Robust Voting Rules on the Interval Domain

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In social choice theory, the domain of single-peaked preference relations has proven invaluable for obtaining positive results as the phantom median rules of Moulin (1980) satisfy numerous desirable properties. However, when we extend this domain to allow voters to vote for more than one alternative, it is no longer clear which voting rule to use. We will thus study voting rules on the interval domain, where the alternatives are arranged according to an externally given strict total order and voters indicate their preferences by reporting subintervals of this order. In more detail, in this paper we introduce and characterize the class of position-threshold rules, which roughly compute a collective position of the voters with respect to every alternative and choose the left-most alternative such that the collective position of the alternative exceeds its threshold value. Our characterization mainly relies on reinforcement, a well-known population consistency condition, and robustness, a new axiom that requires that small changes in the voters' intervals only result in small changes in the outcome. Our main result can thus be seen as an extension of Moulin's (1980) influential characterization of phantom median rules to the interval domain. Furthermore, we characterize an generalization of the median rule to the interval domain, and we give an extension of our main result for the case of selecting a fixed-sized multiset of alternatives.

1. Introduction

A ubiquitous phenomenon in today's societies is collective decision-making: given the possibly conflicting preferences of multiple agents, a joint decision should be reached in a fair and principled way. Such processes of group decision making are formally investigated in the field of social choice theory, where researchers study voting rules from a mathematical perspective. However, despite significant advances in the understanding of voting rules (e.g., Arrow et al., 2011; Brandt et al., 2016), social choice theory fails to give a clear recommendation on which method to use because strong impossibility theorems (e.g., Arrow, 1951; Gibbard, 1973; Satterthwaite, 1975; Moulin, 1988) demonstrate that there are invariable tradeoffs between voting rules.

One of the most successful escape routes to such impossibility theorems is to impose more structure on the voters' preferences. In particular, the notion of *single-peaked preferences*, which goes back to Black (1948), has turned out very fruitful. The basic idea of single-peaked preferences is that there is a strict total order $\rhd$ over the alternatives and that the preference relation of every voter specifies an ideal alternative p such that alternatives become less preferred when moving further away from p with respect to $\rhd$. For single-peaked preferences, it is known that most impossibility theorems vanish and there is large consensus on which voting rules to use (e.g., Border and Jordan, 1983; Sprumont, 1995; Ching, 1997; Ehlers and Storcken, 2008; Weymark, 2011): the median rule and, more generally, the phantom median rules introduced by Moulin (1980) have superior axiomatic properties. Roughly, the median rule sorts the voters with respect to their top-ranked alternatives according to $\rhd$ and then returns the favorite alternative of the median voter. Moreover, phantom median rules generalize the median rule by computing the median rule for the n original voters and $n-1$ phantom voters which always report a fixed single-peaked preference relation.

The appeal of these phantom median rules lies in the fact that they are the only voting rules on the domain of single-peaked preferences that satisfy anonymity (i.e., all voters are treated equally), unanimity (i.e., an alternative is guaranteed to be chosen if it is the favorite alternative of all voters), and strategyproofness (i.e., voters cannot benefit by lying about their true preferences) (Moulin, 1980; Weymark, 2011). This combination of axioms is remarkable because no voting rule satisfies all three properties on the domain of all preference relations (Gibbard, 1973; Satterthwaite, 1975). Moreover, phantom median rules atisfy numerous further desirable properties such as tops-onlyness (i.e., voters only need to report their favorite alternative), reinforcement (i.e., when combining two elections with the same winner, the winner remains the same), and participation (i.e., voters cannot benefit by abstaining) (Moulin, 1984; Jennings et al., 2024).

Given the success of phantom median rules on the domain of single-peaked preferences, it is surprising that rather little is known about voting rules when slightly modifying the setting. In particular, in this paper, we are interested in the case that voters report intervals (i.e., sets of consecutive alternatives with respect to $\rhd$). We refer to this problem as voting on the interval domain and we see at least three convincing explanations of why voters may want to report such intervals instead of single-peaked preference relations.

(1) If there are several identical (or close to identical) alternatives, it seems plausible that voters may be indifferent between such alternatives. When the voters' preferences are moreover consistent with an externally given order $\rhd$, this results in variants of single-peaked preferences that allow for ties between alternatives, such as single-plateaued or weakly single-peaked preferences (e.g., Moulin, 1984; Berga, 1998; Austen-Smith and Banks, 1999). In particular, for all of these models, the set of most preferred alternatives of each voter is an interval, so we may interpret the reported intervals as the sets of the voters' favorite alternatives.

(2) Another motivation for intervals is that, in many settings with single-peaked preferences, it may be cognitively demanding for agents to identify a uniquely most preferred alternative even if it exists. For instance, when voting on budget proposals for an event, it is reasonable to assume that the voters' preferences are single-peaked in the proposed amounts of money. However, it is for voters often hard to grasp the consequences of each budget proposal and they may thus not be able to identify their most preferred proposal. It then seems desirable to ask voters for a budget range instead of a single budget proposal to alleviate their cognitive burden.

(3) Lastly, we believe that there are situations where voters know their most-preferred alternative but are willing to report a larger set of alternatives to reach a socially more acceptable consensus. For instance, when voting for a meeting time, it is frequently the case that the voters have a most-preferred option but are willing to accept other outcomes for the sake of a unanimous compromise.[1] Hence, by reporting larger intervals, voters may hope to arrive at a socially more desirable outcome.

Importantly, in both the second and third example, voters are not necessarily indifferent between the alternatives in their reported interval, but they are either not aware about their own preferences or willingly ignore them. We will thus treat the interval domain as a strategy space instead of the set of the voters' preferences: we assume that the interval of each voter contains the alternatives he likes, but we do not assume that a voter's interval fully describes his preferences.

Contribution. In this paper, we will study voting rules on the interval domain and, in particular, we will introduce and characterize the class of position-threshold rules. Roughly, these rules determine for every alternative a collective position, which quantifies the voters' relative position regarding this alternative, and then choose the left-most alternative whose collective position exceeds its threshold value. In more detail, position-threshold rules are defined by a weight vector $\alpha \in [0,1]^m$ and a threshold vector $\theta \in (0,1)^m$. The weight vector α is used to quantify the relative position of the voters with respect to the alternatives: a voter's relative position to an alternative x_i is 0 if all alternatives in his interval are right of x_i, 1 if all alternatives in his interval are left of or equal to x_i, and α_i otherwise. Then, a position-threshold rule computes the collective position of an alternative by summing up the voters' individual positions to this alternative, and it returns the left-most alternative whose collective position divided by the number of voters exceeds its threshold value. While it may not be clear from this description, position-threshold rules generalize phantom median rules because phantom median rules can also be formulated via collective positions of alternatives and threshold values.

As our main contribution, we characterize position-threshold rules based on a robustness notion, a consistency condition for variable electorates, and some basic auxiliary conditions. In more detail, the main axioms of our characterization are robustness and reinforcement. Robustness formalizes that a small change in the interval of a voter should only result in a

[1] One may argue that such concerns should be incorporated in the voters' preference relation rather than claiming that voters do not exclusively vote for their favorite alternative. We reject this argument because we believe that, in practice, voters often willingly accept alternatives they find less desirable.

small change in the output: if a voter removes his left-most (resp. right-most) alternative from his interval, the outcome is not allowed to change at all or to only change from the old left-most (resp. right-most) alternative of the voter to his new left-most (resp. right-most) alternative. While this axiom is new, it is conceptually related to many prominent conditions such as localizedness (Gibbard, 1977), Maskin-monotonicity (Maskin, 1999), and uncompromisingness (Border and Jordan, 1983). Moreover, to further motivate this axiom, we show in Proposition 1 that robustness is closely related to incentive properties when the voters have weakly single-peaked preference relations because it, e.g., implies strategyproofness. Our second main condition, reinforcement, requires that if an alternative is chosen in two disjoint elections, it should be also chosen in a combined election. Variants of this axiom feature in numerous influential works in social choice theory (e.g., Smith, 1973; Young, 1975; Young and Levenglick, 1978; Fishburn, 1978; Brandl et al., 2016; Lackner and Skowron, 2021; Brandl and Peters, 2022). In our main result, we then show that a voting rule on the interval domain is a position-threshold rule if and only if it satisfies robustness, reinforcement, and three auxiliary conditions called anonymity, unanimity, and right-biased continuity (Theorem 1).

Based on this result, we furthermore aim to extend the median rule to the interval domain. To this end, we note that the median rule is the only phantom median rule that guarantees to select an alternative that is top-ranked by a strict majority of the voters. In the context of intervals, we call this condition the majority criterion and formalize it by requiring that an alternative is chosen if it is uniquely reported by more than half of the voters. We then show that there is only a single position-threshold rule that satisfies the majority criterion and the natural condition of strong unanimity (if the intersection of the intervals of all voters is non-empty, an alternative in this intersection needs to be chosen): the endpoint-median rule (Theorem 2). Intuitively, this rule replaces the interval of each voter with two singleton ballots corresponding to the left-most and right-most alternative in the interval, and then executes the median rule. In combination with Theorem 1, we derive that the endpoint-median rule is the only voting rule on the interval domain that satisfies anonymity, strong unanimity, the majority criterion, robustness, reinforcement, and right-biased continuity.

Finally, we also extend our main result to multi-winner elections, where we need to assign a fixed number of seats to the alternatives instead of choosing a single alternative. When interpreting our alternatives as parties, this model captures the elections of city councils and parliaments and it is thus closely related to apportionment (see, e.g., Balinski and Young, 2001; Pukelsheim, 2014). However, in apportionment, it is commonly assumed that voters only vote for a single party, whereas we allow them to report intervals. We then demonstrate that effectively all our results carry over to this interval-apportionment setting by suitably adapting our axioms. In more detail, based on natural extensions of our axioms, we show that every multi-winner voting rule that satisfies anonymity, unanimity, robustness, right-biased continuity, and a strong form of reinforcement can be decomposed into position-threshold rules (Theorem 3): each such rule assigns each seat of the committee independently based on a position-threshold rule.

Related work. Our paper is closely related to the study of strategyproof voting rules on the domain of single-peaked preference relations, which has garnered significant attention. In particular, Moulin's characterization of phantom median rules (Moulin, 1980) has inspired a large body of follow-up works, including extensions to multi-dimensional variants of single-peakedness (Border and Jordan, 1983; Zhou, 1991; Barberà et al., 1993), non-anonymous variants of phantom median rules (Ching, 1997), and randomized versions of this result (Ehlers et al., 2002; Peters et al., 2014; Pycia and Ünver, 2015). For comprehensive overviews of these early studies on strategyproof voting rules for single-peaked preferences, we refer readers to the surveys by Sprumont (1995) and Weymark (2011). More recent research (e.g., Chatterji et al., 2013; Reffgen, 2015; Chatterji et al., 2016; Chatterji and Massó, 2018; Chatterji and Zeng, 2023) focuses on strategyproof voting rules for somewhat technical extensions of the single-peaked domain, such as semi-single-peaked preferences or lattice single-peaked preferences. In addition to demonstrating the existence of attractive strategyproof voting rules on these domains, these works show that, under various side conditions, the corresponding domains are necessary for the existence of non-dictatorial strategyproof voting rules.

In contrast to the aforementioned papers, we focus on a setting where voters may report an interval of alternatives rather than a single favorite alternative or a ranking of the alternatives. To the best of our knowledge, comparable settings have only been studied by Moulin (1984), Berga (1998), and Berga and Moreno (2009), who investigate voting rules for single-plateaued preference relations, a generalization of single-peaked preferences that allows for indifferences. In more detail, Moulin (1984) characterizes the class of generalized Condorcet winner rules, which roughly compute the median rule with respect to the endpoints of the voters' intervals and some additional parameters depending on the profile, based on two axioms similar to Arrow's independence of irrelevant alternatives. By contrast, Berga (1998) and Berga and Moreno (2009) focus on strategyproof voting rules for the single-plateaued domain but do not provide a closed-form characterization of such rules. Our results differ from these works as we focus on robustness and reinforcement and characterize the class of position-threshold rules based on these axioms.

Furthermore, our paper is related to the problem of facility location, where a public-good facility needs to be placed on the real line based on the voter's preferences over the possible positions. In particular, in facility location it is typically assumed that the voters report their ideal positions for the facility and that their cost for a location is its distance to their ideal position. Put differently, facility location can be seen as voting on the real line with single-peaked preferences. The goal of facility location is to identify strategyproof voting rules that optimize some objective such as the utilitarian or egalitarian social welfare (e.g., Procaccia and Tennenholtz, 2013; Feldman et al., 2016; Chan et al., 2021), a problem for which Moulin's (1980) characterization has proven invaluable. Since recent works on facility location also investigate scenarios where the voters report intervals instead of a single location (Elkind et al., 2022; Zhou et al., 2023), we believe that our results can also provide valuable insights for this setting.

Finally, voting on the interval domain is loosely connected to the problem of interval aggregation (e.g., Farfel and Conitzer, 2011; Klaus and Protopapas, 2020; Endriss et al., 2022), where multiple input intervals need to be aggregated into an output interval.

However, as we aim to choose a single single alternative based on the voters' intervals, we end up with rather different axioms that those considered in interval aggregation.

2. The Model

We will use a variable-electorate framework in this paper and thus let $\mathbb{N} = \{1, 2, \dots\}$ denote an infinite set of voters and $A = \{x_1, \dots, x_m\}$ a finite set of $m \geq 2$ alternatives. Intuitively, $\mathbb{N}$ is the set of all possible voters and a concrete electorate N is a finite and non-empty subset of $\mathbb{N}$. The set of all electorates is therefore defined by $\mathcal{F}(\mathbb{N}) = \{N \subseteq \mathbb{N} \colon N \text{ is non-empty and finite}\}$. The central assumption in this paper is that there is an externally given strict total order $\rhd$ over the alternatives. Throughout the paper, we will assume that $\rhd$ is given by $x_1 \rhd x_2 \rhd \dots \rhd x_m$, and we define the relation $\trianglerighteq$ by $x_i \trianglerighteq x_j$ if and only if $x_i = x_j$ or $x_i \rhd x_j$ for all $x_i, x_j \in A$. Given an electorate $N \in \mathcal{F}(\mathbb{N})$, the voters are asked to report intervals of $\rhd$ to indicate the alternatives they like. Formally, a set of alternatives I is an *interval* (with respect to $\rhd$) if $x_i \in I$ and $x_k \in I$ imply $x_j \in I$ for all alternatives $x_i, x_j, x_k \in A$ with $x_i \rhd x_j \rhd x_k$. Since intervals are fully specified by their endpoints, we define $[x_i, x_k] = \{x_j \in A \colon x_i \trianglerighteq x_j \trianglerighteq x_k\}$ as the interval from x_i to x_k. The set of all intervals, or the *interval domain*, is given by $\Lambda = \{[x_i, x_j] \subseteq A \colon x_i \trianglerighteq x_j\}$.

An *interval profile* $\mathcal{I} = (I_{i_1}, \dots, I_{i_n})$ for a given electorate $N = \{i_1, \dots, i_n\}$ contains the interval of every voter $i \in N$, i.e., it is a function from N to Λ. Next, the set of all interval profiles for a fixed electorate N is defined by Λ^N and the set of all possible interval profiles is $\Lambda^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \Lambda^N$. Given an interval profile $\mathcal{I}$, we will denote by $N_{\mathcal{I}}$ the set of voters that report an interval for this profile and by $n_{\mathcal{I}} = |N_{\mathcal{I}}|$ the size of this set. Our primary goal in this paper is to select a single winning alternative based on such interval profiles. To this end, we will study *voting rules* which are formally functions that map every interval profile $\mathcal{I} \in \Lambda^*$ to a single alternative $x \in A$.

2.1. Relation to Single-peaked Preferences

To relate the interval domain to existing works, we will next introduce the domains of single-peaked and weakly single-peaked preference relations. To this end, we first define *(weak) preference relations* $\succsim$ as complete and transitive binary relations on A, where $x \succsim y$ means that the respective voter weakly prefers x to y. As usual, a preference relation is called *strict* if it is additionally anti-symmetric, i.e., if no ties between alternatives are permitted. We denote by $\mathcal{R}$ the set of all preference relations and by $\mathcal{P}$ the set of all strict preference relations. Now, the idea of (weak) single-peakedness is that preference relations should be consistent with $\rhd$: there is an ideal alternative x and, as we move away from x with respect to $\rhd$, the alternatives become worse. In more detail a preference relation $\succsim \in \mathcal{P}$ (resp. $\succsim \in \mathcal{R}$) is called *single-peaked* (resp. *weakly single-peaked*) if there is an alternative x such that $x \succsim y \succsim z$ for all $y, z \in A$ with $x \rhd y \rhd z$ or $z \rhd y \rhd x$. We note that, since single-peaked preference relations are by definition strict, the ideal alternative x is unique. By contrast, weakly single-peaked preference relations allow for indifferences, so there may be multiple favorite alternatives. The sets of single-peaked and weakly single-peaked preference relations are given by $\mathcal{P}_{\rhd}$ and $\mathcal{R}_{\rhd}$, respectively.

Moreover, we define by $\mathcal{P}_\triangleright^N$ and $\mathcal{R}_\triangleright^N$ the sets of (weakly) single-peaked preference profiles for a fixed electorate N, and by $\mathcal{P}_\triangleright^* = \bigcup_{N\in\mathcal{F}(\mathbb{N})} \mathcal{P}_\triangleright^N$ and $\mathcal{R}_\triangleright^* = \bigcup_{N\in\mathcal{F}(\mathbb{N})} \mathcal{R}_\triangleright^N$ the sets of all (weakly) single-peaked preference profiles.

If we assume that the voters' true preferences are single-peaked or weakly single-peaked, there are at least three plausible ways how they may infer their interval.

(1) Maybe the most direct approach is that each voter reports the the set of his most-preferred alternatives $T(\succsim) = \{x \in A \colon \forall y \in A \colon x \succsim y\}$, which is known to be an interval when $\succsim$ is weakly single-peaked (e.g., Puppe, 2018). We note that this approach has been considered before (e.g., Moulin, 1984; Berga, 1998) and it entails that voters are indifferent between all alternatives in their interval. As a consequence, it may not accurately describe situations where voters are not fully aware of their own preferences or take social considerations into account.

(2) Another method to transform a weakly single-peaked preference relation into an interval is to assume that the reported interval is the set of alternatives that exceed some utility threshold. To make this more precise, let $U(\succsim, x) = \{y \in A \colon y \succsim x\}$ denote the upper contour set of alternative x with respect to the preference relation $\succsim$. Then, we say that an interval I is *consistent* with a weakly single-peaked preference relation $\succsim$ if $I = U(\succsim, x)$ for some alternative $x \in A$. This approach has been studied in the context of approval voting for general preferences (e.g., Brams and Fishburn, 1978; Niemi, 1984; Brams and Fishburn, 2007; Endriss, 2013), and it can be used to describe voters who are not fully aware of their own preferences as only a limited information is needed to infer the set $U(\succsim, x)$.

(3) We believe that even consistency may be too strong for practical purposes. In particular, when voters take additional considerations such as the acceptability of the outcome into account, a voter's interval may not be consistent with his preference relation. We will thus consider an even weaker form of consistency, which only requires that the reported interval contains the voter's most preferred alternatives. More formally, we say an interval I is *top-consistent* with a weakly single-peaked preference relation $\succsim$ if $T(\succsim) \subseteq I$.

2.2. Robustness

We will next introduce the central axiom for our analysis called robustness. The rough idea of this axiom is that small changes in the voters' intervals should only cause small changes in the outcome. In more detail, robustness requires of a voting rule that, if a voter removes the left-most (resp. right-most) alternative from his interval, then the winner cannot change at all or the winner changes from the old left-most (resp. right-most) alternative to the new left-most (resp. right-most) alternative. To formalize this, let $\mathcal{I}^{i\downarrow x}$ denote the interval profile derived from another profile $\mathcal{I}$ by removing alternative x from the interval of voter i, and note that $\mathcal{I}^{i\downarrow x}$ is a valid interval profile only if $|I_i| \geq 2$ and x is the left-most or right-most alternative in I_i. Then, robustness is defined as follows.

Definition 1 (Robustness)**.** A voting rule f is *robust* if, for all interval profiles $\mathcal{I} \in \Lambda^*$, voters $i \in N_{\mathcal{I}}$, and alternatives x_ℓ, x_r such that $I_i = [x_\ell, x_r]$ and $x_\ell \rhd x_r$, it holds that

(i) $f(\mathcal{I}) = f(\mathcal{I}^{i\downarrow x_\ell})$, or $f(\mathcal{I}) = x_\ell$ and $f(\mathcal{I}^{i\downarrow x_\ell}) = x_{\ell+1}$, and

(ii) $f(\mathcal{I}) = f(\mathcal{I}^{i\downarrow x_r})$, or $f(\mathcal{I}) = x_r$ and $f(\mathcal{I}^{i\downarrow x_r}) = x_{r-1}$.

Robustness can equivalently be formulated in terms of adding an alternative to a voter's interval: if we, e.g, add a new left-most alternative to the voter's interval, the winner cannot change or the winner changes from the voter's old left-most alternative to his new left-most alternative. Moreover, we note that similar invariance notions have been studied before (e.g., Gibbard, 1977; Saijo, 1987; Maskin, 1999; Muto and Sato, 2017; Bredereck et al., 2021), with the most prominent examples being localizedness and Maskin-monotonicity. We thus believe that robustness in itself is a desirable property as it prohibits that the outcome changes in an unexpected way. To further motivate this axiom, we will next analyze the relation of robustness to various incentive properties.

Relation to uncompromisingness. First, we explain how robustness relates to a condition called uncompromisingness, which is commonly studied in the context of single-peaked preference relations (e.g., Border and Jordan, 1983; Sprumont, 1995; Ehlers et al., 2002). Uncompromisingness requires of a voting rule on $\mathcal{P}_{\rhd}^N$ that the outcome is not allowed to change if the voter's favorite alternative stays on the same side of the current winner. More formally, a voting rule f on $\mathcal{P}_{\rhd}^N$ is *uncompromising* if $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{P}_{\rhd}^N$ and voters $i \in N$ such that *(i)* $\succsim_j = \succsim'_j$ for all $j \in N \setminus \{i\}$ and *(ii)* $T(\succsim_i) \rhd f(R)$ and $T(\succsim'_i) \unrhd f(R)$, or $f(R) \rhd T(\succsim_i)$ and $f(R) \unrhd T(\succsim'_i)$ (while slightly abusing notation, $T(\succsim_i)$ denotes here the favorite alternative of voter i instead of the corresponding singleton set). Equivalently, uncompromisingness can be formulated by requiring that, if a voter changes his favorite alternative from x_i to x_{i+1} (resp. x_{i-1}), the outcome is not allowed to change or to change from x_i to x_{i+1} (resp. x_{i-1}). It is then easy to see that robustness generalizes this formulation of uncompromisingness from the domain of single-peaked preference relations to the interval domain.

Relation to strategyproofness. Next, we observe that robustness is closely related to strategyproofness when assuming that voters have weakly single-peaked preferences. The rough idea of strategyproofness is that voters cannot benefit by lying about their true preferences. Following Moulin (1980) and Berga (1998), we formalize this by requiring that it is always in the best interest of voters to report their favorite alternatives $T(\succsim_i)$ truthfully: a voting rule f is *strategyproof* if $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$ for all interval profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ with $N_{\mathcal{I}} = N_{\mathcal{I}'}$, voters $i \in N_{\mathcal{I}}$, and weakly single-peaked preference relation $\succsim_i$ such that and $I_i = T(\succsim_i)$ and $I_j = I'_j$ for all $j \in N \setminus \{i\}$. As we show in Proposition 1, robustness implies strategyproofness when voters have weakly single-peaked preferences.

Relation to sincerity. While strategyproofness is the most common incentive property in social choice theory, it may not be the right axiom for our setting. Specificallly, strategyproofness assumes that voters fully adhere to their preference relation $\succsim_i$, whereas one of our motivations for studying intervals was that voters may not be fully aware of

$\succsim_i$ or willingly ignore some preferences. To address such settings, we introduce another incentive property that we call sincerity. For this condition, we assume that every voter has a true interval I_i that is top-consistent with his weakly single-peaked preference relation $\succsim_i \in \mathcal{R}_\triangleright$. Then, we say that a voting rule f is *sincere* if, for all electorates $N \in \mathcal{F}(\mathbb{N})$, voters $i \in N$, interval profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^N$, and weakly single-peaked preference relations $\succsim_i \in \mathcal{R}_\triangleright$ such that $I_j = I'_j$ for all $j \in N \setminus \{i\}$ and I_i is top-consistent with $\succsim_i$, it holds that *(i)* $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$ if $f(\mathcal{I}) \notin I_i$, and *(ii)* $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$ if $I_i \subseteq I'_i$. Put differently, the first condition ensures that a voter cannot manipulate unless an alternative in his interval is chosen. This is reasonable as we assume that voters view I_i as their truthful interval and thus ignore possible preferences within this set. Moreover, it is impossible to prohibit voters from manipulating when taking the preferences between the alternatives in I_i into account. The second condition states that it is never beneficial for a voter to report larger intervals than necessary. Hence, sincerity also incentivizes voters to explore their own preferences to derive the most desirable outcome. As we show in the next proposition, robustness is mathematically equivalent to sincerity.

Proposition 1. *If the voters infer their intervals from weakly single-peaked preference relations, it holds that*

(i) every robust voting rule on Λ^* *is strategyproof, and*

(ii) a voting rule on Λ^* *is robust if and only if it satisfies sincerity.*

Proof. For the proof of Claim *(i)*, we observe that sincerity implies strategyproofness by assuming that the true interval I_i of a voter i coincides with his set of most preferred alternatives, i.e., $I_i = T(\succsim_i)$. If $f(\mathcal{I}) \in I_i$ for some profile $\mathcal{I}$, this means that voter i cannot manipulate as one of his most preferred alternatives is chosen. On the other hand, if $f(\mathcal{I}) \notin I_i$, sincerity requires that $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$ for all interval profiles $\mathcal{I}'$ such that $N_\mathcal{I} = N_{\mathcal{I}'}$ and $I_j = I'_j$ for all $j \in N \setminus \{i\}$. Hence, we focus on proving that robustness is equivalent to sincerity, which also shows that robustness implies strategyproofness. We will prove both directions of this equivalence separately.

($\implies$) We first assume that f is a robust voting rule on Λ^* and will show that it is also sincere. To this end, let $N \in \mathcal{F}(\mathbb{N})$ denote an electorate, $i \in N$ a voter, $\mathcal{I}, \mathcal{I}' \in \Lambda^N$ two interval profiles with $I_j = I'_j$ for all $j \in N \setminus \{i\}$, and $\succsim_i \in \mathcal{R}_\triangleright$ a weakly single-peaked preference relation that is consistent with I_i. For the ease of presentation, we furthermore define the alternatives $x_\ell, x_r, x_{\ell'}, x_{r'}$ such that $I_i = [x_\ell, x_r]$ and $I'_i = [x_{\ell'}, x_{r'}]$. We first assume that $f(\mathcal{I}) \notin I_i$ and will show that $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$. To this end, we suppose that $f(\mathcal{I}) \triangleright x_\ell$ and note that the case that $x_r \triangleright f(\mathcal{I})$ is symmetric. Now, if $f(\mathcal{I}) \trianglerighteq x_{\ell'}$, we infer from robustness that $f(\mathcal{I}) = f(\mathcal{I}')$. In more detail, let y denote the alternative that is directly right of $f(\mathcal{I})$, i.e., $f(\mathcal{I}) \triangleright y$ and there is no other alternative z with $f(\mathcal{I}) \triangleright z \triangleright y$. We first blow up I_i to the interval $[y, x_m]$ by one after another adding alternatives to voter i's interval. Robustness implies for every step that the winner does not change, so $f(\mathcal{I}^1) = f(\mathcal{I})$ for the profile $\mathcal{I}^1$ where voter i reports $[y, x_m]$. Next, if $y \trianglerighteq x_{\ell'}$, we remove alternatives from the interval to arrive at I'_i and robustness implies again that the winner is not allowed to change because voter i does not remove the current winner from his interval. If, by contrast, $x_{\ell'} \triangleright y$, then $x_{\ell'} = f(\mathcal{I})$. In this case, we also add $x_{\ell'}$ to the interval of voter i and note that robustness shows that the winner is not allowed to change for this step. Finally, we remove alternatives on the right end of the voters interval to infer I'_i and robustness shows that the outcome does not change because we again do not remove the current winner $f(\mathcal{I})$ from voter i's interval. Hence, we have that $f(\mathcal{I}) = f(\mathcal{I}')$, and our claim holds in this case.

Next, suppose that $x_{\ell'} \rhd f(\mathcal{I})$. We use a second case distinction and assume additionally that $f(\mathcal{I}) \unrhd x_{r'}$. In this case, consider the profile $\mathcal{I}^2$ where voter i reports $I_i^2 = [f(\mathcal{I}), x_{r'}]$. By the reasoning in the last paragraph, it holds that $f(\mathcal{I}^2) = f(\mathcal{I})$. Now, we one after another add alternatives to the left of $f(\mathcal{I})$ to voter i's interval until he reports I'_i. Robustness implies for all of these steps that the winner can only move to the left, i.e., that $f(\mathcal{I}') \unrhd f(\mathcal{I})$. However, weak single-peakedness then shows that $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$ because $T(\succsim_i) \subseteq I_i$ by top-consistency. In more detail, it holds that $f(\mathcal{I}') \unrhd f(\mathcal{I}) \rhd x_\ell \unrhd x$ for all $x \in T(\succsim_i)$, so $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$ by weak single-peakedness. As the second subcase, suppose that $x_{r'} \rhd f(\mathcal{I})$ and let $\mathcal{I}^3$ denote the profile where voter i reports $[x_{\ell'}, x_r]$ and $I_j^3 = I_j$ for all other voters $j \in N \setminus \{i\}$. By our previous argument, we have that $f(\mathcal{I}^3) \unrhd f(\mathcal{I})$. If $f(\mathcal{I}^3) \unrhd x_{r'}$, we can remove the alternatives right of $x_{r'}$ from voter i's interval and robustness implies that $f(\mathcal{I}') = f(\mathcal{I}^3)$ as we do not touch the current winner. On the other hand, if $x_{r'} \rhd f(\mathcal{I}^3)$, robustness shows that $f(\mathcal{I}') \unrhd f(\mathcal{I}^3)$ as the winner can only move to the left when we remove the right-most alternatives from voter i's interval. In both cases, we have again that $f(\mathcal{I}') \unrhd f(\mathcal{I})$, so weak single-peakedness implies again that $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$. Hence, f satisfies the first condition of sincerity.

Lastly, to prove the second condition of sincerity, we assume that $I_i \subseteq I'_i$ and we will show that $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$. If $f(\mathcal{I}') \notin I'_i$, a repeated application of robustness directly shows that $f(\mathcal{I}') = f(\mathcal{I})$ and the claim holds. Similarly, if $f(\mathcal{I}') \in I_i \subseteq I'_i$, it again holds that $f(\mathcal{I}') = f(\mathcal{I})$ due to robustness. Hence, we assume that $f(\mathcal{I}') \in I'_i \setminus I_i$. To make this more precise, we define $x_\ell, x_r, x_{\ell'}, x_{r'}$ as before and suppose that $x_{\ell'} \unrhd f(\mathcal{I}') \rhd x_\ell$. We first note that robustness implies that $f(\mathcal{I}') = f(\mathcal{I}^4)$ for the profile $\mathcal{I}^4$ where voter i reports $[x_{\ell'}, x_r]$ and $I_j^4 = I_j$ for all $j \in N \setminus \{i\}$. Next, by one after another removing the alternatives left of x_ℓ from voter i's interval, we infer that $f(\mathcal{I}') \unrhd f(\mathcal{I}) \unrhd x_\ell$ due to robustness. The weak single-peakedness of $\succsim_i$ then shows that $f(\mathcal{I}') \succsim_i f(\mathcal{I})$ since $T(\succsim_i) \subseteq I_i$ and $f(\mathcal{I})$ is thus closer to voter i's favorite alternatives than $f(\mathcal{I}')$.

($\Longleftarrow$) Next we will show that, if f fails robustness, then it also fails sincerity. To this end, let N denote an electorate, $\mathcal{I} \in \Lambda^N$ an interval profile, $i \in N$ a voter, and $x_\ell, x_r \in A$ two alternatives such that $x_\ell \rhd x_r$ and $I_i = [x_\ell, x_r]$. Without loss of generality, we assume that f fails robustness when voter i changes his interval to $I'_i = [x_{\ell+1}, x_r]$ as the case that $I_i = [x_\ell, x_{r-1}]$ is symmetric. First, we consider the case that $f(\mathcal{I}) \notin I_i$. Then, violating robustness means that $f(\mathcal{I}) \neq f(\mathcal{I}')$, where $\mathcal{I}'$ is the profile where voter i reports I'_i and all voters $j \in N \setminus \{i\}$ report I_j. To show that f fails sincerity, let $\succsim_i \in \mathcal{P}_\rhd$ denote a single-peaked preference relation such that $T(\succsim_i) = \{x_{\ell+1}\}$, which means that $\succsim_i$ is top-consistent with both I_i and I'_i. Now, if voter i strictly prefers $f(\mathcal{I}')$ to $f(\mathcal{I})$, sincerity is violated because $f(\mathcal{I}) \notin I_i$ and not $f(\mathcal{I}) \succsim_i f(\mathcal{I}')$. Conversely, if voter i strictly prefers $f(\mathcal{I})$ to $f(\mathcal{I}')$, sincerity is violated since $I'_i \subsetneq I_i$ and $f(\mathcal{I}') \succsim_i f(\mathcal{I})$ fails.

Since single-peaked preference relations are strict and $f(\mathcal{I}) \neq f(\mathcal{I}')$, one of these cases must apply, so sincerity is violated if $f(\mathcal{I}) \notin I_i$.

Next, suppose that $f(\mathcal{I}) \in I_i \setminus \{x_\ell\} = I_i'$. In this case, a failure of robustness again implies that $f(\mathcal{I}) \neq f(\mathcal{I}')$. In this case, let $\succsim_i \in \mathcal{P}_\triangleright$ denote a single-peaked preference relation such that $T(\succsim_i) = \{f(\mathcal{I})\}$. Since $f(\mathcal{I}) \in I_i'$, it follows that $\succsim_i$ is top-consistent with I_i'. In turn, the second condition of sincerity requires that $f(\mathcal{I}') \succsim_i f(\mathcal{I})$ since $I_i' \subsetneq I_i$. However, since $f(\mathcal{I}) \neq f(\mathcal{I}')$, $f(\mathcal{I})$ is voter i's favorite alternative in $\succsim_i$, and $\succsim_i$ is strict, voter i strictly prefers $f(\mathcal{I})$ to $f(\mathcal{I}')$, which contradicts sincerity.

Finally, assume that $f(\mathcal{I}) = x_\ell$. In this case, a violation of robustness entails that $f(\mathcal{I}') \notin \{x_\ell, x_{\ell+1}\}$. Now, consider a single-peaked preference relation $\succsim_i \in \mathcal{P}_\triangleright$ with $x_{\ell+1} \succsim_i x_\ell \succsim_i x$ for all $x \in A \setminus \{x_\ell, x_{\ell+1}\}$, which is top-consistent with I_i' since $x_{\ell+1} \in I_i'$. However, since $f(\mathcal{I}') \notin \{x_\ell, x_{\ell+1}\}$, voter i strictly prefers $f(\mathcal{I})$ to $f(\mathcal{I}')$. This contradicts sincerity as $I_i' \subsetneq I_i$. Since we have covered all cases, it follows that if f fails robustness, it also fails sincerity. □

2.3. Further axioms

In this section, we will introduce four standard axioms, namely unanimity, anonymity, reinforcement, and right-biased continuity. Variants of these axioms feature prominently in the analysis of various types of scoring rules (e.g., Smith, 1973; Young, 1975; Young and Levenglick, 1978; Fishburn, 1978; Brandl et al., 2016; Lackner and Skowron, 2021).

Unanimity. A basic requirement of voting rules is that, if all voters agree on a favorite alternative, this alternative is selected. This is formalized by *unanimity*, which requires of a voting rule f on Λ^* that $f(\mathcal{I}) = x_j$ for all interval profiles $\mathcal{I} \in \Lambda^*$ and alternatives $x_j \in A$ such that $I_i = \{x_j\}$ for all $i \in N_\mathcal{I}$.

Anonymity. Anonymity postulates that the selected outcome is invariant under renaming the voters. More formally, a voting rule f is *anonymous* if $f(\mathcal{I}) = f(\tau(\mathcal{I}))$ for all interval profiles $\mathcal{I} \in \Lambda^*$ and bijections $\tau : \mathbb{N} \to \mathbb{N}$. Here, $\mathcal{I}' = \tau(\mathcal{I})$ denotes the profile defined by $N_{\mathcal{I}'} = \{\tau(i) \colon i \in N_\mathcal{I}\}$ and $I'_{\tau(i)} = I_i$ for all $i \in N_\mathcal{I}$.

Reinforcement. The idea of reinforcement is that, if a common outcome is chosen for two disjoint elections, then it should also be chosen for a combined election. The reason for this is that, if an alternative is the "best" outcome for two disjoint profiles, it should also be the "best" outcome for the combined profile. To formalize this, we let $\mathcal{I}'' = \mathcal{I} + \mathcal{I}'$ denote the profile derived from two profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ with $N_\mathcal{I} \cap N_{\mathcal{I}'} = \emptyset$ by setting $N_{\mathcal{I}''} = N_\mathcal{I} \cup N_{\mathcal{I}'}$, $I_i'' = I_i$ for all $i \in N_\mathcal{I}$, and $I_i'' = I_i'$ for all $i \in N_{\mathcal{I}'}$. Then, a voting rule f is *reinforcing* if $f(\mathcal{I} + \mathcal{I}') = f(\mathcal{I})$ for all interval profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ such that $N_\mathcal{I} \cap N_{\mathcal{I}'} = \emptyset$ and $f(\mathcal{I}) = f(\mathcal{I}')$.

Right-biased continuity. As our last axiom, we will introduce a variant of an axiom known as continuity or overwhelming-majority property (Young, 1975; Myerson, 1995). The rough idea of this axiom is that, if we are given two profiles, we can marginalize the effect of one profile on the outcome by sufficiently often cloning the other profile.

More formally, continuity typically requires that, for all profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$, there is $\lambda \in \mathbb{N}$ such that $f(\lambda\mathcal{I} + \mathcal{I}') \subseteq f(\mathcal{I})$ (where $\lambda\mathcal{I}$ denotes a profile that consists of λ copies of $\mathcal{I}$). However, this formulation is incompatible with our model because resolute voting rules require tie-breaking, which is not continuous. We will thus weaken this axiom and, moreover, use it to specify the tie-breaking used in our voting rules. In more detail, we say that a voting rule f on Λ^* satisfies *right-biased continuity* if it holds for all profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ with $N_{\mathcal{I}} \cap N_{\mathcal{I}'} = \emptyset$ that *(i)* if $f(\mathcal{I}') \trianglerighteq f(\mathcal{I})$, there is $\lambda \in \mathbb{N}$ such that $f(\lambda\mathcal{I} + \mathcal{I}') = f(\mathcal{I})$ and *(ii)* if $f(\mathcal{I}) \triangleright f(\mathcal{I}')$, there is an alternative $x_j \in \bigcup_{i \in N_{\mathcal{I}}} I_i$ and $\lambda \in \mathbb{N}$ such that $f(\mathcal{I}) \trianglerighteq f(\lambda\mathcal{I} + \mathcal{I}') \trianglerighteq x_j$. Less formally, right-biased continuity ensures full continuity if $f(\mathcal{I})$ is right of $f(\mathcal{I}')$ and otherwise only guarantees that we cannot completely ignore $\mathcal{I}$ when duplicating this profile sufficiently often. We note, however, that the second condition becomes close to full continuity when the set $\bigcup_{i \in N_{\mathcal{I}}} I_i$ is small.

2.4. Position-Threshold Rules

We will now introduce the central class of voting rules in this paper, which we call position-threshold rules. Since these rules can be seen as a generalization of Moulin's phantom median rules (1980), we will first discuss these rules.

Phantom median rules. Phantom median rules are typically defined on the domain of single-peaked preference profiles for a fixed electorate N and work as follows: given a single-peaked profile $R \in \mathcal{P}_{\triangleright}^N$, we first add $|N| - 1$ phantom voters with fixed preference relations, then order all $2|N| - 1$ voters with respect to their favorite alternatives according to $\triangleright$, and finally return the favorite alternative of $|N|$-th voter in this list. To make this more formal, let p_i denote the number of phantom voters that top-rank alternative x_i and let $q_i(R)$ be the number of regular voters that top-rank alternative x_i in the profile R. Then, a phantom median rule chooses for every profile R the alternative x_i with minimal index i such that $\sum_{j=1}^{i} p_j + q_j(R) \geq |N|$.

We will next reformulate this definition and hence introduce the *(individual) peak position function* $\pi_{SP}(\succsim_i, x_j)$ which states the relative position of a voter i with respect to each alternative x_j: $\pi_{SP}(\succsim_i, x_j) = 1$ if voter i's favorite alternative x_i is weakly left of x_j (i.e., $x_i \trianglerighteq x_j$) and $\pi_{SP}(\succsim_i, x_j) = 0$ otherwise. Moreover, we define the *(collective) peak position function* Π_{SP} of an alternative x_j in a single-peaked profile R by $\Pi_{SP}(R, x_j) = \sum_{i \in N_R} \pi_{SP}(\succsim_i, x_j)$. Put differently, the collective peak position function Π_{SP} counts how many voters in R report a favorite alternative that is weakly left of x_j. Next, we define by $\max_{\triangleright} X$ the left-most alternative x in a given set X, i.e., $x = \max_{\triangleright} X$ satisfies that $x \in X$ and $x \triangleright y$ for all $y \in X \setminus \{x\}$. Then, a voting rule f is a phantom median rule if there are integers $p_1, \dots, p_m \in \mathbb{N}_0$ such that $\sum_{j=1}^{m} p_j = |N| - 1$ and $f(R) = \max_{\triangleright}\{x_i \in A \colon \Pi_{SP}(R, x_i) \geq |N| - \sum_{j=1}^{i} p_j\}$ for all profiles $R \in \mathcal{P}_{\triangleright}^N$.

Finally, to extend the definition of phantom median rules from a fixed electorate to all feasible electorates, we will replace the values p_i with a *threshold vector* θ, which is formally a vector in $(0,1)^m$ such that $\theta_1 \geq \theta_2 \geq \dots \geq \theta_m$. The intuition is that for every electorate N, the value $\theta_i \cdot |N|$ is equal to $|N| - \sum_{j=1}^{i} p_j$, i.e., θ_i states the fraction of the voters that need to report an alternative left of x_i to make x_i an eligible outcome.[2] Based on this notation, we end up with our final definition of phantom median rule for variable electorates.

Definition 2 (Phantom median rules)**.** A voting rule f on $\mathcal{P}^*_{\triangleright}$ is a *phantom median rule* if there is a threshold vector $\theta \in (0,1)^m$ with $\theta_1 \geq \dots \geq \theta_m$ such that $f(R) = \max_{\triangleright}\{x_i \in A \colon \Pi_{SP}(R, x_i) \geq \theta_i |N_R|\}$ for all profiles $R \in \mathcal{P}^*_{\triangleright}$.

We note that, in this definition, the value θ_m is irrelevant since $\Pi_{SP}(R, x_m) = |N_R|$ for all profiles R. Moreover, the restriction that $\theta_i > 0$ for $i \in \{1, \dots, m-1\}$ is necessary to ensure unanimity as $\Pi_{SP}(R, x_i) \geq 0$ for all $i \in \{1, \dots, m\}$, and the condition that $\theta_i < 1$ for all $i \in \{1, \dots, m-1\}$ is necessary to satisfy right-biased continuity.

Position-threshold rules. Position-threshold rules aim to extend phantom median rules to the interval domain by generalizing the peak position function π_{SP} to intervals. The central problem for this is that a voter's position with respect to some alternatives is unclear if his interval contains more than one alternative. For instance, if a voter reports $[x_1, x_2, x_3]$, his relative position with respect to x_1 and x_2 is ambiguous. In this paper, we will solve this issue by using a *weight vector* $\alpha \in [0,1]^m$ which quantifies the relative position of every voter with respect to the alternatives in his interval. In more detail, if an alternative x_k is in the interval I_i of voter i (but it is not the right-most alternative in I_i), then the relative position of voter i with respect to x_k is α_k. By contrast, just as for the peak position function, a voter's position with respect to x_k is 1 if every alternative in his interval is weakly left of x_k and 0 if every alternative in his interval is strictly right of x_k. We formalize this idea with *individual position functions* $\pi_\alpha : \Lambda \times A \to \mathbb{R}$, which depend on a weight vector α and are defined as follows:

$$\pi_\alpha([x_i, x_j], x_k) = \begin{cases} 0 & \text{if } x_k \triangleright x_i \\ \alpha_k & \text{if } x_i \trianglerighteq x_k \triangleright_i x_j \\ 1 & \text{if } x_j \trianglerighteq x_k. \end{cases}$$

We note that every individual position function π_α generalizes the individual peak position function π_{SP} to the interval domain because $\pi_{SP}(\succsim, x_j) = \pi_\alpha(T(\succsim), x_j)$ for all single-peaked preference relations $\succsim \in \mathcal{P}_{\triangleright}$, alternatives $x_j \in A$, and weight vectors α.

Given a weight vector α, we define the *(collective) position function* Π_α of an alternative x_j in an interval profile $\mathcal{I}$ as $\Pi_\alpha(\mathcal{I}, x_j) = \sum_{i \in N_{\mathcal{I}}} \pi_\alpha(I_i, x_j)$. Based on a weight vector α and a threshold vector θ, we can now define position-threshold rules: we simply chose the alternative with the smallest index whose collective position exceeds its threshold. However, while all such rules are well-defined (as $\Pi_\alpha(\mathcal{I}, x_m) = n_{\mathcal{I}}$), we need to impose additional constraints on the weight vector to guarantee robustness. In particular, we say that a weight vector α is *compatible* with a threshold vector θ if, for all $i \in \{1, \dots, m-2\}$,

[2] As we require that $\theta_i < 1$, it is not possible that $\theta_i |N| = |N| - \sum_{j=1}^{i} p_j$ if $\sum_{j=1}^{i} p_j = 0$. However, for every fixed N, it is easy to check that setting $\theta_i = \frac{2(|N| - \sum_{j=1}^{i} p_j) - 1}{2|N|}$ results in the same rule.

it holds that

$$\alpha_{i+1} - \alpha_i \geq (\theta_{i+1} - \theta_i) \cdot \max\left(\frac{\alpha_i}{\theta_i}, \frac{1-\alpha_i}{1-\theta_i}\right).$$

Since $\theta_i \geq \theta_{i+1}$ by the definition of threshold vectors, the right side of this inequality is always less or equal to 0, so this constraint only forbids that α_i is significantly larger than α_{i+1}. Consequently, all weight vectors α with $\alpha_1 \leq \alpha_2 \leq \dots \alpha_m$ are compatible with all threshold vectors θ. These are the most natural weight vectors because the inequality $\alpha_{i+1} \geq \alpha_i$ intuitively captures that a voter is "at least as much left" of x_{i+1} as of x_i. Moreover, as we will show in our proofs, the above inequality is sufficient and necessary to ensure that $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i$ implies $\Pi_\alpha(\mathcal{I}, x_{i+1}) \geq \theta_{i+1}$ for all profiles $\mathcal{I} \in \Lambda^*$ and alternatives $x_i \in A \setminus \{x_m\}$. This condition is naturally satisfied by the peak position function and turns out to be crucial for defining robust voting rules.

We are now ready to state our formal definition of position-threshold rules.

Definition 3 (Position-threshold rules)**.** A voting rule f on Λ^* is a *position-threshold rule* if there are a threshold vector $\theta \in (0,1)^m$ with $\theta_1 \geq \cdots \geq \theta_m$ and a compatible weight vector $\alpha \in [0,1]^m$ such that $f(\mathcal{I}) = \max_{\triangleright}\{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$ for all interval profiles $\mathcal{I} \in \Lambda^*$.

To provide further intuition for these rules, we discuss the roles of the threshold vector θ and the weight vector α in more detail. Moreover, we display in Figure 1 an example illustrating how position-threshold rules work. Now, we first note that the threshold vector θ can be interpreted just as for phantom median rules: for every $i \in \{1, \dots, m-1\}$, θ_i states the fraction of the phantom voters that report an alternative right of x_i. For instance, if $\theta_i = \frac{3}{4}$ for all $i \in \{1, \dots, m\}$, we may imagine that $\frac{1}{4}n_{\mathcal{I}}$ phantom voters report $\{x_1\}$ and $\frac{3}{4}n_{\mathcal{I}}$ phantom voters report $\{x_m\}$. As a second example, if $\theta_i = \frac{m-i}{m}$ for all $i \in \{1, \dots, m-1\}$ and $\theta_m = \frac{1}{m}$, each alternative x_i is reported by a $\frac{1}{m}$ share of the phantom voters. We note here that, even though our voters are allowed to report intervals, it suffices that the phantom voters report single alternatives.

Next, the weight vector can be interpreted as a way of decomposing intervals into singleton ballots: if a voter reports an interval $[x_\ell, x_r]$, we may replace this voter with α_ℓ voters reporting $\{x_\ell\}$, $\alpha_i - \alpha_{i-1}$ voters reporting $\{x_i\}$ for every alternative $x_i \in \{x_{\ell+1}, \dots, x_{r-1}\}$, and $1 - \alpha_r$ voters reporting $\{x_r\}$. When applying this transformation to the interval of every voter, the outcome of the given position-threshold rule for the input profile is the same as for the simplified profile where all (fractional) voters only report singleton ballots. For instance, for the weight vector $\alpha = (1, \dots, 1)$, this means that every interval is represented by a voter that reports only the left-most alternative of the interval. As a second example, if $\alpha = (\frac{1}{2}, \dots, \frac{1}{2})$, every interval $[x_i, x_j]$ is represented by $\frac{1}{2}$ voters reporting $\{x_i\}$ and $\frac{1}{2}$ voters reporting x_j. Moreover, since position-threshold rules satisfy reinforcement, we can scale these numbers and, e.g., equivalently assume that every interval is represented by one voter reporting $\{x_i\}$ and one voter reporting $\{x_j\}$. Since position-threshold rules effectively reduce to phantom median rules when all voters report intervals of size 1, this gives a direct relation between position-threshold rules and phantom median rules.

$\mathcal{I}$: 1: $[x_1, x_2]$
2: $[x_1, x_3]$
3: $[x_2, x_4]$

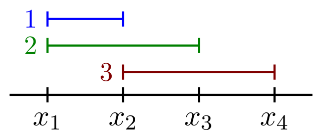

Figure 1: Example of position-threshold rules. The interval profile $\mathcal{I}$ contains 3 voters and 4 alternatives, and it is graphically represented on the right. First, let f_1 denote the position-threshold rule given by the threshold vector $\theta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the weight vector $\alpha = (1, 1, 1, 1)$. This rule selects the median of the voters' left endpoints and it holds that $f_1(\mathcal{I}) = x_1$ since $\Pi_\alpha(\mathcal{I}, x_1) = 2 > \frac{1}{2}n_\mathcal{I}$. By contrast, for the position-threshold rule f_2 defined by the same threshold vector θ and the weight vector $\beta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, it holds that $\Pi_\beta(\mathcal{I}, x_1) = 1 < \frac{1}{2}n_\mathcal{I}$ and $\Pi_\beta(\mathcal{I}, x_2) = \frac{3}{2} \geq \frac{1}{2}n_\mathcal{I}$, so $f_2(\mathcal{I}) = x_2$. As a third example, consider the position-threshold rule induced by the threshold vector vector $\phi = (\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4})$ and the weight vector $\gamma = (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})$. It holds that $\Pi_\gamma(\mathcal{I}, x_1) = \frac{1}{2} < \frac{3}{4}n_\mathcal{I}$, $\Pi_\gamma(\mathcal{I}, x_2) = 1 + \frac{1}{4} < \frac{3}{4}n_\mathcal{I}$, and $\Pi_\gamma(\mathcal{I}, x_3) = 2 + \frac{3}{4} > \frac{1}{4}n_\mathcal{I}$, so $f_3(\mathcal{I}) = x_3$.

3. Results

We are now ready to state our results: in Section 3.1 we discuss our characterization of position-threshold rules, in Section 3.2 we characterize a particular position-threshold rule called the endpoint-median rule, and in Section 3.3 we extend our characterization of position-threshold rules to the case of choosing a multiset of alternatives.

3.1. A Characterization of Position-Threshold Rules

In this section, we discuss our main result, the characterization of position-threshold rule. In more detail, we will prove that position-threshold rules are the only voting rules on Λ^* that satisfy robustness, anonymity, unanimity, reinforcement, and right-biased continuity. We note that it is an easy consequence of Moulin's work (1980) that phantom median rules, as defined in Definition 2, are the only single-winner voting rules on the domain of single-peaked preference relations that satisfy the given axioms when suitably adapting robustness. Hence, our characterization can be seen as an extension of Moulin's result to the interval domain. Since the proof of the subsequent theorem is lengthy, we will only provide a proof sketch in the main body and defer the full proof to the appendix.

Theorem 1. *A single-winner voting rule on Λ^* is robust, anonymous, unanimous, reinforcing, and right-biased continuous if and only if it is a position-threshold rule.*

Proof Sketch. ($\Longleftarrow$) For the direction from right to left, we fix a position-threshold rule f and let θ and α denote the its threshold and weight vectors. Note that, by definition, θ and α are compatible. Now, it is easy to see that f is anonymous as the collective position function Π_α is invariant under renaming the voters. Moreover, f is unanimous as $\Pi_\alpha(\mathcal{I}, x_j) = n_\mathcal{I}$ and $\Pi_\alpha(\mathcal{I}, x_h) = 0$ for all x_h with $x_h \rhd x_i$ if $I_i = \{x_j\}$ for all $i \in N_\mathcal{I}$. Thirdly, our position-threshold rule f is reinforcing as

$\Pi_\alpha(\mathcal{I}+\mathcal{I}', x_i) = \Pi_\alpha(\mathcal{I}, x_i)+\Pi_\alpha(\mathcal{I}', x_i)$. Hence, if $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}$ and $\Pi_\alpha(\mathcal{I}', x_i) \geq \theta_i n_{\mathcal{I}'}$, then $\Pi_\alpha(\mathcal{I}+\mathcal{I}', x_i) \geq \theta_i n_{\mathcal{I}+\mathcal{I}'}$. Moreover, it follows analogously that $\Pi_\alpha(\mathcal{I}, x_i) < \theta_i n_\mathcal{I}$ and $\Pi_\alpha(\mathcal{I}', x_i) < \theta_i n_{\mathcal{I}'}$ imply that $\Pi_\alpha(\mathcal{I}+\mathcal{I}', x_i) < \theta_i n_{\mathcal{I}+\mathcal{I}'}$. Now, if $f(\mathcal{I}) = f(\mathcal{I}') = x_j$ for two profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ with $N_\mathcal{I} \cap N_{\mathcal{I}'} = \emptyset$, then it holds for $\hat{\mathcal{I}} \in \{\mathcal{I}, \mathcal{I}'\}$ that $\Pi_\alpha(\hat{\mathcal{I}}, x_j) \geq \theta_j |N_{\hat{\mathcal{I}}}|$ and $\Pi_\alpha(\hat{\mathcal{I}}, x_h) < \theta_h |N_{\hat{\mathcal{I}}}|$ for all $x_h \in A$ with $x_h \rhd x_j$. Our previous insights thus entail that $f(\mathcal{I}+\mathcal{I}') = x_j$.

Next, for robustness and right-biased continuity, we show in the appendix that, if θ and α are compatible, then $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}$ implies that $\Pi_\alpha(\mathcal{I}, x_{i+1}) \geq \theta_{i+1} n_\mathcal{I}$ for all profiles $\mathcal{I} \in \Lambda^*$ and alternatives $x_i \in A \setminus \{x_m\}$. Now, to prove that f is robust, let $\mathcal{I}$ be an interval profile, $i \in N_\mathcal{I}$ a voter, and $x_\ell, x_r \in A$ the two alternatives such that $x_\ell \rhd x_r$ and $I_i = [x_\ell, x_r]$. We will show that $f(\mathcal{I}) = f(\mathcal{I}^{i\downarrow x_\ell})$ or $f(\mathcal{I}) = x_\ell$ and $f(\mathcal{I}^{i\downarrow x_\ell}) = x_{\ell+1}$ and observe that similar arguments also apply for $\mathcal{I}^{i\downarrow x_r}$. By the definition of Π_α, it holds that $\Pi_\alpha(\mathcal{I}, x_j) = \Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_j)$ for all alternatives $x_j \in A \setminus \{x_\ell\}$ and $\Pi_\alpha(\mathcal{I}, x_\ell) \geq \Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_\ell)$. If $f(\mathcal{I}) \neq x_\ell$, this implies that $f(\mathcal{I}) = f(\mathcal{I}^{i\downarrow x_\ell})$. On the other hand, if $f(\mathcal{I}) = x_\ell$, it follows that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_j) = \Pi_\alpha(\mathcal{I}, x_j) < \theta_j n_\mathcal{I}$ for all alternatives x_j with $x_j \rhd x_i$ and our auxiliary claim in the appendix shows that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_{\ell+1}) = \Pi_\alpha(\mathcal{I}, x_{\ell+1}) \geq \theta_{\ell+1} n_\mathcal{I}$. In combination, this proves that $f(\mathcal{I}^{i\downarrow x_\ell}) \in \{x_\ell, x_{\ell_1}\}$ and f thus is robust.

Finally, to prove that f satisfies right-biased continuity, we consider two profiles $\mathcal{I}$ and $\mathcal{I}'$ and we make a case distinction with respect to the relative position of $x_i = f(\mathcal{I})$ and $x_j = f(\mathcal{I}')$. For instance, if $x_j \rhd x_i$, we infer that $\Pi_\alpha(\mathcal{I}, x_h) < \theta_h n_\mathcal{I}$ for all x_h with $x_h \rhd x_i$. By copying $\mathcal{I}$ sufficiently often, we can make the absolute difference in this inequality arbitrarily big, so there is $\lambda \in \mathbb{N}$ such that $\Pi_\alpha(\lambda\mathcal{I} + \mathcal{I}', x_h) < \theta_h |N_{\lambda\mathcal{I}+\mathcal{I}'}|$ for all x_h with $x_h \rhd x_i$. Furthermore, it holds that $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}$ as $f(\mathcal{I}) = x_i$ and that $\Pi_\alpha(\mathcal{I}', x_i) \geq \theta_i n_{\mathcal{I}'}$ as $f(\mathcal{I}') = x_j$ and $x_j \rhd x_i$. This implies that $\Pi_\alpha(\lambda\mathcal{I}+\mathcal{I}', x_i) \geq \theta_i n_{\lambda\mathcal{I}+\mathcal{I}'}$, so $f(\lambda\mathcal{I} + \mathcal{I}') = x_i$ and right-biased continuity is satisfied in this case.

($\implies$) For the direction from left to right, we assume that f is a voting rule on Λ^* that satisfies anonymity, unanimity, robustness, reinforcement, and right-biased continuity. As the first step, we investigate f on the subset $\mathcal{D}_1^*$ of Λ^* where voters only report a single alternative. This domain is related to the domain of single-peaked preferences by associating the reported alternative with the top-ranked alternative of a single-peaked preference relation. We hence show that f induces a voting rule f' on the domain of single-peaked preference profiles that satisfies anonymity, unanimity, and strategyproofness. The characterization of Moulin (1980) thus shows that f' is a phantom median rule. By using the connection between f and f', we then derive that there is a threshold vector $\theta \in (0,1)^m$ such that $\theta_1 \geq \theta_2 \geq \dots \geq \theta_m$ and $f(\mathcal{I}) = \max_\rhd \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$ for all profiles $\mathcal{I} \in \mathcal{D}_1^*$. (Note that the choice of the weight vector α does not matter here as all voters only report a single alternative.)

Next, we will apply the geometric techniques initially developed by Young (1975) in the context of scoring rules. To this end, we define $q = |\Lambda|$ as the number of intervals over $\rhd$ and assume that the intervals in Λ are enumerated in an arbitrary order $I_1, \dots, I_q$. Based on this enumeration, we can represent each interval profile $\mathcal{I}$ by a vector $v \in \mathbb{N}^q \setminus \{0\}$, where the entry v_i states how often the interval I_i is reported in $\mathcal{I}$. Moreover, since f is anonymous, it can be computed based on the vectors v, i.e., we may interpret f

as function from $\mathbb{N}^q \setminus \{0\}$ to A. Next, we show based on reinforcement that f can be extended to a function $\hat{g} : \mathbb{Q}^q_{\geq 0} \setminus \{0\} \to A$ while preserving its desirable properties. We then define the sets $Q_i = \{v \in \mathbb{Q}^q_{\geq 0} \setminus \{0\} \colon \hat{g}(v) = x_i\}$ and note that these sets are $\mathbb{Q}$-convex (i.e., for all $v, v' \in Q_i$ and $\lambda \in [0,1] \cap \mathbb{Q}$, it holds that $\lambda v + (1-\lambda)v' \in Q_i$) as $\hat{g}$ is reinforcing. This implies that the closure of Q_i with respect to $\mathbb{R}^q$, denoted by $\bar{Q}_i$, is convex. We then show that the interiors of these sets are non-empty and pairwise disjoint. The separating hyperplane theorem for convex sets hence implies that, for all distinct $x_i, x_j \in A$, there is a non-zero vector $u^{i,j} \in \mathbb{R}$ such that $vu^{i,j} \geq 0$ for all $v \in \bar{Q}_i$ and $vu^{i,j} \leq 0$ for all $v \in \bar{Q}_j$ ($vu^{i,j}$ denotes here the standard scalar product between vectors). Moreover, we show that the sets $\bar{Q}_i$ are fully described the these vectors $u^{i,j}$ because $\bar{Q}_i = \{v \in \mathbb{R}^q_{\geq 0} \colon \forall x_j \in A \setminus \{x_i\} \colon vu^{i,j} \geq 0\}$ for all $x_i \in A$.

As the next step, we investigate the vectors $u^{i,j}$ in more detail. In particular, we will prove that $vu^{i,i+1} \geq 0$ implies $vu^{i+1,i+2} \geq 0$ for all $i \in \{1, \dots, m-2\}$ and $v \in \mathbb{R}^q_{\geq 0}$. Based on this insight, we derive a simplified representation of the sets $\bar{Q}_i$: it holds that $\bar{Q}_1 = \{v \in \mathbb{R}^q \colon vu^{1,2} \geq 0\}$, $\bar{Q}_i = \{v \in \mathbb{R}^q \colon vu^{i-1,i} \leq 0 \wedge vu^{i,i+1} \geq 0\}$ for all $i \in \{2, \dots, m-1\}$, and $\bar{Q}_m = \{v \in \mathbb{R}^q \colon vu^{m-1,m} \leq 0\}$. Thus, it suffices to study the vectors $u^{i,i+1}$ for all $i \in \{1, \dots, m-1\}$. To do so, we denote by $u^{i,j}_X$ the entry in $u^{i,j}$ that corresponds to the interval X and we recall that θ is the threshold vector of f for the domain $\mathcal{D}^*_1$. Then, we show that we can scale each vector $u^{i,i+1}$ such that *(i)* $u^{i,i+1}_X = 1 - \theta_i$ for all $X \in \Lambda$ such that $x \trianglerighteq x_i$ for all $x \in X$, *(ii)* $u^{i,i+1}_X = -\theta_i$ for all $X \in \Lambda$ such that $x_i \rhd x$ for all $x \in X$, and *(iii)* $u^{i,i+1}_X = u^{i,i+1}_{\{x_i, x_{i+1}\}}$ for all $X \in \Lambda$ such that $\{x_i, x_{i+1}\} \subseteq X$. We next derive a weight vector α from the vectors $u^{i,i+1}$ by defining $\alpha_i = u^{i,i+1}_{\{x_i, x_{i+1}\}} + \theta_i$ for all $i \in \{i, \dots, m-1\}$ and $\alpha_m = \alpha_{m-1}$. Moreover, we define the individual position function π_α based on this weight vector and we show that $\sum_{X \in \Lambda} v_X \pi_\alpha(X, x_i) = vu^{i,j} + \theta_i \sum_{X \in \Lambda} v_X$. By combining our insights, we then conclude that $\bar{Q}_i = \{v \in \mathbb{R}^q \colon \sum_{X \in \Lambda} v_X \pi_\alpha(X, x_{i-1}) \leq \theta_{i-1} \sum_{X \in \Lambda} v_X \wedge \sum_{X \in \Lambda} v_X \pi_\alpha(X, x_i) \geq \theta_i \sum_{X \in \Lambda} v_X\}$.

Now, fix a profile $\mathcal{I}$ and let v denote the vector such that v_X states how often the interval X is reported in $\mathcal{I}$. It is easy to check that $\sum_{X \in \Lambda} v_X \pi_\alpha(X, x_i)$ is equal to $\Pi_\alpha(\mathcal{I}, x_i)$. We hence conclude for every profile $\mathcal{I} \in \Lambda^*$ with corresponding vector v that

$$\begin{aligned} f(\mathcal{I}) &= g(v) \\ &= \{x_i \in A \colon v \in Q_i\} \\ &\subseteq \{x_i \in A \colon v \in \bar{Q}_i\} \\ &= \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_{i-1}) \leq \theta_{i-1} n_{\mathcal{I}} \wedge \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}. \end{aligned}$$

Based on right-based continuity, we then show that f picks the alternatives with the smallest index in the final set, i.e., $f(\mathcal{I}) = \max_{\rhd}\{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_{i-1}) \leq \theta_{i-1} n_{\mathcal{I}} \wedge \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\} = \max_{\rhd}\{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$. As our last step, we infer based on robustness that θ and α are compatible, which means that f is the position-threshold rule defined by θ and α. □

Remark 1. All axioms of Theorem 1 are necessary for our characterization: if we only drop anonymity, position-threshold rules that weight "even" voters $i \in 2\mathbb{N}$ twice and "odd" voters $i \in 2\mathbb{N}+1$ only once satisfy all given axioms. If we only omit right-biased continuity, we can, for instance, define position-threshold rules that select the left-most alternative x_i such that $\Pi_\alpha(\mathcal{I}, x_i) > \theta_i n_{\mathcal{I}}$. When omitting unanimity, every constant voting rule satisfies the given axioms. When omitting robustness or weakening robustness to strategyproofness, one can extend the class of position-threshold rules by allowing for arbitrary individual position functions (i.e., no consistency between different intervals is required anymore). Finally, if we only drop reinforcement, position-threshold rules that use different weight and threshold vectors depending on $n_{\mathcal{I}}$ satisfy all remaining axioms. For instance, the following rule satisfies all conditions except for reinforcement and it is no position-threshold rule: if $\lceil \log_2 n_{\mathcal{I}} \rceil$ is odd, we choose the median with respect to the left endpoints of the voters' intervals, and if it is even, we we choose the median with respect to the right endpoints of the voters' intervals.

Remark 2. The variable-electorate framework is required for Theorem 1 because we can use profile-dependent weight vectors otherwise. To make this more precise, we define the (profile-dependent) weight vector $\alpha(\mathcal{I})$ by $\alpha_1(\mathcal{I}) = \frac{1}{2} - \frac{|i \in N_{\mathcal{I}} \colon x_1 \notin I_i\}|}{2n_{\mathcal{I}}}$, and $\alpha_j(\mathcal{I}) = 1$ for all $j \in \{2, \dots, m\}$. Then, consider the position-threshold rule f defined by this profile-dependent weight vector α and the threshold vector $\theta = (\frac{1}{2}, \dots, \frac{1}{2})$. We first note that this rule is robust: if the number of voters with $x_1 \notin I_i$ does not change, this is true as f behaves like a position-threshold rule. By contrast, if some voter changes his interval by removing x_1, the collective position of x_1 only decreases while the collective position of all other alternatives remains the same. This means that the winner either changes from x_1 to x_2 or not at all, so robustness is satisfied. However, even when the electorate N is fixed, this rule is no position-threshold rule. In more detail, we suppose subsequently that there are $n = 100$ voters and we assume for contradiction that f is a position-threshold rule. This means that f is defined by a threshold vector ϕ and a weight vector β. Now, first consider the profile $\mathcal{I}$ where 50 voters report $\{x_1\}$ and 50 voters report $\{x_2\}$. By its original definition, it holds that $f(\mathcal{I}) = x_1$, so $\phi_1 \leq \frac{\Pi_\beta(\mathcal{I}, x_1)}{100} = \frac{1}{2}$. Next, consider the profile $\mathcal{I}'$ where all 100 voters report $\{x_1, x_2\}$. Using again the definition of f, we derive that $f(\mathcal{I}') = x_1$ which implies that $\beta_1 \geq \phi_1$. Finally, consider the profile $\mathcal{I}''$ where 50 voters report $\{x_1, x_2\}$, 25 voters report $\{x_1\}$, and 25 voters report $\{x_2\}$. By the definition of f, it follows that $f(\mathcal{I}'') = x_2$ because $\pi_{\alpha(\mathcal{I}'')}(\{x_1, x_2\}, x_1) < \frac{1}{2}$. However, $\Pi_\beta(\mathcal{I}'', x_1) = 25 + 50\beta_1 \geq 100\phi_1$, so f cannot be a position-threshold rule.

Remark 3. We can simplify Theorem 1 when replacing unanimity with weak efficiency, which requires of a voting rule f that an alternative can only be chosen if it is contained in the interval of at least one voter. In particular, this axiom implies for the threshold vector θ of a position-threshold rule that $\theta_i = \theta_j$ for all $i, j \in \{1, \dots, m\}$. In turn, the compatibility condition between the threshold vector θ and the weight vector α then requires that $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m$. Since it is easy to verify that all position-threshold rules that satisfy these additional constraints on the threshold and weight vectors are weakly efficient, this results in a simplified variant of Theorem 1.

3.2. Characterization of the Endpoint-Median Rule

A natural follow-up question to Theorem 1 is which position-threshold rule to use in practice. We will give one possible answer to this question by characterizing the *endpoint-median rule* f_{EM}, which is the position-threshold rule defined by the weight vector $\alpha = (\frac{1}{2}, \dots, \frac{1}{2})$ and the threshold vector $\theta = (\frac{1}{2}, \cdots, \frac{1}{2})$ for all $i \in \{1, \dots, m\}$. We note that the endpoint-median rule has a much simpler formulation when avoiding the formalism of position-threshold rules: we substitute the interval $[x_\ell, x_r]$ of every voter with the intervals $\{x_\ell\}$ and $\{x_r\}$ and compute the median rule on this simplified profile. For our characterization of the endpoint-median rule, we will rely on two axioms which we call the majority criterion and strong unanimity.

Majority criterion. The majority criterion states that an alternative should be chosen if it is uniquely reported by a strict majority of the voters. More formally, a voting rule f on Λ^* satisfies the *majority criterion* if $f(\mathcal{I}) = x_j$ for all profiles $\mathcal{I} \in \Lambda^*$ and alternatives $x_j \in A$ such that $|\{i \in N_\mathcal{I} \colon I_i = \{x_j\}\}| > \frac{n_\mathcal{I}}{2}$. We note that the majority criterion can be seen as a weak variant of Condorcet-consistency for the interval domain.

Strong unanimity. Strong unanimity strengthens unanimity by requiring that, if an alternative is reported by all voters, it should be chosen even if some voters approve additional alternatives. Since multiple alternatives can be reported by all voters in such cases, *strong unanimity* formally postulates of a voting rule f on Λ^* that $f(\mathcal{I}) \in \bigcap_{i \in N_\mathcal{I}} I_i$ for all interval profiles $\mathcal{I} \in \Lambda^*$ with $\bigcap_{i \in N_\mathcal{I}} I_i \neq \emptyset$.

We are now ready to prove our characterization of the endpoint-median rule.

Theorem 2. *The endpoint-median rule is the only position-threshold rule that satisfies the majority criterion and strong unanimity.*

Proof. We will prove both directions of the theorem separately.

($\implies$) We start by showing that f_{EM} satisfies our two axioms and thus define α and θ as the weight and threshold vector of this rule. First, we analyze the majority criterion. For this, consider an interval profile $\mathcal{I}$ and an alternative x_j such that more than $\frac{n_\mathcal{I}}{2}$ voters in $\mathcal{I}$ report $\{x_j\}$. It is straightforward to check that $\Pi_\alpha(\mathcal{I}, x_j) > \frac{n_\mathcal{I}}{2} = \theta_j n_\mathcal{I}$. Moreover, since $\pi_\alpha(\{x_j\}, x_h) = 0$ for all alternatives x_h with $x_h \rhd x_j$, it holds for all such alternatives that $\Pi_\alpha(\mathcal{I}, x_h) < \frac{n_\mathcal{I}}{2} = \theta_h n_\mathcal{I}$. It thus follows that $f_{EM}(\mathcal{I}) = x_j$, so the endpoint-median rule satisfies the majority criterion.

For proving that f_{EM} also satisfies strong unanimity, let $\mathcal{I}$ denote a profile such that $\bigcap_{i \in N_\mathcal{I}} I_i \neq \emptyset$. Moreover, we define x_j as the left-most alternative in $\bigcap_{i \in N_\mathcal{I}} I_i$, i.e., $x_j = \max_\rhd \bigcap_{i \in N_\mathcal{I}} I_i$. By the definition of the weight vector α, we have that $\pi_\alpha(I, x_j) \geq \frac{1}{2}$ for every interval $I \in \Lambda$ with $x_j \in I$. Since $x_j \in I_i$ for all $i \in N_\mathcal{I}$, it thus follows that $\Pi_\alpha(\mathcal{I}, x_j) \geq \frac{1}{2} n_\mathcal{I} = \theta_j n_\mathcal{I}$. Next, let x_h denote an alternative with $x_h \rhd x_j$. Since $x_j \in I_i$ for all voters $i \in N_\mathcal{I}$, x_h is not the right-most approved alternative of any voter. We infer from this insight that $\pi(I_i, x_h) \leq \frac{1}{2}$ for all voters $i \in N_\mathcal{I}$. Next, as x_j is the left-most alternative that is reported by all voters, there is a voter i with $x_h \notin I_i$ and thus $\pi_\alpha(I_i, x_h) = 0$. Combining these insights shows that $\Pi_\alpha(\mathcal{I}, x_h) < \frac{1}{2}n_\mathcal{I} = \theta_h n_\mathcal{I}$. Since this analysis holds for all alternatives x_h with $x_h \rhd x_j$, we derive now that $f_{EM}(\mathcal{I}) = x_j$. This proves that the endpoint-median rule satisfies strong unanimity.

($\Longleftarrow$) Let f denote a position-threshold rule that satisfies the majority criterion and strong unanimity. Moreover, we let α and θ denote the weight and threshold vector of f. We will prove that α and θ are the vectors of the endpoint-median rule as this implies that $f = f_{EM}$. Hence, we first show that $\theta_i = \frac{1}{2}$ for all alternatives $x_i \in A$. To this end, we observe that θ_m has no influence on the outcome of f, so we can always assume that $\theta_m = \frac{1}{2}$. Next, we assume for contradiction that $\theta_i < \frac{1}{2}$ for some $i \in \{1, \dots, m-1\}$. In this case, let $w_1, w_2 \in \mathbb{N}$ denote two integers such that $\theta_i < \frac{w_1}{w_1+w_2} < \frac{1}{2}$. Such integers exist because every rational value $q \in \mathbb{Q} \cap (0, 1)$ can be written as $q = \frac{w_1'}{w_1'+w_2'}$ for two integers $w_1', w_2' \in \mathbb{N}$. Now, consider the profile $\mathcal{I}$ where w_1 voters report $\{x_i\}$ and w_2 voters report $\{x_m\}$. It holds that $f(\mathcal{I}) = x_i$ because $\Pi_\alpha(\mathcal{I}, x_h) = 0$ for all $x_h \in A$ with $x_h \rhd x_i$ and $\Pi_\alpha(\mathcal{I}, x_h) = w_1 > \theta_i n_\mathcal{I}$. However, $\frac{w_1}{w_1+w_2} < \frac{1}{2}$ implies that $w_1 < w_2$, so a strict majority of the voters report $\{x_m\}$. The majority criterion thus postulates that $f(\mathcal{I}) = x_m$, which contradicts that $f(\mathcal{I}) = x_i$. As the second case, suppose that $\theta_i > \frac{1}{2}$ for some $i \in \{1, \dots, m-1\}$. In this case, we can find two integers $w_1, w_2 \in \mathbb{N}$ such that $\frac{1}{2} < \frac{w_1}{w_1+w_2} < \theta_i$ and we consider again the profile $\mathcal{I}$ where w_1 voters report $\{x_i\}$ and w_2 voters report $\{x_m\}$. This time, it can be checked that $x_i \rhd f(\mathcal{I})$ as $\Pi(\mathcal{I}, x_h) = 0$ for all $x_h \in A$ with $x_h \rhd x_i$ and $\Pi(\mathcal{I}, x_i) = w_1 < \theta_i n_\mathcal{I}$. However, since $\frac{1}{2} < \frac{w_1}{w_1+w_2}$, the majority criterion postulates that $f(\mathcal{I}) = x_i$, so we have again a contradiction. Thus, the majority criterion necessitates that $\theta_i = \frac{1}{2}$ for all $i \in \{1, \dots, m-1\}$.

Next, we will show that $\alpha_i = \frac{1}{2}$ for all $i \in \{1, \dots, m-1\}$ and note that α_m is irrelevant for the definition of f. We use again a case distinction for this and first suppose that $\alpha_i < \frac{1}{2}$ for some $i \in \{1, \dots, m-1\}$. In this case we define $\delta = \frac{1}{2} - \alpha_i$ and choose an integer $t \in \mathbb{N}$ such $t\delta > 1$. Now, consider the profile $\mathcal{I}$ where t voters report $\{x_i, x_{i+1}\}$ and a single voter reports $\{x_i\}$. It holds that $\Pi_\alpha(\mathcal{I}, x_i) = 1 + t\alpha_i = 1 + \frac{1}{2}t - \delta t < \frac{1}{2}n_\mathcal{I}$ because $\pi_\alpha(\{x_i, x_{i+1}\}, x_i) = \alpha_i = \frac{1}{2} - \delta$. Since $\theta_i = \frac{1}{2}$ by our previous analysis, this then means that $f(\mathcal{I}) \neq x_i$. However, x_i is the only alternative that is reported by all voters in $\mathcal{I}$, so strong unanimity requires that $f(\mathcal{I}) = x_i$, a contradiction. For the second case, we suppose that $\alpha_i > \frac{1}{2}$ and we define $\delta = \alpha_i - \frac{1}{2}$. Now, we choose $t \in \mathbb{N}$ such that $t\delta > \frac{1}{2}$ and we consider this time the profile $\mathcal{I}$ where t voters report $\{x_i, x_{i+1}\}$ and a single voter reports $\{x_{i+1}\}$. Analogous to the last case, it can be checked that $\Pi_\alpha(\mathcal{I}, x_i) = t\alpha_i = \frac{1}{2}t + \delta t > \frac{1}{2}n_\mathcal{I}$. This means that $f(\mathcal{I}) \trianglerighteq x_i$. However, x_{i+1} is the only alternative that is reported by all voters in $\mathcal{I}$, so strong unanimity requires that $f(\mathcal{I}) = x_{i+1}$. Because we have a contradiction in both cases, we conclude that the assumption that $\alpha_i \neq \frac{1}{2}$ is wrong, i.e., it holds for all $i \in \{1, \dots, m-1\}$ that $\alpha_i = \frac{1}{2}$. This means that α and θ are the weight and threshold vectors of the endpoint-median rule, so f is f_{EM}. □

Remark 4. On the domain of single-peaked preference relations, the median rule is the only phantom median rule that satisfies the majority criterion. Therefore, we interpret Theorem 2 as demonstrating that the endpoint-median rule is the "correct" extension of the median rule to the interval domain. Moreover, in combination, Theorems 1 and 2 show that the endpoint-median rule is the only voting rule on Λ^* that preserves all desirable properties of the median rule because f_{EM} is the unique voting rule on the interval domain that satisfies anonymity, strong unanimity, the majority criterion, robustness, reinforcement, and right-biased continuity.

Remark 5. The endpoint-median rule can also be characterized as the most neutral position-threshold rule. In more detail, this rule is the only position-threshold rule that is shift-symmetric (if we move the interval of every voter one position to the right, the winner will also move one position to the right) and orientation-symmetric (if we exchange x_i with x_{m+1-i} for all $i \in \{1, \dots, m\}$ in the interval of every voter, the winner will change from a_j to a_{m+1-j} unless there is an alternative x_i with $\Pi_\alpha(\mathcal{I}, x_i) = \theta_i n_{\mathcal{I}}$). This again mirrors the behavior of the median rule for single-peaked preference relations because this is the only phantom median rule that satisfies these conditions.

3.3. Characterization of Multi-winner Position-Threshold Rules

As our last contribution, we will extend Theorem 1 to the case of selecting a multiset of size $k > 1$ instead of a single winner. For instance, this model may be interpreted as a variant of apportionment, where we need to distribute the seats of a committee of size k to the alternatives based on the voters' preferences over the alternatives. Similar models have been studied by, e.g., Speroni di Fenizio and Gewurz (2019) and Brill et al. (2022). Furthermore, our model is also closely related to the problem of locating multiple facilities on the real line based on the voters' single-peaked preference relations over the real line (see, e.g., Miyagawa, 2001; Barberà and Beviá, 2005; Procaccia and Tennenholtz, 2013), or, more generally, the provision of multiple public goods based on the voters' single-peaked preferences over these public goods (see, e.g., Ehlers, 2003; Bochet and Gordon, 2012; Heo, 2013).

Now, to formalize our multi-winner setting, we define a *size-k committee* as a function from A to $\mathbb{N}_0$ such that $\sum_{x_i \in A} W(x_i) = k$. Less formally, a size-$k$ committee is a multiset over A and we interpret $W(x_i)$ as the number of seats that are assigned to alternative x_i in the committee W. The set of all size-k committees is given by $\mathcal{W}_k$. Then, a *multi-winner voting rule* F (for a target committee size k) is a function that maps every profile $\mathcal{I} \in \Lambda^*$ to a committee $W \in \mathcal{W}_k$ and we define by $F(\mathcal{I}, x_i)$ the number of seats that a multi-winner voting rule F assigns to an alternative x_i in a profile $\mathcal{I}$. Moreover, we extend this notation to intervals $X \in \Lambda$ by defining $F(\mathcal{I}, X) = \sum_{x_i \in X} F(\mathcal{I}, x_i)$. We note that we will denote multi-winner voting rules by capital letters to clearly distinguish them from single-winner voting rules.

In this section, we will characterize the class of multi-winner position-threshold rules. The rough idea of these rules it to assign each seat s_i of the committee independently by using a single-winner position-threshold rule f_i. Hence, a multi-winner position-threshold rule is specified by k single-winner position-threshold rules $f_1, \dots, f_k$ and, due to technical reasons, we will require that $f_i(\mathcal{I}) \trianglerighteq f_j(\mathcal{I})$ for all $i, j \in \{1, \dots, m\}$ with $i \leq j$ and profiles $\mathcal{I} \in \Lambda^*$. Formally, a multi-winner voting rule F is thus a *multi-winner* *position-threshold rule* if there are k single-winner position-threshold rules $f_1, \dots, f_k$ such that $f_1(\mathcal{I}) \trianglerighteq f_2(\mathcal{I}) \trianglerighteq \dots \trianglerighteq f_k(\mathcal{I})$ and $F(\mathcal{I}, x_i) = |\{j \in \{1, \dots, k\} \colon f_j(\mathcal{I}) = x_i\}|$ for all profiles $\mathcal{I} \in \Lambda^*$. As a simple example, we can define a multi-winner position-threshold rule F for the target committee size $k = 2$ by taking the union of the single-winner position-threshold rule rules induced by the threshold vectors $\theta^1 = (\frac{1}{3}, \dots, \frac{1}{3})$ and $\theta^2 = (\frac{2}{3}, \dots, \frac{2}{3})$ and the weight vectors $\alpha^1 = \alpha^2 = (1, \cdots, 1)$.

Next, we will generalize our axioms to the case of multi-winner voting rules. For anonymity, unanimity, and robustness, the given definitions directly extend the corresponding notions for single-winner voting rules and thus have the same motivation as the original conditions. By contrast, for reinforcement and right-biased continuity, we need to carefully adapt the definitions to account for the larger number of winners.

Anonymity. Just as for single-winner rules, anonymity demands that multi-winner voting rules do not depend on the identities of the voters. Formally, we say a multi-winner voting rule F is *anonymous* if $F(\mathcal{I}) = F(\pi(\mathcal{I}))$ for all interval profiles $\mathcal{I} \in \Lambda^*$ and permutations $\pi : \mathbb{N} \to \mathbb{N}$.

Unanimity. The idea of unanimity is that an alternatives should get all seats in the committee if it is unanimously and exclusively reported by all voters. We thus say that a multi-winner voting rule F is *unanimous* if $F(\mathcal{I}, x_i) = k$ for all interval profiles $\mathcal{I} \in \Lambda^*$ and alternatives $x_i \in A$ such that $I_i = \{x_i\}$ for all voters $i \in N_{\mathcal{I}}$.

Robustness. Robustness requires for multi-winner voting rules that if a voter removes his left-most (resp. right-most) alternative from his interval, we can only reallocate seats from his old-left most (resp. right-most) alternative to his new left-most (resp. right-most) alternative. More formally, a multi-winner voting rule F is *robust* if, for all interval profiles $\mathcal{I} \in \Lambda^*$, voters $i \in N_{\mathcal{I}}$, and intervals $I_i = [x_\ell, x_r]$ with $x_\ell \triangleright x_r$, it holds that *(i)* $F(\mathcal{I}^{i\downarrow x_\ell}, x_j) = F(\mathcal{I}, x_j)$ for all $x_j \in A \setminus \{x_\ell, x_{\ell+1}\}$ and $F(\mathcal{I}^{i\downarrow x_\ell}, x_\ell) \leq F(\mathcal{I}, x_\ell)$, and *(ii)* $F(\mathcal{I}^{i\downarrow x_r}, x_j) = F(\mathcal{I}, x_j)$ for all $x_j \in A \setminus \{x_r, x_{r-1}\}$ and $F(\mathcal{I}^{i\downarrow x_r}, x_r) \leq F(\mathcal{I}, x_r)$.

Range-reinforcement. Next, we extend the concept of reinforcement to multi-winner voting rules. Perhaps the most direct approach for this is to require that $F(\mathcal{I} + \mathcal{I}') = F(\mathcal{I})$ for all profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ with $N_{\mathcal{I}} \cap N_{\mathcal{I}'} = \emptyset$ and $F(\mathcal{I}) = F(\mathcal{I}')$. However, this notion is intuitively much weaker than reinforcement for single-winner voting rules because the precondition of choosing the exact same outcome for two distinct profiles becomes more demanding when multiple winners are chosen. As a result, this variant of reinforcement fails to address situations where we would expect reinforcement to apply: for instance, if $F(\mathcal{I})$ and $F(\mathcal{I}')$ only differ in the allocation of a single seat of the committee, it seems desirable that $F(\mathcal{I} + \mathcal{I}')$ should be similar to the initial two committees. However, our previous notion does not allow for this conclusion. We will thus introduce a stronger reinforcement condition called range-reinforcement. The idea of this condition is that, for every alternative $x_i \in A$, the number of seats assigned to $[x_1, x_i]$ in the joint election is bounded by the number of seats assigned to this interval in each of the disjoint subelections. More formally, a multi-winnner voting rule F on Λ^* is *range-reinforcing* if, for all profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ with $N_{\mathcal{I}} \cap N_{\mathcal{I}'} = \emptyset$ and alternatives $x_i \in A$, it holds that $\min(F(\mathcal{I}, [x_1, x_i]), F(\mathcal{I}', [x_1, x_i])) \leq F(\mathcal{I}+\mathcal{I}', [x_1, x_i]) \leq \max(F(\mathcal{I}, [x_1, x_i]), F(\mathcal{I}', [x_1, x_i]))$. By its definition, range-reinforcement guarantees that the committee $F(\mathcal{I}+\mathcal{I}')$ resembles $F(\mathcal{I})$ and $F(\mathcal{I}')$, especially if the committees $F(\mathcal{I})$ and $F(\mathcal{I}')$ are similar. For instance, if $F(\mathcal{I})$ coincides with $F(\mathcal{I}')$ except for the fact that $F(\mathcal{I}, x_i) = F(\mathcal{I}', x_i) + 1$ and $F(\mathcal{I}, x_{i+1}) = F(\mathcal{I}, x_{i+1}) - 1$, range-reinforcement requires that $F(\mathcal{I}+\mathcal{I}') = F(\mathcal{I})$ or $F(\mathcal{I}+\mathcal{I}') = F(\mathcal{I}')$.

Right-biased continuity. For defining right-biased continuity for multi-winner voting rules, we face the problem that it is not clear when a committee $F(\mathcal{I})$ is right of another committee $F(\mathcal{I}')$. Just as for range-reinforcing, we will address this issue by considering the intervals $[x_1, x_i]$ for all alternatives $x_i \in A$ separately. In more detail, we say that $F(\mathcal{I})$ is at least as right as $F(\mathcal{I}')$ with respect to an alternative x_i if $F(\mathcal{I}, [x_1, x_i]) \leq F(\mathcal{I}', [x_1, x_i])$. We can then directly extend the original idea of right-biased continuity to multiwinner rules: if $F(\mathcal{I})$ is at least as right as $F(\mathcal{I}')$ with respect to an alternative x_i, we can marginalize the effect of $\mathcal{I}'$ on the number of seats assigned to $[x_1, x_i]$ by cloning $\mathcal{I}$ sufficiently often. On the other hand, if $F(\mathcal{I})$ is left of $F(\mathcal{I}')$ with respect to x_i, we can only ensure that the seats in $[x_1, x_i]$ do not move too much to the right by cloning $\mathcal{I}$. More formally, we say a multi-winner voting rule F is *right-biased continuous* if, for all interval profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ and alternatives $x_i \in A$, it holds that *(i)* if $F(\mathcal{I}, [x_1, x_i]) \leq F(\mathcal{I}', [x_1, x_i])$, there is $\lambda \in \mathbb{N}$ such that $F(\lambda\mathcal{I}+\mathcal{I}', [x_1, x_i]) = F(\mathcal{I}, [x_1, x_i])$ and *(ii)* if $F(\mathcal{I}, [x_1, x_i]) > F(\mathcal{I}', [x_1, x_i])$, there are $\lambda \in \mathbb{N}$ and an alternative $x_j \in \bigcup_{i \in N_\mathcal{I}} I_i$ such that $F(\lambda\mathcal{I}+\mathcal{I}', [x_1, x_j]) \geq F(\mathcal{I}, [x_1, x_i])$.

Based on these axioms, we now present our characterization of multi-winner position-threshold rules.

Theorem 3. *A multi-winner voting rule on Λ^* is robust, anonymous, unanimous, range-reinforcing, and right-biased continuous if and only if it is a multi-winner position-threshold rule.*

Proof. We will show both directions of the theorem separately.

($\implies$) We will first prove the direction from left to right and hence assume that F is a multi-winner position-threshold rule for a target committee size $k \geq 1$. Let $f_1, \dots, f_k$ denote the single-winner position-threshold rules of F. By Theorem 1, each rule f_i is anonymous, robust, unanimous, reinforcing, and right-biased continuous. Based on this observation, it is straightforward to verify that F is also anonymous, unanimous, and robust. We will hence focus on range-reinforcement and right-biased continuity.

First, to show that F satisfies range-reinforcement, we consider two interval profiles $\mathcal{I}^1, \mathcal{I}^2 \in \Lambda^*$ with $N_{\mathcal{I}^1} \cap N_{\mathcal{I}^2} = \emptyset$ and an alternative x_i. Moreover, let r denote the maximal index such that $f_r(\mathcal{I}^1) \in [x_1, x_i]$ and r' the maximal index such that $f_{r'}(\mathcal{I}^2) \in [x_1, x_i]$. If no such indices exist for $\mathcal{I}^1$ or $\mathcal{I}^2$, we define $r = 0$ and $r' = 0$, respectively. By the assumption that $f_1(\mathcal{I}) \trianglerighteq f_2(\mathcal{I}) \trianglerighteq \dots f_k(\mathcal{I})$ for all profiles $\mathcal{I} \in \Lambda^*$, it holds that $F(\mathcal{I}^1, [x_1, x_i]) = r$ and $F(\mathcal{I}^2, [x_1, x_i]) = r'$. We will next assume that $r \leq r'$ and we will prove that $r \leq F(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i]) \leq r'$. The case that $r > r'$ follows by exchanging the role of $\mathcal{I}^1$ and $\mathcal{I}^2$ in our proof.

We start by proving that $r \leq F(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i])$. If $r = 0$, there is nothing to show. We hence suppose that $r \geq 1$, and we will show that $f_r(\mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq x_i$. To this end, we define $x_j = f_r(\mathcal{I}^1)$ and $x_{j'} = f_r(\mathcal{I}^2)$, and we note that $x_j \trianglerighteq x_i$ and $x_{j'} \trianglerighteq x_i$ since $r = F(\mathcal{I}^1, [x_1, x_i]) \leq F(\mathcal{I}^2, [x_1, x_i])$. Next, let α denote the weight vector and θ the threshold vector of f_r. By the definition of position-threshold rules, we infer that $\Pi_\alpha(\mathcal{I}^1, x_j) \geq \theta_j n_{\mathcal{I}^1}$ and $\Pi_\alpha(\mathcal{I}^2, x_{j'}) \geq \theta_{j'} n_{\mathcal{I}^2}$. We moreover prove in Step 1 of Lemma 1 (in the appendix) that $\Pi_\alpha(\mathcal{I}, x_h) \geq \theta_h n_{\mathcal{I}}$ implies $\Pi_\alpha(\mathcal{I}, x_{h+1}) \geq \theta_{h+1} n_{\mathcal{I}}$ for all profiles $\mathcal{I}$ and alternatives $x_h \in A \setminus \{x_m\}$. This means that $\Pi_\alpha(\mathcal{I}^1, x_i) \geq \theta_i n_{\mathcal{I}^1}$ and $\Pi_\alpha(\mathcal{I}^2, x_i) \geq \theta_i n_{\mathcal{I}^2}$. In turn, we infer that $\Pi_\alpha(\mathcal{I}^1 + \mathcal{I}^2, x_i) = \Pi_\alpha(\mathcal{I}^1, x_i) + \Pi_\alpha(\mathcal{I}^2, x_i) \geq \theta_i n_{\mathcal{I}^1} + \theta_i n_{\mathcal{I}^2} = \theta_i n_{\mathcal{I}^1 + \mathcal{I}^2}$. This proves that $f_r(\mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq x_i$, which means $r \leq F(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i])$ because $f_i(\mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq f_r(\mathcal{I}^1 + \mathcal{I}^2)$ for all $i \in \{1, \dots, r\}$.

Next, we will show that $F(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i]) \leq r'$. If $r' = k$, this is again trivial, so we suppose that $r' \leq k - 1$. We will prove that $x_i \triangleright f_{r'+1}(\mathcal{I}^1 + \mathcal{I}^2)$ to show our claim. This time, let α denote the weight vector and θ denote the threshold vector of $f_{r'+1}$. By the definition of r and r', we have that $x_i \triangleright f_{r+1}(\mathcal{I}^1) \trianglerighteq f_{r'+1}(\mathcal{I}^1)$ and $x_i \triangleright f_{r'+1}(\mathcal{I}^2)$. Consequently, it holds for all alternative x_h with $x_h \trianglerighteq x_i$ that $\Pi_\alpha(\mathcal{I}^1, x_h) < \theta_h n_{\mathcal{I}^1}$ and $\Pi_\alpha(\mathcal{I}^2, x_h) < \theta_h n_{\mathcal{I}^1}$. Using again the linearity of Π_α, it thus follows that $\Pi_\alpha(\mathcal{I}^1 + \mathcal{I}^2, x_h) < \theta_h n_{\mathcal{I}^1 + \mathcal{I}^2}$ for all such alternatives x_h. This shows that $x_i \triangleright f_{r'+1}(\mathcal{I}^1 + \mathcal{I}^2)$. Since $f_1(\mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq \dots \trianglerighteq f_k(\mathcal{I}^1 + \mathcal{I}^2)$, we now conclude that $F(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i]) \leq r'$. This proves that F is indeed range-reinforcing.

As our last point, we show that F is right-biased continuous and thus consider two profiles $\mathcal{I}^1, \mathcal{I}^2 \in \Lambda^*$ and an alternative $x_i \in A$. Moreover, we define $r = F(\mathcal{I}^1, [x_1, x_i])$ and $r' = F(\mathcal{I}^2, [x_1, x_i])$ and first assume that $r \leq r'$. Since F is range-reinforcing, we derive for every $\lambda \in \mathbb{N}$ that $F(\lambda\mathcal{I}^1, [x_1, x_i]) = F(\mathcal{I}^1, [x_1, x_i])$ and $r \leq F(\lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i]) \leq r'$. If $r = r'$, this completes the proof, so assume that $r < r'$. In this case, we consider the single-winner rule f_{r+1} and we note that $f_{r+1}(\mathcal{I}^1) \notin [x_1, x_i]$ and $f_{r+1}(\mathcal{I}^2) \in [x_1, x_i]$ since $F(\mathcal{I}^1, [x_1, x_i]) < r + 1$ and $F(\mathcal{I}^2, [x_1, x_i]) \geq r + 1$. This means that $f_{r+1}(\mathcal{I}^2) \triangleright f_{r+1}(\mathcal{I}^1)$, so the right-biased continuity of f_{r+1} implies that there is $\lambda^* \in \mathbb{N}$ such that $f_{r+1}(\lambda^*\mathcal{I}^1 + \mathcal{I}^2) = f(\mathcal{I}^1)$. Consequently, $F(\lambda^*\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i]) \leq r$. Combined with our previous insights, this means that $F(\lambda^*\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_i]) = r$ and right-biased continuity holds in this case.

Next, we suppose that $r' < r$. In this case, we consider the single-winner position-threshold rule f_r and note that $f_r(\mathcal{I}^1) \in [x_1, x_i]$ and $f_r(\mathcal{I}^2) \notin [x_1, x_i]$ since $F(\mathcal{I}^1, [x_1, x_i]) = r$ and $F(\mathcal{I}^2, [x_1, x_i]) < r$. Hence, $f_r(\mathcal{I}^1) \triangleright f_r(\mathcal{I}^2)$ and the right-biased continuity of f_r implies that there is a value $\lambda^* \in \mathbb{N}$ and alternative $x_j \in \bigcup_{s \in N_{\mathcal{I}^1}} I^1_s$ such that $f_r(\lambda^*\mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq x_j$. This means that $F(\lambda^*\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_j]) \geq r = F(\mathcal{I}^1, [x_1, x_i])$, so right-biased continuity holds also in this case.

($\Longleftarrow$) Let F denote a multi-winner voting rule on Λ^* that satisfies anonymity, robustness, unanimity, range-reinforcement, and right-biased continuity. We need to prove that there are k singe-winner position-threshold rules $f_1, f_2, \dots, f_k$ such that $f_1(\mathcal{I}) \trianglerighteq f_2(\mathcal{I}) \trianglerighteq \dots \trianglerighteq f_k(\mathcal{I})$ and $F(\mathcal{I}, x_i) = |\{j \in \{1, \dots, k\} \colon f_j(\mathcal{I}) = x_i\}|$ for all profiles $\mathcal{I} \in \Lambda^*$ and alternatives $x_j \in A$. To this end, we define f_i for $i \in \{1, \dots, k\}$ as the single-winner voting rule that returns the i-th left-most alternative in $F(\mathcal{I})$. More formally, f_i returns the alternative x_1 if $F(\mathcal{I}, x_1) \geq i$ and the alternative $x_j \in A \setminus \{x_i\}$ such that $F(\mathcal{I}, [x_1, x_{j-1}]) < i$ and $F(\mathcal{I}, [x_1, x_j]) \geq i$ otherwise. By this definition, it immediately follows that $f_1(\mathcal{I}) \trianglerighteq f_2(\mathcal{I}) \trianglerighteq \dots \trianglerighteq f_k(\mathcal{I})$ for all profiles $\mathcal{I} \in \Lambda^*$. We will subsequently show that each rule f_i satisfies anonymity, robustness, unanimity, reinforcement, and right-biased continuity because Theorem 1 then implies that f_i is a single-winner position-threshold rule.

To this end, we fix an index i and focus on the single-winner voting rule f_i. First, we note that f_i is unanimous because F satisfies this axiom. In more detail, it holds that $F(\mathcal{I}, x_j) = k$ for all profiles $\mathcal{I} \in \Lambda^*$ and alternatives $x_j \in A$ such that $I_i = \{x_j\}$ for all $i \in N_{\mathcal{I}}$. This implies that $f_i(\mathcal{I}) = x_j$ because f_i picks an alternative in $F(\mathcal{I})$. It is also easy to see that f_i inherits anonymity from F: we have that $F(\mathcal{I}) = F(\pi(\mathcal{I}))$ for all profiles $\mathcal{I} \in \Lambda^*$ and all permutations $\pi : \mathbb{N} \to \mathbb{N}$, so $f_i(\mathcal{I}) = f_i(\pi(\mathcal{I}))$.

As the third point, we demonstrate that f_i is robust. To this end, consider a profile $\mathcal{I} \in \Lambda^*$, two alternatives $x_\ell, x_r \in A$ with $x_\ell \triangleright x_r$, and a voter $j \in N_{\mathcal{I}}$ with $I_j = [x_\ell, x_r]$. We focus here on the profile $\mathcal{I}^{j\downarrow x_\ell}$ because the analysis for $\mathcal{I}^{j\downarrow x_r}$ is symmetric. Now, the robustness of F shows that $F(\mathcal{I}, x_\ell) \geq F(\mathcal{I}^{j\downarrow x_\ell}, x_\ell)$, $F(\mathcal{I}, x_{\ell+1}) \leq F(\mathcal{I}^{j\downarrow x_\ell}, x_{\ell+1})$, and $F(\mathcal{I}, x_h) = F(\mathcal{I}^{j\downarrow x_\ell}, x_h)$ for all $x_h \in A \setminus \{x_\ell, x_{\ell+1}\}$. This implies that $F(\mathcal{I}, [x_1, x_\ell]) \geq F(\mathcal{I}^{j\downarrow x_\ell}, [x_1, x_\ell])$ and $F(\mathcal{I}, [x_1, x_h]) = F(\mathcal{I}^{j\downarrow x_h}, [x_1, x_\ell])$ for all $x_h \in A \setminus \{x_\ell\}$. Consequently, $f_i(\mathcal{I}) = f_i(\mathcal{I}^{j\downarrow x_\ell})$ if $f_i(\mathcal{I}) \neq x_\ell$. On the other hand, if $f_i(\mathcal{I}) = x_\ell$, then $f_i(\mathcal{I}^{j\downarrow x_\ell}) \in \{x_\ell, x_{\ell+1}\}$ because $F(\mathcal{I}^{j\downarrow x_\ell}, [x_1, x_{\ell-1}]) = F(\mathcal{I}, [x_1, x_{\ell-1}]) < i$ and $F(\mathcal{I}^{j\downarrow x_\ell}, [x_1, x_{\ell+1}]) = F(\mathcal{I}, [x_1, x_{\ell+1}]) \geq F(\mathcal{I}, [x_1, x_\ell]) \geq i$. This proves that f_i is robust.

As our fourth condition, we turn to reinforcement. To this end, consider two profiles $\mathcal{I}^1$ and $\mathcal{I}^2$ such that $N_{\mathcal{I}^1} \cap N_{\mathcal{I}^2} = \emptyset$ and $f_i(\mathcal{I}^1) = f_i(\mathcal{I}^2) = x_j$ for some alternative $x_j \in A$. First assume that $x_j = x_1$. By the definition of f_i, this means that $F(\mathcal{I}^1, x_1) \geq i$ and $F(\mathcal{I}^2, x_1) \geq i$. Hence, range-reinforcement implies that $F(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_1]) \geq i$, thus proving that $f_i(\mathcal{I}^1 + \mathcal{I}^2) = x_1$, too. Next, assume that $x_1 \triangleright x_j$. In this case, we note that $F(\mathcal{I}^1, [x_1, x_j]) \geq i$ and $F(\mathcal{I}^2, [x_1, x_j]) \geq i$, so $F(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_j]) \geq i$ by range-reinforcement. Put differently, this means that $f_i(\mathcal{I} + \mathcal{I}') \trianglerighteq x_j$. Next, it holds that $F(\mathcal{I}^1, [x_1, x_{j-1}]) < i$ and $F(\mathcal{I}^2, [x_1, x_{j-1}]) < i$ because $f_i(\mathcal{I}^1) = f_i(\mathcal{I}^2) = x_i$, so range-reinforcement requires that $f(\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_{j-1}]) < i$. Consequently, $x_{j-1} \triangleright f_i(\mathcal{I}^1 + \mathcal{I}^2)$, so we conclude that $f(\mathcal{I}^1 + \mathcal{I}^2) = x_j$. This proves that f_i is reinforcing.

Finally, we show that f_i is right-biased continuous. To this end, fix two profiles $\mathcal{I}^1, \mathcal{I}^2 \in \Lambda^*$ and let $x_j = f_i(\mathcal{I}^1)$ and $x_{j'} = f_i(\mathcal{I}^2)$. First, if $x_j = x_{j'}$, the reinforcement of f_i implies that $f_i(\mathcal{I}^1 + \mathcal{I}^2) = x_j$ and right-biased continuity holds. Next, if $x_{j'} \triangleright x_j$, we have that $F(\mathcal{I}^1, [x_1, x_{j-1}]) < i \leq F(\mathcal{I}^2, [x_1, x_{j-1}])$, so the right-biased continuity of F implies that there is $\lambda \in \mathbb{N}$ such that $F(\lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1 x_{j-1}]) = F(\mathcal{I}^1, [x_1, x_{j-1}]) < i$. This means that $x_{j-1} \triangleright f_i(\lambda\mathcal{I}^1 + \mathcal{I}^2)$. Next, we observe that $\min(F(\mathcal{I}^1, [x_1, x_j]), F(\mathcal{I}^2, [x_1, x_j])) \geq i$. In more detail, it holds that $F(\mathcal{I}^1, [x_1, x_j]) \geq i$ since $f_i(\mathcal{I}^1) = x_j$ and that $F(\mathcal{I}^2, [x_1, x_j]) \geq i$ since $f_i(\mathcal{I}^2) = x_{j'} \triangleright x_j$. Hence, range-reinforcement implies that $F(\lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_j]) \geq i$, so $f_i(\lambda\mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq x_j$. Combining our observations shows that $f_i(\lambda\mathcal{I}^1 + \mathcal{I}^2) = x_j$, so right-biased continuity holds in this case.

Lastly, suppose that $x_j \triangleright x_{j'}$, which means that $F(\mathcal{I}^1, [x_1, x_j]) \geq i > F(\mathcal{I}^2, [x_1, x_j])$. By the right-biased continuity of F, there exists $\lambda \in \mathbb{N}$ and an alternative $x_r \in \bigcup_{s \in N_{\mathcal{I}^1}} I^1_s$

such that $F(\lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_r]) \geq F(\mathcal{I}^1, [x_1, x_j])$, which means that $f_i(\lambda\mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq x_r$. Now, if $f_i(\mathcal{I}^1) \trianglerighteq f_i(\lambda\mathcal{I}^1 + \mathcal{I}^2)$, f_i satisfies right-biased continuity. If this was not the case, we have that $f_i(\lambda\mathcal{I}^1 + \mathcal{I}^2) \triangleright f_i(\mathcal{I}^1) = x_j$ and thus $F(\mathcal{I}^1, [x_1, x_{j-1}]) < F(\lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_{j-1}])$. Hence, the right-biased continuity of F implies that there is $\lambda' \in \mathbb{N}$ such that $F(\lambda'\mathcal{I}^1 + \lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_{j-1}]) = F(\mathcal{I}^1, [x_1, x_{j-1}]) < i$. On the other hand, we have that $F(\lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_j]) \geq i$ and $F(\lambda'\mathcal{I}^1, [x_1, x_j]) \geq i$ since $f_i(\lambda\mathcal{I}^1 + \mathcal{I}^2) \triangleright f_i(\mathcal{I}^1) = x_j$. Range-reinforcement hence implies that $F(\lambda'\mathcal{I}^1 + \lambda\mathcal{I}^1 + \mathcal{I}^2, [x_1, x_j]) \geq i$, so we derive $f_i(\lambda'\mathcal{I}^1 + \lambda\mathcal{I}^1 + \mathcal{I}^2) = x_j$. This proves that f_i satisfies right-biased continuity for $\lambda + \lambda'$.

Finally, since all rules f_i satisfy the axioms of Theorem 1, they are single-winner position-threshold rules. This shows that F is a multi-winner position-threshold rule. □

Remark 6. Theorem 3 does not hold if we use standard notion of reinforcement instead of range-reinforcement. To make this more explicit, let f^1 denote the single-winner position-threshold rule induced by the weight vector $\alpha^1 = (0, \dots, 0)$ and the threshold vector $\theta^1 = (\frac{1}{3}, \dots, \frac{1}{3})$, and let f^2 denote the single-winner position-threshold rule induced by the weight vector $\alpha^2 = (1, \dots, 1)$ and the threshold vector $\theta^2 = (\frac{2}{3}, \dots, \frac{2}{3})$. Then, the multi-winner voting rule $F : \Lambda^* \to \mathcal{W}_2$, which returns the committee consisting of $f^1(\mathcal{I})$ and $f^2(\mathcal{I})$ if $f^1(\mathcal{I}) \triangleright f^2(\mathcal{I})$ and the committee containing $f^1(\mathcal{I})$ twice if $f^2(\mathcal{I}) \trianglerighteq f^1(\mathcal{I})$, satisfies anonymity, robustness, reinforcement, right-biased continuity, and unanimity, but it is no multi-winner position-threshold rule. On the other hand, we note that many position-threshold rules fail the more demanding condition that $\min(F(\mathcal{I}, x_i), F(\mathcal{I}', x_i)) \leq F(\mathcal{I} + \mathcal{I}', x_i) \leq \max(F(\mathcal{I}, x_i), F(\mathcal{I}', x_i))$ for all alternatives $x_i \in A$ and profiles $\mathcal{I}, \mathcal{I}' \in \Lambda^*$ with $N_{\mathcal{I}} \cap N_{\mathcal{I}'} = \emptyset$.

Remark 7. When generalizing the idea of the majority criterion to committees, we can extend the characterization of the endpoint-median rule to multi-winner elections. More specifically, we define the *proportionality criterion* for multi-winner rules by requiring that $F(\mathcal{I}, x) > 0$ if more than $\frac{n_{\mathcal{I}}}{k+1}$ voters report $\{x_i\}$. This notion is in its spirit similar to Droop proportionality, which is a commonly studied fairness notion for multi-winner voting rules with strict preferences (Tideman, 1995; Woodall, 1997; Aziz and Lee, 2022). Then, it can be shown that only one multi-winner position-threshold rule satisfies the proportionality criterion and strong unanimity (which requires that $F(\mathcal{I}, \bigcap_{i \in N_{\mathcal{I}}} I_i) = k$ if $\bigcap_{i \in N_{\mathcal{I}}} I_i \neq \emptyset$): this rule is defined by the single-winner position-threshold rules $f_1, \dots, f_k$ such that, for all $i \in \{1, \dots, k\}$, f_i is defined by the threshold vector $\theta^i = (\frac{i}{k+1}, \dots, \frac{i}{k+1})$ and the weight vector $\alpha^i = (\frac{i}{k+1}, \dots, \frac{i}{k+1})$.

4. Conclusion

In this paper, we study voting rule for the interval domain, where voters report subintervals of a set of linearly ordered alternatives to indicate their preferences. As our main contribution, we propose and characterize the class of position-threshold rules, which generalize Moulin's phantom median rules (Moulin, 1980) to the interval domain. In essence, position-threshold rules compute for each alternative a collective position, which quantifies the voters' relative positions with respect to this alternative, and then choose the left-most alternative whose collective position exceeds its threshold value. As our main result, we characterize these rules based on robustness (which demands that small changes to the voters' intervals result in small changes in the outcome), reinforcement (which demands that, if an alternative is chosen for two disjoint elections, it is also chosen when combining these elections), and three mild auxiliary conditions called anonymity, unanimity, and right-biased continuity. Moreover, we propose and characterize the endpoint-median rule, which replaces the interval of each voter with two singleton ballots corresponding to the endpoints of the interval and then computes the median rule. In more detail, we show that his rule is the only position-threshold rule that satisfies the majority criterion (an alternative is guaranteed to be chosen if it is uniquely reported by more than half of the voters) and strong unanimity (if some alternatives are reported by all voters, one such alternative is chosen). Since the median rule is the only phantom median rule that satisfies these conditions, our result suggests that the endpoint-median rule is the "correct" extension of the median rule to the interval domain. Lastly, we extend our characterization of position-threshold rules to the case of selecting multiple alternatives: we prove that every multi-winner rule satisfying variants of our original axioms returns the union of multiple single-winner position-threshold rules.

We note that our paper offers several directions for future work. Firstly, we believe that it is interesting to further analyze the axiomatic properties of position-threshold rules. This may help to identify new desirable voting rules on the interval domain or to strengthen the argument for the endpoint-median rule. Moreover, it may be fruitful to analyze position-threshold rules also in the context of facility location on the real line: an interesting question regarding this is, e.g., how much social welfare position-threshold rules can guarantee. Finally, it seems worthwhile to study voting rules on the interval domain that fail robustness but, e.g., satisfy strategyproofness to give a more complete pictures about the possibilities arising from this domain.

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A. Proof of Theorem 1

In this appendix, we will prove Theorem 1: position-threshold rules are the only single-winner voting rules on Λ^* that satisfy anonymity, unanimity, robustness, reinforcement, and right-biased continuity. More specifically, we will show in Appendix A.1 that position-threshold rules satisfy all desired properties and in Appendix A.2 that these properties indeed characterize position-threshold rules. Since a proof sketch for these claims has been discussed in the main body, we will focus here on the details.

A.1. Axiomatics of Position−Threshold Rules

We start by showing that all position-threshold rules satisfy the five axioms of Theorem 1. To this end, we first discuss an auxiliary claim stating that such rules are robust if and only if the threshold vector θ and the weight vector α are compatible. Recall here that threshold vectors $\theta \in (0,1)^m$ satisfy that $\theta_1 \geq \theta_2 \geq \dots \theta_m$ and that a weight vector α is compatible with a threshold vector θ if $\alpha_{i+1} - \alpha_i \geq (\theta_{i+1} - \theta_i) \max(\frac{\alpha_i}{\theta_i}, \frac{1-\alpha_i}{1-\theta_i})$ for all $i \in \{1, \dots, m-2\}$.

Lemma 1. *Let $\theta \in (0,1)^m$ denote a threshold vector and $\alpha \in [0,1]^m$ a weight vector. The rule f given by $f(\mathcal{I}) = \max_{\rhd}\{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i\}$ is robust if and only if α and θ are compatible.*

Proof. Fix a threshold vector θ and a weight vector α, and let f be defined as in the lemma. We will show both directions independently.

($\Longleftarrow$) We first assume that α and θ are compatible and we will show that f is robust. To this end, we will proceed in two steps. First, we will show that, for all profiles $\mathcal{I} \in \Lambda^*$ and all alternatives $x_i \in A \setminus \{x_m\}$, it holds that $\Pi_\alpha(\mathcal{I}, x_{i+1}) \geq \theta_{i+1} n_{\mathcal{I}}$ if $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}$. Based on this insight, we then show that f is robust.

Step 1: Consider an arbitrary interval profile $\mathcal{I} \in \Lambda^*$ and an alternative $x_i \in A \setminus \{x_m\}$ such that $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}$. We will show that $\Pi_\alpha(\mathcal{I}, x_{i+1}) \geq \theta_{i+1} n_{\mathcal{I}}$. For this, we first observe that this implication holds trivially for x_{m-1} since $\Pi_\alpha(\mathcal{I}, x_m) = n_{\mathcal{I}}$ for all profiles $\mathcal{I}$. We thus assume that $x_i \neq x_{m-1}$ and we partition the voters in three sets regarding their position with respect to x_i: $L = \{j \in N_{\mathcal{I}} \colon \forall x \in I_j \colon x \unrhd x_i\}$ are the voters that are weakly left of x_i, $R = \{j \in N_{\mathcal{I}} \colon \forall x \in I_j \colon x_i \rhd x\}$ are the voters that are fully right of x_i, and $Z = N \setminus (L \cup R)$ is the set of voters who report an interval $[x_\ell, x_r]$ with $x_\ell \unrhd x_i \rhd x_r$ By the definition of these sets, we have that $\pi_\alpha(I_j, x_i) = 1$ for all $j \in L$, $\pi_\alpha(I_j, x_i) = 0$ for all $j \in R$, and $\pi_\alpha(I_j, x_i) = \alpha_i$ for all $j \in Z$. Moreover, it holds that $\pi_\alpha(I_j, x_{i+1}) = 1$ for all $j \in L$, $\pi_\alpha(I_j, x_{i+1}) \geq \alpha_{i+1}$ for all $j \in Z$, and $\pi_\alpha(I_j, x_{i+1}) \geq 0$ for all $j \in R$. Now, we define $\ell_{\mathcal{I}} = \frac{|L|}{n_{\mathcal{I}}}$, $r_{\mathcal{I}} = \frac{|R|}{n_{\mathcal{I}}}$, and $z_{\mathcal{I}} = \frac{|Z|}{n_{\mathcal{I}}}$. Since $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}$, it holds that $\ell_{\mathcal{I}} + z_{\mathcal{I}}\alpha_i \geq \theta_i$ and we aim to show that $\ell_{\mathcal{I}} + z_{\mathcal{I}}\alpha_{i+1} \geq \theta_{i+1}$ because this implies that $\Pi_\alpha(\mathcal{I}, x_{i+1}) \geq \theta_{i+1} n_{\mathcal{I}}$. First, if $\alpha_{i+1} \geq \alpha_i$, this follows immediately as $\theta_i \geq \theta_{i+1}$. We hence assume that $\alpha_{i+1} < \alpha_i$. Since $\alpha_{i+1} \geq 0$ by the definition of a weight vector, this further implies that $\alpha_i > 0$.

Next, we proceed with a case distinction with respect to whether $\frac{\alpha_i}{\theta_i} \geq \frac{1-\alpha_i}{1-\theta_i}$.

Case 1: We first assume that $\frac{\alpha_i}{\theta_i} \geq \frac{1-\alpha_i}{1-\theta_i}$. In this case, we will minimize the term $\ell + z\alpha_{i+1}$ subject to $\ell + z\alpha_i \geq \theta_i$, $\ell \geq 0$, and $z \geq 0$. To this end, we observe that we can set $\ell = 0$: if $\ell > 0$, we define $z' = z + \frac{\ell}{\alpha_i}$ and $\ell' = 0$. It is easy to see that our constraints are still satisfied. Moreover, since $\alpha_{i+1} < \alpha_i$, it holds that

$$\begin{aligned}
\ell + z\alpha_{i+1} - \ell' - z'\alpha_{i+1} &= \ell + z\alpha_{i+1} - \left(z + \frac{\ell}{\alpha_i}\right)\alpha_{i+1} \\
&= \ell\left(1 - \frac{\alpha_{i+1}}{\alpha_i}\right) \\
&> 0.
\end{aligned}$$

Hence, we need to set $\ell = 0$ to minimize $\ell + z\alpha_{i+1}$. In turn, to satisfy that $\ell + z\alpha_i \geq \theta_i$, we set $z = \frac{\theta_i}{\alpha_i}$, i.e., as the minimal value such that $z\alpha_i \geq \theta_i$ is satisfied. As a consequence of this analysis, the optimal value our linear program is $\frac{\theta_i\alpha_{i+1}}{\alpha_i}$. Since $\ell_{\mathcal{I}}$ and $z_{\mathcal{I}}$ are a feasible solution to this linear program, it follows that $\ell_{\mathcal{I}} + z_{\mathcal{I}}\alpha_{i+1} \geq \frac{\theta_i\alpha_{i+1}}{\alpha_i}$. Finally, because α and θ are compatible and $\frac{\alpha_i}{\theta_i} \geq \frac{1-\alpha_i}{1-\theta_i}$ by assumption, we have that $\alpha_{i+1} - \alpha_i \geq (\theta_{i+1} - \theta_i)\max(\frac{\alpha_i}{\theta_i}, \frac{1-\alpha_i}{1-\theta_i}) = (\theta_{i+1} - \theta_i)\frac{\alpha_i}{\theta_i}$. We now conclude that $\Pi_\alpha(\mathcal{I}, x_{i+1}) \geq \theta_{i+1}n_{\mathcal{I}}$ as

$$\frac{\Pi_\alpha(\mathcal{I}, x_{i+1})}{n_{\mathcal{I}}} \geq \ell_{\mathcal{I}} + z_{\mathcal{I}}\alpha_{i+1} \geq \frac{\theta_i\alpha_{i+1}}{\alpha_i} = \theta_i + (\alpha_{i+1} - \alpha_i)\frac{\theta_i}{\alpha_i} \geq \theta_i + (\theta_{i+1} - \theta_i) = \theta_{i+1}.$$

Case 2: For our second case, we suppose that $\frac{\alpha_i}{\theta_i} < \frac{1-\alpha_i}{1-\theta_i}$. Equivalently, this means that $\theta_i > \alpha_i$, which implies that $1 - \alpha_i > 0$. We will use a similar approach as in the first case here and minimize the term $\ell + z\alpha_{i+1}$ subject to $\ell + z\alpha_i \geq \theta_i$, $\ell \geq 0$, $z \geq 0$, and $\ell + z \leq 1$. Just as in Case 1, it can be shown that the objective value of this linear program is minimal if ℓ is chosen to be minimal. However, since $\theta_i > \alpha_i$ and we require that $\ell + z \leq 1$, we cannot set ℓ to 0 without violating that $\ell + z\alpha_i \geq \theta_i$. Instead, by using a similar reasoning as in the first case, one can compute that the values of ℓ and z that minimize $\ell + z\alpha_{i+1}$ subject to our constraints are $\ell = 1 - \frac{1-\theta_i}{1-\alpha_i}$ and $z = 1 - \ell$. For these values, our objective value is

$$\begin{aligned}
1 - \frac{1-\theta_i}{1-\alpha_i} + \frac{1-\theta_i}{1-\alpha_i}\alpha_{i+1} &= 1 - (1-\alpha_i)\frac{1-\theta_i}{1-\alpha_i} + \frac{1-\theta_i}{1-\alpha_i}(\alpha_{i+1} - \alpha_i) \\
&= \theta_i + \frac{1-\theta_i}{1-\alpha_i}(\alpha_{i+1} - \alpha_i).
\end{aligned}$$

In particular, since $\ell_{\mathcal{I}}$ and $z_{\mathcal{I}}$ form a feasible solution to our linear program, we infer that $\frac{\Pi_\alpha(\mathcal{I}, x_{i+1})}{n_{\mathcal{I}}}$ is lower-bounded by this expression. Finally, since α and θ are compatible and $\frac{\alpha_i}{\theta_i} < \frac{1-\alpha_i}{1-\theta_i}$, we derive that $\alpha_{i+1} - \alpha_i \geq (\theta_{i+1} - \theta_i)\max(\frac{\alpha_i}{\theta_i}, \frac{1-\alpha_i}{1-\theta_i}) = (\theta_{i+1} - \theta_i)\frac{1-\alpha_i}{1-\theta_i}$. Combined with our previous insight, this means that

$$\frac{\Pi_\alpha(\mathcal{I}, x_{i+1})}{n_{\mathcal{I}}} \geq \theta_i + \frac{1-\theta_i}{1-\alpha_i}(\alpha_{i+1} - \alpha_i) \geq \theta_i + (\theta_{i+1} - \theta_i) \cdot \frac{1-\alpha_i}{1-\theta_i} \cdot \frac{1-\theta_i}{1-\alpha_i} = \theta_{i+1}.$$

Step 2: We will now show that f is robust. To this end, consider an arbitrary interval profile $\mathcal{I}$, let i denote a voter in $N_\mathcal{I}$, and let x_ℓ and x_r denote the alternatives such that $x_\ell \rhd x_r$ and $I_i = [x_\ell, x_r]$. First, we analyze the profile $\mathcal{I}^{i\downarrow x_\ell}$ and note that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_\ell) \leq \Pi_\alpha(\mathcal{I}, x_\ell)$ because $\pi_\alpha(\mathcal{I}, x_\ell) = \alpha_i \geq 0 = \pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_\ell)$. Moreover, $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_h) = \Pi_\alpha(\mathcal{I}, x_h)$ for all $x_h \in A \setminus \{x_\ell\}$ because the relative position of no voter changed with respect to x_h. Now, if $f(\mathcal{I}) = x_j$ for some alternative $x_j \neq x_\ell$, it is easy to show that $f(\mathcal{I}^{i\downarrow x_\ell}) = x_j$, too. In more detail, if $x_j \rhd x_\ell$, then the value $\Pi_\alpha(\mathcal{I}, x_\ell)$ does not matter as $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_j) = \Pi_\alpha(\mathcal{I}, x_j) \geq \theta_j n_\mathcal{I}$ and $x_j \rhd x_\ell$. On the other hand, if $x_\ell \rhd x_j$, then $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_\ell) \leq \Pi_\alpha(\mathcal{I}, x_\ell) < \theta_\ell n_\mathcal{I}$ and the outcome again remains the same. Finally, if $f(\mathcal{I}) = x_\ell$, then $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_h) = \Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_h) < \theta_h n_\mathcal{I}$ for all $x_h \rhd x_\ell$ and $x_\ell \unrhd f(\mathcal{I}^{i\downarrow x_\ell})$. Moreover, it holds by Step 1 that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_\ell}, x_{\ell+1}) = \Pi_\alpha(\mathcal{I}, x_{\ell+1}) \geq \theta_{i+1} n_\mathcal{I}$ because $f(\mathcal{I}) = x_\ell$ implies that $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}$. This means that $f(\mathcal{I}^{i\downarrow x_\ell}) \unrhd x_{i+1}$, so $f(\mathcal{I}^{i\downarrow x_\ell}) \in \{x_\ell, x_{\ell+1}\}$. This proves that f is robust in this case.

Next, consider the profile $\mathcal{I}^{i\downarrow x_r}$, for which $\pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_{r-1}) = 1 \geq \pi_\alpha(\mathcal{I}, x_{r-1})$ and $\pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_r) = 1 = \pi_\alpha(\mathcal{I}, x_r)$. This means that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_{r-1}) \geq \Pi_\alpha(\mathcal{I}, x_{r-1})$ and $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_h) = \Pi_\alpha(\mathcal{I}, x_h)$ for all $x_h \in A \setminus \{x_{r-1}\}$. We first assume that $f(\mathcal{I}) = x_j \neq x_r$. If $x_j \rhd x_r$, then $f(\mathcal{I}^{i\downarrow x_r}) = f(\mathcal{I})$ because $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_h) = \Pi_\alpha(\mathcal{I}, x_h)$ for all x_h with $x_h \rhd x_{r-1}$ and $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_{r-1}) \geq \Pi_\alpha(\mathcal{I}, x_{r-1})$. On the other hand, if $x_r \rhd x_j$, we infer that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_r) = \Pi_\alpha(\mathcal{I}, x_r) < \theta_r n_\mathcal{I}$. By the contrapositive of Step 1, this means also that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_{r-1}) < \theta_{r-1} n_\mathcal{I}$. Hence, it is now easy to derive that $f(\mathcal{I}) = f(\mathcal{I}^{i\downarrow x_r})$ since $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_h) = \Pi_\alpha(\mathcal{I}, x_h)$ for all alternatives $x_h \in A \setminus \{x_{r-1}\}$. Finally, assume that $f(\mathcal{I}) = x_r$. This means that $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_h) = \Pi_\alpha(\mathcal{I}, x_h) < \theta_h n_\mathcal{I}$ for all x_h with $x_h \rhd x_{r-1}$ and $\Pi_\alpha(\mathcal{I}^{i\downarrow x_r}, x_r) = \Pi_\alpha(\mathcal{I}, x_r) \geq \theta_r n_\mathcal{I}$. It follows that $f(\mathcal{I}^{i\downarrow x_r}) \in \{x_{r-1}, x_r\}$ and robustness holds again.

($\implies$) Next, we will show that f fails robustness if α and θ are not compatible. To this end, we assume that there is an index $i \in \{1, \dots, m-2\}$ with $\alpha_{i+1} - \alpha_i < (\theta_{i+1} - \theta_i) \max(\frac{\alpha_i}{\theta_i}, \frac{1-\alpha_i}{1-\theta_i})$. We proceed with a case distinction depending on $\max(\frac{\alpha_i}{\theta_i}, \frac{1-\alpha_i}{1-\theta_i})$.

Case 1: First assume that $\frac{\alpha_i}{\theta_i} \geq \frac{1-\alpha_i}{1-\theta_i}$. Equivalently, this assumption means that $\alpha_i \geq \theta_i$, so it follows that $\alpha_i > 0$. We define $\delta = (\theta_{i+1} - \theta_i) \cdot \frac{\alpha_i}{\theta_i} - (\alpha_{i+1} - \alpha_i)$ and we choose a value $\epsilon > 0$ such that $\delta \frac{\theta_i}{\alpha_i} > \epsilon \alpha_{i+1}$. Moreover, let $v \in \mathbb{Q} \cap (0, 1]$ denote a rational value such that $\frac{\theta_i}{\alpha_i} \leq v \leq \frac{\theta_i}{\alpha_i} + \epsilon$. Finally, let $w_1, w_2 \in \mathbb{N}_0$ denote two integers such that $v = \frac{w_1}{w_1 + w_2}$ and we consider the interval profile $\mathcal{I}$ where w_1 voters report $\{x_i, x_{i+1}, x_{i+2}\}$ and w_2 voters report $\{x_m\}$. It can be easily computed that

$$\Pi_\alpha(\mathcal{I}, x_i) = w_1 \cdot \alpha_i = v \cdot n_\mathcal{I} \cdot \alpha_i \geq \frac{\theta_i}{\alpha_i} \cdot n_\mathcal{I} \cdot \alpha_i = \theta_i n_\mathcal{I}.$$

Since $\Pi_\alpha(\mathcal{I}, x_h) = 0$ for all x_h with $x_h \rhd x_i$, we conclude that $f(\mathcal{I}) = x_i$.

Next, let $\mathcal{I}'$ denote the profile where w_1 voters report $\{x_{i+1}, x_{i+2}\}$ and w_2 voters report $\{x_m\}$. Robustness from $\mathcal{I}$ to $\mathcal{I}'$ postulates that $f(\mathcal{I}') \in \{x_i, x_{i+1}\}$. Moreover, it holds that $\Pi_\alpha(\mathcal{I}', x_i) = 0$ as no voter approves an alternative left of x_{i+1} in $\mathcal{I}'$, so $f(\mathcal{I}') \neq x_i$.

However, the subsequent computations show that $\Pi_\alpha(\mathcal{I}', x_{i+1}) = \Pi_\alpha(\mathcal{I}, x_{i+1}) < \theta_{i+1} n_{\mathcal{I}}$. This means that $f(\mathcal{I}') \neq x_{i+1}$ and robustness is violated.

$$\begin{aligned}
\Pi_\alpha(\mathcal{I}', x_{i+1}) &= v \cdot n_{\mathcal{I}} \cdot \alpha_{i+1} \\
&\leq \left(\frac{\theta_i}{\alpha_i} + \epsilon\right) \cdot n_{\mathcal{I}} \cdot \alpha_{i+1} \\
&= \theta_i \cdot n_{\mathcal{I}} + (\alpha_{i+1} - \alpha_i) \cdot \frac{\theta_i n_{\mathcal{I}}}{\alpha_i} + \epsilon \cdot n_{\mathcal{I}} \cdot \alpha_{i+1} \\
&= \theta_i \cdot n_{\mathcal{I}} + \left((\theta_{i+1} - \theta_i) \cdot \frac{\alpha_i}{\theta_i} - \delta\right) \cdot \frac{\theta_i n_{\mathcal{I}}}{\alpha_i} + \epsilon \cdot n_{\mathcal{I}} \cdot \alpha_{i+1} \\
&= n_{\mathcal{I}} \left(\theta_i + \theta_{i+1} - \theta_i - \delta \frac{\theta_i}{\alpha_i} + \epsilon \cdot \alpha_{i+1}\right) \\
&< \theta_{i+1} n_{\mathcal{I}}.
\end{aligned}$$

Here, the second line uses the definition of v, the third one rearranges the terms, and the fourth one applies the definition of δ. The fifth line is again simple calculus, and the last inequality follows because $\delta \frac{\theta_i}{\alpha_i} > \epsilon \alpha_{i+1}$.

Case 2: As the second case, we assume that $\frac{\alpha_i}{\theta_i} < \frac{1-\alpha_i}{1-\theta_i}$. This is equivalent to $\theta_i > \alpha_i$, so we derive that $0 < 1 - \theta_i < 1 - \alpha_i$. Next, we define $\delta = (\theta_{i+1} - \theta_i) \cdot \frac{1-\alpha_i}{1-\theta_i} - (\alpha_{i+1} - \alpha_i)$ and we choose $\epsilon > 0$ such that $\delta \frac{1-\theta_i}{1-\alpha_i} > \epsilon(1 - \alpha_{i+1})$. Moreover, we observe that $\frac{1-\theta_i}{1-\alpha_i} > 0$ since $0 < \theta_i < 1$ and $1 - \alpha_i > 0$, and that $\frac{1-\theta_i}{1-\alpha_i} < 1$ since $0 < 1 - \theta_i < 1 - \alpha_i$. Hence, there is a rational value $v \in \mathbb{Q} \cap (0, 1)$ with $\frac{1-\theta_i}{1-\alpha_i} - \epsilon \leq v \leq \frac{1-\theta_i}{1-\alpha_i}$. Finally, let $w_1, w_2 \in \mathbb{N}$ denote two integers such that $v = \frac{w_1}{w_1+w_2}$ and consider the profile $\mathcal{I}$ where w_1 voters report $\{x_i, x_{i+1}, x_{i+2}\}$ and w_2 voters report $\{x_i\}$. We first compute that

$$\Pi_\alpha(\mathcal{I}, x_i) = w_2 + w_1 \alpha_i = n_{\mathcal{I}}(1 - v + v\alpha_i) \geq n_{\mathcal{I}} \left(1 - (1 - \alpha_i) \frac{1 - \theta_i}{1 - \alpha_i}\right) = \theta_i n_{\mathcal{I}}.$$

Here, the inequity in the third step follows because $v \leq \frac{1-\theta_i}{1-\alpha_i}$ and $(1 - \alpha_i) \geq 0$. Since no voter reports an alternative left of x_i, this shows that $f(\mathcal{I}) = x_i$.

Next, let $\mathcal{I}'$ denote the profile where w_1 voters report $\{x_{i+1}, x_{i+2}\}$ and w_2 voters report $\{x_{i+1}\}$. First, repeatedly applying robustness from $\mathcal{I}$ to $\mathcal{I}'$ shows that $f(\mathcal{I}') \in \{x_i, x_{i+1}\}$. In particular, for the voters deviating from $\{x_i\}$ to $\{x_{i+1}\}$, we can make an intermediate step by expanding the interval to $\{x_i, x_{i+1}\}$. On the other hand, we derive that $f(\mathcal{I}') \neq x_i$ because no voter reports an alternative x_h with $x_h \trianglerighteq x_i$. Finally, as the following inequality shows, it holds $\Pi_\alpha(\mathcal{I}', x_{i+1}) < \theta_{i+1} n_{\mathcal{I}'}$. This proves that $f(\mathcal{I}) \neq x_{i+1}$, so f

fails robustness.

$$\begin{aligned}
\Pi_\alpha(\mathcal{I}', x_{i+1}) &= w_2 + w_1\alpha_{i+1} \\
&= n_\mathcal{I}(1 - v + v\alpha_{i+1}) \\
&\leq n_\mathcal{I}\left(1 - (1-\alpha_{i+1})\left(\frac{1-\theta_i}{1-\alpha_i} - \epsilon\right)\right) \\
&= n_\mathcal{I}\left(1 - (1-\alpha_i)\cdot\frac{1-\theta_i}{1-\alpha_i} + (\alpha_{i+1}-\alpha_i)\cdot\frac{1-\theta_i}{1-\alpha_i} + (1-\alpha_{i+1})\epsilon\right) \\
&= n_\mathcal{I}\left(\theta_i + \left((\theta_{i+1}-\theta_i)\cdot\frac{1-\alpha_i}{1-\theta_i} - \delta\right)\cdot\frac{1-\theta_i}{1-\alpha_i} + (1-\alpha_{i+1})\epsilon\right) \\
&= n_\mathcal{I}\left(\theta_{i+1} - \delta\cdot\frac{1-\theta_i}{1-\alpha_i} + (1-\alpha_{i+1})\epsilon\right) \\
&< \theta_{i+1}n_\mathcal{I}.
\end{aligned}$$

The first line uses the definition of Π_α, the second the definition of w_1 and w_2, and the third inequality that $\frac{1-\theta_i}{1-\alpha_i} - \epsilon < v$. Next, we rearrange our formula and substitute the definition of δ in the fifth line. The remaining two lines follow from simple calculus and the definition of ϵ. □

Based on Lemma 1, it is now easy to check that position-threshold rules indeed satisfy all required axioms.

Lemma 2. *Every position-threshold rule satisfies anonymity, unanimity, robustness, reinforcement, and right-biased continuity.*

Proof. Fix a threshold vector $\theta \in (0,1)^m$ and a compatible weight vector $\alpha = (\alpha_1, \dots, \alpha_m)$ and let f denote the position-threshold rule induced by these vectors. We first note that Π_α is anonymous, so f also satisfies this property. Moreover, if $I_i = \{x_j\}$ for all voters in a profile $\mathcal{I}$, then $\Pi_\alpha(\mathcal{I}, x_j) = n_\mathcal{I} \geq \theta_j|N_\mathcal{I}|$ and $\Pi_\alpha(\mathcal{I}, x_h) = 0 < \theta_h n_\mathcal{I}$ for all $x_h \rhd x_j$. This means that $f(\mathcal{I}) = x_j$, so f is unanimous. Next, Lemma 1 implies that f is robust since α and θ are compatible.

As the fourth axiom, we will show that f is reinforcing. For this, let $\mathcal{I}^1$ and $\mathcal{I}^2$ denote two profiles in Λ^* such that $f(\mathcal{I}^1) = f(\mathcal{I}^2) = x_i$ for some alternative $x_i \in A$ and $N_{\mathcal{I}^1} \cap N_{\mathcal{I}^2} = \emptyset$. By definition of f, it holds for $\mathcal{I} \in \{\mathcal{I}^1, \mathcal{I}^2\}$ that $\Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}$ and $\Pi_\alpha(\mathcal{I}, x_h) < \theta_h n_\mathcal{I}$ for all $x_h \in A$ with $x_h \rhd x_i$. Moreover, we have for all $x \in A$ that $\Pi_\alpha(\mathcal{I}^1 + \mathcal{I}^2, x) = \Pi_\alpha(\mathcal{I}^1, x) + \Pi_\alpha(\mathcal{I}^2, x)$. Hence, it follows that $\Pi_\alpha(\mathcal{I}^1 + \mathcal{I}^2, x_h) < \theta_h n_{\mathcal{I}^1} + \theta_h n_{\mathcal{I}^2} = \theta_h n_{\mathcal{I}^1+\mathcal{I}^2}$ for all x_h with $x_h \rhd x_i$ and $\Pi_\alpha(\mathcal{I}^1 + \mathcal{I}^2, x_i) \geq \theta_i n_{\mathcal{I}^1} + \theta_i n_{\mathcal{I}^2} = \theta_i n_{\mathcal{I}^1+\mathcal{I}^2}$. This means that $f(\mathcal{I}^1 + \mathcal{I}^2) = x_i$ and f thus is reinforcing.

Finally, we will prove that f satisfies right-biased continuity. For this, we consider two profiles $\mathcal{I}^1, \mathcal{I}^2 \in \Lambda^*$. First, if $f(\mathcal{I}^2) = f(\mathcal{I}^1)$, it follows by reinforcement that $f(\mathcal{I}^1 + \mathcal{I}^2) = f(\mathcal{I}^1)$ and right-biased continuity is satisfied. Next, we assume that $f(\mathcal{I}^2) \rhd f(\mathcal{I}^1)$ and we will show that there is $\lambda \in \mathbb{N}$ such that $f(\lambda\mathcal{I}^1 + \mathcal{I}^2) = f(\mathcal{I}^1)$. By the definition of f, we derive that $\Pi_\alpha(\mathcal{I}^1, x_h) < \theta_h n_{\mathcal{I}^1}$ for all $x_h \in A$ with $x_h \rhd x_i$ and $\Pi_\alpha(\mathcal{I}^1, x_i) \geq \theta_i|N_{\mathcal{I}^1}|$. Moreover, it holds that $\Pi_\alpha(\mathcal{I}^2, x_j) \geq \theta_j n_{\mathcal{I}^2}$ for the alternative

$x_j = f(\mathcal{I}^2)$. We have shown in the proof of Lemma 1 (see Step 1) that $\Pi_\alpha(\mathcal{I}, x_h) \geq \theta_h n_\mathcal{I}$ implies $\Pi_\alpha(\mathcal{I}, x_{h+1}) \geq \theta_{h+1} n_\mathcal{I}$ for all interval profiles $\mathcal{I}$ and all alternatives $x_h \in A \setminus \{x_m\}$ if α and θ are compatible. Based on this insight, we infer that $\Pi_\alpha(\mathcal{I}^2, x_i) \geq \theta_i n_{\mathcal{I}^2}$ as $x_j \rhd x_i$. Now, let $\delta_h = \theta_h n_{\mathcal{I}^1} - \Pi_\alpha(\mathcal{I}^1, x_h)$ for all $h \in \{1, \dots, i-1\}$ and let $\lambda \in \mathbb{N}$ denote an integer such that $\lambda \delta_h > \Pi_\alpha(\mathcal{I}^2, x_h)$ for all such h. By the definition of λ, we derive for all x_h with $x_h \rhd x_i$ that

$$\begin{aligned}
\Pi_\alpha(\lambda \mathcal{I}^1 + \mathcal{I}^2, x_h) &= \lambda \Pi_\alpha(\mathcal{I}^1, x_h) + \Pi_\alpha(\mathcal{I}^2, x_h) \\
&= \lambda(\theta_h n_{\mathcal{I}^1} - \delta_h) + \Pi_\alpha(\mathcal{I}^2, x_h) \\
&< \theta_h n_{\lambda \mathcal{I}^1 + \mathcal{I}^2}
\end{aligned}$$

This then implies that $x_i \trianglerighteq f(\lambda \mathcal{I}^1 + \mathcal{I}^2)$. On the other hand, it is holds that

$$\begin{aligned}
\Pi_\alpha(\lambda \mathcal{I}^1 + \mathcal{I}^2, x_i) &= \lambda \Pi_\alpha(\mathcal{I}^1, x_i) + \Pi_\alpha(\mathcal{I}^2, x_i) \\
&\geq \lambda \theta_i n_{\mathcal{I}^1} + \theta_i n_{\mathcal{I}^2} \\
&= \theta_i n_{\lambda \mathcal{I}^1 + \mathcal{I}^2}.
\end{aligned}$$

Hence, $f(\lambda \mathcal{I}^1 + \mathcal{I}^2) = x_i$ and right-biased continuity is satisfied.

For the second case, suppose that $f(\mathcal{I}^1) \rhd f(\mathcal{I}^2)$ and let x_r denote the right-most alternative that is reported by some voter in $\mathcal{I}^1$. We moreover let $x_i = f(\mathcal{I}^1)$, and we will show that there is $\lambda \in \mathbb{N}$ such that $x_i \trianglerighteq f(\lambda \mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq x_r$. To this end, we first observe that $\Pi_\alpha(\mathcal{I}, x_j) < \theta_j n_\mathcal{I}$ for all $j < i$ and $\mathcal{I} \in \{\mathcal{I}^1, \mathcal{I}^2\}$, so analogous arguments as before show that $x_i \trianglerighteq f(\lambda \mathcal{I}^1 + \mathcal{I}^2)$ for all $\lambda \in \mathbb{N}$. Next, we choose λ such that $\theta_r \leq \frac{\lambda n_{\mathcal{I}^1}}{\lambda n_{\mathcal{I}^1} + n_{\mathcal{I}^2}}$. We note that such a λ exists as $\theta_r < 1$ and $\frac{\lambda n_{\mathcal{I}^1}}{\lambda n_{\mathcal{I}^1} + n_{\mathcal{I}^2}}$ converges to 1 as λ increases. By the choice of x_r, we have that $\Pi_\alpha(\mathcal{I}^1, x_r) = n_{\mathcal{I}^1}$. Hence, we compute that

$$\begin{aligned}
\Pi_\alpha(\lambda \mathcal{I}^1 + \mathcal{I}^2, x_r) &= \lambda \Pi_\alpha(\mathcal{I}^1, x_r) + \Pi_\alpha(\mathcal{I}^2, x_r) \\
&\geq \lambda n_{\mathcal{I}^1} \\
&\geq \theta_r(\lambda n_{\mathcal{I}^1} + n_{\mathcal{I}^2}) \\
&= \theta_r n_{\lambda \mathcal{I}^1 + \mathcal{I}^2}.
\end{aligned}$$

This proves that $f(\lambda \mathcal{I}^1 + \mathcal{I}^2) \trianglerighteq x_r$ and thus completes the proof that f satisfies right-biased continuity. □

A.2. Derivation of Weight and Threshold Vectors

We will next show that every voting rule on Λ^* that satisfies anonymity, unanimity, robustness, reinforcement, and right-biased continuity is a position-threshold rule. To this end, we suppose throughout this section that f is a voting rule on Λ^* that satisfies all considered axioms, and we aim to represent f as a position-threshold rule by deriving the weight and threshold vectors that induce f.

As a first step, we will show that f coincides with a phantom median rule on the domain $\mathcal{D}_1^N$ where all voters of a fixed electorate N report a single alternative. More formally, the domain $\mathcal{D}_1^N$ is the subset of Λ^* given by $\mathcal{D}_1^N = \{\mathcal{I} \in \Lambda^N : \forall i \in N : |I_i| = 1\}$. We will prove our claim by showing that f induces a voting rule on the domain of single-peaked preferences $\mathcal{P}_\triangleright^N$ that satisfies anonymity, unanimity, and strategyproofness. By the characterization of Moulin (1980), we then infer that f is a phantom median rule for $\mathcal{P}_\triangleright^N$ (see also Border and Jordan (1983) or Weymark (2011) as Moulin uses slightly stronger axioms than we do), which will then imply the desired representation of f on $\mathcal{D}_1^N$. To make our proof precise, we will next present the definitions of anonymity, unanimity, and strategyproofness for the domain of single-peaked preferences $\mathcal{P}_\triangleright^N$. We say that a voting rule f on $\mathcal{P}_\triangleright^N$ is

- *anonymous* if $f(\pi(R)) = f(R)$ for all preference profiles $\mathcal{P}_\triangleright^N$ and permutations $N \to N$.
- *unanimous* if $f(R) = x_i$ for all preference profiles $R \in \mathcal{P}_\triangleright^N$ and alternatives $x_i \in A$ such that all voters in R report x_i as their favorite alternative.
- *strategyproof* if $f(R) \succsim f(R')$ for all profiles $R, R' \in \mathcal{R}_\triangleright^N$ and voters $i \in N$ such that $\succsim_j = \succsim_j'$ for all $j \in N \setminus \{i\}$.

Then, the characterization of Moulin (1980) states that a voting rule f on $\mathcal{P}_\triangleright^N$ satisfies anonymity, unanimity, and strategyproofness if and only if it is a phantom median rule, i.e., there is a threshold vector $\theta \in (0,1)^m$ such that $\theta_1 \geq \dots \geq \theta_m$ and $f(R) = \max_\triangleright \{x_i \in A : \Pi_{SP}(R, x_i) \geq \theta_i |N_R|\}$ for all profiles $R \in \mathcal{P}_\triangleright^N$.[3] We are now ready to show our first lemma. For this lemma, we extend the definition of the individual and collective peak position functions to interval profiles $\mathcal{I} \in \mathcal{D}_1^N$ by defining $\pi_{SP}(\{x_i\}, x_j) = 1$ if $x_i \trianglerighteq x_j$ and $\pi_{SP}(\{x_i\}, x_j) = 0$ if $x_j \triangleright x_i$ for all $x_i, x_j \in A$, and $\Pi_{SP}(\mathcal{I}, x_j) = \sum_{i \in N_\mathcal{I}} \pi_{SP}(I_i, x_j)$ for all $x_j \in A$ and $\mathcal{I} \in \mathcal{D}_1^N$.

Lemma 3. *For every electorate $N \in \mathcal{F}(\mathbb{N})$, there is a threshold vector $\theta = (\theta_1, \dots, \theta_m) \in (0,1)^m$ such that $\theta_1 \geq \dots \geq \theta_m$ and $f(\mathcal{I}) = \max_\triangleright \{x_i \in A : \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$ for all interval profiles $\mathcal{I} \in \mathcal{D}_1^N$.*

Proof. We will show the lemma by reducing f to a voting rule f' on $\mathcal{P}_\triangleright^N$ that is strategyproof, anonymous, and unanimous. Then, the characterization of Moulin (1980) shows that there is a vector $\theta \in (0,1)^m$ such that $\theta_1 \geq \dots \geq \theta_m$ and $f'(R) = \max_\triangleright \{x_i \in A : \Pi_{SP}(R, x_i) \geq \theta_i |N_R|\}$ for all profiles $R \in \mathcal{P}_\triangleright^N$. We thus define the interval profile $\mathcal{I}(R)$ given a single-peaked profile $R \in \mathcal{P}_\triangleright^N$ by $I(R)_i = \{x \in A : \forall y \in A \setminus \{x\} : x \succ_i y\}$ for all $i \in N$, i.e., the interval of each voter only contains his favorite alternative in R. Then, we set $f'(R) = f(\mathcal{I}(R))$ for all profiles $R \in \mathcal{P}_\triangleright^N$. We first note that it is straightforward to check that f' is anonymous and unanimous as f satisfies these axioms.

[3] The standard way to state Moulin's result is that a voting rule f on $\mathcal{P}_\triangleright^N$ satisfies anonymity, unanimity, and strategyproofness if and only if there are $|N| - 1$ phantom voters who report fixed single-peaked preference relations and that f chooses the top-ranked alternative of the median voter with respect to our $|N|$ original voters and the $|N| - 1$ phantom voters. To arrive at our representation, we define p_i as the number of phantom voters that top-rank alternative x_i. Then, it can be checked that, for all profiles $R \in \mathcal{P}_\triangleright^N$, it holds that $f(R) = \max_\triangleright \{x_i \in A : \Pi_{SP}(R, x_i) \geq \theta_i |N_R|\}$ for the threshold vector θ given by $\theta_k = \frac{1 + 2\sum_{i=k+1}^m p_i}{2|N|}$ for all $k \in \{1, \dots, m\}$.

We hence focus on showing that f' is strategyproof. For this, we consider two preference profiles $R, R' \in \mathcal{P}_\rhd^N$ and a voter $i \in N$ such that $\succsim_j = \succsim'_j$ for all $j \in N \setminus \{i\}$. We will show that, $f'(R) \succsim_i f'(R')$. To this end, let x_i denote voter i's favorite alternative in R and x'_i denote his favorite alternative in R'. First, if $f'(R) = x_i$, voter i cannot manipulate as his favorite alternative is chosen in R. Without loss of generality, we will hence assume that $x_i \rhd f'(R)$. Now, if $x'_i \trianglerighteq x_i$, it follows from the robustness of f that $f'(R) = f'(R')$. In more detail, let $\mathcal{I}^* \in \Lambda^N$ denote the interval profile such that $I^*_i = [x'_i, x_i]$ and $I^*_j = \{x \in A \colon \forall y \in A \setminus x \colon x \succ_i y\}$ for all $j \in N \setminus \{i\}$. First, we note that we can transform $\mathcal{I}(R)$ to $\mathcal{I}^*$ by one after another adding alternatives left of x_i to voter i's interval. Since $x_i \rhd f'(R) = f(\mathcal{I}(R))$, robustness implies for all of these steps that the outcome does not change. Hence, we have that $f(\mathcal{I}^*) = f(\mathcal{I}(R))$. Finally, we can then transform $\mathcal{I}^*$ to $\mathcal{I}(R')$ by one after another deleting the alternatives right of x'_i from voter i's interval. Since $f(\mathcal{I}^*) \notin I^*_i$, robustness implies that the outcome is again not allowed to change, so we now conclude that $f'(R) = f(\mathcal{I}^*) = f'(R')$.

Next, if $x_i \rhd x'_i \trianglerighteq f'(R)$, we can use an analogous argument based on the interval $I^*_i = [x_i, x'_i]$ as robustness still implies that the winner is not allowed to change. Finally, if $f'(R) \rhd x'_i$, it follows from robustness that $f'(R) \trianglerighteq f'(R')$. To see this, we consider first the profile $\mathcal{I}^1$ where voter i reports $[x_i, f'(R)]$ and every other voter only reports his favorite alternative in R. Repeatedly applying robustness from $\mathcal{I}^1$ to $\mathcal{I}(R)$ shows that $f(\mathcal{I}(R)) = f(\mathcal{I}^1)$. Next, let $\mathcal{I}^2$ denote the profile derived from $\mathcal{I}^1$ by assigning voter i the interval $[x_i, x'_i]$. We can transform $\mathcal{I}^1$ to $\mathcal{I}^2$ by adding one after another the alternatives right of $f'(R)$ to voter i's interval. Robustness implies that the winner can only move to the right, i.e., that $f(\mathcal{I}^1) \trianglerighteq f(\mathcal{I}^2)$. As the last step, we transform $\mathcal{I}^2$ to $\mathcal{I}(R')$ by one after another deleting alternatives left of x'_i from voter i's interval. Robustness implies for such actions again that the winner can only move to the right, so $f(\mathcal{I}^2) \trianglerighteq f(\mathcal{I}(R'))$. By chaining these insights and using the definition of f', it follows now that $f'(R) \trianglerighteq f'(R')$. Finally, since $x_i \rhd f'(R) \trianglerighteq f'(R')$, the single-peakedness of $\succsim_i$ implies that $f'(R) \succsim_i f'(R')$, so f' is indeed strategyproof.

By applying Moulin's characterization, we now derive that there is a threshold vector $\theta \in (0,1)^m$ such that $\theta_1 \geq \dots \geq \theta_m$ and $f'(R) = \max_\rhd \{x_i \in A \colon \Pi_{SP}(R, x_i) \geq \theta_i |N_R|\}$ for all profiles $R \in \mathcal{P}_\rhd^N$. Due to the relation between f and f', it then follows that $f(\mathcal{I}) = \max_\rhd \{x_i \in A \colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$ for all interval profiles $\mathcal{I} \in \mathcal{D}_1^N$ because there is a profile $R \in \mathcal{P}_\rhd^N$ such that $\mathcal{I} = \mathcal{I}(R)$. □

Next, we will generalize Lemma 3 from a fixed electorate N to the domain of all electorates. To this end, we set $\mathcal{D}_1^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \mathcal{D}_1^N$.

Lemma 4. *There is a threshold vector $\theta \in (0,1)^m$ such that $\theta_1 \geq \dots \geq \theta_m$ and $f(\mathcal{I}) = \max_\rhd \{x_i \in A \colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$ for all interval profiles $\mathcal{I} \in \mathcal{D}_1^*$.*

Proof. To prove this lemma, we denote by $N(z)$ an arbitrary electorate with z voters. Because of the anonymity of f, the choice of $N(z)$ does not matter. By Lemma 3, there is for every electorate $N(z)$ a threshold vector $\theta^z \in (0,1)^m$ such that $\theta^z_1 \geq \dots \geq \theta^z_m$ and $f(\mathcal{I}) = \max_\rhd \{x_i \in A \colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta^z_i n_{\mathcal{I}}\}$ for all profiles $\mathcal{I} \in \mathcal{D}_1^{N(z)}$. We note, however, that these vectors are not unique: instead of θ^z, we can represent f on

$\mathcal{D}_1^{N(z)}$ by every vector $q \in (0,1]^m$ such that $q_m = q_{m-1}$ and $q_i \in (\frac{v_i^z}{z}, \frac{v_i^z+1}{z}]$ for all $i \in \{1\ldots,m-1\}$, where $v_i^z \in \mathbb{N}_0$ is chosen such $\frac{v_i^z}{z} < \theta_i^z \leq \frac{v_i^z+1}{z}$. The reason for this is that, $\Pi_{SP}(\mathcal{I},x_i) \in \{0,\ldots,z\}$ for every alternative $x_i \in A$ and profile $\mathcal{I} \in \mathcal{D}_1^{N(z)}$. Also, the exact choice of q_m has no influence as $\Pi_{SP}(\mathcal{I},x_m) = n_{\mathcal{I}}$ for all $\mathcal{I} \in \mathcal{D}_1^{N(z)}$. We hence define the interval $I_i^z = (\frac{v_i^z}{z}, \frac{v_i^z+1}{z}]$ for all $z \in \mathbb{N}$ and $i \in \{1,\ldots,m-1\}$ and emphasize that $f(\mathcal{I}) = \max_{\rhd}\{x_i \in A\colon \Pi_{SP}(\mathcal{I},x_i) \geq q_i n_{\mathcal{I}}\}$ for for all $\mathcal{I} \in \mathcal{D}_1^{N(z)}$ and every vector q with $q_i \in I_i^z$ for all $i \in \{1,\ldots,m\}$.

We next define $\bar{I}_i^z = \bigcap_{s\in\{1,\ldots,z\}} I_i^z$ as the intersection of the first z intervals I_i^s for some alternative $x_i \in A$, and we will show that $\bar{I}_i^z \neq \emptyset$ for all $z \in \mathbb{N}$. Assume for contradiction that there are $z \in \mathbb{N}$ and $i \in \{1,\ldots,m-1\}$ such that $\bar{I}_i^z = \emptyset$, and moreover suppose that z is chosen minimal, i.e., $\bar{I}_i^{z-1} \neq \emptyset$ but $\bar{I}_i^z = \emptyset$. Since $\bar{I}_i^{z-1}$ is the non-empty intersection of intervals that are all closed to the right, it is itself an interval that is closed to the right. Next, we denote every interval I_i^s by $I_i^s = (\ell_i^s, r_i^s]$ and define $\bar{r}_i^{z-1} = \min_{s\in\{1,\ldots,z-1\}} r_i^s$ and $\bar{\ell}_i^{z-1} = \max_{s\in\{1,\ldots,z-1\}} \ell_i^s$. It holds that $\bar{I}_i^{z-1} = (\bar{\ell}_i^{z-1}, \bar{r}_i^{z-1}]$ since every point left of $\bar{\ell}_i^{z-1}$ and right of $\bar{r}_i^{z-1}$ is not included in some interval I_i^s. Because $\bar{I}_i^z = \emptyset$, it either holds that $r_i^z \leq \bar{\ell}_i^{z-1}$ or $\bar{r}_i^{z-1} \leq \ell_i^z$. We subsequently assume that $r_i^z \leq \bar{\ell}_i^{z-1}$ as both cases are symmetric. Now, let $s \in \{1,\ldots,z-1\}$ denote the index of an interval I_i^s with $r_i^s = \bar{r}_i^{z-1}$, which means that $\bar{r}_i^{z-1} = \frac{c}{s}$ for some $c \in \{1,\ldots,s\}$.

We consider the profile $\mathcal{I}$ where $s \cdot \bar{r}_i^{z-1}$ voters report x_i and $s \cdot (1-\bar{r}_i^{z-1})$ voters report x_{i+1}. By the definition of I_i^s, we have that $f(\mathcal{I}) = \max_{\rhd}\{x_j \in A\colon \Pi_{SP}(\mathcal{I},x_j) \geq \theta_j^s \cdot s) = x_i$ since $\Pi_{SP}(\mathcal{I},x_j) = 0$ for all $x_j \in A$ with $x_j \rhd x_i$ and $\Pi_{SP}(\mathcal{I},x_i) = s \cdot \bar{r}_i^{z-1} \geq \theta_i^s n_{\mathcal{I}}$. Moreover, by reinforcement, it also holds that $f(z\mathcal{I}) = x_i$ for the profile $z\mathcal{I}$ that consists of z copies of $\mathcal{I}$. Next, consider the profile $\mathcal{I}'$, which consists of $z \cdot \ell_i^z$ voters reporting x_i and $z \cdot (1-\ell_i^z)$ voters reporting x_{i+1} (note that $z \cdot \ell_i^z$ and $z \cdot (1-\ell_i^z)$ are integers since $\ell_i^z = \frac{c}{z}$ for some $c \in \{0,\ldots,z-1\}$). Using the definition of ℓ_i^z, it follows that $f(\mathcal{I}') = \max_{\rhd}\{x_j \in A\colon \Pi_{SP}(\mathcal{I}',x_j) \geq \theta_j^z \cdot z\} = x_{i+1}$ since $\Pi(\mathcal{I}',x_i) = z \cdot \ell_i^z < \theta_i^z \cdot n_{\mathcal{I}}$ and $\Pi(\mathcal{I}',x_i) = z \geq \theta_{i+1}^z n_{\mathcal{I}}$. By reinforcement, it then follows for the profile $s\mathcal{I}'$, which consists of s copies of $\mathcal{I}'$, that $f(s\mathcal{I}') = x_{i+1}$. Finally, there is also a threshold vector θ^{sz} such that $f(\hat{\mathcal{I}}) = \max_{\rhd}\{x_j \in A\colon \Pi_{SP}(\hat{\mathcal{I}},x_j) \geq \theta_j^{sz} \cdot s \cdot z\}$ for all profiles $\mathcal{I} \in \mathcal{D}_1^{N(sz)}$. Now, since $f(s\mathcal{I}') = x_{i+1}$ and there are $z \cdot s \cdot \ell_i^z$ voters reporting x_i in $z\mathcal{I}$, we infer that $\theta_i^{sz} > \frac{\Pi_{SP}(s\mathcal{I}',x_i)}{sz} = \ell_i^z$. Analogously, it holds that $\theta_i^{sz} \leq \bar{r}_i^{z-1}$ since $s \cdot z \cdot \bar{r}_i^{z-1}$ voters report x_i in $z\mathcal{I}$ and $f(z\mathcal{I}) = x_i$. However, this means that $\theta_i^{sz} \leq \bar{r}_i^{z-1} \leq \ell_i^s < \theta_i^{sz}$. This contradiction proves that our assumption that $\bar{I}_i^z = \emptyset$ is wrong.

We now define the threshold vector θ. To this end, we observe that $\bar{I}_i^z \subseteq \bar{I}_i^{z+1}$ for all $z \in \mathbb{N}$ and $i \in \{1,\ldots,m-1\}$. Finally, since $\bar{r}_i^z - \bar{\ell}_i^z \leq r^z - \ell^z = \frac{1}{z}$ for all $z \in \mathbb{N}$, the series $\bar{r}_i^1, \bar{r}_i^2, \ldots$ is guaranteed to converge for all $i \in \{1,\ldots,m-1\}$. We hence define the vector θ by $\theta = \lim_{z\to\infty} \bar{r}_i^z$ for all $i \in \{1,\ldots,m-1\}$ and $\theta_m = \theta_{m-1}$.

We will first show that $\theta_i \geq \theta_{i+1}$ for all $i \in \{1,\ldots,m-1\}$. For $i = m-1$, this is clear from the definition. For $i < m-1$, we have by definition that $\theta_i^z \geq \theta_{i+1}^z$ for all $z \in \mathbb{N}$. This implies that $r_i^z \geq r_{i+1}^z$ for all $z \in \mathbb{N}$ and consequently also that $\bar{r}_i^z \geq \bar{r}_{i+1}^z$. This then shows that $\theta_i = \lim_{z\to\infty} \bar{r}_i^z \geq \lim_{z\to\infty} \bar{r}_{i+1}^z = \theta_{i+1}$.

Next, we will prove that $\theta_i \in (0,1)$ for all $i \in \{1, \dots, m-1\}$. To this end, assume for contradiction that $\theta_i \notin (0,1)$ for some $i \in \{1, \dots, m-1\}$. This means that $\theta_i = 0$ or $\theta_i = 1$. We first consider the case that $\theta_i = 0$ for some alternative x_i and we assume that x_i is the alternative with minimal index such that $x_i = 0$, i.e., $x_j > 0$ for all $j \in \{1, \dots, i-1\}$. Since $\theta_i = 0$, we infer that $\bar{\ell}_i^z = 0$ for all $z \in \mathbb{N}$ because $\bar{\ell}_i^z < \bar{r}_i^z$ for all $z \in \mathbb{N}$. In particular, this means that $f(\mathcal{I}) = x_i$ for all profiles $\mathcal{I} \in \mathcal{D}_1^*$ where $\{x_i\}$ is reported by one voter and all other voters report $\{x_m\}$. Now, consider the profile $\mathcal{I}$ that consists of one voter reporting $\{x_m\}$, and the profile $\mathcal{I}'$ that consists of one voter reporting $\{x_i\}$. By unanimity, we have that $f(\mathcal{I}) = x_m$ and $f(\mathcal{I}') = x_i$. Hence, right-biased continuity requires that there is a $\lambda \in \mathbb{N}$ such that $f(\lambda\mathcal{I} + \mathcal{I}') = \{x_m\}$. However, this contradicts with our previous insight, so the assumption that $\theta_i = 0$ must have been wrong. Next, consider the case that $\theta_i = 1$. This is only possible if $\bar{r}_i^z = 1$ for all $z \in \mathbb{N}$, so $f(\mathcal{I}) = x_i$ requires for all profiles $\mathcal{I} \in \mathcal{D}_1^*$ that no voter reports $\{x_m\}$. Now, consider again the profiles $\mathcal{I}$ and $\mathcal{I}'$ where one voter reports $\{x_m\}$ and one voter reports $\{x_i\}$, respectively. By right-biased continuity, there must be a $\lambda \in \mathbb{N}$ such that $f(\lambda\mathcal{I}' + \mathcal{I}) \trianglerighteq x_i$. However, this contradicts with the our previous observation, so we conclude that $\theta_i \neq 1$.

As our last point, we will verify that $f(\mathcal{I}) = \max_{\triangleright}\{x_i \in A \colon \Pi_{SP}(\mathcal{I}, x_j) \geq \theta_j n_{\mathcal{I}}\}$ for all profiles $\mathcal{I} \in \mathcal{D}_1^*$. To this end, fix a set of voters $N(z)$. By the definition of θ, it holds that $\theta_i \in [\bar{\ell}_i^z, \bar{r}_i^z] \subseteq [\ell_i^z, r_i^z]$ for all $i \in \{1, \dots, m-1\}$. If $\theta_i \in (\ell_i^z, r_i^z]$ for all $i \in \{1, \dots, m-1\}$, then $f(\mathcal{I}) = \max_{\triangleright}\{x_i \in A \colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$ because all values in $(\ell_i^z, r_i^z]$ result in the same rule. We hence will show that $\theta_i \neq \ell_i^z$ for all $i \in \{1, \dots, m-1\}$ and assume for contradiction that $\theta_i = \ell_i^z$ for some $i \in \{1, \dots, m-1\}$. Since $\ell_i^s < r_i^s$ for all $s \in \mathbb{N}$, this is only possible if $\ell_i^z \geq \ell_i^s$ for all $s \in \mathbb{N}$. Now, consider the profile $\mathcal{I}$ with $z \cdot \ell_i^z$ voters reporting $\{x_i\}$ and $z \cdot (1 - \ell_i^z)$ voters reporting $\{x_m\}$. By the definition of ℓ_z^i, we have that $f(\mathcal{I}) \neq x_i$. Moreover, because $\theta_j^z > 0$ for all $j \in \{1, \dots, m\}$, we conclude that $x_i \triangleright f(\mathcal{I})$. Next, let $\mathcal{I}'$ denote the profile where a single voter reports $\{x_1\}$. By unanimity, we have that $f(\mathcal{I}') = x_1$. Finally, by right-biased continuity, we infer that there is a value $\lambda \in \mathbb{N}$ such that $f(\lambda\mathcal{I} + \mathcal{I}') = f(\mathcal{I})$. However, for each $\lambda \in \mathbb{N}$, it holds that $\Pi_{SP}(\lambda\mathcal{I}' + \mathcal{I}, x_i) = 1 + \lambda \cdot z \cdot \ell_i^z > (1 + \lambda \cdot z)\ell_i^z \geq (1 + \lambda \cdot z)\ell_i^{1+\lambda \cdot z}$. By the definition of $\ell_i^{1+\lambda z}$, this means that $f(\lambda\mathcal{I}' + \mathcal{I}) \trianglerighteq x_i$ for all $\lambda \in \mathbb{N}$, which contradicts right-biased continuity. Hence, we conclude that $\theta_i \neq \ell_i^z$, which completes the proof of this lemma. □

As it will turn out, the threshold vector θ derived in Lemma 4 is the threshold vector that defines f. We will hence focus next on deriving the weight vector of f, for which we will mainly rely on reinforcement. In more detail, to employ the full power of reinforcement, we will change the domain of f. To this end, we define $q = |\Lambda|$ as the number of intervals with respect to $\triangleright$ and we enumerate the intervals by $I^1, \dots, I^q$. This allows us to represent each interval profile $\mathcal{I}$ by a vector $v \in \mathbb{N}_0^q \setminus \{0\}$: the i-th entry of v states how often the interval I^i is reported. For the ease of notation, we write $v(\mathcal{I})$ to indicate the vector corresponding to the profile $\mathcal{I}$, and v_I to indicate the entry in v corresponding to an interval I. Because f is anonymous, there is a function g from $\mathbb{N}^q \setminus \{0\}$ to X such that $f(\mathcal{I}) = g(v(\mathcal{I}))$ for all profiles $\mathcal{I} \in \Lambda^*$. We moreover note that g inherits all desirable properties of f. We will next generalize g to a function $\hat{g} : \mathbb{Q}_{\geq 0}^q \setminus \{0\} \to X$ while preserving the desirable properties of f. In particular, we will show that $\hat{g}$ extends f (i.e., $f(\mathcal{I}) = \hat{g}(v(\mathcal{I}))$ for all profiles $\mathcal{I} \in \Lambda^*$) and satisfies reinforcement (i.e., $g(v + v') = g(v)$ for all $v, v' \in \mathbb{Q}^q_{\geq 0} \setminus \{0\}$ with $g(v) = g(v')$).

Lemma 5. *There is a functions $\hat{g} : \mathbb{Q}^q_{\geq 0} \setminus \{0\} \to C$ that extends f and satisfies reinforcement.*

Proof. Let g denote the function from $\mathbb{N}^q \setminus \{0\} \to A$ that computes f based on anonymized profiles. We first note that g clearly satisfies both conditions of the lemma as it is only a different representation of f. Next, we define the function $\hat{g}$ by $\hat{g}(v) = g(\lambda v)$ for all $v \in \mathbb{Q}^q_{\geq 0} \setminus \{0\}$, where $\lambda \in \mathbb{N}$ is an arbitrary scalar such that $\lambda v \in \mathbb{N}^q_0 \setminus \{0\}$.

We first show that $\hat{g}$ is well-defined despite not fully specifying the parameter λ. To this end, let $v \in \mathbb{Q}^q_{\geq 0}$ denote an arbitrary vector and let $\lambda_1, \lambda_2 \in \mathbb{N}$ denote two scalars such that $\lambda_1 v, \lambda_2 v \in \mathbb{N}^q_0 \setminus \{0\}$. We will show that $g(\lambda_1 v) = g(\lambda_2 v)$ as this implies that $\hat{g}$ is well-defined. For this, we note that reinforcement implies that $g(\lambda_1 v) = g(\lambda_1 \lambda_2 v)$ and that $g(\lambda_2 v) = g(\lambda_1 \lambda_2 v)$. Hence, $g(\lambda_1 v) = g(\lambda_2 v)$ as desired. Moreover, observe that this proves that $\hat{g}(v(\mathcal{I})) = g(1 \cdot v(\mathcal{I})) = f(\mathcal{I})$ for all profiles $\mathcal{I} \in \Lambda^*$, so $\hat{g}$ indeed extends f.

Next, we show that $\hat{g}$ is reinforcing. To this end, consider two vectors $v^1, v^2 \in \mathbb{Q}^q_{\geq 0} \setminus \{0\}$ and let $\lambda_1, \lambda_2 \in \mathbb{N}$ denote scalars such that $\lambda_1 v^1, \lambda_2 v^2 \in \mathbb{N}^q_0 \setminus \{0\}$. We suppose that $\hat{g}(v^1) = \hat{g}(v^2)$ as there is otherwise nothing to show. By the definition of $\hat{g}$ and the reinforcement of g, it holds that $\hat{g}(v^1) = g(\lambda_1 v^1) = g(\lambda_1 \lambda_2 v^1)$ and $\hat{g}(v^2) = g(\lambda_1 v^2) = g(\lambda_1 \lambda_2 v^2)$. Because of the reinforcement of g, we next conclude that $\hat{g}(v^1 + v^2) = g(\lambda_1 \lambda_2 (v^1 + v^2)) = g(\lambda_1 \lambda_2 v^1) = \hat{g}(v^1)$. This proves that $\hat{g}$ is reinforcing, too. $\square$

Next, we define for every alternative $x_i \in A$ the set $Q_i = \{v \in \mathbb{Q}^q_{\geq 0} \setminus \{0\} \colon \hat{g}(v) = x_i\}$ as the subset of $\mathbb{Q}^q_{\geq 0} \setminus \{0\}$ such that $\hat{g}$ chooses x_i for every point in Q_i. We note that $Q_i \cap Q_j = \emptyset$ for all $i \neq j$ as $\hat{g}$ returns for every point in $\mathbb{Q}^q_{\geq 0} \setminus \{0\}$ only a single alternative and that $\bigcup_{x_i \in A} Q_i = \mathbb{Q}^q_{\geq 0} \setminus \{0\}$. Moreover, Q_i is $\mathbb{Q}$-convex (i.e., for all $v^1, v^2 \in Q_i$ and $\lambda \in \mathbb{Q} \cap [0, 1]$, it holds that $\lambda v^1 + (1 - \lambda) v^2 \in Q_i$) because $\hat{g}$ is reinforcing. Next, we let $\bar{Q}_i$ denote the closure of Q_i with respect to $\mathbb{R}^q$. Using standard arguments from Young (1975), it can be shown that $\bar{Q}_i$ is convex for all $x_i \in A$ and that $\bigcup_{x_i \in A} \bar{Q}_i = \mathbb{R}^q_{\geq 0}$. We will next show that the sets $\bar{Q}_i$ are polytopes. In the following, uv will denote the standard scalar product between two vectors $u, v \in \mathbb{R}^q$, i.e., $uv = \sum_{i=1}^q u_i v_i$.

Lemma 6. *For every alternative $x_i \in A$, the following claims hold:*

(1) $\bar{Q}_i$ is fully dimensional.

(2) For every alternative $x_j \in A \setminus \{x_i\}$, there is a non-zero vector $u^{i,j} \in \mathbb{R}^q$ such that $vu^{i,j} \geq 0$ for all $v \in \bar{Q}_i$ and $vu^{i,j} \leq 0$ for all $v \in \bar{Q}_j$.

(3) For every $x_j \in A \setminus \{x_i\}$, let $u^{i,j}$ denote a non-zero vector such that $vu^{i,j} \geq 0$ if $v \in \bar{Q}_i$ and $vu^{i,j} \leq 0$ if $v \in \bar{Q}_j$. It holds that $\bar{Q}_i = \{v \in \mathbb{R}^q_{\geq 0} \colon \forall x_j \in A \setminus \{x_i\} \colon vu^{i,j} \geq 0\}$.

Proof. Fix an alternative x_i and consider the corresponding set $\bar{Q}_i$. We will prove each of the three claims separately.

Claim (1): We will first show that $\bar{Q}_i$ is fully dimensional by studying f in more detail. Thus, let $\theta \in (0,1)^m$ denote the threshold vector such that $f(\mathcal{I}) = \max_{\triangleright}\{x_i \in A\colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$ for all $\mathcal{I} \in \mathcal{D}_1^*$; such a vector exists due to Lemma 4. Moreover, we define $\delta = \min(\theta_i, 1-\theta_i)$ and we choose $w \in \mathbb{N}$ such that $\frac{2(q-1)}{w} < \delta$. We claim that $f(\mathcal{I}) = x_i$ for all profiles $\mathcal{I}$ such that each interval $I \in \Lambda \setminus \{\{x_i\}\}$ is reported by at most two voters and at least w voters report the interval $\{x_i\}$. Assume for contradiction that this is not the case, i.e., that $f(\mathcal{I}) = x_j$ for such a profile $\mathcal{I}$ and some alternative $x_j \neq x_i$. Next, let n_X denote the number of voters who report the interval X in the profile $\mathcal{I}$ and recall that $n_{\mathcal{I}}$ is the total number of voters in $\mathcal{I}$. We consider the profile $\mathcal{I}'$ where $\sum_{X \in \Lambda \setminus \{\{x_i\}\}} n_X$ voters report $\{x_j\}$ and w voters report $\{x_i\}$. A repeated application of robustness shows that if $f(\mathcal{I}) = x_j$, then $f(\mathcal{I}') = x_j$. In more detail, for each voter k with $x_j \notin I_k$, we can first extend his interval to include x_j and then delete the other alternatives. For each of these steps, robustness implies that the winner is not allowed to change. Similarly, if $x_j \in I_k$, we can one after another remove all alternatives but x_j from the voter's interval, and robustness again demands that the outcome does not change. Next, we observe that $\mathcal{I}' \in \mathcal{D}_1^*$, so $f(\mathcal{I}') = \max_{\triangleright}\{x_i \in A\colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$. Now, if $x_j \triangleright x_i$, it holds that

$$\Pi_{SP}(\mathcal{I}', x_j) \leq \sum_{X \in \Lambda \setminus \{\{x_i\}\}} n_X \leq 2(q-1) < \delta w \leq \theta_i n_{\mathcal{I}'}.$$

Since $x_j \triangleright x_i$, it holds that $\theta_j \geq \theta_i$, so $f(\mathcal{I}') \neq x_j$. On the other hand, if $x_i \triangleright x_j$, then

$$\Pi_{SP}(\mathcal{I}', x_i) = w \geq n_{\mathcal{I}}(1 - \frac{2(q-1)}{n_{\mathcal{I}}}) > n_{\mathcal{I}}(1-\delta) \geq n_{\mathcal{I}}(1-(1-\theta_i)) = \theta_i n_{\mathcal{I}}.$$

Hence, we derive that $f(\mathcal{I}) \trianglerighteq x_i$, which means again that $f(\mathcal{I}) \neq x_j$. Since we have a contradiction in both cases, it follows that $f(\mathcal{I}) = x_i$ for all profiles $\mathcal{I}$ where at least w voters report $\{x_i\}$ and every other interval is reported by at most 2 voters. Since $f(\mathcal{I}) = \hat{g}(v(\mathcal{I}))$, we conclude that $v \in Q_i \subseteq \bar{Q}_x$ for all vectors $v \in \mathbb{Q}^q_{\geq 0}$ such that $v_{\{x_i\}} \geq w$ and $v_I \leq 2$ for all other intervals $I \in \Lambda \setminus \{\{x_i\}\}$. Combined with the convexity of $\bar{Q}_i$, this shows that this set is indeed fully dimensional.

Claim (2): We next let $x_j \in A \setminus \{x_i\}$ denote a second alternative, and we aim to apply the separating hyperplane theorem for convex sets to infer a non-zero vector $u^{i,j}$ such that $vu^{i,j} \geq 0$ for all $v \in \bar{Q}_i$ and $vu^{i,j} \leq 0$ for all $v \in \bar{Q}_j$. To this end, we need to show that the interiors of $\bar{Q}_i$ and $\bar{Q}_j$ are disjoint, i.e., that int $\bar{Q}_i \cap$ int $\bar{Q}_j = \emptyset$. Assume for contradiction that this is not the case, i.e., int $\bar{Q}_i \cap$ int $\bar{Q}_j \neq \emptyset$. Since int $\bar{Q}_i$ and int $\bar{Q}_j$ are open and convex, this means that there is a point v such that $v \in$ int $\bar{Q}_i \cap$ int $\bar{Q}_j \cap \mathbb{Q}^q_{\geq 0}$. We will show that this implies that $v \in Q_i \cap Q_j$, which is a contradiction as these sets are disjoint by definition. We will focus subsequently on showing that $v \in Q_i$ as the argument for Q_j is symmetric. Now, because $Q_i \subseteq \bar{Q}_i$ and the latter set is convex, it holds that the convex hull of Q_i, i.e., $Conv(Q_i)$, is a subset of $\bar{Q}_i$. Since $Q_i \subseteq Conv(Q_i) \subseteq \bar{Q}_i$, this means that $\bar{Q}_i \subseteq \overline{Conv(Q_i)} \subseteq \bar{\bar{Q}}_i$, so $\overline{Conv(Q_i)} = \bar{Q}_i$. Next, it holds for convex sets Y with non-empty interior that int $Y =$ int $\bar{Y}$, so int $\bar{Q}_i =$ int $\overline{Conv(Q_i)} =$ int $Conv(Q_i)$. Finally, Lemma

1 of Young (1975) shows that $Conv(Q_i) \cap \mathbb{Q}^q$ is the same as Q_i due to the $\mathbb{Q}$-convexity of this set. Hence, we have that int $\bar{Q}_i \cap \mathbb{Q}^q =$ int $Conv(Q_i) \cap \mathbb{Q}^q \subseteq Conv(Q_i) \cap \mathbb{Q}^q = Q_i$. This means that, if $v \in$ int $\bar{Q}_i \cap \mathbb{Q}^q$, then $v \in Q_i$. Since an analogous argument works for Q_j, this gives the desired contradiction.

We next apply the separation theorem for convex sets to infer that there is a non-zero vector $u^{i,j} \in \mathbb{R}^q$ and a constant $c \in \mathbb{R}$ such that $vu^{i,j} > c$ for all $v \in$ int $\bar{Q}_i$ and $vu^{i,j} < c$ for all $v \in \bar{Q}_j$. By taking the closure, it then follows that $vu^{i,j} \geq c$ for all $v \in \bar{Q}_i$ and $vu^{i,j} \leq c$ for all $v \in \bar{Q}_j$. Moreover, since these sets are closed under multiplication with a scalar, it is easy to infer that c must be 0. This completes the proof of this step.

Claim (3): For all $x_j \in A \setminus \{x_i\}$, let $u^{i,j} \in \mathbb{R}^q$ denote a non-zero vector such that $vu^{i,j} \geq 0$ if $v \in \bar{Q}_i$ and $vu^{i,j} \leq 0$ if $v \in \bar{Q}_j$. Moreover, we define define $S_i = \{v \in \mathbb{R}^q_{\geq 0} \colon \forall x_j \in A \setminus \{x_i\} \colon vu^{i,j} \geq 0\}$, and we will show that $\bar{Q}_i = S_i$ by considering the two set inclusions between these sets. First, by the definition of the vectors $u^{i,j}$, it follows immediately that if $v \in \bar{Q}_i$, then $vu^{i,j} \geq 0$ for all $x_j \in A \setminus \{x_i\}$, so $\bar{Q}_i \subseteq S_i$. For the other direction, we first observe that int $S_i \neq \emptyset$ as int $\bar{Q}_i \neq \emptyset$. Now, it holds by the definition of the interior that, if $v \in$ int S_i, then $vu^{i,j} > 0$ for all $x_j \in A \setminus \{x_i\}$. By the definition of these vectors, this means that $v \notin \bar{Q}_j$ for all $x_j \in A \setminus \{x_i\}$ since this would imply that $vu^{i,j} \leq 0$. Because $\bigcup_{x_k \in A} \bar{Q}_k = \mathbb{R}^q_{\geq 0}$, we conclude that $v \in \bar{Q}_i$. Hence int $S_i \subseteq \bar{Q}_i$, and taking the closure of both sets then implies that $S_i \subseteq \bar{Q}_i$. □

Motivated by Lemma 6, we will next study the vector $u^{i,j}$ in more detail. To this end, we define the relation $\rhd^*$ by $x_i \rhd^* x_j$ if and only if $x_i \rhd x_j$ and there is no alternative x_k such that $x_i \rhd x_k \rhd x_j$. By our assumption that $\rhd$ is given by $x_1 \rhd x_2 \rhd \dots \rhd x_k$, this means that $x_i \rhd^* x_j$ if and only if $j = i + 1$. For the next lemmas, we denote by $\theta \in (0, 1)^m$ the threshold vector inferred in Lemma 4. We will next show that the robustness of f severely restricts the vectors $u^{i,j}$ when $x_i \rhd^* x_j$.

Lemma 7. *Fix two alternatives $x_i, x_j \in A$ such that $x_i \rhd^* x_j$ and let $u^{i,j} \in \mathbb{R}^q$ denote a non-zero vector such that $vu^{i,j} \geq 0$ for all $v \in \bar{Q}_i$ and $vu^{i,j} \leq 0$ for all $v \in \bar{Q}_j$. It holds that*

(1) $\theta_i \cdot u^{i,j}_{\{x_i\}} = -(1 - \theta_i) \cdot u^{i,j}_{\{x_j\}} > 0$,

(2) $u^{i,j}_{\{x_i\}} \geq u^{i,j}_X \geq u^{i,j}_{\{x_j\}}$ *for all intervals $X \subseteq \Lambda$, and*

(3) $u^{i,j}_X = u^{i,j}_Y$ *for all intervals $X = [\ell, r]$, $Y = [\ell', r']$ such that either (i) $r \unrhd x_i$ and $r' \unrhd x_i$, (ii) $x_j \unrhd \ell$ and $x_j \unrhd \ell'$, or (iii) $\{x_i, x_j\} \subseteq X$.*

Proof. Fix two alternatives $x_i, x_j \in A$ with $x_i \rhd^* x_j$ and let $u^{i,j}$ denote the non-zero vector derived in Lemma 6 that satisfies that $vu^{i,j} \geq 0$ for all $v \in \bar{Q}_i$ and $vu^{i,j} \leq 0$ for all $v \in \bar{Q}_j$. We will prove each claim separately, but we will always use the same strategy: if the given equation is violated, we will construct a profile $\mathcal{I}$ such that $v(\mathcal{I})u^{i,j} > 0$ (or $v(\mathcal{I})u^{i,j} < 0$) but $f(\mathcal{I}) = x_j$ (or $f(\mathcal{I}) = x_i$). This is a contradiction since $f(\mathcal{I}) = x_j$ implies that $v(\mathcal{I}) \in \bar{Q}_i$ and therefore $v(\mathcal{I})u^{i,j} \geq 0$.

Claim (1): We will first show that $u^{i,j}_{\{x_i\}} > 0$. To this end, we first observe that, by Claim (1) of Lemma 6, both $\bar{Q}_i$ and $\bar{Q}_j$ are fully dimensional. In particular, this implies for $\bar{Q}_i$ that $\text{int } \bar{Q}_i \neq \emptyset$. Hence, there is a vector v such that $vu^{i,j} > 0$. This requires that there is an interval X such that $u^{i,j}_X > 0$. Moreover, if it was the case that $u^{i,j}_Y \geq 0$ for all $Y \in \Lambda$, then $\bar{Q}_j$ would not be fully dimensional because $vu^{i,j} > 0$ if $v_X > 0$ and therefore $v \notin \bar{Q}_j$. This proves that there is also an interval $Z \in \Lambda$ such that $u^{i,j}_Z < 0$.

Now, let $\delta = \min(\theta_i, 1-\theta_i)$ and define $w \in \mathbb{N}$ such that $\frac{1}{w+1} < \delta$. We consider the profile $\mathcal{I}$ where a single voter reports Z and w voters report $\{x_i\}$, and we will show that $f(\mathcal{I}) = x_i$. Assume for contradiction that this is not the case and let $x_k = f(\mathcal{I})$ denote the chosen alternative. We consider next the profile $\mathcal{I}'$ derived from $\mathcal{I}$ by assigning the interval $\{x_k\}$ to the voter who initially reported Z. Since this voter does not remove x_k from his interval (but he may add it), robustness implies that $f(\mathcal{I}') = f(\mathcal{I}) = x_k$. Next, since $\mathcal{I}' \in \mathcal{D}_1^*$, it holds that $f(\mathcal{I}') = \max_{\rhd}\{x_t \in A \colon \Pi_{SP}(\mathcal{I}', x_t) \geq \theta_t n_{\mathcal{I}'}\}$ by Lemma 4. However, if $x_k \rhd x_i$, we compute that

$$\Pi(\mathcal{I}', x_k) = 1 < \delta(w+1) \leq \theta_i n_{\mathcal{I}} \leq \theta_k n_{\mathcal{I}}.$$

Here, the last inequality uses that $\theta_k \geq \theta_i$ as $x_k \rhd x_i$, and the second to last one uses the definition of δ. This contradicts that $\max_{\rhd}\{x_i \in A \colon \Pi_{SP}(\mathcal{I}', x_i) \geq \theta_i n_{\mathcal{I}'}\} = x_k$ if $x_k \rhd x_i$. As the second case, suppose that $x_i \rhd x_k$. We observe that

$$\Pi_{SP}(\mathcal{I}', x_i) = w = n_{\mathcal{I}'}(1 - \frac{1}{w+1}) > n_{\mathcal{I}'}(1-\delta) \geq n_{\mathcal{I}'}(1-(1-\theta_i)) \geq \theta_i n_{\mathcal{I}'}.$$

This shows that $\max_{\rhd}\{x_i \in A \colon \Pi_{SP}(\mathcal{I}', x_i) \geq \theta_i n_{\mathcal{I}'}\} \trianglerighteq x_i$, which contradicts $f(\mathcal{I}') = x_k$. Since we have a contradiction in both cases, it follows that the assumption that $f(\mathcal{I}) = x_k \neq x_i$ is wrong, i.e., it holds that $f(\mathcal{I}) = x_i$. This implies that $v(\mathcal{I}) \in \bar{Q}_i$, so $v(\mathcal{I})u^{i,j} = wu^{i,j}_{\{x_i\}} + u^{i,j}_Z \geq 0$. From this, we finally infer that $u^{i,j}_{\{x_i\}} > 0$ because $u^{i,j}_Z < 0$.

Next, we note that $u^{i,j}_{\{x_i\}} > 0$ implies that $u^{i,j}_{\{x_j\}} < 0$. To see this, we can consider a profile $\mathcal{I}''$ where a single voter reports $\{x_i\}$ and w' voters report $\{x_j\}$. Just as before, if w' is large enough, then $f(\mathcal{I}'') = x_j$. This implies that $\mathcal{I}'' \in \bar{Q}_j$ and thus $v(\mathcal{I}'')u^{i,j} = u^{i,j}_{\{x_i\}} + w'u^{i,j}_{\{x_j\}} \leq 0$, which is only possible if $u^{i,j}_{\{x_j\}} < 0$.

Next, we will prove that $\theta_i u^{i,j}_{\{x_i\}} = -(1-\theta_i)u^{i,j}_{\{x_j\}}$ and we assume for contradiction that this is not the case. We subsequently focus on the case that $\theta_i u^{i,j}_{\{x_i\}} < -(1-\theta_i)u^{i,j}_{\{x_j\}}$; the case that $\theta_i u^{i,j}_{\{x_i\}} > -(1-\theta_i)u^{i,j}_{\{x_j\}}$ follows analogously by exchanging the role of x_i and x_j. By reformulating our assumption, we obtain that $\theta_i < \frac{-u^{i,j}_{\{x_j\}}}{u^{i,j}_{\{x_i\}} - u^{i,j}_{\{x_j\}}}$, so there is a value $\lambda \in (\theta_i, \frac{-u^{i,j}_{\{x_j\}}}{u^{i,j}_{\{x_i\}} - u^{i,j}_{\{x_j\}}}) \cap \mathbb{Q} \subseteq (0,1) \cap \mathbb{Q}$. Moreover, there are two integers $w_1, w_2 \in \mathbb{N}$ such that $\lambda = \frac{w_1}{w_1+w_2}$. Now, consider the profile $\mathcal{I}$ such that w_1 voters report $\{x_i\}$ and w_2 voters report $\{x_j\}$. Since $\mathcal{I} \in \mathcal{D}_1^*$, $\Pi_{SP}(\mathcal{I}, x_i) = w_1 > \theta_i n_{\mathcal{I}}$, and $\Pi_{SP}(\mathcal{I}, x_k) = 0$ for all x_k with $x_k \rhd x_i$, we derive $f(\mathcal{I}) = \max_{\rhd}\{x_i \in A \colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\} = x_i$ and thus

$v(\mathcal{I}) \in \bar{Q}_i$. On the other hand, it holds that

$$\begin{aligned}
v(\mathcal{I})u^{i,j} &= w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_j\}} \\
&= (w_1+w_2)\left(\lambda u^{i,j}_{\{x_i\}} + (1-\lambda)u^{i,j}_{\{x_j\}}\right) \\
&< (w_1+w_2)\left(\frac{-u^{i,j}_{\{x_j\}}}{u^{i,j}_{\{x_i\}} - u^{i,j}_{\{x_j\}}} \cdot u^{i,j}_{\{x_i\}} + (1 + \frac{u^{i,j}_{\{x_j\}}}{u^{i,j}_{\{x_i\}} - u^{i,j}_{\{x_j\}}}) \cdot u^{i,j}_{\{x_j\}}\right) \\
&= (w_1+w_2)\left(\frac{-u^{i,j}_{\{x_j\}}}{u^{i,j}_{\{x_i\}} - u^{i,j}_{\{x_j\}}} \cdot u^{i,j}_{\{x_i\}} + \frac{u^{i,j}_{\{x_i\}}}{u^{i,j}_{\{x_i\}} - u^{i,j}_{\{x_j\}}} \cdot u^{i,j}_{\{x_j\}}\right) \\
&= 0.
\end{aligned}$$

This contradicts that $v(\mathcal{I}) \in \bar{Q}_i$ and the assumption that $\theta_i u^{i,j}_{\{x_i\}} < -(1-\theta_i)u^{i,j}_{\{x_j\}}$ hence is wrong. Since the case that $\theta_i u^{i,j}_{\{x_i\}} > -(1-\theta_i)u^{i,j}_{\{x_j\}}$ is symmetric, we conclude that $\theta_i u^{i,j}_{\{x_i\}} = -(1-\theta_i)u^{i,j}_{\{x_j\}} > 0$, which completes the proof of our first claim.

Claim (2): Consider an arbitrary interval X and assume for contradiction that $u^{i,j}_X \notin [u^{i,j}_{\{x_j\}}, u^{i,j}_{\{x_i\}}]$. We will assume here that $u^{i,j}_X > u^{i,j}_{\{x_i\}}$ since our cases are again symmetric. Moreover, we let $\delta = u^{i,j}_X - u^{i,j}_{\{x_i\}}$ and choose $t \in \mathbb{N}$ such that $t \cdot \delta > u^{i,j}_{\{x_i\}}$. Next, we choose integers $j_1, j_2 \in \mathbb{N}$ such that $j_2 > j_1 \geq t$, $\frac{t}{j_2} < 1 - \theta_i$, and $\frac{j_1}{j_2} < \theta_i \leq \frac{j_1+1}{j_2}$. Moreover, we define $w_1 = j_1$ and $w_2 = j_2 - j_1$ and consider the profile $\mathcal{I}$ in which w_1 voters report $\{x_i\}$ and w_2 voters report $\{x_j\}$. It holds that $f(\mathcal{I}) = \max_{\rhd}\{x_k \in A \colon \Pi_{SP}(\mathcal{I}, x_k) \geq \theta_k n_{\mathcal{I}}\}$ since $\mathcal{I} \in \mathcal{D}^*_1$. We thus conclude that $f(\mathcal{I}) = x_j$ because $\Pi_{SP}(\mathcal{I}, x_k) = 0$ for all x_k with $x_k \rhd x_i$, $\Pi_{SP}(\mathcal{I}, x_i) = w_1 < \theta_i n_{\mathcal{I}}$, and $\Pi_{SP}(\mathcal{I}, x_j) = n_{\mathcal{I}}$. Note that we use here also that $x_i \rhd^* x_j$ as otherwise some alternative x_k with $x_i \rhd x_k \rhd x_j$ could be chosen.

Next, let $\mathcal{I}'$ denote the profile where $w_1 - t$ voters report $\{x_i\}$, t voters report X, and w_2 voters report $\{x_j\}$. We claim that $f(\mathcal{I}') = x_j$ due to robustness. To this end, let $X = [\ell, r]$. Now, if $r \rhd x_j$, we can transform the interval $\{x_i\}$ to X by sequentially adding and removing alternatives without touching x_j. Hence, robustness implies immediately that $f(\mathcal{I}') = x_j$ in this case. On the other hand, if $x_j \unrhd r$ and thus $x_i \rhd r$, we first expand the interval $\{x_i\}$ to the right until our t voters report $[x_i, r]$. Repeatedly applying robustness during this process shows that these steps can only move the winner to the right, i.e., that an alternative y with $x_j \unrhd y$ is now chosen. Finally, we transform the intervals $[x_i, r]$ into $[\ell, r]$ by either adding more alternatives to the left of x_i (if $\ell \rhd x_i$) or by deleting alternatives from the interval (if $x_i \rhd \ell$). In the first case, robustness implies that the winner cannot change as the current winner is right of x_i, and in the second case, robustness only allows the winner to move further to the right. Hence, it follows that $x_j \unrhd f(\mathcal{I}')$. Finally, assume for contradiction that $f(\mathcal{I}') = x_k$ for some x_k with $x_j \rhd x_k$. In this case, we consider the profile $\mathcal{I}''$ where $w_1 - t$ voters report $\{x_i\}$, t voters report $\{x_k\}$, and w_2 voters report $\{x_j\}$. For this profile, robustness from $\mathcal{I}'$ implies that $f(\mathcal{I}'') = x_k$. However, $\mathcal{I}'' \in \mathcal{D}^*_1$, so $f(\mathcal{I}'') = \max_{\rhd}\{x_t \in A \colon \Pi_{SP}(\mathcal{I}'', x_t) \geq \theta_t n_{\mathcal{I}''}\}$. Hence, we compute that $\Pi_{SP}(\mathcal{I}'', x_j) = w_1 - t + w_2 = n_{\mathcal{I}''}(1 - \frac{t}{n_{\mathcal{I}''}}) > n_{\mathcal{I}''}(1 - (1-\theta_i)) \geq \theta_i n_{\mathcal{I}''} \geq \theta_j n_{\mathcal{I}''}$.

The strict inequality here uses that $\frac{t}{n_{\mathcal{I}''}} = \frac{t}{j_2} < 1 - \theta_i$ and the last inequality that $\theta_i \geq \theta_j$ as $x_i \rhd x_j$. This contradicts that $x_j \rhd f(\mathcal{I}'')$, so we conclude that $f(\mathcal{I}') = x_j$.

Finally, we will next compute $vu^{i,j}$ for the vector $v = v(\mathcal{I}')$.

$$\begin{aligned}
vu^{i,j} &= (w_1 - t)u^{i,j}_{\{x_i\}} + tu^{i,j}_X + w_2 u^{i,j}_{\{x_j\}} \\
&= w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_j\}} + t\delta \\
&> (w_1 + 1)u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_j\}} \\
&= (w_1 + w_2)\left(\frac{w_1 + 1}{w_1 + w_2} u^{i,j}_{\{x_i\}} + \frac{w_2}{w_1 + w_2} u^{i,j}_{\{x_j\}}\right) \\
&= (w_1 + w_2)\left(\frac{j_1 + 1}{j_2} u^{i,j}_{\{x_i\}} + (1 - \frac{j_1}{j_2}) u^{i,j}_{\{x_j\}}\right) \\
&> (w_1 + w_2)\left(\theta_i u^{i,j}_{\{x_i\}} + (1 - \theta_i) u^{i,j}_{\{x_j\}}\right) \\
&= 0
\end{aligned}$$

The first equality here uses the definition of v (resp. $\mathcal{I}'$), the second one uses that $\delta = u^{i,j}_X - u^{i,j}_{\{x_i\}}$, and the third one that we choose t such that $t\delta > u^{i,j}_{\{x_i\}}$. The fourth line is a simple transformation, and the fifth one uses the definition of w_1 and w_2. The sixth inequality uses that, by definition, $\frac{j_1+1}{j_2} > \theta_i$ and $u^{i,j}_{\{x_i\}} > 0$, as well as $1 - \frac{j_1}{j_2} > 1 - \theta_i > 0$ and $u^{i,j}_{\{x_j\}} < 0$ and the last step follows from Claim (1). However, the observation that $vu^{i,j} > 0$ contradicts that $f(\mathcal{I}') = x_j$ as the latter implies that $v \in \bar{Q}_j$ and thus $vu^{i,j} \leq 0$. This is the desired contradiction and we thus infer that $u^{i,j}_{\{x_i\}} \geq u^{i,j}_X$. Finally, a symmetric argument shows that $u^{i,j}_X \geq u^{i,j}_{\{x_j\}}$, thus completing the proof of Claim 2).

Claim (3): Finally, we will show that $u^{i,j}_X = u^{i,j}_Y$ for all intervals $X = [\ell, r]$, $Y = [\ell', r']$ such that either $r \trianglerighteq x_i$ and $r' \trianglerighteq x_i'$, $x_j \trianglerighteq \ell$ and $r_j \trianglerighteq \ell'$, or $\{x_i, x_j\} \subseteq X \cap Y$. We focus here on the last case, i.e., we assume that $\{x, y\} \subseteq X \cap Y$, and note that all three cases follow from analogous arguments. Moreover, we suppose that $Y \setminus X = \{x_k\}$, i.e., Y arises from X by adding one more alternative x_k. This assumption is without loss of generality, because for all intervals X', Y' with $\{x, y\} \subseteq X' \cap Y'$, we can transform X' to Y' by one after another adding and deleting alternatives. Finally, we will assume that $x_j \rhd x_k$; the case that $x_k \rhd x_i$ is symmetric.

Now, assume for contradiction that $u^{i,j}_X \neq u^{i,j}_Y$ and first consider the case that $u^{i,j}_X < u^{i,j}_Y$. We define $\delta = u^{i,j}_Y - u^{i,j}_X$ and let $t \in \mathbb{N}$ denote an integer such that $\delta t > 2u^{i,j}_{\{x_i\}}$. Moreover, let $w_1, w_2 \in \mathbb{N}$ denote integers such that $\frac{t}{w_1+w_2+t} < \min(\theta_i, 1 - \theta_i)$ and $w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_j\}} + tu^{i,j}_X < 0 < w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_j\}} + tu^{i,j}_Y$. Such integers exist because we can first set w_2 to an arbitrarily large number such that $tu^{i,j}_X + w_2 u^{i,j}_{\{x_j\}} < 0$ and then choose w_1 such that $-u^{i,j}_{\{x_i\}} \leq w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_j\}} + tu^{i,j}_X < 0$. Next, let $\mathcal{I}^X$ (resp. $\mathcal{I}^Y$) denote the profile where w_1 voters report $\{x_i\}$, w_2 voters report $\{x_j\}$, and t voters report X (resp. Y), and let $v^X = v(\mathcal{I}^X)$ and $v^Y = v(\mathcal{I}^Y)$. We first observe that, by construction, $v^X u^{i,j} < 0$, which means that $v^X \notin \bar{Q}_i$ and hence $f(\mathcal{I}^X) \neq x_i$. Similarly, $v^Y u^{i,j} > 0$ and hence $v^Y \notin \bar{Q}_j$ and $f(\mathcal{I}^Y) \neq x_j$.

Next, we will show that $f(\mathcal{I}^X) \in \{x_i, x_j\}$ and $f(\mathcal{I}^Y) \in \{x_i, x_j\}$. Since the argument for both profiles is symmetric, we assume for contradiction that $f(\mathcal{I}^X) = x_k \notin \{x_i, x_j\}$. In this case, let $\hat{\mathcal{I}}^X$ denote the profile where all voters reporting X change their interval to $\{x_k\}$. By repeatedly applying robustness, we infer that $f(\hat{\mathcal{I}}^X) = x_k$. On the other hand, $\hat{\mathcal{I}}^X \in \mathcal{D}_1^*$, which implies that $f(\hat{\mathcal{I}}^X) = \max_{\rhd}\{x_\ell \in A\colon \Pi_{SP}(\hat{\mathcal{I}}^X, x_\ell) \geq \theta_\ell n_{\hat{\mathcal{I}}^X}\}$. Now, if $x_k \rhd x_i$, this results in a contradiction as $\Pi_{SP}(\hat{\mathcal{I}}^X, x_k) = t < \theta_i(w_1 + w_2 + t) \leq \theta_k n_{\hat{\mathcal{I}}^X}$. For the last inequality, we recall that $\theta_k \geq \theta_i$ if $x_k \rhd x_i$. By contrast, if $x_j \rhd x_k$, we derive a similar contradiction because $\Pi_{SP}(\hat{\mathcal{I}}^X, x_j) = w_1 + w_2 = n_{\hat{\mathcal{I}}^X}(1 - \frac{t}{w_1+w_2+t}) > n_{\hat{\mathcal{I}}^X}(1-(1-\theta_i)) \geq \theta_j n_{\hat{\mathcal{I}}^X}$. Hence, the assumption that $f(\mathcal{I}^X) \notin \{x_i, x_j\}$ is wrong and an analogous argument shows that $f(\mathcal{I}^Y) \in \{x_i, x_j\}$. Combined with our previous insights, this means that $f(\mathcal{I}^X) = x_j$ and $f(\mathcal{I}^Y) = x_i$. However, robustness rules out such a deviation because, if $f(\mathcal{I}^X) = x_j$ and we add an alternative to the right (recall that $Y = X \cup \{x_k\}$ and $x_j \rhd x_k$), then the winner can only move to the right. Hence, our assumption that $u_X^{i,j} < u_Y^{i,j}$ must have been wrong.

As second case suppose that $u_X^{i,j} > u_Y^{i,j}$. In this case, we define $\delta = u_X^{i,j} - u_Y^{i,j}$ and let $t \in \mathbb{N}$ again denote an integer such that $\delta t > 2u_{\{x_i\}}^{i,j}$. Moreover, we choose two integers $w_1, w_2 \in \mathbb{N}$ such that $\frac{t}{w_1+w_2+t} < \min(\theta_i, 1-\theta_i)$ and $w_1 u_{\{x_i\}}^{i,j} + w_2 u_{\{x_j\}}^{i,j} + t u_X^{i,j} > 0 > w_1 u_{\{x_i\}}^{i,j} + w_2 u_{\{x_j\}}^{i,j} + t u_Y^{i,j}$, and define the profiles $\mathcal{I}^X$ and $\mathcal{I}^Y$ as before. Analogous arguments as before show that $f(\mathcal{I}^X) = x_i$ and $f(\mathcal{I}^Y) = x_j$. However, as we only made changes to the right of x_j, this contradicts with robustness and we infer also that $u_X^{i,j} > u_Y^{i,j}$ is not possible. This means that $u_X^{i,j} = u_Y^{i,j}$. Finally, we note that we never used the fact that $x_i, x_j \in X \cap Y$, but only that the modifications from X to Y does not affect x_i or x_j. As a consequence, it is straightforward to extend the analysis to the remaining cases. □

Motivated by Claim (1) of Lemma 7, we will assume from now on that $u_{\{x_i\}}^{i,j} = (1-\theta_i)$ and $u_{\{x_j\}}^{i,j} = -\theta_i$ for all $x_i, x_j \in A$ with $x_i \rhd^* x_j$. This is without loss of generality because we can scale the vector $u^{i,j}$ arbitrarily and it still separates $\bar{Q}_i$ from $\bar{Q}_j$. We will next use our insights to severely simplify the representation of $\bar{Q}_i$ as polytopes.

Lemma 8. *The following claims are true:*

(1) $\bar{Q}_1 = \{v \in \mathbb{R}^q\colon vu^{1,2} \geq 0\}$.

(2) $\bar{Q}_i = \{v \in \mathbb{R}^q\colon vu^{i-1,i} \leq 0 \wedge vu^{i,i+1} \geq 0\}$ *for all* $i \in \{2, \dots, m-1\}$.

(3) $\bar{Q}_m = \{v \in \mathbb{R}^q\colon vu^{m-1,m} \leq 0\}$.

Proof. For proving this claim, we will first show an auxiliary claim: for all alternatives $x_i, x_j, x_k \in A$ with $x_i \rhd^* x_j \rhd^* x_k$, the vectors $u^{i,j}$ and $u^{j,k}$ given by Lemmas 6 and 7, and all vectors $v \in \mathbb{R}^q_{\geq 0}$, it holds that $vu^{i,j} \geq 0$ implies that $vu^{j,k} \geq 0$. In a second step, we then prove the lemma.

Step 1: Let $x_i, x_j, x_k \in A$ denote alternatives such that $x_i \rhd^* x_j \rhd^* x_k$ and assume for contradiction that there is a vector $v \in \mathbb{R}^q_{\geq 0}$ such that $vu^{i,j} \geq 0$ and $vu^{j,k} < 0$. Now, if such a vector v exists, there is also a vector v' such that $v'u^{i,j} > 0$ and $vu^{j,k} < 0$. In more detail, since $u^{i,j}_{\{x_i\}} > 0$, we can derive v' from v by marginally increasing $v_{\{x_i\}}$. This shows that the set $\{x \in \mathbb{R}^q_{\geq 0} \colon xu^{i,j} > 0 \wedge xu^{j,k} < 0\}$ is non-empty, so there also is a vector $v^0 \in \mathbb{Q}^q_{\geq 0} \setminus \{0\}$ such that $v^0 u^{i,j} > 0$ and $v^0 u^{j,k} < 0$.

We will next simplify the presentation of v^0 by employing the insights of Lemma 7. To this end, we first define the vector v^1 by $v^1_{\{x_j\}} = 0$ and $v^1_X = v^0_X$ for all intervals $X \in \Lambda \setminus \{\{x_j\}\}$. Since $u^{i,j}_{\{x_j\}} < 0$ and $u^{j,k}_{\{x_j\}} > 0$ by Claim (1) of Lemma 7, it holds for this vector that $v^1 u^{i,j} \geq v^0 u^{i,j} > 0$ and that $v^1 u^{j,k} \leq v^0 u^{j,k} < 0$.

Next, let $\Lambda_1 = \{I \in \Lambda \colon \forall x_\ell \in I \colon x_\ell \trianglerighteq x_i\}$ denote the set of intervals that contain only alternatives weakly left of x_i, $\Lambda_2 = \{I \in \Lambda \colon \forall x_\ell \in I \colon x_k \trianglerighteq x_\ell\}$ denote the set of intervals that are weakly right of x_k, and $\Lambda_3 = \{I \in \Lambda \colon \{x_i, x_j, x_k\} \subseteq I\}$ denote the set of intervals that contain x_i, x_j, and x_k. By Claim (3) of Lemma 7, we have that *(i)* $u^{i,j}_X = u^{i,j}_{\{x_i\}}$ and $u^{j,k}_X = u^{j,k}_{\{x_i\}}$ for all $X \in \Lambda_1$, *(ii)* $u^{i,j}_X = u^{i,j}_{\{x_k\}}$ and $u^{j,k}_X = u^{j,k}_{\{x_k\}}$ for all $X \in \Lambda_2$, and *(iii)* $u^{i,j}_X = u^{i,j}_{\{x_i,x_j,x_k\}}$ and $u^{j,k}_X = u^{j,k}_{\{x_i,x_j,x_k\}}$ for all $X \in \Lambda_3$. We hence define the vector v^2 by *(i)* $v^2_{\{x_i\}} = \sum_{X \in \Lambda_1} v^1_X$ and $v^2_X = 0$ for all $X \in \Lambda_1 \setminus \{\{x_i\}\}$, *(ii)* $v^2_{\{x_k\}} = \sum_{X \in \Lambda_2} v^1_X$ and $v^2_X = 0$ for all $X \in \Lambda_2 \setminus \{\{x_k\}\}$, *(iii)* $v^2_{\{x_i,x_j,x_k\}} = \sum_{X \in \Lambda_3} v^1_X$ and $v^2_X = 0$ for all $X \in \Lambda_3 \setminus \{\{x_i, x_j, x_k\}\}$, and *(iv)* $v^2_X = v^1_X$ for all $X \in \Lambda \setminus (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)$. By our previous insights, it holds that $v^2 u^{i,j} = v^1 u^{i,j} > 0$ and $v^2 u^{j,k} = v^1 u^{j,k} < 0$.

For our third modification, let $\Lambda_4 = \{I \in \Lambda \colon \{x_i, x_j\} \subseteq I, x_k \notin I\}$ denote the intervals that contain x_i and x_j but not x_k, and let $\Lambda_5 = \{I \in \Lambda \colon \{x_j, x_k\} \subseteq I, x_i \notin I\}$ denote the intervals that contain x_j and x_k but not x_i. By Claim (2) of Lemma 7, it holds that $u^{i,j}_{\{x_i\}} \geq u^{i,j}_X$ for all $X \in \Lambda_4$ and $u^{j,k}_{\{x_k\}} \leq u^{j,k}_X$ for all $X \in \Lambda_5$. Moreover, Claim (3) of this lemma shows that $u^{j,k}_{\{x_i\}} = u^{j,k}_X$ for all $X \in \Lambda_4$ and $u^{i,j}_{\{x_k\}} = u^{i,j}_X$ for all $X \in \Lambda_5$. We now define our final vector v^3: $v^3_{\{x_i\}} = v^2_{\{x_i\}} + \sum_{X \in \Lambda_4} v^2_X$, $v^3_{\{x_k\}} = v^2_{\{x_k\}} + \sum_{X \in \Lambda_5} v^2_X$, and $v^3_X = v^2_X$ for all $X \in \Lambda \setminus (\Lambda_4 \cup \Lambda_5)$. Based on our insights from Lemma 7, it holds that $v^3 u^{i,j} \geq v^2 u^{i,j} > 0$ and $v^3 u^{j,k} \leq v^2 u^{j,k} < 0$. Moreover, it can be checked that $\Lambda = \{\{x_j\}\} \cup \bigcup_{\ell \in \{1,\dots,5\}} \Lambda_\ell$, so we have by construction that $v^3_X = 0$ for all $X \notin \{\{x_i\}, \{x_k\}, \{x_i, x_j, x_k\}\}$. Finally, we note that $v^3 \in \mathbb{Q}^q_{\geq 0}$ since $v^0 \in \mathbb{Q}^q_{\geq 0}$.

Because $v^3 \in \mathbb{Q}^q_{\geq 0}$ (and $v^3 \neq 0$ as $v^3 u^{i,j} > 0$), there is a scalar $\lambda \in \mathbb{N}$ such that $\lambda v^3 \in \mathbb{N}^q_0 \setminus \{0\}$. Since $\lambda v^3 u^{i,j} > 0$, $\lambda v^3 u^{j,k} < 0$ and $v^3_X = 0$ for all $X \notin \{\{x_i\}, \{x_k\}, \{x_i, x_j, x_k\}\}$, this shows that there are integers $w^1, w^2, t \in \mathbb{N}_0$ such that such that $w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_k\}} + t u^{i,j}_{\{x_i,x_j,x_k\}} > 0$ and $w_1 u^{j,k}_{\{x_i\}} + w_2 u^{j,k}_{\{x_k\}} + t u^{j,k}_{\{x_i,x_j,x_k\}} < 0$. Now, let $\mathcal{I}$ denote the profile where w_1 voters report $\{x_i\}$, w_2 voters report $\{x_k\}$, and t voters report $\{x_i, x_j, x_k\}$, and let $v^* = v(\mathcal{I})$ denote the corresponding vector. First, it is easy to see that $f(\mathcal{I}) \in \{x_i, x_j, x_k\}$. Indeed, if $f(\mathcal{I}) \notin \{x_i, x_j, x_k\}$, then all our voter can deviate to report, e.g., $\{x_i\}$ and robustness implies that the outcome is not allowed to change. However, for the resulting profile $\bar{\mathcal{I}}$, unanimity requires that $f(\bar{\mathcal{I}}) = x_i$, a contradiction. Next, since $v^* u^{i,j} > 0$

and $v^*u^{j,k} < 0$, we conclude that $f(\mathcal{I}) \in \{x_i, x_k\}$. If $f(\mathcal{I}) = x_k$, we consider the profile $\mathcal{I}'$ derived from $\mathcal{I}$ by changing the intervals of the t voters who report $\{x_i, x_j, x_k\}$ to $\{x_i, x_j\}$ and the intervals of the w_2 voters reporting $\{x_k\}$ to $\{x_j\}$. The conjunction of unanimity and robustness implies that $f(\mathcal{I}') = x_j$. On the other hand, Claim (3) of Lemma 7 shows that

$$v(\mathcal{I}')u^{i,j} = w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_j\}} + t u^{i,j}_{\{x_i,x_j\}} = w_1 u^{i,j}_{\{x_i\}} + w_2 u^{i,j}_{\{x_k\}} + t u^{i,j}_{\{x_i,x_j,x_k\}} > 0.$$

This implies that $f(\mathcal{I}') \neq x_j$. However, then there is no feasible choice left for this profile, so the assumption that $vu^{i,k} < 0$ must have been wrong.

Conversely, $f(\mathcal{I}) = x_i$, we derive a contradiction by considering the profile $\mathcal{I}''$ where w_1 voters report $\{x_j\}$, w_2 voters report $\{x_k\}$, and t voters report $\{x_j, x_k\}$. In particular, unanimity and robustness imply for this profile that $f(\mathcal{I}'') = x_j$ but $v(\mathcal{I}'')u^{j,k} < 0$, thus yielding the desired contradiction. Since we have a contradiction in both cases, we finally conclude that if $vu^{i,j} \geq 0$ for some vector $v \in \mathbb{R}^q_{\geq 0}$, then $vu^{j,k} \geq 0$.

Step 2: Next, we will prove the lemma. To this end, fix an arbitrary alternative $x_i \in A$, let $S_i = \{v \in \mathbb{R}^q \colon \forall x_j \in A \setminus \{x_i\} \colon vu^{i,j} \geq 0\}$ and $T_i = \{v \in \mathbb{R}^q \colon vu^{i-1,i} \leq 0 \land vu^{i,i+1} \geq 0\}$. Note that, for T_i, we define the vectors $u^{0,1}$ (if $i = 1$) and $u^{m,m+1}$ (if $i = m$) by $u^{0,1}_X = u^{m,m+1}_X = 0$ for all $X \in \Lambda$. First, by Lemma 6, it holds that $\bar{Q}_i = S_i$, so it suffices to show that $S_i = T_i$. To this end, we first note that we can suppose that $u^{i,i-1} = -u^{i-1,i}$ because Claim (3) of Lemma 6 allows to replace the vector $u^{i,i-1}$ with any non-zero vector $u \in \mathbb{R}^q$ such that $vu \geq 0$ if $v \in \bar{Q}_i$ and $vu \leq 0$ if $v \in \bar{Q}_{i-1}$. Since $-u^{i-1,i}$ satisfies this condition, we derive that $T_i = \{v \in \mathbb{R}^q \colon vu^{i,i-1} \geq 0 \land vu^{i,i+1} \geq 0\}$. By this insight, it is clear that $S_i \subseteq T_i$ because we only remove constraints to infer T_i from S_i.

Now, assume for contradiction that there is an point $v \in T_i \setminus S_i$. Since $v \in T_i$, we have that $vu^{i,i-1} \geq 0$ and $vu^{i,i+1} \geq 0$. On the other hand, because $v \notin S_i$, there is an index $k \notin \{i-1, i, i+1\}$ such that $vu^{i,k} < 0$. Next, let v' denote the vector such that $v'_{\{x_i\}} = 1$ and $v'_X = 0$ for all $X \in \Lambda \setminus \{\{x_i\}\}$. Moreover, we define $v^* = v + \epsilon v'$, where $\epsilon > 0$ is so small that $v^*u^{i,k} < 0$ still holds, and observe that $v^*u^{i,i-1} > 0$ and $v^*u^{i,i+1} > 0$ by Claim (1) of Lemma 7. By Step 1, we derive from $v^*u^{i,i+1} > 0$ that $v^*u^{i+1,i+2} \geq 0$, too. Moreover, if $v^*u^{i+1,i+2} = 0$, it we could marginally increase the value of $v^*_{\{x_{i+2}\}}$ to construct a vector $\bar{v}$ with $\bar{v}u^{i,i+1} > 0$ and $\bar{v}u^{i+1,i+2} < 0$, which contradicts Step 1. Hence, we derive from $v^*u^{i,i+1} > 0$ also that $v^*u^{i+1,i+2} > 0$. Moreover, by repeatedly applying this reasoning, we conclude that $v^*u^{j,j+1} > 0$ for all $j \in \{i+1, \dots, m-1\}$, which means that $v^* \notin \bar{Q}_{j+1}$ for all such $j \in \{i, \dots, m-1\}$. Next, we observe that $v^*u^{i,i-1} > 0$ means that $v^*u^{i-1,i} < 0$. By the contraposition of Step 1, we infer that if $v^*u^{i-1,i} < 0$, then $v^*u^{i-2,i-1} < 0$. Moreover, by repeating this argument, it follows that for all $j \in \{2, \dots, i\}$ that $v^*u^{j-1,j} < 0$, so $v^* \notin \bar{Q}_{j-1}$. Finally, since $v^*u^{i,k} < 0$, we also have that $v^* \notin \bar{Q}_i$. However, this means that $v^* \notin \bar{Q}_j$ for any $j \in \{1, \dots, m\}$. This contradicts that $\bigcup_{x_j \in A} \bar{Q}_j = \mathbb{R}^q_{\geq 0}$ (which is implied by the basic insight that $\bigcup_{x_j \in A} Q_j = \mathbb{Q}^q_{\geq 0} \setminus \{0\}$). Hence, we have now a contradiction, so there is no point $v \in T_i \setminus S_i$. Put differently, it holds that $T_i \subseteq S_i$, which completes the proof of the lemma. □

Based on our observations so far, we will now define a weight vector α such that its induced collective position function Π_α satisfies that $\Pi_\alpha(v, x_i) \geq \theta_i$ if and only if $vu^{i,i+1} \geq 0$. For the sake of completeness, we extend here the definition of collective position functions from interval profiles to vectors $v \in \mathbb{R}^q_{\geq 0}$: $\Pi_\alpha(v, x_i) = \sum_{X\in\Lambda} v_X \pi_\alpha(X, x_i)$. Recall for the subsequent lemma that we scale our vectors $u^{i,i+1}$ such that $u^{i,i+1}_{\{x_i\}} = 1 - \theta_i$ and $u^{i,i+1}_{\{x_{i+1}\}} = -\theta_i$.

Lemma 9. *There is a weight vector $\alpha \in [0, 1]^m$ and a collective position function Π_α such that $\Pi_\alpha(v, x_i) = vu^{i,i+1} + \theta_i \sum_{X\in\Lambda} v_X$ for all $v \in \mathbb{R}^q_{\geq 0}$ and $x_i \in A$.*

Proof. We define the weight vector $\alpha = (\alpha_1, \dots, \alpha_m)$ by $\alpha_i = u^{i,i+1}_{\{x_i,x_{i+1}\}} + \theta_i$ for all $i \in \{1, \dots, m-1\}$ and $\alpha_m = 1$. We moreover note that the value α_m does not matter as $\pi_\alpha(X, x_m) = 1$ for all intervals $X \in \Lambda$. First, we note that, since $-\theta_i \leq u^{i,i+1}_{\{x_i,x_{i+1}\}} \leq 1 - \theta$ by Claim (2) of Lemma 7, it holds that $\alpha_i \in [0, 1]$. Next, let $v \in \mathbb{R}^q_{\geq 0}$ denote an arbitrary vector and fix an alternative $x_i \neq x_m$. We next partition the set of intervals Λ with respect to x_i: the set $L = \{X \in \Lambda \colon X \subseteq [x_1, x_i]\}$ contains all intervals $X = [\ell, r]$ that are (weakly) left of x_i, $M = \{X \in \Lambda \colon \{x_i, x_{i+1}\} \subseteq X\}$ is the set of intervals containing both x_i and x_{i+1}, and $R = \{X \in \Lambda \colon X \subseteq [x_{i+1}, x_m]\}$ are the intervals that are (weakly) right of x_{i+1}. By Claim (3) in Lemma 7, we have that $u^{i,i+1}_X = u^{i,i+1}_{\{x_i\}} = 1 - \theta_i$ for all $X \in L$, $u^{i,i+1}_X = u^{i,i+1}_{\{x_i,x_{i+1}\}} = \alpha_i - \theta_i$ for all $X \in M$, and $u^{i,i+1}_X = u^{i,i+1}_{\{x_{i+1}\}} = -\theta_i$ for all $X \in R$. Moreover, for the individual position function induced by α, it holds that $\pi_\alpha(X, x_i) = 1$ if $X \in L$, $\pi_\alpha(X, x_i) = \alpha_i$ if $X \in M$, and $\pi_\alpha(X, x_i) = 0$ if $X \in R$. We thus compute that

$$\begin{aligned}\Pi_\alpha(v, x_i) &= \sum_{X\in L} v_X + \alpha_i \sum_{X\in M} v_X \\ &= (1 - \theta_i) \sum_{X\in L} v_X + u^{i,i+1}_{\{x_i,x_{i+1}\}} \sum_{x\in M} v^X - \theta_i \sum_{X\in R} v_X + \theta_i \sum_{X\in\Lambda} v_X \\ &= vu^{i,i+1}_{\{x_i,x_{i+1}\}} + \theta_i \sum_{X\in\Lambda} v_X.\end{aligned}$$

This completes the proof of this lemma. □

We are finally ready to prove Theorem 1.

Theorem 1. *A single-winner voting rule on Λ^* is robust, anonymous, unanimous, reinforcing, and right-biased continuous if and only if it is a position-threshold rule.*

Proof. We hae shown the direction from left to right in Lemma 2, so we focus here on the converse. Thus, let f denote a single winner voting rule on Λ^* that satisfies our five axioms. Now, by Lemmas 3 and 4, there is a threshold vector $\theta \in (0, 1)^m$ such that $\theta_1 \geq \theta_2 \geq \dots \geq \theta_m$ and $f(\mathcal{I}) = \max_\triangleright\{x_i \in A \colon \Pi_{SP}(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$ for all profiles $\mathcal{I} \in \mathcal{D}^*_1$. Next, we note that we can represent interval profiles $\mathcal{I}$ as vectors $v \in \mathbb{N}^q_0 \setminus \{0\}$, where the entry v_i states how often the i-th interval is submitted. Moreover, there is a (unique) function $g : \mathbb{N}^q_0 \setminus \{0\} \to A$ such that $f(\mathcal{I}) = g(v(\mathcal{I}))$ for all $\mathcal{I} \in \Lambda^*$. In

Lemma 5, we extend this function to the domain $\mathbb{Q}^q_{\geq 0} \setminus \{0\}$, i.e., we show that there is a function $\hat{g} : \mathbb{Q}^q_{\geq 0} \setminus \{0\} \to C$ that is reinforcing and satisfies that $f(\mathcal{I}) = \hat{g}(v(\mathcal{I}))$ for all $\mathcal{I} \in \Lambda^*$. Based on this function, we then define the sets $Q_i = \{v \in \mathbb{Q}^q_{\geq 0} \setminus \{0\} \colon \hat{g}(v) = x\}$ and let $\bar{Q}_i$ denote the closure of Q_i with respect to $\mathbb{R}^q$. In a sequence of lemmas (Lemmas 6 to 8), we then derive that there are non-zero vectors $u^{1,2}, u^{2,3}, \dots, u^{m-1,m} \in \mathbb{R}^q$ such that $\bar{Q}_i = \{v \in \mathbb{R}^q_{\geq 0} \colon vu^{i-1,i} \leq 0 \wedge vu^{i,i+1} \geq 0\}$ for all $i \in \{1, \dots, m\}$ (where $u^{0,1} = u^{m,m+1} = 0$ for notational simplicity). Finally, we show in Lemma 9 that there is a weight vector α such that the corresponding collective position function Π_α satisfies for all $i \in \{1, \dots, m-1\}$ and $v \in \mathbb{R}^q_{\geq 0}$ that $\Pi_\alpha(v, x_i) = vu^{x_i,x_{i+1}} + \theta_i \sum_{X \in \Lambda} v_X$. We derive from this that $\bar{Q}_i = \{v \in \mathbb{R}^q_{\geq 0} \colon \Pi_\alpha(v, x_{i-1}) \leq \theta_{i-1} \cdot \sum_{X \in \Lambda} v_X \wedge \Pi_\alpha(v, x_i) \geq \theta_i \cdot \sum_{X \in \Lambda} v_X\}$ for all $i \in \{1, \dots, m\}$ (where we define $\theta_0 = \Pi_\alpha(\mathcal{I}, x_0) = 0$ for notational simplicity). Since $\Pi_\alpha(\mathcal{I}, x_i) = \Pi_\alpha(v(\mathcal{I}), x_i)$ and $\sum_{X \in \Lambda} v(\mathcal{I})_X = n_\mathcal{I}$, this shows for all interval profiles $\mathcal{I} \in \Lambda^*$ that

$$\begin{aligned} f(\mathcal{I}) &= \hat{g}(v(\mathcal{I})) \\ &= \{x_i \in A \colon v(\mathcal{I}) \in Q_i\} \\ &\subseteq \{x_i \in A \colon v(\mathcal{I}) \in \bar{Q}_i\} \\ &= \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_{i-1}) \leq \theta_{i-1} n_\mathcal{I} \wedge \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}. \end{aligned}$$

Now, we define $O(\mathcal{I}) = \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_{i-1}) \leq \theta_{i-1} n_\mathcal{I} \wedge \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$ as the set of possible winners of f at the profile $\mathcal{I}$ and we note that $\max_\rhd O(\mathcal{I}) = \max_\rhd \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$. To see this, let $x_j = \max_\rhd O(\mathcal{I})$. By definition, we have that $\Pi_\alpha(\mathcal{I}, x_j) \geq \theta_j n_\mathcal{I}$, so $x_j \in \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$. This proves that $\max_\rhd \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\} \trianglerighteq \max_\rhd O(\mathcal{I})$. Next, let $x_j = \max_\rhd \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$. If $x_j = x_1$, the $x_j \in O(\mathcal{I})$ as the condition on θ_0 is trivial. Otherwise, it holds $x_{j-1} \notin \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$, so $\Pi_\alpha(\mathcal{I}, x_{j-1}) < \theta_{j-1} n_\mathcal{I}$. This proves again that $x_j \in O(\mathcal{I})$ and we thus conclude that $\max_\rhd O(\mathcal{I}) \trianglerighteq \max_\rhd \{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_\mathcal{I}\}$. Combining these two observations then gives the desired equality.

Based on our last insight, we will next show that $f(\mathcal{I}) = \max_\rhd O(\mathcal{I})$ for all profiles $\mathcal{I} \in \Lambda^*$. To this end, we assume for contradiction that there is a profile $\mathcal{I}$ such that $f(\mathcal{I}) = x_j \neq x_i = \max_\rhd O(\mathcal{I})$. Because $f(\mathcal{I}) \subseteq O(\mathcal{I})$, this means that $x_i \rhd x_j$. Next, we partition the voters in $N_\mathcal{I}$ into three sets: $L = \{k \in N_\mathcal{I} \colon \forall x \in I_k \colon x \rhd x_j\}$ contains all voters whose interval is fully left of x_j, $M = \{k \in N_\mathcal{I} \colon \{x_i, x_j\} \subseteq I_k\}$ contains all voters who report both x_i and x_j, and $R = N_\mathcal{I} \setminus (L \cup M)$ contains all voters who do not approve x_i but an alternative that is weakly right of x_j. Now, consider the profile $\mathcal{I}^1$ where all voters in L report $\{x_i\}$, all voters in M report $[x_i, x_j]$, and all voters in R report $\{x_j\}$. Repeatedly applying robustness shows that $f(\mathcal{I}^1) = x_j$ because we can transform $\mathcal{I}$ to $\mathcal{I}^1$ without removing x_j of the interval of any voter. Now, assume that $j \geq i+2$; otherwise, we can skip the next step. In this case, we consider the profile $\mathcal{I}^2$ where all voters in L report $\{x_i\}$, all voters in M report $[x_i, x_{j-1}]$, and all voters in R report $\{x_{j-1}\}$. Using robustness from $\mathcal{I}^1$, we infer that $f(\mathcal{I}^2) \in \{x_{j-1}, x_j\}$. Moreover, if $f(\mathcal{I}^2) = x_j$, our voters can deviate to, e.g., unanimously report $\{x_i\}$. Since none of these modifications touches on x_j, this alternative has to remain the winner by robustness, but unanimity postulates that x_i is now chosen. This contradiction proves that $f(\mathcal{I}^2) = x_{j-1}$. Furthermore, by repeating this argument, we derive a profile $\mathcal{I}^*$ such that all voters in L report $\{x_i\}$, all voters in M report $\{x_i, x_{i+1}\}$, all voters in R report $\{x_{i+1}\}$, and $f(\mathcal{I}^*) = x_{i+1}$.

Next, we compute that $\Pi_\alpha(\mathcal{I}^*, x_i) = L + \alpha_i M \geq \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}^*}$ because $\pi_\alpha(I_k, x_i) \leq 1$ for all $k \in L$, $\pi_\alpha(I_k, x_i) \leq \alpha_i$ for all $k \in M$ (as $x_j \in I_k$), and $\pi_\alpha(I_k, x_i) = 0$ for all $k \in R$ (as all these voters report intervals fully right of x_i). On the other hand, since $f(\bar{\mathcal{I}}) \in O(\bar{\mathcal{I}})$ for all interval profiles $\bar{\mathcal{I}}$ and $f(\mathcal{I}^*) = x_{i+1}$, we conclude that $\Pi_\alpha(\mathcal{I}^*, x_i) \leq \theta_i n_{\mathcal{I}^*}$. This proves that $\Pi_\alpha(\mathcal{I}^*, x_i) = \theta_i n_{\mathcal{I}^*}$. Now, let $\mathcal{I}'$ denote the profile where a single voter report $\{x_i\}$. By unanimity, we have that $f(\mathcal{I}') = x_i$ and, in turn, right-biased continuity implies that there must be a $\lambda \in \mathbb{N}$ such that $f(\lambda\mathcal{I}^* + I') = x_{i+1}$. However, it holds for every $\lambda \in \mathbb{N}$ that $\Pi_\alpha(\lambda\mathcal{I}^* + I', x_i) = \lambda\theta_i n_{\mathcal{I}^*} + 1 > \theta_i(\lambda n_{\mathcal{I}^*} + 1) = \theta_i n_{\lambda\mathcal{I}^* + I'}$. This shows that $x_{i+1} \notin O(\lambda\mathcal{I}^* + I')$ because the membership of x_{i+1} in this set requires that $\Pi_\alpha(\lambda\mathcal{I}^* + I', x_i) \leq \theta_i n_{\lambda\mathcal{I}^* + \lambda I'}$. This is the desired contradiction, so we conclude that $f(\mathcal{I}) = \max_\rhd O(\mathcal{I}) = \max_\rhd\{x_i \in A \colon \Pi_\alpha(\mathcal{I}, x_i) \geq \theta_i n_{\mathcal{I}}\}$ for all profiles $\mathcal{I} \in \Lambda^*$. Hence, f is induced by the weight vector α and the threshold vector θ and, since f is robust, Lemma 1 shows that these vectors must be compatible. This proves that f is indeed the position-threshold rule defined by α and θ. $\square$