sum_{i\in N} u_i(p) \geq \frac{1}{\alpha}$. Hence, it follows that UES only satisfies α-AFS for $\alpha \geq \frac{n^2}{2n-1} > \frac{n}{2}$.

Moreover, it is also easy to see that the utility of each agent improves by a factor $\frac{n^2}{2n-1}$ when moving from p to the distribution q with $q(x^*) = 1$, which gives the corresponding lower bound for α-core. □

Remark 2. The proof of Proposition 3.6 uses roughly n^2 candidates. By using more candidates, it is straightforward to adapt the proof to give higher bounds on the approximation ratio of UES to α-AFS and α-core. Conversely, when we restrict ourselves to instances with $m \leq n$ candidates, one can still show a lower bound of $\Omega(\sqrt{n})$ for the approximation ratio of UES to AFS and core.

4 Sequential Payment Rules

We will now introduce the central class of rules in this paper which we call sequential payment rules. The idea of these rules is that every voter has a virtual budget of 1 and that voters sequentially spend this budget on the candidates. In particular, in each round, voters indicate, depending on their remaining budget, how much they are willing to spend on the next candidate in their approval set. The candidate that maximizes the total payment willingness is then chosen and the

Algorithm 1: Sequential payment rules

input : An approval profile $\mathcal{A}$
output: A distribution $p \in \Delta(C)$

$X \leftarrow \emptyset$;
for $j \in [m]$ **do**
  $x \leftarrow \arg\max_{x \in C \setminus X} \sum_{i \in N_x(\mathcal{A})} \pi(|A_i|, |A_i \cap X| + 1)$;
  $X \leftarrow X \cup \{x\}$;
  $p(x) \leftarrow \frac{1}{n} \sum_{i \in N_x(\mathcal{A})} \pi(|A_i|, |A_i \cap X| + 1)$;
return p

voters who are willing to pay for this candidate send the corresponding amount to this candidate. Finally, we remove this candidate from consideration, and repeat this process until all voters have spent all their budget.

Our key ingredient to formalize these rules are *payment willingness functions* $\pi : \mathbb{N} \times [0,1] \to [0,1]$, which specify how much each voter is willing to spend on the next candidate depending on the number of his approved candidates and his remaining budget. We note that, from a technical point of view, payment willingness functions can also be seen as functions of the type $\pi : \mathbb{N} \times \mathbb{N} \to [0,1]$ as it suffices to know how many times a voter paid for candidates to infer his remaining budget and thus his next payment willingness. In more detail, when defining b_t as the remaining budget of a voter i after he has spent parts of his budget on t candidates, then $\pi(|A_i|, t+1) = \pi(|A_i|, b_t)$ and $b_{t+1} = b_t - \pi(|A_i|, t+1)$. By repeatedly applying this reasoning starting from $b_0 = 1$, we can substitute the remaining budget with the number of payments made by the voter. Since this description is easier to handle from a technical perspective, we will view payment willingness functions from now on as mappings from $\mathbb{N} \times \mathbb{N}$ to $[0,1]$ and interpret $\pi(|A_i|, t)$ as the payment willingness of voter i for his t-th candidate. Finally, payment willingness functions need to satisfy two more conditions:

(1) Voters have to spend their whole budget: $\sum_{j=1}^{|A_i|} \pi(|A_i|, j) = 1$ for all approval ballots A_i. Since $\pi(|A_i|, j)$ is always non-negative, this implies that the payment willingness of a voter is 0 if he spent his entire budget.

(2) The payment willingness is weakly decreasing in the budget: $\pi(|A_i|, j) \geq \pi(|A_i|, j+1)$ for all approval ballots A_i and all integers $j \in [|A_i| - 1]$.

Each sequential payment rule f is fully specified by its payment willingness function π. In particular, when letting X denote the set of candidates to whom we already allocated their share, then the total payment willingness for a candidate $x \in C \setminus X$ is $\Pi(\mathcal{A}, x, X) = \sum_{i \in N_x(\mathcal{A})} \pi(|A_i|, |A_i \cap X| + 1)$. A sequential payment rule f then works as follows: starting from $X = \emptyset$, we identify in each round the candidate $x \in C \setminus X$ that maximizes $\Pi(\mathcal{A}, x, X)$ (with ties broken lexicographically), assign a share of $\frac{1}{n}\Pi(\mathcal{A}, x, X)$ to this candidate, and add x to the set X. We then run this process for m rounds to compute the final distribution. A pseudocode description of sequential payment rules is given in Algorithm 1.

In this paper, we will often focus on specific sequential payment functions. In particular, we will discuss in Section 4.1 the *maximum payment rule (*MP*)*, where voters are willing to spend their whole budget on the first approved candidate. More formally, MP is defined by the payment willingness function π with $\pi(x, 1) = 1$ and $\pi(x, y) = 0$ for all $x \in \mathbb{N}$ and $y \in [x] \setminus \{1\}$. We note that this rule can be seen as an analog of the majoritarian portioning rule that is studied in approval-based apportionment [Speroni di Fenizio and Gewurz, 2019; Brill et al., 2024]. Another natural rule in our class is UES, which is defined by the payment willingness function π with $\pi(x,y) = \frac{1}{x}$ for all $x \in \mathbb{N}$ and $y \in [x]$. Note here that, if the payment willingness function of every voter is constant, the order in which we pick the candidates does not matter. Lastly, MP and UES can be seen as the extremes of the class of *γ-multiplicative sequential payment rules* (γ–MSP). The idea of these rules is that the payment willingness of a voter is always discounted by a factor of $\gamma \in [0,1]$ if he pays for a candidate. More formally, the payment willingness function π of γ–MSP is defined by the equations $\sum_{y\in[x]} \pi(x,y) = 1$ for all $x \in \mathbb{N}$ and $\pi(x,y+1) = \gamma\pi(x,y)$ for all $x \in \mathbb{N}$ and $y \in [x-1]$. For $\gamma \in (0,1]$, this means that $\pi(x,y) = \frac{\gamma^y}{\sum_{i\in[x]}\gamma^i}$. It can be checked that the maximum payment rule is equivalent to 0–MSP and that UES is 1–MSP.

Example 4.1 (Sequential payment rules)**.** To further illustrate sequential payment rules, we will next compute MP and $\frac{1}{3}$–MSP for the profile $\mathcal{A}$ shown below.

$$\mathcal{A}: \quad 4\colon \{a,b\} \quad 4\colon \{a\} \quad 2\colon \{b,c\} \quad 1 : \{c,d\} \quad 1 : \{d\}$$

In this profile, the total payment willingness for each candidate according to MP initially is $\Pi(\mathcal{A},a,\emptyset) = 8, \Pi(\mathcal{A},b,\emptyset) = 6, \Pi(\mathcal{A},c,\emptyset) = 3$, and $\Pi(\mathcal{A},d,\emptyset) = 2$. Hence, MP assigns in the first step a share of $\frac{8}{12}$ to a. After this, all voters approving a have no budget left, so $\Pi(\mathcal{A},b,\{a\}) = 2$, $\Pi(\mathcal{A},c,\{a\}) = 3$, and $\Pi(\mathcal{A},d,\{a\}) = 2$. Consequently, MP assigns $\frac{3}{12}$ to c. In the third step, the total payment willingness for b is $\Pi(\mathcal{A},b,\{a,c\}) = 0$ and for d is $\Pi(\mathcal{A},d,\{a,c\}) = 1$, so MP assigns a share of $\frac{1}{12}$ to d. Since all voters have now spent their entire budget, no part of the budget will be allocated to b and MP selects the distribution p with $p(a) = \frac{8}{12}$, $p(b) = 0$, $p(c) = \frac{3}{12}$, and $p(d) = \frac{1}{12}$.

Using analogous computations for $\frac{1}{3}$–MSP shows that this rule processes the candidates in the order a,b,d,c. In more detail, note that the payment willingness function π of $\frac{1}{3}$–MSP satisfies that $\pi(1,1) = 1$, $\pi(2,1) = \frac{3}{4}$, and $\pi(2,2) = \frac{1}{4}$. At the steps the candidates are assigned their shares, the total payment willingness for them is $\Pi(\mathcal{A},a,\emptyset) = 4+4\cdot\frac{3}{4} = 7$, $\Pi(\mathcal{A},b,\{a\}) = 4\cdot\frac{1}{4}+2\cdot\frac{3}{4} = \frac{5}{2}$, $\Pi(\mathcal{A},d,\{a,b\}) = 1+1\cdot\frac{3}{4} = \frac{7}{4}$, and $\Pi(\mathcal{A},c,\{a,b,d\}) = \frac{3}{4}$. Hence, $\frac{1}{3}$–MSP returns for $\mathcal{A}$ the distribution q with $q(a) = \frac{28}{48}$, $q(b) = \frac{10}{48}$, $q(c) = \frac{3}{48}$ and $q(d) = \frac{7}{48}$.

Next, we will prove that all sequential payment rules satisfy RPC, thus giving a strong argument in favor of this class. We moreover note that sequential payment rules are decomposable by definition. We denote subsequently the strict part of the distribution ranking $\succsim^p$ for a given distribution p by $\succ^p$ (i.e., $x \succ^p y$ if and only if $p(x) > p(y)$) and the indifference part by $\sim^p$ (i.e., $x \sim^p y$ if and only if $p(x) = p(y)$).

Proposition 4.2. *Every sequential payment rule satisfies RPC.*

Proof. Let f denote a sequential payment rule and let π denote its corresponding payment willingness function. Moreover, let $\mathcal{A}$ and $\mathcal{A}'$ denote two voter-disjoint profiles, let $p = f(\mathcal{A})$ and $q = f(\mathcal{A}')$ be the distributions chosen by f, and suppose that $\succsim^p = \succsim^q$. We need to show that $\min\{f(\mathcal{A},x), f(\mathcal{A}',x)\} \leq f(\mathcal{A}+\mathcal{A}',x) \leq \max\{f(\mathcal{A},x), f(\mathcal{A}',x)\}$ for all $x \in C$. To this end, let $x_1,\dots,x_m$ and $x'_1,\dots,x'_m$ denote the sequences according to which f assigns the shares to the candidates in $\mathcal{A}$ and $\mathcal{A}'$, respectively. As a first step, we will show that $x_i = x'_i$ for all i and that f processes the candidates in $\mathcal{A}+\mathcal{A}'$ in the same order as well. For this, we first note that

$$\Pi(\mathcal{A},x_i,\{x_1,\dots,x_{i-1}\}) \geq \Pi(\mathcal{A},x_{i+1},\{x_1,\dots,x_{i-1}\}) \geq \Pi(\mathcal{A},x_{i+1},\{x_1,\dots,x_i\})$$

for all $i \in \{1, \dots, m-1\}$, where the first inequality follows from the fact that x_i maximizes the total payment willingness given that the candidates $\{x_1, \dots, x_{i-1}\}$ are already assigned their shares, and the second one holds because the voters' payment willingness is weakly decreasing in their budget. This means that $p(x_1) \geq p(x_2) \geq \dots \geq p(x_m)$ and an analogous argument for q shows that $q(x'_1) \geq q(x'_2) \geq \dots \geq q(x'_m)$.

Now, if $p(x_i) > p(x_j)$, then $x_i \succ^p x_j$ by definition. Since $\succsim^p = \succsim^q$, it thus also follows that $x_i \succ^q x_j$ and thus also $q(x_i) > q(x_j)$. By our previous insight, this means that x_i is processed before x_j for both $\mathcal{A}$ and $\mathcal{A}'$. Next, consider the case that $p(x_i) = p(x_j)$ for some $i, j \in [m]$ with $i < j$, which means that $\Pi(\mathcal{A}, x_i, \{x_1, \dots, x_{i-1}\}) = \Pi(\mathcal{A}, x_j, \{x_1, \dots, x_{j-1}\})$. By using the definition of x_i and the fact that the payment willingness is decreasing in the budget, we infer from this that $\Pi(\mathcal{A}, x_i, \{x_1, \dots, x_{i-1}\}) = \Pi(\mathcal{A}, x_j, \{x_1, \dots, x_{i-1}\})$, too. Hence, x_i is assigned its share of the budget before x_j in $\mathcal{A}$ because of the lexicographic tie-breaking. Moreover, $p(x_i) = p(x_j)$ implies that $x_i \sim^p x_j$. Since $\succsim^p = \succsim^q$, this means that $x_i \sim^q x_j$ and thus $q(x_i) = q(x_j)$. Because our tie-breaking favors x_i over x_j, this means again that x_i is processed before x_j in $\mathcal{A}'$. By combining our observations so far, we conclude now that $x_i = x'_i$ for all $i \in [m]$, i.e., f processes the candidates in the same order for $\mathcal{A}$ and $\mathcal{A}'$.

Next, we observe that $\Pi(\mathcal{A} + \mathcal{A}', x, Y) = \Pi(\mathcal{A}, x, Y) + \Pi(\mathcal{A}', x, Y)$ for all $x \in C$, $Y \subseteq C \setminus \{x\}$. Since $x_i = x'_i$ for all $i \in [m]$, it follows for all such x_i and all $y \notin \{x_1, \dots, x_{i-1}\}$ that

$$\begin{aligned}\Pi(\mathcal{A} + \mathcal{A}', x_i, \{x_1, \dots, x_{i-1}\}) &= \Pi(\mathcal{A}, x_i, \{x_1, \dots, x_{i-1}\}) + \Pi(\mathcal{A}', x_i, \{x_1, \dots, x_{i-1}\}) \\ &\geq \Pi(\mathcal{A}, y, \{x_1, \dots, x_{i-1}\}) + \Pi(\mathcal{A}', y, \{x_1, \dots, x_{i-1}\}) \\ &= \Pi(\mathcal{A} + \mathcal{A}', y, \{x_1, \dots, x_{i-1}\}).\end{aligned}$$

If this inequality is tight, it holds that $\Pi(\mathcal{A}, x_i, \{x_1, \dots, x_{i-1}\}) = \Pi(\mathcal{A}, y, \{x_1, \dots, x_{i-1}\})$ and that $\Pi(\mathcal{A}', x_i, \{x_1, \dots, x_{i-1}\}) = \Pi(\mathcal{A}', y, \{x_1, \dots, x_{i-1}\})$. We then infer that x_i is lexicographically favored to y, so x_i will be processed before y in $\mathcal{A} + \mathcal{A}'$. It thus follows for the sequence $x''_1, \dots, x''_m$ according to which f processes the candidates in $\mathcal{A} + \mathcal{A}'$ that $x''_i = x_i$ for all $i \in [m]$.

Finally, assume there are n voters in $\mathcal{A}$ and n' in voters in $\mathcal{A}'$. By our insights so far, we infer for every candidate $x_i \in C$ that

$$\begin{aligned}f(\mathcal{A} + \mathcal{A}', x_i) &= \frac{1}{n+n'} \Pi(\mathcal{A} + \mathcal{A}', x_i, \{x_1, \dots, x_{i-1}\}) \\ &= \frac{1}{n+n'} (\Pi(\mathcal{A}, x_i, \{x_1, \dots, x_{i-1}\}) + \Pi(\mathcal{A}', x_i, \{x_1, \dots, x_{i-1}\})) \\ &= \frac{n}{n+n'} \cdot \frac{1}{n} \cdot \Pi(\mathcal{A}, x_i, \{x_1, \dots, x_{i-1}\}) + \frac{n'}{n+n'} \cdot \frac{1}{n'} \cdot \Pi(\mathcal{A}', x_i, \{x_1, \dots, x_{i-1}\}) \\ &= \frac{n}{n+n'} f(\mathcal{A}, x_i) + \frac{n'}{n+n'} f(\mathcal{A}', x_i).\end{aligned}$$

This shows that $f(\mathcal{A} + \mathcal{A}', x_i)$ is a convex combination of $f(\mathcal{A}, x_i)$ and $f(\mathcal{A}', x_i)$, so we derive now that $\min\{f(\mathcal{A}, x_i), f(\mathcal{A}', x_i)\} \leq f(\mathcal{A} + \mathcal{A}', x_i) \leq \max\{f(\mathcal{A}, x_i), f(\mathcal{A}', x_i)\}$ for all candidates $x_i \in C$. This completes the proof that f satisfies RPC. □

Remark 3. The basic idea of sequential payment rules is related to that of sequential Thiele rules, which are studied in the context of approval-based committee elections [Lackner and Skowron, 2023; Dong and Lederer, 2023]. In more detail, sequential Thiele rules also sequentially select candidates that maximize a score. However, sequential Thiele rules only add these candidates to the winning committee, whereas sequential payment rules assign shares of the budget depending on the total payment willingness. This significantly affects the behavior of the rules: for instance, the payment willingness function of MP corresponds to the score function of Chamberlin-Courant approval voting, but the properties of both rules are very different.

4.1 The Maximum Payment Rule

We will next discuss the maximum payment rule in more detail and show that it satisfies monotonicity and strong fairness guarantees. In combination with its simplicity, we believe this makes a strong case for using MP in practice.

We will first prove that MP is monotonic and that it is in fact the only monotonic sequential payment rule other than UES. We note that this result both underlines the appeal of MP and demonstrates how challenging monotonicity is in budget division with dichotomous preferences.

Theorem 4.3. *A sequential payment function is monotonic if and only if it is* MP *or* UES*.*

Proof. ($\Longleftarrow$) We first show that MP and UES satisfy monotonicity. For UES, it suffices to observe that the share of a candidate x in a profile $\mathcal{A}$ is $\frac{1}{n}\sum_{i\in N_x(\mathcal{A})}\frac{1}{|A_i|}$. Since this term is strictly increasing if a voter additionally approves candidate x, UES is monotonic.

We now turn to MP. Let $\mathcal{A}$ and $\mathcal{A}'$ be two profiles, i a voter, and x^* a candidate such that $\mathcal{A}'$ is derived from $\mathcal{A}$ by adding x^* to the approval ballot of voter i. We need to show that $q(x^*) \geq p(x^*)$ for the distributions $p = \mathsf{MP}(\mathcal{A})$ and $q = \mathsf{MP}(\mathcal{A}')$. To this end, we observe for all profiles $\bar{\mathcal{A}}$, candidates $x \in C$, and sets $Y \subseteq C \setminus \{x\}$ that $\Pi(\bar{\mathcal{A}}, x, Y) = |\{i \in N\colon x \in A_i \wedge A_i \cap Y = \emptyset\}|$. This means that $\Pi(\mathcal{A}, x, Y) = \Pi(\mathcal{A}', x, Y)$ for all $x \in C \setminus \{x^*\}$ and $Y \subseteq C \setminus \{x, x^*\}$, and that $\Pi(\mathcal{A}, x^*, Y) \leq \Pi(\mathcal{A}', x^*, Y)$ for all $Y \subseteq C \setminus \{x^*\}$. Now, let $x_1, \dots, x_m$ and $x'_1 \dots, x'_m$ denote the sequences according to which MP assigns the shares to the candidates in $\mathcal{A}$ and $\mathcal{A}'$, and let ℓ and k denote the indices such that $x_\ell = x^*$ and $x'_k = x^*$. First, we observe that $\ell \geq k$ because the payment willingness for x^* in $\mathcal{A}'$ is always weakly higher than in $\mathcal{A}$. Moreover, since $\Pi(\mathcal{A}, x, Y) = \Pi(\mathcal{A}', x, Y)$ for all $x \in C \setminus \{x^*\}$ and $Y \subseteq C \setminus \{x, x^*\}$, it follows that $x_i = x'_i$ for all $i < k$. This means that $\{x'_1, \dots, x'_{k-1}\} = \{x_1, \dots, x_{k-1}\} \subseteq \{x_1, \dots, x_{\ell-1}\}$, so we now conclude that

$$\Pi(\mathcal{A}', x^*, \{x'_1, \dots, x'_{k-1}\}) \geq \Pi(\mathcal{A}, x^*, \{x_1, \dots, x_{k-1}\}) \geq \Pi(\mathcal{A}, x^*, \{x_1, \dots, x_{\ell-1}\}).$$

This shows that MP satisfies monotonicity because $q(x^*) = \frac{1}{n}\Pi(\mathcal{A}', x^*, \{x'_1, \dots, x'_{k-1}\})$ and $p(x^*) = \frac{1}{n}\Pi(\mathcal{A}, x^*, \{x_1, \dots, x_{\ell-1}\})$.

($\Longrightarrow$) Let f denote a sequential payment rule other than MP and UES and let π denote its payment willingness function. Since f is not MP, there is an integer ℓ_1 such that $\pi(\ell_1, 1) = 1 > \pi(\ell_1 + 1, 1)$. Moreover, since f is not UES, there is another pair of indices ℓ_2, k with $k \in [\ell_2 - 1]$ such that $\pi(\ell_2, k) \neq \pi(\ell_2, k+1)$. Because payment willingness functions are non-increasing in their second parameter, this means that $\pi(\ell_2, k) > \pi(\ell_2, k+1)$. Based on these parameters, we will now construct a profile where f fails monotonicity. To this end, let $\ell = \max(\ell_1, \ell_2)$. The rough idea of our construction is as follows: there will be three "active" candidates x, y, z and $2\ell - 2$ inactive candidates. In the original profile, f will process the active candidates in the order y, x, z. By contrast, if some voters additionally approve x, this will decrease the payment willingness for y as $\pi(\ell_1, 1) > \pi(\ell_1 + 1, 1)$. In particular, z will then be chosen as the first candidate among x, y, and z. However, choosing z before x will reduce the payment willingness towards x as $\pi(\ell_2, k) > \pi(\ell_2, k+1)$, thus resulting in a smaller share for x despite the fact that it gained more approvals. To formalize this, we will use $2\ell + 1$ candidates which are partitioned into three groups $B = \{b_1, \dots, b_{\ell-1}\}$, $D = \{d_1, \dots, d_{\ell-1}\}$, and x, y, z. For convenience, we let by $B(s) = \{b_1, \dots, b_s\}$ and $D(s) = \{d_1, \dots, d_s\}$ denote subsets of B and D of size s. Then, we consider the following approval profile $\mathcal{A}$.

- Let $\delta_1 = \pi(\ell_1, 1) - \pi(\ell_1 + 1, 1)$ and define t_1 as an integer such that $t_1\delta_1 > 1$. We add t_1 voters approving the set $D(\ell_1 - 1) \cup \{y\}$ and we call this set of voters N_1.
- Let $\delta_2 = \pi(\ell_2, k) - \pi(\ell_2, k+1)$ and choose an integer t_2 such that $t_2\delta_2 > t_1\pi(\ell_1 + 1, 1)$. We add t_2 voters who report the set $B(k-1) \cup D(\ell_2 - k - 1) \cup \{x, z\}$.
- Choose an integer t_3 such that $t_3\delta_2 > t_1\pi(\ell_1 + 1, 1) + 1$. We add t_3 voters who approve the set $B(k-1) \cup D(\ell_2 - k - 1) \cup \{y, z\}$.
- For each $w \in \{x, y, z\}$, we add at least $t_1 + t_2 + t_3$ voters who only approve w to guarantee that the candidates in $\{x, y, z\}$ will be assigned their shares before the candidates in D. This works because the definition of payment willingness functions require that $\pi(1, 1) = 1$. Moreover, we can add more such voters to ensure that $\Pi(\mathcal{A}, z, B) + 1 \geq \Pi(\mathcal{A}, y, B) > \Pi(\mathcal{A}, z, B)$ and $\Pi(\mathcal{A}, x, B) + t_1 \cdot \pi(\ell_1 + 1, 1) + 1 \geq \Pi(\mathcal{A}, z, B) > \Pi(\mathcal{A}, x, B) + t_1 \cdot \pi(\ell_1 + 1, 1)$.
- Let T denote the total number of voters introduced so far. For each $w \in B$, we add $2T$ voters who only approve w. This ensures that these candidates will be assigned their shares before all other candidates.

By construction, f will first assign shares to the candidates in B for the profile $\mathcal{A}$. Moreover, since we add sufficiently many voters who only approve y, this candidate will be processed next. Let n_x and n_z denote the number of voters who report $\{x\}$ and $\{z\}$, respectively. Now, it holds for the payment willingness for x and z that

$$\begin{aligned}
\Pi(\mathcal{A}, z, B \cup \{y\}) &= t_2 \cdot \pi(\ell_2, k) + t_3 \cdot \pi(\ell_2, k+1) + n_z \\
&= t_2 \cdot \pi(\ell_2, k) + t_3 \cdot \pi(\ell_2, k) - t_3\delta_2 + n_z \\
&= \Pi(\mathcal{A}, z, B) - t_3\delta_2 \\
&< \Pi(\mathcal{A}, z, B) - t_1\pi(\ell_1 + 1, 1) - 1 \\
&\leq \Pi(\mathcal{A}, x, B) \\
&= \Pi(\mathcal{A}, x, B \cup \{y\}).
\end{aligned}$$

Here, the first equality follows from the definition of Π, the second one uses the definition of δ_2, and the third equation uses that $\Pi(\mathcal{A}, z, B) = t_2 \cdot \pi(\ell_2, k) + t_3 \cdot \pi(\ell_2, k) + n_z$. The next line follows from the definition of t_3 and the fifth line because we ensure that $\Pi(\mathcal{A}, z, B) \leq \Pi(\mathcal{A}, x, B) + t_1\pi(\ell_1 + 1, 1) + 1$. Finally, the last equation holds since there is no voter in $\mathcal{A}$ that approves both x and y. Because of this inequality, we now infer that f next assigns a share of $\frac{t_2 \cdot \pi(\ell_2, k) + n_x}{n}$ to x.

Now, let $\mathcal{A}'$ denote the profile where all voters in N_1 additionally approve x. We will show that the share of x decreases, which contradicts monotonicity. To this end, we note that in $\mathcal{A}'$, the candidates in B are still the first that will be assigned their shares by f. After this, f assigns a share to z. In more detail, the payment willingness for z is higher than that for y because

$$\begin{aligned}
\Pi(\mathcal{A}', y, B) &= t_1 \cdot \pi(\ell_1 + 1, 1) + t_3 \cdot \pi(\ell_2, k) + n_y \\
&= \Pi(\mathcal{A}, y, B) - t_1\delta_1 \\
&< \Pi(\mathcal{A}, y, B) - 1 \\
&\leq \Pi(\mathcal{A}, z, B) \\
&= \Pi(\mathcal{A}', z, B).
\end{aligned}$$

Here, we use the definitions of Π, δ_1, t_1, and that $\Pi(\mathcal{A}, z, B) + 1 \geq \Pi(\mathcal{A}, y, B)$ in this order. The final equation holds since all voters who approve z report the same preferences in $\mathcal{A}$ and $\mathcal{A}'$.

Moreover, using similar reasoning, we also infer that the payment willingness for z in $\mathcal{A}'$ is higher than that for x because

$$\begin{aligned}\Pi(\mathcal{A}', x, B) &= t_1 \cdot \pi(\ell_1 + 1, 1) + t_2 \cdot \pi(\ell_2, k) + n_x \\ &= \Pi(\mathcal{A}, x, B) + t_1 \cdot \pi(\ell_1 + 1, 1) \\ &< \Pi(\mathcal{A}, z, B) \\ &= \Pi(\mathcal{A}', z, B).\end{aligned}$$

Hence, we indeed first choose z among $\{x, y, z\}$. Now, assume that x is chosen in the next step; if y is chosen next, the share of x only decreases further. The total payment willingness for x is

$$\begin{aligned}\Pi(\mathcal{A}', x, B \cup \{z\}) &= t_1 \cdot \pi(\ell_1 + 1, 1) + t_2 \cdot \pi(\ell_2, k + 1) + n_x \\ &= t_1 \cdot \pi(\ell_1 + 1, 1) + t_2 \cdot \pi(\ell_2, k) - t_2\delta_2 + n_x \\ &< t_2 \cdot \pi(\ell_2, k) + n_x.\end{aligned}$$

The last inequality follows because we choose t_2 such that $t_2\delta_2 > t_1\pi(\ell_1 + 1, 1)$. This shows that the share of x in $\mathcal{A}'$ is less than $\frac{t_2 \cdot \pi(\ell_2, k) + n_x}{n}$, so f fails monotonicity. □

As our second point on MP, we will show that MP also satisfies strong fairness guarantees. In more detail, MP guarantees 2-AFS and $\Theta(\log n)$-core. This means that MP is a fair, monotonic, ranked population consistent, and simple distribution rule, thus satisfying all of our original goals.

Theorem 4.4. MP *satisfies* 2*-AFS and* $\Theta(\log n)$*-core. Moreover, these bounds are tight.*

Proof. To prove the theorem, we will first show that MP satisfies 2-AFS and give a matching lower bound. By Proposition 2.9, it then follows that MP satisfies $O(\log n)$-core because every rule that satisfies 2-AFS satisfies $2(1 + \log n)$-core. Finally, we complete the proof by giving a family of profiles where MP only satisfies $\Omega(\log n)$-core.

Claim 1: MP **satisfies** 2**-AFS.**

Fix some approval profile $\mathcal{A}$, and let $p = \text{MP}(\mathcal{A})$ denote the distribution returned by MP. Moreover, let $S \subseteq N$ denote a subset of voters and $x^* \in C$ a candidate such that $x^* \in A_i$ for all $i \in S$. Lastly, let $x_1, \dots, x_m$ denote the sequence according to which MP assigns shares to the candidates, and let k denote the index such that $x_k = x^*$. The total utility of the voters in S is

$$\sum_{i \in S} u_i(p) = \frac{1}{n} \sum_{j \in [m]} |\{i \in S \colon x_j \in A_i\}| \cdot \Pi(\mathcal{A}, x_j, \{x_1, \dots, x_{j-1}\}).$$

By the definition of sequential payment rules, candidate x_j maximizes $\Pi(\mathcal{A}, x, \{x_1, \dots, x_{j-1}\})$, so it holds for all $j \in [k]$ that $\Pi(\mathcal{A}, x_j, \{x_1, \dots, x_{j-1}\}) \geq \Pi(\mathcal{A}, x^*, \{x_1, \dots, x_{j-1}\})$. Next, let $\hat{S}_j = \{i \in S \colon x_j \in A_i \wedge \{x_1, \dots, x_{j-1}\} \cap A_i = \emptyset\}$ denote the set of voters in S for whom x_j is the first approved candidate in our sequence, and note that $|\{i \in S \colon x_j \in A_i\}| \geq |\hat{S}_j|$. We conclude that

$$\sum_{i \in S} u_i(p) \geq \frac{1}{n} \sum_{j \in [k]} |\hat{S}_j| \cdot \Pi(\mathcal{A}, x^*, \{x_1, \dots, x_{j-1}\}).$$

Next, it holds that $\Pi(\mathcal{A}, x^*, \{x_1, \dots, x_{j-1}\}) \geq |S| - \sum_{\ell \in [j-1]} |\hat{S}_\ell|$ since the payment willingness of a voter is 0 once one of his approved candidates has been assigned its share. This means that

$$\sum_{i \in S} u_i(p) \geq \frac{1}{n} \sum_{j \in [k]} |\hat{S}_j| \cdot (|S| - \sum_{\ell \in [j-1]} |\hat{S}_\ell|).$$

We will next derive a lower bound on $\sum_{j\in[k]} |\hat{S}_j| \cdot (|S| - \sum_{\ell\in[j-1]} |\hat{S}_\ell|)$. To this end, we note that $\sum_{j\in[k]} |\hat{S}_j| = |S|$ because the sets $\hat{S}_j$ are disjoint and $\hat{S}_k = S \setminus \bigcup_{j\in[k-1]} \hat{S}_j$. Hence, it follows that $\sum_{j\in[k]} |\hat{S}_j| \cdot |S| = |S|^2$. Next, it holds for all $\hat{S}_j$ that $|\hat{S}_j| \cdot \sum_{\ell\in[j-1]} |\hat{S}_\ell| \leq \sum_{r=0}^{|\hat{S}_j|-1} (r + \sum_{\ell\in[j-1]} |\hat{S}_\ell|)$. By applying this for all these sets, we derive that $\sum_{j\in[k]} |\hat{S}_j| \cdot \sum_{\ell\in[j-1]} |\hat{S}_\ell| \leq \sum_{j=0}^{|S|-1} j = \frac{(|S|-1)|S|}{2}$. Finally, we infer that

$$\sum_{i\in S} u_i(p) \geq \frac{1}{n} \sum_{j\in[k]} |\hat{S}_j| \cdot (|S| - \sum_{\ell\in[j-1]} |\hat{S}_\ell|) \geq \frac{1}{n} \left(|S|^2 - \frac{(|S|-1)|S|}{2} \right) \geq \frac{|S|^2}{2n}.$$

Since this holds for all groups of voters S that approve a common candidate, this proves that MP indeed satisfies 2-AFS.

Claim 2: MP **fails** $(2-\epsilon)$**-AFS for every** $\epsilon > 0$**.**

Assume for contradiction that MP satisfies $(2-\epsilon)$-AFS for some $\epsilon > 0$ and choose $\ell \in \mathbb{N}$ such that $(2-\epsilon)(\ell+3) < 2\ell$. To derive a contradiction, we consider the following profile $\mathcal{A}$ with $\ell+1$ candidates $C = \{x_1, \dots, x_\ell, x^*\}$ and $n = \ell + \frac{\ell(\ell+1)}{2}$ voters.

- For each candidate $x_i \in \{x_1, \dots, x_\ell\}$, there is one voter who reports $\{x_i, x^*\}$. We will refer to this group of voters as S and observe that $|S| = \ell$.
- For each candidate $x_i \in \{x_1, \dots, x_\ell\}$, there are $\ell + 1 - i$ voters who only approve x_i.

For this profile, MP will return the distribution p defined by $p(x_i) = \frac{\ell+2-i}{n}$ for all candidates $x_i \in \{x_1, \dots, x_\ell\}$ and $p(x^*) = 0$. In more detail, MP processes the candidates for $\mathcal{A}$ in the order $x_1, x_2, \dots, x_\ell, x^*$ because, for all $i \in [\ell]$, it holds for the total payment willingness that $\Pi(\mathcal{A}, x_i, \{x_1, \dots, x_{i-1}\}) = \ell + 2 - i$ and $\Pi(\mathcal{A}, x^*, \{x_1, \dots, x_{i-1}\}) = \ell + 1 - i$. Since each candidate $x_i \in C \setminus \{x^*\}$ is approved by exactly one voter in S, the total utility of these voters is

$$\sum_{i\in S} u_i(p) = \sum_{j\in[\ell]} \frac{\ell+2-j}{n} = \frac{1}{n} \sum_{j\in[\ell]} (j+1) = \frac{1}{n} \left(\ell + \frac{\ell(\ell+1)}{2} \right) = 1.$$

On the other hand, because all voters in S approve x^*, $(2-\epsilon)$-AFS requires that

$$\sum_{i\in S} u_i(p) \geq \frac{1}{2-\epsilon} \cdot \frac{|S|^2}{n} = \frac{1}{2-\epsilon} \cdot \frac{\ell^2}{\ell + \frac{\ell(\ell+1)}{2}} = \frac{1}{2-\epsilon} \cdot \frac{2\ell}{\ell+3} > 1.$$

The last inequality follows by the choice of ℓ. Since our two equations contradict each other, it follows that MP fails $(2-\epsilon)$-AFS.

Claim 3: MP **only satisfies** $\Omega(\log n)$**-core.**

To prove this claim, we will construct a family of approval profiles $\mathcal{A}^k$ with $n = k \cdot 2^{k-1}$ voters and $3 \cdot 2^k - 1$ candidates for all $k \in \mathbb{N}$ with $k \geq 2$ such that the distribution $p^k = \text{MP}(\mathcal{A}^k)$ only satisfies α-core for $\alpha \geq \frac{k}{2}$. Since $\log_2 n = \log_2(k \cdot 2^{k-1}) = \log_2 k + k - 1 \leq 2k$, this shows that MP only satisfies $\Omega(\log n)$-core.

Now, fix an integer $k \in \mathbb{N}$ with $k \geq 2$. To describe the profile $\mathcal{A}^k$, we will denote the set of voters of this profile by N^k and we assume that $N^k = [k] \times [2^{k-1}]$. That is, every voter is indicated by a unique tuple (i, j), where i is best interpreted as "row index" and j as "column index". Moreover, there are two types of candidates $X^k = \{x_1, \dots, x_{2^{k-1}}\}$ and $Y^k = \{y^i_\ell \colon i \in [k], \ell \in [2^{i-1}]\}$ and we assume that our tie-breaking prefers all candidates in Y^k to those in X^k. The profile $\mathcal{A}^k$ is defined as follows:

- Every voter (i, j) approves exactly one candidate in Y^k, namely the candidate y^i_ℓ for $\ell = \lceil \frac{j}{2^{k-i}} \rceil$. Less formally, the voters in the i-th row are partitioned into 2^{i-1} sets of size 2^{k-i} such that all voters in a set approve candidate y^i_ℓ.
- Every voter (k, j) only approves x_j among all candidates in X^k. Less formally, the voters in the k-th row only approve the candidate in X^k that matches their column entry.
- Every voter (i, j) with $i < k$ approves 2^{k-i-1} candidates from X^k, namely all x_ℓ with $2^{k-i-1} \cdot (\lceil \frac{j}{2^{k-i-1}} \rceil - 1) \leq \ell \leq 2^{k-i-1} \cdot \lceil \frac{j}{2^{k-i-1}} \rceil$. Less formally, for $i < k$, each voter in the i-th row approves 2^{k-i-1} candidates from X^k and all candidates in X^k are approved by 2^{k-i-1} voters of this row.

We note that the candidates in Y^k are unique for every row and partition the rows into subgroups. By contrast, the candidates in X^k cover the voters across all rows and voters in rows with smaller indices approve more candidates from X^k. An example of $\mathcal{A}^k$ for $k = 3$ is shown below.

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	$\{y^1_1, x_1, x_2\}$	$\{y^1_1, x_1, x_2\}$	$\{y^1_1, x_3, x_4\}$	$\{y^1_1, x_3, x_4\}$
$i = 2$	$\{y^2_1, x_1\}$	1: $\{y^2_1, x_2\}$	1: $\{y^2_2, x_3\}$	1: $\{y^2_2, x_4\}$
$i = 3$	$\{y^3_1, x_1\}$	1: $\{y^3_2, x_2\}$	1: $\{y^3_3, x_3\}$	1: $\{y^3_4, x_4\}$

In the profile $\mathcal{A}^k$, each candidate y^i_ℓ is approved by 2^{k-i} voters and each candidate in X^k is approved by $1 + \sum_{r=1}^{k-1} 2^{k-1-r} = 2^{k-1}$ voters. By our tie-breaking, MP first assigns a share of $\frac{1}{k}$ to candidate y^1_1. After this step, the total payment willingness for the candidates in y^i_ℓ with $i \geq 2$ remains unchanged, and the total payment willingness for the candidates in X^k is $1 + \sum_{r=2}^{k-1} 2^{k-1-r} = 2^{k-2}$. Hence, by our tie-breaking, MP will assign next a share of $\frac{1}{2k}$ to both y^2_1 and y^2_2. After these two steps, the total payment willingness for every candidate in X^k is $1 + \sum_{r=3}^{k-1} 2^{k-1-r} = 2^{k-3}$ and the total payment willingness for the candidates y^i_ℓ with $i \geq 3$ is still 2^{k-i}. By repeating this reasoning, we derive that MP returns the distribution p^k given by $p^k(y^i_\ell) = \frac{1}{k \cdot 2^{i-1}}$ for all $y^i_\ell \in Y^k$ and $p^k(x) = 0$ for all $x \in X^k$. Consequently, the utility of every voter (i, j) for p^k is $u_{(i,j)}(p^k) = \frac{1}{k \cdot 2^{i-1}}$.

Now, consider the distribution q defined by $q(x) = \frac{1}{2^{k-1}}$ for every candidate $x \in X^k$. Every voter (i, j) with $i < k$ approves 2^{k-i-1} candidates in X^k, so $u_{(i,j)}(q) = \frac{1}{2^{k-1}} \cdot 2^{k-i-1} = \frac{1}{2^i}$. Moreover, the voters (i, j) with $i = k$ approve a single candidate $x \in X^k$, so their utility is $u_{(i,j)}(q) = \frac{1}{2^{k-1}}$. Hence, it holds that $u_{(i,j)}(q) \geq \frac{k}{2} \cdot u_{(i,j)}(p^k)$ for every voter (i, j), which shows that MP fails α-core for every $\alpha < \frac{k}{2}$. □

Remark 4. The profiles in Claim 3 of the proof of Theorem 4.4 show that MP violates efficiency because all voters can improve their utility by an $\Omega(\log n)$-factor in these profiles. However, since it is known that no strategyproof and efficient rule can satisfy minimal fairness conditions [Brandl et al., 2021] and that monotonicity and strategyproofness are closely related, we conjecture that there is an inherent tradeoff between fairness and efficiency for monotonic rules.

Remark 5. At a quick glance, MP may seem like a very utilitarian rule. While MP indeed maximizes the utilitarian social welfare among sequential payment rule in some important special cases (such as laminar profiles [Peters and Skowron, 2020]), no such statement holds in general. For a counterexample, assume that there are k voters approving $\{a^*, a_1, \ldots, a_\ell\}$, two voters approving only a^*, and for each $i \in [\ell]$, there is one voter approving $\{a_i, b_i\}$ and one voter approving $\{b_i\}$. In this instance, MP will assign a share of $\frac{k+2}{n}$ to a^* and a share of $\frac{2}{n}$ to each candidate b_i.

However, if k is large, assigning a share to a_i instead of b_i results in a higher social welfare, and every sequential payment rules other than MP will assign parts of the budget to the candidates a_i if k is sufficiently large. This shows that the approximation ratio of the utilitarian social welfare of MP can be arbitrarily bad and that it is not superior to that of other sequential payment rules.

Remark 6. We note that MP shows that the implications between α-AFS and α-core in Proposition 2.9 are asymptotically tight. In more detail, as MP satisfies 2-AFS and $\Theta(\log n)$-core, it is not possible to show a sub-logarithmic implication from approximate AFS to approximate core. Moreover, in the profiles for the proof of Claim 2 of Theorem 4.4, the approximation ratio of MP for core converges to 1, thus showing that it is not possible to show a better implication from approximate core to approximate AFS than the one given in Proposition 2.9.

4.2 Optimizing for Fairness

Motivated by the positive result on the AFS approximation ratio of MP, we will next aim to optimize fairness within the class of sequential payment rules. In more detail, we will first analyze the AFS approximation ratio for γ-multiplicative sequential payment rules and, in particular, show that $\frac{1}{3}$–MSP satisfies $\frac{3}{2}$-AFS. Furthermore, we will show that this is optimal within the class of sequential payment rules. By employing a more fine-grained analysis that takes the maximum ballot size of a profile into account, we even derive a characterization of $\frac{1}{3}$–MSP as the uniquely fairest sequential payment rule. Since the proofs of all results in this section are rather technical, we will only discuss proof sketches in the main body and defer the full proofs to the appendix.

To facilitate the proofs of our lower bounds, we will next present necessary conditions on the payment willingness function of every sequential payment rule that satisfies α-AFS. More specifically, the next proposition directly relates the payment willingness function π of a sequential payment rule with its approximation ratio α to AFS.

Proposition 4.5. *Let f denote a sequential payment rule that satisfies α-AFS for some $\alpha \in \mathbb{R}$, and let $t \in \mathbb{N}$. It holds for the payment willingness function π of f that*

$$\begin{aligned} \pi(t,1) &\geq \frac{1}{\alpha} \\ \frac{1}{2}\pi(t,1) + \frac{3}{2}\pi(t,2) &\geq \frac{1}{\alpha} \\ \frac{1}{2}\pi(t,1) + \pi(t,2) + \frac{3}{2}\pi(t,3) &\geq \frac{1}{\alpha} \\ &\cdots \\ \frac{1}{2}\pi(t,1) + \pi(t,2) + \cdots + \pi(t,t-1) + \frac{3}{2}\pi(t,t) &\geq \frac{1}{\alpha}. \end{aligned}$$

Proof Sketch. For the proof of this proposition, we fix a sequential payment rule f, its payment willingness function π, and an integer $t \in \mathbb{N}$. We then construct t profiles $\mathcal{A}^k$ for $k \in [t]$ based on the idea that there is group of voters S, all of which approve t candidates in total but only one candidate x^* is approved by all of the voters in S. Moreover, in $\mathcal{A}^k$, f assigns its share to x^* only after each voter in S spent his budget on $k-1$ other candidates. In more detail, the approval ballot A_i^k of each voter $i \in S$ is given by $A_i = \{x^*, y_1^i, \dots, y_{t-1}^i\}$, where the candidates y_j^i satisfy that $A_u \cap A_v = \{x^*\}$ for all distinct $u, v \in S$. Moreover, there are several auxiliary voters in $\mathcal{A}^k$, none of which approves x^*, who ensure that f assigns the shares to the candidates $\{y_j^i \colon i \in S \wedge j \in [k-1]\}$ before x^* and that the total payment willingness for these candidates is only marginally higher than for x^* when they are assigned their shares. Finally, by analyzing the α-AFS guarantee for S and the distribution chosen by f, we derive from the profile $\mathcal{A}^k$ our k-th inequality. $\square$

We note that Proposition 4.5 is a powerful tool to infer lower bounds on the AFS approximation ratios of sequential payment rules. For instance, if $\pi(t,1) = \frac{2}{3}$ for some $t \in \mathbb{N}$, the first inequality of this proposition implies that the corresponding sequential payment rule only satisfies α-AFS for $\alpha \geq \frac{3}{2}$. Moreover, we will next use this proposition to fully determine the AFS approximation ratio of γ-multiplicative sequential payment (γ–MSP) rules.

Theorem 4.6. *γ–MSP satisfies $\frac{2}{1+\gamma}$-AFS if $0 \leq \gamma \leq \frac{1}{3}$ and $\frac{1}{1-\gamma}$-AFS if $\frac{1}{3} \leq \gamma < 1$. Moreover, these bounds are tight.*

Proof Sketch. Let f denote a γ–MSP rule for $\gamma \in [0,1)$ and let π denote its corresponding payment function. In this proof sketch, we will assume that $\gamma \geq \frac{1}{3}$ and note that the case that $\gamma \leq \frac{1}{3}$ follows from a similar analysis. Now, for the lower bound, we note that Proposition 4.5 shows that $\pi(t,1) \geq \frac{1}{\alpha}$ for all $t \in \mathbb{N}$. Since $\pi(t,1) = \frac{\gamma}{\sum_{i\in[t]} \gamma^i}$, we derive that $\alpha \geq \sum_{i\in[t]} \gamma^{i-1}$. Finally, the right hand side of this inequality is the geometric sum which converges from below to $\sum_{i\in\mathbb{N}_0} \gamma^i = \frac{1}{1-\gamma}$. Hence, f cannot satisfy α-AFS for any $\alpha < \frac{1}{1-\gamma}$ as we can otherwise find an integer $t \in \mathbb{N}$ such that the first constraint in Proposition 4.5 is violated.

To show that f satisfies $\frac{1}{1-\gamma}$-AFS, we fix a profile $\mathcal{A}$, let $p = f(\mathcal{A})$ be the distribution chosen by f, and let $x_1, \dots, x_m$ denote the order in which f assigns the shares to the candidates. Moreover, let S denote a set of s voters and x_k a candidate such that $x_k \in \bigcap_{i\in S} A_i$. Similar to the proof of Theorem 4.4, it can be shown that

$$\begin{aligned}\sum_{i\in S} u_i(p) &= \frac{1}{n} \sum_{j\in[m]} |\{i \in S \colon x_j \in A_i\}| \cdot \Pi(\mathcal{A}, x_j, \{x_1, \dots, x_{j-1}\}) \\ &\geq \frac{1}{n} \sum_{j\in[k]} |\{i \in S \colon x_j \in A_i\}| \cdot \Pi(\mathcal{A}, x_k, \{x_1, \dots, x_{j-1}\}).\end{aligned}$$

Next, we define t as the maximal ballot size in $\mathcal{A}$ and $T_k = \sum_{\ell\in[k-1]} |\{i \in S \colon x_\ell \in A_i\}|$ as the total number of approvals of the voters in S for the candidates in $\{x_1, \dots, x_{k-1}\}$. Put differently, T_k measures how often the voters in S payed for candidates prior to x_k. Moreover, we let $u = \lfloor T_k/|S| \rfloor + 1$ and $v = T_k \mod |S|$. In a series of technical steps, we prove that

$$\begin{aligned}\sum_{j\in[k]} |\{i \in S \colon x_j \in A_i\}| \cdot \Pi(\mathcal{A}, x_k, \{x_1, \dots, x_{j-1}\}) \geq &\sum_{j=1}^{u-1} \sum_{\ell=0}^{s-1} \Big((s-\ell)\pi(t,j) + \ell\pi(t,j+1) \Big) \\ &+ \sum_{\ell=0}^{v-1} \Big((s-\ell)\pi(t,u) + \ell\pi(t,u+1) \Big) \\ &+ s((s-v)\pi(t,u) + v\pi(t,u+1)).\end{aligned}$$

Using that $\gamma\pi(t,j) = \pi(t,j+1)$ and $\pi(t,j) = \gamma^{j-1}\pi(t,1)$, we then compute based on the Gaussian and geometric sum formulas that

$$\sum_{j=1}^{u-1} \sum_{\ell=0}^{s-1} (s-\ell)\pi(t,j) + \ell\pi(t,j+1) = s^2 \cdot \pi(t,1) \cdot \frac{1+\gamma}{2} \cdot \frac{1-\gamma^{u-1}}{1-\gamma}.$$

Moreover, since $\gamma \geq \frac{1}{3}$, it can be shown that

$$\sum_{\ell=0}^{v-1}(s-\ell)\pi(t,u) + \ell\pi(t,u+1) + s((s-v)\pi(t,u) + v\pi(t,u+1)) \geq s^2 \cdot \pi(t,1) \cdot \gamma^{u-1}$$

Next, we prove that $\frac{1+\gamma}{2} \cdot \frac{1-\gamma^{u-1}}{1-\gamma} + \gamma^{u-1} \geq 1$ if $\gamma \geq \frac{1}{3}$. By combining all insights, we then infer that

$$\sum_{i \in S} u_i(p) \geq \frac{s^2}{n} \cdot \pi(t,1) \cdot \left(\frac{1+\gamma}{2} \cdot \frac{1-\gamma^{u-1}}{1-\gamma} + \gamma^{u-1}\right) \geq \frac{s^2}{n} \cdot \pi(t,1).$$

Finally, it holds that $\pi(t,1) \geq 1-\gamma$ as $\frac{\gamma}{\sum_{i \in t} \gamma^i} \geq \frac{1}{\sum_{i \in \mathbb{N}_0} \gamma^i}$ and $\sum_{i \in \mathbb{N}_0} \gamma^i = \frac{1}{1-\gamma}$. It thus follows that f satisfies $\frac{1}{1-\gamma}$-AFS by rearranging the previous inequality. □

Theorem 4.6 implies that $\frac{1}{3}$–MSP is the fairest multiplicative sequential payment rule: it is a $\frac{3}{2}$-approximation to AFS, whereas every other γ–MSP rule has a larger approximation ratio to AFS. A natural follow-up question in light of this is whether other sequential payment rules can achieve a better AFS approximation than $\frac{1}{3}$–MSP. We will subsequently answer this question in the negative and, even more, we will show that $\frac{1}{3}$–MSP is the uniquely fairest sequential payment rule. For proving this claim, we will use a more fine-grained analysis that parameterizes the AFS approximation ratio based on the maximum ballot size in a profile. In more detail, we say that a distribution rule satisfies α-AFS *for a maximal ballot size t* if it satisfies α-AFS for all profiles where all voters approve at most t candidates. We will next show that, if we only allow for ballots of size at most t, then no sequential payment rule satisfies α-AFS for $\alpha < \frac{3}{2}(1-3^{-t})$, and that $\frac{1}{3}$–MSP is the only sequential payment rule that matches this bound for all maximal ballot sizes t.

Theorem 4.7. *The following statements hold:*

(1) Let $t \in \mathbb{N}$. No sequential payment rule satisfies α-AFS for the maximal ballot size t and $\alpha < \frac{3}{2}(1-3^{-t})$.

(2) $\frac{1}{3}$–MSP is the only sequential payment rule that satisfies $\frac{3}{2}(1-3^{-t})$-AFS for all maximal ballot sizes $t \in \mathbb{N}$.

Proof Sketch. For showing Claim (1), we solve the inequality system given by Proposition 4.5. To this end, let I_k denote the k-th inequality of this system (e.g., I_1 is $\pi(t,1) \geq \frac{1}{\alpha}$, I_2 is $\frac{1}{2}\pi(t,1) + \frac{3}{2}\pi(t,2) \geq \frac{1}{\alpha}$), etc.). We define the modified set of inequalities I'_k which is given by $I'_1 = I_1$ and $I'_k = 2I_k + I_1$ (i.e., I'_k states the condition $2\sum_{i \in [k-1]} \pi(t,i) + 3\pi(t,k) \geq \frac{3}{\alpha}$). It can then be shown that $I'_1 + I'_2 + 3I'_3 + 9I'_4 + \cdots + 3^{t-2}I'_t$ corresponds to the inequality $3^{t-1}\sum_{i \in [t]} \pi(t,i) \geq \frac{1}{\alpha}\sum_{i=0}^{t-1} 3^i$. Finally, since $\sum_{i \in [t]} \pi(t,i) = 1$ and $\sum_{i=0}^{t-1} 3^i = \frac{1}{2}(3^t - 1)$, our lower bound follows by solving for α.

We next turn to Claim (2) and note for this that the proof of Theorem 4.6 implies that $\frac{1}{3}$–MSP satisfies $\frac{3}{2}(1-3^{-t})$-AFS for all maximal ballot sizes $t \in \mathbb{N}$. To make this more precise, let π be the payment willingness function of $\frac{1}{3}$–MSP, consider a profile $\mathcal{A}$ with maximal ballot size t, fix a set of voters S with $\bigcap_{i \in S} A_i \neq \emptyset$, and let p denote the distribution chosen by $\frac{1}{3}$–MSP for $\mathcal{A}$. In the proof of Theorem 4.6, we show that $\sum_{i \in S} u_i(p) \geq \frac{|S|^2}{n}\pi(t,1)$. Finally, since $\pi(t,1) = \frac{3^{-1}}{\sum_{i \in [t]} 3^{-i}} = \frac{1}{\sum_{i=0}^{t-1} 3^{-i}} = \frac{1}{\frac{3}{2}(1-3^{-t})}$, it then follows that $\frac{1}{3}$–MSP satisfies $\frac{3}{2}(1-3^{-t})$-AFS for the profile $\mathcal{A}$. □

Remark 7. Theorem 4.7 implies that, when there is no maximum feasible ballot size, no sequential payment rule can have a better approximation ratio than $\frac{3}{2}$. While $\frac{1}{3}$–MSP is the only sequential payment rule that matches our lower bound for all maximal ballot sizes, there are multiple rules that match the general lower bound. A particularly simple example of such a rule is the $\frac{1}{3}$-additive sequential payment rule, where the payment willingness of every agent decreases by $\frac{1}{3}$ whenever he pays for a candidate. More formally, this rule is defined by the payment willingness function π with $\pi(1,1) = 1$, $\pi(x,1) = \frac{2}{3}$, $\pi(x,2) = \frac{1}{3}$, and $\pi(x,y) = 0$ for all $x \in \mathbb{N} \setminus \{1\}$, $y \in [x] \setminus \{1,2\}$. Additionally to satisfying $\frac{3}{2}$-AFS, this rule is also monotonic for all instances where voters approve at least two candidates, i.e., if the minimal ballot size is 2. Thus, this rule is an attractive alternative to MP if voters are required to approve more than one candidate.

Remark 8. Theorem 4.6 combined with Proposition 2.9 shows that all γ–MSP rules with $\gamma \in [0,1)$ satisfy $O(\log n)$-core. We leave it open whether this ratio is tight because core requires us to compare the output distribution p of γ–MSP to every other distribution q and our techniques are not suitable for this comparison.

5 Conclusion

In this paper, we introduce and analyze the class of sequential payment rules with the aim of finding simple, well-behaved, and fair distribution rules. We show that all sequential payment rules satisfy a strong form of population consistency, which we call ranked population consistency. Moreover, we identify two particularly appealing rules within our class: the maximum payment rule (MP), where voters are willing to spend their whole budget on their first approved candidate, and the $\frac{1}{3}$-multiplicative payment rule ($\frac{1}{3}$–MSP), where the payment willingness of voters is always discounted by a factor of $\frac{1}{3}$ when they pay for a candidate. In more detail, we show that MP is, except for the uncoordinated equal shares rule, the only sequential payment rule that satisfies monotonicity. Moreover, it guarantees a 2-approximation to a fairness notion called average fair share (AFS), and a $\Theta(\log n)$-approximation to core, thus demonstrating that MP is indeed a simple, well-behaved, and fair rule. On the other hand, we identify $\frac{1}{3}$–MSP as the fairest sequential payment rule by showing that this rule is a $\frac{3}{2}$-approximation to AFS and that no other sequential payment has a better approximation ratio. We refer to Table 1 for an overview of our results and a comparison to existing rules.

Our work points to several interesting follow-up questions. For instance, a straightforward drawback of sequential payment rules is that they fail Pareto efficiency, leading to the question of whether one can define similar spending dynamics that are approximately fair and efficient. More generally, it seems interesting to study other types of spending dynamics that, e.g., could take the current utility of voters into account or allow voters to pay multiple times for a candidate.

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A Omitted Proofs

In this appendix, we present the proofs omitted from the main body.

A.1 Proof of Proposition 4.5

Proposition 4.5. *Let f denote a sequential payment rule that satisfies α-AFS for some $\alpha \in \mathbb{R}$, and let $t \in \mathbb{N}$. It holds for the payment willingness function π of f that*

$$\begin{aligned} \pi(t,1) &\geq \frac{1}{\alpha} \\ \frac{1}{2}\pi(t,1) + \frac{3}{2}\pi(t,2) &\geq \frac{1}{\alpha} \\ \frac{1}{2}\pi(t,1) + \pi(t,2) + \frac{3}{2}\pi(t,3) &\geq \frac{1}{\alpha} \\ &\dots \\ \frac{1}{2}\pi(t,1) + \pi(t,2) + \dots + \pi(t,t-1) + \frac{3}{2}\pi(t,t) &\geq \frac{1}{\alpha}. \end{aligned}$$

Proof. Let f denote a sequential payment rule that satisfies α-AFS for some finite α, let π be its payment willingness function, and let $t \in \mathbb{N}$ denote an arbitrary integer. Throughout this proof, we will interpret t as the maximal feasible ballot size to ensure that we can use this lemma also in the proof of Theorem 4.7. We moreover define $\pi_i = \pi(t,i)$ for all $i \in [t]$ to simplify notation and note that $\pi_1 \geq \pi_2 \geq \dots \geq \pi_t$ by the definition of payment willingness functions.

In this proof, it will be easier to discuss fractional profiles as this allows us to avoid complicated computations. Formally, a fractional profile W is a function from the set of approval ballots $2^C \setminus \{\emptyset\}$ to $\mathbb{Q}_{\geq 0}$, with the interpretation that $W(A)$ indicates how many voters report the ballot A in the profile W. That is, rather than having $x \in \mathbb{N}_0$ voters reporting an approval ballot A, it is now possible that x is a rational number. The size of a fractional profile W is defined by $|W| = \sum_{A \in 2^C \setminus \{\emptyset\}} W(A)$ and we will assume that $|W| > 0$ for all fractional profiles. It is straightforward to generalize f to fractional profiles: we can extend the definition of the total payment willingness Π to fractional profiles by letting $\Pi(W,x,Y) = \sum_{A \in 2^C \setminus \{\emptyset\} \colon x \in A} W(A) \cdot \pi(|A|, |A \cap Y| + 1)$ for every fractional profile W, candidate x, and set of candidates $Y \subseteq C \setminus \{x\}$. Then, sequential payment rules work on fractional profiles as described in Algorithm 1. From the definition of sequential payments rules, it follows that $f(W) = f(\ell W)$ for every $\ell \in \mathbb{N}$, where the fractional profile $W' = \ell W$ is defined by $W'(A) = \ell \cdot W(A)$ for all approval ballots A.

Next, we extend the definition of α-AFS to fractional profiles as follows: a distribution p satisfies α-AFS for a fractional profile W if

$$\frac{\alpha}{|S|} \cdot \sum_{A \in 2^C \setminus \{\emptyset\}} \left(S(A) \cdot \sum_{y \in A} p(y) \right) \geq \frac{|S|}{|W|}$$

for all fractional (sub)profiles S and candidates $x \in C$ such that $S(A) \leq W(A)$ for all $A \in \mathcal{A}$ and $S(A) > 0$ implies that $x \in A$. Intuitively, S can be seen here a set of (fractional) voters such that $x \in A_i$ for all voters $i \in S$. It holds that f satisfies α-AFS for regular approval profiles if and only if it satisfies α-AFS for fractional profiles. In more detail, f satisfies α-AFS for regular approval profiles if it satisfies this property for fractional profiles, because every regular approval profile corresponds to a fractional profile. On the other hand, if f fails α-AFS for a fractional profile W, there is a fractional subprofile S and a candidate $x \in C$ such that $S(A) \leq W(A)$ for all

$A \in 2^C \setminus \{\emptyset\}$, $S(A) > 0$ implies $x \in A$, and $\frac{\alpha}{|S|} \cdot \sum_{A\in 2^C\setminus\{\emptyset\}} \left(S(A) \cdot \sum_{y\in A} f(W,y)\right) < \frac{|S|}{|W|}$. Since $f(\ell W) = f(W)$ for all $\ell \in \mathbb{N}$, it follows that f also fails α-AFS for every profile ℓW, which is witnessed by the fractional subprofile ℓS. However, because both S and W are fractional profiles and thus functions of the type $2^C \setminus \{\emptyset\} \to \mathbb{Q}$, there is an integer ℓ^* such that $\ell^* W(A) \in \mathbb{N}_0$ and $\ell^* S(A) \in \mathbb{N}_0$ for all approval ballots A. Hence, $\ell^* W$ corresponds to an approval profile and ℓS^* to a group of voters in this approval profile, so f fails α-AFS for approval profiles, too.

We will next derive our upper bounds for $\frac{1}{\alpha}$ by investigating several fractional profiles. To this end, we fix some integer $\ell \geq 2$ and use $1 + \ell(t-1)$ candidates $\{x^*\} \cup \{y^i_j \colon j \in [t-1], i \in [\ell]\}$. We then define the ballots $B^1, \dots, B^\ell$ by $B^i = \{x^*, y^i_1, \dots, y^i_{t-1}\}$ for all $i \in [\ell]$ and note that $|B^i| = t$ for all $i \in [\ell]$. First, we consider the fractional profile W^0 defined by $W^0(B^i) = 1$ for all $i \in [\ell]$ and $W^0(A) = 0$ for all ballots $A \notin \{B^1, \dots, B^\ell\}$. The total payment willingness for the candidate x^* in the first round is $\ell\pi_1$ and the total payment willingness for all other candidates is π_1. Hence, f chooses x^* in the first round and assigns it a share of $\frac{\ell\pi_1}{|W^0|} = \pi_1$. Moreover, since there is no overlap between the approval ballots $B^1, \dots, B^\ell$ except for x^* and voters only spend their budget on their approved candidates, it holds for the distribution $p = f(W^0)$ that $\sum_{x\in B^i\setminus\{x^*\}} p(x) = \frac{1-\pi_1}{|W^0|} = \frac{1-\pi_1}{\ell}$ for all $i \in [\ell]$. Since every voter in W^0 approves x^*, α-AFS requires that

$$\sum_{A\in 2^C\setminus\{\emptyset\}} W^0(A) \sum_{x\in A} p(x) = \sum_{i=1}^{\ell} \sum_{x\in B^i} p(x) = \ell\pi_1 + 1 - \pi_1 \geq \frac{\ell}{\alpha}.$$

Put differently, this means that $\pi_1 + \frac{1-\pi_1}{\ell} \geq \frac{1}{\alpha}$. Now, if $\pi_1 < \frac{1}{\alpha}$, there is an integer ℓ such that $\pi_1 + \frac{1-\pi_1}{\ell} < \frac{1}{\alpha}$, which implies that α-AFS is violated. Since f satisfies this property by assumption, we derive that $\pi_1 \geq \frac{1}{\alpha}$.

Next, we consider $t-1$ more profiles W^z_ϵ with $z \in [t-1]$ and $\epsilon \in (0,1)$, where the rough idea is that each candidate in $\{y^i_j \colon j \in [z], i \in [\ell]\}$ is chosen before x^* with a total payment willingness that is marginally higher than that of x^*. To this end, we let $K(u,v) = (\ell - v) \cdot \pi_u + v \cdot \pi_{u+1}$ for $u \in [t-1]$ and $v \in [\ell] \cup \{0\}$ denote the payment willingness for x^* when v of the ℓ voters reporting $B^1, \dots, B^\ell$ have payed for u candidates and the remaining $\ell - v$ voters have payed for $u-1$ candidates. We moreover note that $K(u,\ell) = \ell\pi_{u+1} = K(u+1, 0)$ for all $u \in [t-1]$ and $K(u,v) \geq K(u, v+1)$ for all $u \in [t-1]$ and $v \in [\ell-1] \cup \{0\}$ since $\pi_1 \geq \pi_2 \geq \dots \geq \pi_t$. Next, we define $K^\epsilon(u,v)$ for all $u \in [t-1]$, $v \in [\ell] \cup \{0\}$ as a function such that

1) $K^\epsilon(u,v) \in \mathbb{Q}$,
2) $K(u,v) < K^\epsilon(u,v) \leq K(u,v) + \epsilon$, and
3) $K^\epsilon(u,v) > K^\epsilon(u',v')$ for all u', v' with $u' > u$, or $u' = u$ and $v' > v$.

Less formally, K^ϵ is an approximation of K which takes care of two issues: firstly, $K(u,v)$ may be irrational while $K^\epsilon(u,v)$ is rational. Secondly, K may assign the same score to multiple combinations of u and v, whereas K^ϵ breaks these ties. We furthermore note that K^ϵ exists for every $\epsilon \in (0,1]$ due to the density of $\mathbb{Q}$ in $\mathbb{R}$.

We now derive the profile W^z_ϵ from W^0 by adding $K^\epsilon(u, v-1)$ voters for each candidate y^v_u with $u \in [z]$ and $v \in [\ell]$ that uniquely approve y^v_u. These voters ensure that f processes the candidates in the order $y^1_1, \dots, y^\ell_1, y^1_2, \dots, y^\ell_2, \dots, y^1_z, \dots, y^\ell_z, x^*$ for the profile W^z_ϵ. To see this, fix two values $u \in [z]$ and $v \in [\ell]$ and assume that all candidates in $X = \{y^i_j \colon i \in [\ell], j \in [u-1]\} \cup \{y^i_u \colon i \in [v-1]\}$ have been assigned their shares. In this case, the total payment willingness for x^* is

$$\Pi(W^z_\epsilon, x^*, X) = \sum_{i\in[\ell]} \pi(t, |X \cap B^i| + 1) = (\ell - v + 1)\pi_u + (v-1)\pi_{u+1} = K(u, v-1).$$

By contrast, the total payment willingness for a candidate $y_j^i \notin X$ with $j \leq z$ is $\Pi(W_\epsilon^z, y_j^i, X) = K^\epsilon(j, i-1) + \pi(t, |B^i \cap X| + 1)$, which is $K^\epsilon(j, i-1) + \pi_u$ if $i \geq v$ or $K^\epsilon(j, i-1) + \pi_{u+1}$ if $i \leq v-1$. Now, by the definition of K^ϵ and the fact that $\pi_u \geq \pi_{u+1}$, we have that $\Pi(W_\epsilon^z, y_u^v, X) > \Pi(W_\epsilon^z, y, X)$ for all candidates $y \notin X \cup \{x^*, y_u^v\}$. Moreover, since $K^\epsilon(u, v-1) > K(u, v-1)$, we also have that $\Pi(W_\epsilon^z, y_u^v, X) > \Pi(W_\epsilon^z, x^*, X)$, which means that y_u^v is the next candidate that will be assigned its share. Iteratively applying this reasoning results in the given sequence.

We next consider again the sets of voters S that report $B^1, \dots, B^\ell$, i.e., S is the subprofile of W_ϵ^z such that $S(B^i) = 1$ for all $i \in [\ell]$ and $S(A) = 0$ for all other ballots. In particular, we will next compute the total utility of the set of voters in S. To this end, let $p = f(W_\epsilon^z)$ denote the distribution returned by f for the profile W_ϵ^z. Since each candidate y_j^i is approved by a single voter in S, it holds that

$$\sum_{A \in 2^C \setminus \{\emptyset\}} S(A) \sum_{x \in A} p(x) = \ell \cdot p(x^*) + \sum_{j \in [t-1]} \sum_{i \in [\ell]} p(y_j^i).$$

First, we note that, by our previous analysis, it is easy to see that x^* is chosen at a total payment willingness of $\Pi(W_\epsilon^z, x^*, \{y_j^i \colon j \in [z], i \in [\ell]\}) = \ell \pi_{z+1}$. Next, every candidate y_j^i with $j \leq z$ has a total payment willingness of $K^\epsilon(j, i-1) + \pi_j$ when it is assigned its share. Since $K^\epsilon(j, i-1) \leq K(j, i-1) + \epsilon$, we can compute that

$$\begin{aligned}
\sum_{j \in [z]} \sum_{i \in [\ell]} K^\epsilon(j, i-1) &\leq \sum_{j \in [z]} \sum_{i \in [\ell]} K(j, i-1) + \epsilon \\
&= z \cdot \ell \cdot \epsilon + \sum_{j \in [z]} \sum_{i \in [\ell]} (\ell - (i-1))\pi_j + (i-1)\pi_{j+1} \\
&= z \cdot \ell \cdot \epsilon + \sum_{i \in [\ell]} (\ell - (i-1))\pi_1 + \sum_{i \in [\ell]} (i-1)\pi_{z+1} \\
&\quad + \sum_{j=2}^{z} \sum_{i \in [\ell]} (\ell - (i-1))\pi_j + (i-1)\pi_j \\
&= z \cdot \ell \cdot \epsilon + \pi_1 \frac{\ell(\ell+1)}{2} + \pi_{z+1} \frac{(\ell-1)\ell}{2} + \sum_{j=2}^{z} \ell^2 \pi_j
\end{aligned}$$

Moreover, it is easy to see that $\sum_{j \in [z]} \sum_{i \in [\ell]} \pi_j = \ell \sum_{j \in [z]} \pi_j$, and that $p(y_j^i) = \pi_{j+1}$ for all $j > z$ and $i \in [\ell]$ because these candidates are chosen after $y_{j'}^i$ for $j' \in [z]$ and x^*. Hence, we conclude that

$$\begin{aligned}
\sum_{A \in 2^C \setminus \{\emptyset\}} S(A) \sum_{x \in A} p(x) &= \ell \cdot p(x^*) + \sum_{j \in [t-1]} \sum_{i \in [\ell]} p(y_j^i) \\
&\leq \frac{1}{|W_\epsilon^z|} \left(\ell^2 \pi_{z+1} + \sum_{j \in [z]} \sum_{i \in [\ell]} (K^\epsilon(j, i-1) + \pi_j) + \sum_{j=z+2}^{t} \sum_{i \in [\ell]} \pi_j \right) \\
&\leq \frac{1}{|W_\epsilon^z|} \Bigg(\ell^2 \pi_{z+1} + z \cdot \ell \cdot \epsilon + \pi_1 \frac{\ell(\ell+1)}{2} + \pi_{z+1} \frac{(\ell-1)\ell}{2} \\
&\quad + \sum_{j=2}^{z} \ell^2 \pi_j + \sum_{j=1}^{z} \ell \pi_j + \ell(1 - \sum_{j=1}^{z+1} \pi_j) \Bigg) \\
&\leq \frac{\ell^2}{|W_\epsilon^z|} \left(\frac{\pi_1}{2} + \frac{\pi_1}{2\ell} + \sum_{j=2}^{z} \pi_j + \frac{3\pi_{z+1}}{2} + \frac{1}{\ell} + \frac{z\epsilon}{\ell} \right).
\end{aligned}$$

Furthermore, α-AFS requires that $\sum_{A \in 2^C \setminus \{\emptyset\}} S(A) \sum_{x \in A} p(x) \geq \frac{1}{\alpha} \frac{\ell^2}{|W_\epsilon^z|}$. Hence, we conclude that

$$\frac{\pi_1}{2} + \frac{\pi_1}{2\ell} + \sum_{j=2}^{z} \pi_j + \frac{3\pi_{z+1}}{2} + \frac{1}{\ell} + \frac{z\epsilon}{\ell} \geq \frac{1}{\alpha}.$$

Now, if $\frac{\pi_1}{2} + \sum_{j=2}^{z} \pi_j + \frac{3\pi_{z+1}}{2} < \frac{1}{\alpha}$, we can find a sufficiently large ℓ (and sufficiently small ϵ) such that the above inequality is violated. This contradicts α-AFS, so $\frac{\pi_1}{2} + \sum_{j=2}^{z} \pi_j + \frac{3\pi_{z+1}}{2} \geq \frac{1}{\alpha}$. Since this holds for every $z \in [t-1]$, we now derived all inequalities given by the proposition. □

A.2 Proof of Theorem 4.6

Theorem 4.6. γ–MSP *satisfies* $\frac{2}{1+\gamma}$*-AFS if* $0 \leq \gamma \leq \frac{1}{3}$ *and* $\frac{1}{1-\gamma}$*-AFS if* $\frac{1}{3} \leq \gamma < 1$. *Moreover, these bounds are tight.*

Proof. Since the 0-multiplicative sequential spending rule is the maximum payment rule, the theorem follows for $\gamma = 0$ from Theorem 4.4. Hence, we assume that $\gamma \in (0, 1)$ and we will subsequently prove an upper and lower bound on the approximation ratio of γ–MSP. As usual, we will denote by π the payment willingness function of the γ-MPS rule, i.e., $\pi(x, y) = \frac{\gamma^y}{\sum_{i=1}^{x} \gamma^i}$ for all $x \in \mathbb{N}$, $y \in [x]$. We start by showing the lower bound on the AFS approximation ratio of f.

Claim 1: Lower Bound

To prove our lower bound, we will heavily rely on Proposition 4.5 and use a case distinction with respect to γ. First, consider the case that $\frac{1}{3} \leq \gamma < 1$. In this case, we use the first inequality of Proposition 4.5, which shows that, if f satisfies α-AFS for some $\alpha \in \mathbb{R}$, then $\pi(t, 1) \geq \frac{1}{\alpha}$ for all $t \in \mathbb{N}$. Since $\pi(t, 1) = \frac{\gamma}{\sum_{i \in [t]} \gamma^i}$, this means that, for all $t \in \mathbb{N}$, it holds that $\alpha \geq \sum_{i=0}^{t-1} \gamma^i$. The right hand side of this inequality is the geometric series which converges to $\sum_{i \in \mathbb{N}_0} \gamma^i = \frac{1}{1-\gamma}$, so it follows that α cannot be smaller than $\frac{1}{1-\gamma}$. In more detail, for every $\epsilon > 0$, there is some $t \in \mathbb{N}$ such that $\frac{1}{1-\gamma} - \epsilon < \sum_{i=0}^{t-1} \gamma^i$ and Proposition 4.5 would thus be violated.

Next, for the case that $0 < \gamma < \frac{1}{3}$, we consider the last inequality given by Proposition 4.5, which shows that, for all $t \in \mathbb{N}$, $\frac{1}{2}\pi(t, 1) + \frac{3}{2}\pi(t, t) + \sum_{i=2}^{t-1} \pi(t, i) \geq \frac{1}{\alpha}$ if f satisfies α-AFS. Using the fact that $\sum_{i \in [t]} \pi(t, i) = 1$, we can rewrite this inequality as

$$1 - \frac{1}{2}\pi(t, 1) + \frac{1}{2}\pi(t, t) \geq \frac{1}{\alpha}$$
$$\iff \alpha \geq \frac{2}{2 - \pi(t, 1) + \pi(t, t)}.$$

Since $\pi(t, 1) = \frac{\gamma}{\sum_{i \in [t]} \gamma^i} \geq \frac{\gamma}{\sum_{i \in \mathbb{N}} \gamma^i} = 1 - \gamma$, this means that $\alpha \geq \frac{2}{1+\gamma+\pi(t,t)}$. Finally, we note that $\pi(t, t) = \frac{\gamma^t}{\sum_{i \in [t]} \gamma^i} \leq \gamma^{t-1}$, which converges to 0 as t goes to infinity. Hence, for every $\epsilon > 0$, there is a $t \in \mathbb{N}$ such that $\frac{2}{1+\gamma} - \epsilon < \frac{2}{1+\gamma+\pi(t,t)}$, thus showing that f fails α-AFS for every $\alpha < \frac{2}{1+\gamma}$.

Claim 2: Upper Bound

To show our upper bound on the AFS approximation ratio of f, let $\mathcal{A}$ denote an approval profile, let $S \subseteq N$ be a group of voters, x^* a candidate such that $x^* \in \bigcap_{i \in S} A_i$, and p the distribution chosen by γ–MSP for $\mathcal{A}$. We moreover define $s = |S|$ and $n = |N|$ for a simple notation, and let $t = \max_{i \in N} |A_i|$ denote the maximal ballot size in $\mathcal{A}$. Furthermore, let $x_1, \ldots, x_m$ denote the sequence in which the γ-MPS rule processes the candidates for $\mathcal{A}$ and let k denote the index such that $x_k = x^*$. It holds that

$$\sum_{i\in S} u_i(p) = \frac{1}{n} \sum_{j\in[m]} |N_{x_j}(\mathcal{A}) \cap S| \cdot \Pi(\mathcal{A}, x_j, \{x_1, \ldots, x_{j-1}\}).$$

By the definition of sequential payment rules, we have for every candidate x_j with $j \leq k$ that

$$\begin{aligned}\Pi(\mathcal{A}, x_j, \{x_1, \ldots, x_{j-1}\}) &\geq \Pi(\mathcal{A}, x^*, \{x_1, \ldots, x_{j-1}\}) \\ &\geq \sum_{i\in S} \pi(|A_i|, |A_i \cap \{x_1, \ldots, x_{j-1}\}| + 1).\end{aligned}$$

Moreover, it holds that $\pi(|A_i|, |A_i \cap \{x_1, \ldots, x_{j-1}\}| + 1) \geq \pi(t, |A_i \cap \{x_1, \ldots, x_{j-1}\}| + 1)$ because $|A_i| \leq t$ and

$$\pi(x, y) = \frac{\gamma^y}{\sum_{i=1}^{x} \gamma^i} > \frac{\gamma^y}{\sum_{i=1}^{x+1} \gamma^i} = \pi(x+1, y)$$

for all $x < t$, $y \in [x]$. Consequently, we conclude that

$$\sum_{i\in S} u_i(p) \geq \frac{1}{n} \sum_{j\in[k]} |N_{x_j}(\mathcal{A}) \cap S| \sum_{i\in S} \pi(t, |A_i \cap \{x_1, \ldots, x_{j-1}\}| + 1).$$

For the next step, we define $\pi_i = \pi(t, i)$ for all $i \in [t]$ and $T_j = \sum_{\ell=1}^{j-1} |N_{x_{j-1}}(\mathcal{A}) \cap S|$ as the total number of approvals that the voters in S give to the candidates in $\{x_1, \ldots, x_{j-1}\}$. We will now derive a lower bound for $\sum_{i\in S} \pi(t, |A_i \cap \{x_1, \ldots, x_{j-1}\}| + 1)$ depending on T_j. To this end, let w^j denote the vector defined by $w^j_\ell = |\{i \in S \colon |A_i \cap \{x_1, \ldots, x_{j-1}\}| + 1 = \ell\}|$ for all $\ell \in [t]$, i.e., w^j_ℓ states the number of voters in S that approve $\ell - 1$ of the candidates in $\{x_1, \ldots, x_{j-1}\}$. It holds by definition that $\sum_{\ell\in[t]} w^j_\ell \cdot (\ell - 1) = T_j$ and $\sum_{\ell\in[t]} w^j_\ell = s$. Moreover, we have that

$$\sum_{i\in S} \pi(t, |A_i \cap \{x_1, \ldots, x_{j-1}\}| + 1) = \sum_{\ell\in[t]} w^j_\ell \pi_\ell.$$

Now, assume that there are two indices ℓ_1 and ℓ_2 such that $\ell_2 - \ell_1 \geq 2$, $w^j_{\ell_1} > 0$, and $w^j_{\ell_2} > 0$. We consider next the vector $\hat{w}$ with $\hat{w}_z = w^j_z$ for all $z \in [t] \setminus \{\ell_1, \ell_1 + 1, \ell_2 - 1, \ell_2\}$, $\hat{w}_{\ell_1} = w^j_{\ell_1} - 1$, $\hat{w}_{\ell_1+1} = w^j_{\ell_1+1} + 1$, $\hat{w}_{\ell_2-1} = w^j_{\ell_2-1} + 1$, and $\hat{w}_{\ell_2} = w^j_{\ell_2} - 1$. It holds that

$$\begin{aligned}\sum_{\ell\in[t]} w^j_\ell \pi_\ell - \sum_{\ell\in[t]} \hat{w}_\ell \pi_\ell &= (\pi_{\ell_1} - \pi_{\ell_1+1}) + (\pi_{\ell_2} - \pi_{\ell_2-1}) \\ &= (1-\gamma)\pi_{\ell_1} - (1-\gamma)\pi_{\ell_2-1} \\ &> 0.\end{aligned}$$

The second equation uses that $\gamma\pi_\ell = \pi_{\ell+1}$ for all $\ell \in [m-1]$ and the final inequality that $\pi_{\ell_1} > \pi_{\ell_2-1}$ (which holds since $\ell_1 < \ell_2 - 1$). Moreover, our construction ensures that $\sum_{\ell\in[t]} \hat{w}_\ell \cdot (\ell - 1) = T_j$ and $\sum_{\ell\in[t]} \hat{w}_\ell = s$. Now, we can repeat this process until we arrive at a vector w^* such that there are no indices ℓ_1 and ℓ_2 with $\ell_2 - \ell_1 \geq 2$, $w^*_{\ell_1} > 0$, and $w^*_{\ell_2} > 0$. In particular, for this vector, it still holds that $\sum_{\ell\in[t]} w^*_\ell \cdot (\ell - 1) = T_j$ and $\sum_{\ell\in[t]} w^*_\ell = s$ and that $\sum_{\ell\in[t]} w^j_\ell \pi_\ell > \sum_{\ell\in[t]} w^*_\ell \pi_\ell$.

Due to our constraints, we can moreover fully specify $\bar{w}$. To this end, let $\ell^* = \lfloor \frac{T_j}{s} \rfloor + 1$. Since w^* can only have two non-zero entries and $\sum_{\ell \in [t]} w^*_\ell \cdot (\ell - 1) = T_j$ and $\sum_{\ell \in [t]} w^*_\ell = s$, it must be that $w^*_{\ell^*} = s - (T_j \mod s)$, $w^*_{\ell^*+1} = T_j \mod s$, and $w^*_\ell = 0$ for all $\ell \in [t] \setminus \{\ell^*, \ell^*+1\}$. Finally, we infer that

$$\begin{aligned}\sum_{i \in S} \pi(t, |A_i \cap \{x_1, \dots, x_{j-1}\}| + 1) &= \sum_{\ell \in [t]} w^j_\ell \pi_\ell \\ &\geq \sum_{\ell \in [t]} w^*_\ell \pi_\ell \\ &= (s - (T_j \mod s))\pi_{\ell^*} + (T_j \mod s)\pi_{\ell^*+1}.\end{aligned}$$

We next define $K(u, v) = (s - v)\pi_u + v\pi_{u+1}$ for all $u \in [t-1]$ and $v \in [s] \cup \{0\}$ and note that $K(\lfloor \frac{T_j}{s} \rfloor + 1, T_j \mod s) = \sum_{\ell \in [t]} w^*_\ell \pi_\ell$. By our previous analysis, it holds that $\sum_{i \in S} \pi(t, |A_i \cap \{x_1, \dots, x_{j-1}\}| + 1) \geq K(\lfloor \frac{T_j}{s} \rfloor + 1, T_j \mod s)$ for all $j \in [k]$ and thus,

$$\sum_{i \in S} u_i(p) \geq \frac{1}{n} \sum_{j \in [k]} |N_{x_j}(\mathcal{A}) \cap S| \cdot K(\lfloor \frac{T_j}{s} \rfloor + 1, T_j \bmod s).$$

By the definition of K, we have that $K(u, v) > K(u, v+1)$ for all $u \in [t-1]$ and $v \in [s-1] \cup \{0\}$ and $K(u, s) = K(u+1, 0)$. This implies that

$$\begin{aligned}|N_{x_j}(\mathcal{A}) \cap S| \cdot K(\lfloor \frac{T_j}{s} \rfloor + 1, T_j \bmod s) &= \sum_{T = T_j}^{T_j + |N_{x_j}(\mathcal{A}) \cap S| - 1} K(\lfloor \frac{T_j}{s} \rfloor + 1, T_j \bmod s) \\ &\geq \sum_{T = T_j}^{T_j + |N_{x_j}(\mathcal{A}) \cap S| - 1} K(\lfloor \frac{T}{s} \rfloor + 1, T \bmod s).\end{aligned}$$

Moreover, it holds hat $|N_{x_k}(\mathcal{A}) \cap S| = s$ as all voters in S approve $x_k = x^*$. This means that

$$\sum_{i \in S} u_i(p) \geq \frac{1}{n} \sum_{T=0}^{T_k - 1} K(\lfloor \frac{T}{s} \rfloor + 1, T \bmod s) + \frac{s}{n} \cdot K(\lfloor \frac{T_k}{s} \rfloor + 1, T_k \bmod s).$$

Next, we define $u = \lfloor \frac{T_k}{s} \rfloor + 1$ and $v = T_k \mod s$. Based on this notation, we infer that

$$\sum_{T=0}^{T_k - 1} K(\lfloor \frac{T}{s} \rfloor + 1, T \bmod s) = \sum_{j=1}^{u-1} \sum_{\ell=0}^{s-1} K(j, \ell) + \sum_{\ell=0}^{v-1} K(u, \ell).$$

We recall that $\gamma \pi_j = \pi_{j+1}$, so we calculate that

$$\begin{aligned}\sum_{\ell=0}^{s-1} K(j, \ell) &= \sum_{\ell=0}^{s-1} (s - \ell) \cdot \pi_j + \ell \cdot \pi_{j+1} \\ &= \sum_{\ell=0}^{s-1} (s - (1-\gamma)\ell) \cdot \pi_j \\ &= s^2 \cdot \pi_j - (1-\gamma) \cdot \frac{(s-1)s}{2} \cdot \pi_j \\ &\geq \frac{1+\gamma}{2} \cdot s^2 \cdot \pi_j.\end{aligned}$$

Next, we observe that $\pi_j = \gamma^{j-1}\pi_1$ by the definition of the γ-MPS rule. Using the geometric sum, we derive that

$$\begin{aligned}\sum_{j=1}^{u-1}\sum_{\ell=0}^{s-1} K(j,\ell) &\geq \sum_{j=1}^{u-1} \frac{1+\gamma}{2}\cdot s^2 \cdot \pi_j \\ &= \frac{1+\gamma}{2}\cdot s^2\cdot \pi_1 \cdot \sum_{j=0}^{u-2}\gamma^j \\ &= s^2\cdot\pi_1\cdot\frac{1+\gamma}{2}\cdot\frac{1-\gamma^{u-1}}{1-\gamma}.\end{aligned}$$

We will next use a case distinction with respect to γ to complete the proof.

Case 1: $\gamma \geq \frac{1}{3}$. In this case, we compute the following lower bound for $\sum_{\ell=0}^{v-1} K(u,\ell) + s\cdot K(u,v)$:

$$\begin{aligned}\sum_{\ell=0}^{v-1} K(u,\ell) + s\cdot K(u,v) &= \sum_{\ell=0}^{v-1}\left((s-\ell)\cdot\pi_u + \ell\cdot\pi_{u+1}\right) + s\cdot\left((s-v)\cdot\pi_u + v\cdot\pi_{u+1}\right) \\ &= \sum_{\ell=0}^{v-1}\left((s-(1-\gamma)\ell)\cdot\pi_u\right) + s\cdot\left((s-(1-\gamma)v)\cdot\pi_u\right) \\ &= \pi_u\cdot\left(v\cdot s - (1-\gamma)\cdot\frac{(v-1)v}{2} + s^2 - (1-\gamma)v\cdot s\right) \\ &\geq \pi_u\cdot\left(s^2 + v\cdot s\cdot(1-\frac{3}{2}(1-\gamma))\right) \\ &\geq \pi_u\cdot s^2 \\ &= s^2\cdot\gamma^{u-1}\cdot\pi_1.\end{aligned}$$

In our second to last line, we use that $\gamma \geq \frac{1}{3}$, so $1-\frac{3}{2}(1-\gamma)\geq 0$. Based on our observations, we now conclude that

$$\begin{aligned}\sum_{i\in S} u_i(p) &\geq \frac{1}{n}\sum_{T=0}^{T_k-1} K(\lfloor\frac{T}{s}\rfloor+1, T \bmod s) + \frac{s}{n}\cdot K(\lfloor\frac{T_k}{s}\rfloor+1, T_k \bmod s) \\ &= \frac{1}{n}\left(\sum_{j=1}^{u-1}\sum_{\ell=0}^{s-1}K(j,\ell) + \sum_{\ell=0}^{v-1}K(u,\ell) + s\cdot K(u,v)\right) \\ &\geq \frac{s^2}{n}\pi_1\left(\frac{1+\gamma}{2}\cdot\frac{1-\gamma^{u-1}}{1-\gamma} + \gamma^{u-1}\right) \\ &= \frac{s^2}{n}\pi_1\cdot\frac{1+\gamma-\gamma^{u-1}-\gamma^u+2\gamma^{u-1}-2\gamma^u}{2(1-\gamma)} \\ &= \frac{s^2}{n}\pi_1\cdot\frac{1+\gamma+\gamma^{u-1}(1-3\gamma)}{2(1-\gamma)} \\ &\geq \frac{s^2}{n}\pi_1\cdot\frac{1+\gamma+1-3\gamma}{2(1-\gamma)} \\ &= \frac{s^2}{n}\pi_1.\end{aligned}$$

The first five inequalities here use our previous bounds and some standard transformation. For the sixth line (where we replace γ^{u-1} with 1), we use the fact that $1-3\gamma \leq 0$ and $\gamma^{u-1} \leq 1$ as $\frac{1}{3} \leq \gamma < 1$ and $u \geq 1$. Finally, employing the geometric series, we infer that

$$\pi_1 = \frac{\gamma}{\sum_{i=1}^{t} \gamma^i} \geq \frac{\gamma}{\sum_{i \in \mathbb{N}} \gamma^i} = \frac{1}{\sum_{i \in \mathbb{N}_0} \gamma^i} = \frac{1}{1/(1-\gamma)} = 1-\gamma.$$

By combining this insight with the fact that $\sum_{i \in S} u_i(p) \geq \frac{s^2}{n}\pi_1$, it finally follows that γ–MSP satisfies $\frac{1}{1-\gamma}$-AFS if $\gamma \geq \frac{1}{3}$.

Case 2: $\gamma < \frac{1}{3}$. Next, assume that $\gamma < \frac{1}{3}$. In this case, it holds that

$$\begin{aligned}
\sum_{\ell=0}^{v-1} K(u,\ell) + s \cdot K(u,v) &\geq \pi_u \cdot \left(s^2 + v \cdot s \cdot (1 - \frac{3}{2}(1-\gamma))\right) \\
&\geq \pi_u \cdot \left(s^2 + s^2 \cdot (1 - \frac{3}{2}(1-\gamma))\right) \\
&= \pi_u \cdot s^2 \cdot (\frac{1}{2} + \frac{3}{2}\gamma) \\
&= \gamma^{u-1} \cdot \pi_1 \cdot s^2 \cdot (\frac{1}{2} + \frac{3}{2}\gamma)
\end{aligned}$$

Note here that the first inequality can be derived analogously to the last case. However, we now have that $(1 - \frac{3}{2}(1-\gamma)) < 0$, which means that $v \cdot s \cdot (1 - \frac{3}{2}(1-\gamma)) \geq s^2 \cdot (1 - \frac{3}{2}(1-\gamma))$. In a similar fashion to the first case, we can now derive that

$$\begin{aligned}
\sum_{i \in S} u_i(p) &\geq \frac{s^2}{n}\pi_1 \left(\frac{1+\gamma}{2} \cdot \frac{1-\gamma^{u-1}}{1-\gamma} + \gamma^{u-1} \cdot (\frac{1}{2} + \frac{3}{2}\gamma) \right) \\
&= \frac{s^2}{n}\pi_1 \cdot \frac{1+\gamma-\gamma^{u-1}-\gamma^u+\gamma^{u-1}+3\gamma^u-\gamma^u-3\gamma^{u+1}}{2(1-\gamma)} \\
&= \frac{s^2}{n}\pi_1 \cdot \frac{1+\gamma+\gamma^u(1-3\gamma)}{2(1-\gamma)} \\
&\geq \frac{s^2}{n}\pi_1 \cdot \frac{1+\gamma}{2(1-\gamma)}.
\end{aligned}$$

We use here for the last inequality that $1-3\gamma > 0$. Finally, by using again that $\pi_1 \geq 1-\gamma$, it follows that $\sum_{i \in S} u_i(p) \geq \frac{s^2}{n} \cdot \frac{1+\gamma}{2}$, which proves that γ–MSP satisfies $\frac{2}{1+\gamma}$-AFS if $0 < \gamma < \frac{1}{3}$. □

A.3 Proof of Theorem 4.7

Theorem 4.7. *The following statements hold:*

(1) Let $t \in \mathbb{N}$. No sequential payment rule satisfies α-AFS for the maximal ballot size t and $\alpha < \frac{3}{2}(1-3^{-t})$.

(2) $\frac{1}{3}$–MSP *is the only sequential payment rule that satisfies $\frac{3}{2}(1-3^{-t})$-AFS for all maximal ballot sizes $t \in \mathbb{N}$.*

Proof. We split the theorem into three statements. Firstly, we will show that no sequential payment rule satisfies α-AFS for $\alpha < \frac{3}{2}(1-3^{-t})$ when t denotes the maximum feasible ballot size. Secondly, we will prove that each sequential payment rule except for the $\frac{1}{3}$-MSP rule also fails α-AFS for $\alpha = \frac{3}{2}(1-3^{-t})$ for some maximal feasible ballot size t. As the last point, we will show that the $\frac{1}{3}$-multiplicative sequential payment rule indeed matches this lower bound, thus singling out this rule as the fairest sequential payment rule.

Claim 1: No sequential payment rule satisfies α-AFS for $\alpha < \frac{3}{2}(1-3^{-t})$ when the maximal feasible ballot size is t.

Let f denote a sequential payment rule that satisfies α-AFS for some $\alpha \in \mathbb{R}$, let π be its payment willingness function, and let $t \in \mathbb{N}$ denote the maximum feasible ballot size. We moreover define $\pi_i = \pi(t,i)$ for all $i \in [t]$ and note that $\pi_1 \geq \pi_2 \geq \dots \geq \pi_t$ by definition. Using Proposition 4.5, we infer the following inequalities:

$$\begin{aligned}\pi_1 &\geq \frac{1}{\alpha}\\ \frac{1}{2}\pi_1 + \frac{3}{2}\pi_2 &\geq \frac{1}{\alpha}\\ \frac{1}{2}\pi_1 + \pi_2 + \frac{3}{2}\pi_3 &\geq \frac{1}{\alpha}\\ &\dots\\ \frac{1}{2}\pi_1 + \pi_2 + \dots + \pi_{t-1} + \frac{3}{2}\pi_t &\geq \frac{1}{\alpha}.\end{aligned}$$

Based on these inequalities, we will now derive an upper bound on $\frac{1}{\alpha}$ by using the fact that $\pi_1 + \pi_2 + \dots + \pi_t = 1$. In more detail, we first note that our inequalities imply that

$$\begin{aligned}\pi_1 &\geq \frac{1}{\alpha}\\ 2\pi_1 + 3\pi_2 &\geq \frac{3}{\alpha}\\ 2\pi_1 + 2\pi_2 + 3\pi_3 &\geq \frac{3}{\alpha}\\ &\dots\\ 2\pi_1 + 2\pi_2 + \dots + 2\pi_{t-1} + 3\pi_t &\geq \frac{3}{\alpha}.\end{aligned}$$

Now, by adding up the first and second constraint, we get that $3\pi_1 + 3\pi_2 \geq \frac{4}{\alpha}$. Using this constraint and adding in three times the third constraint, we derive that $9\pi_1 + 9\pi_2 + 9\pi_3 \geq \frac{13}{\alpha}$. As next step, we can add 9 times our fourth constraint to get that $27\pi_1 + 27\pi_2 + 27\pi_3 + 27\pi_4 \geq \frac{40}{\alpha}$. More generally, denoting the left hand sides of our modified inequalities by $I^1, \dots, I^t$, it holds that $I^1 + I^2 + 3I^3 + 9I^4 + \cdot + 3^{t-2}I^t = 3^{t-1}(\pi_1 + \pi_2 + \dots \pi_t) = 3^{t-1}$. Moreover, when adding up the right hand sides with the same coefficients, we get that $\frac{1}{\alpha} + \frac{3}{\alpha} + \frac{9}{\alpha} + \dots + \frac{3^{t-1}}{\alpha} = \frac{1}{\alpha}\sum_{i=0}^{t-1} 3^i$. Finally, solving for α shows that $\alpha \geq \frac{\sum_{i=0}^{t-1} 3^i}{3^{t-1}} = \frac{3}{2}(1-3^{-t})$ as $\sum_{i=0}^{t-1} 3^i = \frac{1}{2}(3^t - 1)$, which proves our lower bound.

Claim 2: No sequential payment rule other than the $\frac{1}{3}$-MSP rule satisfies $\frac{3}{2}(1-3^{-t})$-AFS for all maximum feasible ballot sizes $t \in \mathbb{N}$.

Let f denote a sequential payment rule, let π denote its payment willingness function, fix some arbitrary integer $t \in \mathbb{N}$ with $t \geq 2$, and define π_i as in Claim 1. We will show that $\pi_i = \frac{3^{-i}}{\sum_{j\in[t]} 3^{-j}}$ for all $i \in [t]$ if f satisfies $\frac{3}{2}(1-3^{-t})$-AFS for the maximal ballot size t. Because this holds for all $t \in \mathbb{N}$, this then proves that f is the $\frac{1}{3}$-MPS rule. Put differently, this shows that no other rule satisfies our fairness criteria. Now, to show this claim, we use again the inequalities of Proposition 4.5. In particular, we note here that $\frac{3}{2}(1-3^{-t}) = \sum_{i=0}^{t-1} 3^{-i}$. Hence, Proposition 4.5 shows that

$$\begin{aligned}
\pi_1 &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}\\
\frac{1}{2}\pi_1 + \frac{3}{2}\pi_2 &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}\\
\frac{1}{2}\pi_1 + \pi_2 + \frac{3}{2}\pi_3 &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}\\
&\dots\\
\frac{1}{2}\pi_1 + \pi_2 + \dots + \pi_{t-1} + \frac{3}{2}\pi_t &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}
\end{aligned}$$

Moreover, it holds by definition that $\sum_{i\in[t]} \pi_i = 1$. We will next show that the only payment willingness function that satisfies these conditions is the one of $\frac{1}{3}$-MSP, i.e., that $\pi_i = \frac{3^{-i}}{\sum_{j\in[t]} 3^{-j}}$ for all $i \in [t]$. By our first inequality, we immediately derive that $\pi_1 \geq \frac{1}{\sum_{i=0}^{t-1} 3^{i}} = \frac{3^{-1}}{\sum_{i\in[t]} 3^{-i}}$. Next, using the fact that $\sum_{i\in[t]} \pi_i = 1$, we can rewrite our last inequality by

$$\begin{aligned}
\frac{1}{2}\pi_1 + \pi_2 + \dots + \pi_{t-1} + \frac{3}{2}\pi_t &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}\\
\iff 1 - \frac{1}{2}\pi_1 + \frac{1}{2}\pi_t &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}\\
\iff \pi_t &\geq \frac{2}{\sum_{i=0}^{t-1} 3^{-i}} + \pi_1 - 2.
\end{aligned}$$

Using the fact that $\pi_1 \geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}$ and that $\sum_{i=0}^{t-1} 3^{-i} = \frac{3}{2}(1-3^{-t})$, we get that

$$\pi_t \geq \frac{3}{\sum_{i=0}^{t-1} 3^{-i}} - \frac{2\sum_{i=0}^{t-1} 3^{-i}}{\sum_{i=0}^{t-1} 3^{-i}} \geq \frac{3^{-(t-1)}}{\sum_{i=0}^{t-1} 3^{-i}} = \frac{3^{-t}}{\sum_{i=1}^{t} 3^{-i}}.$$

Next, assume inductively that there is $j \in \{3, \dots, t\}$ such that $\pi_\ell \geq \frac{3^{-(\ell-1)}}{\sum_{i=0}^{t-1} 3^{-j}}$ for all $\ell \in \{j, \dots, t\}$. We will show that the same holds for $j-1$. To this end, we first note that

$$\begin{aligned}
\frac{1}{2}\pi_1 + \pi_2 + \dots + \pi_{j-2} + \frac{3}{2}\pi_{j-1} &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}\\
\iff (1 - \sum_{i=j}^{t} \pi_i) + \frac{1}{2}\pi_{j-1} - \frac{1}{2}\pi_1 &\geq \frac{1}{\sum_{i=0}^{t-1} 3^{-i}}\\
\iff \pi_{j-1} &\geq \frac{2}{\sum_{i=0}^{t-1} 3^{-i}} + \pi_1 + 2\sum_{i=j}^{t} \pi_i - 2.
\end{aligned}$$

Using the same computations as for the lower bound of π_t shows that $\frac{2}{\sum_{i=0}^{t-1} 3^{-i}} + \pi_1 - 2 \geq \frac{3^{-(t-1)}}{\sum_{i=0}^{t-1} 3^{-i}}$. Combining this with the lower bounds on π_i for $i \geq j$ given by the induction hypothesis, we conclude that

$$\begin{aligned}\pi_{j-1} &\geq \frac{3^{-(t-1)}}{\sum_{i=0}^{t-1} 3^{-i}} + 2\sum_{i=j}^{t} \frac{3^{-(i-1)}}{\sum_{i=0}^{t-1} 3^{-i}} \\ &= \frac{3^{-(j-1)}}{\sum_{i=0}^{t-1} 3^{-i}} (3^{-(t-j)} + 2\sum_{i=0}^{t-j} 3^{-i}) \\ &= \frac{3^{-(j-1)}}{\sum_{i=0}^{t-1} 3^{-i}} (3^{-(t-j)} + 3 - 3^{-(t-j)}) \\ &= \frac{3^{-(j-2)}}{\sum_{i=0}^{t-1} 3^{-i}}.\end{aligned}$$

Hence, it holds that $\pi_j \geq \frac{3^{-(j-1)}}{\sum_{i=0}^{t-1} 3^{i}} = \frac{3^{-j}}{\sum_{i\in[t]} 3^{-i}}$ for all $j \in [t]$. Since $\sum_{j\in[t]} \pi_j = 1$, all of these inequalities must be tight, which means that the values π_j correspond with the payment willingness of $\frac{1}{3}$–MSP. Because this analysis holds for all t, it follows that $\frac{1}{3}$–MSP is the only sequential payment rule that can satisfy our lower bound tightly for all maximum feasible ballot sizes.

Claim 3: $\frac{1}{3}$**–MSP satisfies** $\frac{3}{2}(1-3^{-t})$**-AFS for all profiles** $\mathcal{A}$ **with maximum ballot size** $t \in \mathbb{N}$**.**

Let $\mathcal{A}$ denote a profile such that $|A_i| \leq t$ for all $i \in N$, and let $S \subseteq N$ be a group of voters and x^* a candidate such that $x^* \in \bigcap_{i\in S} A_i$. We need to show for the distribution $p = \frac{1}{3}$–MSP$(\mathcal{A})$ that

$$\sum_{i\in S} u_i(p) \geq \frac{1}{\frac{3}{2}(1-3^{-t})} \cdot \frac{|S|^2}{n}.$$

To this end, we recall that we have shown in the proof of Theorem 4.6 that every γ–MSP rule with $\gamma \geq \frac{1}{3}$ satisfies that $\sum_{i\in S} u_i(q) \geq \pi(\ell, 1) \cdot \frac{|S|^2}{n}$, where π denotes the payment willingness function of the corresponding distribution rule, q the distribution chosen by the rule, and $\ell = \max_{i\in N} |A_i| \leq t$ the maximal ballot size in $\mathcal{A}$. By applying this inequality to $\frac{1}{3}$–MSP and using that $\pi(\ell, 1) \geq \pi(t, 1)$ for its payment willingness function, it follows that

$$\sum_{i\in S} u_i(p) \geq \frac{3^{-1}}{\sum_{j\in[t]} 3^{-j}} \cdot \frac{|S|^2}{n} = \frac{1}{\sum_{j=0}^{t-1} 3^{-j}} \cdot \frac{|S|^2}{n}.$$

Finally, using that $\sum_{j=0}^{t-1} 3^{-j} = \frac{3}{2}(1-3^{-t})$ then shows that $\frac{1}{3}$–MSP indeed satisfies $\frac{3}{2}(1-3^{-t})$-AFS for profiles with a maximal ballot size of t. $\square$