

Settling the Score: Portioning with Cardinal Preferences*

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Abstract

We study a portioning setting in which a public resource such as time or money is to be divided among a given set of candidates, and each agent proposes a division of the resource. We consider two families of aggregation rules for this setting—those based on coordinate-wise aggregation and those that optimize some notion of welfare—as well as the recently proposed independent markets rule. We provide a detailed analysis of these rules from an axiomatic perspective, both for classic axioms, such as strategyproofness and Pareto-optimality, and for novel axioms, some of which aim to capture proportionality in this setting. Our results indicate that a simple rule that computes the average of the proposals satisfies many of our axioms and fares better than all other considered rules in terms of fairness properties. In addition, we complement these results by presenting two characterizations of the average rule.

1 Introduction

A town council has just received its annual funding from the government, and it needs to determine how to split the budget among constructing new facilities, maintaining clean streets, and ensuring public safety. The mayor is in favor of making decisions democratically, so she asks each resident of the town to propose a division of the budget. After collecting the proposals, how should the council aggregate them into an actual allocation?

In the problem of *portioning*, the aim is to divide a homogeneous resource among a given set of candidates. Besides dividing money, another important application of portioning is the division of time—for example, a conference needs to distribute its time among research talks, panels, and social gatherings. Several prior works on portioning assumed that each agent submits her preferences in the form of either an approval ballot [Bogomolnaia et al., 2005, Duddy, 2015, Aziz et al., 2020] or an ordinal ranking [Airiau et al., 2023]. However, in many portioning scenarios, these preference formats are not expressive enough to fully describe agents’ intentions. For instance, if a citizen wants the budget to be used both for constructing new facilities and for cleaning the streets, but with twice as much money spent on the former than the latter, her preference cannot be captured by a ranking or an approval set. Likewise, a conference attendee who would like 75% of the time to be spent on research talks, 15% on panels, and 10% on social gatherings ranks these activities in the same way as another attendee who prefers a 40%–35%–25% split, but the actual preferences of these two attendees are quite different.

*A preliminary version of this paper appeared in *Proceedings of the 26th European Conference on Artificial Intelligence* [Elkind et al., 2023]. This version is substantially enhanced: it fills in all missing results (Table 1) and improves the organization of the results, strengthens the case for using the average rule by providing characterizations of the rule (Section 7), expands the discussion of related work (Section 1.2), and includes all proofs omitted from the conference version.

In an important recent work, Freeman et al. [2021] studied portioning with cardinal preferences, where every agent is asked to propose a division of the resource. This input format allows each agent to specify what she views as the ideal portioning outcome, and is therefore much more descriptive than the two formats discussed earlier. Assuming that an agent’s disutility is given by the ℓ_1 distance between her ideal distribution and the actual outcome,¹ Freeman et al. observed that even though the rule that maximizes the utilitarian social welfare is known to be strategyproof (for a specific tie-breaking convention) [Lindner et al., 2008, Goel et al., 2019], it tends to put too much weight on majority preferences. In light of this observation, they introduced the *independent markets* (IM) rule, which is strategyproof and, in some sense, more proportional. Their work inspired a number of follow-up papers in this fundamental social choice setting, mostly focusing on strategyproofness [Caragiannis et al., 2022, Brandt et al., 2024, de Berg et al., 2024, Freeman and Schmidt-Kraepelin, 2024]. However, while strategyproofness is an important consideration, there may be scenarios where other features of aggregation rules are just as—if not more—desirable. Thus, to help decision-makers identify suitable aggregation rules for their applications, it would be useful to (a) build catalogues of axioms and popular aggregation rules for the portioning setting, (b) determine which of these axioms are satisfied by each of the aggregation rules, and (c) characterize some of the most important rules in terms of these axioms.

1.1 Overview of Contributions

We consider a diverse set of axioms for portioning with cardinal preferences. Besides classic axioms such as strategyproofness and Pareto optimality, we put forward new axioms including score-unanimity and score-representation (see Section 2 for definitions). Several of our axioms are independent of the underlying utility functions; for those where the utility functions matter, following most prior works in this domain, we assume ℓ_1 utilities. We then conduct a systematic study of aggregation rules with respect to these axioms. We focus on two families of portioning rules—those that are based on coordinate-wise aggregation and those that optimize some notion of welfare—as well as the recently proposed IM rule [Freeman et al., 2021]. We also include observations regarding relationships between the axioms. Table 1 summarizes our results.

Our findings offer several insights on portioning rules. As shown in Table 1, the most promising rules with respect to the axioms that we study are the average rule (AVG), which simply returns the average of all the proposals, and the utilitarian welfare-maximizing rule (UTIL), with the trade-off being that AVG fails strategyproofness and Pareto optimality whereas UTIL fails fairness and consistency notions such as single-minded proportionality,² score-representation, and independence. While IM satisfies both strategyproofness and single-minded proportionality, it fails other intuitive properties such as score-unanimity and score-representation; these failures can lead to highly counterintuitive outcomes. The axiomatic properties of various rules can be used to inform decision-making in a wide range of settings. For instance, consider again the scenario where a conference organizer needs to divide time among different activities at a conference. In this case, it is likely difficult for an attendee to accurately predict what other attendees’ preferences are, making strategyproofness arguably less relevant as a consideration. On the other hand, strategyproofness could be more important in smaller-scale settings where agents know each other well, e.g., portioning within a family or a small organization. Moreover, intuitive properties such as score-unanimity and range-respect may be essential in settings where votes are revealed: for example, if all agents allocate 80% of the budget to a certain activity, but the rule allocates 60% to it, this may well lead to dissatisfaction among agents regarding the use of that rule.

¹Freeman et al. [2021] noted that ℓ_1 preferences arise naturally when agents are endowed with separable uniform utilities over candidates together with a funding cap.

²This property was simply called “proportionality” by Freeman et al. [2021]. However, the property only applies to instances in which all agents are “single-minded”, thereby making it rather weak compared to proportionality notions in other settings (e.g., fair division [Procaccia, 2016]). Hence, in this paper we call this property “single-minded proportionality”.

F	Coordinate-wise					Welfare-based		Other
	AVG	MAX	MIN	MED	GEO	UTIL	EGAL	IM
Pareto-optimality	X [‡]	X [†]	X [†]	X [‡]	X [†]	✓	✓	X [†]
Range-respect	✓	X [†]	X [†]	X [‡]	X [†]	✓	✓	X [†]
Score-unanimity	✓	X [†]	X [†]	X [‡]	X [†]	✓	✓	X [†]
Score-representation	✓	X	X	X*	X	X	X*	X [†]
Single-minded Proportionality	✓	X*	X	X*	X	X*	X*	✓
Independence	✓	X [†]	X [†]	X [‡]	X [†]	X [†]	X [‡]	X [†]
Score-monotonicity	✓	✓	✓	✓	✓	✓	X [‡]	✓
Reinforcement	✓	✓	✓	X	✓	✓	✓	✓
Strategyproofness	X	X	X	X	X	✓	X	✓
Participation	✓	✓	✓	X	✓	✓	✓	✓

Table 1: Summary of our results. The asterisk symbol (*) means that the axiom is satisfied for $n = 2$, but may fail when $n \geq 3$ (even if $m = 2$). The dagger symbol (†) indicates that the axiom is satisfied for $m = 2$, but may fail when $m \geq 3$ (even if $n = 2$). The double dagger symbol (‡) indicates that the axiom is satisfied when $\min(n, m) = 2$. Some of the results on UTIL and IM were obtained by Freeman et al. [2021].

In addition to fulfilling several axioms, the average rule is intuitive and easy to explain to laypeople. We further strengthen the case for using this rule by providing two characterizations of it. Specifically, we show that the average rule is the only aggregation rule satisfying score-unanimity (i.e., if all agents allocate a γ fraction of the resource to a candidate, then the rule also assigns a γ fraction of the resource to it), independence (i.e., the fraction allocated to a candidate only depends on the fractions that the agents allocate to this candidate), and a mild fairness condition called anonymity, when there are at least three candidates. We also prove that, within the class of coordinate-wise rules, the average rule is the unique rule that satisfies score-unanimity, anonymity, and continuity whenever the number of candidates is at least four.

1.2 Further Related Work

Portioning can be viewed as a variant of *participatory budgeting*, a framework that allows citizens to democratically decide how the public budget should be spent. Participatory budgeting has been used in over 7,000 cities around the world [Participatory Budgeting Project, 2024] and received much recent interest in computational social choice—see, for example, the surveys by Aziz and Shah [2021] and Rey and Maly [2023]. Nevertheless, most of the participatory budgeting literature focuses on the discrete setting, where each project is either implemented in full or not implemented at all, and projects may have varying costs (see, however, recent work of Goyal et al. [2023]). This makes the nature of the problem quite different from that of portioning.

As mentioned earlier, Freeman et al. [2021] investigated portioning with cardinal preferences and introduced IM, which is strategyproof and single-minded proportional under ℓ_1 utilities. In fact, IM belongs to a class of *moving phantoms* mechanisms, all of which are strategyproof.³ Caragiannis et al. [2022] followed up on their work by examining the deviation of moving phantoms mechanisms from the average rule according to the ℓ_1 distance, while Freeman and Schmidt-Kraepelin [2024] explored a

³However, not all strategyproof rules are moving phantoms mechanisms [de Berg et al., 2024].

similar question using the ℓ_∞ distance. Brandt et al. [2024] showed that no rule can simultaneously be strategyproof, single-minded proportional, and Pareto-optimal under ℓ_1 or ℓ_∞ utilities, but such a rule exists for an alternative utility model that they called “minimal quotient” utilities.

Portioning also bears similarity to the domain of *probabilistic social choice*, where the output is likewise a fractional allocation. However, unlike in portioning, in probabilistic social choice the fraction allocated to each candidate is interpreted as the probability of eventually choosing this candidate as the unique winner. As a consequence, it is usually desirable for a rule to minimize the use of randomness, thereby making notions of fairness and proportionality less important. Moreover, much of the work in probabilistic social choice assumes that agents’ preferences are given as ordinal rankings (see, e.g., the survey by Brandt [2017]). A notable exception is the work of Intriligator [1973], which postulates that each agent has an ideal distribution over the candidates, but does not consider utility functions of agents. While Intriligator gives a characterization of the average rule, his characterization relies on a rather esoteric axiom regarding the sensitivity to probability changes.⁴

Finally, another related topic is *probabilistic opinion pooling*, which aims to aggregate probabilistic beliefs representing, for example, weather forecasts [Genest and Zidek, 1986, Clemen, 1989]. The focus of probabilistic opinion pooling is mainly to preserve epistemic and stochastic properties, which again leads to different axioms being considered.

2 Preliminaries

We present the model of portioning with cardinal preferences, and introduce the rules and axioms that we will study.

2.1 Model

Let $[t] := \{1, \dots, t\}$ for any positive integer t . Assume that there is a set of $n \geq 2$ agents, $N = [n]$, who report their preferences as ideal distributions of a homogeneous resource among a set $C = \{c_1, \dots, c_m\}$ of $m \geq 2$ candidates. Specifically, letting $\Delta^m := \{\mathbf{x} \in \mathbb{R}_{\geq 0}^m \mid \sum_{j \in [m]} x_j = 1\}$ denote the set of probability distributions over C , we assume that each agent $i \in N$ reports her preferences as a distribution $\mathbf{s}_i \in \Delta^m$. We typically refer to \mathbf{s}_i as agent i ’s *score vector*, and write $\mathbf{s}_i = (s_{i,1}, \dots, s_{i,m})$ to specify this vector. An *instance* \mathcal{I} of our problem is the collection of the preferences of all agents, i.e., $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$. For each vector $\mathbf{x} = (x_1, \dots, x_m)$, agent i ’s *disutility* is defined as $d_i(\mathbf{x}) := \sum_{j \in [m]} |s_{i,j} - x_j|$, which is the ℓ_1 distance between the agent’s score vector \mathbf{s}_i and \mathbf{x} . Given an instance \mathcal{I} , we aim to find a vector $\mathbf{x} \in \Delta^m$ that reflects the agents’ collective preferences. To this end, we use *aggregation rules*, which are defined as follows.

Definition 2.1 (Aggregation rule). An *aggregation rule* F is a function $F : \Delta^{m \times n} \rightarrow \Delta^m$ that maps every instance $\mathcal{I} \in \Delta^{m \times n}$ to an outcome vector $\mathbf{x} \in \Delta^m$.

We will frequently use the notation $F(\mathcal{I})_j$ to denote the probability that the aggregation rule F assigns to candidate c_j on instance \mathcal{I} .

2.2 Aggregation Rules

In this paper, we will focus on two natural classes of aggregation rules, namely, coordinate-wise rules and welfare-optimizing rules. In addition, we will also consider the independent markets rule of Freeman et al. [2021].

⁴Rice [1977] later pointed out that Intriligator’s characterization was wrong, and proposed a fix.

2.2.1 Coordinate-wise Aggregation Rules

We first introduce the class of *coordinate-wise* aggregation rules. The idea behind these rules is to aggregate the reported scores for each candidate individually and then normalize the aggregated scores so that they sum up to 1.

Definition 2.2. An aggregation rule F is *coordinate-wise* if, for each n and each $j \in [m]$ there exist *coordinate-aggregation functions* $f_j^n : (\mathbb{R}_{\geq 0})^n \rightarrow \mathbb{R}_{\geq 0}$ such that $F(\mathcal{I})_j = \frac{f_j^n(s_{1,j}, \dots, s_{n,j})}{\sum_{k \in [m]} f_k^n(s_{1,k}, \dots, s_{n,k})}$ for all instances \mathcal{I} and all $j \in [m]$.

We extend Definition 2.2 to allow for the possibility that, for certain instances, $f_j^n(s_{1,j}, \dots, s_{n,j}) = 0$ for all $j \in [m]$: in this case, we assign each candidate the same probability, i.e., we interpret $\frac{0}{0}$ as $\frac{1}{m}$. We remark that our negative results do not depend on this tie-breaking convention. When f_j^n is the same for all $j \in [m]$, we omit the subscript j and write f^n . Furthermore, we may omit the superscript n when it is clear from the context, and simply write f_j or f .

We will focus on five natural coordinate-wise aggregation rules, where f is the *average*, *maximum*, *minimum*, *median* (if the number of agents is even, we take the average of the two middle scores), or *geometric mean* function. For brevity, we refer to these rules as AVG, MAX, MIN, MED, and GEO, respectively. The advantage of these rules is that they are intuitive and easily computable.

2.2.2 Welfare-based Aggregation Rules

For our second class, we consider rules that are based on welfare optimization. In particular, we focus on two popular welfare criteria:⁵ the *utilitarian welfare* $-\sum_{i \in N} d_i(\mathbf{x})$ and the *egalitarian welfare* $\min_{i \in N}(-d_i(\mathbf{x}))$. The minus sign ensures that these definitions indeed capture the welfare, as d_i measures the disutility (rather than the utility) of agent i . The *utilitarian rule* (UTIL) and the *egalitarian rule* (EGAL) then return an outcome that minimizes the utilitarian and egalitarian welfare, respectively. More formally, for each instance \mathcal{I} , UTIL chooses an outcome \mathbf{x}_{UTIL} such that $\mathbf{x}_{\text{UTIL}} \in \arg \max_{\mathbf{x}}(-\sum_{i \in N} d_i(\mathbf{x}))$, while EGAL returns \mathbf{x}_{EGAL} such that $\mathbf{x}_{\text{EGAL}} \in \arg \max_{\mathbf{x}} \min_{j \in N}(-d_j(\mathbf{x}))$.

For both of these rules, the tie-breaking is important. Following Freeman et al. [2021], we consider the variant of UTIL that breaks ties in favor of the maximum entropy division. Specifically, we assume that UTIL outputs the utilitarian welfare-maximizing outcome \mathbf{x} that minimizes the quantity $\sum_{j \in [m]} (x_j - \frac{1}{m})^2$, i.e., the ℓ_2 distance to the uniform distribution $\mathbf{x}_u = (\frac{1}{m}, \dots, \frac{1}{m})$. This tie-breaking choice is neutral for candidates and ensures strategyproofness [Lindner et al., 2008].

For EGAL, if there are multiple outcomes that maximize the egalitarian welfare, then we break ties in a “leximin” manner. That is, we minimize the largest disutility, then subject to that, minimize the second-largest disutility, and so on. This type of leximin tie-breaking is standard when dealing with egalitarian welfare [e.g., Bogomolnaia and Moulin, 2004, Kurokawa et al., 2018]. However, even after this tie-breaking process, there may still be multiple EGAL outcomes. We show next that if $n = 2$, AVG always returns an EGAL outcome, so we assume that the EGAL rule coincides with AVG in this case. By contrast, our results for $n \geq 3$ will not depend on this choice, and we allow EGAL to break ties in any consistent manner (i.e., if \mathbf{x} and \mathbf{x}' are both EGAL outcomes for two instances \mathcal{I} and \mathcal{I}' , then we assume that if EGAL chooses \mathbf{x} for \mathcal{I} , it does not choose \mathbf{x}' for \mathcal{I}').

Proposition 2.3. *When $n = 2$, the output of AVG is an EGAL outcome.*

Proof. Let \mathbf{x} be the output of AVG for the case $n = 2$. Then $x_j = \frac{s_{1,j} + s_{2,j}}{2}$ for each $j \in [m]$, so $d_1(\mathbf{x}) = d_2(\mathbf{x})$, and $d_1(\mathbf{x}) + d_2(\mathbf{x}) = \sum_{j \in [m]} |s_{1,j} - s_{2,j}|$. Since $d_1(\mathbf{x}') + d_2(\mathbf{x}') \geq \sum_{j \in [m]} |s_{1,j} - s_{2,j}|$ for any outcome \mathbf{x}' , it follows that \mathbf{x} is an EGAL outcome. \square

⁵There is a third popular welfare criterion called *Nash welfare*, which is defined based on the product of utilities. However, this welfare notion is not well-defined in our setting, as we are considering disutilities. For example, it has been observed that there is no natural equivalent of Nash welfare in the fair allocation of chores [Freeman et al., 2020, Ebadian et al., 2022].

We also note that both UTIL and EGAL (with the given tie-breaking conventions) can be computed in polynomial time. For UTIL, this follows from the results of Freeman et al. [2021], and for EGAL, we prove this claim in the appendix (see Proposition A.1).

2.2.3 Independent Markets Rule

The last aggregation rule that we will study is the *independent markets* (IM) rule of Freeman et al. [2021]. This rule belongs to the class of *moving phantoms* rules, which take for each candidate the median of the agents' reports and $n + 1$ phantom values. However, because the final scores of all candidates must sum up to 1, the phantom values cannot be constant. Instead, there are $n + 1$ phantom functions $f_0, \dots, f_n : [0, 1] \rightarrow [0, 1]$ that are weakly increasing and satisfy $f_k(0) = 0$ and $f_k(1) = 1$ for all $k \in \{0, \dots, n\}$. Then, a moving phantoms rule determines the smallest value t^* such that $\sum_{j \in [m]} \text{med}(s_{1,j}, \dots, s_{n,j}, f_0(t^*), \dots, f_n(t^*)) = 1$, and returns the vector \mathbf{x} given by $x_j = \text{med}(s_{1,j}, \dots, s_{n,j}, f_0(t^*), \dots, f_n(t^*))$ for each $j \in [m]$. Finally, IM is defined by setting $f_k^{\text{IM}}(t) = \min(kt, 1)$ for all $k \in \{0, \dots, n\}$.

Before proceeding further, let us present a simple example demonstrating how the different aggregation rules work.

Example 2.4. Consider an instance with $n = 2$ agents and $m = 3$ candidates. The first agent has a preferred distribution $\mathbf{s}_1 = (0.8, 0.2, 0)$ and the second agent has a preferred distribution $\mathbf{s}_2 = (0.8, 0, 0.2)$.

AVG and MED output $(0.8, 0.1, 0.1)$, MIN and GEO output $(1, 0, 0)$, and MAX outputs $(2/3, 1/6, 1/6)$. For UTIL, the distribution $(0.8, x, 0.2 - x)$ maximizes the utilitarian welfare for any $x \in [0, 0.2]$; among these distributions, $(0.8, 0.1, 0.1)$ minimizes the ℓ_2 distance to the uniform distribution $(1/3, 1/3, 1/3)$, so UTIL returns this distribution. The same distribution is also returned by EGAL. Finally, for IM, we have $\sum_{j \in [m]} \text{med}(s_{1,j}, s_{2,j}, f_0^{\text{IM}}(t^*), f_1^{\text{IM}}(t^*), f_2^{\text{IM}}(t^*)) = 1$ when $t^* = 0.3$, and IM returns the distribution $(0.6, 0.2, 0.2)$.

2.3 Axioms

We next introduce the axioms that we will use to evaluate our aggregation rules. We roughly group the axioms into four categories: efficiency properties, fairness properties, consistency properties, and incentive properties. However, we note that the boundaries between these categories are fluid, particularly when considering weak axioms.

2.3.1 Efficiency Properties

We start by introducing efficiency properties, which intuitively require the outcome chosen by an aggregation rule to be guided by the agents' preferences. Perhaps the most prominent such property is Pareto-optimality, which postulates that it should not be possible to make one agent better off without making another agent worse off.

Definition 2.5 (Pareto-optimality). An outcome \mathbf{x} is *Pareto-optimal* in an instance \mathcal{I} if there is no other outcome \mathbf{x}' such that $d_i(\mathbf{x}') \leq d_i(\mathbf{x})$ for all agents $i \in N$ and $d_i(\mathbf{x}') < d_i(\mathbf{x})$ for some agent $i \in N$. An aggregation rule F is *Pareto-optimal* if $F(\mathcal{I})$ is Pareto-optimal for every instance \mathcal{I} .

As it will turn out, Pareto-optimality is a rather restrictive property in our setting, as only UTIL and EGAL satisfy it among the considered rules. We thus introduce two further axioms based on the agents' scores, which can be viewed as weaker efficiency axioms. The first axiom is *range-respect*, previously studied by Freeman et al. [2021]. This axiom states that the score assigned to a candidate should always lie between the minimum and the maximum scores that agents assign to this candidate.

Definition 2.6 (Range-respect). An aggregation rule F is *range-respecting* if $\min_{i \in N} s_{i,j} \leq F(\mathcal{I})_j \leq \max_{i \in N} s_{i,j}$ for all instances \mathcal{I} and all $j \in [m]$.

Next, we introduce a new property which we call *score-unanimity*. This property demands that if all agents report the same score for some candidate, then this candidate should receive exactly that score. Score-unanimity resembles a property called unanimity in single-winner voting, which states that if all agents agree on a favorite candidate, this candidate should be chosen. In single-winner voting, this property appears almost indispensable, and we believe that much of its appeal carries over to score-unanimity.

Definition 2.7 (Score-unanimity). An aggregation rule F is *score-unanimous* if $F(\mathcal{I})_j = \gamma$ for all instances \mathcal{I} and all $j \in [m]$ for which there exists $\gamma \in [0, 1]$ such that $s_{i,j} = \gamma$ for all $i \in N$.

We show that our three efficiency notions are logically related.

Proposition 2.8. *The following claims hold.*

- (1) *Pareto-optimality implies range-respect.*
- (2) *Range-respect implies Pareto-optimality if and only if $m = 2$ or $n = 2$.*
- (3) *Range-respect implies score-unanimity.*

Proof. We prove each of the claims separately.

Claim 1: Suppose for contradiction that there is an outcome \mathbf{x} that is Pareto-optimal but not range-respecting for an instance \mathcal{I} . Without loss of generality, this means that there exists some $j \in [m]$ such that $x_j > \max_{i \in N} s_{i,j}$ (the case that $x_j < \min_{i \in N} s_{i,j}$ allows for a symmetric argument). Since $\sum_{k \in [m]} x_k = \sum_{k \in [m]} s_{1,k} = 1$, there exists $\ell \in [m]$ such that $x_\ell < s_{1,\ell}$. Next, we define $\varepsilon = \min(x_j - \max_{i \in N} s_{i,j}, s_{1,\ell} - x_\ell)$ and consider the outcome \mathbf{x}' defined by $x'_j = x_j - \varepsilon$, $x'_\ell = x_\ell + \varepsilon$, and $x'_k = x_k$ for all $k \in [m] \setminus \{j, \ell\}$. Since $x_j - \varepsilon \geq s_{i,j}$ for all $i \in N$, it holds that $|(x_j - \varepsilon) - s_{i,j}| = |x_j - s_{i,j}| - \varepsilon$. By applying the triangle inequality and this observation, we find that for every agent $i \in N$,

$$\begin{aligned} d_i(\mathbf{x}') &= |(x_j - \varepsilon) - s_{i,j}| + |(x_\ell + \varepsilon) - s_{i,\ell}| + \sum_{k \in [m] \setminus \{j, \ell\}} |x_k - s_{i,k}| \\ &\leq |x_j - s_{i,j}| - \varepsilon + |x_\ell - s_{i,\ell}| + \varepsilon + \sum_{k \in [m] \setminus \{j, \ell\}} |x_k - s_{i,k}| = d_i(\mathbf{x}). \end{aligned}$$

Moreover, for agent 1, it additionally holds that $s_{1,\ell} \geq x_\ell + \varepsilon$, so we have that $|(x_\ell + \varepsilon) - s_{1,\ell}| = |x_\ell - s_{1,\ell}| - \varepsilon$. Using analogous computations as before, we then infer that $d_1(\mathbf{x}') = d_1(\mathbf{x}) - 2\varepsilon < d_1(\mathbf{x})$. This means that \mathbf{x} is not Pareto-optimal, a contradiction. It follows that Pareto-optimality implies range-respect.

Claim 2: For our second claim, we show that range-respect implies Pareto-optimality if and only if $m = 2$ or $n = 2$. First, we assume that $n = 2$ and let \mathbf{s}_1 and \mathbf{s}_2 denote the preferences of the agents. Using the triangle inequality, we derive for every outcome \mathbf{x} that $d_1(\mathbf{x}) + d_2(\mathbf{x}) \geq \sum_{j \in [m]} |s_{1,j} - s_{2,j}|$. Moreover, if \mathbf{x} is range-respecting, it holds that $d_1(\mathbf{x}) + d_2(\mathbf{x}) = \sum_{j \in [m]} |s_{1,j} - s_{2,j}|$ because $\min(s_{1,j}, s_{2,j}) \leq x_j \leq \max(s_{1,j}, s_{2,j})$ implies that $|s_{1,j} - x_j| + |s_{2,j} - x_j| = |s_{1,j} - s_{2,j}|$. This shows that every range-respecting outcome is Pareto-optimal, because any outcome that Pareto-dominates a range-respecting outcome would need to have strictly less total disutility.

Next, consider the case $m = 2$ and let \mathbf{x} denote an outcome that is range-respecting for an instance \mathcal{I} . Furthermore, let \mathbf{x}' denote another outcome and assume without loss of generality that $x_1 > x'_1$. Since $m = 2$, this means that $x_2 < x'_2$. Let i be an agent such that $s_{i,1} \geq x_1$, which implies that $s_{i,2} \leq x_2$; such an agent exists since \mathbf{x} is range-respecting. Observe that $d_i(\mathbf{x}) = (s_{i,1} - x_1) + (x_2 - s_{i,2}) <$

$(s_{i,1} - x'_1) + (x'_2 - s_{i,2}) = d_i(\mathbf{x}')$. Consequently, agent i strictly prefers \mathbf{x} to every outcome \mathbf{x}' with $x_1 > x'_1$. Since an analogous argument holds if $x_1 < x'_1$, we infer that \mathbf{x} is Pareto-optimal.

We now show that range-respect does not imply Pareto-optimality if $m \geq 3$ and $n \geq 3$. We focus on the case where $m = 3$, as it is trivial to extend the counterexample to larger m by universally assigning probability 0 to additional candidates. Now, consider the following instance and the outcome $\mathbf{x} = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$. It is easy to verify that \mathbf{x} is range-respecting for \mathcal{I} . However, the outcome $(0, \frac{1}{2}, \frac{1}{2})$ strictly benefits agent 2 and does not hurt the other agents, so \mathbf{x} is not Pareto-optimal.

\mathcal{I}	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
1	0	0	1
2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i \in \{3, \dots, n\}$	$\frac{1}{2}$	$\frac{1}{2}$	0

Claim 3: Finally, we show that range-respect implies score-unanimity. To this end, let $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$ denote an instance and $j \in [m]$ be such that there exists $\gamma \in [0, 1]$ with $s_{i,j} = \gamma$ for all $i \in N$. Range-respect of an outcome \mathbf{x} implies that $\gamma = \min_{i \in N} s_{i,j} \leq x_j \leq \max_{i \in N} s_{i,j} = \gamma$, so $x_j = \gamma$ and score-unanimity is satisfied. \square

2.3.2 Fairness Properties

For the second type of axioms, we turn to fairness concepts, which intuitively demand that every group of agents with similar preferences is proportionally represented by the outcome. A rather mild axiom based on this idea has been formulated by Freeman et al. [2021]. Specifically, these authors call an agent *single-minded* if she assigns probability 1 to some candidate. Their proportionality notion then requires that, if all agents are single-minded, each candidate should receive a probability proportional to the number of agents that assign probability 1 to it. To formalize this, we define $\mathcal{N}(\mathcal{I}, c_j, \gamma) := |\{i \in N : s_{i,j} \geq \gamma\}|$ as the number of agents who assign a probability of at least γ to candidate c_j in the instance \mathcal{I} .

Definition 2.9 (Single-minded Proportionality). An aggregation rule F satisfies *single-minded proportionality* if $F(\mathcal{I})_j = \frac{\mathcal{N}(\mathcal{I}, c_j, 1)}{n}$ for all candidates $c_j \in C$ and instances \mathcal{I} in which all agents are single-minded.

However, agents are rarely single-minded in several applications of portioning (such as dividing time between different activities at a conference), so an appropriate notion of proportionality for general preferences is needed. We formulate one such notion for the cardinal preference setting, and refer to it as *score-representation*. The idea of this notion is that if a $\frac{k}{n}$ fraction of the agents assign a portion of at least γ to a candidate c_j , then this candidate should receive a probability of at least $\gamma \cdot \frac{k}{n}$.

Definition 2.10 (Score-representation). An aggregation rule F satisfies *score-representation* if $F(\mathcal{I})_j \geq \gamma \cdot \frac{\mathcal{N}(\mathcal{I}, c_j, \gamma)}{n}$ for all instances \mathcal{I} , candidates $c_j \in C$, and $\gamma \in [0, 1]$.

It follows directly from the definitions that score-representation is strictly stronger than single-minded proportionality.

2.3.3 Consistency Properties

As the third type of axioms, we consider consistency properties, which aim to ensure that voting rules behave, in some sense, consistently across instances. The first such axiom that we examine is *independence*, which has previously been studied by Intriligator [1973]. The idea of this axiom is that the score

assigned to a candidate by an aggregation rule should only depend on the scores that the agents assign to this candidate, and is therefore independent of the scores assigned to other candidates. We remark that this axiom is similar in spirit to Arrow’s independence of irrelevant alternatives [Arrow, 1951], as it postulates that we can compute the outcome for a candidate without taking into account the remaining candidates. The definition is as follows.

Definition 2.11 (Independence). An aggregation rule F satisfies *independence* if $F(\mathcal{I})_j = F(\mathcal{I}')_j$ for all instances \mathcal{I} and \mathcal{I}' and all $j \in [m]$ such that $s_{i,j} = s'_{i,j}$ for all $i \in N$.

Independence is a demanding axiom and, for example, implies that the considered rule must be coordinate-wise, because we can define the j -th coordinate-aggregation function by $f_j^n(s_{1,j}, \dots, s_{n,j}) = F(\mathcal{I})_j$. Nevertheless, we believe that this axiom is appealing in practice, as it is intuitive and greatly simplifies the task of aggregating the agents’ score vectors. Moreover, a violation of independence can lead to complaints from candidates that receive a smaller portion of the resource despite getting the same score from every agent as before.

Next, we introduce *score-monotonicity*. This axiom requires that, if an agent increases the score of some candidate, then the aggregated score of this candidate is also weakly increasing. Score-monotonicity was previously studied by Freeman et al. [2021],⁶ and similar monotonicity notions are omnipresent in social choice theory.

Definition 2.12 (Score-monotonicity). An aggregation rule F is *score-monotone* if $F(\mathcal{I})_j \leq F(\mathcal{I}')_j$ for all instances \mathcal{I} , \mathcal{I}' and all $j \in [m]$ for which there is an agent $i \in N$ such that (i) $s_{i'} = s'_{i'}$ for all $i' \in N \setminus \{i\}$, (ii) $s_{i,j} < s'_{i,j}$, and (iii) $s_{i,j'} \geq s'_{i,j'}$ for all $j' \in [m] \setminus \{j\}$.

The final consistency notion that we study is *reinforcement*, which demands that if an aggregation rule chooses the same outcome for two instances with disjoint sets of agents, then it also chooses that outcome when combining these two instances. Variants of this axiom feature prominently in numerous results in social choice theory [e.g., Young, 1975, Fishburn, 1978, Young and Levenglick, 1978, Brandl et al., 2016].

Definition 2.13 (Reinforcement). An aggregation rule F satisfies *reinforcement* if, for all instances $\mathcal{I} = (s_1, \dots, s_n)$ and $\mathcal{I}' = (s'_1, \dots, s'_{n'})$ with $F(\mathcal{I}) = F(\mathcal{I}')$, it holds that $F(s_1, \dots, s_n, s'_1, \dots, s'_{n'}) = F(\mathcal{I})$.

Independence, score-monotonicity, and reinforcement are logically unrelated, as they formalize rather different notions of consistency. However, we emphasize that variants of these three axioms are well-established in the literature and we therefore believe that it is important to study all of them.

2.3.4 Incentive Properties

Our last category of axioms is concerned with the incentives of agents: aggregation rules should incentivize agents to participate and to report their preferences truthfully. These ideas lead to the well-known notions of *participation* and *strategyproofness*. We start by defining strategyproofness, which stipulates that agents should not be able to benefit from lying about their true preferences.

Definition 2.14 (Strategyproofness). An aggregation rule F is *strategyproof* if $d_i(F(\mathcal{I})) \leq d_i(F(\mathcal{I}'))$ for all instances \mathcal{I} and \mathcal{I}' and agents $i \in N$ such that $s_{i'} = s'_{i'}$ for all $i' \in N \setminus \{i\}$.

While it is known that IM and UTIL satisfy strategyproofness [Goel et al., 2019, Freeman et al., 2021], this property is in general rather demanding, as demonstrated by the impossibility theorem of

⁶Freeman et al. [2021] simply called this notion “monotonicity”.

Brandt et al. [2024] stating that no aggregation rule simultaneously satisfies strategyproofness, Pareto-optimality, and single-minded proportionality.

Participation is a property closely related to strategyproofness—it dictates that agents should not be able to profit by abstaining. Put differently, participation ensures that it is always weakly better for every agent to express her preference.

Definition 2.15 (Participation). An aggregation rule F satisfies *participation* if $d_i(F(\mathcal{I})) \leq d_i(F(\mathcal{I}'))$ for all instances \mathcal{I} and \mathcal{I}' such that \mathcal{I} is obtained from \mathcal{I}' by adding agent i .

We note that participation and strategyproofness are logically independent in general. However, when imposing reinforcement and the very mild condition that $F(\mathbf{x}) = \mathbf{x}$ (i.e., if there is a single agent, we choose her ideal distribution), it can be shown that strategyproofness implies participation.

3 Efficiency Properties

We now analyze our aggregation rules with respect to the axioms defined in Section 2.3. In this section, we study the three efficiency properties and show that, while UTIL and EGAL satisfy Pareto-optimality, all other rules except AVG fail even score-unanimity. Recall from Proposition 2.8 that Pareto-optimality implies range-respect which in turn implies score-unanimity, and that range-respect implies Pareto-optimality if $m = 2$ or $n = 2$.

Theorem 3.1. *The following claims hold.*

- (1) UTIL and EGAL are Pareto-optimal (and thus range-respecting and score-unanimous).
- (2) AVG is range-respecting (and thus Pareto-optimal when $m = 2$ or $n = 2$, and score-unanimous for any m, n), but fails Pareto-optimality for all $m \geq 3$ and $n \geq 3$.
- (3) MED is range-respecting when $m \leq 3$ or $n = 2$ (and thus Pareto-optimal when $m = 2$ or $n = 2$). If $m = 3$, it is Pareto-optimal if $n \geq 3$ is odd, but fails Pareto-optimality if $n \geq 4$ is even. It fails score-unanimity for all $m \geq 4$ and $n \geq 3$.
- (4) MAX, MIN, GEO, and IM are Pareto-optimal when $m = 2$, but fail score-unanimity for all $m \geq 3$ and $n \geq 2$.

Proof. We prove each of the claims separately.

Claim 1: The claim follows directly from the definitions of UTIL and EGAL, since a Pareto improvement would also give rise to an improvement with respect to the welfare measure. In more detail, for EGAL, we use the fact that its definition lexicographically minimizes the maximum disutility of the agents. In particular, if an outcome \mathbf{x} returned by EGAL was Pareto-dominated by another outcome \mathbf{x}' , then \mathbf{x}' would have a better egalitarian welfare in this leximin optimization, which contradicts the definition of EGAL. By Proposition 2.8, both rules are range-respecting and score-unanimous as well.

Claim 2: Let \mathbf{x} be the output of AVG for some instance \mathcal{I} . It holds for all $j \in [m]$ that $\min_{i \in N} s_{i,j} \leq \frac{1}{n} \sum_{i \in N} s_{i,j} \leq \max_{i \in N} s_{i,j}$, which shows that AVG is range-respecting. By Claim 2 of Proposition 2.8, this also means that AVG is Pareto-optimal if $m = 2$ or $n = 2$.

To show that AVG fails Pareto-optimality when $m \geq 3$ and $n \geq 3$, consider the following instance \mathcal{I}^1 , where all candidates in $C \setminus \{c_1, c_2, c_3\}$ receive probability 0 from all agents and can thus be ignored. For this instance, AVG outputs the vector $(\frac{1}{2n}, \frac{2}{2n}, 1 - \frac{3}{2n})$. However, the outcome $\mathbf{x}' = (0, \frac{3}{2n}, 1 - \frac{3}{2n})$ decreases the disutility of agent 1 without decreasing the disutility of the other agents, so AVG is not Pareto-optimal.

\mathcal{I}^1	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{2}$	0
$i \in \{3, \dots, n\}$	0	0	1

Claim 3: Since MED is equivalent to AVG when $n = 2$, it is range-respecting for $n = 2$ by Claim 2, and also Pareto-optimal by Claim 2 of Proposition 2.8. For the case $m = 2$, consider an instance \mathcal{I} and assume without loss of generality that $s_{1,1} \leq s_{2,1} \leq \dots \leq s_{n,1}$. Since $m = 2$, we infer that $s_{i,2} = 1 - s_{i,1}$, so $s_{1,2} \geq s_{2,2} \geq \dots \geq s_{n,2}$. Hence, MED returns $\mathbf{x} = (s_{(n+1)/2,1}, s_{(n+1)/2,2})$ if n is odd, and $\mathbf{x} = (\frac{s_{n/2,1} + s_{(n+2)/2,1}}{2}, \frac{s_{n/2,2} + s_{(n+2)/2,2}}{2})$ if n is even. This verifies that MED is range-respecting for $m = 2$, and Claim 2 of Proposition 2.8 entails that it is also Pareto-optimal.

Now, when $m = 3$, suppose for contradiction that there exists an instance \mathcal{I} such that the outcome \mathbf{x} chosen by MED fails range-respect. Without loss of generality, we assume that $x_1 < \min_{i \in N} s_{i,1}$ (the choice of the candidate does not matter, and if $x_1 > \max_{i \in N} s_{i,1}$, we can reverse all inequalities in the proof). Moreover, let m_1, m_2, m_3 denote the medians (before normalization) for the candidates c_1, c_2, c_3 , respectively. This means that $\mathbf{x} = (\frac{m_1}{m_1+m_2+m_3}, \frac{m_2}{m_1+m_2+m_3}, \frac{m_3}{m_1+m_2+m_3})$. Since $m_1 \geq \min_{i \in N} s_{i,1}$, we infer from $x_1 < \min_{i \in N} s_{i,1}$ that $m_1 + m_2 + m_3 > 1$. Moreover, it holds that $s_{i,2} + s_{i,3} = 1 - s_{i,1} < 1 - \frac{m_1}{m_1+m_2+m_3}$ for all agents $i \in N$. We will next show that $m_2 + m_3 \leq 1 - \min_{i \in N} s_{i,1}$. This then implies that $\frac{m_1+m_2+m_3}{m_1+m_2+m_3} < m_2 + m_3 + \frac{m_1}{m_1+m_2+m_3} < 1$, which yields the desired contradiction.

To show that $m_2 + m_3 \leq 1 - \min_{i \in N} s_{i,1}$, we note that each median by itself is monotone (i.e., if we increase some value $s_{i,j}$, then m_j does not decrease). Hence, we consider modified values $\hat{s}_{i,2}$ and $\hat{s}_{i,3}$ which satisfy $\hat{s}_{i,2} \geq s_{i,2}$, $\hat{s}_{i,3} \geq s_{i,3}$, and $\hat{s}_{i,2} + \hat{s}_{i,3} = 1 - \min_{i' \in N} s_{i',1}$ for all $i \in N$. By the monotonicity of the medians, we have that the corresponding medians \hat{m}_2 and \hat{m}_3 satisfy $\hat{m}_2 \geq m_2$ and $\hat{m}_3 \geq m_3$. Moreover, it holds that $\hat{s}_{i,2} = 1 - \min_{i' \in N} s_{i',1} - \hat{s}_{i,3}$. This means that, for all agents i and i' , we have $\hat{s}_{i,2} \leq \hat{s}_{i',2}$ if and only if $\hat{s}_{i,3} \geq \hat{s}_{i',3}$. Consequently, the median agent(s) for c_2 are also the median agent(s) for c_3 , which implies that $\hat{m}_2 + \hat{m}_3 = 1 - \min_{i \in N} s_{i,1}$. This completes the proof of our auxiliary claim.

Next, we prove that MED satisfies Pareto-optimality for $m = 3$ and any odd $n \geq 3$. Let m_1, m_2, m_3 again denote the medians (before normalization) for the candidates c_1, c_2, c_3 , respectively. Without loss of generality, assume that $m_1 + m_2 + m_3 \leq 1$. For $k \in \{1, 2, 3\}$, let $T_k \subseteq N$ be the set of agents i such that $s_{i,k} \geq m_k$ and $s_{i,k'} \leq m_{k'}$ for all other $k' \neq k$. Fix some $k \in \{1, 2, 3\}$. We first show that $T_k \neq \emptyset$. Suppose for contradiction that $T_k = \emptyset$, i.e., for every agent $i \in N$ such that $s_{i,k} \geq m_k$, there exists $k' \neq k$ with $s_{i,k'} > m_{k'}$. Furthermore, for every agent $i' \in N$ with $s_{i',k} < m_k$, there also exists $k' \neq k$ with $s_{i',k'} > m_{k'}$; otherwise $s_{i',1} + s_{i',2} + s_{i',3} < m_1 + m_2 + m_3 \leq 1$, a contradiction. Thus, for all agents $i \in N$, there exists $k' \neq k$ with $s_{i,k'} > m_{k'}$. However, since $\sum_{k' \neq k} |\{i \in N : s_{i,k'} > m_{k'}\}| \leq \frac{n-1}{2} + \frac{n-1}{2} = n-1$, we get a contradiction. Hence, $T_k \neq \emptyset$.

Since $m_1 + m_2 + m_3 \leq 1$, for each $j \in \{1, 2, 3\}$, we have $x_j = \frac{m_j}{m_1+m_2+m_3} \geq m_j$. This means that for all $i \in T_k$ and $k' \neq k$, it holds that $s_{i,k'} \leq x_{k'}$. Together with the fact that $s_{i,1} + s_{i,2} + s_{i,3} = 1$, we must have that $s_{i,k} \geq x_k$. Then, in order to construct an outcome \mathbf{x}' such that $d_i(\mathbf{x}') \leq d_i(\mathbf{x})$ for every agent $i \in N$, we must have that $x'_k \geq x_k$; otherwise agent $i \in T_k$ will have a strictly higher disutility. However, since this holds for all $k \in \{1, 2, 3\}$, we get that $\mathbf{x}' = \mathbf{x}$, which means that \mathbf{x} is Pareto-optimal. The argument for the case where $m_1 + m_2 + m_3 > 1$ is symmetric, by reversing the signs (including the definition of T_k).

On the other hand, we show that MED fails Pareto-optimality for $m = 3$ and any even $n \geq 4$. To this end, consider the following instance \mathcal{I}^2 for an even number of agents n . In this instance, we have that $\text{med}_{i \in N} s_{i,1} = \frac{9}{20}$, $\text{med}_{i \in N} s_{i,2} = \frac{3}{10}$, and $\text{med}_{i \in N} s_{i,3} = \frac{3}{8}$, so MED returns the outcome

$\mathbf{x} = \left(\frac{4}{10}, \frac{4}{15}, \frac{1}{3}\right)$. However, the outcome $\left(\frac{4}{10}, \frac{5}{20}, \frac{7}{20}\right)$ strictly benefits agent 2 and does not hurt the other agents, so \mathbf{x} is not Pareto-optimal.

\mathcal{I}^2	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
1	$\frac{3}{10}$	$\frac{7}{20}$	$\frac{7}{20}$
2	$\frac{6}{10}$	0	$\frac{4}{10}$
$i \in \{3, \dots, \frac{n}{2} + 1\}$	$\frac{7}{10}$	$\frac{1}{4}$	$\frac{1}{20}$
$i \in \{\frac{n}{2} + 2, \dots, n\}$	$\frac{1}{4}$	$\frac{7}{20}$	$\frac{4}{10}$

Finally, we show that MED fails even score-unanimity when $m \geq 4$ and $n \geq 3$. To this end, consider the following instance \mathcal{I}^3 for an odd number of agents n , where all candidates c_j with $5 \leq j \leq m$ receive probability 0 from all agents. In this instance, we have that $\text{med}_{i \in N} s_{i,1} = \frac{3}{10}$, $\text{med}_{i \in N} s_{i,2} = \frac{5}{10}$, and $\text{med}_{i \in N} s_{i,3} = \text{med}_{i \in N} s_{i,4} = 0$, so MED assigns probability $\frac{3}{8}$ to c_1 . To obtain a counterexample for even n , we can duplicate agent 1 to get a similar counterexample.

\mathcal{I}^3	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$	$s_{i,4}$
1	$\frac{3}{10}$	$\frac{5}{10}$	0	$\frac{2}{10}$
$i \in \{2, \dots, \frac{n+1}{2}\}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{6}{10}$	0
$i \in \{\frac{n+3}{2}, \dots, n\}$	$\frac{3}{10}$	$\frac{7}{10}$	0	0

Claim 4: We consider each rule separately.

GEO: First, assume that $m = 2$. We will show that GEO is range-respecting and hence Pareto-optimal. To this end, fix some instance \mathcal{I} and let \mathbf{x} be the outcome returned by GEO. Without loss of generality, we focus on c_1 and show that $\min_{i \in N} s_{i,1} \leq x_1 \leq \max_{i \in N} s_{i,1}$. Let $y_j = \sqrt[n]{\prod_{i \in N} s_{i,j}}$ and observe that $\min_{i \in N} s_{i,j} \leq y_j \leq \max_{i \in N} s_{i,j}$. By multiplying with $(1 - \max_{i \in N} s_{i,1})$ and $(1 - \min_{i \in N} s_{i,1})$, this leads to the following inequalities:

$$y_1 \cdot (1 - \max_{i \in N} s_{i,1}) \leq \max_{i \in N} s_{i,1} \cdot (1 - \max_{i \in N} s_{i,1});$$

$$y_1 \cdot (1 - \min_{i \in N} s_{i,1}) \geq \min_{i \in N} s_{i,1} \cdot (1 - \min_{i \in N} s_{i,1}).$$

Equivalently, this means that

$$\frac{y_1}{y_1 + 1 - \max_{i \in N} s_{i,1}} \leq \max_{i \in N} s_{i,1} \quad \text{and} \quad \frac{y_1}{y_1 + 1 - \min_{i \in N} s_{i,1}} \geq \min_{i \in N} s_{i,1}.$$

Since $m = 2$, it holds that $1 - \min_{i \in N} s_{i,1} = \max_{i \in N} s_{i,2}$ and $1 - \max_{i \in N} s_{i,1} = \min_{i \in N} s_{i,2}$. Hence, we derive that

$$\min_{i \in N} s_{i,1} \leq \frac{y_1}{y_1 + \max_{i \in N} s_{i,2}} \leq \frac{y_1}{y_1 + y_2} \leq \frac{y_1}{y_1 + \min_{i \in N} s_{i,2}} \leq \max_{i \in N} s_{i,1}.$$

Because $x_1 = \frac{y_1}{y_1 + y_2}$, this shows that GEO is range-respecting if $m = 2$.

To see that GEO fails score-unanimity when $m \geq 3$ and $n \geq 2$, consider the following instance \mathcal{I}^4 . For this instance, GEO outputs the vector

$$\mathbf{x} = \left(\frac{1}{1 + 2^{\frac{1}{n}} + 2^{\frac{n-1}{n}}}, \frac{2^{\frac{1}{n}}}{1 + 2^{\frac{1}{n}} + 2^{\frac{n-1}{n}}}, \frac{2^{\frac{n-1}{n}}}{1 + 2^{\frac{1}{n}} + 2^{\frac{n-1}{n}}} \right).$$

This outcome fails score-unanimity as $2^{\frac{1}{n}} + 2^{\frac{n-1}{n}} < 3$ for all $n \geq 2$; this follows from the inequality $(2^{\frac{n-1}{n}} - 1)(2^{\frac{1}{n}} - 1) > 0$. To extend the counterexample to larger m , one can add candidates that receive a score of 0 from all agents.

\mathcal{I}^4	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$i \in \{2, \dots, n\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

MAX: Next, we consider MAX and first show that it is range-respecting (and hence Pareto-optimal) if $m = 2$. For this, we fix an instance \mathcal{I} and focus on candidate c_1 . We observe that $\max_{i \in N} s_{i,1} + \max_{i \in N} s_{i,2} \geq 1$, which implies that $x_1 = \frac{\max_{i \in N} s_{i,1}}{\max_{i \in N} s_{i,1} + \max_{i \in N} s_{i,2}} \leq \max_{i \in N} s_{i,1}$. For the other direction, we note that

$$\begin{aligned} \min_{i \in N} s_{i,1} (1 + \max_{i \in N} s_{i,1} - \min_{i \in N} s_{i,1}) &= \min_{i \in N} s_{i,1} + \min_{i \in N} s_{i,1} (\max_{i \in N} s_{i,1} - \min_{i \in N} s_{i,1}) \\ &\leq \min_{i \in N} s_{i,1} + (\max_{i \in N} s_{i,1} - \min_{i \in N} s_{i,1}) = \max_{i \in N} s_{i,1}. \end{aligned}$$

Since $\max_{i \in N} s_{i,2} = 1 - \min_{i \in N} s_{i,1}$ when $m = 2$, we derive that

$$\min_{i \in N} s_{i,1} \leq \frac{\max_{i \in N} s_{i,1}}{\max_{i \in N} s_{i,1} + \max_{i \in N} s_{i,2}} = x_1.$$

This shows that MAX is range-respecting if $m = 2$.

To show that MAX fails score-unanimity if $m \geq 3$ and $n \geq 2$, we consider the instance \mathcal{I}^4 which was used to show that GEO also fails score-unanimity. For this instance, MAX returns the outcome $\mathbf{x} = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$, which violates score-unanimity for c_1 .

MIN: We now turn to MIN and show that it is range-respecting and therefore Pareto-optimal if $m = 2$. Hence, we consider an instance \mathcal{I} and focus again on candidate c_1 . We first observe that $\min_{i \in N} s_{i,1} + \min_{i \in N} s_{i,2} \leq 1$, so $\min_{i \in N} s_{i,1} \leq \frac{\min_{i \in N} s_{i,1}}{\min_{i \in N} s_{i,1} + \min_{i \in N} s_{i,2}} = x_1$. For the other direction, we note analogously to MAX that

$$\begin{aligned} \max_{i \in N} s_{i,1} (1 + \min_{i \in N} s_{i,1} - \max_{i \in N} s_{i,1}) &= \max_{i \in N} s_{i,1} + \max_{i \in N} s_{i,1} (\min_{i \in N} s_{i,1} - \max_{i \in N} s_{i,1}) \\ &\geq \max_{i \in N} s_{i,1} + (\min_{i \in N} s_{i,1} - \max_{i \in N} s_{i,1}) = \min_{i \in N} s_{i,1}. \end{aligned}$$

Using that $\min_{i \in N} s_{i,2} = 1 - \max_{i \in N} s_{i,1}$, we thus have that

$$\max_{i \in N} s_{i,1} \geq \frac{\min_{i \in N} s_{i,1}}{\min_{i \in N} s_{i,1} + \min_{i \in N} s_{i,2}} = x_1.$$

This completes the proof that MIN is range-respecting.

Finally, we consider again the instance \mathcal{I}^4 to prove that MIN fails score-unanimity. MIN chooses for this instance the outcome $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which violates score-unanimity for c_1 .

IM: As the last rule, we consider IM and first show that it is range-respecting (and Pareto-optimal) when $m = 2$. Consider an instance \mathcal{I} and let \mathbf{x} be the output vector of IM. We first note that $x_j \leq \max_{i \in N} s_{i,j}$ for $j \in \{1, 2\}$ as one phantom is always at 0. Next, if $x_j < \min_{i \in N} s_{i,j}$ for some $j \in \{1, 2\}$, then it holds for $j' \neq j$ that $x_{j'} > \max_{i \in N} s_{i,j'}$, since $s_{i,j} = 1 - s_{i,j'}$ for all $i \in N$ and $x_j = 1 - x_{j'}$. However, this is impossible due to our previous observation, so we conclude that IM is range-respecting when $m = 2$.

To show that IM fails score-unanimity when $m \geq 3$ and $n \geq 2$, we consider the instance \mathcal{I}^5 shown below, where all candidates c_j with $j \geq 4$ receive a probability of 0 from all agents. For this instance, score-unanimity requires that $x_1 = \frac{n+1}{n+2}$. However, it can be checked that IM assigns probability $\frac{n}{n+2}$ to c_1 (at $t^* = \frac{1}{n+2}$), thus showing that score-unanimity is violated.

\mathcal{I}^5	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
1	$\frac{n+1}{n+2}$	$\frac{1}{n+2}$	0
$i \in \{2, \dots, n\}$	$\frac{n+1}{n+2}$	0	$\frac{1}{n+2}$

This completes the proof. \square

Remark 3.2. Freeman et al. [2021, p. 22] suggested a variant of IM where the last moving phantom is fixed to 1, i.e., $f_n^{\text{IM}'}(t) = 1$. This modified rule satisfies range-respect, single-minded proportionality, and strategyproofness. However, it is unclear whether it inherits other desirable properties of IM such as reinforcement and participation.

Remark 3.3. In some settings, it is computationally challenging to determine whether an outcome is efficient [e.g., Aziz et al., 2019]. This is not the case in our context: it is straightforward to check whether an outcome satisfies score-unanimity and range-respect, and we give a linear programming formulation for deciding whether an outcome is Pareto-optimal in the appendix (see Proposition A.2).

4 Fairness Properties

We next turn to fairness properties and study our rules with respect to single-minded proportionality and the more demanding notion of score-representation. In particular, we will show that none of our rules satisfies score-representation except AVG, thus making a strong case for this rule. Note that some of our results follow from the work of Freeman et al. [2021]: these authors have shown that IM satisfies single-minded proportionality and that UTIL fails this property (without specifying the boundaries for when this is the case).

Theorem 4.1. *The following claims hold.*

- (1) AVG satisfies score-representation (and therefore single-minded proportionality).
- (2) IM satisfies score-representation when $m = 2$ and single-minded proportionality for any m , but fails score-representation for all $n \geq 2$ and $m \geq 3$.
- (3) MAX and UTIL satisfy score-representation when $n = m = 2$ and single-minded proportionality when $n = 2$. Both rules violate score-representation for all $m \geq 3$ and $n \geq 2$, and single-minded proportionality for all $m \geq 2$ and $n \geq 3$.
- (4) MED and EGAL satisfy score-representation when $n = 2$, but fail single-minded proportionality for all $m \geq 2$ and $n \geq 3$.
- (5) MIN and GEO satisfy single-minded proportionality when $n = m = 2$, but fail to do so for all $n \geq 3$ or $m \geq 3$. Both rules fail score-representation for all $m \geq 2$ and $n \geq 2$.

Proof. We prove each of the claims separately.

Claim 1: Consider an instance \mathcal{I} , and fix a candidate c_j and a value $\gamma \in [0, 1]$. We need to show that $x_j \geq \gamma \cdot \frac{\mathcal{N}(\mathcal{I}, c_j, \gamma)}{n}$ for the vector \mathbf{x} chosen by AVG. For this, let $S := \{i \in N : s_{i,j} \geq \gamma\}$ denote the set of agents that report a score of at least γ for c_j . It holds that $x_j = \frac{1}{n} \sum_{i \in N} s_{i,j} \geq \frac{1}{n} \sum_{i \in S} s_{i,j} \geq \gamma \cdot \frac{\mathcal{N}(\mathcal{I}, c_j, \gamma)}{n}$, which shows that AVG satisfies score-representation.

Claim 2: First, it was shown by Freeman et al. [2021] that IM satisfies single-minded proportionality. Next, we show that IM also satisfies score-representation if $m = 2$. To this end, consider an instance \mathcal{I} and assume for contradiction that the outcome \mathbf{x} of IM fails score-representation. Without loss of generality, suppose that there exists $\gamma \in [0, 1]$ such that $x_1 < \gamma \cdot \frac{\mathcal{N}(\mathcal{I}, c_1, \gamma)}{n}$. Clearly, this means that $\gamma > x_1$. Now, consider the instance \mathcal{I}' such that all agents with $s_{i,1} > x_1$ report $s'_{i,1} = 1$. It holds for the modified instance \mathcal{I}' still that $x'_1 = x_1$ because increasing the score of c_1 for agents with $s_{i,1} > x_1$ (and simultaneously decreasing the score of c_2 for agents with $s_{i,2} < x_2$) does not change the position of the medians. Since $\gamma > x_1$, we have that $x_1 < \gamma \cdot \frac{\mathcal{N}(\mathcal{I}, c_1, \gamma)}{n} \leq \frac{\mathcal{N}(\mathcal{I}', c_1, 1)}{n}$. Next, let \mathcal{I}'' denote the instance derived from \mathcal{I}' by setting $s''_{i,1} = 0$ for all agents $i \in N$ with $s_{i,1} \leq x_1$. Since IM is known to be score-monotone [see Freeman et al., 2021, Thm. 3], it follows that $x''_1 \leq x'_1$. Moreover, it holds that $\mathcal{N}(\mathcal{I}', c_1, 1) = \mathcal{N}(\mathcal{I}'', c_1, 1)$, so we can again infer that $x''_1 < \frac{\mathcal{N}(\mathcal{I}'', c_1, 1)}{n}$. However, in \mathcal{I}'' , all agents are single-minded, so $x''_1 = \frac{\mathcal{N}(\mathcal{I}'', c_1, 1)}{n}$ because IM satisfies single-minded proportionality. This yields the desired contradiction, which means that IM satisfies score-representation for $m = 2$.

Finally, to show that IM fails score-representation for all $m \geq 3$ and $n \geq 2$, it suffices to consider the instance \mathcal{I}^5 in the proof of Theorem 3.1. In this instance, all agents assign probability $\frac{n+1}{n+2}$ to c_1 , but IM only assigns a probability of $\frac{n}{n+2}$ to this candidate, so score-representation is violated.

Claim 3: We consider MAX and UTIL separately.

MAX: We first prove that MAX satisfies score-representation when $n = m = 2$. Consider an instance \mathcal{I} and assume without loss of generality that $s_{1,1} \geq s_{2,1}$ and $s_{2,2} \geq s_{1,2}$. Then, score-representation demands that (i) $x_1 \geq \frac{s_{1,1}}{2}$ and $x_2 \geq \frac{s_{2,2}}{2}$, and (ii) $x_1 \geq s_{2,1}$ and $x_2 \geq s_{1,2}$. Property (ii) follows from the fact that MAX is range-respecting when $m = 2$ (see Theorem 3.1). On the other hand, $x_1 = \frac{s_{1,1}}{s_{1,1} + s_{2,2}} \geq \frac{s_{1,1}}{2}$ and $x_2 = \frac{s_{2,2}}{s_{1,1} + s_{2,2}} \geq \frac{s_{2,2}}{2}$ since $s_{i,j} \leq 1$ for all $i \in N, j \in [m]$. This demonstrates that (i) also holds.

Next, we show that MAX satisfies single-minded proportionality for $n = 2$ and $m \geq 3$. Let \mathcal{I} denote an instance where both agents are single-minded. If both agents put probability 1 on the same candidate, then MAX puts probability 1 on this candidate, too. If the two agents put probability 1 on different candidates, both of these candidates receive a probability of $\frac{1}{2}$ from MAX. In both cases, MAX satisfies single-minded proportionality.

To show that MAX fails score-representation when $n \geq 2$ and $m \geq 3$, it is sufficient to consider our counterexample showing that it fails score-unanimity (instance \mathcal{I}^4 in the proof of Theorem 3.1). In this instance, all agents assign probability $\frac{1}{4}$ to candidate c_1 , but MAX only assigns probability $\frac{1}{5}$ to this candidate. We now show that MAX fails single-minded proportionality if $m \geq 2$ and $n \geq 3$. Consider the following instance \mathcal{I}^1 (where $s_{i,j} = 0$ for all $j \geq 3$ and $i \in N$). In this instance, MAX assigns probability $\frac{1}{2}$ to both c_1 and c_2 , but single-minded proportionality requires that $x_2 = \frac{n-1}{n} > \frac{1}{2}$.

\mathcal{I}^1	$s_{i,1}$	$s_{i,2}$
1	1	0
$i \in \{2, \dots, n\}$	0	1

UTIL: We first show that UTIL satisfies score-representation when $n = m = 2$ and thus consider an instance \mathcal{I} . Just as for MAX, we assume that $s_{1,1} \geq s_{2,1}$ and $s_{2,2} \geq s_{1,2}$, and we will prove that (i) $x_1 \geq \frac{s_{1,1}}{2}$ and $x_2 \geq \frac{s_{2,2}}{2}$, and (ii) $x_1 \geq s_{2,1}$ and $x_2 \geq s_{1,2}$. Property (ii) follows immediately as UTIL is Pareto-optimal and therefore also range-respecting. As for (i), if $s_{1,1} \geq \frac{1}{2} \geq s_{1,2}$ and $s_{2,2} \geq \frac{1}{2} \geq s_{2,1}$, then UTIL will pick the vector $(\frac{1}{2}, \frac{1}{2})$ and score-representation is satisfied. Now, if $s_{1,1} < \frac{1}{2}$, then also $s_{2,1} < \frac{1}{2}$ since $s_{2,1} \leq s_{1,1}$. In this case, UTIL will choose the outcome $(s_{1,1}, s_{1,2})$, for which it holds that $x_1 \geq \frac{s_{1,1}}{2}$ and $x_2 \geq \frac{s_{2,2}}{2}$ since $x_2 = s_{1,2} > \frac{1}{2}$. Lastly, if $s_{2,2} < \frac{1}{2}$, a symmetric argument applies, so UTIL indeed satisfies score-representation if $n = m = 2$.

Next, an analogous argument to that for MAX shows that UTIL satisfies single-minded proportionality when $n = 2$.

Finally, we prove that UTIL fails score-representation when $m \geq 3$ and $n \geq 2$, and single-minded proportionality when $n \geq 3$. For the latter, it suffices to consider the instance \mathcal{I}^1 shown for MAX. In this instance, UTIL will assign probability 1 to c_2 , which violates single-minded proportionality as c_1 deserves probability $\frac{1}{n}$. It remains to show that UTIL fails score-representation when $n = 2$ and $m \geq 3$. To this end, consider the following instance \mathcal{I}^2 , where UTIL chooses the vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This vector fails score-representation since the axiom requires x_1 to be at least $\frac{1}{2}$. As usual, the counterexample can be extended to larger m by adding candidates that receive probability 0 from all agents.

\mathcal{I}^2	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
1	1	0	0
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Claim 4: We note that if $n = 2$, then MED and EGAL coincide with AVG, so they satisfy score-representation in this case due to Claim 1. By contrast, if $n \geq 3$, we consider the instance \mathcal{I}^1 used to show that MAX fails single-minded proportionality. For this instance, MED returns the outcome $\mathbf{x} = (0, 1)$ while EGAL returns the outcome $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$. Both of these outcomes fail single-minded proportionality.

Claim 5: We first prove that MIN and GEO satisfy single-minded proportionality when $n = m = 2$. There are two cases to consider. If both agents give a score of 1 to the same candidate, it will be assigned a score of 1 by both rules. On the other hand, if both agents give a score of 1 to different candidates, then both rules return $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$, which also satisfies single-minded proportionality.

To see that MIN and GEO fail single-minded proportionality for the case where $n \geq 3$ or $m \geq 3$, we again consider the instance \mathcal{I}^1 used to show that MAX fails single-minded proportionality (recall that all candidates c_j with $j \geq 3$ receive probability 0 from all agents). In this instance, both rules assign probability $\frac{1}{m}$ to all candidates, which fails single-minded proportionality unless $m = n = 2$. Finally, to see that these rules also fail score-representation for the case $n = m = 2$, consider the instance \mathcal{I} with $\mathbf{s}_1 = (1, 0)$ and $\mathbf{s}_2 = (\frac{1}{2}, \frac{1}{2})$. Then, score-representation mandates that $x_2 \geq \frac{1}{4}$, but both rules return $\mathbf{x} = (1, 0)$. \square

Remark 4.2. In the instance used for showing that MIN and GEO fail single-minded proportionality, it holds that $\min_{i \in N} s_{i,j} = 0$ for all $j \in [m]$. This is unavoidable due the definition of single-minded proportionality. Because $\min_{i \in N} s_{i,j} = 0$ for all $j \in [m]$, both rules assign a probability of $\frac{1}{m}$ to every candidate as we defined that $\frac{0}{0} = \frac{1}{m}$. However, note that single-minded proportionality will be violated regardless of how we define $\frac{0}{0}$, so it is still true that our negative results do not depend on this particular assumption.

5 Consistency Properties

In this section, we analyze our aggregation rules with respect to various consistency notions. In particular, we will show that score-monotonicity and reinforcement are satisfied by almost all of our rules, whereas AVG is the only one that fulfills independence. We note that Freeman et al. [2021] have already shown that IM and UTIL satisfy reinforcement and score-monotonicity.

Theorem 5.1. *The following claims hold.*

- (1) *AVG satisfies independence. MED and EGAL satisfy independence when $m = 2$ or $n = 2$ but fail this condition for all $m \geq 3$ and $n \geq 3$. MAX, MIN, GEO, UTIL, and IM satisfy independence when $m = 2$ but fail to do so for all $m \geq 3$ and $n \geq 2$.*

- (2) All five coordinate-wise aggregation rules, UTIL, and IM satisfy score-monotonicity. EGAL is score-monotone when $m = 2$ or $n = 2$ but fails to be so for all $m \geq 4$ and $n \geq 4$.
- (3) AVG, MIN, MAX, and GEO as well as UTIL, EGAL, and IM satisfy reinforcement. MED satisfies reinforcement when $m = 2$ but fails to do so for all $m \geq 3$.

We remark that while the bounds on m and n are tight for almost all results in our paper, it remains open whether EGAL satisfies score-monotonicity when $m = 3$ or $n = 3$.

Proof. We prove each of the claims separately.

Claim 1: We consider independence and first note that this axiom is trivial when $m = 2$: if $s_{i,j} = s'_{i,j}$ for some candidate c_j and all $i \in N$, then $\mathcal{I} = \mathcal{I}'$, which means that every aggregation rule F satisfies independence. Moreover, if $n = 2$, then MED and EGAL coincide per definition with AVG. Hence, all of our positive claims follow by showing that AVG satisfies independence. We show this next and then explain why the other rules fail independence for the given values of m and n .

AVG: Consider two instances \mathcal{I} and \mathcal{I}' and a candidate c_{j^*} such that $s_{i,j^*} = s'_{i,j^*}$ for all $i \in N$. This clearly means that $\frac{1}{n} \sum_{i \in N} s_{i,j^*} = \frac{1}{n} \sum_{i \in N} s'_{i,j^*}$. Moreover, it holds that $\sum_{j \in [m]} \left(\frac{1}{n} \sum_{i \in N} s_{i,j} \right) = \frac{1}{n} \sum_{i \in N} \sum_{j \in [m]} s_{i,j} = 1$ and $\sum_{j \in [m]} \left(\frac{1}{n} \sum_{i \in N} s'_{i,j} \right) = \frac{1}{n} \sum_{i \in N} \sum_{j \in [m]} s'_{i,j} = 1$. This implies that AVG returns the same outcome for c_j in \mathcal{I} and \mathcal{I}' .

MAX, MIN, GEO, and IM: To show that all of these rules fail independence, we observe that they satisfy score-unanimity in the special case where there are constants γ_j for $j \in [m]$ such that $s_{i,j} = \gamma_j$ for all $j \in [m]$ and $i \in N$. However, since MAX, MIN, GEO, and IM fail score-unanimity in general when $m \geq 3$ and $n \geq 2$ (see Theorem 3.1), this immediately means that independence is violated for these cases.

UTIL: First, if $m \geq 3$ and $n \geq 3$ is odd, consider the following instances \mathcal{I} and \mathcal{I}' : in \mathcal{I} , $\frac{n-1}{2}$ agents assign probability 1 to c_1 , $\frac{n-1}{2}$ agents assign probability 1 to c_2 , and 1 agent assigns probability 1 to c_3 . In \mathcal{I}' , the same agents assign probability 1 to c_1 as in \mathcal{I} , but all remaining agents assign probability 1 to c_2 . If $n = 3$, then $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, while if $n \geq 5$, then $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, 0)$. On the other hand, $\mathbf{x}' = (0, 1, 0)$, thus showing that independence fails for c_1 .

Next, if $m \geq 3$ and $n \geq 4$ is even, we consider the instance \mathcal{I} , where $\frac{n}{2}$ agents assign probability 1 to c_1 , $\frac{n}{2} - 1$ agents assign probability 1 to c_2 , and 1 agent assigns probability 1 to c_3 , and \mathcal{I}' , where the same $\frac{n}{2}$ as in \mathcal{I} assign probability 1 to c_1 and the other $\frac{n}{2}$ agents assign probability 1 to c_2 . UTIL put probability 1 on c_1 in \mathcal{I} , but probability $\frac{1}{2}$ on both c_1 and c_2 in \mathcal{I}' , so independence is violated for c_1 .

Finally, if $n = 2$ and $m \geq 3$, we consider the instance \mathcal{I} , where agent 1 puts probability 1 on c_1 and agent 2 put probability 1 on c_2 , and \mathcal{I}' , where agent 1 still puts probability 1 on c_1 but agent 2 puts probability $\frac{1}{2}$ on both c_2 and c_3 . UTIL assigns a probability of $\frac{1}{2}$ to both c_1 and c_2 in \mathcal{I} , and a probability of $\frac{1}{3}$ to each of c_1 , c_2 , and c_3 in \mathcal{I}' due to the maximum entropy tie-breaking. Hence, we have again that independence is violated for c_1 .

EGAL: To show that EGAL fails independence for all $m \geq 3$ and $n \geq 3$, we consider the instances \mathcal{I} and \mathcal{I}' defined as follows: in \mathcal{I} , agent 1 assigns probability 1 to c_1 , agent 2 assigns probability 1 to c_2 , and every other agent assigns probability 1 to c_3 . Furthermore, in \mathcal{I}' , agent 1 assigns probability 1 to c_1 and every other agent assigns probability 1 to c_2 . It is easy to check that EGAL chooses the outcome $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for \mathcal{I} and the outcome $\mathbf{x}' = (\frac{1}{2}, \frac{1}{2}, 0)$ for \mathcal{I}' , thus showing that independence is violated for c_1 . As usual, we can extend the instances to larger m by adding candidates that are unanimously assigned probability 0.

MED: Finally, we turn to MED and consider the following instances \mathcal{I}^1 and \mathcal{I}^2 , where $s_{i,j}^1 = s_{i,j}^2 = 0$ for all $j \geq 4$ and $i \in N$. It holds that $\text{med}_{i \in N} s_{i,1}^1 = \frac{1}{2}$, $\text{med}_{i \in N} s_{i,2}^1 = \frac{1}{2}$ if n is odd and $\text{med}_{i \in N} s_{i,2}^1 = \frac{1}{4}$ if n is even, and $\text{med}_{i \in N} s_{i,3}^1 = \frac{1}{2}$. This means that MED assigns a probability of less than $\frac{1}{2}$ to c_1 in \mathcal{I} . By contrast, in \mathcal{I}^2 , the medians are $\text{med}_{i \in N} s_{i,1}^2 = \frac{1}{2}$, $\text{med}_{i \in N} s_{i,2}^2 = 0$, and $\text{med}_{i \in N} s_{i,3}^2 = \frac{1}{2}$. Hence, MED assigns a probability of $\frac{1}{2}$ to c_1 in this instance, and independence is violated.

\mathcal{I}^1	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$	\mathcal{I}^2	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$
$i \in \{2, \dots, \lceil \frac{n+1}{2} \rceil\}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$i \in \{2, \dots, \lceil \frac{n+1}{2} \rceil\}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$i \in \{\lceil \frac{n+1}{2} \rceil + 1, \dots, n\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$i \in \{\lceil \frac{n+1}{2} \rceil + 1, \dots, n\}$	0	$\frac{1}{2}$	$\frac{1}{2}$

Claim 2: We first note that Theorem 3 of Freeman et al. [2021] shows that IM and UTIL are score-monotone. We next deal with the coordinate-wise aggregation rules and EGAL separately.

Coordinate-wise aggregation rules: To show that our five coordinate-wise aggregation rules satisfy score-monotonicity, we first prove a more general auxiliary claim: a coordinate-wise aggregation rule F is score-monotone if all coordinate-aggregation functions f_j satisfy that $f_j(s_{1,j}, \dots, s_{i,j}, \dots, s_{n,j}) \geq f_j(s_{1,j}, \dots, s'_{i,j}, \dots, s_{n,j})$ for all candidates $c_j \in C$ and scores $s_{1,j}, \dots, s_{n,j}, s'_{i,j}$ such that $s_{i,j} \geq s'_{i,j}$. To see this, consider two instances \mathcal{I} and \mathcal{I}' , an agent $i \in N$, and a candidate c_j such that $s_{i'} = s'_{i'}$ for all $i' \in N \setminus \{i\}$, $s_{i,j} > s'_{i,j}$, and $s_{i',j'} \leq s'_{i',j'}$ for all $c_{j'} \in C \setminus \{c_j\}$. For ease of presentation, we will slightly abuse notation and write $f_\ell(\mathcal{I})$ to mean $f_\ell(s_{1,\ell}, \dots, s_{n,\ell})$.

We first consider the case that $f_{j'}(\mathcal{I}) = 0$ for all $j' \in [m]$ or that $f_{j'}(\mathcal{I}') = 0$ for all $j' \in [m]$. If $f_{j'}(\mathcal{I}) = f_{j'}(\mathcal{I}') = 0$ for all $j' \in [m]$, then $F(\mathcal{I})_j = F(\mathcal{I}')_j = \frac{1}{m}$ by definition and score-monotonicity holds. Next, if $f_{j'}(\mathcal{I}) = 0$ for all $j' \in [m]$ but there is some ℓ with $f_\ell(\mathcal{I}') > 0$, then $\ell \neq j$ because $s_{i,j} > s'_{i,j}$ implies that $f_j(\mathcal{I}) \geq f_j(\mathcal{I}')$. Hence, we have in this case that $F(\mathcal{I})_j = \frac{1}{m} > 0 = F(\mathcal{I}')_j$ and score-monotonicity holds. As the third case, assume that $f_{j'}(\mathcal{I}') = 0$ for all $j' \in [m]$ but there is some ℓ with $f_\ell(\mathcal{I}) > 0$. Since $f_{j'}(\mathcal{I}) \leq f_{j'}(\mathcal{I}')$ for all $j' \in [m] \setminus \{j\}$, it must be that $\ell = j$. This implies that $F(\mathcal{I})_j = 1 > \frac{1}{m} = F(\mathcal{I}')_j$ and score-monotonicity holds again. As the last case, suppose that neither $f_{j'}(\mathcal{I}) = 0$ for all $j' \in [m]$ nor $f_{j'}(\mathcal{I}') = 0$ for all $j' \in [m]$. Our assumptions imply that $f_j(\mathcal{I}) \geq f_j(\mathcal{I}')$ and $f_{j'}(\mathcal{I}) \leq f_{j'}(\mathcal{I}')$ for all $j' \in [m] \setminus \{j\}$. It follows that

$$F(\mathcal{I})_j = \frac{f_j(\mathcal{I})}{\sum_{j' \in [m]} f_{j'}(\mathcal{I})} \geq \frac{f_j(\mathcal{I}')}{f_j(\mathcal{I}') + \sum_{j' \in [m] \setminus \{j\}} f_{j'}(\mathcal{I})} \geq \frac{f_j(\mathcal{I}')}{\sum_{j' \in [m]} f_{j'}(\mathcal{I}')} = F(\mathcal{I}')_j.$$

This shows that our condition indeed implies score-monotonicity. Finally, for each of AVG, MAX, MIN, GEO, and MED, it is easy to verify that $f_j(s_{1,j}, \dots, s_{i,j}, \dots, s_{n,j}) \geq f_j(s_{1,j}, \dots, s'_{i,j}, \dots, s_{n,j})$ for all $j \in [m]$ and $s_{i,j}, s'_{i,j}$ with $s_{i,j} \geq s'_{i,j}$, so these rules are indeed score-monotone.

EGAL: When $n = 2$, EGAL is equivalent to AVG (by Proposition 2.3), for which we have just shown that it satisfies score-monotonicity. Thus, we only consider the case $m = 2$ and assume that $n \geq 3$. Let \mathcal{I} and \mathcal{I}' denote instances such that \mathcal{I}' is identical to \mathcal{I} except that $s_{i,1} < s'_{i,1}$ and $s_{i,2} > s'_{i,2}$ for some agent $i \in N$. Moreover, suppose that \mathbf{x} and \mathbf{x}' are the outcome vectors returned by EGAL for instances \mathcal{I} and \mathcal{I}' , respectively. Without loss of generality, we assume that the agents in \mathcal{I} are ordered such that $s_{1,1} \leq \dots \leq s_{n,1}$ and $s_{1,2} \geq \dots \geq s_{n,2}$. Then, EGAL will return $\mathbf{x} = \left(\frac{s_{1,1} + s_{n,1}}{2}, \frac{s_{n,2} + s_{1,2}}{2} \right)$. For the instance \mathcal{I}' , we have $\min_{i \in N} s'_{i,1} \geq s_{1,1}$ and $\max_{i \in N} s'_{i,1} \geq s_{n,1}$. So $x_1 = \frac{s_{1,1} + s_{n,1}}{2} \leq \frac{\min_{i \in N} s'_{i,1} + \max_{i \in N} s'_{i,1}}{2} = x'_1$, which means that score-monotonicity is satisfied.

Next, we prove that when $m \geq 4$ and $n \geq 4$, EGAL fails score-monotonicity. As usual, we will show our counterexample only for $m = 4$ candidates as it is straightforward to add dummy candidates.

Moreover, we focus on the case $n = 4$ as EGAL is invariant under duplicating agents. Now, consider the following two instances \mathcal{I}^3 and \mathcal{I}^4 with their respective EGAL outcomes \mathbf{x}^3 and \mathbf{x}^4 . In \mathcal{I}^3 , all agents get disutility 1 under \mathbf{x}^3 . Thus, to show that EGAL indeed returns \mathbf{x}^3 , it suffices to show that any other outcome would increase the disutility of at least one agent. If the share of the first candidate decreases, agent 1 would be worse off. Similarly, if x_4^3 decreases, agent 3 or 4 would be worse off, so \mathbf{x}^3 is returned in \mathcal{I}^3 . Turning to \mathcal{I}^4 , note that $d_i(\mathbf{x}^4) = 4/5$ for all agents i . For any distribution \mathbf{x} with $d_1(\mathbf{x}) \leq 4/5$, we have $x_1 + x_4 \geq 3/5$. Analogously, we get $x_2 + x_4 \geq 3/5$ and $x_3 + x_4 \geq 3/5$ from agents 3 and 4, respectively. Summing up these three inequalities and using the fact that \mathbf{x} is a distribution shows that $x_4 \geq 2/5$. If $x_4 > 2/5$ then $d_2(\mathbf{x}) > 4/5$, so in order to have $d_2(\mathbf{x}) \leq 4/5$, we must have $x_4 = 2/5$. By the first three inequalities, the remaining probability of $3/5$ needs to be distributed uniformly over the first three candidates, leading to $\mathbf{x} = \mathbf{x}^4$. Hence, \mathbf{x}^4 is indeed returned in \mathcal{I}^4 .

We see that $x_4^3 = \frac{1}{2} > \frac{2}{5} = x_4^4$ despite the fact that $s_{1,4}^3 < s_{1,4}^4$. It follows that score-monotonicity is violated.

\mathcal{I}^3	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$	$s_{i,4}$	\mathcal{I}^4	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$	$s_{i,4}$
1	1	0	0	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0
3	0	$\frac{1}{2}$	0	$\frac{1}{2}$	3	0	$\frac{1}{2}$	0	$\frac{1}{2}$
4	0	0	$\frac{1}{2}$	$\frac{1}{2}$	4	0	0	$\frac{1}{2}$	$\frac{1}{2}$
\mathbf{x}^3	$\frac{1}{2}$	0	0	$\frac{1}{2}$	\mathbf{x}^4	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

Claim 3: We next show that among all the rules considered in this paper, only MED fails reinforcement. For this, we first note that Freeman et al. [2021, Thms. 9 and 13] have shown that IM and UTIL satisfy reinforcement. We hence focus on the remaining rules and consider three instances $\mathcal{I} = (s_1, \dots, s_n)$ (defined for the electorate N), $\mathcal{I}' = (s'_1, \dots, s'_{n'})$ (defined for the electorate N'), and $\mathcal{I}'' = (s_1, \dots, s_n, s'_1, \dots, s'_{n'})$ (defined for the electorate $N \cup N'$; we suppose that $N \cap N' = \emptyset$). We consider each rule separately.

AVG: Suppose that \mathbf{x} , \mathbf{x}' , and \mathbf{x}'' are the outcomes chosen by AVG for \mathcal{I} , \mathcal{I}' , and \mathcal{I}'' , respectively. We observe that $\sum_{j \in [m]} \left(\frac{1}{|N|} \sum_{i \in N} s_{i,j} \right) = \sum_{j \in [m]} \left(\frac{1}{|N'|} \sum_{i \in N'} s'_{i,j} \right) = 1$. Hence, if $x_j = x'_j$ for some $j \in [m]$, then $\frac{1}{|N|} \sum_{i \in N} s_{i,j} = \frac{1}{|N'|} \sum_{i \in N'} s'_{i,j}$. Consequently,

$$x''_j = \frac{1}{|N \cup N'|} \sum_{i \in N \cup N'} s''_{i,j} = \frac{|N|}{|N \cup N'|} \cdot \frac{1}{|N|} \sum_{i \in N} s_{i,j} + \frac{|N'|}{|N \cup N'|} \cdot \frac{1}{|N'|} \sum_{i \in N'} s'_{i,j} = x_j,$$

which shows that reinforcement is satisfied.

MAX: Let \mathbf{x} , \mathbf{x}' , and \mathbf{x}'' denote the outcomes chosen by MAX for the respective instances and assume that $\mathbf{x} = \mathbf{x}'$. This means that there is a constant $\alpha > 0$ such that $\max_{i \in N} s_{i,j} = \alpha \cdot \max_{i \in N'} s'_{i,j}$ for all $j \in [m]$. Moreover, we assume that $\alpha \geq 1$; otherwise we can exchange the roles of \mathbf{x} and \mathbf{x}' . Consequently, $\max_{i \in N \cup N'} s''_{i,j} = \max_{i \in N} s_{i,j}$ for all $j \in [m]$, which shows that $\mathbf{x}'' = \mathbf{x}$.

MIN: Denote by \mathbf{x} , \mathbf{x}' , and \mathbf{x}'' the outcomes chosen by MIN for the respective instances and assume that $\mathbf{x} = \mathbf{x}'$. First, if $\min_{i \in N} s_{i,j} = 0$ for all $j \in [m]$ or $\min_{i \in N'} s'_{i,j} = 0$ for all $j \in [m]$, then $\min_{i \in N \cup N'} s''_{i,j} = 0$ for all $j \in [m]$ and $x''_j = \frac{1}{m} = x_j = x'_j$ for all $j \in [m]$. Next, assume that $\min_{i \in N} s_{i,j} > 0$ and $\min_{i \in N'} s'_{i,j} > 0$ for some $j, j' \in [m]$. Then, $\mathbf{x} = \mathbf{x}'$ implies that there is a constant $\alpha > 0$ such that $\min_{i \in N} s_{i,j} = \alpha \cdot \min_{i \in N'} s'_{i,j}$ for all $j \in [m]$. We assume without loss of generality that $\alpha \leq 1$ as we can otherwise exchange the roles of \mathbf{x} and \mathbf{x}' in our argument. This assumption means

that $\min_{i \in N} s_{i,j} \leq \min_{i \in N'} s'_{i,j}$ for all $j \in [m]$ and hence $\min_{i \in N \cup N'} s''_{i,j} = \min_{i \in N} s_{i,j}$ for all $j \in [m]$. This proves that $\mathbf{x}'' = \mathbf{x}$ and MIN therefore satisfies reinforcement.

GEO: Define \mathbf{x} , \mathbf{x}' , and \mathbf{x}'' as the outcomes selected by GEO for the respective instances and assume that $\mathbf{x} = \mathbf{x}'$. If $\min_{i \in N} s_{i,j} = 0$ for all $j \in [m]$ or $\min_{i \in N'} s'_{i,j} = 0$ for all $j \in [m]$, the same holds for \mathcal{I}'' and we derive that $x''_j = \frac{1}{m} = x_j = x'_j$ for all $j \in [m]$. Hence, suppose that there are indices $j, j' \in [m]$ such that $\min_{i \in N} s_{i,j} > 0$ and $\min_{i \in N'} s'_{i,j'} > 0$. Next, we define $Z(\mathcal{I}) = \sum_{j \in [m]} (\prod_{i \in N} s_{i,j})^{1/|N|}$ and $Z(\mathcal{I}') = \sum_{j \in [m]} (\prod_{i \in N'} s'_{i,j})^{1/|N'|}$. For each $j \in [m]$, we have that

$$\begin{aligned} \left(\prod_{i \in N \cup N'} s''_{i,j} \right)^{\frac{1}{|N|+|N'|}} &= \left(\prod_{i \in N} s_{i,j} \cdot \prod_{i \in N'} s'_{i,j} \right)^{\frac{1}{|N|+|N'|}} \\ &= \left((x_j \cdot Z(\mathcal{I}))^{|N|} \cdot (x'_j \cdot Z(\mathcal{I}'))^{|N'|} \right)^{\frac{1}{|N|+|N'|}} \\ &= x_j \cdot \left(Z(\mathcal{I})^{|N|} \cdot Z(\mathcal{I}')^{|N'|} \right)^{\frac{1}{|N|+|N'|}}. \end{aligned}$$

Since this holds for all $j \in [m]$, GEO satisfies reinforcement because

$$x''_j = \frac{x_j \cdot \left(Z(\mathcal{I})^{|N|} \cdot Z(\mathcal{I}')^{|N'|} \right)^{\frac{1}{|N|+|N'|}}}{\sum_{j' \in [m]} \left(x_{j'} \cdot \left(Z(\mathcal{I})^{|N|} \cdot Z(\mathcal{I}')^{|N'|} \right)^{\frac{1}{|N|+|N'|}} \right)} = x_j.$$

EGAL: Let \mathbf{x} , \mathbf{x}' , and \mathbf{x}'' denote the outcomes chosen by EGAL for \mathcal{I} , \mathcal{I}' , and \mathcal{I}'' , respectively. Assume for contradiction that $\mathbf{x} = \mathbf{x}'$ but $\mathbf{x}'' \neq \mathbf{x}$. For a set of agents T , we define $v_T(\mathbf{y})$ as the vector that states the disutilities of all agents $i \in X$ for the outcome \mathbf{y} in non-increasing order. Moreover, we write $v >_{lex} v'$ if there is an integer k such that $v_\ell = v'_\ell$ for all $\ell \in [k-1]$ and $v_k > v'_k$. Since $\mathbf{x} = \mathbf{x}'$ is chosen for \mathcal{I} and \mathcal{I}' , we get that $v_N(\mathbf{x}) <_{lex} v_N(\mathbf{x}'')$ or $v_N(\mathbf{x}) = v_N(\mathbf{x}'')$, and $v_{N'}(\mathbf{x}) <_{lex} v_{N'}(\mathbf{x}'')$ or $v_{N'}(\mathbf{x}) = v_{N'}(\mathbf{x}'')$. On the other hand, $v_{N \cup N'}(\mathbf{x}) >_{lex} v_{N \cup N'}(\mathbf{x}'')$ or $v_{N \cup N'}(\mathbf{x}) = v_{N \cup N'}(\mathbf{x}'')$ as \mathbf{x}'' is chosen for \mathcal{I}'' . It is easy to see that these conditions can only be true if $v_N(\mathbf{x}) = v_N(\mathbf{x}'')$ and $v_{N \cup N'}(\mathbf{x}) = v_{N \cup N'}(\mathbf{x}'')$. However, consistent tie-breaking requires that if EGAL returns \mathbf{x} for \mathcal{I} , then \mathbf{x}'' is not chosen for \mathcal{I}'' , which contradicts our assumptions.

MED: Finally, we show that MED satisfies reinforcement if $m = 2$ but fails this property when $m \geq 3$. First, assume that $m = 2$ and let \mathbf{x} , \mathbf{x}' , and \mathbf{x}'' denote the outcomes chosen for our instances \mathcal{I} , \mathcal{I}' , and \mathcal{I}'' . As usual, we assume that $\mathbf{x} = \mathbf{x}'$. Note that $\text{med}_{i \in N} s_{i,1} = 1 - \text{med}_{i \in N} s_{i,2}$ and $\text{med}_{i \in N'} s'_{i,1} = 1 - \text{med}_{i \in N'} s'_{i,2}$ as $m = 2$, which implies that $\text{med}_{i \in N} s_{i,1} = x_1 = x'_1 = \text{med}_{i \in N'} s'_{i,1}$. We hence need to show that $\text{med}_{i \in N \cup N'} s''_{i,1} = \text{med}_{i \in N} s_{i,1}$. To this end, we make a case distinction with respect to the parity of $|N|$ and $|N'|$. First, if both $|N|$ and $|N'|$ are odd, there are at least $\frac{|N|+1}{2}$ agents $i \in N$ with $s_{i,1} \leq x_1$ and at least $\frac{|N|+1}{2}$ agents $i \in N$ with $s_{i,1} \geq x_1$. Similarly, there are at least $\frac{|N'|+1}{2}$ agents $i \in N'$ with $s'_{i,1} \leq x'_1 = x_1$ and at least $\frac{|N'|+1}{2}$ agents $i \in N'$ with $s'_{i,1} \geq x'_1 = x_1$. Thus, there are at least $\frac{|N|+|N'|}{2} + 1$ agents $i \in N \cup N'$ with $s''_{i,1} \leq x_1$ and at least $\frac{|N|+|N'|}{2} + 1$ agents $i \in N \cup N'$ with $s''_{i,1} \geq x_1$. This shows that $\text{med}_{i \in N \cup N'} s''_{i,1} = x_1$ and proves our claim. For the remaining cases, similar arguments show that the median does not change. In total, this means that $\text{med}_{i \in N \cup N'} s''_{i,1} = x_1$ (which also implies that $\text{med}_{i \in N \cup N'} s''_{i,2} = x_2$), so MED satisfies reinforcement if $m = 2$.

Now, for the case $m \geq 3$, consider the following instances \mathcal{I}^5 (defined for an odd number of agents $n \geq 5$) and \mathcal{I}^6 (defined for a single agent $n + 1$). All candidates but c_1 , c_2 , and c_3 receive probability 0 from all agents and can be ignored. It holds that $\text{med}_{i \in N} s_{i,1}^5 = \frac{1}{2}$, $\text{med}_{i \in N} s_{i,2}^5 = \frac{1}{3}$, and $\text{med}_{i \in N} s_{i,3}^5 = \frac{1}{2}$, so MED returns the outcome $\mathbf{x} = (\frac{3}{8}, \frac{2}{8}, \frac{3}{8})$ for \mathcal{I}^5 . On the other hand, \mathcal{I}^6 consists of a single

agent with preference $(\frac{3}{8}, \frac{2}{8}, \frac{3}{8})$, so MED returns again this vector. However, when combining these two instances, our new medians are $\text{med}_{i \in NU\{n+1\}s_{i,1}} = \frac{1}{2}(\frac{1}{2} + \frac{3}{8}) = \frac{7}{16}$, $\text{med}_{i \in NU\{n+1\}s_{i,2}} = \frac{1}{3}$, and $\text{med}_{i \in NU\{n+1\}s_{i,3}} = \frac{1}{2}(\frac{1}{2} + \frac{3}{8}) = \frac{7}{16}$. This means that for the new outcome \mathbf{x}' , we have $x'_1 = \frac{21}{58} < \frac{3}{8}$ and reinforcement is violated.

\mathcal{I}^5	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
$i \in \{1, \dots, \frac{n-1}{2}\}$	$\frac{2}{3}$	$\frac{1}{3}$	0
$i \in \{\frac{n+1}{2}, \dots, n-1\}$	0	$\frac{1}{3}$	$\frac{2}{3}$
n	$\frac{1}{2}$	0	$\frac{1}{2}$

\mathcal{I}^6	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
$n+1$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{3}{8}$

This completes the proof. □

Remark 5.2. We do not specify the boundary on n for which MED satisfies reinforcement. The reason for this is that reinforcement is a variable electorate property that is usually studied under the assumption that there is an infinite set of possible agents. The same reasoning will apply for participation in the next section.

6 Incentive Properties

As the last evaluation criteria for our rules, we study their incentive properties. To this end, we first note that Freeman et al. [2021] have shown that both IM and UTIL satisfy strategyproofness and participation. By contrast, we will show that none of our other rules is strategyproof, whereas all except for MED satisfy participation.

Theorem 6.1. *The following claims hold.*

- (1) IM and UTIL satisfy strategyproofness and participation.
- (2) AVG, MAX, MIN, GEO, and EGAL fail strategyproofness for all $n \geq 2$ and $m \geq 2$ but satisfy participation.
- (3) MED satisfies strategyproofness when $m = 2$ and n is odd, but fails this property when $m = 2$ and n is even, or when $m \geq 3$. Moreover, MED satisfies participation if $m = 2$ but fails it for all $m \geq 3$.

Proof. Claim 1 directly follows from the work of Freeman et al. [2021, Thms. 2, 8, and 12], so we only prove Claims 2 and 3 here.

Claim 2: We first consider participation and then show that all considered rules fail strategyproofness by giving a common counterexample.

Participation: To show this for AVG, MIN, MAX, and GEO, we first proof an auxiliary claim (the proof for EGAL will be independent of this claim). Consider two instances \mathcal{I} and \mathcal{I}' such that \mathcal{I}' is derived from \mathcal{I} by adding an agent i with preference s_i . Moreover, let \mathbf{x} and \mathbf{x}' denote the respective outcomes chosen by an aggregation rule F , and define $X^+ = \{j \in [m]: x'_j > x_j\}$ and $X^- = \{j \in [m]: x'_j < x_j\}$. We claim that, if $s_{i,j} \geq x'_j$ for all $j \in X^+$ or $s_{i,j} \leq x'_j$ for all $j \in X^-$, then F satisfies participation on \mathcal{I} and \mathcal{I}' . We will prove this claim only for the case that $s_{i,j} \geq x'_j$ for all $j \in X^+$, as the case that $s_{i,j} \leq x'_j$ for all $j \in X^-$ is symmetric. Now, if $s_{i,j} \geq x'_j$ for all $j \in X^+$, then it holds for these candidates that

$$|s_{i,j} - x'_j| - |s_{i,j} - x_j| = (s_{i,j} - x'_j) - (s_{i,j} - x_j) = x_j - x'_j.$$

On the other hand, we observe that

$$\sum_{j \in [m] \setminus X^+} (|s_{i,j} - x'_j| - |s_{i,j} - x_j|) \leq \sum_{j \in [m] \setminus X^+} |x_j - x'_j| = \sum_{j \in [m] \setminus X^+} (x_j - x'_j) = \sum_{j \in X^+} (x'_j - x_j).$$

The first inequality follows from the triangle inequality, the first equality uses the definition of X^+ , and the second equality uses that $\sum_{j \in [m]} x_j = \sum_{j \in [m]} x'_j = 1$. Based on these observations, we conclude that F satisfies participation for \mathcal{I} and \mathcal{I}' since

$$\begin{aligned} d_i(\mathbf{x}') - d_i(\mathbf{x}) &= \sum_{j \in [m]} |s_{i,j} - x'_j| - \sum_{j \in [m]} |s_{i,j} - x_j| \\ &= \sum_{j \in X^+} (|s_{i,j} - x'_j| - |s_{i,j} - x_j|) + \sum_{j \in [m] \setminus X^+} (|s_{i,j} - x'_j| - |s_{i,j} - x_j|) \\ &\leq \sum_{j \in X^+} (x_j - x'_j) + \sum_{j \in X^+} (x'_j - x_j) \\ &= 0. \end{aligned}$$

To show that our rules satisfy participation, it therefore suffices to prove that they satisfy $s_{i,j} \geq x'_j$ for all $j \in X^+$ or $s_{i,j} \leq x'_j$ for all $j \in X^-$. For this, we consider each rule individually.

AVG: For AVG, it is straightforward to reason that $s_{i,j} \geq x'_j$ for all $j \in X^+$. Indeed, otherwise we have that $x'_j > s_{i,j}$ and $x'_j > x_j$, which implies that

$$x'_j = \frac{1}{n+1} \sum_{i' \in N \cup \{i\}} s_{i',j} = \frac{1}{n+1} (s_{i,j} + nx_j) \leq \max(s_{i,j}, x_j) < x'_j.$$

Since this is a contradiction, we infer that AVG satisfies participation.

MAX: For MAX, it holds that $s_{i,j} \geq x'_j$ for all $j \in X^+$. To see this, note that $s_{i,j} > \max_{i' \in N} s_{i',j}$ for all $j \in X^+$, because otherwise $\max_{i' \in N \cup \{i\}} s_{i',j} = \max_{i' \in N} s_{i',j}$ and hence

$$x'_j = \frac{\max_{i' \in N \cup \{i\}} s_{i',j}}{\sum_{j' \in [m]} \max_{i' \in N \cup \{i\}} s_{i',j'}} \leq \frac{\max_{i' \in N} s_{i',j}}{\sum_{j' \in [m]} \max_{i' \in N} s_{i',j'}} = x_j.$$

Moreover, since $\sum_{j' \in [m]} \max_{i' \in N \cup \{i\}} s_{i',j'} \geq 1$ and therefore $s_{i,j} \geq \frac{s_{i,j}}{\sum_{j' \in [m]} \max_{i' \in N \cup \{i\}} s_{i',j'}}$, we conclude that $s_{i,j} \geq x'_j$ for all $j \in X^+$. This proves that MAX satisfies participation.

MIN: For MIN, we first consider the case that $\min_{i' \in N} s_{i',j} = 0$ for all $j \in [m]$ or $\min_{i' \in N \cup \{i\}} s_{i',j} = 0$ for all $j \in [m]$. If $\min_{i' \in N} s_{i',j} = 0$ for all $j \in [m]$, then the same holds for \mathcal{I}' and $\mathbf{x} = \mathbf{x}' = (\frac{1}{m}, \dots, \frac{1}{m})$, so participation is satisfied in this case. Next, if $\min_{i' \in N} s_{i',j} \neq 0$ for some $j \in [m]$ but $\min_{i' \in N \cup \{i\}} s_{i',j} = 0$, then $x_j > 0$ implies $s_{i,j} = 0$, and $s_{i,j} > 0$ implies that $x_j = 0$. Consequently, $d_i(\mathbf{x}) = \sum_{j \in [m]} |s_{i,j} - x_j| = \sum_{j \in [m]} (s_{i,j} + x_j) = 2$, which is the largest possible distance between two score vectors. Hence, $d_i(\mathbf{x}') \leq d_i(\mathbf{x})$ and participation holds again.

Now, if there is a candidate c_j such that $\min_{i' \in N \cup \{i\}} s_{i',j} > 0$, we claim that $s_{i,j} \leq x'_j$ for all $j \in X^-$. To see this, note that if $j \in X^-$, then $s_{i,j} < \min_{i' \in N} s_{i',j}$. Indeed, otherwise it holds that $\min_{i' \in N \cup \{i\}} s_{i',j} = \min_{i' \in N} s_{i',j}$ and hence

$$x'_j = \frac{\min_{i' \in N \cup \{i\}} s_{i',j}}{\sum_{j' \in [m]} \min_{i' \in N \cup \{i\}} s_{i',j'}} \geq \frac{\min_{i' \in N} s_{i',j}}{\sum_{j' \in [m]} \min_{i' \in N} s_{i',j'}} = x_j,$$

contradicting the definition of X^- . Since $\sum_{j' \in [m]} \min_{i' \in N \cup \{i\}} s_{i',j'} \leq 1$, we thus derive that $s_{i,j} \leq \frac{s_{i,j}}{\sum_{j' \in [m]} \min_{i' \in N \cup \{i\}} s_{i',j'}} = x'_j$ for all $j \in X^-$, which shows that MIN satisfies participation.

GEO: We first note that an analogous analysis as for MIN shows that GEO satisfies participation if $\sqrt[n]{\prod_{i' \in N} s_{i',j}} = 0$ for all $j \in [m]$ or $\sqrt[n+1]{\prod_{i' \in N \cup \{i\}} s_{i',j}} = 0$ for all $j \in [m]$. We hence assume that there is a candidate c_{j^*} such that $\sqrt[n+1]{\prod_{i' \in N \cup \{i\}} s_{i',j^*}} > 0$. Now, let $Z(\mathcal{I}) = \sum_{j \in [m]} \sqrt[n]{\prod_{i' \in N} s_{i',j}}$ and $Z(\mathcal{I}') = \sum_{j \in [m]} \sqrt[n+1]{\prod_{i' \in N \cup \{i\}} s_{i',j}}$. By the definition of x_j , we have for each $j \in [m]$ that

$$\prod_{i' \in N} s_{i',j} = (x_j \cdot Z(\mathcal{I}))^n.$$

Consequently, it holds for all $j \in [m]$ that

$$x'_j = \frac{(s_{i,j})^{1/(n+1)} \cdot (x_j \cdot Z(\mathcal{I}))^{n/(n+1)}}{Z(\mathcal{I}')} = (s_{i,j})^{1/(n+1)} \cdot (x_j)^{n/(n+1)} \cdot \frac{Z(\mathcal{I})^{n/(n+1)}}{Z(\mathcal{I}')}$$

Next, we use a case distinction with respect to whether $Z(\mathcal{I})^{n/(n+1)} \geq Z(\mathcal{I}')$ or $Z(\mathcal{I})^{n/(n+1)} \leq Z(\mathcal{I}')$ and start with the case that $Z(\mathcal{I})^{n/(n+1)} \geq Z(\mathcal{I}')$. In this case, it holds that $s_{i,j} \leq x'_j$ for all $j \in X^-$, as otherwise

$$x'_j \geq (s_{i,j})^{1/(n+1)} \cdot (x_j)^{n/(n+1)} > (x'_j)^{1/(n+1)} \cdot (x'_j)^{n/(n+1)} = x'_j,$$

a contradiction. On the other hand, if $Z(\mathcal{I})^{n/(n+1)} \leq Z(\mathcal{I}')$, we infer for all $j \in X^+$ that $s_{i,j} \geq x'_j$, as otherwise

$$x'_j \leq (s_{i,j})^{1/(n+1)} \cdot (x_j)^{n/(n+1)} < (x'_j)^{1/(n+1)} \cdot (x'_j)^{n/(n+1)} = x'_j.$$

We can thus conclude that GEO satisfies participation based on our auxiliary claim.

EGAL: For EGAL, we will use a direct argument instead of our auxiliary claim to show that it satisfies participation. Consider two instances \mathcal{I} and \mathcal{I}' such that \mathcal{I}' is derived from \mathcal{I} by adding an agent i with preference s_i , and let \mathbf{x} and \mathbf{x}' denote the outcome of EGAL for these instances. Assume for contradiction that $d_i(\mathbf{x}) < d_i(\mathbf{x}')$. For a set of agents T , let $v_T(\mathbf{y})$ be a vector that contains the disutilities of all agents $i' \in T$ for a given score vector \mathbf{y} in non-increasing order. Moreover, we write $v >_{lex} v'$ if there is an integer k such that $v_\ell = v'_\ell$ for all $\ell \in [k-1]$ and $v_k > v'_k$. Now, since EGAL chooses \mathbf{x} for \mathcal{I} , it either holds that $v_N(\mathbf{x}) = v_N(\mathbf{x}')$ or $v_N(\mathbf{x}') >_{lex} v_N(\mathbf{x})$. In both cases, it is simple to derive that $v_{N \cup \{i\}}(\mathbf{x}') >_{lex} v_{N \cup \{i\}}(\mathbf{x})$ since $d_i(\mathbf{x}) < d_i(\mathbf{x}')$. However, this contradicts the assumption that EGAL chooses \mathbf{x}' for \mathcal{I}' . Hence, our assumption that $d_i(\mathbf{x}) < d_i(\mathbf{x}')$ must have been wrong, which means that EGAL satisfies participation.

Strategyproofness: To prove the claim on strategyproofness, we consider the following instances \mathcal{I} and \mathcal{I}' . All candidates c_j with $j \geq 3$ receive probability 0 from all agents and are ignored for the rest of the proof.

\mathcal{I}	$s_{i,1}$	$s_{i,2}$	\mathcal{I}'	$s_{i,1}$	$s_{i,2}$
1	$\frac{4}{5}$	$\frac{1}{5}$	1	1	0
$i \in \{2, \dots, n\}$	$\frac{1}{5}$	$\frac{4}{5}$	$i \in \{2, \dots, n\}$	$\frac{1}{5}$	$\frac{4}{5}$

AVG chooses for these instances the outcomes $\mathbf{x} = (\frac{1}{5} + \frac{3}{5n}, \frac{4}{5} - \frac{3}{5n})$ and $\mathbf{x}' = (\frac{1}{5} + \frac{4}{5n}, \frac{4}{5} - \frac{4}{5n})$, MAX chooses $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ and $\mathbf{x}' = (\frac{5}{9}, \frac{4}{9})$, MIN chooses $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ and $\mathbf{x}' = (1, 0)$, GEO chooses $\mathbf{x} = (\frac{\sqrt[3]{16}}{4 + \sqrt[3]{16}}, \frac{4}{4 + \sqrt[3]{16}})$ and $\mathbf{x}' = (1, 0)$, and EGAL chooses $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ and $\mathbf{x}' = (\frac{3}{5}, \frac{2}{5})$. In each case, it can be checked that agent 1 benefits by deviating from \mathcal{I} to \mathcal{I}' , so strategyproofness is violated.

Claim 3: Finally, we turn to MED and again consider participation and strategyproofness separately.

Participation: First, we show that MED satisfies participation when $m = 2$. Consider two instances \mathcal{I} and \mathcal{I}' such that \mathcal{I}' is derived from \mathcal{I} by adding an agent i with preference s_i . Moreover, let \mathbf{x} and \mathbf{x}' denote the vectors returned by MED for these instances, respectively. Without loss of generality, assume that $x'_1 > x_1$. We show that this is only possible if $s_{i,1} \geq x'_1$. Hence, assume for contradiction that $s_{i,1} < x'_1$. Since $x'_1 > x_1$ and there are at least $\lceil \frac{n}{2} \rceil$ agents $i' \in N$ with $x_1 \geq s_{i',1}$, there are at least $\lceil \frac{n}{2} \rceil + 1$ agents i' in \mathcal{I}' with $x'_1 > s_{i',1}$. This means that $\text{med}_{i' \in N \cup \{i\}} s_{i',1} < x'_1$. However, since we have only two candidates, it holds that $\text{med}_{i' \in N \cup \{i\}} s_{i',1} + \text{med}_{i' \in N \cup \{i\}} s_{i',2} = 1$, so it must be that $x'_1 = \text{med}_{i' \in N \cup \{i\}} s_{i',1}$. This is the desired contradiction and we thus conclude that MED satisfies participation when $m = 2$.

Next, to see that MED fails participation when $m \geq 3$, we consider the instances \mathcal{I}^5 and \mathcal{I}^6 in the proof of Theorem 5.1 that were used to show that MED fails reinforcement. In this example, \mathcal{I}^6 consists of a single agent whose ideal distribution coincides with the outcome of MED for \mathcal{I}^5 . However, in the combined instance, the outcome changes, which means that participation is violated for this agent.

Strategyproofness: Next, we focus on strategyproofness. First, we will show that, if $m = 2$ and n is odd, MED is strategyproof. To this end, we note that MED simply returns the distribution of the agent who assigns the $(\frac{n+1}{2})$ -th highest score to c_1 . Since $d_{i'}(\mathbf{x}) = |s_{i',1} - x_1| + |s_{i',2} - x_2| = 2|s_{i',1} - x_1|$ for all agents i' , the strategyproofness follows directly from the well-known result by Moulin [1980]. On the other hand, if $m = 2$ and n is even, MED returns the average between the outcome of two agents, which allows agents to benefit by exaggerating their report. For instance, if one agent reports $(\frac{2}{3}, \frac{1}{3})$, $\frac{n}{2} - 1$ agents report $(1, 0)$, and $\frac{n}{2}$ agents report $(0, 1)$, MED chooses the outcome $(\frac{1}{3}, \frac{2}{3})$. However, if the first agent reports $(1, 0)$, MED returns $(\frac{1}{2}, \frac{1}{2})$, which constitutes a beneficial manipulation for this agent. This example can be generalized to $m \geq 3$ by adding candidates that receive a score of 0 from every agent, so we next focus on the case of $m \geq 3$ and odd $n \geq 3$. In this case, consider the instance \mathcal{I} below. One can check that MED chooses $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This means for agent n that $d_n(\mathbf{x}) = (\frac{2}{5} - \frac{1}{3}) + (\frac{3}{5} - \frac{1}{3}) + \frac{1}{3} = \frac{2}{3}$. Next, suppose that agent n reports the distribution $(\frac{8}{15}, \frac{7}{15}, 0)$. It can be verified that MED now picks the distribution $(\frac{2}{5}, \frac{3}{10}, \frac{3}{10})$, which means that $d_n(\mathbf{x}') = \frac{2}{5} < \frac{2}{3}$. Hence, MED is indeed manipulable if $m \geq 3$ and $n \geq 3$ is odd.

\mathcal{I}	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$	\mathcal{I}'	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$
$i \in \{1, \dots, \frac{n-1}{2}\}$	0	$\frac{2}{5}$	$\frac{3}{5}$	$i \in \{1, \dots, \frac{n-1}{2}\}$	0	$\frac{2}{5}$	$\frac{3}{5}$
$i \in \{\frac{n+1}{2}, \dots, n-1\}$	$\frac{3}{5}$	0	$\frac{2}{5}$	$i \in \{\frac{n+1}{2}, \dots, n-1\}$	$\frac{3}{5}$	0	$\frac{2}{5}$
n	$\frac{2}{5}$	$\frac{3}{5}$	0	n	$\frac{8}{15}$	$\frac{7}{15}$	0

This completes the proof. \square

7 Characterizations of the Average Rule

Thus far, we have conducted an extensive analysis of specific natural rules with respect to a number of desirable axioms. In this section, we complement those findings by presenting two characterizations of

the AVG rule. These results highlight the strong appeal of AVG by showing that it is the only rule within large classes of rules that, for example, satisfies score-unanimity and independence.

To state these results, we will introduce two more axioms. Firstly, we say an aggregation function F is *anonymous* if the identities of the agents do not matter, i.e., if $F(\mathcal{I}) = F(\pi(\mathcal{I}))$ for all instances \mathcal{I} and permutations $\pi : N \rightarrow N$, where $\mathcal{I}' = \pi(\mathcal{I})$ is the instance defined by $s'_{\pi(i)} = s_i$ for all $i \in N$. Secondly, an aggregation function F is *continuous* if it is a continuous function, i.e., $F(\lim_{t \rightarrow \infty} \mathcal{I}^t) = \lim_{t \rightarrow \infty} F(\mathcal{I}^t)$ for every sequence of instances $\mathcal{I}^1, \mathcal{I}^2, \dots$ such that $\lim_{t \rightarrow \infty} \mathcal{I}^t$ exists. We remark that both of these conditions are very weak: all of the rules considered in this paper are anonymous, and all except MIN and GEO are continuous.⁷

Theorem 7.1. *The following claims hold.*

- (1) *AVG is the only aggregation rule that satisfies anonymity, score-unanimity, and independence if $m \geq 3$.*
- (2) *AVG is the only coordinate-wise aggregation rule that satisfies anonymity, continuity, and score-unanimity if $m \geq 4$.*

Proof. It is easy to verify that AVG satisfies all stated axioms, so we focus on showing that the given sets of axioms indeed characterize AVG. We will establish the two claims separately.

Claim 1: Let F be an aggregation rule that satisfies anonymity, score-unanimity, and independence. In particular, this means that $F(\mathcal{I})_k = \gamma$ whenever $s_{i,k} = \gamma$ for all $i \in N$. If $n = 1$, this already proves our claim, so we subsequently assume that $n \geq 2$. We establish the result in three steps. Firstly, we will show that for every $\gamma \in (0, 1]$, there is a constant C_γ such that $F(\mathcal{I})_j = C_\gamma$ for all instances \mathcal{I} on n agents and all candidates $c_j \in C$ such that a single agent puts probability γ on c_j and every other agent puts probability 0 on this candidate. In the second step, we will prove that $C_\gamma = \frac{\gamma}{n}$. Based on this insight, we will derive that F corresponds to AVG in the last step.

Step 1: Our first goal is to show that, for every $\gamma \in (0, 1]$, there is a constant C_γ such that $F(\mathcal{I})_j = C_\gamma$ for all instances \mathcal{I} and candidates c_j such that $s_{i,j} = \gamma$ for a single agent $i \in N$ and $s_{i',j} = 0$ for all other agents $i' \in N \setminus \{i\}$. To prove this claim, consider two instances \mathcal{I}^1 and \mathcal{I}^2 for which there are two candidates c_{j_1} and c_{j_2} and two agents i_1 and i_2 such that (i) $s_{i_1,j_1}^1 = \gamma$ and $s_{i',j_1}^1 = 0$ for all $i' \in N \setminus \{i_1\}$, and (ii) $s_{i_2,j_2}^2 = \gamma$ and $s_{i',j_2}^2 = 0$ for all $i' \in N \setminus \{i_2\}$. Our goal is to show that $F(\mathcal{I}^1)_{j_1} = F(\mathcal{I}^2)_{j_2}$. Since \mathcal{I}^1 and \mathcal{I}^2 are arbitrarily chosen, this implies that there is a constant C_γ such that $F(\mathcal{I})_j = C_\gamma$ for all instances \mathcal{I} in which a single agent puts probability γ on a candidate c_j .

To prove that $F(\mathcal{I}^1)_{j_1} = F(\mathcal{I}^2)_{j_2}$, let $\hat{\mathcal{I}}^2$ denote the instance which is derived from \mathcal{I}^2 by exchanging the preferences of agents i_2 and i_1 . In particular, this means that $\hat{s}_{i_1,j_2}^2 = \gamma$ and $\hat{s}_{i',j_2}^2 = 0$ for all other agents $i' \in N \setminus \{i_1\}$. Clearly, anonymity requires that $F(\hat{\mathcal{I}}^2) = F(\mathcal{I}^2)$ as we derive $\hat{\mathcal{I}}^2$ by renaming the agents in \mathcal{I}^2 . Now, if $j_1 = j_2$, it holds that $\hat{s}_{i,j_1}^2 = s_{i,j_1}^1$ for all $i \in N$, so independence implies that $F(\mathcal{I}^1)_{j_1} = F(\hat{\mathcal{I}}^2)_{j_2}$ in this case. Hence, if $j_1 = j_2$, it follows that $F(\mathcal{I}^1)_{j_1} = F(\mathcal{I}^2)_{j_2}$.

Next, assume that $j_1 \neq j_2$. In this case, we let c_{j_3} denote an arbitrary candidate in $C \setminus \{c_{j_1}, c_{j_2}\}$ and consider the instances $\tilde{\mathcal{I}}^1$ and $\tilde{\mathcal{I}}^2$.

$\tilde{\mathcal{I}}^1$	s_{i,j_1}	s_{i,j_2}	s_{i,j_3}	$s_{i,j}$ for $j \in [m] \setminus \{j_1, j_2, j_3\}$
i_1	γ	$1 - \gamma$	0	0
$i' \in N \setminus \{i_1\}$	0	$1 - \gamma$	γ	0

⁷MIN and GEO fail continuity because they return the distribution $(\frac{1}{m}, \dots, \frac{1}{m})$ when $\min_{i \in N} s_{i,j} = 0$ for all $j \in [m]$. Apart from this corner case, these rules are also continuous.

$\tilde{\mathcal{I}}^2$	s_{i,j_1}	s_{i,j_2}	s_{i,j_3}	$s_{i,j}$ for $j \in [m] \setminus \{j_1, j_2, j_3\}$
i_1	$1 - \gamma$	γ	0	0
$i' \in N \setminus \{i_1\}$	$1 - \gamma$	0	γ	0

We infer from score-unanimity that $F(\tilde{\mathcal{I}}^1)_{j_2} = F(\tilde{\mathcal{I}}^2)_{j_1} = 1 - \gamma$ and $F(\tilde{\mathcal{I}}^1)_j = F(\tilde{\mathcal{I}}^2)_j = 0$ for all $j \in [m] \setminus \{j_1, j_2, j_3\}$. We therefore have $F(\tilde{\mathcal{I}}^1)_{j_3} = \gamma - F(\tilde{\mathcal{I}}^1)_{j_1}$ and $F(\tilde{\mathcal{I}}^2)_{j_3} = \gamma - F(\tilde{\mathcal{I}}^2)_{j_2}$. Moreover, independence implies that $F(\mathcal{I}^1)_{j_1} = F(\tilde{\mathcal{I}}^1)_{j_1}$ and $F(\tilde{\mathcal{I}}^2)_{j_2} = F(\tilde{\mathcal{I}}^2)_{j_2}$ because $s_{i,j_1}^1 = \tilde{s}_{i,j_1}^1$ and $\hat{s}_{i,j_2}^2 = \tilde{s}_{i,j_2}^2$ for all $i \in N$. Now, independence between $\tilde{\mathcal{I}}^1$ and $\tilde{\mathcal{I}}^2$ implies that $F(\tilde{\mathcal{I}}^1)_{j_3} = F(\tilde{\mathcal{I}}^2)_{j_3}$. This means that $F(\tilde{\mathcal{I}}^1)_{j_1} = F(\tilde{\mathcal{I}}^2)_{j_2}$ and we hence conclude that $F(\mathcal{I}^1)_{j_1} = F(\hat{\mathcal{I}}^2)_{j_2} = F(\mathcal{I}^2)_{j_2}$.

Step 2: For our second step, we fix a value $\gamma \in (0, 1]$ and let C_γ denote the constant derived in Step 1. The goal of this step is to show that $C_\gamma = \frac{\gamma}{n}$. To prove this claim, let c_{j_1} , c_{j_2} , and c_{j_3} denote three distinct candidates. We will inductively show that $F(\mathcal{I}^k)_{j_1} = k \cdot C_\gamma$ for the instances \mathcal{I}^k shown below.

\mathcal{I}^k	s_{i,j_1}	s_{i,j_2}	s_{i,j_3}	$s_{i,j}$ for $j \in [m] \setminus \{j_1, j_2, j_3\}$
$i \in \{1, \dots, k\}$	γ	0	$1 - \gamma$	0
$i \in \{k + 1, \dots, n\}$	0	γ	$1 - \gamma$	0

In particular, this means that $F(\mathcal{I}^{n-1})_{j_1} = (n - 1) \cdot C_\gamma$. Moreover, we infer from Step 1 that $F(\mathcal{I}^{n-1})_{j_2} = C_\gamma$ and score-unanimity shows that $F(\mathcal{I}^{n-1})_{j_3} = 1 - \gamma$ and $F(\mathcal{I}^{n-1})_j = 0$ for all $j \in [m] \setminus \{j_1, j_2, j_3\}$. Hence, we can now compute that $C_\gamma = \frac{\gamma}{n}$ because $\sum_{j \in [m]} F(\mathcal{I}^{n-1})_j = 1$.

For the proof that $F(\mathcal{I}^k)_{j_1} = k \cdot C_\gamma$ for all $k \in \{1, \dots, n - 1\}$, we first note that $F(\mathcal{I}^1)_{j_1} = C_\gamma$ by Step 1. Next, we inductively assume that $F(\mathcal{I}^k)_{j_1} = k \cdot C_\gamma$ for some $k \in \{1, \dots, n - 2\}$ and aim to show that $F(\mathcal{I}^{k+1})_{j_1} = (k + 1) \cdot C_\gamma$. To this end, we consider the instances $\hat{\mathcal{I}}^k$ and $\tilde{\mathcal{I}}^k$ shown below.

$\hat{\mathcal{I}}^k$	s_{i,j_1}	s_{i,j_2}	s_{i,j_3}	$s_{i,j}$ for $j \in [m] \setminus \{j_1, j_2, j_3\}$
$i \in \{1, \dots, k\}$	γ	0	$1 - \gamma$	0
$k + 1$	0	γ	$1 - \gamma$	0
$i \in \{k + 2, \dots, n\}$	0	0	1	0

$\tilde{\mathcal{I}}^k$	s_{i,j_1}	s_{i,j_2}	s_{i,j_3}	$s_{i,j}$ for $j \in [m] \setminus \{j_1, j_2, j_3\}$
$i \in \{1, \dots, k + 1\}$	γ	0	$1 - \gamma$	0
$i \in \{k + 2, \dots, n\}$	0	0	1	0

By independence and the induction hypothesis, we have that $F(\hat{\mathcal{I}}^k)_{j_1} = F(\mathcal{I}^k)_{j_1} = k \cdot C_\gamma$. Moreover, Step 1 shows that $F(\hat{\mathcal{I}}^k)_{j_2} = C_\gamma$ and score-unanimity requires that $F(\hat{\mathcal{I}}^k)_j = 0$ for all $j \in [m] \setminus \{j_1, j_2, j_3\}$. So we derive that $F(\hat{\mathcal{I}}^k)_{j_3} = 1 - (k + 1) \cdot C_\gamma$. Next, we turn to the instance $\tilde{\mathcal{I}}^k$. By score-unanimity, we have that $F(\tilde{\mathcal{I}}^k)_j = 0$ for all $j \in [m] \setminus \{j_1, j_3\}$ and independence implies that $F(\tilde{\mathcal{I}}^k)_{j_3} = F(\hat{\mathcal{I}}^k)_{j_3} = 1 - (k + 1) \cdot C_\gamma$. It thus holds that $F(\tilde{\mathcal{I}}^k)_{j_1} = (k + 1) \cdot C_\gamma$ as the probabilities must sum up to 1. Finally, independence implies that $F(\mathcal{I}^{k+1})_{j_1} = F(\tilde{\mathcal{I}}^k)_{j_1} = (k + 1) \cdot C_\gamma$, which completes the induction step.

Step 3: For our last step, we will show that F corresponds to AVG. Consider an arbitrary candidate c_{j_1} and an instance \mathcal{I} . Our goal is to show that $F(\mathcal{I})_{j_1} = \frac{1}{n} \sum_{i \in N} s_{i,j_1}$. To prove this claim, we take a candidate $c_{j_2} \neq c_{j_1}$, and consider the following instances \mathcal{I}^k for $k \in \{1, \dots, n\}$ defined by (i) $s_{i,j_1}^k = s_{i,j_1}$ and $s_{i,j_2}^k = 1 - s_{i,j_1}$ for all $i \in \{1, \dots, k\}$, (ii) $s_{i,j_1}^k = 0$ and $s_{i,j_2}^k = 1$ for all

$i \in \{k+1, \dots, n\}$, and (iii) $s_{i,j}^k = 0$ for all $i \in N$ and $j \in [m] \setminus \{j_1, j_2\}$. In particular, it holds by independence that $F(\mathcal{I})_{j_1} = F(\mathcal{I}^n)_{j_1}$, so our goal is to show that $F(\mathcal{I}^n)_{j_1} = \frac{1}{n} \sum_{i \in N} s_{i,j_1}$.

We will prove by induction on $k \in \{1, \dots, n\}$ that $F(\mathcal{I}^k)_{j_1} = \frac{1}{n} \sum_{i=1}^k s_{i,j_1}$. For the induction base $k=1$, we observe that $F(\mathcal{I}^1)_{j_1} = \frac{s_{1,j_1}^1}{n} = \frac{1}{n} \sum_{i=1}^1 s_{i,j_1}$ due to Step 2 (if $s_{1,j_1} > 0$) or score-unanimity (if $s_{1,j_1} = 0$). Next, we inductively assume that $F(\mathcal{I}^k)_{j_1} = \frac{1}{n} \sum_{i=1}^k s_{i,j_1}$ for some $k \in \{1, \dots, n-1\}$, and aim to show the same for $k+1$. If $s_{k+1,j_1} = 0$, this follows due to independence as $F(\mathcal{I}^{k+1})_{j_1} = F(\mathcal{I}^k)_{j_1} = \frac{1}{n} \sum_{i=1}^{k+1} s_{i,j_1}$. We thus assume that $s_{k+1,j_1} > 0$ and consider the instance $\bar{\mathcal{I}}^k$ derived from \mathcal{I}^k by setting $\bar{s}_{k+1,j_3}^k = s_{k+1,j_1}$ for some arbitrary $j_3 \in [m] \setminus \{j_1, j_2\}$ and $\bar{s}_{k+1,j_2}^k = 1 - s_{k+1,j_1}$. By independence, it holds that $F(\bar{\mathcal{I}}^k)_{j_1} = F(\mathcal{I}^k)_{j_1} = \frac{1}{n} \sum_{i=1}^k s_{i,j_1}$. Furthermore, $F(\bar{\mathcal{I}}^k)_{j_3} = \frac{s_{k+1,j_1}}{n}$ by Step 2. Since $F(\bar{\mathcal{I}}^k)_j = 0$ for all $j \in [m] \setminus \{j_1, j_2, j_3\}$ due to score-unanimity, we infer that $F(\bar{\mathcal{I}}^k)_{j_2} = 1 - F(\bar{\mathcal{I}}^k)_{j_1} - F(\bar{\mathcal{I}}^k)_{j_3} = 1 - \frac{1}{n} \sum_{i=1}^{k+1} s_{i,j_1}$.

Next, we derive from independence between $\bar{\mathcal{I}}^k$ and \mathcal{I}^{k+1} that $F(\mathcal{I}^{k+1})_{j_2} = F(\bar{\mathcal{I}}^k)_{j_2} = 1 - \frac{1}{n} \sum_{i=1}^{k+1} s_{i,j_1}$. Since $F(\mathcal{I}^{k+1})_j = 0$ for all $j \in [m] \setminus \{j_1, j_2\}$ due to score-unanimity, it follows that $F(\mathcal{I}^{k+1})_{j_1} = \frac{1}{n} \sum_{i=1}^{k+1} s_{i,j_1}$. This completes the induction step, and it thus follows that $F(\mathcal{I})_{j_1} = F(\mathcal{I}^n)_{j_1} = \frac{1}{n} \sum_{i \in N} s_{i,j_1}$. We finally conclude that F is AVG.

Claim 2: Let F be a coordinate-wise aggregation rule that satisfies anonymity, continuity, and score-unanimity. Moreover, let f_j denote the coordinate-aggregation function of F for the j -th coordinate. In slight abuse of notation, we will frequently write $f_j(\mathcal{I})$ to mean $f_j(s_{1,j}, \dots, s_{n,j})$. Since F is invariant under scaling all functions f_j with a constant, we can assume without loss of generality that $f_1(0.5, \dots, 0.5) = 0.5$.

In the remainder of the proof, we will show that $f_j(\mathcal{I}) = \frac{1}{n} \sum_{i \in N} s_{i,j}$ for all instances \mathcal{I} and candidates c_j . To this end, we first prove this claim for the case where all agents put the same probability $\gamma \in [0, 1)$ on a candidate c_j . In the second step, we then use our first characterization of AVG to show that $f_j(\mathcal{I}) = \frac{1}{n} \sum_{i \in N} s_{i,j}$ for all instances \mathcal{I} and candidates c_j with $\max_{i \in N} s_{i,j} < 1$. Finally, we infer using continuity that F corresponds to AVG in the last step.

Step 1: As the first step, we show that $f_j(\gamma, \dots, \gamma) = \gamma$ for all $j \in [m]$ and $\gamma \in [0, 1)$. To this end, we first observe that $f_j(0, \dots, 0) = 0$ for all $j \in [m]$ because of score-unanimity. To prove Step 1, we will first show that $f_j(0.5, \dots, 0.5) = 0.5$ for all $j \in [m]$. Thus, fix some candidate $c_j \neq c_1$ and consider the instance \mathcal{I}^1 such that $s_{i,1}^1 = s_{i,j}^1 = 0.5$ for all $i \in N$ and $s_{i,j'} = 0$ for all $i \in N$ and $j' \in [m] \setminus \{1, j\}$. By score-unanimity, we infer that $f_{j'}(\mathcal{I}^1) = 0$ for all $j' \in [m] \setminus \{1, j\}$. Moreover, score-unanimity requires that $F(\mathcal{I}^1)_1 = F(\mathcal{I}^1)_j = 0.5$. Since we assume that $f_1(0.5, \dots, 0.5) = 0.5$ and $F(\mathcal{I}^1)_1 = \frac{f_1(\mathcal{I}^1)}{\sum_{j' \in [m]} f_{j'}(\mathcal{I}^1)}$, we infer now that $\sum_{j' \in [m]} f_{j'}(\mathcal{I}^1) = 1$. By combining this with our previous observations, it follows that $f_j(\mathcal{I}^1) = 0.5$. This proves that $f_j(0.5, \dots, 0.5) = 0.5$ for all $j \in [m]$.

Next, we show that $f_j(\gamma, \dots, \gamma) = \gamma$ for all $\gamma \in (0, 0.5)$ and $j \in [m]$. To this end, we fix three distinct candidates c_{j_1} , c_{j_2} , and c_{j_3} and consider the instance \mathcal{I}^2 such that $s_{i,j_1}^2 = \gamma$, $s_{i,j_2}^2 = 0.5$, and $s_{i,j_3}^2 = 0.5 - \gamma$ for all $i \in N$. All other candidates receive probability 0 from all agents. By score-unanimity, $f_j(\mathcal{I}^2) = 0$ for all $j \in [m] \setminus \{j_1, j_2, j_3\}$. Moreover, we infer that $F(\mathcal{I}^2)_{j_2} = 0.5 = f_{j_2}(\mathcal{I}^2)$ by score-unanimity and our previous insights, which implies that $\sum_{j \in [m]} f_j(\mathcal{I}^2) = 1$. Since score-unanimity also requires that $F(\mathcal{I}^2)_{j_1} = \gamma$, we conclude now that $f_{j_1}(\gamma, \dots, \gamma) = \gamma$.

Finally, we consider the case that $\gamma \in (0.5, 1)$. Fix two candidates c_{j_1} and c_{j_2} and consider the instance \mathcal{I}^3 such that $s_{i,j_1}^3 = \gamma$ and $s_{i,j_2}^3 = 1 - \gamma$ for all $i \in N$. All other candidates again obtain probability 0 from all agents. Since $1 - \gamma < 0.5$, we derive from our previous insights and score-unanimity that $F(\mathcal{I}^3)_{j_2} = 1 - \gamma = f_{j_2}(\mathcal{I}^3)$. Hence, it holds that $\sum_{j \in [m]} f_j(\mathcal{I}^3) = 1$. Since score-unanimity implies that $f_j(\mathcal{I}^3) = 0$ for all $j \in [m] \setminus \{j_1, j_2\}$, we infer that $f_{j_1}(\gamma, \dots, \gamma) = \gamma$. This completes the proof of Step 1.

Step 2: Next, we prove that $f_j(\mathcal{I}) = \frac{1}{n} \sum_{i \in N} s_{i,j}$ for all instances \mathcal{I} and candidates c_j with $\max_{i \in N} s_{i,j} < 1$. Fix an arbitrary instance \mathcal{I}^* and a candidate c_{j_1} that satisfy our requirements. Moreover, we define $\varepsilon = 1 - \max_{i \in N} s_{i,j_1}^*$ and note that $\varepsilon > 0$ as $\max_{i \in N} s_{i,j_1}^* < 1$. If $\varepsilon = 1$, then all agents assign probability 0 to c_{j_1} in \mathcal{I}^* , and score-unanimity immediately implies that $f_{j_1}(\mathcal{I}^*) = 0 = \frac{1}{n} \sum_{i \in N} s_{i,j_1}^*$.

Hence, we assume that $\varepsilon < 1$ and consider three more candidates $c_{j_2}, c_{j_3}, c_{j_4}$. Furthermore, we define another aggregation rule G for the candidates $\{c_{j_1}, c_{j_2}, c_{j_3}\} \subsetneq C$ as follows: Given an instance \mathcal{I} on these three candidates, we define the extended instance \mathcal{I}^E on C by (i) $s_{i,j}^E = (1 - \varepsilon)s_{i,j}$ for all $i \in N$ and $j \in \{j_1, j_2, j_3\}$, (ii) $s_{i,j_4}^E = \varepsilon$ for all $i \in N$, and (iii) $s_{i,j}^E = 0$ for all $i \in N$ and $j \in [m] \setminus \{j_1, j_2, j_3, j_4\}$. Then, $G(\mathcal{I})_j = \frac{1}{1-\varepsilon} F(\mathcal{I}^E)_j$ for all $j \in \{j_1, j_2, j_3\}$.

Our goal is to show that G corresponds to AVG since this implies that $F(\mathcal{I}^E)_{j_1} = (1 - \varepsilon)G(\mathcal{I})_{j_1} = (1 - \varepsilon)\frac{1}{n} \sum_{i \in N} s_{i,j_1}$ for all instances \mathcal{I} on $\{c_{j_1}, c_{j_2}, c_{j_3}\}$. Because $s_{i,j}^E = (1 - \varepsilon)s_{i,j}$ for $j \in \{j_1, j_2, j_3\}$, it then follows that $F(\mathcal{I}^E)_{j_1} = \frac{1}{n} \sum_{i \in N} s_{i,j_1}^E$. Furthermore, by Step 1 and score-unanimity, we have that $F(\mathcal{I}^E)_{j_4} = \varepsilon = f_{j_4}(\mathcal{I}^E)$, which implies that $\sum_{j \in [m]} f_j(\mathcal{I}^E) = 1$. In turn, this means that $F(\mathcal{I}^E)_j = f_j(\mathcal{I}^E)$ for all $j \in [m]$, so it follows that $f_{j_1}(\mathcal{I}^E) = \frac{1}{n} \sum_{i \in N} s_{i,j_1}^E$ for all instances \mathcal{I} on $\{c_{j_1}, c_{j_2}, c_{j_3}\}$. Finally, since we assumed that $0 < \varepsilon = 1 - \max_{i \in N} s_{i,j_1}^*$, there exists an instance \mathcal{I} on $\{c_{j_1}, c_{j_2}, c_{j_3}\}$ such that $s_{i,j_1}^* = s_{i,j_1}^E$ for all $i \in N$, so we conclude that $f_{j_1}(\mathcal{I}^*) = \frac{1}{n} \sum_{i \in N} s_{i,j_1}^*$.

It remains to show that G is indeed AVG. To this end, we aim to employ our first characterization and show that G is a well-defined aggregation rule that satisfies anonymity, score-unanimity, and independence. First, it is easy to verify that G is well-defined: it holds that $G(\mathcal{I})_j \geq 0$ for all instances \mathcal{I} and $j \in \{j_1, j_2, j_3\}$ as the same is true for $F(\mathcal{I}^E)$. Moreover, score-unanimity implies that $F(\mathcal{I}^E)_{j_4} = \varepsilon$ and $F(\mathcal{I}^E)_j = 0$ for all $j \in [m] \setminus \{j_1, j_2, j_3, j_4\}$, so $F(\mathcal{I}^E)_{j_1} + F(\mathcal{I}^E)_{j_2} + F(\mathcal{I}^E)_{j_3} = 1 - \varepsilon$. It follows that $G(\mathcal{I})_{j_1} + G(\mathcal{I})_{j_2} + G(\mathcal{I})_{j_3} = 1$ for all instances \mathcal{I} .

Next, we show that G is anonymous. Consider two instances \mathcal{I} and $\hat{\mathcal{I}}$ such that $\hat{\mathcal{I}}$ is derived from \mathcal{I} by permuting the agents. Consequently, the instances \mathcal{I}^E and $\hat{\mathcal{I}}^E$ can be derived from each other by permuting the agents, so the anonymity of F implies that $F(\mathcal{I}^E) = F(\hat{\mathcal{I}}^E)$. This also means that $G(\mathcal{I}) = G(\hat{\mathcal{I}})$, so G is anonymous.

To show that G is score-unanimous, we consider an instance \mathcal{I} such that all agents $i \in N$ put the same probability γ on some candidate c_j . Hence, all agents put probability $(1 - \varepsilon)\gamma$ on c_j in the extended instance \mathcal{I}^E . The score-unanimity of F then implies that $F(\mathcal{I}^E)_j = (1 - \varepsilon)\gamma$, which entails that $G(\mathcal{I})_j = \frac{1}{1-\varepsilon} F(\mathcal{I}^E)_j = \gamma$. So G satisfies this axiom, too.

For the last point, we show that G satisfies independence. To this end, consider two instances $\mathcal{I}, \hat{\mathcal{I}}$ on $\{c_{j_1}, c_{j_2}, c_{j_3}\}$ and a candidate c_j such that $s_{i,j} = \hat{s}_{i,j}$ for all $i \in N$. This means that $s_{i,j}^E = \hat{s}_{i,j}^E$ for all $i \in N$, so $f_j(\mathcal{I}^E) = f_j(\hat{\mathcal{I}}^E)$. Moreover, it holds by Step 1 and score-unanimity that $F(\mathcal{I}^E)_{j_4} = \varepsilon = f_{j_4}(\mathcal{I}^E)$ and $F(\hat{\mathcal{I}}^E)_{j_4} = \varepsilon = f_{j_4}(\hat{\mathcal{I}}^E)$. We hence infer that $\sum_{j' \in [m]} f_{j'}(\mathcal{I}^E) = \sum_{j' \in [m]} f_{j'}(\hat{\mathcal{I}}^E) = 1$. This means that $F(\mathcal{I}^E)_j = f_j(\mathcal{I}^E) = f_j(\hat{\mathcal{I}}^E) = F(\hat{\mathcal{I}}^E)_j$, so G satisfies independence because $G(\mathcal{I})_j = \frac{1}{1-\varepsilon} F(\mathcal{I}^E)_j = \frac{1}{1-\varepsilon} F(\hat{\mathcal{I}}^E)_j = G(\hat{\mathcal{I}})_j$.

Finally, since the rule G satisfies all axioms of Claim 1, we conclude that it is AVG, which completes the proof of this step.

Step 3: By the insights of Step 2, we get that $f_j(\mathcal{I}) = \frac{1}{n} \sum_{i \in N} s_{i,j}$ for all instances \mathcal{I} and candidates c_j such that $\max_{i \in N} s_{i,j} < 1$. This means that F is equal to AVG for all instances \mathcal{I} with $\max_{i \in N, j \in [m]} s_{i,j} < 1$. In order to extend the result to instances where some agents assign probability 1 to some candidate, we use the continuity of F . Specifically, consider an instance \mathcal{I}^* such that $s_{i,j}^* = 1$ for some agents $i \in N$ and candidate $c_j \in C$. Moreover, let \mathcal{I} denote the instance where every agent assigns probability $\frac{1}{m}$ to every candidate. We can now consider the sequence of instances \mathcal{I}^k defined by $s_{i,j}^k = \frac{1}{2^k} s_{i,j} + (1 - \frac{1}{2^k}) s_{i,j}^*$ for all $i \in N, j \in [m]$. Clearly, this sequence converges to \mathcal{I}^* . Moreover, for every instance \mathcal{I}^k and all $j \in [m]$, it holds that $F(\mathcal{I}^k)_j = \frac{1}{n} \sum_{i \in N} s_{i,j}^k$ due to Step 2. Hence, we

can infer by continuity that $F(\mathcal{I}^*)_j = \lim_{k \rightarrow \infty} F(\mathcal{I}^k)_j = \frac{1}{n} \sum_{i \in N} s_{i,j}^*$. This shows that F is AVG, as desired. \square

Remark 7.2. For both of our characterizations, all axioms are necessary. Every dictatorial aggregation rule, which always returns the score vector of a specific agent, satisfies all given axioms except anonymity. Every constant aggregation rule satisfies all given axioms except score-unanimity. UTIL satisfies all given axioms except independence and coordinate-wiseness. Furthermore, the coordinate-wise rule defined by the coordinate-aggregation function $f_j(\mathcal{I}) = 1$ if there is an agent i with $s_{i,j} = 1$ and $f_j(\mathcal{I}) = \frac{1}{n} \sum_{i \in N} s_{i,j}$ otherwise satisfies all axioms except independence and continuity. Finally, the condition that $m \geq 3$ is necessary for our first characterization as independence becomes otherwise trivial and, e.g., UTIL satisfies all given axioms, and the condition that $m \geq 4$ is necessary for the second characterization because MED satisfies all conditions if $m \leq 3$.

Remark 7.3. One can check that in our first characterization, it is possible to replace score-unanimity and anonymity with score-representation. That is, AVG is the only aggregation rule that satisfies score-representation and independence. The reason for this is that all key steps of our proof still work when using score-representation instead of anonymity and score-unanimity. In particular, for instances of the form $\tilde{\mathcal{I}}^1$, we obtain from score-representation that candidate c_{j_1} receives probability at least $\frac{\gamma}{n}$, candidate c_{j_2} receives probability at least $1 - \gamma$, and candidate c_{j_3} receives probability at least $\frac{(n-1)\gamma}{n}$. Since the probabilities must sum up to 1, this is only possible if all of these bounds are tight. By independence and the fact that this argument does not depend on the identities of agents or candidates, this immediately completes the proof of Steps 1 and 2.

Perhaps surprisingly, this claim does not hold for our second characterization as the coordinate-wise rule, whose coordinate-aggregation functions assign to each candidate the minimum probability that meets the requirements imposed by score-representation, satisfies score-representation, anonymity, and continuity.

Remark 7.4. Another natural way to characterize AVG is to rely on convexity. Specifically, we say that an aggregation rule F is *weakly convex* if $F(\mathcal{I}'') = F(\mathcal{I})$ for all instances $\mathcal{I}, \mathcal{I}'$, and \mathcal{I}'' such that $F(\mathcal{I}) = F(\mathcal{I}')$ and there exists $\lambda \in (0, 1)$ with $s''_{i,j} = \lambda s_{i,j} + (1 - \lambda) s'_{i,j}$ for all $i \in N, j \in [m]$. Then, one can show that AVG is the only aggregation rule that satisfies weak convexity, anonymity, and score-unanimity. The idea is that for every instance \mathcal{I} , we can permute it with every permutation $\pi : N \rightarrow N$ and then take the average of all permuted instances. This results in an instance \mathcal{I}' where all agents assign probability $\frac{1}{n} \sum_{i \in N} s_{i,j}$ to each candidate c_j . Hence, score-unanimity requires $F(\mathcal{I}')_j = \frac{1}{n} \sum_{i \in N} s_{i,j}$, and weak convexity and anonymity imply that the same holds for \mathcal{I} .

Remark 7.5. All coordinate-wise (and anonymous) rules from Table 1 violate Pareto-optimality. Intuitively, this does not come as a surprise as correlations between scores and Pareto improvements cannot be taken into account by coordinate-wise rules. Formally, the proof of Theorem 7.1 shows that any coordinate-wise, anonymous, and score-unanimous (which is necessary for Pareto-optimality) rule has to coincide with AVG on all instances with $s_{i,j} \neq 1$ for all i, j . It is not difficult to see—e.g., by adapting the instance \mathcal{I}^1 in the proof of Theorem 3.1—that no such rule satisfies Pareto-optimality.

8 Conclusion

In this paper, we have analyzed aggregation rules for portioning with cardinal preferences from an axiomatic perspective. Specifically, we considered a natural model in which each agent reports her ideal distribution of a homogeneous resource over a set of candidates and her disutility for a distribution corresponds to the ℓ_1 distance from her ideal distribution. We investigated rules based on coordinate-wise aggregation or welfare aggregation as well as the independent markets rule of Freeman et al. [2021] with

respect to efficiency, fairness, consistency, and incentive properties. Our results, which are summarized in Table 1, show that the rule that simply returns the average of the agents’ reports satisfies most of the studied axioms. In particular, even though this rule violates strategyproofness and Pareto-optimality, it is the only rule among the ones we considered that fulfills the strong fairness notion of score-representation as well as the strong consistency property of independence. To further strengthen this point, we provided two characterizations demonstrating that the average rule is the only rule within large classes of rules that satisfies, for example, independence and score-unanimity at the same time.

We believe that our paper can serve as a basis for extensive future research in the domain of cardinal portioning. For instance, the insights that our findings offer may be helpful toward characterizations of further rules. It would also be interesting to examine additional axioms, especially those concerning fairness, which is important but arguably not yet well-understood in this setting. One such axiom is membership in the *core*, which intuitively means that no subset of agents can guarantee a better outcome (in the sense of a Pareto improvement) by using their proportional share of the resource. The core strengthens both Pareto-optimality (since the latter makes this requirement only for the set of all agents) and single-minded proportionality, so Table 1 immediately implies that none of the rules we considered always returns an outcome in the core. Whether any such rule exists is therefore an intriguing question which we leave for future work.

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A Appendix: Computational Aspects

In this appendix, we prove two claims about computational aspects that were made in the main body. First, we show that EGAL (including the leximin tie-breaking) can be computed in polynomial time.

Proposition A.1. *EGAL can be computed in polynomial time.*

Proof. Similar to Airiau et al. [2023, Alg. 1], we formulate a series of linear programs (LP) for finding an EGAL outcome. Let the objective function be

$$\text{minimize } \xi$$

subject to the following constraints.

- (1) $\sum_{j \in [m]} x_j = 1$.
- (2) $x_j \geq 0$ for each $j \in [m]$ and $\xi \geq 0$.
- (3) $z_{i,j} \geq s_{i,j} - x_j$ and $z_{i,j} \geq x_j - s_{i,j}$ for each $i \in N, j \in [m]$.
- (4) $\sum_{j \in [m]} z_{i,j} \leq \xi$ for each $i \in N$.

This allows us to minimize the largest disutility ξ . There is an agent i that has disutility ξ in every leximin outcome: indeed, if for every $i \in N$ there is an outcome in which i incurs disutility less than ξ and every other agent incurs disutility at most ξ , then by averaging these outcomes across all $i \in N$, we obtain an outcome in which every agent's disutility is less than ξ , contradicting the choice of ξ . To find such an agent, for each $i \in N$, we formulate an LP that computes the maximum δ for which there exists an outcome such that agent i incurs disutility at most $\xi - \delta$ while every other agent incurs disutility at most ξ ; an agent with the desired property will return $\delta = 0$. We fix the disutility of this agent to ξ , and continue by finding the second largest disutility, and so on. The total number of LPs is $\mathcal{O}(n^2)$. \square

Next, we prove that checking whether an outcome is Pareto-optimal can be done in polynomial time.

Proposition A.2. *Determining whether an outcome \mathbf{x}' is Pareto-optimal can be done in polynomial time.*

Proof. Suppose we are given an outcome \mathbf{x}' for an instance $\mathcal{I} = (\mathbf{s}_1, \dots, \mathbf{s}_n)$ and we want to determine whether \mathbf{x}' is Pareto-optimal.

For each $i \in N$ and $j \in [m]$, let $z'_{i,j} = |x'_j - s_{i,j}|$. The quantities $z'_{i,j}$ and $s_{i,j}$ can be computed from the input, and will appear in the constraints of the linear program below.

We formulate a linear program as follows:

$$\text{minimize } \sum_{i \in N} \sum_{j \in [m]} z_{i,j},$$

subject to the following constraints.

- (1) $\sum_{j \in [m]} x_j = 1$.
- (2) $x_j \geq 0$ for each $j \in [m]$.
- (3) $z_{i,j} \geq s_{i,j} - x_j$ and $z_{i,j} \geq x_j - s_{i,j}$ for each $i \in N, j \in [m]$.
- (4) $\sum_{j \in [m]} z_{i,j} \leq \sum_{j \in [m]} z'_{i,j}$ for each $i \in N$.

Note that $z_{i,j}$ is an upper bound on i 's disutility for candidate c_j , so the smaller it is, the better. Then, for the optimal solution $(z_{i,j}^*)_{i \in N, j \in [m]}$ returned, if $\sum_{j \in [m]} z_{i,j}^* < \sum_{j \in [m]} z'_{i,j}$ for some $i \in N$, we know that \mathbf{x}' is not Pareto-optimal; otherwise it is. \square