Department of Informatics
Technical University of Munich

Master's Thesis in Informatics

# Kelly-strategyproof Social Choice Functions

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Kelly-strategiebeständige Sozialwahlverfahren

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I confirm that this master's thesis is my own work and I have documented all sources and material used.

# Abstract

In this thesis, we discuss when set-valued social choice functions are prone to strategic manipulations according to Kelly's set extension. We derive both Kelly-strategyproof social choice functions and impossibility results by analyzing the combination of various axioms paired with Kelly-strategyproofness. First, we discuss implications between various types of monotonicity and strategyproofness. One of the strongest results of this type is that set-monotonic social choice functions are even strategyproof if preferences are allowed to be intransitive and a slightly weakened variant of Kelly-strategyproofness is used. Next, we consider the combination of rank-basedness and Kelly-strategyproofness. On the one hand, we prove that there are several interesting social choice functions that satisfy these axioms in the strict domain. On the other hand, we derive a strong impossibility result stating that there is no Kelly-strategyproof, rank-based and Pareto-optimal social choice function if preferences may contain ties. Moreover, we analyze when social choice functions in C2 are Kelly-strategyproof if preferences may contain ties. As there are almost no social choice functions that satisfy these axioms and Pareto-optimality, we derive strong necessary conditions for Kelly-strategyproof C2-functions. The strongest one states that every Kelly-strategyproof and Pareto-optimal C2-function must choose one of the most preferred alternatives of every voter. Finally, we consider the combination of Kelly-strategyproofness and responsive efficiency as there is the conjecture that no anonymous social choice function satisfies these axioms. Even though we cannot prove this conjecture, we provide strong evidence that it is true by showing that there is no Kelly-strategyproof and responsively efficient social choice function in C2. Thereby, our results provide many new insights on Kelly-strategyproofness.

# Contents


**1 Introduction** **1**
1.1 Motivation . . . . . . . . . . 1
1.2 Problem Definition and Notation . . . . . . . . . . 2
1.2.1 Problem Definition . . . . . . . . . . 2
1.2.2 Axioms for Social Choice Functions . . . . . . . . . . 5
1.3 Literature Review . . . . . . . . . . 8

**2 Monotonicity and Strategyproofness** **11**
2.1 Monotonicity Axioms and Set Extensions . . . . . . . . . . 11
2.2 Results in the Weak Domain . . . . . . . . . . 16
2.3 Generalizations to the Intransitive Domain . . . . . . . . . . 24
2.4 Generalizations to Group-Strategyproofness . . . . . . . . . . 27

**3 Rank-based Social Choice Functions** **33**
3.1 Introduction to Rank-based Social Choice Function . . . . . . . . . . 33
3.2 Independent Rank-based Social Choice Functions . . . . . . . . . . 36
3.2.1 Threshold Rules and Multi-Threshold Rules . . . . . . . . . . 37
3.2.2 $\tilde{P}$-strategyproofness and Independence of Ranks of Other Alternatives . . . . . . . . . . 47
3.3 General Rank-based Social Choice Functions . . . . . . . . . . 56
3.3.1 Rank-based Refinements of the OMNI-Rule . . . . . . . . . . 56
3.3.2 Scoring Rules . . . . . . . . . . 60
3.3.3 Dominance Rules . . . . . . . . . . 64
3.4 Rank-based Social Choice Functions in the Weak Domain . . . . . . . . . . 67
3.4.1 Ranks of Alternatives in the Weak Domain . . . . . . . . . . 68
3.4.2 Rank-basedness and $\tilde{P}$-strategyproofness in the Weak Domain 74

**4 Social Choice Functions in C2** **81**
4.1 Introduction to Social Choice Functions in C2 . . . . . . . . . . 81
4.2 $\tilde{P}$-strategyproof Social Choice Functions in C2 . . . . . . . . . . 83
4.2.1 Social Choice Functions Violating Pareto-optimality . . . . . . . . . . 83
4.2.2 Social Choice Functions Satisfying Pareto-optimality . . . . . . . . . . 87
4.3 Requirements for $\tilde{P}$-strategyproof C2-Functions . . . . . . . . . . 93
4.3.1 $\tilde{P}$-strategyproofness and Neutrality . . . . . . . . . . 93
4.3.2 $\tilde{P}$-strategyproofness and Pareto-optimality . . . . . . . . . . 102
4.3.3 A Characterization of the Pareto-rule . . . . . . . . . . 117

**5 Responsively Efficient Social Choice Functions** **127**
5.1 Introduction to Responsive Efficiency . . . . . . . . . . . . . . . . . . . 127
5.2 Analysis of Responsive Efficiency . . . . . . . . . . . . . . . . . . . . . 130
5.3 $\tilde{P}$-strategyproofness and Responsive Efficiency . . . . . . . . . . . . . 136
5.3.1 $\tilde{P}$-Strategyproof and Responsively Efficient Social Choice Functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 137
5.3.2 Impossibility Results based on $\tilde{P}$-strategyproofness and Responsive Efficiency . . . . . . . . . . . . . . . . . . . . . . . . 141

**6 Conclusion** **149**

Chapter 1

# Introduction

## 1.1 Motivation

Voting is a common tool for making group decisions. It has already been used by the old Greek poleis and is still used in today's modern society. However, we are recently confronted with more and more problems in elections: voting fraud, manipulations and ballot schemes for instance. These problems encourage the thorough analysis of voting schemes from a mathematical standpoint, which is the focus of a research field called computational social choice.

One main question in this research field asks when a voting scheme is manipulable. Informally, we call a voting scheme manipulable if a voter can improve the outcome of the election from his individual perspective by lying about his preferences. This question has been answered independently by Gibbard [Gib73, Gib77] and Satterthwaite [Sat75] who have shown that every reasonable voting scheme is manipulable if it chooses always a single winner. Even though this result is considered a milestone in computational social choice, it is criticized for the assumption of single-valuedness. This assumption is from a theoretical standpoint not reasonable because alternatives often seem equally good. For instance, consider an election with two voters and three alternatives in which the voters do not agree on the most preferred alternative but on the least preferred one. In this situation, it seems unreasonable to pick a single winner, but it is required for the Gibbard-Satterthwaite-Theorem.

As consequence, current research analyzes voting schemes allowing for multiple winners. We call such voting schemes social choice functions. Unfortunately, it is not apparent how to define manipulations when multiple winners are possible as it is not clear on how to compare sets of alternatives. For instance, assume that a voter prefers $a$ to $b$ and $b$ to $c$. In this situation, it is unclear whether he prefers the set $\{a, c\}$ over the set $\{b\}$. There are multiple approaches for dealing with this problem, see, e.g., [Gär79, BBP04]. We focus in this thesis on set extensions: This term refers to functions that extend a voter's preference over single alternatives to sets of alternatives. The main advantage of set extensions is that we do not need any information but the voter's preference on the alternatives. On the downside, set extensions are usually incomplete, i.e., not all sets can be compared with a set extension. Furthermore, there are multiple different set extensions, see, e.g., [Fis72, Kel77, Neh00] among others, and all of them are reasonable under suitable assumptions.

In this thesis, we focus on Kelly's set extension first presented in [Kel77] and some of its variations. This extension weakly prefers a set of alternatives $X$ to another set of alternatives $Y$ if no alternative in $Y$ is strictly preferred to an alternative in $X$. This definition is motivated by the assumption that a voter has no information about the tie breaking algorithm which is used for eventually selecting a unique winner. In this situation, a voter can only be sure to improve the outcome through a manipulation if all alternatives that are chosen after the manipulation are weakly preferred to the alternatives that were chosen before the manipulation. Otherwise, it is possible that a worse alternative is eventually the winner after tie breaking.

In this thesis, we analyze when voting schemes are manipulable with respect to Kelly's extension. In the rest of this chapter, we introduce the basic terminology in Section 1.2 and provide an overview of known results on strategyproofness in Section 1.3. The remaining chapters are organized as follows: First, we discuss implications between monotonicity and strategyproofness in Chapter 2. The goal of this chapter is to find simple criteria that imply strategyproofness. In the subsequent chapters, we focus on special classes of social choice functions. In particular, we discuss rank-based social choice functions with respect to strategyproofness in Chapter 3 and social choice functions in C2 in Chapter 4. The results in these chapters affect almost all important social choice functions as C2 and rank-basedness are the two main approaches for designing election schemes. In Chapter 5, we discuss social choice functions satisfying responsive efficiency with respect to strategyproofness. Finally, Chapter 6 concludes this thesis.

## 1.2 Problem Definition and Notation

In this section, we discuss basic terminology and notation used in this thesis. First, we introduce the general problem setting considered in this thesis in Section 1.2.1. Thereafter, we discuss well-established axioms for social choice functions in Section 1.2.2. Note that we always stick to our terminology, even if we cite a result of another author who originally used his own notation.

### 1.2.1 Problem Definition

The goal of this section is to provide a general overview of the problems considered in this thesis. Therefore, we first aim to formalize the concept of elections, which leads to the definition of social choice functions. For introducing this term, recall how an election works: Multiple individuals state their preferences over some alternatives and in the end, a set of winners is determined. This means that we consider a finite set of voters $N = \{1, ..., n\}$ and each voter submits a preference over a finite set of

alternatives $A = \{a_1, ..., a_m\}$. If only few alternatives are required in a proof or an example, we usually refer to the alternatives as $a$, $b$, $c$, etc.
The preference $R_i$ is modeled as binary relation on $A$ for every voter $i \in N$. Furthermore, the preference relation satisfies additional requirements depending on the domain we are working in:

- We say $R_i$ is intransitive, denoted by $R_i \in \mathcal{I}$, if $R_i$ is complete.
- We say $R_i$ is in the weak domain, denoted by $R_i \in \mathcal{W}$, if $R_i$ is complete and transitive.
- We say $R_i$ is strict, denoted by $R_i \in \mathcal{S}$, if $R_i$ is complete, transitive and anti-symmetric.

Note that we usually write $a \succeq_i b$ instead of $(a, b) \in R_i$, which means that voter $i$ prefers alternative $a$ weakly to alternative $b$. If $(a, b) \in R_i$ and $(b, a) \in R_i$, we write $a \sim_i b$, which indicates that voter $i$ is indifferent between $a$ and $b$. Furthermore, we write $a \succ_i b$ if $(a, b) \in R_i$ and $(b, a) \notin R_i$. In this case, we say that $a$ is strictly preferred to $b$ by voter $i$. Moreover, if we work in the strict or weak domain, we concatenate multiple preferences, e.g., $R_i = \{(a, b), (a, c), (b, c), (c, b)\}$ is abbreviated by $a \succ_i b \sim_i c$.
In an election, we do not get a single preference as input. Instead, every individual voter submits a preference relation. This leads to a tuple $R = (R_1, ..., R_n)$ which we call a preference profile. Just as individual preferences, a preference profile $R$ is

- in the intransitive domain if $R \in \mathcal{I}^n$.
- in the weak domain if $R \in \mathcal{W}^n$.
- in the strict domain if $R \in \mathcal{S}^n$.

Observe that we often consider multiple preference profiles for a proof or even in examples. We distinguish the various profiles by superscripts or primes, e.g., $R$ and $R'$ denote different profiles. Furthermore, all properties related to a preference profile are identified by the same superscript, e.g., $a \succ'_i b$ denotes that voter $i$ prefers alternative $a$ strictly over $b$ in the preference profile $R'$. Additionally, we sometimes consider profiles in which the preferences of some voters are ignored. Thus, given a profile $R = (R_1, R_2, ..., R_n)$, we define $R_{-i} = (R_1, ..., R_{i-1}, R_{i+1}, ..., R_n)$ as the profile in which the preference of the single voter $i$ is not considered. Similarly, we denote with $R_{-I}$ the preference profile derived from $R$ by ignoring the preferences of the voters in $I \subseteq N$.
We always present preference profiles in one of the forms shown in Figure 1.1. We use the left representation if we require many preference profiles. In this representation, the left-most column contains the name of the preference profile, whereas the remaining columns contain preferences over the alternatives. Furthermore, the

| | 1 | 1 |
|---|---|---|
| $R^1$ | $a \succ b \sim c$ | $b \succ a \sim c$ |

| | 1 | 1 |
|---|---|---|
| $R^1$ : | $a$ | $b$ |
| | $b, c$ | $a, c$ |

Figure 1.1: Example for the representations of preference profiles

number over each column indicates how many voters submit the corresponding preference. Finally, a row forms a preference profile.
The right representation is usually used for better readability and if only few preference profiles are required. It shows only a single preference profile where each column corresponds to the preference of some voters. The meaning behind this notation is the following: If an alternative $a$ is placed above another alternative $b$, then $a \succ b$, and if they are in the same cell, then $a \sim b$. The first row states again how many voters submit a specific preference.
Finally, we introduce social choice functions which are mappings from a set of preference profiles to a non-empty subset of alternatives. This intuition is formalized in the next definition.

**Definition 1.1 (Social choice functions).** *Consider a set of voters $N$, a set of alternatives $A$ and a domain of preferences $\mathcal{D}$ and assume that every voter $i \in N$ submits a preference $R_i \in \mathcal{D}$. A social choice function $f$ is a mapping from the set of preference profiles with respect to $\mathcal{D}$ and $N$ to the set of non-empty subsets of $A$, i.e., $f : \mathcal{D}^{|N|} \mapsto 2^A \setminus \emptyset$.*

We often refer to social choice functions as rule or abbreviate the term with SCF. An instance of a social choice function is $f : \mathcal{W}^3 \mapsto 2^{\{a,b,c\}} \setminus \emptyset : (R_1, R_2, R_3) \to \{a\}$. This function takes the preferences of 3 voters over the alternatives $\{a, b, c\}$ where ties are allowed and returns always $a$ as winner. Note that this example is defined on exactly 3 voters. However, many social choice functions can be defined on an arbitrary number of voters and alternatives. Therefore, we do often not specify the size of the electorate as it does not affect the definition of the corresponding social choice function. In contrast, the domain on which a social function is defined influences its properties and therefore, it is always clarified which types of preferences are used.
Our main goal in this thesis is to analyze various social choice functions with respect to strategyproofness. Thus, we aim to formalize this term next. As social choice functions are set-valued, we first have to define a method for comparing sets of alternatives based on the preference of a voter. For this reason, we introduce a set extension. This term refers to a function generalizing a voter's preference defined on the set of alternatives $A$ to an incomplete binary relation on $2^A \setminus \emptyset$. Note that there are various set extensions discussed in the literature. In this thesis, we focus on the set extension introduced in [Kel77] which is defined in the sequel.

**Definition 1.2 (Kelly's set extension).** *Consider the individual preference $R_i$ of an arbitrary voter $i \in N$. The set extension $\tilde{P}$ is defined as*

$$X \; \tilde{P}_i \; Y \iff \forall x \in X, y \in Y : \; x \succeq_i y \; \wedge \; \exists x \in X, y \in Y : \; x \succ_i y \quad .$$

Informally, a voter prefers a set $X$ over a set $Y$ according to the set extension $\tilde{P}$ if every alternative in $X$ is at least as good as every alternative in $Y$ and at least one alternative in $X$ is strictly preferred to at least one alternative in $Y$. Note that many sets are not comparable by this set extension. For instance, assume that $R_i = a \succ_i b \succ_i c \succ_i d$; then $\{a, d\}$ is not comparable to any other set but $\{d\}$ even though $\{a, c\}$ also seems preferable to $\{a, d\}$.
Next, we define strategyproofness based on the set extension $\tilde{P}$. Intuitively, a social choice function is $\tilde{P}$-strategyproof if no voter can obtain a better outcome with respect to the set extension $\tilde{P}$ by lying about his true preference. Formally, this is defined as follows.

**Definition 1.3 ($\tilde{P}$-strategyproofness).** *A social choice function $f$ is $\tilde{P}$-strategyproof if there are no voter $i \in N$ and preference profiles $R$, $R'$ such that $R_{-i} = R'_{-i}$ and $f(R') \; \tilde{P}_i \; f(R)$.*

We call a social choice function $\tilde{P}$-manipulable if it is not $\tilde{P}$-strategyproof. Note that $\tilde{P}$-strategyproofness is a rather weak variant of strategyproofness because the set extension $\tilde{P}$ allows to compare only few sets. Nevertheless, $\tilde{P}$-strategyproofness is well-established and often considered in the literature for two reasons. On the one hand, there are several interesting $\tilde{P}$-strategyproof social choice functions in the strict domain, which is for many stronger forms of strategyproofness not true. On the other hand, there are also various impossibility results with respect to $\tilde{P}$-strategyproofness in the weak domain which are very strong as only a weak variant of strategyproofness is required. The goal of this thesis is to continue this line of work: We aim to find social choice functions that are $\tilde{P}$-strategyproof or prove that no such functions exist if we assume additional axioms ensuring that the considered social choice functions are reasonable.

### 1.2.2 Axioms for Social Choice Functions

Observe that we use various axioms throughout this thesis for analyzing and characterizing social choice functions. Therefore, we define in this section many standard axioms used in various chapters. Note that the axioms introduced in this section should be known by an expert in the field of computational social choice. Less common Axioms are discussed in more detail in subsequent chapters.
The first terms that we introduce are neutrality and anonymity. The intention of the former axiom is that all alternatives should be treated identically, whereas the latter one states that the voters should be treated identically. The idea for formalizing

$$R^1: \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline a & b & b \\ b & a & a \end{array} \qquad R^2: \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline b & a & a \\ a & b & b \end{array} \qquad R^3: \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline a & b & a \\ b & a & b \end{array}$$

Figure 1.2: Preference profiles used for explaining neutrality and anonymity

neutrality is that renaming alternatives in the preference profile leads to renaming them in the choice set. Formally, this is defined as follows.

**Definition 1.4 (Neutrality).** *A social choice function $f$ satisfies neutrality if for all permutations $\pi : A \mapsto A$ and preference profiles $R$, $R'$ such that $a \succeq_i b$ if and only if $\pi(a) \succeq'_i \pi(b)$ for all $i \in N$ and alternatives $a, b \in A$, it holds that $\pi(f(R)) = f(R')$.*

Even though the definition of neutrality seems rather technical, it is a very desirable axiom that is satisfied by many real elections. The main idea is that if an alternative $a$ is chosen by a social choice function $f$ for a preference profile $R$, another alternative $b$ should be chosen by $f$ if we exchange the roles of $a$ and $b$ in the preference profile $R$. This means that $a$ and $b$ are treated equally, which is usually desired in an election. An example of an application of neutrality is displayed in Figure 1.2 where the preference profile $R^2$ is obtained from the preference profile $R^1$ by renaming $a$ to $b$ and $b$ to $a$. This means that every neutral social choice function $f$ with $f(R^1) = \{b\}$ satisfies that $f(R^2) = \{a\}$.
Our next goal is to introduce anonymity which models that voters are treated equally. As it defines the same property for voters as neutrality does for alternatives, we use the same idea: If we rename the voters, the choice set should not change. Formally, it is defined as follows.

**Definition 1.5 (Anonymity).** *A social choice function $f$ satisfies anonymity if for all permutations $\pi : N \mapsto N$ and preference profiles $R$, $R'$ with $R_{\pi(i)} = R'_i$ for all $i \in N$, it holds that $f(R) = f(R')$.*

Anonymity is a very desirable property that is satisfied by many elections in the real world. Its main idea is that it should not matter who submits a specific vote but only the votes themselves are important. This is reached by allowing to rename the voters without affecting the choice set. An example of an application of anonymity can be seen in Figure 1.2 where the preference profile $R^3$ is obtained from the preference profile $R^2$ by exchanging the preferences of voter 2 and 3. Thus, it follows that $f(R^2) = f(R^3)$ for every anonymous social choice function $f$. Furthermore, if $f$ is also neutral and $f(R^1) = \{a\}$, the choice sets for $R^2$ and $R^3$ are already determined.
While the goal of anonymity and neutrality is to treat voters and alternatives identically, there are also axioms which intend to model that social choice functions should obey the preferences of the voters. Even though this sounds trivial, this property is not formalized by the definition of social choice functions as these functions can return arbitrary choice sets. Furthermore, even though this condition sounds easy at first, it is surprisingly hard to model it properly. A common approach for this issue

is Pareto-optimality which states that a social choice function should not choose an alternative as winner if there is another alternative that is preferred by every voter.

**Definition 1.6 (Pareto-optimality).** *An alternative $a$ is Pareto-dominated by an alternative $b$ in a preference profile $R$ if it holds for all voters $i \in N$ that $b \succeq_i a$ and there exists a voter $i^* \in N$ with $b \succ_{i^*} a$. If $a$ is not Pareto-dominated by any alternative, it is Pareto-optimal. A social choice function $f$ is called Pareto-optimal if $f(R) \subseteq \{a \in A \mid a \text{ is Pareto-optimal in } R\}$ for all preference profiles $R$.*

Pareto-optimality is a very desirable property as it seems not reasonable to choose an alternative as winner if every voter agrees that there is a better alternative. Note that the definition of Pareto-optimality leads straightforwardly to a social choice function which we call Pareto-rule. This rule returns always all Pareto-optimal alternatives and we abbreviate it with PO, especially if we discuss its choice set. For an instance of Pareto-optimality, consider the profile $R^4$ displayed in Figure 1.3. In this profile, every voter prefers $c$ strictly to $d$, which means that $d$ is Pareto-dominated by $c$. Every other alternative is Pareto-optimal as a uniquely first-ranked alternative cannot be Pareto-dominated. Thus, $PO(R^4) = \{a, b, c\}$. Note that the last observation leads to another prominent social choice function which is known as Omninomination-rule. This SCF, abbreviated by OMNI, chooses all alternatives that are first-ranked by at least one voter and clearly, it is another example of a Pareto-optimal SCF in the strict domain.
A similar intention as Pareto-optimality is modeled with an axiom called Condorcet-consistency. The intuition behind this axiom is that it is often easy to decide on a best alternative in a preference profile. This idea is formalized with the notion of the Condorcet winner which refers to an alternative that is preferred to every other alternative by a majority of the voters. For explaining this concept, we first introduce majorities between alternatives.

**Definition 1.7 (Majority for $a$ against $b$).** *Consider a preference profile $R$ and two alternatives $a, b \in A$. The majority for $a$ against $b$ in the profile $R$ is denoted by $n_{ab} = |\{i \in N \mid a \succ_i b\}|$.*

The majority $n_{ab}$ for alternative $a$ against alternative $b$ is simply the number of voters who prefer $a$ strictly to $b$. Note that the concept of majorities leads to many interesting social choice functions and even to a hierarchy among these functions. In fact, the concept of majorities has significantly influenced the research on social choice functions. Based on these majorities, it is easy to define the Condorcet winner.

**Definition 1.8 (Condorcet winner).** *An alternative $a \in A$ is the Condorcet winner in a preference profile $R$ if $n_{ab} > n_{ba}$ for all alternatives $b \in A \setminus \{a\}$.*

Note that a Condorcet winner does not exist in every preference profile. For instance, there is no Condorcet winner in the profile $R^5$ shown in Figure 1.3 because $n_{ab} > n_{ba}$,

$R^4$ :

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ | $d$ |
| $c$ | $d$ | $c$ | $a$ |
| $d$ | $b$ | $d$ | $b$ |

$R^5$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |

Figure 1.3: Preference profiles used for explaining Pareto-optimality and Condorcet-consistency

$n_{bc} > n_{cb}$ and $n_{ca} > n_{ac}$. It should be mentioned that $R^5$ is a well-known profile usually referred to as Condorcet cycle. In contrast, the preference profile $R^4$ in the same figure has a Condorcet winner: The alternative $a$ satisfies $n_{ax} = 3 > n_{xa} = 1$ for all $x \in A \setminus \{a\}$. There are several arguments in favor of always choosing a Condorcet winner if one exists. Most importantly, if we try to choose another alternative as winner, we always find a majority rejecting this idea. This leads to the notion of Condorcet-consistency which demands that a social choice function picks the Condorcet winner as unique winner if it exists.

**Definition 1.9 (Condorcet-consistency).** *A social choice function $f$ satisfies Condorcet-consistency if it holds for all preference profiles $R$ that $f(R) = \{a\}$ if $a$ is the Condorcet winner in $R$.*

Note that we refer to social choice functions satisfying Condorcet-consistency as Condorcet extensions. Furthermore, there are many interesting social choice functions satisfying this axiom. A simple example of a Condorcet extension is the Condorcet-rule which returns the Condorcet winner if it exists; otherwise, it returns all alternatives.

Note that Condorcet-consistency and Pareto-optimality are axioms that ensure small choice sets. Unfortunately, this is often in conflict with $\tilde{P}$-strategyproofness as it becomes easier to manipulate if only few alternatives are chosen. Nevertheless, we desire social choice functions with small choice sets and therefore, we often consider $\tilde{P}$-strategyproofness combined with one of these axioms in subsequent chapters. This leads usually either to interesting social choice functions or to an impossibility result.

## 1.3 Literature Review

In this section, we review existing results on $\tilde{P}$-strategyproofness. Thus, we discuss $\tilde{P}$-strategyproof social choice functions and present well-established impossibility results.

First, we discuss $\tilde{P}$-strategyproof social choice functions in the strict domain. One of the most important results with respect to this setting is a criterion which implies $\tilde{P}$-strategyproofness: Every set-monotonic social choice function defined on the strict domain is $\tilde{P}$-strategyproof [Bra15]. Informally, set-monotonicity means that a social choice function is invariant under the weakening of unchosen alternatives. A more formal definition is discussed in Section 2.1. Note that this result is very helpful for proving the $\tilde{P}$-strategyproofness of social choice functions.

Additionally, Brandt [Bra15] provides two more results on the $\tilde{P}$-strategyproofness of social choice functions. Firstly, if a SCF $f$ is $\tilde{P}$-strategyproof and there is a SCF $g$ such that $f(R) \subseteq g(R)$ and $f(R) = g(R)$ if $|f(R)| = 1|$ for all preference profiles $R \in \mathcal{S}^n$, then $g$ is also $\tilde{P}$-strategyproof. Secondly, the author observes that the strong superset property and monotonicity imply set-monotonicity. Informally, the strong superset property means that a social choice function $f$ is invariant under removing unchosen alternatives from the preference profile, whereas monotonicity is a weakening of set-monotonicity.

We can deduce from the last two remarks that many tournament solutions are $\tilde{P}$-strategyproof in the strict domain. It can be shown that the bipartisan set ($BP$) defined in [LLLB93] satisfies the strong superset property and monotonicity and therefore, this social choice function is $\tilde{P}$-strategyproof. Furthermore, it has also been proven that the minimal covering set ($MC$) introduced in [Dut88], the uncovered set ($UC$) proposed in [Fis77] and [Mil80], and the top cycle ($TC$) (see, e.g., [Bor76]) are $\tilde{P}$-strategyproof because $BP(R) \subsetneq MC(R) \subsetneq UC(R) \subsetneq TC(R)$ for all preference profiles $R \in \mathcal{W}^n$ and all these rules return a single winner if and only if it is the Condorcet winner. It should be mentioned that the $\tilde{P}$-strategyproofness of the top cycle has been independently proven in [MP81]. Furthermore, note that the previous chain of inclusions leads to the question whether the bipartisan set is the finest tournament solution that satisfies $\tilde{P}$-strategyproofness. However, this conjecture has been disproved in [BG16]. Further social choice functions that are known to be $\tilde{P}$-strategyproof in the strict domain are the Condorcet-rule [Gär76, Neh00], the Omninomination-rule [Gär76] and the Pareto-rule [Fel79]. Additionally, it has been shown that the Omninomination-rule, the Pareto-rule and the intersection of these rules are also $\tilde{P}$-strategyproof in the weak domain [BSS].

Next, we discuss impossibility results which have a longer history than the possibility results. The first result of this kind has been shown independently by Gibbard [Gib73] and Satterthwaite [Sat75]. It states that there is no strategyproof, non-dictatorial, non-imposing (i.e., $\forall x \in A : \exists R \in \mathcal{S}^n : f(R) = \{x\}$) and single-valued (i.e., $\forall R \in \mathcal{S}^n : |f(R)| = 1$) social choice function, even if strict preferences are used. The main criticism of this theorem is the single-valuedness as it is unreasonable from a theoretic standpoint.

Thus, impossibility results for set-valued social choice functions have been developed. The first results of this kind have been discussed by Kelly [Kel77] and Barbera [Bar77b]. Both of these results rely heavily on an axiom called quasi-transitive rationalizability which is itself very restrictive. Further research, such as

[Ban82, Ban83, MP81], requires weaker rationalizability axioms for deriving impossibility results. These works lead to new axioms such as minimal binariness and quasi-binariness. However, many social choice functions also fail these axioms and therefore, other approaches have been investigated.

In particular, it is in [Bar77a] shown that every unanimous and positive responsive social choice function cannot be simultaneously non-dictatorial and $\tilde{P}$-strategyproof. While unanimity is a weak assumption, positive responsiveness means that if multiple winners are chosen for a preference profile and a voter reinforces one of them, this alternative becomes the unique winner. As consequence, social choice functions satisfying this axiom are almost single-valued. Furthermore, it should be mentioned that the definition of strategyproofness used in [Bar77a] is even weaker than $\tilde{P}$-strategyproofness.

More recently, computer-aided proofs have been used to show the non-existence of $\tilde{P}$-strategyproof social choice functions in various important settings. With this approach, it has been shown in [BSS] that no pairwise and Pareto-optimal social choice function is $\tilde{P}$-strategyproof and in [Bra11] that no Condorcet extension is $\tilde{P}$-strategyproof. Note that pairwiseness requires that a SCF only depends on the majority margins $n_{ab} - n_{ba}$ of all pairs of alternatives $a, b \in A$. Both of these results require the weak domain, whereas many of the previously mentioned theorems are also true in the strict domain. However, the axioms used for deducing these results are significantly less restrictive than the ones used in earlier impossibility results. Thus, it seems that it is significantly harder to find $\tilde{P}$-strategyproof social choice functions if preferences may contain ties.

Chapter 2

# Monotonicity and Strategyproofness

The main theorem of [Bra15] provides a simple condition for the $\tilde{P}$-strategyproofness of social choice functions. In detail, it states that set-monotonicity implies $\tilde{P}$-strategyproofness in the strict domain. This theorem turns out to be very helpful for proving the $\tilde{P}$-strategyproofness of multiple social choice functions such as the bipartisan set. However, not every $\tilde{P}$-strategyproof social choice function satisfies set-monotonicity. Thus, we analyze in this chapter whether other axioms than set-monotonicity also imply $\tilde{P}$-strategyproofness. Furthermore, even if we do not find such axioms, it might be possible that weaker axioms are sufficient to prove strategyproofness based on weaker set extensions. Therefore, we analyze in this chapter various types of monotonicity and their relations to different variants of strategyproofness.

First, we discuss the required monotonicity axioms and set extensions in Section 2.1. We relate these monotonicity axioms with the various types of strategyproofness in Section 2.2. Note that we work in this section with the weak domain, which means that we generalize the results in [Bra15]. Furthermore, we can even adapt our results to the intransitive domain in Section 2.3. Finally, we consider a strengthening of strategyproofness called group-strategyproofness in Section 2.4. This axiom asks whether a group of voters can manipulate and we show that all our results also hold for the respective variant of group-strategyproofness.

## 2.1 Monotonicity Axioms and Set Extensions

In this section, we discuss various monotonicity axioms and set extensions that are required for the main theorems of this chapter. Note that many axioms introduced here are currently only rarely considered in the literature.

We start with the introduction of monotonicity axioms. The intention of monotonicity axioms is to ensure that social choice functions behave reasonable. This is necessary as the mere mathematical description of these functions allows for arbitrary choice sets. However, in practice, the outcome of a social choice function should be predictable in many situations. For instance, if an alternative is in the choice set for a given preference profile, this alternative should still be chosen if a voter reinforces it in his preference. This is exactly the behavior which is formalized by monotonicity.

**Definition 2.1 (Monotonicity).** *Consider two arbitrary preference profiles $R$, $R'$, a voter $i \in N$ and an alternative $a \in A$ such that $R_{-i} = R'_{-i}$, $x \succeq_i y$ if and only if $x \succeq'_i y$, $a \succeq_i y$ implies $a \succeq'_i y$ and $a \succ_i y$ implies $a \succ'_i y$ for all alternatives $x, y \in A \setminus \{a\}$. We call a social choice function $f$ monotonic if $a \in f(R)$ implies that $a \in f(R')$.*

The intuitive meaning of monotonicity is that a chosen alternative should also be chosen after a voter reinforces it. However, note that this standard definition can be criticized from a practical standpoint: It only ensures that the alternative $a$ is still chosen after it is reinforced, but it does not affect the other alternatives. Therefore, it is possible that other alternatives are additionally chosen, which is often not desirable. For instance, assume that alternative $a$ is the unique winner in a preference profile. Next, a voter reinforces this alternative and as a result, many alternatives can be the winner without violating monotonicity.
To prohibit such a behavior, stronger monotonicity axioms such as set-monotonicity have been designed. This property has been introduced in [Bra15] and generalized to the weak domain in [BBGH15]. The intuition of this axiom is that the weakening of unchosen alternatives should not affect the choice set.

**Definition 2.2 (Set-monotonicity).** *Consider two arbitrary preference profiles $R$, $R'$ such that $x \succeq_i y$ implies $x \succeq'_i y$ and $y \succeq'_i x$ implies $y \succeq_i x$ for all voters $i \in N$. We call a social choice function $f$ set-monotonic if $f(R) = f(R')$.*

This rather strong axiom requires that, unless a chosen alternative is weakened, the choice set does not change. However, this definition is in conflict with the decisiveness of a social choice function: Regardless of how much a voter reinforces a chosen alternative, it is impossible for him to make the choice set smaller without changing his preference between chosen alternatives. For fixing this flaw, we introduce another monotonicity axiom called weak set-monotonicity.

**Definition 2.3 (Weak set-monotonicity).** *Consider two arbitrary preference profiles $R$, $R'$ such that $x \succeq_i y$ implies $x \succeq'_i y$ and $y \succeq'_i x$ implies $y \succeq_i x$ for all voters $i \in N$. We call a social choice function $f$ weakly set-monotonic if $f(R') \subseteq f(R)$.*

Weak set-monotonicity states that, unless a chosen alternative is weakened, no unchosen alternative can become chosen. The most important difference between set-monotonicity and weak set-monotonicity is that the latter allows for refining the choice set. Thus, a voter may be able to remove some alternatives from the choice set by reinforcing a chosen alternative.
For an example that illustrates the differences between the various types of monotonicity, consider Figure 2.1. This figure shows three preference profiles $R^1$, $R^2$ and $R^3$ that are each defined on 3 voters and 4 alternatives. Furthermore, we introduce three social choice function $f_1$, $f_2$ and $f_3$ whose domain is $\{R^1, R^2, R^3\}$.

$R^1$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a, c$ | $c$ |
| $b$ | $b$ | $a, b$ |
| $c, d$ | $d$ | $d$ |

$R^2$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a, c$ | $c$ |
| $b, c$ | $b$ | $a, b$ |
| $d$ | $d$ | $d$ |

$R^3$

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a, c$ | $c$ |
| $b$ | $d$ | $a, b$ |
| $c, d$ | $b$ | $d$ |

Figure 2.1: Preference profiles used for explaining the different monotonicity axioms

Additionally, we assume that $f_1$ is set-monotonic, $f_2$ is weakly set-monotonic and $f_3$ is monotonic. Finally, let $f_1(R^1) = f_2(R^1) = f_3(R^1) = \{a, c\}$. Because of set-monotonicity, it follows for $f_1$ that $f_1(R^2) = f_1(R^3) = f_1(R^1) = \{a, c\}$ as no chosen alternative is weakened if we compare $R^1$ with $R^2$ and $R^3$. We can deduce from weak set-monotonicity for the same reason that $f_2(R^2) \subseteq f_2(R_1)$, e.g., $f(R_2) = \{a\}$ is a valid choice. Furthermore, it holds that $f_2(R^3) = f_2(R^1)$ as the profiles $R^1$ and $R^3$ only differ in preferences between unchosen alternatives. Thus, we can deduce from weak set-monotonicity that $f(R^3) \subseteq f(R^1)$ and $f(R^1) \subseteq f(R^3)$, which means that the choice set cannot change. Finally, consider the social choice function $f_3$: It holds that $c \in f_3(R^2)$ as this alternative is reinforced by the first voter. For instance, $f_3(R^2) = \{b, c\}$ is a valid choice. In contrast, monotonicity cannot be applied for deriving information about $f_3(R^3)$ as $R^3$ is derived from $R^1$ by reinforcing an unchosen alternative. Hence, many outcomes are possible, e.g., $f_3(R^3) = \{a, c, d\}$.
In the previous example, we see that neither $f_1$ nor $f_2$ are allowed to change the choice set if only unchosen alternatives are reordered. In contrast, monotonicity does not imply this property and $f_3$ can choose almost all choice sets for the profile $R^3$. As this property plays an important role in the proof of the main theorem of [Bra15], it seems reasonable to define it as an own axiom called independence of unchosen alternatives.

**Definition 2.4 (Independence of unchosen alternatives).** *A social choice function $f$ satisfies independence of unchosen alternatives if for all preference profiles $R$, $R'$ such that $x \succeq_i y$ if and only if $x \succeq'_i y$ and $y \succeq_i x$ if and only if $y \succeq'_i x$ for all alternatives $x \in f(R)$, $y \in A$ and voters $i \in N$, it holds that $f(R) = f(R')$.*

The intuitive meaning of independence of unchosen alternatives is that a social choice function returns the same choice set if no preferences involving a chosen alternative are modified. As we have seen in the previous example, set-monotonicity and weak set-monotonicity seem related to this axiom. We prove that there is even an implication between these properties.

**Lemma 2.1.** *Every weakly-set monotonic social choice function satisfies independence of unchosen alternatives.*

Proof: Consider an arbitrary weakly set-monotonic social choice function $f$ and two preference profiles $R$ and $R'$ such that $R_i|_{\{x,y\}} = R'_i|_{\{x,y\}}$ for all alternatives $x \in f(R)$, $y \in A$ and voters $i \in N$. It follows from this assumption that we can apply weak set-monotonicity to deduce that $f(R') \subseteq f(R)$. Furthermore, this means that $R_i|_{\{x,y\}} = R'_i|_{\{x,y\}}$ for all $x \in f(R')$, $y \in A$ and $i \in N$ and therefore, weak set-monotonicity implies that $f(R) \subseteq f(R')$. Thus, $f(R) = f(R')$ which shows that $f$ satisfies independence of unchosen alternatives. □

As consequence of this lemma, it follows that every set-monotonic social choice function satisfies independence of unchosen alternatives as set-monotonicity implies weak set-monotonicity. Furthermore, set-monotonicity also implies monotonicity as shown in [Bra15]. In contrast, it can be shown that weak set-monotonicity does not imply monotonicity. This claim follows from the previously considered example: The weakly set-monotonic social choice function $f_2$ satisfies $f_2(R^2) = \{a\}$ even though $c \in f_2(R^1)$ and $c$ is reinforced to derive $R^2$ from $R^1$. This contradicts monotonicity as every monotonic social choice function selects $c$ in the profile $R^2$ if it is selected in $R^1$.

Next, we focus on various set extensions related to Kelly's extension. The reason for this is that our monotonicity axioms do not imply $\tilde{P}$-strategyproofness in the weak domain, but they can be used to prove strategyproofness with respect to weaker set extensions.

**Definition 2.5 (Set extensions).** *Consider the individual preference $R_i$ of an arbitrary voter $i \in N$. We define the following set extensions:*

1. $X\ \bar{P}_i\ Y \iff \forall x \in X, y \in Y:\ x \succ_i y$
2. $X\ \hat{P}_i\ Y \iff \forall x \in X, y \in Y, x \neq y:\ x \succ_i y\ \wedge\ \exists x \in X, y \in Y:\ x \succ_i y$
3. $X\ \tilde{P}_i\ Y \iff \forall x \in X, y \in Y:\ x \succeq_i y\ \wedge\ \exists x \in X, y \in Y:\ x \succ_i y$

For convenience, we include also Kelly's extension in this list even though it has already been introduced in Definition 1.2. The reason for this is that we need Kelly's extension in the subsequent sections. Furthermore, we define two weaker set extensions. The first one is introduced by Nehring in [Neh00] and requires that every alternative in $X$ is strictly preferred to every alternative in $Y$. This set extension is one of the weakest considered in the literature as it only allows to compare very few sets. For instance, $X\ \bar{P}\ Y$ requires that $X$ and $Y$ are disjoint. The second set extension has not been considered in the literature yet. Intuitively, it states that a set $X$ is preferable to a set $Y$ if every alternative in $X$ is strictly better than every alternative in $Y$ with the exception that a single alternative can be in both $X$ and $Y$. This means that the set extensions $\hat{P}$ and $\tilde{P}$ are equal in the strict domain. However, $\hat{P}$ is weaker than $\tilde{P}$ in the weak domain as $\tilde{P}$ allows that multiple alternatives are in the intersection of $X$ and $Y$, whereas $\hat{P}$ only allows for a single one. Furthermore, $\hat{P}$ is stronger than $\bar{P}$ as it can additionally compare sets that intersect in one alternative. This makes this set extension interesting as it lies between $\tilde{P}$ and $\hat{P}$. The notion of $\hat{P}$- and $\bar{P}$-strategyproofness follows from replacing $\tilde{P}$ in Definition 1.3 with the respective set extension.

$R^1$ :

| 1 | 1 |
|---|---|
| $a$ | $d$ |
| $b, c$ | $c$ |
| $d$ | $a, b$ |

$R^2$ :

| 1 | 1 |
|---|---|
| $a$ | $d$ |
| $b$ | $c$ |
| $c, d$ | $a, b$ |

Figure 2.2: Preference profiles used for explaining the various types of strategyproofness

Next, we discuss an example that shows the difference between these notions of strategyproofness. Thus, consider the preference profiles $R^1$ and $R^2$ shown Figure 2.2. Furthermore, let $f_1$, $f_2$ and $f_3$ denote three social choice functions that satisfy $f_1(R^2) = f_2(R^2) = f_3(R^2) = \{a, b\}$, $f_1(R^1) = \{b, c, d\}$, $f_2(R^1) = \{b, d\}$ and $f_3(R^1) = \{d\}$. This means that $f_1$ is $\tilde{P}$-manipulable by the first voter because $f_1(R^2) = \{a, b\}\ \tilde{P}_1\ \{b, c, d\} = f(R^1)$. In contrast, this modification is no $\hat{P}$-manipulation for voter 1 and therefore, it is also no $\bar{P}$-manipulation. The reason for this is that $b \in f_1(R^2)$ is not strictly preferred to $c \in f_1(R^1)$. Moreover, $f_2$ is $\hat{P}$-manipulable because voter 1 prefers every alternative $x \in f_2(R^2)$ strictly to every alternative $y \in f_2(R^1)$, $x \neq y$. However, this is no $\bar{P}$-manipulation as $b \in f_2(R^1) \cap f_2(R^2)$ and an alternative cannot be strictly preferred to itself. Finally, the SCF $f_3$ is $\bar{P}$-manipulable by voter 1 since he prefers both $a$ and $b$ strictly to $d$. Therefore, he can $\bar{P}$-manipulate $f_3$ by switching from $R^1$ to $R^2$.
Note that this example shows on the one hand the differences between the set extensions introduced in Definition 2.5. On the other hand, we see that often only very few preference profiles suffice to discuss strategyproofness. This can also be observed in subsequent sections in which we analyze implications between monotonicity axioms and different types of strategyproofness. For formalizing this observation, we introduce comprehensive subsets of a domain.

**Definition 2.6 (Comprehensive Domain).** *We call a domain $\mathcal{D}$ comprehensive if for every triple of individual preferences $R_i, R'_i, R''_i$ with $R_i \cap R'_i \subseteq R''_i \subseteq R_i \cup R'_i$, it holds that $R''_i \in \mathcal{D}$ if $R_i, R'_i \in \mathcal{D}$.*

Note that this definition differs from the original one suggested in [Neh00] which only requires that $R_i \cap R'_i \subseteq R''_i$. However, our definition is equivalent to the original one in the strict domain and slightly stronger in the weak domain. The reason for this is that the original definition allows for arbitrarily introducing ties which is not possible with Definition 2.6 because of $R''_i \subseteq R_i \cup R'_i$. Furthermore, note that both constraints in Definition 2.6 are independent: $R''_i \subseteq R_i \cup R'_i$ prohibits the arbitrary introduction of ties in $R''_i$ and $R_i \cap R'_i \subseteq R''_i$ prohibits the arbitrary removal of ties in $R''_i$. An example for a comprehensive domain is the set of all preferences extending a partial order on $A$ to a complete and transitive relation.

## 2.2 Results in the Weak Domain

In this section, we present variations of the main theorem in [Bra15] stating that set-monotonicity implies $\tilde{P}$-strategyproofness in the strict domain. It is easy to see that this implication does not hold in the weak domain anymore. Therefore, we try to find similar implications in the weak domain for various types of monotonicity and strategyproofness. Note that we use comprehensive subsets of the weak domain for proving positive results and the strict domain for proving negative results. This strengthens the results as we only require subsets of the weak domain.
As we consider many combinations of various types of monotonicity and strategyproofness, a short overview of the results in this section are presented in Table 2.1. It should be mentioned that this table also illustrates the results of the following sections, i.e., if there is an implication, we can generalize it to the intransitive domain and group-strategyproofness as discussed in Section 2.3 and Section 2.4.
We start the discussion of our results with a conjecture arising from [Bra15]: In this paper, it is observed that set-monotonicity implies monotonicity and independence of unchosen alternatives. Thus, one might conjecture that these two properties entail $\tilde{P}$-strategyproofness. We disprove this conjecture by showing that monotonicity combined with independence of unchosen alternatives does not even imply $\bar{P}$-strategyproofness.

**Theorem 2.1.** *There is a social choice function $f_1 : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that satisfies monotonicity and independence of unchosen alternatives and that violates $\bar{P}$-strategyproofness if $m \geq 4$ and $n \geq 1$.*

Proof: For proving this theorem, we construct a social choice function $f_1$ that satisfies monotonicity and independence of unchosen alternatives but not $P^N$-strategyproofness. For simplicity we design a quasi-dictatorial SCF, i.e., $f_1$ depends only on the preference of a single voter $i$. In the sequel, we focus on the case that $m = 4$ as we

| | $\bar{P}$-strategypr. | $\hat{P}$-strategypr. | $\tilde{P}$-strategypr. |
|---|---|---|---|
| Monotonicity + Independence of unchosen alternatives | $\times$ | $\times$ | $\times$ |
| Weak set-monotonicity | $\surd$ | $\times$ | $\times$ |
| Set-monotonicity | $\surd$ | $\surd$ | $\times$ |

Table 2.1: Overview of the results in Section 2.2. The rows of this table represent the different types of monotonicity and the columns are associated with the different types of strategyproofness. A tick in a cell means that the corresponding variant of monotonicity implies the considered type of strategyproofness in every comprehensive subset of the weak domain. A cross indicates that there is no implication between the type of monotonicity and strategyproofness.

| $R_i$ | $f_1(R)$ | $f_2(R)$ |
|---|---|---|
| $a \succ b \succ c \succ d$ | $\{c, d\}$ | $\{a\}$ |
| $a \succ b \succ d \succ c$ | $\{a, d\}$ | $\{a\}$ |
| $a \succ c \succ b \succ d$ | $\{a, c\}$ | $\{a\}$ |
| $a \succ c \succ d \succ b$ | $\{a, c\}$ | $\{a\}$ |
| $a \succ d \succ b \succ c$ | $\{a, d\}$ | $\{a\}$ |
| $a \succ d \succ c \succ b$ | $\{a, d\}$ | $\{a\}$ |
| $b \succ a \succ c \succ d$ | $\{c, d\}$ | $\{a\}$ |
| $b \succ a \succ d \succ c$ | $\{b, d\}$ | $\{a\}$ |
| $b \succ c \succ a \succ d$ | $\{b, c\}$ | $\{a, d\}$ |
| $b \succ c \succ d \succ a$ | $\{b, c\}$ | $\{a, b, c, d\}$ |
| $b \succ d \succ a \succ c$ | $\{b, d\}$ | $\{a, c\}$ |
| $b \succ d \succ c \succ a$ | $\{b, d\}$ | $\{a, b, c, d\}$ |
| $c \succ a \succ b \succ d$ | $\{a, c\}$ | $\{a\}$ |
| $c \succ a \succ d \succ b$ | $\{a, c\}$ | $\{a\}$ |
| $c \succ b \succ a \succ d$ | $\{b, c\}$ | $\{a, d\}$ |
| $c \succ b \succ d \succ a$ | $\{b, c\}$ | $\{a, b, c, d\}$ |
| $c \succ d \succ a \succ b$ | $\{a, b\}$ | $\{a, b\}$ |
| $c \succ d \succ b \succ a$ | $\{b, c\}$ | $\{a, b, c, d\}$ |
| $d \succ a \succ b \succ c$ | $\{a, d\}$ | $\{a\}$ |
| $d \succ a \succ c \succ b$ | $\{a, d\}$ | $\{a\}$ |
| $d \succ b \succ a \succ c$ | $\{b, d\}$ | $\{a, c\}$ |
| $d \succ b \succ c \succ a$ | $\{b, d\}$ | $\{a, b, c, d\}$ |
| $d \succ c \succ a \succ b$ | $\{a, b\}$ | $\{a, b\}$ |
| $d \succ c \succ b \succ a$ | $\{d, b\}$ | $\{a, b, c, d\}$ |

Table 2.2: Social choice functions used for proving Theorem 2.1 and Theorem 2.3

can easily generalize $f_1$ to more alternatives by adding dummy alternatives which that are never chosen and that do not affect the social choice function. Under these assumptions, we define $f_1$ as follows: If $R_i = a \succ b \succ c \succ d$ or $R_i = b \succ a \succ c \succ d$, then $f_1(R) = \{c, d\}$ and if $R_i = c \succ d \succ a \succ b$ or $R_i = d \succ c \succ a \succ b$, then $f_1(R) = \{a, b\}$. Otherwise, $f_1$ returns the winner of the comparison of $a$ against $b$ and the winner of the comparison of $c$ against $d$ in the preference of voter $i$. The complete social choice function $f_1$ is displayed in the second column of Table 2.2.

First, we analyze the social choice function $f_1$ with respect to $\bar{P}$-strategyproofness, monotonicity and independence of unchosen alternatives. Thus, note that $f_1$ is $\bar{P}$-manipulable: If the true preference of voter $i$ is $a \succ b \succ c \succ d$, he can $\bar{P}$-manipulate by submitting $c \succ d \succ a \succ b$ as $\{a, b\}\ \bar{P}_i\ \{c, d\}$.

Next, we show that this rule also satisfies monotonicity and independence of unchosen alternatives. As only the preference of voter $i$ affects the choice set of $f_1$, we focus on this voter. Even more, if we ignore the special cases of $f_1$, it is clear that monotonicity and independence of unchosen alternatives are satisfied as $f_1$ always returns the winners of the pairwise comparison of $a$ against $b$ and $c$ against $d$ in the preference $R_i$.

Thus, we consider the special cases and assume that $R_i = a \succ b \succ c \succ d$ because all special cases are symmetric. First note that $R_i^* = b \succ a \succ c \succ d$ is the only other preference such that $R_i$ is related to it by independence of unchosen alternatives. As $f_1$ returns the same choice for every preference profile $R$ in which voter $i$ submits $R_i$ or $R_i^*$, independence of unchosen alternatives is satisfied. Next, we prove that no other preference $R_i' \in \mathcal{S} \setminus \{R_i, R_i^*\}$ is related to $R_i$ by independence of unchosen alternatives. Therefore, we consider an arbitrary preference profile $R' \in \mathcal{S}^n$ with $R_i' \notin \{R_i, R_i^*\}$ and make a case distinction with respect to $f_1(R')$. If $d \in f_1(R')$, then $d \succ_i' c$, which contradicts that $R_i'$ and $R_i$ are related by independence of unchosen alternatives. If $c \in f(R')$, then $c$ is preferred to at least two alternatives by voter $i$ as $c \succ_i' d$ and $R_i' \notin \{R_i, R_i^*\}$. This contradicts again that $R_i$ and $R_i'$ are related by independence of unchosen alternatives. Finally, if neither $c$ nor $d$ is returned, then we are in the second special case in which $b$ is the least preferred alternative of voter $i$. It follows again that $R_i$ and $R_i'$ are not related by independence of unchosen alternatives and we can deduce that $f_1$ satisfies this axiom.

Next, we discuss why the SCF $f_1$ satisfies monotonicity in the special cases. We focus again on $R_i = a \succ b \succ c \succ d$ since the other special cases are symmetric. First note that whenever voter $i$ reinforces $c$ or $d$ this alternative is still in the choice set as it wins the pairwise comparison. Thus, monotonicity holds if we reinforce a winning alternative in $R_i$. Moreover, we show that this axiom is also satisfied if voter $i$ switches from an arbitrary preference $R_i'$ to $R_i$. Therefore, observe that, unless $R_i' = b \succ a \succ c \succ d$, a winning alternative is weakened in $R_i'$ to switch to $R_i$. The reason for this is that in every other preference, at least one alternative in $\{c, d\}$ is strictly preferred to at least one alternative in $\{a, b\}$ by voter $i$ or he prefers $d$ over $c$. As this is not true in $R_i$, a winning alternative is weakened to go from $R_i'$ to $R_i$ and therefore, monotonicity does not relate the choice sets $f(R)$ and $f(R')$. Note that this argument does not hold for the second special case $c \succ d \succ a \succ b$ and $d \succ c \succ a \succ b$. However, we have to reinforce both $a$ and $b$ to derive $R_i$ from this preference. This means that the special cases are not related by monotonicity either. Furthermore, if $R_i' = b \succ a \succ c \succ d$, then $f_1$ returns the same choice sets for the corresponding preference profiles $R$ and $R'$. Hence, we can finally conclude that $f_1$ is monotonic, which proves this theorem. □

Observe that the social choice function used in the proof of Theorem 2.1 can be extended to the weak domain. It still compares $a$ against $b$ and $c$ against $d$ in the preference of voter $i$ and chooses the winners unless $a$ is uniquely-third ranked and $b$ is uniquely last-ranked or $c$ is uniquely third-ranked and $d$ is uniquely last-ranked. If voter $i$ submits a tie between $a$ and $b$ or $c$ and $d$, then both alternatives are chosen unless we are in a special case. In the special cases, we still choose $\{a, b\}$ or $\{c, d\}$ depending on the last-ranked alternatives. It can be easily checked that this

leads to a social choice function in the weak domain that satisfies monotonicity and independence of unchosen alternatives but violates $\bar{P}$-strategyproofness.
Note that Theorem 2.1 leads to the question whether there is an implication between a stronger monotonicity axiom and $\bar{P}$-strategyproofness. We give a positive answer to this question by showing that weak set-monotonicity implies $\bar{P}$-strategyproofness in the weak domain. Note that this proves the sixth remark of [Bra15]. However, for the proof of the following theorem, we need to restrict preferences to a subset of alternatives. Therefore, we define the restriction of $R_i$ to the set of alternatives $X \subseteq A$ as $R_i|_X = R_i \cap X^2$.

**Theorem 2.2.** *Every weakly set-monotonic social choice function $f$ defined on a comprehensive subset of the weak domain is $\bar{P}$-strategyproof.*

Proof: Assume for contradiction that there is a social choice function $f$ that is defined on a comprehensive subset $D^n$ of the weak domain and that is weakly set-monotonic but not $\bar{P}$-strategyproof. Thus, there are preference profiles $R$, $R' \in D^n$ and a voter $i \in N$ such that $f(R')\ \bar{P}_i\ f(R)$ and $R_{-i} = R'_{-i}$. It follows from the definition of $\bar{P}$-strategyproofness that $f(R') \cap f(R) = \emptyset$ and that $Y\ \bar{P}_i\ X$ for all $X \subseteq f(R)$, $Y \subseteq f(R')$. We show that there is a preference profile $R^*$ such that $f(R) = f(R^*) \subseteq f(R')$ and $R_{-i} = R^*_{-i} = R'_{-i}$, which contradicts that $f(R')\ \bar{P}_i\ f(R)$. For constructing this preference profile $R^*$, we partition the alternatives into two sets $U$ and $L$. For this step, we denote with $m$ one of voter $i$'s least preferred alternative in $f(R')$, i.e., $\forall x \in f(R') : x \succeq_i m$. Then, $U = \{x \in A \mid x \succeq_i m\}$ is the set of all alternatives weakly preferred to $m$ by voter $i$ in the preference profile $R$ and $L = A \setminus U$. With the help of these sets, we define the preference $R^*_i$ as follows.

$$R^*_i = R'_i|_U \ \cup\ R_i|_L \ \cup\ \{(x,y) \mid x \in U, y \in L\}$$

Next, we explain why $f(R) = f(R^*)$ holds. For this reason, observe that the transitivity of $R_i$ implies that $y \succ_i x$ for all $y \in U$ and $x \in L$ because $y \succeq_i m$ and $m \succ_i x$. Furthermore, it holds that $f(R) \subseteq L$ because $m \succ_i x$ for all $x \in f(R)$. This means that voter $i$'s preference between alternatives in $U$ and alternatives in $f(R)$ is the same in $R_i$ and $R^*_i$, i.e., for all $x \in U$ and $y \in f(R)$ it holds that $x \succ_i y$ and $x \succ^*_i y$. Moreover, we have by construction that $R_i|_L = R^*_i|_L$. Hence, we can apply independence of unchosen alternatives to deduce that $f(R) = f(R^*)$.
Furthermore, we discuss why $f(R^*) \subseteq f(R')$. For proving this inclusion, we define the auxiliary preference profile $R^{aux} = (R'_{-i}, R^{aux}_i)$ where $R^{aux}_i$ differs from $R^*_i$ only in the fact that the alternatives in $L$ are ordered according to $R'_i$ instead of $R_i$. This leads to the following formal definition.

$$R^{aux}_i = R'_i|_U \cup R'_i|_L \cup \{(x,y) \mid x \in U, y \in L\}$$

Observe that $f(R^{aux}) \subseteq f(R')$ as we can apply weak set-monotonicity in $R'_i$ to weaken the alternatives in $L$ until we reach $R^{aux}_i$. Furthermore, we can deduce from independence of unchosen alternatives that $f(R^*) = f(R^{aux})$ because $R^*_i$ and $R^{aux}_i$

only differ in preferences between alternatives in $L$. This is no problem as every alternative in $f(R^{aux})$ is strictly preferred to every alternative in $L$. Consequently, it holds that $f(R) = f(R^*) \subseteq f(R')$ which contradicts $f(R') \; \bar{P}_i \; f(R)$.
Finally, we show that $R_i^*$ is an element of the considered comprehensive subset $D$ of the weak domain. This is true since we have for all $x, y \in A$ with $x \succeq_i^* y$ that either $x \succeq_i y$ or $x \succeq_i' y$: If $x, y \in U$, this observation follows from $R_i^*|_U = R_i'|_U$; if $x \in L, y \in A$, we have $x \succeq_i^* y$ if and only if $x \succeq_i y$. Thus, it is obvious that $R_i \cap R_i' \subseteq R_i^* \subseteq R_i \cup R_i'$, which implies that $R_i^* \in D$. □

Note that we are not aware of commonly discussed social choice functions that are defined on the weak domain and that satisfy weak set-monotonicity but violate set-monotonicity. However, there are multiple simple examples: For instance, consider the social choice function that returns $a$ and $b$ for all profiles but those in which only one alternative in $\{a, b\}$ is top-ranked by all voters; in this case we only return the alternative that is top-ranked by everyone. This social choice function is clearly weakly set-monotonic and therefore, we can deduce from Theorem 2.2 that it is $\bar{P}$-strategyproof. However, it is not set-monotonic since it might happen that a voter submits $c \succ a \succ b$ and after weakening $c$, every voter prefers $a$ the most. In this scenario, the constructed social choice function returns initially $\{a, b\}$ and after weakening $c$, it only returns $\{a\}$, which contradicts set-monotonicity.
Furthermore, note that Theorem 2.1 and Theorem 2.2 give a tight bound on the strength of monotonicity axioms required for proving $\bar{P}$-strategyproofness. Nevertheless, it might be possible to generalize Theorem 2.2 by showing that weak set-monotonicity implies also stronger variants of strategyproofness. Unfortunately, this conjecture is false because we can construct a social choice function that is weakly set-monotonic and $\hat{P}$-manipulable.

**Theorem 2.3.** *There is a social choice function $f_2 : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that satisfies weak set-monotonicity but not $\hat{P}$-strategyproofness if $m \geq 4$ and $n \geq 1$.*

Proof: We construct a social choice function $f_2$ that is weakly set-monotonic but not $\hat{P}$-strategyproof. As in the proof of Theorem 2.1, we focus on the case that $m = 4$ because we can always add dummy alternatives that are never chosen and that do not affect the behavior of $f_2$. Furthermore, the SCF $f_2$ that we construct in the sequel is quasi-dictatorial, i.e., its outcome only depends on a single voter $i$. The social choice function $f_2$ is defined as follows: If alternative $a$ is the most preferred alternative or the second most preferred alternative of voter $i$ in the preference profile $R$, then $f_2(R) = \{a\}$; if alternative $a$ is ranked third by voter $i$, the choice set consists of alternative $a$ and the least preferred alternative of voter $i$; if alternative $a$ is ranked last by voter $i$, then $f(R) = A$. The last column of Table 2.2 shows an explicit description of $f_2$ depending only on the preference $R_i$.
First note that voter $i$ can $\hat{P}$-manipulate $f_2$: If his true preference is $c \succ b \succ a \succ d$, then $f_2(R) = \{a, d\}$. However, $\{a, c\} \; \hat{P}_i \; \{a, d\}$ and voter $i$ can achieve this outcome by submitting $d \succ b \succ a \succ c$.

Next, we show that the SCF $f_2$ is indeed weakly set-monotonic. Note for this that $f_2$ only depends on the preference of voter $i$ and therefore, it is trivially weakly set-monotonic for all voters $j \neq i$. Hence, we focus in the sequel on the preference submitted by voter $i$. If alternative $a$ is the least preferred alternative of voter $i$, then all alternatives are chosen. Hence, every choice set is a subset and weak set-monotonicity trivially holds. Next, assume that $a$ is ranked third and that an unchosen alternative is weakened by voter $i$. If $a$ is ranked third afterwards, then the least preferred alternative is also the same, which means that the choice set has not changed. Otherwise, alternative $a$ is now the most preferred or second most preferred alternative of voter $i$, which means that $a$ is the unique winner. Thus, it follows that $f_2$ satisfies weak set-monotonicity in this case. Finally, if alternative $a$ is among the two most preferred alternatives of voter $i$ and we weaken an unchosen alternative, $a$ is still preferred to at least two other alternatives. Hence, the choice set does not change and we can deduce that $f_2$ satisfies weak set-monotonicity in all cases. □

Note that the social choice function $f_2$ can be extended to the weak domain. Therefore, we define the rank of an alternative $x$ in the preference $R_i$ of voter $i$ as $r(x, R_i) = |A| - |\{a \in A \mid x \succ_i a\}|$. Based on this definition, we extend $f_2$ as follows: If $r(a, R_i) = |A|$, we return every alternative; if $r(a, R_i) = |A| - 1$, we return $a$ and the alternative $b$ that satisfies $a \succ_i b$; otherwise, we return $a$. It is easy to see that this approach leads to a weakly set-monotonic SCF in the weak domain that is $\hat{P}$-manipulable. Furthermore, it should be mentioned that the social choice function $f_2$ is also monotonic. Thus, not even weak set-monotonicity paired with monotonicity implies $\hat{P}$-strategyproofness.
In contrast, we can show that set-monotonicity implies $\hat{P}$-strategyproofness in the weak domain. This gives again a tight bound for the weakest monotonicity axiom required for proving $\hat{P}$-strategyproofness.

**Theorem 2.4.** *Every set-monotonic social choice function $f$ defined on a comprehensive subset of the weak domain is $\hat{P}$-strategyproof.*

Proof: Assume for contradiction that there is a social choice function $f$ that is defined on a comprehensive subset $D^n$ of the weak domain and that is set-monotonic but not $\hat{P}$-strategyproof. Hence, there are preference profiles $R, R' \in D^n$ and a voter $i \in N$ such that $R_{-i} = R'_{-i}$ and $f(R') \ \hat{P}_i \ f(R)$. We derive a contradiction to this assumption by constructing a preference profile $R^*$ such that $R^*_{-i} = R_{-i} = R'_{-i}$ and $f(R) = f(R^*) = f(R')$ contradicting that $f(R') \ \hat{P}_i \ f(R)$.
As in the proof of Theorem 2.2, we partition the alternatives $A$ into two sets $U$ and $L$. For this step, we denote with $m$ one of voter $i$'s least preferred alternatives in $f(R')$, i.e., $\forall x \in f(R') : x \succeq_i m$. Furthermore, let $U = \{x \in A \mid x \succeq_i m\}$ and $L = A \setminus U$. We use these sets to define $R^*_i$: The alternatives of $U$ are ordered according to $R'_i$ and they are strictly preferred to all alternatives in $L$. Furthermore, the alternatives of $L$ are ordered according to $R_i$. Formally,

$$R_i^* = R_i'|_U \ \cup \ R_i|_L \ \cup \ \{(x,y) \mid x \in U, y \in L\} \quad .$$

As next step, we explain why the equation $f(R') = f(R^*)$ is true. For this reason, we consider the auxiliary profile $R^{aux1} = (R'_{-i}, R_i^{aux1})$ in which the preference of voter $i$ is defined as

$$R_i^{aux1} = R_i'|_U \ \cup \ R_i'|_L \ \cup \ \{(x,y) \mid x \in U, y \in L\} \quad .$$

Observe that $f(R') = f(R^{aux1})$ since we can use set-monotonicity to weaken the alternatives in $L$ in the preference $R_i'$ until we arrive at $R_i^{aux1}$. Moreover, it holds that $f(R^{aux1}) = f(R^*)$ as we can apply independence of unchosen alternatives to $R_i^{aux1}$ to reorder the alternatives of $L$ in $R_i^{aux1}$ according to $R_i^*$. Thus, independence of unchosen alternatives and set-monotonicity imply that $f(R') = f(R^{aux1}) = f(R^*)$. Furthermore, we show that $f(R) = f(R^*)$. It is important for the proof of this equivalence that $f(R') \ \hat{P}_i \ f(R)$ implies that $|f(R) \cap f(R')| \leq 1$. This observation results in a case distinction with respect to the size of $f(R) \cap f(R')$. First, we assume that $f(R) \cap f(R') = \emptyset$. In this case, voter $i$ can directly apply independence of unchosen alternatives to reorder the alternatives in $U$ in the same way as in $R_i^*$. The reason for this is that it holds before and after this change that $y \succ_i x$ for all $x \in f(R)$, $y \in U$. It follows that $f(R) = f(R^*) = f(R')$, which contradicts the initial assumption of $f(R') \ \hat{P}_i \ f(R)$.
For the second case, we assume that there is an alternative $m$ in $f(R) \cap f(R')$. Note that this alternative satisfies that $m \succ_i x$ for all $x \in f(R)$ and $y \succ_i m$ for all $y \in f(R')$. Otherwise, there are alternatives $x \in f(R)$, $y \in f(R')$, $x \neq y$, such that $x \succeq_i y$ contradicting that $f(R') \ \hat{P}_i \ f(R)$. We use this fact to define an auxiliary preference profile $R^{aux2}$ with $R_{-i}^{aux2} = R_{-i}$ and $f(R) = f(R^{aux2}) = f(R^*)$. Formally, we define $R_i^{aux2}$ as

$$R_i^{aux2} = R_i'|_{U \setminus \{m\}} \ \cup \ R_i|_{L \cup \{m\}} \ \cup \ \{(x,y) \mid x \in U \setminus \{m\}, y \in L \cup \{m\})\} \quad .$$

It follows from independence of unchosen alternatives that $f(R) = f(R^{aux2})$ because we only reorder the alternatives $U \setminus \{m\}$ in $R_i$. Next, observe that $R_i^{aux2}$ and $R_i^*$ only differ in the position of alternative $m$. Hence, we can now apply set-monotonicity to move $m$ up in the preference $R_i^{aux}$ until we reach $R_i^*$. This implies that $f(R) = f(R^{aux2}) = f(R^*) = f(R')$, which is a contradiction to $f(R') \ \hat{P}_i \ f(R)$. Consequently, $f$ is $\hat{P}$-strategyproof.
Finally, we show that $R_i^*$ is also element of $D$. This claim follows from the same argument as presented in the proof of Theorem 2.2: It holds that $x \succeq_i^* y$ if and only if $x \succeq_i' y$ for all $x, y \in U$ and that $x \succeq_i^* y$ if and only if $x \succeq_i y$ for all $x \in A$, $y \in L$. Thus, it is easy to see that $R_i^* \in D$ because $D$ is a comprehensive domain. ☐

As a consequence of this theorem, there are many $\hat{P}$-strategyproof social choice functions in the weak domain. For instance, it has been shown in [BBH18] that

$R^1$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a$ | $b$ |
| $c$ | $c$ | $d$ |
| $d$ | $d$ | $c$ |
| $b$ | $b$ | $a$ |

$R^2$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a$ | $b$ |
| $c$ | $c$ | $c$ |
| $d$ | $d$ | $d$ |
| $b$ | $b$ | $a$ |

Figure 2.3: Preference profiles used to show that $f_3$ does not satisfy independence of unchosen alternatives

every tournament solution that is set-monotonic in the strict domain can be generalized with the conservative extension to the weak domain while maintaining set-monotonicity. Together with Theorem 2.4, this implies that many tournament solutions, such as the bipartisan set and the top cycle, can be generalized to social choice functions that are $\hat{P}$-strategyproof in the weak domain. This observation is remarkable as $\hat{P}$-strategyproofness is only slightly weaker than $\tilde{P}$-strategyproofness, but it is a consequence of theorem 2 in [BSS] that no tournament solution is both Pareto-optimal and $\tilde{P}$-strategyproof.

Finally, we consider set-monotonicity paired with $\tilde{P}$-strategyproofness in the weak domain. In this domain, it is easy to show that these two axioms are independent. It is obvious that $\tilde{P}$-strategyproofness does not imply set-monotonicity. For the other direction, we consider the Condorcet-rule. This social choice function picks the Condorcet winner if there is one; otherwise, it returns all alternatives. This rule is not $\tilde{P}$-strategyproof in the weak domain as shown in [BB11], but it is set-monotonic: If all alternatives are picked, set-monotonicity does not allow for any change. If the Condorcet winner is chosen, this alternative clearly stays the winner if a voter reinforces it or reorders unchosen alternatives. As set-monotonicity implies $\hat{P}$-strategyproofness also in the strict domain, it seems more accurate to state that set-monotonicity implies $\hat{P}$-strategyproofness instead of $\tilde{P}$-strategyproofness.

Furthermore, observe that no implication in this section can be inverted. We consider the social choice function $f_3$ that is a variation of the OMNI-rule in order to prove this claim. The OMNI-rule returns all alternatives that are first-ranked by at least one voter. The SCF $f_3$ does the same unless every voter prefers an alternative in $\{a, b\}$ uniquely the most, both $a$ and $b$ are at least once first-ranked and every voter prefers $c$ uniquely second most. For such profiles, $f_3$ returns $\{a, b, c\}$. This rule violates independence of unchosen alternatives if there are at least 4 alternatives as $f_3(R^1) = \{a, b\}$ and $f_3(R^2) = \{a, b, c\}$ where the profiles $R^1$ and $R^2$ are shown in Figure 2.3. This contradicts independence of unchosen alternatives and therefore, neither weak set-monotonicity nor set-monotonicity are satisfied by $f_3$. However, it is easy to check that the SCF $f_3$ is $\tilde{P}$-strategyproof. Even more, Theorem 4.1 discussed in a subsequent chapter provides a simple criterion which can be used to prove the $\tilde{P}$-strategyproofness of $f_3$ with the help of the OMNI-rule. Thus, there is a $\tilde{P}$-strategyproof social choice function that violates independence of unchosen alternatives and therefore, no result of this section can be inverted.

## 2.3 Generalizations to the Intransitive Domain

In the following, we discuss generalizations of the theorems presented in the last section to the intransitive domain. This domain is only rarely considered in the literature as the absence of transitivity makes many theorems invalid. Nevertheless, we show that variants of Theorem 2.2 and Theorem 2.4 are also true in the intransitive domain. However, we require the complete intransitive domain instead of discussing comprehensive subsets.
We start the discussion of our results by generalizing Theorem 2.2. Note that even though Theorem 2.5 and its proof seem similar to this theorem, there are many important differences. For instance, we require the complete intransitive domain instead of a comprehensive subset for the following result.

**Theorem 2.5.** *Every weakly set-monotonic social choice function defined on the intransitive domain is $\bar{P}$-strategyproof.*

Proof: Assume for contradiction that there is a social choice function $f : \mathcal{I}^n \mapsto 2^A \setminus \emptyset$ that is weakly set-monotonic but not $\bar{P}$-strategyproof. Thus, there are preference profiles $R, R' \in \mathcal{I}^n$ and a voter $i \in N$ such that $R_{-i} = R'_{-i}$ and $f(R')\ \bar{P}_i\ f(R)$. Note that the definition of $\bar{P}$-strategyproofness implies that $f(R') \cap f(R) = \emptyset$ and that all alternatives in $f(R')$ are strictly preferred to those in $f(R)$ by voter $i$. Hence, we can derive a contradiction by constructing a preference profile $R^*$ such that $R^*_{-i} = R_{-i}$ and $f(R) = f(R^*) \subseteq f(R')$. Therefore, we define $R^*_i$ as follows:

$$R^*_i = R'_i|_{f(R')} \cup R_i|_{A \setminus f(R')} \cup \{(x, y) \mid x \in f(R'),\ y \in A \setminus f(R')\} \quad .$$

Observe that $f(R) = f(R^*)$ holds because we can apply independence of unchosen alternatives to reorder the preferences of voter $i$ that involve alternatives in $f(R')$. More precisely, for every $x \in f(R'), y \in A \setminus f(R')$ with $x \succeq_i y$ we change the preference to $y \succ_i x$ and we order the alternatives in $f(R')$ according to $R'_i$. None of these modifications involves an element of $f(R)$ because $x \succ_i y$ is true for all $x \in f(R'), y \in f(R)$; otherwise, $f(R')$ is no $\bar{P}$-improvement over $f(R)$. As a consequence, it follows that $f(R) = f(R^*)$.
Finally, we explain why $f(R^*) \subseteq f(R')$ is true. Note for this that we can switch from $x \succeq_i y$ to $y \succ_i x$ for all $x \in A \setminus f(R')$, $y \in f(R')$ while ensuring that no unchosen alternatives becomes chosen because of weak set-monotonicity. It also suffices to make only these modifications since the preference of voter $i$ is allowed to be intransitive. This leads to an auxiliary preference profile $R^{aux} = (R'_{-i}, R^{aux}_i)$ such that $f(R^{aux}) \subseteq f(R^*)$ and the alternatives in $f(R')$ are strictly preferred to all other alternatives. Formally, $R^{aux}_i$ is defined as

$$R^{aux}_i = R'_i|_{f(R')} \ \cup\ R'_i|_{A \setminus f(R')} \ \cup\ \{(x, y) \mid x \in f(R'), y \in A \setminus f(R')\} \quad .$$

Next, we can apply independence of unchosen alternatives to reorder the alternatives in $A \setminus f(R')$ in the preference $R^{aux}_i$ according to $R_i$. After this step, we arrive at $R^*_i$

and therefore, it results from independence of unchosen alternatives and weak set-monotonicity that $f(R^*) = f(R^{aux}) \subseteq f(R')$. This contradicts the initial assumption of $f(R')\ \bar{P}_i\ f(R)$. Thus, this assumption is wrong and $f$ is $\bar{P}$-strategyproof. □

Note that the proof of Theorem 2.2 cannot be used to show Theorem 2.5 as the latter result works in the intransitive domain. For showing the result in the weak domain, we have defined the set $U$ containing all alternatives that are weakly preferred to voter $i$'s least preferred alternative in $f(R')$. Furthermore, it is important for deriving the preference $R_i^*$ that voter $i$ prefers every alternative in $U$ strictly to every alternative in $f(R)$. However, it might happen that there is an alternative $x$ that voter $i$ prefers to his worst alternative in $f(R')$ and an alternative $y \in f(R)$ with $y \succ_i x$ as his preference can be intransitive. Hence, we have to modify a preference involving a chosen alternative to go from $R_i$ to $R_i^*$ and consequently, weak set-monotonicity does not relate the choice sets $f(R)$ and $f(R^*)$. However, this is only possible if we allow intransitive preferences as otherwise all alternatives in $U$ are strictly preferred to those in $f(R)$.
The same problem occurs if we try to generalize Theorem 2.4 to the intransitive domain without adapting the proof. Therefore, we provide another argument similar to the one presented in the proof of Theorem 2.5 which shows that set-monotonic social choice functions are $\hat{P}$-strategyproof in the intransitive domain.

**Theorem 2.6.** *Every set-monotonic social choice function defined on the intransitive domain is $\hat{P}$-strategyproof.*

Proof: Assume for contradiction that there is a social choice function $f : \mathcal{I}^n \mapsto 2^A \setminus \emptyset$ that is set-monotonic but not $\hat{P}$-strategyproof. Thus, there are preference profiles $R, R' \in \mathcal{I}^n$ and a voter $i \in N$ such that $R_{-i} = R'_{-i}$ and $f(R')\ \hat{P}_i\ f(R)$. We construct in the sequel a preference profile $R^*$ such that $R_{-i} = R^*_{-i} = R'_{-i}$ and $f(R) = f(R^*) = f(R')$, which contradicts that $f(R')\ \hat{P}_i\ f(R)$. Therefore, we define $R_i^*$ as

$$R_i^* = R'_i|_{f(R')} \ \cup\ R_i|_{A\setminus f(R')} \ \cup\ \{(x,y) \mid x \in f(R'), y \in A \setminus f(R')\} \quad .$$

Next, we show that $f(R') = f(R^*)$. For proving this equality, we use set-monotonicity to switch for all alternatives $x \in f(R'), y \in A \setminus f(R')$ with $y \succeq_i x$ to $x \succ_i y$ in the preference $R'_i$. It suffices to make only these modifications as voter $i$'s preference is allowed to be intransitive. This step leads to an auxiliary preference profile $R^{aux1}$ such that $R^{aux1}_{-i} = R'_{-i}$ and all alternatives in $f(R')$ are preferred to those in $A\setminus f(R')$ in $R_i^{aux1}$. Formally, $R_i^{aux1}$ is defined as

$$R_i^{aux1} = R'_i|_{f(R')} \ \cup\ R'_i|_{A\setminus f(R')} \ \cup\ \{(x,y) \mid x \in f(R'), y \in A \setminus f(R')\} \quad .$$

As we apply set-monotonicity to derive the preference $R_i^{aux1}$ from $R'_i$, it follows that $f(R') = f(R^{aux1})$. Thereafter, we reorder the alternatives in $A \setminus f(R')$ in the preference $R_i^{aux1}$ according to $R_i$, which results in $R_i^*$. As we only change preferences

between unchosen alternatives, it follows from independence of unchosen alternatives that $f(R') = f(R^{aux1}) = f(R^*)$.
Finally, we discuss why $f(R) = f(R^*)$ is true. For proving this equality, it is important that $|f(R) \cap f(R')| \leq 1$, which follows directly form the definition of $\hat{P}$-strategyproofness. This observation leads to a case distinction on the size of the intersection of the choice sets. First, assume that $f(R) \cap f(R') = \emptyset$. In this case, it follows for all $x \in f(R)$ and $y \in f(R')$ that $y \succ_i x$ as otherwise $f(R')$ is no $\hat{P}$-manipulation to $f(R)$. Thus, we can use independence of unchosen alternatives to reorder the alternatives in $f(R')$ in the preference $R_i$ according to $R'_i$ and to switch from $x \succeq_i y$ to $y \succ_i x$ for all pairs of alternatives $x \in A \setminus f(R')$, $y \in f(R')$. After applying these modifications, we arrive at $R^*_i$ and as no change involves an alternative in $f(R)$, it follows from independence of unchosen alternatives that $f(R) = f(R^*) = f(R')$. However, this contradicts $f(R') \ \hat{P}_i \ f(R)$ and therefore, $f$ is not $\hat{P}$-manipulable in this case.
For the next case, assume that $f(R) \cap f(R') = \{m\}$ for an alternative $m \in A$. Observe that for all $x \in f(R') \setminus \{m\}$ and $y \in f(R) \setminus \{m\}$, it holds that $x \succ_i m$, $m \succ_i y$ and $x \succ_i y$. Otherwise, there are alternatives $x \in f(R')$ and $y \in f(R)$, $x \neq y$, such that $y \succeq_i x$, which contradicts that $f(R') \ \hat{P}_i \ f(R)$. Thus, we can use the auxiliary profile $R^{aux2} = (R_{-i}, R^{aux2}_i)$ in which voter $i$ prefers all alternatives in $U = f(R') \setminus \{m\}$ strictly to those in $L = A \setminus U$, the alternatives in $U$ are ordered according to $R'_i$ and the alternatives in $L$ are ordered according to $R_i$. Formally, $R^{aux2}_i$ is defined as follows.

$$R^{aux2}_i = R'_i|_U \ \cup \ R_i|_L \ \cup \ \{(x, y) \mid x \in U, y \in L\}$$

Observe that we can derive $R^{aux2}_i$ from $R_i$ by applying independence of unchosen alternatives as no preferences involving alternatives in $f(R)$ are changed. This means that $f(R) = f(R^{aux2})$. Moreover, note that $R^*_i$ and $R^{aux}_i$ only differ from each other in the preferences involving $m$. Hence, we can apply set-monotonicity to switch all preferences with $x \succeq_i m$ to $m \succ_i x$ for all $x \in A \setminus f(R')$. This step does not involve any alternatives in $f(R) \setminus \{m\}$ as $m \succ_i x$ holds for all $x \in f(R) \setminus \{m\}$ in $R_i$ and $R^{aux2}_i$. Thus, we can deduce $R^*_i$ from $R^{aux2}_i$ by applying set-monotonicity, which implies that $f(R) = f(R^*) = f(R')$. This contradicts the initial assumption stating that $f(R') \ \hat{P}_i \ f(R)$ and therefore, it follows that $f$ is $\hat{P}$-strategyproof. □

A consequence of the last theorem is that many tournament solutions are even in the intransitive domain $\hat{P}$-strategyproof. The reason for this is that the conservative extension can also be applied to intransitive preferences [BBH18]. Thus, it follows from Theorem 2.6 that many social choice functions, such as generalizations of the bipartisan set and the top cycle, are even in the intransitive domain $\hat{P}$-strategyproof. To the best of our knowledge, the theorems in this section are the first results discussing strategyproofness in the intransitive domain. Note that the results are also rather surprising as positive theorems on strategyproofness often turn into impossibility results if we try to generalize them from the strict domain to the weak domain.

In contrast, it seems that generalizing social choice functions from the weak domain to the intransitive one maintains many positive results such as the theorems discussed in this section. Thus, the intransitive domain does not appear to be more difficult to handle than the weak one with respect to strategyproofness.

## 2.4 Generalizations to Group-Strategyproofness

In this section, we discuss a natural extension of strategyproofness called group-strategyproofness. While our standard definition of strategyproofness is only concerned with a single voter, group-strategyproofness asks whether an arbitrarily large group of voters can manipulate. This leads to the question whether the implications presented in the previous sections also hold for this stronger variant of strategyproofness. As we show in the sequel, every result presented in the previous two sections is true for the corresponding version of group-strategyproofness.
For proving this claim, we formally introduce group-strategyproofness first. This term has first been discussed in [Bra11] and informally, a social choice function is group-strategyproof if no group of voters can lie about their preferences such that the outcome of an election is for every voter in the group preferable to the outcome before the manipulation. As usual, we use set extensions to define which sets a voter considers as preferable to the current choice set. This leads to the following formal definition.

**Definition 2.7 (Group-Strategyproofness).** *Consider an arbitrary set extension $\mathcal{P}$. A social choice function $f$ is $\mathcal{P}$-group-strategyproof if there are no preference profiles $R$, $R'$ and a non-empty subset of voters $I \subseteq N$ such that $R_{-I} = R'_{-I}$ and $f(R')\ \mathcal{P}_i\ f(R)$ for all voters $i \in I$.*

Next, we present a short example illustrating the difference between strategyproofness and group-strategyproofness. For this reason, consider the profiles $R^1$ and $R^2$ depicted in Figure 2.4 and the social choice function OMNI that chooses all most preferred alternatives of all voters. This social choice function is known to be $\tilde{P}$-strategyproof, i.e., it cannot be $\tilde{P}$-manipulated by a single voter. However, it is easy to see that it can be $\tilde{P}$-manipulated by a group of voters. For instance, observe that $\text{OMNI}(R^1) = \{a, b, c, d\}$ as every alternative is first-ranked by at least one voter. This means that voter 3 and 4 can manipulate if they weaken $c$ and $d$, which leads to the profile $R^2$. For this profile, OMNI returns $\{a, b\}$ which is a subset of the most preferred alternatives of these voters. This is a $\tilde{P}$-group-manipulation and therefore, OMNI is not $\tilde{P}$-group-strategyproof.
It follows from this example that group-strategyproofness is indeed stronger than strategyproofness. Thus, it is not clear whether the results discussed in Section 2.2 and Section 2.3 also hold for group-strategyproofness. However, if we consider the

$$R^1: \begin{array}{c|c|c|c} 1 & 1 & 1 & 1 \\ \hline a,b & b & a,b,c & a,b,d \\ c,d & a,c,d & d & c \end{array} \qquad R^2: \begin{array}{c|c|c|c} 1 & 1 & 1 & 1 \\ \hline a,b & b & a,b & a,b \\ c,d & a,c,d & c,d & c,d \end{array}$$

Figure 2.4: Preference profiles used for explaining the difference between strategyproofness and group-strategyproofness

proofs of the corresponding theorems, we see that we always use the same strategy: We assume that a voter can manipulate, i.e., there are two profiles $R$, $R'$ which only differ in the preference of a single voter. Next, we construct a third profile $R^*$ by assigning a new preference to the manipulator. Finally, we derive that $f(R) = f(R^*)$ and that $f(R^*) \subseteq f(R')$ for a weakly set-monotonic SCF $f$, and that $f(R) = f(R^*)$ and that $f(R^*) = f(R')$ for a set-monotonic SCF $f$. Note that the last observations do not rely on the fact that we go from $R$ and $R'$ to the same profile $R^*$. Instead, the only important point is how we change the preference of the manipulator. Thus, it seems reasonable that we can adapt the preference of every voter of a group in a similar way to prove group-strategyproofness.
For formalizing this idea, we introduce a new axiom called interpolation. This axiom means that if there are two profiles $R$, $R'$ that differ in the preferences of a group of voters $I$, we can find two other profiles $R^1$ and $R^2$ that differ in the preferences of less voters and whose choice sets are related to $R$ and $R'$. Formally, this axiom is defined as follows.

**Definition 2.8 (Interpolation).** *A social choice function $f$ is interpolating if for all preference profiles $R$, $R'$ with $R_{-I} = R'_{-I}$ for a group of voters $I \subseteq N$ there are two profile $R^1$ and $R^2$ such that $R^1_{-I'} = R^2_{-I'}$ for a group of voters $I' \subsetneq I$ with $|I'| = |I| - 1$, $f(R) = f(R^1)$ and $f(R^2) \subseteq f(R')$. Furthermore, we call a social choice function $f$ strongly interpolating if it is interpolating and $f(R^2) = f(R')$.*

Note that it follows directly from the proofs of Theorem 2.2 and Theorem 2.5 that every weakly set-monotonic social choice function is interpolating on every comprehensive subset of the weak domain and on the intransitive domain. Furthermore, Theorem 2.4 and Theorem 2.6 imply that set-monotonic social choice functions are even strongly interpolating on the corresponding domains. Finally, the main theorem of [Bra15] shows that set-monotonic social choice functions defined on the strict domain are also strongly interpolating.
Our next goal is to prove that every interpolating social choice function is $\bar{P}$-group-strategyproof and that every strongly interpolating social choice function is $\hat{P}$-group-strategyproof. Thus, we can deduce from these statements that all previously explained results also hold for group-strategyproofness. We start the discussion of these results by proving the implication between interpolation and $\bar{P}$-group-strategyproofness.

**Theorem 2.7.** *If a social choice function $f : \mathcal{D}^n \mapsto 2^A \setminus \emptyset$ is interpolating on its domain $\mathcal{D}^n$, then $f$ is $\bar{P}$-group-strategyproof.*

Proof: Consider a social choice function $f : \mathcal{D}^n \mapsto 2^A \setminus \emptyset$ that is interpolating on its domain $\mathcal{D}^n$ and assume for contradiction that $f$ is not $\bar{P}$-group-strategyproof. Thus, there is a non-empty set of voters $I \subseteq N$ and two profiles $R$, $R'$ such that $R_{-I} = R'_{-I}$ and $f(R')\ \bar{P}_i\ f(R)$ for all voters $i \in I$. We show that there is a profile $R^*$ such that $R_{-I} = R^*_{-I} = R'_{-I}$ and $f(R) = f(R^*) \subseteq f(R')$ by an induction on $|I|$. This contradicts that $f(R')\ \bar{P}_i\ f(R)$ for any voter $i \in I$ as the set extension $\bar{P}$ can only compare disjoint sets. Thus, it only remains to find the profile $R^*$.
First, we consider the induction basis assuming that $|I| = 0$. This assumption implies that $R = R^* = R'$ if $|I| = 0$ and that $f(R) = f(R^*) = f(R')$. Therefore, the induction basis trivially holds. Next, we consider the induction step assuming that $|I| = k$ and that we can find the profile $R^*$ if two profiles differ only on the preferences of $k-1$ voters. Observe that there are two profiles $R^1$, $R^2$ such that $R^1_{-I'} = R^2_{-I'}$ for a group of voters $I' \subsetneq I$ with $|I'| = |I| - 1$ and $f(R) = f(R^1)$ and $f(R^2) \subseteq f(R')$ as $f$ is interpolating. Thus, we can use the induction hypothesis on $R^1$ and $R^2$ to deduce that there is a profile $R^*$ such that $f(R) = f(R^1) = f(R^*) \subseteq f(R^2) \subseteq f(R')$. This implies that there is indeed a profile $R^*$ with $f(R) = f(R^*) \subseteq f(R')$, which contradicts that $f(R')\ \bar{P}_i\ f(R)$ for all voters $i \in I$. Therefore, the initial assumption is wrong and $f$ is $\bar{P}$-group-strategyproof. □

As already mentioned, it follows directly from the proofs of Theorem 2.2 and Theorem 2.5 that weakly set-monotonic social choice functions are in the weak and the intransitive domain interpolating. Thus, we deduce the following corollary from Theorem 2.7.

**Corollary 2.1.** *Every weakly set-monotonic social choice function defined on a comprehensive subset of the weak domain or on the intransitive domain is $\bar{P}$-group-strategyproof.*

Proof: We show that weakly set-monotonic social choice functions are interpolating on the considered domains because the $\bar{P}$-group-strategyproofness follows from this axiom and Theorem 2.7. Therefore, consider two arbitrary profiles $R$ and $R'$ such that $R_{-I} = R'_{-I}$ for a non-empty set of voters $I \subseteq N$. Next, we choose an arbitrary voter $i \in I$ and let him change his preference in both profiles to the preference $R^*_i$ described in the proof of Theorem 2.2 if $f$ is defined on a comprehensive subset of the weak domain, or to the preference $R^*_i$ discussed in the proof of Theorem 2.5 if $f$ is defined on the intransitive domain. This leads to new profiles $R^1$ and $R^2$ with $R^1 = (R_{-i}, R^*_i)$ and $R^2 = (R'_{-i}, R^*_i)$. Furthermore, as $i \in I$, $R^1$ and $R^2$ only differ in the preferences of $|I| - 1$ voters. Finally, it follows from the same arguments presented in proofs of Theorem 2.2 and Theorem 2.5 that $f(R) = f(R^1)$ and $f(R^2) \subseteq f(R')$. Consequently, $f$ is indeed interpolating such that Theorem 2.7 implies this corollary. □

It follows from this corollary that weak set-monotonicity even implies $\bar{P}$-group-strategyproofness on important domains such as the weak and the intransitive one.

Furthermore, it should be stressed that the step from $\bar{P}$-strategyproofness to $\bar{P}$-group-strategyproofness is remarkably simple as we only have to apply the construction designed for a single voter inductively for a set of voters.
Therefore, it seems reasonable that we can prove similar results for set-monotonic social choice functions. As we discuss next, set-monotonicity is closely related to strong interpolation which implies $\hat{P}$-strategyproofness. Thus, it is indeed true that set-monotonic social choice functions are $\hat{P}$-group-strategyproof in many important domains such as the weak or the intransitive one.

**Theorem 2.8.** *If a social choice function $f : \mathcal{D}^n \mapsto 2^A \setminus \emptyset$ is strongly interpolating on its domain $\mathcal{D}^n$, then $f$ is $\hat{P}$-group-strategyproof.*

Proof: Consider a social choice function $f : \mathcal{D}^n \mapsto 2^A \setminus \emptyset$ that is strongly interpolating on its domain $\mathcal{D}^n$ and assume for contradiction that $f$ is not $\hat{P}$-group-strategyproof. Thus, there is a non-empty set of voters $I \subseteq N$ and two profiles $R$, $R'$ such that $R_{-I} = R'_{-I}$ and $f(R')\ \hat{P}_i\ f(R)$ for all voters $i \in I$. We show that there is a profile $R^*$ such that $R_{-I} = R^*_{-I} = R'_{-I}$ and $f(R) = f(R^*) = f(R')$ by an induction on $|I|$. This contradicts that $f(R')\ \hat{P}_i\ f(R)$ for any voter $i \in I$, which means that $f$ is $\hat{P}$-group-strategyproof. Thus, it only remains to find $R^*$.
Therefore, note that if $|I| = 0$, then $R = R^* = R'$ and $f(R) = f(R^*) = f(R')$. Hence, the induction basis trivially holds. Next, consider the induction step assuming that $|I| = k$ and that there is such a profile $R^*$ for all profiles that only differ in the preferences of $k-1$ voters. Note that there are two profiles $R^1$, $R^2$ such that $R^1_{-I'} = R^2_{-I'}$ for a group of voters $I' \subsetneq I$ with $|I'| = |I| - 1$, $f(R) = f(R^1)$ and $f(R^2) = f(R')$ as $f$ is strongly interpolating. Therefore, we can use the induction hypothesis on $R^1$ and $R^2$ to deduce that there is a profile $R^*$ such that $f(R) = f(R^1) = f(R^*) = f(R^2) = f(R')$. Hence, we have found $R^*$ for profiles $R$ and $R'$ that differ in the preferences of $k$ voters. As $f(R) = f(R^*) = f(R')$, it follows that our initial assumption is wrong and therefore, $f$ is $\hat{P}$-group-strategyproof. □

Similar to the arguments on weak set-monotonicity and interpolation, we can show that every set-monotonic social choice function is strongly interpolating on the weak and the intransitive domain. Therefore, it follows that functions satisfying set-monotonicity are even $\hat{P}$-group-strategyproof.

**Corollary 2.2.** *Every set-monotonic social choice function defined on a comprehensive subset of the weak domain or on the intransitive domain is $\hat{P}$-group-strategyproof.*

Proof: We prove that set-monotonicity implies strong interpolation on comprehensive subsets of the weak domain and on the intransitive domain. Then, this corollary follows from Theorem 2.8. Thus, consider an arbitrary set-monotonic social choice function $f$ and two profiles $R$, $R'$ such that $R_{-I} = R'_{-I}$ for a non-empty set of voters $I \subseteq N$. We show how to find the profiles $R^1$ and $R^2$ such that $R^1_{-I'} = R^2_{-I'}$ for a group of voters $I' \subsetneq I$ with $|I'| = |I| - 1$, $f(R) = f(R^1)$ and $f(R^2) = f(R')$. For this reason, consider an arbitrary voter $i \in I$ and let $R_i^*$ denote the preference discussed in the proof of Theorem 2.4 if $f$ is defined on a comprehensive subset of the weak domain, or the preference discussed in the proof of Theorem 2.6 if $f$ is defined on the intransitive domain. We set $R^1 = (R_{-i}, R_i^*)$ and $R^2 = (R'_{-i}, R_i^*)$ and observe that these profiles only differ in the preferences of voters in $I \setminus \{i\}$. Furthermore, it follows from the arguments presented in the proofs of Theorem 2.4 and Theorem 2.6 that $f(R^1) = f(R)$ and $f(R^2) = f(R')$. Thus, we derive from Theorem 2.8 that $f$ is $\hat{P}$-strategyproof on its domain. □

As a consequence of this theorem, it follows that many important social choice functions are even $\hat{P}$-group-strategyproof. For instance, many tournament solutions, such as the bipartisan set or the top cycle, can be generalized to the weak and even to the intransitive domain with the help of the conservative extension while maintaining the set-monotonicity of these functions. Thus, there are $\hat{P}$-group-strategyproof social choice functions in these domains. Furthermore, note that $\hat{P}$-group-strategyproofness is equal to $\tilde{P}$-group-strategyproofness on the strict domain and therefore, it follows from this result that set-monotonic social choice functions defined on a comprehensive subset of the strict domain are $\tilde{P}$-group-strategyproof. This proves the third remark in [Bra15].

Furthermore, it should be mentioned that the implications in this section are not invertible. This claim follows from analyzing the social choice function $f_3$ which has been used at the end of Section 2.2 to prove that the results of this section are not invertible. The main problem is that social choice functions with large choice sets are often $\hat{P}$- or even $\tilde{P}$-group-strategyproof while violating independence of unchosen alternatives.

Chapter 3

# Rank-based Social Choice Functions

Social choice functions are often categorized by their behavior. This approach leads to different classes of social choice functions such as tournament solutions or C2-functions. Another interesting class are rank-based social choice functions. These social choice functions have been discussed first in [Las96]. However, after this publication, rank-based rules have not been considered in the literature anymore and therefore, almost no results are known yet. Nevertheless, there are many important rank-based social choice functions such as scoring rules. Thus, we analyze these functions in detail. In particular, we focus on three goals in this chapter: Firstly, we introduce various classes of rank-based social choice functions that seem interesting because of their properties. We discuss different ideas for designing these functions with the hope to stimulate further research on rank-based social choice functions. Secondly, we provide insights on the $\tilde{P}$-strategyproofness of rank-based social choice functions by analyzing the introduced rules thoroughly. Thirdly, we suggest extensions of rank-based social choice functions to the weak domain. This is necessary as rank-based social choice functions have hitherto only been defined in the strict domain.

For achieving these goals, we first provide a short introduction to rank-based social choice functions in Section 3.1. After that, we analyze various classes of rank-based social choice functions with respect to $\tilde{P}$-strategyproofness in Section 3.2 and Section 3.3. Finally, we discuss various extensions of rank-based social choice functions to the weak domain in Section 3.4 and we derive an impossibility result stating that none of these approaches leads to a $\tilde{P}$-strategyproof, Pareto-optimal and rank-based social choice function in the weak domain.

## 3.1 Introduction to Rank-based Social Choice Function

In this section, we formally introduce rank-based social choice functions as we need a strong understanding of these rules in the subsequent sections. It should be mentioned that rank-based social choice functions have only been introduced for the

strict domain in [Las96]. Therefore, we focus first in all sections but the last one on the strict domain.
As we want to discuss rank-based social choice functions, we first define the rank of an alternative. Intuitively, the rank of an alternative in the preference of a voter states how much a voter prefers this alternative, i.e., if alternative $a$ is the $i$-th most preferred alternative of a voter, it has rank $i$. Since we work in the strict domain, this approach assigns a unique rank to each alternative. A formal definition is presented in the sequel.

**Definition 3.1 (Rank of an alternative).** *The rank of an alternative $a \in A$ in the preference $R_i \in \mathcal{S}$ of voter $i$ is defined as $r(R_i, a) = 1 + |\{b \in A \mid b \succ_i a\}|$.*

Given the rank of every alternative with respect to every voter, we can reconstruct the preference profile. Therefore, only working with the ranks does not restrict the set of possible social choice functions. Hence, we consider the ranks of an alternative with respect to all individual voters and sort them in ascending order. This idea leads to the rank vector of an alternative.

**Definition 3.2 (Rank vector of an alternative).** *The rank vector of alternative $a$ in the profile $R \in \mathcal{S}^n$ is defined as $r^*(R, a) = (r(R_{i_1}, a), r(R_{i_2}, a), ..., r(R_{i_n}, a))$, where $r(R_{i_1}, a) \leq r(R_{i_2}, a) \leq ... \leq r(R_{i_n}, a)$.*

Note that a key idea of rank vectors is that they are ordered, i.e., $v_1 \leq v_2 \leq ... \leq v_n$ if $v$ is a rank vector. We always assume this property throughout this chapter implicitly, even for vectors that are no rank vectors. Furthermore, observe that ordering the entries in a rank vector ensures anonymity as it is no longer possible to find out the individual preferences of a voter. However, a single rank vector is usually insufficient to decide which alternatives are in the choice set. Therefore, we consider all rank vectors simultaneously, which leads to the rank matrix.

**Definition 3.3 (Rank matrix).** *The rank matrix of the preference profile $R \in \mathcal{S}^n$ is defined as*

$$r^*(R) = \begin{pmatrix} r^*(R, a_1) \\ r^*(R, a_2) \\ \vdots \\ r^*(R, a_m) \end{pmatrix} .$$

The rank matrix contains the rank vectors of all alternatives as rows. It is usually not possible to associate a rank matrix with a unique preference profile. Instead, many profiles can have the same rank matrix. Even more, we cannot reconstruct the preference of a single voter from the rank matrix as it is not clear which voter assigns which rank to an alternative. Thus, the rank matrix disregards information about its original preference profile. Nevertheless, the ranks of the alternatives are often sufficient to decide the winners of an election. This is the main idea of rank-based social choice functions which only rely on the rank matrix to compute the choice set.

$R^1$ :

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $b$ | $d$ |
| $a$ | $c$ | $c$ |
| $d$ | $d$ | $b$ |
| $c$ | $a$ | $a$ |

$R^2$ :

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $b$ | $d$ |
| $c$ | $c$ | $a$ |
| $d$ | $d$ | $b$ |
| $a$ | $a$ | $c$ |

Figure 3.1: Preference profiles used for explaining rank-basedness

**Definition 3.4 (Rank-based social choice functions).** *A social choice function $f : \mathcal{S}^n \mapsto 2^A \backslash \emptyset$ is rank-based if it only depends on the rank matrix, i.e., $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{S}^n$ with $r^*(R) = r^*(R')$.*

Before we discuss properties of rank-based social choice functions, we first provide an example which illustrates all previous definitions. Thus, consider the preference profiles $R^1$ and $R^2$ shown in Figure 3.1. If we compute the rank of alternative $c$ in $R^1$ for every voter, we obtain that $r(R^1_1, c) = 4$, $r(R^1_2, c) = 2$ and $r(R^1_3, c) = 2$. This means that $r^*(R^1, c) = (2, 2, 4)$. If we repeat this for all alternatives and both profiles, we obtain that

$$r^*(R^1) = r^*(R^2) = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 1 & 3 \\ 2 & 2 & 4 \\ 1 & 3 & 3 \end{pmatrix} \quad .$$

Thus, every rank-based social choice function $f$ satisfies that $f(R^1) = f(R^2)$.
This example shows that different preference profiles can have the same rank matrix. Additionally, we can observe that rank-based social choice functions cannot be based on the majorities between alternatives anymore. The reason for this is simple: We do not know how an individual voter ranks the alternatives and therefore, we cannot recompute the majorities. This is also the case in the example: While $c$ is preferred to $b$ by the third voter in $R^1$, $c$ is Pareto-dominated by $b$ in $R^2$. Hence, the social choice functions satisfying Fishburn's C2 [Fis77] are almost never rank-based. Note that many well-established social choice functions satisfy C2, such as tournament solutions and many variants of the Pareto-rule. Even more, this example shows that many rank-based social choice functions violate desirable axioms based on majorities such as Pareto-optimality and Condorcet-consistency.
Nevertheless, there are also some well-known rank-based social choice functions. One of the most prominent examples is Borda's rule which has been designed in the 19th century [Bor81]. This rule gives an alternative $m-i$ points for every voter who ranks it at position $i$ and chooses the alternatives with maximal score as winners. Borda's rule can be generalized to the well-established class of scoring rules. These rules give an alternative $s_i$ points for every voter who ranks it at position $i$ and choose the alternatives with maximal score as winners. Another important rank-based social choice function is the OMNI-rule which chooses all alternatives that

are first-ranked by at least one voter. This rule is particularly interesting as is $\tilde{P}$-strategyproof [Gär76].

In the following sections, we discuss on the one hand these well-established rank-based SCFs. On the other hand, we also design new approaches for defining rank-based social choice functions such as independent rank-based SCFs. These functions decide for every alternative whether it is in the choice set only based on its own rank vector. The beauty of this approach is that we completely avoid comparing alternatives which each other, which makes these social choice functions easy to analyze and efficient to compute. Furthermore, these rules are used in practice as they model hurdles such as the five-percent hurdle used in German elections. We discuss independent rank-based social choice functions in more detail in Section 3.2. Note that we can think of many more approaches for defining rank-based social choice functions. However, these rules do not show so clear characteristics as independent rank-based SCFs. Therefore, we simply refer to all rank-based social choice functions that are not independent ones as general rank-based social choice functions. These rules are analyzed in Section 3.3 where we discuss various interesting subclasses of general rank-based social choice functions. The goal is to characterize $\tilde{P}$-strategyproof social choice functions or to find simple criteria that prove that a social choice function is $\tilde{P}$-manipulable.

## 3.2 Independent Rank-based Social Choice Functions

In this section, we discuss independent rank-based social choice functions. These rules form an interesting subclass of rank-based social choice functions and decide whether an alternative is in the choice set only based its own rank vector. This idea makes independent rank-based SCFs easy to handle and efficient to compute. Furthermore, these social choice functions are used in practice. For instance, the five-percent hurdle used in German elections implements this approach. Formally, these social choice functions are defined by the following axiom called independence of ranks of other alternatives.

**Definition 3.5 (Independence of ranks of other alternatives).** *A social choice function* $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ *satisfies independence of ranks of other alternatives if* $a \in f(R)$ *implies that* $a \in f(R')$ *for all alternatives* $a \in A$ *and preference profiles* $R, R' \in \mathcal{S}^n$ *with* $r^*(R, a) = r^*(R', a)$.

It follows from this definition that for every independent rank-based social choice function $f$ and alternative $a \in A$, there is a decision function $g_a$ that takes the rank vector of $a$ as input and returns 1 if and only if $a$ is in the choice set; otherwise $g_a$ returns 0. For instance, the decision function $g_a$ of the OMNI-rule only checks for every alternative $a$ if the first entry of its rank vector is equal to 1. This view on independent rank-based social choice functions is sometimes helpful for formalizing observations.

As consequence of the previous observation, it follows that OMNI is an independent rank-based social choice function. Even more, there are many interesting rules within this class. Therefore, we discuss social choice functions satisfying independence of ranks of other alternatives in Section 3.2.1. Unfortunately, it turns out that only few of them are $\tilde{P}$-strategyproof. Thus, we thoroughly analyze independent rank-based social choice functions with respect of $\tilde{P}$-strategyproofness in Section 3.2.2. This leads to a characterization of the OMNI-rule based on independence of ranks of other alternatives and $\tilde{P}$-strategyproofness.

### 3.2.1 Threshold Rules and Multi-Threshold Rules

In this section, we introduce two large classes of independent rank-based social choice functions: Threshold rules and multi-threshold rules. We provide characterizations for both classes that show that almost all reasonable independent rank-based social choice functions are threshold rules or multi-threshold rules.

The rough intuition of threshold rules and multi-threshold rules is that of a hurdle: An alternative is chosen if its rank vector is good enough. For mathematically formalizing this intuition, we have to define a way on evaluating how good a rank vector is. For this reason, we introduce a dominance relation that allows to compare vectors.

**Definition 3.6 (Dominance relation on vectors).** *A vector $u = (u_1, u_2, ..., u_n)$ dominates another vector $v = (v_1, v_2, ..., v_n)$, denoted by $u\ D\ v$, if $u_i \leq v_i$ for all $i \in \{1, 2, ..., n\}$.*

This definition states that a vector $u$ dominates another vector $v$ if no entry $u_i$ is larger than the corresponding entry $v_i$. It is obvious that this dominance relation is incomplete, transitive and anti-symmetric. Incompleteness is very difficult to avoid by transitive relations on vectors, whereas the latter two properties are required for a reasonable quality measure on vectors. Furthermore, the chosen dominance relation is rather simple to handle and still strong enough to prove various interesting results. Because of these reasons, we focus on this relation even though different methods for comparing vectors exist.

With the help of this dominance relation, we formalize the intuition of threshold rules and multi-threshold rules next.

**Definition 3.7 (Threshold rules).** *A social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ is a threshold rule if there is an ordered vector $v \in \{1, 2, ..., m\}^n$ called threshold vector such that $a \in f(R)$ if and only if $r^*(R, a)\ D\ v$ for all alternatives $a \in A$ and preference profiles $R \in \mathcal{S}^n$.*

It follows from this definition that threshold rules deem an alternative good enough to be chosen if its rank vector dominates the threshold vector. Many social choice functions used in practice implement this idea, for instance the five-percent hurdle. Furthermore, the OMNI-rule which chooses all first-ranked alternatives is also a threshold rule defined by the vector $v = (1, m, ..., m)$.
Next, we generalize threshold rules to multi-threshold rules. These social choice functions are defined by a set of threshold vectors instead of a single one and an alternative is chosen if its rank vector dominates at least one of these threshold vectors. The main motivation for introducing these rules is that many rank vectors seem equally good but are often not comparable with respect to the dominance relation. This means that threshold rules often do not choose alternatives even though their rank vector seems intuitively better than the threshold vector. This problem can be solved by using multiple threshold vectors because this allows to define more precisely which rank vectors suffice to include the corresponding alternatives in the choice set.

**Definition 3.8 (Multi-threshold rules).** *A social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ is a multi-threshold rule if there is a set of threshold vectors $V = \{v_1, v_2, ..., v_k\}$ such that $a \in f(R)$ if and only if there is a vector $v_i \in V$ with $r^*(R, a)\ D\ v_i$ for all alternatives $a \in A$ and preference profiles $R \in \mathcal{S}^n$.*

Note that we can assume without loss of generality that no vectors in $V$ can be compared by the dominance relation. This assumption is valid as an alternative is chosen if its rank vector dominates a single vector $v \in V$. If there are vectors $v_1, v_2 \in V$ with $v_1\ D\ v_2$, then it follows from the transitivity of the domination relation that if $r^*(R, a)\ D\ v_1$, then also $r^*(R, a)\ D\ v_2$. Thus, we can remove the vector $v_1$ from $V$ without changing the corresponding social choice function. Furthermore, observe that every threshold rule is a multi-threshold rule whose set of threshold vectors contains only a single element.
Next, we illustrate threshold rules and multi-threshold rules with the example shown in Figure 3.2. In this figure the preference profiles $R^1$ and $R^2$ and their corresponding rank matrices are displayed. Moreover, consider the threshold rule $f_1$ defined by the vector $v_1 = (2, 3, 4, 4, 5)$. It follows that $f_1(R^1) = \{b, c, d, e\}$. However, note that $r^*(R^1, b)$, $r^*(R^1, c)$ and $r^*(R^1, e)$ dominate $r^*(R^1, d)$. Thus, a threshold rule that only chooses $\{b, c, e\}$ for $R^1$ seems interesting. Unfortunately, it is not possible for a threshold rule to choose this set as the rank vectors of $b$ and $c$ are not comparable and therefore, one of them is excluded if we choose a more restrictive threshold vector. We can fix this by using a multi-threshold rule. For instance, consider the multi-threshold rule $f_2$ defined by the set $V_2 = \{(2, 3, 3, 4, 5), (1, 3, 4, 4, 5)\}$ and note that $f_2(R^1) = \{b, c, e\}$. Another critique on the social choice functions $f_1$ and $f_2$ is that they do not choose $a$ in $R^1$ even though it is the Condorcet winner. Even more, $b$ is chosen even though it is Pareto-dominated by $a$. Hence, a more restrictive threshold rule seems interesting. For instance, consider the threshold rule $f_3$ defined by the vector $v_3 = (1, 3, 3, 5, 5)$. It holds that $f_3(R^1) = \{a, e\}$, which seems more desirable

$$R^1: \begin{array}{c|c|c|c|c} 1 & 1 & 1 & 1 & 1 \\ \hline c & e & a & a & a \\ d & b & e & e & e \\ e & d & c & b & b \\ b & c & d & d & c \\ a & a & b & c & d \end{array} \qquad R^2: \begin{array}{c|c|c|c|c} 1 & 1 & 1 & 1 & 1 \\ \hline c & b & b & a & a \\ e & e & e & e & c \\ d & d & d & d & e \\ b & c & c & c & d \\ a & a & a & b & b \end{array}$$

$$r^*(R^1) = \begin{pmatrix} 1 & 1 & 1 & 5 & 5 \\ 2 & 3 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 5 \\ 2 & 3 & 4 & 4 & 5 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \qquad r^*(R^2) = \begin{pmatrix} 1 & 1 & 5 & 5 & 5 \\ 1 & 1 & 4 & 5 & 5 \\ 1 & 2 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 & 3 \end{pmatrix}$$

Figure 3.2: Preference profiles used for explaining threshold rules

than the outcome of the previous two functions. However, observe that $f_3(R^2) = \emptyset$ as no rank vector dominates $v_3$. Thus, the vector $v_3$ does not even define a proper social choice function. We can fix this problem by using a multi-threshold rule. For instance, the multi-threshold rule $f_4$ defined by $V_4 = \{(1,3,3,5,5),(2,3,3,4,4)\}$ is a feasible social choice function. Furthermore, $f_4(R^1) = \{a, e\}$ and $f_4(R^2) = \{e\}$ and therefore, $f_4$ seems to select reasonable choice sets.

Our next goal is to analyze a problem observed in the previous example: Not every vector defines a proper threshold rule. Unfortunately, it seems that there is no simple mathematical criterion for deciding whether a vector defines a feasible threshold rule. Therefore, we design Algorithm 1 which decides this problem efficiently with a greedy approach. If a vector does not define a feasible threshold rule $f$, then this algorithm constructs a preference profile $R$ such that $f(R) = \emptyset$ and it returns false. If no such profile exists, then the algorithm returns true indicating that $f$ is a feasible social choice function.

The main idea of this algorithm is to ensure that, given an ordered vector $v$, as many alternatives as possible are not chosen while minimizing for each alternative the number of voters that place it at a position with large rank. For defining this formally, we introduce the violation index $l(R, a)$ that denotes for a preference profile $R$ and an alternative $a \in A$ the largest index $i$ such that $r^*(R, a)_i > v_i$; if there is no such index $i$, then $l(R, a) = 0$. Hence, the formal goal of the algorithm is to find a preference profile $R^*$ that maximizes $|\{a \in A \mid l(R^*, a) \geq i\}|$ for every index $i$.

To achieve that goal, Algorithm 1 iterates over all alternatives and maximizes the violation index $l(R, a)$ for every alternative individually. As only the indices $i \geq l(R, a)$ are relevant for determining the violation index, this maximization is achieved by placing an alternative repeatedly at a position with maximal rank until it cannot dominate the vector $v$ anymore. Therefore, the algorithm places the current alternative $a$ at the position with the highest available rank. If there are multiple such positions, we choose the left-most one. This is repeated until the alternative $a$ can-

Iteration 1:

$R$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| $a$ | $a$ | | | |

Iteration 2:

$R$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| $a$ | $a$ | $b$ | $b$ | |

Iteration 3:

$R$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| $c$ | $c$ | | | |
| $a$ | $a$ | $b$ | $b$ | $c$ |

Iteration 4:

$R$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| $c$ | $c$ | $d$ | $d$ | $d$ |
| $a$ | $a$ | $b$ | $b$ | $c$ |

Iteration 5:

$R$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| | | | | |
| | | | | |
| $e$ | $e$ | $e$ | $e$ | $e$ |
| $c$ | $c$ | $d$ | $d$ | $d$ |
| $a$ | $a$ | $b$ | $b$ | $c$ |

Filling the remaining positions arbitrarily

$R$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $d$ | $d$ | $c$ | $c$ | $a$ |
| $b$ | $b$ | $a$ | $a$ | $b$ |
| $e$ | $e$ | $e$ | $e$ | $e$ |
| $c$ | $c$ | $d$ | $d$ | $d$ |
| $a$ | $a$ | $b$ | $b$ | $c$ |

Figure 3.3: Example for Algorithm 1 with $A = \{a, b, c, d, e\}$, $n = 5$, $v = (1, 3, 3, 4, 5)$

not be chosen anymore, i.e., until there is an index $l_a$ such that $r^*(R, a)_{l_a} > v_{l_a}$, or until every voter has placed this alternative and it is still chosen. The latter case means that $a$ cannot be removed from the choice set anymore, which implies that $v$ defines a feasible threshold rule. This procedure fills the rank vector of an alternative from the right since we always add the largest possible entry. It follows from this observation that this procedure leads indeed to the largest violation indices.

Note that after iterating over all alternatives, usually not all positions in $R$ are used. The remaining positions in $R$ can be filled arbitrarily because they do not influence the outcome of the algorithm. If the considered vector $v$ does not define a feasible threshold rule, no alternative can be chosen anymore in $R$ regardless of how the free positions are filled.

In Figure 3.3, an example for the execution of Algorithm 1 is provided. For the example we choose $A = \{a, b, c, d, e\}$, $n = 5$ and $v = (1, 3, 3, 4, 5)$. We always display the profile that is constructed during the exeuction of the algroithm after an alternative is removed from the choice set. Observe that we only have to ensure that the rank vector for every alternative does not dominate the vector $v = (1, 3, 3, 4, 5)$. This is achieved for alternatives $a$ and $b$ since $r^*(R, a)_4 = r^*(R, b)_4 = 5 > 4 = v_4$, for alternatives $c$ and $d$ since $r^*(R, c)_3 = r^*(R, d)_3 = 4 > 3 = v_3$ and for alternative $e$ since $r^*(R, e)_1 = 3 > 1 = v_1$. Thus, it follows that the vector $v$ does not define a feasible threshold rule.

After discussing Algorithm 1 in detail, we prove its correctness, i.e., we show that an ordered vector $v$ defines a feasible threshold rule if and only if the algorithm returns true.

```
input  : A threshold vector v, a set of alternatives A and the number of
         voters n
output: true if v defines a feasible threshold rule for A and n; false
         otherwise
m ← |A|
c ← m          // number of alternatives that can still be chosen
x ← 1              // pointer to the current voter (=column of R)
y ← m                 // pointer to the current rank (=row of R)
for a ∈ A do
    r*(a) ← zeros(m)  // current approximation of rank vector of a
    l_a = n                     // index of r*(a) that is filled next
    while r*(a) D v ∧ l_a > 0 do
        r*(a)_{l_a} = y                     // Update the rank vector of a
        if y > v_{l_a} then          // Check if r*(a) can still dominate v
            c ← c − 1                       // a cannot be chosen anymore
        end
        l_a ← l_a − 1               // Update pointer to the rank vector
        R(x, y) ← a                    // Update the preference profile
        x ← x + 1       // Update pointers to the preference profile
        if x > n then
            x ← 1
            y ← y − 1
        end
    end
end
Fill up the free position in R arbitrarily such that R is a feasible preference
 profile
if c = 0 then
    return false
else
    return true
end
```

**Algorithm 1:** Decision procedure for the feasibility of threshold rules

**Theorem 3.1.** *Consider an ordered vector $v$, a set of alternatives $A$ and let $n$ denote the number of voters. The vector $v$ defines a feasible threshold rule on $A$ and $n$ voters if and only if Algorithm 1 returns true.*

Proof: Consider an arbitrary threshold vector $v$, the corresponding set of alternatives $A$ and the number of voters $n$. Furthermore, let $f$ denote the threshold rule defined by $v$. It is straightforward that the vector $v$ does not define a feasible threshold rule if Algorithm 1 returns false. The reason for this is that $f(R) = \emptyset$ for the preference profile $R$ constructed during the execution of Algorithm 1. This is true as the counter $c$ in the algorithm equals 0 after the construction of $R$ only if there is an index $l_a$ for every alternative $a \in A$ such that $r^*(R,a)_{l_a} > v_{l_a}$. Otherwise, $c$ is not 0 as this is the requirement for decreasing it. Thus, $f(R) = \emptyset$, which proves that $f$ is indeed no feasible threshold rule if Algorithm 1 returns false.
Next, we focus on the inverse direction: If there is a preference profile $R'$ such that no rank vector $r^*(R',a)$ of an alternative $a \in A$ dominates the threshold vector $v$, then Algorithm 1 returns false. We prove this by showing that the preference profile $R^*$ constructed during the execution of Algorithm 1 leads to maximal violation indices. Thus, consider the preference profile $R^*$ constructed during the execution of Algorithm 1 and an arbitrary second preference profile $R'$. Let $a^1, ..., a^m$ denote the alternatives in $A$ ordered in decreasing order of their violation indices $l(R^*, a)$ and let $b^1, ..., b^m$ denote the alternatives ordered in decreasing order of their violation indices $l(R', a)$. We claim that $l(R^*, a^i) \geq l(R', b^i)$ for every $i \in \{1, ..., m\}$. If we assume that $f(R') = \emptyset$ for a preference profile $R'$, we can deduce from this statement that $f(R^*) = \emptyset$, too. The reason for this is that $f(R') = \emptyset$ implies that the violation index of every alternative in $R'$ is larger than 0. By applying our claim, we can deduce that the violation index of every alternative in $R^*$ is also larger than 0, which implies that $f(R^*) = \emptyset$. This is detected by the algorithm and therefore, it returns indeed false.
Hence, it remains to prove the claim. Therefore, assume for contradiction that there is a preference profile $R'$ and an index $i$ such that $l(R', b^i) > l(R^*, a^i)$. If there is such an index $i$, we can also consider the minimal index $i^*$ meeting the condition, i.e., $l(R', b^{i^*}) > l(R^*, a^{i^*})$ and $l(R', b^j) \leq l(R^*, a^j)$ for all $j < i^*$. This implies the following inequality.

$$\sum_{j=1}^{i^*-1} l(R', b^j) \leq \sum_{j=1}^{i^*-1} l(R^*, a^j)$$

Furthermore, as $b^{i^*}$ satisfies $r^*(R', b^{i^*})_{l(R', b^{i^*})} > v_{l(R', b^{i^*})}$ and $l(R', b^{i^*}) \leq l(R', b^j)$ for every $j < i^*$, it follows that

$$n(m - v_{i^*}) - \sum_{j=1}^{i^*} n + 1 - l(R', b^j) \geq 0 \quad .$$

We obtain this equation as there are $n(m - v_{i^*})$ positions with rank larger than $v_{i^*}$ and every alternative $b^j$, $j \leq i^*$, must be placed at a position with a rank larger than $v_{l(R',b^j)} \geq v_{l(R',b^{i^*})}$ by at least $n + 1 - l(R', b^j)$ voters. We can deduce from the previous inequalities that

$$n(m - v_{i^*}) - n + 1 - l(R', b^{i^*}) - \sum_{j=1}^{i^*-1} n + 1 - l(R^*, a^j) \geq 0 \quad .$$

This means that our algorithm has enough positions to remove alternative $a^{i^*}$ also at the index $l(R', b^{i^*})$. Furthermore, Algorithm 1 fills up the profile from bottom to top and therefore, an alternative is always removed with the largest possible violation index. Hence, the initial assumption is a contradiction as $l(R^*, a^{i^*})$ cannot be smaller than $l(R', b^{i^*})$. This means that $l(R^*, a^i) \geq l(R', b^i)$ for all preference profiles $R'$ and indices $i \in \{1, ..., m\}$. This proves the claim and therefore also the theorem. □

Note that we actually have the same problem for multi-threshold rules as for threshold rules: Not every set of vectors defines a feasible social choice function. However, it seems unlikely that Algorithm 1 can be extended to check whether a set of vectors defines a feasible multi-threshold rule. Instead, we conjecture that it is NP-hard to decide this problem. The reason for this conjecture is that the problem seems related to scheduling problems on multiple machines with deadlines. Furthermore, many straightforward extensions of Algorithm 1 such as greedily removing alternatives by placing them at the positions with the least rank do not lead to a correct decision procedure.
Observe that we can deduce from the last remarks and Theorem 3.1 that it is preferable to work with threshold rules instead of multi-threshold rules. Hence, the questions which social choice functions can be modeled with threshold rules and when multi-threshold rules are required arise naturally. We answer these questions by providing characterizations for both classes.
However, before we discuss the characterization of either of these classes, we first make an important observation between independent rank-based social choice functions that satisfy monotonicity (see Definition 2.1) and the dominance relation. This observation is required for the proof of both characterizations as threshold rules and multi-threshold rules are monotonic.

**Lemma 3.1.** *Consider an arbitrary social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that satisfies monotonicity and independence of ranks of other alternatives. It holds for all alternatives $a \in A$ that if $a \in f(R)$ for a preference profile $R \in \mathcal{S}^n$, then $a \in f(R')$ if $r^*(R', a)\ D\ r^*(R, a)$.*

Proof: Let $f$ denote an arbitrary social choice function that satisfies all axioms required by the lemma. Furthermore, consider two preference profiles $R, R' \in \mathcal{S}^n$ such that $a \in f(R)$ and $r^*(R', a)\ D\ r^*(R, a)$. We prove in the sequel that $a \in f(R')$.

Observe for this that $r^*(R', a)_i \le r^*(R, a)_i$ for all $i \in \{1, ..., n\}$. If this inequality is for every index $i \in \{1, ..., n\}$ tight, the lemma follows directly from independence of ranks of other alternatives. Thus, assume that there is an index $i$ such that $r^*(R', a)_i < r^*(R, a)_i$ and let $i^*$ denote the smallest index satisfying this condition. By the definition of the rank vector, there is a voter $j$ with $r(R_j, a) = r^*(R, a)_{i^*}$. Next, consider the preference profile $R^*$ in which voter $j$ reinforces alternative $a$ such that $r^*(R^*_j, a) = r^*(R', a)_{i^*}$ and nothing else changes. It follows from monotonicity that $a \in f(R^*)$. Furthermore, it holds that $r^*(R', a)\ D\ r^*(R^*, a)$ as the only difference between $r^*(R^*, a)$ and $r^*(R, a)$ is that $r^*(R^*, a)_{i^*} = r^*(R', a)_{i^*}$. This means that we can repeat this step and as $i^*$ is growing larger in each iteration, we eventually arrive at a profile $\tilde{R}$ with $a \in f(\tilde{R})$ and $r^*(R', a) = r^*(\tilde{R}, a)$. This implies that $a \in f(R')$ because of independence of ranks of other alternatives. □

We aim to use Lemma 3.1 to characterize threshold rules. Unfortunately, it seems that a stronger axiom than independence of ranks of other alternatives is required to uniquely characterize these rules. Therefore, we introduce the following strengthening of this property which we call max-closure.

**Definition 3.9 (Max-closure).** *Given two rank vectors $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, we define the max-operation as*

$$a \circ_{\max} b = (\max(a_1, b_1), \max(a_2, b_2), ..., \max(a_n, b_n)) \quad .$$

*We call a social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ max-closed if there is a function $g_a(v)$ such that $a \in f(R)$ if and only if $g_a(r^*(R, a)) = 1$ for every alternatives $a \in A$ and every preference profile $R \in \mathcal{S}^n$ and if $g_a(u) = g_a(v) = 1$ implies $g_a(u \circ_{\max} v) = 1$ for all vectors $u, v$.*

The intuitive meaning of max-closure is that there is a unique worst rank vector that is still accepted. Clearly, this is characteristic for threshold rules for which the threshold vector is this unique worst vector. However, it seems that we cannot weaken max-closure without violating the following characterization of threshold rules.

**Theorem 3.2 (Characterization of threshold rules).** *A social choice function is a threshold rule if and only if it is neutral, max-closed and monotonic.*

Proof: Consider an arbitrary threshold rule $f$ defined by a threshold vector $v$. We prove in the sequel that $f$ satisfies all the axioms required by the theorem. It is obvious that $f$ is neutral as we compare the rank vector of each alternative with the same threshold vector $v$ to decide whether it is chosen.
Next, we explain why $f$ satisfies max-closure. This follows from the observation that for every alternative $a \in A$, there is a decision function $g_a$ that takes the rank vector of $a$ as input and returns 1 if and only if this rank vector dominates the threshold vector $v$. Thus, consider two arbitrary preference profiles $R, R' \in \mathcal{S}^n$ such

that $g_a(r^*(R,a)) = g_a(r^*(R',a)) = 1$ for an alternative $a \in A$. We can deduce from the definition of the dominance relation that $r_i^*(R,a) \leq v_i$ and $r_i^*(R',a) \leq v_i$ for all $i \in \{1,...,n\}$. This implies that $\max(r_i^*(R,a), r_i^*(R,a)) \leq v_i$ for all $i \in \{1,...,n\}$ and therefore, $g_a(r^*(R,a) \circ_{\max} r^*(R',a)) = 1$. Hence, $f$ satisfies max-closure.

Finally, we show that $f$ satisfies monotonicity as defined in Definition 2.1. To prove this claim, consider two preference profiles $R, R' \in \mathcal{S}^n$ such that $R'$ is obtained from $R$ by letting a single voter reinforce an arbitrary alternative $a \in A$. This implies that the new rank vector $r^*(R',a)$ dominates $r^*(R,a)$. Thus, if $a \in f(R)$, then $r^*(R,a)$ dominates the threshold vector $v$ which implies that $r^*(R',a)$ also dominates $v$ because of the transitivity of the dominance relation. Hence, we can conclude that every threshold rule satisfies neutrality, monotonicity and max-closure.

Next, assume that we are given a social choice function $f$ that satisfies neutrality, max-closure and monotonicity. We prove in the sequel that there is a vector $v$ whose corresponding threshold rule is $f$. Since max-closure implies independence of ranks of other alternatives, it suffices to consider the rank vector of an alternative to decide whether it is in the choice set. Formally, this means that there is for every alternative $a \in A$ a decision function $g_a$ such that $a \in f(R)$ if and only if $g_a(r^*(R,a)) = 1$. Furthermore, we know that $g_a(r^*(R,a) \circ_{\max} r^*(R',a)) = 1$ holds for all preference profiles $R, R' \in \mathcal{S}^n$ with $g_a(r^*(R,a)) = g_a(r^*(R',a)) = 1$ as $f$ satisfies max-closure. Hence, it follows that there is a unique vector $v_a = (v_1, ..., v_n)$ such that $g_a(v_a) = 1$ and $g_a(v_1, ..., v_i + 1, v_{i+1}, ..., v_n) = 0$ for all $i \in \{1, 2, ..., n\}$. By neutrality, $g_a(v) = g_b(v)$ for all alternatives $a, b \in A$ and rank vectors $u$ and thus, $v_a = v_b$ for all alternatives $a, b \in A$. Finally, we can deduce from Lemma 3.1 that an alternative $a$ is chosen in every preference profile $R$ such that $r^*(R,a)\ D\ v_a$. Thus, the social choice function $f$ is a threshold rule defined by the threshold vector $v_a$. □

Note that all axioms required for Theorem 3.2 are independent: A social choice function that is monotonic and max-closed but not neutral can be defined by using different threshold vectors for different alternatives. An instance for a social choice function that is neutral and max-closed but not monotonic is the function that returns all alternatives in all preference profiles except those for which OMNI returns a single alternative $x$. For these profiles, the SCF returns $A \setminus \{x\}$. Finally, a social choice function that satisfies monotonicity and neutrality but not max-closure is the multi-threshold rule defined by the set $\{(m-2, m-1, ..., m-1, m), (m-1, ..., m-1)\}$. This multi-threshold rule violates max-closure as alternatives with the rank vectors $(m-2, m-1, ..., m-1, m)$ and $(m-1, ..., m-1)$ are chosen, but an alternative with the rank vector $(m-1, ..., m-1, m)$ is not chosen.

Note that the last observation is actually the only real difference between multi-threshold rules and threshold rules. As we prove in the sequel, this small difference is reflected in the characterization of multi-threshold rules by weakening max-closure to independence of ranks of other alternatives.

**Theorem 3.3 (Characterization of multi-threshold rules).** *A social choice function is a multi-threshold rule if and only if it satisfies neutrality, monotonicity and independence of ranks of other alternatives.*

Proof: Consider an arbitrary multi-threshold rule $f$ that is specified by the set of threshold vectors $V = \{v_1, ..., v_k\}$. This multi-threshold rule is neutral as the rank vector of every alternative is compared with the same set of threshold vectors. Thus, if the rank vector of an alternative dominates a threshold vector, this is also true after renaming the alternative.
Next, we prove the monotonicity of $f$ with an argument similar to the one in the proof for threshold rules: If the rank vector $r^*(R, a)$ of alternative $a$ in the preference profile $R$ dominates a threshold vector $v_i$, then this vector is still dominated if a single voter reinforces $a$. The reason for this is the transitivity of the dominance relation and the fact that reinforcing an alternative means that its new rank vector dominates its old one. Thus, $a \in f(R)$ implies $a \in f(R')$ for all profiles $R, R'$ that satisfy that $R'$ is derived from $R$ by reinforcing $a$.
Finally, we prove that $f$ also satisfies independence of ranks of other alternatives. This is true since $a \in f(R)$ if the rank vector $r^*(R, a)$ dominates a threshold vector $v_i \in V$. Clearly, this property does not depend on other alternatives or their ranks and thus, this axiom is satisfied. As consequence, every multi-threshold rule satisfies monotonicity, neutrality and independence of ranks of other alternatives.
It remains to prove that every social choice function that satisfies neutrality, monotonicity and independence of ranks of other alternatives is a multi-threshold rule. Therefore, consider an arbitrary social choice function $f$ that satisfies all these axioms. Since $f$ satisfies independence of ranks of other alternatives, it suffices to consider for every alternative its rank vector to decide whether the alternative is in the choice set. Furthermore, if we have found a preference profile $R$ with $a \in f(R)$, it follows from Lemma 3.1 that $a$ is chosen in every preference profile $R'$ with $r^*(R', a) \ D \ r^*(R, a)$. This observation means that we can find for every alternative $a \in A$ a set of threshold vectors $V_a$ such that alternative $a$ is chosen in a preference profile $R$ if its rank vectors $r^*(R, a)$ dominates at least one vector in $V_a$. Finally, these sets of threshold vectors must be equal for all alternatives because of neutrality. Thus, we can represent the social choice function $f$ as multi-threshold rule. □

All axioms in the characterization of multi-threshold rules are independent. Note that we only have to show that independence of rank of other alternatives is not implied by neutrality and monotonicity as the remaining points follow from the examples discussed for threshold rules. This independence results from considering tournament solutions such as the bipartisan set or the top cycle, see, e.g., [BBH16]. Both rules are known to be neutral and monotonic, but they clearly fail to satisfy independence of ranks of other alternatives.
Furthermore, note that it follows from this characterization that almost all reasonable independent rank-based social choice functions are threshold rules or multi-

threshold rules. The reason for this is that neutrality and monotonicity are very desirable axioms which should be satisfied in order to deem a social choice function reasonable.

### 3.2.2 $\tilde{P}$-strategyproofness and Independence of Ranks of Other Alternatives

In this section, we focus on social choice functions that satisfy both independence of ranks of other alternatives and $\tilde{P}$-strategyproofness. Even though there are many interesting social choice functions that satisfy independence of ranks of other alternatives, there are almost none that additionally satisfy $\tilde{P}$-strategyproofness. Thus, we first discuss two ideas for defining social choice functions that satisfy both axioms. Unfortunately, all social choice functions in these classes return rather large choice sets. We use this observation to characterize the OMNI-rule as one of the finest social choice functions that satisfies both independence of ranks of other alternatives and $\tilde{P}$-strategyproofness.

The first approach for defining social choice functions that satisfy both independence of ranks of other alternatives and $\tilde{P}$-strategyproofness is inspired by the OMNI-rule. Instead of only picking the most preferred alternative of every voter, we pick the $k$ most-preferred alternatives. This is formalized by the following class of threshold rules which we call I1-functions.

**Definition 3.10 (I1-function).** *A social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ is an I1-function if it is a threshold rule defined by a threshold vector $v = (k, m, m..., m)$ with $k \in \{1, ..., m\}$.*

We can deduce from the definition that OMNI is the finest I1-function, i.e., it holds for all I1-functions $f$ and preference profiles $R \in \mathcal{S}^n$ that $\text{OMNI}(R) \subseteq f(R)$. This idea for comparing SCFs is rather important and therefore, we introduce it formally.

**Definition 3.11 (Refinement).** *A social choice function $f$ defined on the domain $D^n$ is a refinement of a social choice function $g$ defined on the same domain if $f(R) \subseteq g(R)$ for all preference profiles $R \in D^n$ and there is a preference profile $R^* \in D^n$ with $f(R^*) \subsetneq g(R^*)$.*

We can deduce the notion of a finest social choice function within a class from this definition. A social choice function is a finest one in a set of SCFs if no other SCF in this set refines it. For example, no I1-function refines the OMNI-rule and therefore, it is one of the finest social choice function in this class. Observe that there can be multiple finest social choice functions in a class. Furthermore, if a social choice function $f$ refines another social choice function $g$, we call $g$ a coarsening of $f$.

As all I1-functions are coarsenings of the OMNI-rule, it is straightforward that all these SCFs return for every preference profile a non-empty choice set. Furthermore, we can also show that all I1-functions are $\tilde{P}$-strategyproof.

**Theorem 3.4.** *Every I1-function is $\tilde{P}$-strategyproof*

Proof: Consider an arbitrary I1-function $f : \mathcal{S}^n \mapsto 2^A \backslash \emptyset$. This function is a threshold rule defined by a vector $v = (k, m, ..., m)$ with $k \in \{1, ..., m\}$. If $k = 1$, then $f$ is the OMNI-rule which is known to be $\tilde{P}$-strategyproof [Gär76]. Thus, assume that $k > 1$. This means that the most preferred and the second most preferred alternatives of every voter are included in the choice set. Hence, a voter can only $\tilde{P}$-manipulate if he can ensure that his most preferred alternative is the unique winner. However, this is impossible as $|f(R)| > 1$ by definition. Thus, our initial assumption is wrong and $f$ is $\tilde{P}$-strategyproof. □

Note that Theorem 3.4 is not surprising as we only considered coarsenings of the OMNI-rule. Nevertheless, it shows that OMNI is not the only $\tilde{P}$-strategyproof social choice function that satisfies independence of ranks of other alternatives. However, these remarks lead to the question whether there are social choice functions that are not related to the OMNI-rule and that satisfy independence of ranks of other alternatives and $\tilde{P}$-strategyproofness. For answering this question, we introduce another class of social choice functions called I2-functions.

**Definition 3.12 (I2-functions).** *A social choice function $f : \mathcal{S}^n \mapsto 2^A \backslash \emptyset$ is an I2-function if it is a threshold rule defined by a vector $v = (k, k, ...k, m, m, ..., m)$ such that $k \in \{1, 2, ..., m\}$ and the largest index $l$ with $v_l = k$ satisfies that $\frac{(k-1)n}{m-1} + 1 > l$.*

The idea behind I2-functions is that an alternative should be chosen if sufficiently many voters agree that it is not too bad. This is formalized by demanding that an alternative is only chosen if there are at least $l$ voters that place it at position with rank less or equal to $k$. The second condition stating that $l$ should not be too large is required for the feasibility and $\tilde{P}$-strategyproofness of I2-functions. Without this condition, it is possible that an I2-function is not well-defined as it might return an empty set for a preference profile. An instance of an I2-function is given by the threshold rule defined by the vector $v = (2, 2, m, ..., m)$ if there are $n \geq m$ voters.
Note that we can derive from Theorem 3.1 that I2-functions are indeed feasible threshold rules. However, we can prove an even stronger result stating that every I2-function returns for every preference profile at least 2 alternatives.

**Lemma 3.2.** *Every I2-function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ satisfies that $|f(R)| \geq 2$ for every preference profile $R \in \mathcal{S}^n$.*

Proof: Consider an arbitrary I2-function $f : \mathcal{S}^n \mapsto 2^A \backslash \emptyset$ and let $v = (k, .., k, m, ..., m)$ denote its corresponding threshold vector. Furthermore, let $l$ denote the largest index with $v_l = k$. This lemma can be deduced from the condition on $l$ stating that $\frac{(k-1)n}{m-1} + 1 > l$. This condition is inspired from the following scoring scheme: Every alternative $a \in A$ receives in a preference profile $R$ one point from each voter $i \in N$ with $r(R_i, a) \leq k$. It follows that an alternative is chosen by $f$ if it has at least $l$ points. Thus, we need to show that there are at least two alternatives with at least $l$ points for every preference profile $R \in \mathcal{S}^n$.

We can prove this claim by a simple counting argument. Thus, observe that we have to distribute in total $kn$ points to all alternatives. Furthermore, every alternative can receive at most $n$ points. Hence, the maximal number of points that can be given to all alternatives while only selecting a single one is $n + (m-1) \cdot (l-1)$ as the single chosen alternative can get $n$ points and the remaining $m-1$ unchosen alternative have at most $l-1$ points. Thus, if $kn > n + (m-1) \cdot (l-1)$, at least two alternatives have at least $l$ points regardless of the preference profile $R$. As this term is equivalent to $\frac{(k-1)n}{m-1} + 1 > l$, it follows that $|f(R)| \geq 2$ for all preference profiles $R \in \mathcal{S}^n$. □

This lemma tells us on the one hand that I2-functions violate various axioms that demand decisiveness such as Pareto-optimality. On the other hand, this observation turns out to be crucial for proving that I2-functions are $\tilde{P}$-strategyproof, which is the goal of the next theorem.

**Theorem 3.5.** *Every I2-function is $\tilde{P}$-strategyproof.*

Proof: Consider an arbitrary I2-function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that is defined by the threshold vector $v = (k, ..., k, m, ..., m)$ and let $l$ denote the largest index with $v_l = k$. Assume for contradiction that there are two preference profiles $R, R' \in \mathcal{S}^n$ and a voter $i \in N$ such that $R_{-i} = R'_{-i}$ and $f(R')\ \tilde{P}_i\ f(R)$. Moreover, let $z$ denote voter $i$'s most preferred alternative in $f(R)$, i.e., $z \succ_i x$ for all $x \in f(R) \setminus \{z\}$. We make a case distinction with respect to the rank of $z$ in the preference of voter $i$ to prove this theorem.
First, assume that the rank of $z$ in $R_i$ is larger than $k$, i.e., $r(R_i, z) > k$. In this case, it holds for all alternatives $x \in f(R)$ that they have a rank larger than $k$ in $R_i$ as $z \succ_i x$ for all $x \in f(R) \setminus \{z\}$. This means that for all alternatives $a \in f(R)$, there are at least $l$ voters in $R_{-i}$ who assign $a$ a rank of at most $k$. Thus, these alternatives are chosen regardless of the preference of voter $i$, which implies that $f(R) \subseteq f(R')$. Furthermore, we know that $|f(R)| \geq 2$ as shown in Lemma 3.2. Hence, there is an alternative $y \in f(R')$ such that $z \succ_i y$ contradicting that $f(R')\ \tilde{P}_i\ f(R)$.
Next, assume that the rank of $z$ in $R_i$ is at most $k$, i.e., $r(R_i, z) \leq k$. This implies that no alternative $a \in A$ with $a \succ_i z$ is in $f(R)$ or $f(R')$. The reason for this is that $r(R_i, a) < k$ for all $a \in A$ with $a \succ_i z$, but these alternatives are not chosen. Thus, there are at most $l-2$ voters in $R_{-i}$ who assign these alternatives a rank less or equal to $k$. This makes it impossible for voter $i$ to force these alternatives in the choice set. Consequently, every alternative $x \in f(R')$ satisfies that $z \succeq_i x$. Furthermore, this preference is strict for at least one alternative as $|f(R')| \geq 2$. This contradicts that $f(R')\ \tilde{P}_i\ f(R)$ and therefore, it is impossible for voter $i$ to $\tilde{P}$-manipulate. □

Note that it is crucial for the proof of Theorem 3.5 that there are always at least two alternatives that win. Otherwise, it is possible that $f(R) = \{z\}$ and that there is a voter $i$ who prefers $z$ the least. This voter $i$ might be able to $\tilde{P}$-manipulate by placing an alternative $a$ with $k < r(R_i, a) < r(R_i, z)$ first because this can suffice to include it.

We can even generalize this observation. As we show in the sequel, OMNI is the only $\tilde{P}$-strategyproof social choice function that satisfies independence of ranks of other alternatives and for which there is a profile $R$ for every alternative $a \in A$ such that $f(R) = \{a\}$. The last criterion is called non-imposition and it is formally introduced in the sequel.

**Definition 3.13 (Non-imposition).** *A social choice function $f$ is non-imposing if there is a preference profile $R$ for all alternatives $a \in A$ such that $f(R) = \{a\}$.*

Note that non-imposition is a rather weak axiom. For instance, it is even weaker than Pareto-optimality as every Pareto-optimal social choice function chooses an alternative uniquely if it is top-ranked by every voter. This weakness makes it a very desirable axiom that is satisfied by many social choice functions. In the sequel, our goal is to prove that OMNI is the only $\tilde{P}$-strategyproof social choice function that satisfies non-imposition and independence of ranks of other alternatives. We start this proof by showing that we can focus on refinements of the OMNI-rule.

**Lemma 3.3.** *Every non-imposing social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that satisfies $\tilde{P}$-strategyproofness and independence of ranks of other alternatives satisfies $f(R) \subseteq OMNI(R)$ for all preference profiles $R \in \mathcal{S}^n$.*

Proof: Let $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ denote an arbitrary social choice function that satisfies all axioms required by the lemma. Additionally, we assume that $m \geq 2$ because if $m = 1$, every social choice function can only return the single available alternative. We show under these assumptions that $f(R) \subseteq \text{OMNI}(R)$ for all preference profiles $R \in \mathcal{S}^n$.
As first step of the proof, we choose an arbitrary alternative $a \in A$. As $f$ is non-imposing, there is a profile $R$ such that $f(R) = \{a\}$. Thereafter, we iterate over all voters $i \in N$ and let them switch from their current preference $R_i$ to an arbitrary preference $R_i'$ such that $a \succ_i' x$ for all $x \in A \setminus \{a\}$. This process defines a sequence of profiles $R^0$, $R^1$, ..., $R^n$ such that $R^0 = R$ and $R^i$ differs from $R^{i-1}$ only by the discussed manipulation of voter $i$. Note that $f(R^i) = \{a\}$ for all $i \in \{0, ..., n\}$ as otherwise there is a minimal index $i^* \geq 1$ such that $f(R^{i^*}) \neq \{a\}$ and $f(R^{i^*-1}) = \{a\}$ and voter $i^*$ prefers $a$ the most in $R^{i^*}$. Thus, voter $i^*$ can $\tilde{P}$-manipulate by switching form $R^{i^*}$ to $R^{i^*-1}$. This contradicts the $\tilde{P}$-strategyproofness of $f$ and therefore, we can deduce that every voter prefers $a$ the most in $R^n$ and $f(R^n) = \{a\}$.
Furthermore, every voter $i \in N$ can reorder the alternatives $b \in A$ with $r(R_i^n, b) > 1$ arbitrarily without changing the choice set; otherwise, there is a profile $R^+$ such that $R_{-i}^n = R_{-i}^+$, voter $i$ prefers $a$ in both profiles the most and $f(R^n) = \{a\} \neq f(R^+)$. Thus, voter $i$ can $\tilde{P}$-manipulate by switching from $R^+$ to $R^n$. As this holds for all preference profiles in which all voters prefer the unique winner the most, it follows that no alternative in $A \setminus \{a\}$ can be chosen after these modifications. Finally, as $f$ satisfies independence of ranks of other alternatives, no alternative $b \in A \setminus \{a\}$ can be chosen in a preference profile $R$ unless $r^*(R, b)_1 = 1$. As we can repeat this argument also for another alternative $b \neq a$, it follows that an alternative can only

$R$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |

$R^n$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a$ | $a$ |
| $b$ | $b$ | $c$ |
| $c$ | $c$ | $b$ |

$R^+$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a$ | $a$ |
| $c$ | $c$ | $c$ |
| $b$ | $b$ | $b$ |

Figure 3.4: Preference profiles illustrating the proof of Lemma 3.3

be chosen by $f$ if it is first-ranked by at least a single voter. Thus, $f(R) \subseteq \text{OMNI}(R)$ for all preference profiles $R \in \mathcal{S}^n$. □

Note that the main steps of the proof are illustrated with the help of the profiles $R$, $R^n$ and $R^+$ shown in Figure 3.4. Furthermore, assume that $f$ is a $\tilde{P}$-strategyproof, non-imposing and independent rank-based social choice function and that satisfies $f(R) = \{a\}$. Thus, $R$ is the profile where the proof of Lemma 3.3 starts. Next, every voter pushes $a$ to the top, which leads to the profile $R^n$. It follows from the $\tilde{P}$-strategyproofness that $f(R^n) = \{a\}$. Finally, the voters can arbitrarily reorder the alternatives $b$ and $c$ without affecting the choice set. This means for instance that $f(R^+) = \{a\}$. As consequence of this observation and independence of ranks of other alternatives, $b$ and $c$ are only chosen by $f$ if they are first-ranked by a voter.
It follows from Lemma 3.3 that we only have to show that no refinement of the OMNI-rule satisfies both independence of ranks of other alternatives and $\tilde{P}$-strategyproofness. Therefore, we discuss social choice functions satisfying all these properties in more detail and prove that they are also monotonic.

**Lemma 3.4.** *Every $\tilde{P}$-strategyproof social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that refines the OMNI-rule and that satisfies independence of ranks of other alternatives is monotonic.*

Proof: Consider a social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ as specified in the lemma, two arbitrary preference profiles $R, R' \in \mathcal{S}^n$, $R \neq R'$, an alternative $a \in f(R)$ and a voter $i \in N$ such that $R_{-i} = R'_{-i}$, $x \succ_i y$ if and only if $x \succ'_i y$ for all $x, y \in A \setminus \{a\}$ and $a \succ_i x$ implies $a \succ'_i x$ for all $x \in A \setminus \{a\}$. Additionally, we assume that there are $n \geq 2$ voters as there is no refinement of the OMNI-rule if $n = 1$. We have to show that $a \in f(R')$ to prove that $f$ satisfies monotonicity.
As first step, we choose an arbitrary alternative $b \neq a$ and let every voter $j$ with $r(R_j, a) > 1$ rank $b$ first. Furthermore, we ensure that the rank of $a$ does not change in the preference of any voter during these modifications. This leads to a preference profile $R^1$ in which only the alternatives $a$ and $b$ are first-ranked. It follows that $\{a\} \subseteq f(R^1) \subseteq \{a, b\}$, where the first inclusion follows from independence of ranks of other alternatives and the second one results from the fact that $f(R^1) \subseteq \text{OMNI}(R^1)$. Furthermore, observe that voter $i$ prefers $b$ the most in $R^1$ as he cannot reinforce $a$ if it is already his most preferred alternative in $R$.
Next, assume for contradiction that $a \notin f(R')$. We prove that this assumption implies that voter $i$ can $\tilde{P}$-manipulate, which contradicts the $\tilde{P}$-strategyproofness

$R$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |

$R^1$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $b$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $c$ |

$R^2$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $b$ |
| $b$ | $a$ | $a$ |
| $c$ | $c$ | $c$ |

$R'$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $a$ | $a$ |
| $c$ | $c$ | $b$ |

Figure 3.5: Preference profiles illustrating the proof of Lemma 3.4

of $f$. Therefore, consider the profile $R^2$ that only differs from $R^1$ in the fact that $r(R_i^2, a) = r(R_i', a)$. This implies that $r^*(R', a) = r^*(R^2, a)$, which means that $a \notin f(R^2)$ because of independence of ranks of other alternatives. Furthermore, as $a$ and $b$ are the only first-ranked alternatives in $R^2$, it follows that $f(R^2) = \{b\}$. Hence, switching from $R^1$ to $R^2$ is a $\tilde{P}$-manipulation for voter $i$ as $b$ is his most preferred alternative in $R^1$. However, this contradicts the $\tilde{P}$-strategyproofness of $f$ and therefore, $a \in f(R')$ is true. Hence, we can conclude that $f$ is monotonic. □

Figure 3.5 illustrates the proof of the last lemma with an example. This figure shows four preference profiles $R$, $R^1$, $R^2$ and $R'$. As in the proof, we assume that $a \in f(R)$ for an independent rank-based and $\tilde{P}$-strategyproof social choice function $f$ that refines the OMNI-rule. It follows from independence of ranks of other alternatives that $a \in f(R^1)$ as $r^*(R, a) = r^*(R^1, a)$. Furthermore, observe that only $a$ and $b$ are first-ranked in $R^1$, which means that $f(R^1) \subseteq \{a, b\}$ as $f$ refines the OMNI-rule. Thereafter, voter 2 reinforces $a$ in his preference to derive $R^2$. It holds that $a$ is in $f(R^2)$ as $f(R^2) \subseteq \text{PO}(R^2) = \{a, b\}$ and $f$ is $\tilde{P}$-strategyproof. Finally, independence of ranks of other alternatives implies that $a \in f(R')$ as $r^*(R^2, a) = r^*(R', a)$, which shows that $f$ satisfies monotonicity.
With the help of the previous two lemmas, we can show that OMNI is indeed the only social choice function that satisfies $\tilde{P}$-strategyproofness, non-imposition and independence of ranks of other alternatives. First, we consider the simple scenario with at most 2 voters.

**Lemma 3.5.** *There is no social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that refines the OMNI-rule and that satisfies $\tilde{P}$-strategyproofness, independence of ranks of other alternatives and non-imposition if $m \geq 3$ and $n \leq 2$.*

Proof: First note that Lemma 3.3 even holds if we have only a single voter. This means that every social choice function $f$ that satisfies all axioms of this lemma is a refinement of the OMNI-rule or it is the OMNI-rule. Furthermore, if there is only a single voter, then $|\text{OMNI}(R)| = 1$ for all preference profiles $R \in \mathcal{S}^n$. Therefore, we can directly deduce that no social choice function refines OMNI in this case.
Next, consider the case with $n = 2$ voters and assume for contradiction that there is a social choice function $f$ that satisfies all axioms of the lemma and that refines the OMNI-rule. Thus, there is a preference profile $R \in \mathcal{S}^n$ such that $f(R) \subsetneq \text{OMNI}(R)$. Note that this implies that $|\text{OMNI}(R)| = 2$ and $|f(R)| = 1$, which means that the voters rank two different alternatives first but only one of them is chosen by $f$. In

$R$:

| 1 | 1 |
|---|---|
| $a$ | $b$ |
| $b$ | $a$ |
| $c$ | $c$ |

$R^1$:

| 1 | 1 |
|---|---|
| $a$ | $b$ |
| $c$ | $a$ |
| $b$ | $c$ |

$R^2$:

| 1 | 1 |
|---|---|
| $a$ | $b$ |
| $c$ | $c$ |
| $b$ | $a$ |

$R^3$:

| 1 | 1 |
|---|---|
| $c$ | $b$ |
| $a$ | $a$ |
| $b$ | $c$ |

$R^4$:

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $c$ | $b$ |
| $b$ | $a$ |

Figure 3.6: Preference profiles illustrating the proof of Lemma 3.5

the sequel, we assume without loss of generality that voter 1 prefers $a$ the most, voter 2 prefers $b$ the most and $f(R) = \{b\}$. In this case, voter 1 can make $b$ his least preferred alternative and in the resulting profile $R^1$, it still holds that $f(R^1) = \{b\}$ because $f(R^1) \subseteq \text{OMNI}(R^1) = \{a, b\}$ and every other outcome is a $\tilde{P}$-improvement for voter 1.

Moreover, there is at least one alternative $c$ that is not first-ranked by voter 1 or voter 2 in $R^1$ since $m \geq 3$. Let voter 2 place alternative $c$ at the position with rank 2 and alternative $a$ at the position with rank 3 to obtain the preference profile $R^2$. It holds that $f(R^2) = \{b\}$ as otherwise voter 2 can $\tilde{P}$-manipulate by switching back to $R^1$. Note that $r^*(R^2, a) = (1, 3)$ and $a$ is not chosen in $R^2$. Next, consider the preference profile $R^3$ which is derived from $R^2$ by letting voter 1 place $c$ first and note that $f(R^3) = \{b\}$; otherwise, voter 1 can $\tilde{P}$-manipulate by switching from $R^2$ to $R^3$ because $b$ is his worst alternative in $R^2$. In particular, note that $r^*(R^3, c) = \{1, 2\}$ and $c \notin f(R^3)$. Finally, consider the preference profile $R^4$ in which voter 1 ranks $a$ first and $c$ second and voter 2 ranks $c$ first and $a$ third. The remaining alternatives can be placed arbitrarily. As $f$ is a refinement of the OMNI-rule, it follows that $f(R^4) \subseteq \{a, c\}$. However, independence of ranks of other altneratives implies that $a \notin f(R^4)$ and $c \notin f(R^4)$ because $r^*(R^4, a) = (1, 3) = r^*(R^2, a)$, $r^*(R^4, c) = (1, 2) = r^*(R^3, c)$ and $a \notin f(R^2)$, $c \notin f(R^3)$. Therefore, no valid choice for the profile $R^4$ is left. This is a contradiction, which means that the initial assumption is wrong. Thus, no refinement of the OMNI-rule satisfies $\tilde{P}$-strategyproofness, non-imposition and independence of ranks of other alternatives if $m \geq 3$ and $n \leq 2$. □

First, we discuss an example for the constructions in the proof of Lemma 3.5. Thus, consider the preference profiles shown in Figure 3.6 and assume that $f$ denotes a $\tilde{P}$-strategyproof and independent rank-based refinement of the Pareto-rule with $f(R) = \{b\}$. As a first step, voter 1 makes $b$ his least preferred alternative to derive $R^1$. Because $f$ is $\tilde{P}$-strategyproof and it refines the Pareto-rule, we can deduce that $f(R^1) = \{b\}$. Next, voter 2 modifies his preference such that $c$ has rank 2, which results in the profile $R^2$. We can derive from $\tilde{P}$-strategyproofness that $f(R^2) = \{b\}$. After that voter 1 manipulates such that $c$ is his most preferred alternative. This steps leads to the profile $R^3$ and it holds that $f(R^3) = \{b\}$ because of $\tilde{P}$-strategyproofness. Finally, we consider the profile $R^4$. As $f$ refines the OMNI-rule, it must hold that $f(R^4) \subseteq \{a, c\}$. However, this contradicts independence

of unchosen alternatives as $r^*(R^4, a) = r^*(R^2, a)$ and $r^*(R^4, c) = r^*(R^3, c)$ and $a \notin f(R^2)$, $c \notin f(R^3)$. Thus, $f$ does not satisfy all required axioms.
Observe that the approach used in this lemma relies heavily on the fact that there are only two voters, which implies that every refinement $f$ of the OMNI-rule yields a profile $R$ such that $|f(R)| = 1$ even though two alternatives are first ranked by the voters. This is no longer true if there are more voters and therefore, this proof cannot be adapted to the situation with $n \geq 3$ voters. Nevertheless, we show in the next theorem that the characterization also holds in this case.

**Theorem 3.6 (Characterization of OMNI).** *OMNI is the only non-imposing social choice function that satisfies $\tilde{P}$-strategyproofness and independence of ranks of other alternatives if $m \geq 3$.*

Proof: First note that OMNI satisfies all three axioms. We have already seen that OMNI is a threshold rule and therefore, it satisfies independence of ranks of other alternatives. Furthermore, if all voters agree on the most preferred alternative, this alternative is the unique winner, which means that OMNI is indeed non-imposing. Finally, it has been shown in [Gär76] that OMNI is $\tilde{P}$-strategyproof.
Thus, it only remains to show that no other social choice function satisfies these axioms if $m \geq 3$. First note that we know by Lemma 3.3 that it suffices to show that no refinement of the OMNI-rule satisfies all axioms required by this theorem. Furthermore, we can additionally assume that $n \geq 3$ as we have already shown in Lemma 3.5 that there is no such social choice function if $n < 3$. Thus, assume for contradiction that there is a social choice function $f$ defined on $m \geq 3$ alternatives and $n \geq 3$ voters that refines the OMNI-rule and that satisfies $\tilde{P}$-strategyproofness and independence of ranks of other alternatives. Note that we omit non-imposition as this axiom is implied by the fact that $f$ refines the OMNI-rule. Furthermore, we know by Lemma 3.4 that $f$ is also monotonic. In the sequel, we construct two preference profiles $R, R' \in \mathcal{S}^n$ such that voter 1 can $\tilde{P}$-manipulate by switching from $R$ to $R'$. This contradicts that $f$ is $\tilde{P}$-strategyproof and therefore, it shows that $f$ does not satisfy all required axioms.
For this construction, let $c$ denote an arbitrary alternative that is not chosen if its rank vector $r^*(R, c)$ equals $(1, m, ..., m)$. Note that such an alternative $c$ exists because $f$ satisfies independence of ranks of other alternatives and monotonicity and therefore, Lemma 3.1 can be used. It follows from this lemma that if an alternative is chosen with rank vector $v = (1, m, ..., m)$, then it is chosen for every rank vector $v'$ with $v'_1 = 1$ as all of these rank vectors dominate $v$. Thus, if all alternatives are chosen with the rank vector $(1, m, ..., m)$, then $f$ does not refine the OMNI-rule. Therefore, an alternative $c$ exists that is not chosen if its rank vector equals $(1, m, ..., m)$.
Next, let $a \neq c$ denote an arbitrary alternative and let $v$ denote a vector such that the index $l$ with $v_l = 1$ and $v_{l+1} > 1$ is minimal and $a$ is chosen if $v$ is its rank vector. Note that the index $l$ is well-defined as an alternative is only chosen if the first entry of its rank vector equals 1. This is true as $f$ refines OMNI. Thus, if the

$R$:

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $c$ | $a$ | $b$ | $b$ | $b$ |
| $a$ | $b$ | $a$ | $a$ | $a$ |
| $b$ | $c$ | $c$ | $c$ | $c$ |

$R'$:

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ | $b$ |
| $c$ | $b$ | $a$ | $a$ | $a$ |
| $b$ | $c$ | $c$ | $c$ | $c$ |

Figure 3.7: Preference profiles illustrating the proof of Theorem 3.6

index $l$ is not well-defined, then alternative $a$ is only chosen if every voter prefers $a$ the most. However, this means that no alternative is chosen if voter 1 prefers $c$ the most and all other voters prefer $a$ the most and $c$ the least. This is a contradiction and therefore, $v_n > 1$. This means that $l$ is well-defined. Furthermore, observe that the alternative $a$ is not chosen if $k < l$ voters prefer $a$ the most because of the definition of $l$. Finally, let $b \notin \{a, c\}$ denote another arbitrary alternative.

We construct the preference profile $R$ as follows: The first voter prefers $c$ first, $a$ second and $b$ third. The voters $i$ with $2 \leq i \leq l$ prefer $a$ first, $b$ second and $c$ the least. Finally, the voters $j$ with $l < j \leq n$ prefer $b$ first, $a$ second and $c$ the least. The remaining alternatives can be placed arbitrarily. Note that no other alternative but $a$, $b$ and $c$ can be chosen by $f$ as every voter prefers one of these alternatives the most. Moreover, observe that $c$ is not in $f(R)$ because $r^*(R, c) = (1, m, ..., m)$ and by construction, $c$ does not win with this rank vector. Furthermore, $a$ is not in the choice set as only $l - 1$ voters prefer $a$ the most. However, the definition of $l$ requires that there must be at least $l$ voters who prefer $a$ the most to make it win. This implies that $f(R) = \{b\}$. However, voter 1 prefers $a$ strictly to $b$ and he can make alternative $a$ win by ranking $a$ first and $c$ second. In this new profile $R'$, alternative $a$ is first-ranked by $l$ voters and second-ranked by the rest. As the rank vector $r^*(R', a)$ dominates the vector $v$, it follows from Lemma 3.1 that $a \in f(R')$. Furthermore, as $f$ refines OMNI, it follows that $f(R') \in \{\{a\}, \{a, b\}\}$. Thus, we can deduce that voter 1 can $\tilde{P}$-manipulate by switching from $R$ to $R'$ because he prefers both sets to $\{b\} = f(R)$. Thus, we can deduce that $f$ is $\tilde{P}$-manipulable, which contradicts the initial assumption. Hence, no other social choice function but the Pareto-rule satisfies non-imposition, $\tilde{P}$-strategyproofness and independence of ranks of other alternatives if $m \geq 3$. □

An example of the constructions used in the proof of Theorem 3.6 is displayed in Figure 3.7. Assume for this example that $f$ is a threshold rule defined by the vector $v = (1, 1, 3, 3, 3)$. This means that $f(R) = \{b\}$ as only the rank vector of $b$ dominates the threshold vector. Thus, voter 1 can $\tilde{P}$-manipulate by swapping $c$ and $a$, which means that $f$ is $\tilde{P}$-manipulable.

Note that all axioms used in the characterization of OMNI are independent: All I2-functions satisfy independence of ranks of other alternatives and $\tilde{P}$-strategyproofness but are not non-imposing. The bipartisan set (see, e.g., [BBH16]) satisfies non-imposition and $\tilde{P}$-strategyproofness but not independence of ranks of other alternatives. Furthermore, every threshold rule $f$ refining OMNI satisfies independence of ranks of other alternatives and non-imposition but not $\tilde{P}$-strategyproof. Finally, if $m = 2$, the majority rule that chooses the alternative that is first-ranked by the most voters satisfies all axioms. Hence, even $m \geq 3$ is required.

We can deduce from this characterization that OMNI is the most preferable $\tilde{P}$-strategyproof social choice function that satisfies independence of ranks of other alternatives. The reason for this is that an alternative $a$ should be the unique winner of an election if all voters agree that $a$ is the best alternative. However, we can deduce from Theorem 3.6 that this is not true for every other independent rank-based and $\tilde{P}$-strategyproof social choice function as such a rule cannot be non-imposing.

## 3.3 General Rank-based Social Choice Functions

After discussing independent rank-based social choice functions in the last section, we focus on general rank-based SCFs next. Note that there are many well-established social choice functions within this class, such as scoring rules. Furthermore, we also discuss new ideas for designing general rank-based social choice functions. Hitherto, neither the known nor the new social choice functions have been analyzed with respect to $\tilde{P}$-strategyproofness. Therefore, we discuss various subclasses of rank-based social choice functions with respect to this axiom.

In particular, we try to find $\tilde{P}$-strategyproof and rank-based refinements of the OMNI-rule in Section 3.3.1. The existence of these functions is interesting as it shows that the characterization of the OMNI-rule based on $\tilde{P}$-strategyproofness, non-imposition and independence of ranks of other alternatives cannot be generalized. Furthermore, we analyze the well-established class of scoring rules with respect to $\tilde{P}$-strategyproofness in Section 3.3.2. Finally, we consider in Section 3.3.3 a new class of general rank-based social choice functions based on the idea of comparing rank vectors of alternatives with each other. In this section, we prove a criterion that shows that a large number of these rules violates $\tilde{P}$-strategyproofness.

### 3.3.1 Rank-based Refinements of the OMNI-Rule

The goal in this section is to find rank-based refinements of the OMNI-rule which are $\tilde{P}$-strategyproof. The question on the existence of such social choice functions arises naturally from the characterization of the OMNI-rule presented in Theorem 3.6. This theorem states that there is no $\tilde{P}$-strategyproof, non-imposing and independent rank-based social choice function but the OMNI-rule. Thus, we are interested in the question whether such functions exist if we weaken independence of ranks of other alternatives to rank-basedness. As we can prove in this section, there are

indeed social choice functions that satisfy rank-basedness, non-imposition and $\tilde{P}$-strategyproofness.
Our first approach for defining a $\tilde{P}$-strategyproof refinement of the OMNI-rule is motivated by the observation that there are many $\tilde{P}$-strategyproof social choice functions in the strict domain that are Condorcet extensions. Thus, the idea is to refine the OMNI-rule by returning the Condorcet winner if there is one; otherwise, we do not change the OMNI-rule. Unfortunately, it is impossible to detect the existence of the Condorcet winner always correctly by a rank-based social choice function. Therefore, we use the following rank-based SCF, which we refer to as $f^+$, as approximation of the Condorcet-rule: $f^+(R) = \{a\}$ if $a$ is first-ranked by more than $n/2$ voters; if no such alternative exists in $R$, then $f^+(R) = A$. Note that $f^+$ is well-defined as there can only be a single alternative that is first-ranked by more than half of the voters. Furthermore, if $f^+(R) = \{a\}$, then $a$ is the Condorcet winner in $R$. This implies that $f^+$ is a coarsening of the Condorcet-rule.
With the help of the SCF $f^+$, we define the first $\tilde{P}$-strategyproof refinement of the OMNI-rule called OMNI$^+$. This social choice function returns the intersection of the OMNI-rule and $f^+$, i.e., $\text{OMNI}^+(R) = \text{OMNI}(R) \cap f^+(R)$. Thus, it is clear that this SCF indeed refines the OMNI-rule. As we show next, OMNI$^+$ is also $\tilde{P}$-strategyproof.

**Theorem 3.7.** *The social choice function OMNI$^+$ is $\tilde{P}$-strategyproof in the strict domain.*

Proof: Consider an arbitrary preference profile $R$ defined on $n$ voters. We prove in the sequel that no voter in $R$ can $\tilde{P}$-manipulate the OMNI$^+$-rule. Thus, let $k = \lceil \frac{n+1}{2} \rceil$ denote the number of voters which must agree on a most preferred alternative $a$ such that $f^+$ returns $a$ as unique winner. Furthermore, let $l_R$ denote the largest number of voters that agree on a first-ranked alternative in the preference profile $R$. In the sequel, we make a case distinction with respect to $l_R$.
First, we consider the case that $l_R < k-1$. In this case, it is impossible for a voter to manipulate the outcome such that $l_{R'} \geq k$. This means that OMNI$^+$ coincides with OMNI, even if a voter tries to manipulate. As OMNI is $\tilde{P}$-strategyproof, it follows directly that OMNI$^+$ cannot be $\tilde{P}$-manipulated in this case.
Next, consider a preference profile $R$ with $l_R \geq k$. Thus, there is an alternative $a$ such that at least $k$ voters prefer this alternative the most. As $f^+(R) = \{a\}$, it follows that $\text{OMNI}^+(R) = \{a\}$. Thus, no voter who prefers $a$ the most can improve the outcome as his most preferred alternative is the unique winner. Furthermore, every voter $i$ who prefers $a$ not the most cannot affect the outcome as there are at least $l_R \geq k$ voters who prefer $a$ the most in $R_{-i}$. This means that $\text{OMNI}^+(R') = \{a\}$ for all preference profiles $R' = (R_{-i}, R'_i)$ and therefore, voter $i$ cannot $\tilde{P}$-manipulate. Thus, OMNI$^+$ is also in this case $\tilde{P}$-strategyproof.
Finally, consider an arbitrary preference profile $R$ with $l_R = k-1$. In this case, it is possible for a voter to make an alternative win uniquely by lying about his true preferences. Hence, consider a profile $R'$ with $l_{R'} = k$ and $R_{-i} = R'_{-i}$ for a

voter $i \in N$. It follows that $\text{OMNI}^+(R') = \{a\}$ for an arbitrary alternative $a \in A$. However, $a$ cannot be voter $i$'s most preferred alternative in $R$ as otherwise $l_{R'} > l_R$ cannot be true. Furthermore, voter $i$'s most preferred alternative is in $\text{OMNI}^+(R)$ since this rule coincides with OMNI if $l_R < k$. This means that switching from $R$ to $R'$ is no $\tilde{P}$-manipulation for voter $i$. Furthermore, every modification that does not lead to a preference profile $R'$ with $l_{R'} = k$ cannot be a $\tilde{P}$-manipulation as $\text{OMNI}^+$ coincides for both $R$ and $R'$ with OMNI under these assumptions. Therefore, no $\tilde{P}$-manipulation is possible in this case either. Thus, $\text{OMNI}^+$ is indeed $\tilde{P}$-strategyproof. □

Note we can deduce from this theorem that OMNI is not the finest social choice function that satisfies rank-basedness and $\tilde{P}$-strategyproofness. Furthermore, it also follows that we cannot generalize independence of ranks of other alternatives to rank-basedness in the characterization of OMNI as $\text{OMNI}^+$ satisfies all required axioms. However, $\text{OMNI}^+$ chooses still a rather large choice set in many situations. Therefore, a sparser refinement of the OMNI-rule is desirable.

For finding such a social choice function, we consider the Plurality-rule in more detail. This rule chooses the alternatives that are first-ranked by the most voters. Formally, let $n(R, a)$ denote the number of voters who prefer $a$ the most in the preference profile $R$. Then, the Plurality-rule chooses all alternatives $a \in A$ that maximize $n(R, a)$. Note that this rule is well-known, see, e.g., [Chi96, Lep92]. It is clear that the Plurality-rule is rank-based and refines the OMNI-rule. Furthermore, it is obvious that this social choice function returns for many preference profiles very small choice sets. Unfortunately, it is a well-established result that the Plurality-rule is $\tilde{P}$-manipulable. Therefore, we aim to find a slight variation of this social choice function that fixes this problem.

For finding this variant, we first analyze exactly when the plurality rule is $\tilde{P}$-manipulable. Thus, observe that the plurality rule can be $\tilde{P}$-manipulated if it returns a single winner $a$ in the preference profile $R$ and there is an alternative $b$ with $n(R, b) = n(R, a) - 1$. Furthermore, assume that there is a voter in $R$ who submits $c \succ b \succ a$. Then this voter can $\tilde{P}$-manipulate by changing his preference to $b \succ c \succ a$. Similarly, a voter whose most preferred alternative is not chosen can also manipulate if the Plurality-rule chooses multiple winners. He can reinforce the alternative that he prefers the most among the winners and make it the unique winner. Thus, a voter can $\tilde{P}$-manipulate by adding an alternative to the choice set if only a single alternative is chosen, or by removing all alternatives but one from the choice set if multiple alternatives are chosen. Formally, this means that a voter $\tilde{P}$-manipulates by switching from a profile $R$ to a profile $R'$ with $f(R) \subsetneq f(R')$ or $f(R') \subsetneq f(R)$. Even more, no other types of manipulations are possible. The reason for this is that a single voter can only decrease $n(R, a)$ by 1 and increase $n(R, b)$ by 1 for two alternatives $a, b \in A$. Thus, we only have to prohibit these manipulative inclusions and exclusions of alternatives. The simplest way to do this is to ensure that always two or more alternatives are chosen. This modification does not really solve the problem that a voter can add or remove some alternatives, but it makes the sets with respect to the set extension $\tilde{P}$ incomparable. This idea leads to the following social choice function which we call 2-Plurality.

**Definition 3.14 (2-Plurality).** *Consider an arbitrary preference profile $R \in \mathcal{S}^n$ and let $a_1$, $a_2$, ..., $a_m$ denote the alternatives ordered in decreasing order of their plurality score, i.e., $n(R, a_1) \geq n(R, a_2) \geq ... \geq n(R, a_m)$. Then, 2-Plurality, abbreviated by $p_2$, is defined as $p_2(R) = \{a \in A \mid n(R, a) \geq n(R, a_2) \wedge n(R, a) > 0\}$.*

The intuition behind 2-Plurality is to pick the best and the second best alternative with respect to the plurality score. If there are more alternatives $b \in A$ with $n(R, b) = n(R, a_2)$, we additionally include them in the choice set to ensure neutrality. Furthermore, in the special case that all voters agree on a most preferred alternative, 2-Plurality returns this alternative as unique winner. As it holds for almost all preference profiles $R$ that $|p_2(R)| \geq 2$, it is easy to see that $p_2$ is not $\tilde{P}$-manipulable by switching from a profile $R$ to a profile $R'$ with $f(R) \subseteq f(R')$ or $f(R') \subseteq f(R)$. We formalize this intuition to prove that 2-Plurality is indeed $\tilde{P}$-strategyproof.

**Theorem 3.8.** *2-Plurality is $\tilde{P}$-strategyproof in the strict domain.*

Proof: Consider an arbitrary preference profile $R \in \mathcal{S}^n$ and a voter $i \in N$. We prove in the sequel that voter $i$ cannot $\tilde{P}$-manipulate 2-Plurality in $R$. We use for this proof a case distinction with respect to $p_2(R)$ and the most preferred alternative of voter $i$ which we refer to as $a$. First, assume that $a \in p_2(R)$. In this case, voter $i$ can only $\tilde{P}$-manipulate by switching to a profile $R'$ with $p_2(R') = \{a\}$. However, 2-Plurality returns $a$ only as unique winner if all voters agree that $a$ is the best alternative. Thus, if $f(R) \neq \{a\}$, voter $i$ cannot $PK$-manipulate as there is another voter who does not prefer $a$ the most. Furthermore, if $f(R) = \{a\}$, voter $i$ cannot $\tilde{P}$-manipulate either as he already obtains his most preferred alternative as unique winner.

Thus, assume next that the most preferred alternative $a$ of voter $i$ is not in $p_2(R)$. This means that there are two other alternatives $b$ and $c$ with $n(R, b) > n(R, a)$ and $n(R, c) > n(R, a)$ and $\{b, c\} \subseteq p_2(R)$. As consequence, voter $i$ cannot enforce that $a$ is chosen as he cannot increase the plurality score of $a$ or reduce the plurality score of $b$ and $c$. This means that the only possibility to $\tilde{P}$-manipulate is to increase the plurality score of another alternative $d$. First, we consider the case that $d \notin p_2(R)$ and we additionally assume without loss of generality that $n(R, b) \geq n(R, c)$. This means that $n(R, c) > n(R, d)$ and voter $i$ can only enforce that $d$ is chosen if $n(R, c) = n(R, d) + 1$. In this case, he can try to manipulate by placing $d$ first. In the resulting preference profile $R'$, it holds that $n(R', c) = n(R', d)$ and $c$ still has the second highest plurality score. This implies that $\{b, c\} \subseteq p_2(R')$ and therefore, this is no $\tilde{P}$-manipulation for voter $i$. Thus, voter $i$ cannot $\tilde{P}$-manipulate by ranking an unchosen alternative first.

Finally, assume that voter $i$'s most preferred alternative $a$ is not chosen and that this voter tries to manipulate by ranking an alternative $d \in p_2(R)$ first. We denote the profile derived from this modification as $R'$. Observe that no alternative $x \in A \backslash f(R)$ is in $p_2(R')$ as we do not increase their plurality scores. Thus, switching from $R$ to $R'$ is only a $\tilde{P}$-manipulation for voter $i$ if $d$ is the only winner after the manipulation. However, there is at least one other alternative that is first-ranked by a voter $j \neq i$ in $R$ as otherwise the most preferred alternative of voter $i$ is in $p_2(R)$. This implies that $|p_2(R')| > 1$, which shows that voter $i$ cannot $\tilde{P}$-manipulate in this case either. Hence, 2-Plurality is $\tilde{P}$-strategyproof in each case. □

Note that it is straightforward that 2-Plurality is a refinement of the OMNI-rule. Even more, it returns for most preference profiles only two alternatives, which means that it is one of the most discriminating $\tilde{P}$-strategyproof social choice functions. It seems even reasonable to conjecture that the average size of the choice set of $p_2$ is approximately 2 if we use sufficiently many voters and alternatives. Furthermore, 2-Plurality also satisfies many other desirable properties such as Pareto-optimality and neutrality. Thus, 2-Plurality seems to be one of the most preferable $\tilde{P}$-strategyproof social choice functions in the strict domain. However, it should be mentioned that this SCF only exploits a weakness of the set extension $\tilde{P}$ to achieve $\tilde{P}$-strategyproofness. This means that it still suffers from the same disadvantageous as the Plurality-rule in practice. Therefore, we do not deem 2-Plurality as important social choice function from a pracitcal standpoint. Nevertheless, it shows that there can be very discriminating social choice functions that satisfy $\tilde{P}$-strategyproofness.

### 3.3.2 Scoring Rules

In this section, we discuss the well-known class of scoring rules with respect to $\tilde{P}$-strategyproofness. These rules have often been considered in the literature, see, e.g., [Smi73, You75, NR81, CS98, Mer03]. These results discuss various interesting scoring rules and characterize them. Furthermore, there are also results that analyze scoring rules with respect to specific axioms. However, we are not aware of a publication discussing scoring rules with respect to $\tilde{P}$-strategyproofness. Therefore, we analyze scoring rules with respect to this axiom and show that all non-trivial scoring rules are $\tilde{P}$-manipulable. Note that we use in the proof of this result a large number of voters. This is no problem as one of the most important results about scoring rules characterizes them with weak axioms but a variable electorate [Smi73, You75]. This means that a scoring rule can be used for an arbitrary number of voters. Consequently, it is often assumed that there is a huge number of voters to prove a result.

Before we discuss scoring rules with respect to $\tilde{P}$-strategyproofness, we formalize their idea first. Intuitively, a scoring rule gives an alternative $s_j$ points for every voter who places it at rank $j$ and chooses eventually the alternatives with maximal score. This leads to the following formal definition.

**Definition 3.15 (Scoring rules).** *Let $c(v, i)$ denote a function that counts how often the entry $i$ appears in the vector $v$, and let $sc(R, a, s) = \sum_{i=1}^{m} c(r^*(R, a), i) \cdot s_i$ denote the score of alternative $a$ in the profile $R$ with respect to the scoring vector $s = (s_1, s_2, ..., s_m)$. A social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ is a scoring rule if there is a scoring vector $s = (s_1, s_2, ..., s_m)$ such that $f$ chooses the alternatives with maximal score, i.e., $f(R) = \{a \in A \mid \nexists b \in A : sc(R, b, s) > sc(R, a, s)\}$ for all preference profiles $R \in \mathcal{S}^n$.*

A well-known example of a scoring rule is Borda's rule which is defined by the scoring vector $s = (m-1, m-2, ..., 0)$. For more details see, e.g., [Bor81, NR81, Saa]. Another important scoring rule is the Plurality-rule which is defined by the scoring vector $(1, 0, ..., 0)$ [Lep92, Chi96]. Note that this rule has already been discussed in Section 3.3.1.

Furthermore, scoring rules can behave rather unexpected depending on their scoring vectors. For instance, if there is an index $i$ with $s_i < s_{i+1}$, an alternative might not be chosen after it is reinforced by a voter. Thus, a scoring rule is monotonic if and only if $s_1 \geq s_2 \geq ... \geq s_m$. Unfortunately, this observation does not exclude the scoring rule defined by the vector $s$ with $s_1 = s_2 = ... = s_m$. This scoring rule returns always all alternatives and therefore, we call it trivial. Hence, we can avoid this unreasonable social choice function by discussing non-trivial ones. Another venue for defining reasonable scoring rules is to discuss strictly monotonic ones, i.e., scoring rules that are defined by a scoring vector $s$ with $s_1 > s_2 > ... > s_m$. This definition also excludes the trivial scoring rule.

Note that the monotonicity of a scoring rule strongly influences its behavior. Therefore, we distinguish scoring rules with respect to their scoring vectors in order to analyze their $\tilde{P}$-strategyproofness. As consequence, the proof of the next theorem stating that every non-trivial scoring rule is $\tilde{P}$-manipulable if there are sufficiently many voters and at least three alternatives requires a case distinction.

**Theorem 3.9.** *Every non-trivial scoring rule is $\tilde{P}$-manipulable if there are sufficiently many voters and $m \geq 3$ alternatives.*

Proof: Consider an arbitrary non-trivial scoring rule $f$ defined by the scoring vector $s = (s_1, s_2, ..., s_m)$. We prove the theorem by constructing a preference profile in which a voter can $\tilde{P}$-manipulate.

As the first step of the proof, we construct a preference profile $R^a \in \mathcal{S}^{(m-1)!}$ such that $f(R^a) = \{a\}$ for an arbitrary alternative $a \in A$. Thus, let $j$ denote an index with $s_j \geq s_k$ for all $k \in \{1, ..., m\}$. Then, every voter places $a$ at rank $j$ in $R^a$, which implies that $a$ receives the maximal score. Hence, we have to ensure that no other alternative receives the same number of points. Therefore, we place the remaining alternatives such that no voter orders them in the same way, i.e., all

$(m-1)!$ permutations of these alternatives are used. In the profile $R^a$, alternative $a$ obtains $(m-1)! \cdot s_j$ points, whereas every other alternative obtains $(m-2)! \cdot \sum_{k=1, k \neq j}^{m} s_i$ points. This is strictly less than the score of $a$ as $s_j > s_i$ for at least one index $i$ since $f$ is a non-trivial scoring rule. Moreover, let $R^b$ denote a profile constructed in the same way as $R^a$ such that $b$ wins uniquely. It follows that $f(R^{ab}) = \{a, b\}$ where the profile $R^{ab} \in \mathcal{S}^{2 \cdot (m-1)!}$ is the concatenation of $R^a$ and $R^b$. We increase the difference between the score of $a$ and $b$ and the remaining alternatives by adding copies of $R^{ab}$ to the profile until a single voter cannot make an alternative $c \in A \setminus \{a, b\}$ win. This leads to the profile $R^*$ which we use as basis for all subsequent profiles.

Next, we make a case distinction with respect to the monotonicity of $f$. Firstly, assume that $f$ is non-monotonic. This means that there is an $i \in \{1, ..., m-1\}$ such that $s_i < s_{i+1}$. In this case, we add a single voter $i^*$ with $r(R_{i^*}, a) = i$ and $r(R_{i^*}, b) = i + 1$ to the profile $R^*$ to obtain our final profile $R^1$. Because $a$ and $b$ have the same score in $R^*$ and $s_i < s_{i+1}$, $f(R^1) = \{b\}$. However, if voter $i^*$ places alternative $a$ at rank $i + 1$ and $b$ at rank $i$, then the unique winner is $a$. This is a $\tilde{P}$-manipulation for voter $i^*$ as he prefers $a$ to $b$. Thus, every non-monotonic scoring rules is $\tilde{P}$-manipulable.

Secondly, assume that $f$ is a monotonic but not strictly monotonic scoring rule. Thus, $f$ is defined by a scoring vector $s$ such that $s_i \geq s_{i+1}$ for all $i \in \{1, ..., m-1\}$ and there is an index $j \in \{1, ..., m-1\}$ with $s_j = s_{j+1}$. Furthermore, we can deduce from the non-triviality of $f$ that there is an index $j^*$ such that $s_{j^*-1} > s_{j^*} = s_{j^*+1}$ or $s_{j^*} = s_{j^*+1} > s_{j^*+2}$. In this case, we add a new voter $i^*$ with $r(R_{i^*}, a) = j^*$ and $r(R_{i^*}, b) = j^* + 1$ to the profile $R^*$ to obtain the new profile $R^2$. As $s_j^* = s_{j^*+1}$, it follows that $f(R^2) = \{a, b\}$. If $s_{j^*-1} > s_{j^*}$, voter $i^*$ can $\tilde{P}$-manipulate by placing $a$ at rank $j^* - 1$. If $s_{j^*+1} > s_{j^*+2}$, he can $\tilde{P}$-manipulate by placing $b$ at rank $i + 2$. In both cases, alternative $a$ wins uniquely as it receives more points than $b$ and voter $i^*$ cannot make any other alternative win by construction. As $\{a\}\ \tilde{P}_{i^*}\ \{a, b\}$, this is indeed a $\tilde{P}$-manipulation for voter $i^*$. Thus, every non-trivial and monotonic but not strictly monotonic social choice function is $\tilde{P}$-manipulable.

Finally, assume that $f$ is strictly monotonic, i.e., $s_1 > s_2 > ... > s_m$. In this case, we use the preference profile $R^3 = R^*$ to show that a voter can $\tilde{P}$-manipulate. It follows from the construction of $R^*$ that half of the voters prefer $a$ the most and half of the voters prefer $b$ the most as $s_1$ is maximal. Furthermore, as $m \geq 3$ and all permutations are used, there is a voter $i^*$ with $r(R^3_{i^*}, a) = 1$ and $r(R^3_{i^*}, b) = 2$. This voter can $\tilde{P}$-manipulate by weakening $b$. If he switches to a preference $R'_{i^*}$ with $r(R'_{i^*}, b) = 3$, the score of $b$ decreases as $f$ is strictly monotonic. This leads to a preference profile $R'$ in which $a$ receives strictly more points than $b$. Furthermore, the alternative which has been swapped with $b$ by voter $i^*$ cannot win as it receives still a smaller score than $a$ in the profile $R^a$ and the same score as $a$ in the profile $R^b$. Thus, it holds that $f(R') = \{a\}$. Therefore, it is a $\tilde{P}$-manipulation for voter $i^*$ to switch from $R^3$ to $R'$ as he prefers $a$ strictly to $b$. Thus, also strictly monotonic scoring rules are $\tilde{P}$-manipulable and therefore, we can conclude the theorem. $\square$

$R^1$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $b$ | $c$ | $a$ | $c$ | $a$ |
| $a$ | $a$ | $b$ | $b$ | $b$ |
| $c$ | $b$ | $c$ | $a$ | $c$ |

$R^2$ :

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ | $c$ |
| $b$ | $c$ | $a$ | $c$ | $a$ |
| $c$ | $b$ | $c$ | $a$ | $b$ |

$R^3$ :

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ |
| $b$ | $c$ | $a$ | $c$ |
| $c$ | $b$ | $c$ | $b$ |

Figure 3.8: Preference profiles illustrating the proof of Theorem 3.9

First, we discuss Figure 3.8 which shows preference profiles illustrating the proof of the last theorem. Thus, consider the scoring rule $f_1$ defined by the scoring vector $s^1 = (1, 2, 0)$ and the profile $R^1$. This choice means that $s^1_2$ is the maximal entry in $s^1$ and that $f(R^1) = \{b\}$. Moreover, note that the first four voters in $R^1$ form the profile $R^* = R^{ab}$, i.e., $R^1_{-5} = R^*$. Finally, voter 5 can $\tilde{P}$-manipulate by placing $b$ first and $a$ second. As consequence of this change, $f_1$ chooses alternative $a$ as unique winner which is the most preferred alternative of voter 5. This shows how to $\tilde{P}$-manipulate a non-monotonic scoring rule. Next, consider the preference profile $R^2$ and the scoring rule $f_2$ defined by the scoring vector $s^2 = (2, 1, 1)$. As consequence of this choice, $f_2$ is a monotonic but not strictly monotonic scoring rule and $f_2(R^2) = \{a, b\}$. Furthermore, voter 5 can $\tilde{P}$-manipulate by exchanging the alternatives $a$ and $c$ in his preference. This modification leads to a new preference profile $R'$ with $f_2(R') = \{a\}$. This is a $\tilde{P}$-manipulation as $\{a\}\ \tilde{P}_5\ \{a, b\}$. Finally, we discuss the profile $R^3$ together with the scoring rule $f_3$ defined by the scoring vector $s^3 = (2, 1, 0)$. Note that $f_3$ is a strictly monotonic scoring rule as $s^3_1 > s^3_2 > s^3_1$ and that $f_3(R^3) = \{a, b\}$. Furthermore, voter 1 can $\tilde{P}$-manipulate $f_3$ by exchanging $b$ and $c$ in his preference. After this modification $f_3$ chooses $a$ as unique winner, which is a more preferable outcome for voter $i$. Thus, this example shows how to $\tilde{P}$-manipulate the various types of scoring rules.

Furthermore, note that the condition that $m \geq 3$ is indeed required for the validity of this theorem. If $m = 2$, the majority rule defined by the scoring vector $s = (1, 0)$ can be shown to satisfy $\tilde{P}$-strategyproofness. Furthermore, it should be mentioned that the preference profiles $R^1$ and $R^2$ constructed in the proof of Theorem 3.9 require often a huge number of voters. However, this number of voters is for our proof necessary to show that non-monotonic and monotonic but not strictly monotonic scoring rules are $\tilde{P}$-manipulable. Nevertheless, it might be possible to find other constructions that rely on less voters. In contrast, we can use smaller preference profiles to show that strictly monotonic scoring rules are $\tilde{P}$-manipulable. In particular, every profile in which both $a$ and $b$ are ranked first by half of the voters and second by the other half of the voters suffices for the proof.

Finally, observe that the proof of Theorem 3.9 shows important ideas that can be used for many social choice functions. In particular, its displays one of the most common situations in which rank-based social choice functions are $\tilde{P}$-manipulable: Two alternatives are chosen and a voter can break this tie by reinforcing his most preferred winner or weakening his least preferred winner. This idea can be used to prove the $\tilde{P}$-manipulability of many SCFs and we also see it in subsequent sections.

### 3.3.3 Dominance Rules

In this section, we introduce a new idea for defining rank-based social choice functions based on comparing the rank vectors of alternatives to determine the choice set. This idea leads to a class of rank-based social choice functions which we call dominance rules. This class contains many rank-based social choice functions and therefore, it seems interesting to analyze it in more detail.

The intuition of dominance rules is to use a transitive and anti-symmetric relation on the rank vectors to determine the choice set. In more detail, we choose the alternatives whose rank vectors are maximal with respect to the considered relation. For instance, we can define a dominance rule with the help of the dominance relation $D$ discussed in Definition 3.6. In contrast to independent rank-based social choice functions, we do not compare the rank vectors with a pre-defined threshold vector but with each other to determine the best alternatives. A formal definition of dominance rules is presented in the sequel.

**Definition 3.16 (Dominance rules).** *A social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ is a dominance rule if there is a transitive and anti-symmetric relation $\mathcal{D}$ on the rank vectors such that $f(R) = \{a \in A \mid \nexists b \in A \setminus \{a\} : b\mathcal{D}a\}$.*

Informally, this definition means that a dominance rule only chooses alternatives that are maximal with respect to the considered relation. We pick these alternatives as they are uncontroversially the best ones with respect to the chosen relation. Observe that this idea is inspired by a well-known axiom called transitive rationalizatiblity which states that a social choice function implies for a given preference profile a complete and transitive relation on the alternatives. We relax this axiom by dismissing the completeness of the relation and adapt it to rank-based social choice functions which results in the idea of dominance rules.

Note that we have already seen many dominance rules in the previous sections. For instance, scoring rules are dominance rules that compare rank vectors with the help of their scores. Furthermore, even threshold and multi-threshold rules are dominance rules as we can define a suitable relation. In particular, a vector $a$ is preferred to another vector $b$ with respect to this relation if $a$ dominates the threshold vector $v$ and $b$ does not dominate $v$. Thus, we see that this class contains many important social choice functions. However, formalizing scoring rules or threshold rules as dominance rules seems unnatural and therefore, we do not intend to discuss them during this section.

In contrast, we focus on dominance rules that arise by considering natural relations on vectors such as the dominance relation introduced in Definition 3.6. Another example of such a dominance rule is induced by the stochastic dominance relation $D^S$. A vector $v$ dominates another vector $u$ with respect to $D^S$ if for all $j \in \{1, ..., n\}$ it holds that $\sum_{i=1}^{j} v_i \leq \sum_{i=1}^{j} u_i$. This relation is important in many settings as it allows

to compare many vectors with each other. Furthermore, there are many more well-known relations on vectors that lead to dominance rules.
However, as we see in the sequel, most of these approaches fail to be $\tilde{P}$-strategyproof. The intuitive reason for this is that many relations are intended to compare as many vectors as possible. As consequence, the corresponding dominance rules choose rather small choice sets and therefore, it is easy to $\tilde{P}$-manipulate them. For formalizing this intuition, we focus on dominance extensions of $D$ which are defined as follows.

**Definition 3.17 (Dominance extension of $D$).** *Let $f_D$ denote the dominance rule defined by the dominance relation $D$. A social choice function $f$ is a dominance extension of $D$ if $f(R) \subseteq f_D(R)$ for all preference profiles $R$.*

Note that every dominance rule that is defined by a transitive and anti-symmetric relation $\tilde{D}$ with $D \subseteq \tilde{D}$ is a domination extension of $D$. This is also the intuition in the name of these social choice functions as they are defined by a relation $\tilde{D}$ that extends $D$. For instance, the stochastic dominance relation $D^S$ satisfies that $D \subseteq D^S$ and therefore, its corresponding dominance rule is a dominance extension of $D$. Further important instances of dominance extensions of $D$ are strictly monotonic scoring rules. Thus, we see that many interesting social choice functions are dominance extensions of $D$.
The reason why we introduce dominance extensions of $D$ is that they allow us to discuss many dominance rules by focusing on the dominance rule $f_D$ defined by the dominance relation $D$. As we prove next, this dominance rule turns out to be $\tilde{P}$-manipulable, which implies that all dominance extensions of $D$ are also $\tilde{P}$-manipulable as they choose similar choice sets for many profiles.

**Theorem 3.10.** *No dominance extension of $D$ is $\tilde{P}$-strategyproof if $m \geq 3$ and $n \geq 4$.*

Proof: Consider an arbitrary social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ that is a dominance extension of $D$. In the sequel, we construct a preference profile $R$ in which a voter can $\tilde{P}$-manipulate $f$. Furthermore, we focus on the case that $m = 3$ as we can assume that all other alternatives are not among the three most preferred alternatives of any voter in $R$. This implies that these alternatives are dominated with respect to $D$ and therefore, they are not chosen by any dominance extension of $D$.
We make a case distinction with respect to the number of voters $n$ to prove this theorem. First, assume that $n$ is even. In this case, consider the profile $R^1$ shown in Figure 3.9. In this profile, one half of the voters submits $a \succ b \succ c$ and the other half of the voters submits $b \succ a \succ c$. It follows that $r^*(R^1, a) = r^*(R^1, b)$ and that the rank vector of $c$ is dominated by the rank vector of both $a$ and $b$. This implies that $f(R^1) \subseteq \{a, b\}$. In the sequel, we assume that $a \in f(R^1)$ because the case $b \in f(R^1)$ is symmetric. If $a$ is chosen, then the voters with $b \succ a \succ c$ can improve by switching the position of $a$ and $c$. As result of this manipulation, the rank vector

$R^1$ :

| $n/2$ | $n/2$ |
|---|---|
| $a$ | $b$ |
| $b$ | $a$ |
| $c$ | $c$ |

$R^2$:

| $(n-3)/2$ | $(n-3)/2$ | 1 | 1 | 1 |
|---|---|---|---|---|
| a | b | c | b | a |
| b | a | b | a | c |
| c | c | a | c | b |

Figure 3.9: Preference profiles used in the proof of Theorem 3.10

of $b$ dominates both the rank vector of $a$ and $c$ with respect to $D$. Therefore, $b$ is the unique winner after this modification. This is a $\tilde{P}$-manipulation as the manipulator prefers $b$ strictly the most. Thus, no dominance extension of $D$ is $\tilde{P}$-strategyproof if $m \geq 3$, $n \geq 2$ and $n$ is even.

Next, we focus on the case that $n$ is odd. In this case, consider the profile $R^2$ shown in Figure 3.9. This profile consists of $(n-3)/2$ voters who submit $a \succ b \succ c$, another $(n-3)/2$ voters who submit $b \succ a \succ c$ and three voters who submit a Condorcet cycle. Observe that $r^*(R^1, a) = r^*(R^1, b)$ and that the rank vector of $c$ is dominated by the rank vector of both $a$ and $b$. Thus, $f(R^2) \subseteq \{a, b\}$. We assume again without loss of generality that $a \in f(R^2)$ as the case $b \in f(R^2)$ is symmetric. In this situation, every voter $i$ with $b \succ a \succ c$ can $\tilde{P}$-manipulate by exchanging $a$ and $c$ in his preference. As consequence, the rank vector of $b$ dominates all other rank vectors with respect to $D$ and therefore, $b$ is the unique winner. As $b$ is voter $i$'s most preferred alternative, this is a $\tilde{P}$-manipulation. Therefore, we can conclude that every dominance extension of $D$ is $\tilde{P}$-manipulable if $m \geq 3$, $n \geq 5$ and $n$ is odd. □

Observe that the proof does not work if $n = 3$ because the profile $R^2$ reduces to a Condorcet cycle in this case. This means that $c$ is not dominated with respect to $D$ anymore and might be chosen. Even more, the Condorcet cycle is the only profile in which the rank vectors of $a$ and $b$ are equal if $n = 3$ and $m = 3$. As this is a key idea of the proof, dominance extensions of $D$ may be $\tilde{P}$-strategyproof if $n = 3$. In contrast, the profile $R^1$ still suffices to show that every domination extension of $D$ is $\tilde{P}$-manipulable if $n = 2$ and $m \geq 3$. The reason for this that it is easier to ensure that $a$ and $b$ have the same rank vector if the number of voters is even. Furthermore, if $m < 3$, almost all dominance extensions of $D$ are equal to the majority rule which chooses the alternatives that are first-ranked by the most voters. As it is a well-known result that this social choice function is $\tilde{P}$-strategyproof, it follows that many dominance extensions of $D$ are $\tilde{P}$-strategyproof if $m \leq 2$. Thus, the axioms used for Theorem 3.10 are independent.

Note that the last theorem leads to the question whether there is any natural relation on vectors that leads to a $\tilde{P}$-strategyproof dominance rule. As it is clear that this relation must be rather weak, we try to find one that is contained in $D$. This approach leads to the weak dominance relation defined next.

**Definition 3.18 (Weak dominance relation).** *A vector $u = (u_1, ..., u_n)$ weakly dominates another vector $v = (v_1, ..., v_n)$, denoted by $u\ D^W\ v$, if $u_i < v_i$ for all $i \in \{1, ..., n\}$.*

The weak dominance relation demands that every component of the dominating vector is strictly smaller than the corresponding component of the dominated vector. In contrast, the dominance relation $D$ only requires that every component in the first vector is not larger than the corresponding alternative in the second vector. Thus, it follows directly that $D^W \subsetneq D$. On the one side, this means that $D^W$ leads to a rather indecisive dominance rule. On the other hand, we can prove that this dominance rule is indeed $\tilde{P}$-strategyproof.

**Theorem 3.11.** *The dominance rule $f_{D^W} : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ defined by the weak dominance relation $D^W$ is $\tilde{P}$-strategyproof.*

Proof: This theorem follows from the observation that $f_{D^W}$ is a coarsening of the OMNI-rule. This is true because an alternative $a$ is chosen by $f_{D^W}$ if it is first-ranked by at least one voter. If a voter prefers $a$ the most, then $r^*(R, a)_1 = 1$ and therefore, there is no alternative $b$ with $r^*(R, b)_1 < r^*(R, a)_1$. Furthermore, if $\text{OMNI}(R) = \{a\}$ for an alternative $a \in A$, then every voter prefers $a$ the most. This means that $f_{D^W}(R) = \{a\}$ as $r^*(R, a) = (1, ..., 1)$ and the rank vector of every other alternative is in $\{2, ..., m\}^n$. Thus, we can use a remark of [Bra15] to derive the $\tilde{P}$-strategyproofness of $f_{D^W}$. This remark states that a coarsening $g$ of a $\tilde{P}$-strategyproof SCF $f$ is also $\tilde{P}$-strategyproof if $f(R) = g(R)$ for all profiles $R$ with $|f(R)| = 1$. If we choose the OMNI-rule as $f$ and $f_{D^W}$ as $g$, all requirements of this result are met and therefore, $f_{D^W}$ is $\tilde{P}$-strategyproof. □

Note that this observation is interesting for various reasons. First of all, it shows that there are $\tilde{P}$-strategyproof dominance rules that are not independent rank-based SCFs. Furthermore, $f_{D^w}$ is another rank-based and $\tilde{P}$-strategyproof social choice function that is non-imposing. Even more, it follows from results in [Las96] that the weak dominance rule $f_{D^W}$ is Pareto-optimal. Therefore, $f_{D^W}$ is the first social choice function discussed in this chapter that is rank-based, $\tilde{P}$-strategyproof and Pareto-optimal and that does not refine the OMNI-rule. Thus, $f_{D^W}$ might be the coarsest $\tilde{P}$-strategyproof and rank-based social choice function that is Pareto-optimal.

## 3.4 Rank-based Social Choice Functions in the Weak Domain

Rank-based social choice functions have hitherto not been considered in the weak domain. However, ties are often necessary for properly modelling preferences. Even more, weak preferences often allow for stronger results. Therefore, we discuss in this section ideas on how to generalize the rank to the weak domain. However, it is not straightforward to extend the definition of the rank of an alternative from the strict domain to the weak one as it is not clear how to handle ties. Therefore, we propose different rank extensions, i.e., generalizations of the rank to the weak domain, in Section 3.4.1 and discuss their properties. Next, we try to find rank-based social choice functions in the weak domain that are both Pareto-optimal and $\tilde{P}$-strategyproof in Section 3.4.2. Unfortunately, it turns out that there is no such social choice function if there are at least four alternatives and three voters.

### 3.4.1 Ranks of Alternatives in the Weak Domain

In this section, we discuss various ideas on how to extend the concept of ranks and rank-based social choice functions to the weak domain. As there are multiple feasible ideas for approaching this problem, we propose various methods for generalizing the rank to the weak domain. We refer to those ideas as rank extensions to clearly distinguish them from the rank defined for the strict domain.
Before we discuss ideas for implementing rank extensions, we present a short list of criteria that a rank extension should satisfy. We do not claim that this list is complete or formally correct as its main goal is to provide an intuition about rank extensions. Firstly, a rank extension should satisfy that if a voter is indifferent between two alternatives, they obtain the same rank. Secondly, if $a \succ_i b$ in the preference of voter $i$, then $a$ should have a smaller rank than $b$. Lastly, a rank extension should coincide with the rank in the strict domain. As these conditions are rather weak, there are many possible rank extensions. Therefore, we propose four different ideas that satisfy these conditions in the sequel.

**Definition 3.19 (Rank extensions).** *Given an arbitrary preference $R_i \in \mathcal{W}$ and an arbitrary alternative $a \in A$, we define the following rank extensions:*

- $r_+(R_i, a) = 1 + |\{x \in A \mid x \succ_i a\}|$.
- $r_r(R_i, a) = 1 + |\{X \subseteq A \mid \forall x \in X : x \succ_i a \wedge \forall x, y \in X : x \sim_i y \wedge \forall x \in X, y \in A \setminus X : x \succ_i y \vee y \succ_i x\}|$.
- $r_-(R_i, a) = 1 + |\{x \in A \mid x \succeq_i a\}|$.
- $r_=(R_i, a) = 1 + |\{x \in A \mid x \succ_i a\}| + \frac{1}{2}|\{x \in A \setminus \{a\} \mid x \sim_i a\}|$.

Based on a rank extension, we can define the rank vector, the rank matrix and rank-based social choice functions as presented in Definition 3.2, Definition 3.3 and Definition 3.4 where we use the rank extension instead of the rank. The used rank extension is always indicated in the index, e.g., $r^*_+(R)$ denotes the rank matrix of $R$ where $r_+$ is used as rank extension. Furthermore, instead of stating a social choice function is rank-based, we write $r_+$-based, $r_-$-based, etc. to indicate which rank extension is used.

Next, we discuss the intuition behind the different rank extensions. The idea of $r_+$ is to treat every alternative $a$ as the best alternative within its indifference class $\{x \in A \mid x \sim_i a\}$. Thus, it is irrelevant for the rank of an alternative $a$ whether it is strictly or weakly preferred to an alternative $b$. A benefit of this rank extension is that the best alternatives of every voter have rank 1. This property allows for a straightforward extension of threshold rules to the weak domain as it ensures that these rules are feasible. For instance, we can define an extension of the OMNI-rule to the weak domain with the rank vector $v = (1, m, ..., m)$ and $r_+$ as rank extension. This social choice function returns all alternatives that are among the most preferred ones of at least one voter. Note that many theorems proven in the strict domain also hold in the weak domain if we use $r_+$ as rank extension. For instance, the characterization of OMNI carries over to the weak domain if we use $r_+$. One disadvantage of this rank extension is that it does not always react to the weakening of an alternative. For instance, consider the case where $A = \{a, b\}$. Then, the rank of alternative $a$ is 1 if a voter submits $a \succ b$ and if he submits $a \sim b$ even though the position of $a$ in the second preference seems weaker. Finally, it should be mentioned that this rank extension is often used in practice. For instance, $r_+$ is often used in tables of sport competitions if two teams are equally good.

The second rank extension $r_r$ arises naturally if we consider the preference of a voter as a linear list of indifference classes, i.e., sets $X \subseteq A$ such that $x \sim y$ for all alternatives $x, y \in X$ and $x \succ y$ or $y \succ x$ for all $x \in X$, $y \in A \setminus X$. The idea of this definition is to assign the rank directly to the indifference classes instead of the alternatives. If $X$ denotes the $j$-th indifference class in the preference of voter $i$, the set $X$ is assigned rank $j$. This means that every alternative in $X$ also has rank $j$. More informally, the rank of an alternative with respect to $r_r$ denotes the row in which it is written if we represent a preference profile as table. Note that this rank extension is rather similar to $r_+$ and it holds for all preferences $R_i \in \mathcal{W}$ that $r_r(R_i, a) \leq r_+(R_i, a)$. The difference between $r_r$ and $r_+$ is that the former one avoids large gaps in the rank, e.g., if $R_i = a_1 \sim a_2 \sim ... \sim a_k \succ b$, then $r_+(R_i, b) = k+1$, whereas $r_r(R_i, b) = 2$. It clearly depends on the application of the rank extension whether this behavior is advantageous. Furthermore, observe that the rank extension $r_r$ suffers from the same drawback as $r_+$: If we switch from $a \succ b$ to $a \sim b$, the rank of $a$ does not change. Finally, it should be mentioned that this rank extension is also often used in practice. For instance, if we sort items, such as books or clothes, according to our preference, we usually refer to the different stacks as most preferred items, second most preferred items, etc. This is exactly the idea of $r_r$.

The intuition behind the third rank extension $r_-$ is similar to $r_+$. However, instead of treating every alternative as the best alternative within its indifference class, we treat it as the worst one. Thus, the rank of an alternative $a$ with respect to $r_-$ only depends on the number of alternatives that are strictly less preferred than $a$

$R^1$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a$ | $b, c$ |
| $b$ | $b$ | $d$ |
| $c$ | $c$ | $a$ |
| $d$ | $d$ | |

$R^2$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a, b$ | $c$ |
| $b$ | $c$ | $b$ |
| $c$ | $d$ | $d$ |
| $d$ | | $a$ |

$R^3$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a$ | $b, c$ |
| $b$ | $b, c$ | $d$ |
| $c$ | $d$ | $a$ |
| $d$ | | |

$R^4$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a, b$ | $c$ |
| $b$ | $c$ | $b, d$ |
| $c$ | $d$ | $a$ |
| $d$ | | |

$R^5$ :

| 1 | 1 | 1 |
|---|---|---|
| $a, b, c, d$ | $a$ | $a$ |
| | $b$ | $c$ |
| | $c, d$ | $b, d$ |

$R^6$ :

| 1 | 1 | 1 |
|---|---|---|
| $b, c$ | $a$ | $a$ |
| $a, d$ | $b, c, d$ | $b, c, d$ |

$R^7$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $b$ |
| $b, c, d$ | $a$ | $d$ |
| | $d$ | $a$ |
| | $c$ | $c$ |

$R^8$ :

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $b$ | $a$ |
| $a, c, d$ | $a$ | $d$ |
| | $d$ | $b$ |
| | $c$ | $c$ |

Figure 3.10: Preference profiles showing the independence of all rank extensions

and it is irrelevant whether a voter submits $a \sim b$ or $b \succ a$. Note that this rank extension cannot be used to define reasonable threshold rules as every alternative has rank $m$ if a voter is indifferent between all alternatives. This is also reflected in the following observation: If a voter switches from $a \succ b$ to $a \sim b$, the rank of $b$ does not increase. This clearly contradicts the intuition of a rank. Nevertheless, this rank is sometimes used for generalizing scoring rules to the weak domain and it is useful if the quality of an alternative $a$ only depends on the number of alternatives that are strictly worse than $a$.

The last rank extension $r_=$ can be seen as a mixture of $r_-$ and $r_+$. It associates an alternative with the average rank that it obtains if we transform a preference in the weak domain to a preference in the strict domain by breaking ties uniformly at random. The main advantage of this approach is that the rank of $a$ decreases and the rank of $b$ increases if we switch from $a \succ b$ to $a \sim b$. Thus, it seems that this definition results in the most reasonable rank extension as it formalizes the intuition of the rank of an alternative very well. Unfortunately, this definition leads to a rather weak variant of rank-basedness as only few preference profiles have the same rank matrix with respect to $r_=$. Another problem is that we cannot derive the number of alternatives that are preferred to $a$ from $r_=(R_i, a)$. For instance, the rank of $c$ with respect to $r_=$ is 3 in both $R_1 = a \succ b \succ c \succ d \succ e$ and $R_2 = a \sim b \sim c \sim d \sim e$. Nevertheless, it follows from a result in [Bra17] that this rank extension can be used to define Borda's rule in the weak domain.

Next, we illustrate the different rank extensions with an example which also shows that the corresponding definitions of rank-basedness are all independent from each other. Therefore, consider the preference profiles $R^1$ to $R^8$ shown in Figure 3.10. First, observe that $r^*_+(R^1) = r^*_+(R^2)$ because the only difference in these profiles is the rank of $b$ in the preferences of voter 2 and 3. Voter 2 places $b$ second in $R^1$ and first in $R^2$ and voter 3 moves $b$ vice versa. Thus, $r^*_+(R^1, b) = r^*_+(R^2, b)$ and the rank vectors of the remaining alternatives with respect to $r_+$ do not change either. In contrast, $r^*_-(R^1, a) \neq r^*_-(R^2, a)$ and $r^*_=(R^1, a) \neq r^*_=(R^2, a)$ as the rank of $a$ gets larger with respect to $r_-$ and $r_=$ when voter 2 switches from $a \succ b$ to $a \sim b$. Furthermore, $r^*_r(R^1, a) \neq r^*_r(R^2, a)$ as $a$ appears in $R^1$ three times in the first row and once in the third, whereas it is placed in the fourth row in $R^2_3$. Thus, we see that $R^1$ and $R^2$ only have the same rank matrix with respect to $r_+$.

Moreover, note that $r^*_r(R^3) = r^*_r(R^4)$ as every alternative appears in the same rows of $R^3$ and $R^4$. The only difference between $R^3$ and $R^4$ is the position of $b$ in the preferences of voter 2 and 3. For going from $R^3$ to $R^4$, voter 2 reinforces $b$ from rank 2 to rank 1 and voter 3 equalizes this swap. Thus, $r^*_r(R^3, b) = r^*_r(R^4, b)$ and it is easy to see that this also holds for the other alternatives. In contrast, $r^*_-(R^3, a) \neq r^*_-(R^4, a)$ and $r^*_=(R^1, a) \neq r^*_=(R^2, a)$ because voter 2 switches from $a \succ b$ to $a \sim b$, which implies that the rank of $a$ increases with respect to $r_-$ and $r_=$. Finally, $r^*_+(R^3, d) \neq r^*_+(R^4, d)$ because the rank of $d$ with respect to $r_+$ decreases from 3 to 2 in the preference of voter 3 and no other change involves $d$. Thus, $r^*_r(R^3) = r^*_r(R^4)$ and no other rank extension allows this equality.

Furthermore, it holds that $r^*_-(R^5) = r^*_-(R^6)$ and that this equivalence is not true for any other rank extension. The equality follows as $r_-(R^5_1, x) = 4$ for all $x \in A$ as voter 1 is indifferent between all alternatives. In contrast, $r_-(R^6_1, b) = r_-(R^6_1, c) = 2$ and the other two alternatives still have rank 4 in $R^6_1$. This change is possible due to voter 2 and 3 who ensure that the rank vector of no alternative changes. More precisely, it holds that $r_-(R^5_2, b) = r_-(R^5_3, c) = 2$ and $r_-(R^6_2, b) = r_-(R^6_3, c) = 4$ as $b$ and $c$ are not strictly preferred to any other alternative in $R^6_2$ and $R^6_3$. Thus, $r^*_-(R^5, x) = r^*_-(R^6, x)$ for $x \in \{b, c\}$ and since $r_-(R^5_i, y) = r_-(R^6_i, y)$ for all voters $i \in N$ and alternatives $y \in \{a, d\}$, it holds that $r^*_-(R^5) = r^*_-(R^6)$. Moreover, observe that $r^*_+(R^5, d) \neq r^*_+(R^6, d)$ and $r^*_r(R^5, d) \neq r^*_r(R^6, d)$ because $d$ is only first or third ranked in $R^5$ and second ranked in $R^6_2$ with respect to these rank extensions. Finally, $r^*_=(R^5, b) \neq r^*_=(R^6, b)$ as $b$ receives a rank of 1.5 in $R^6_1$ and $r(R^5_i, b) \geq 2$ for all voters $i \in N$. Thus, $R^5$ and $R^6$ are only related by $r_-$-basedness.

Finally, we show that $r^*_=(R^7) = r^*_=(R^8)$ and that this equality does hold for any other rank extension. First note that the only difference between $R^7$ and $R^8$ are the preferences of voter 1 and 3: In $R^7$ voter 1 prefers $a$ the most and is indifferent between all other alternatives, which means that $r_=(R^7_1, a) = 1$ and $r_=(R^7_1, x) = 3$ for all $x \in A \setminus \{a\}$. In $R^8$ voter 1 exchanges the roles of $a$ and $b$ and therefore, it holds that $r_=(R^8_1, b) = 1$ and $r_=(R^8_1, x) = 3$ for all $x \in A \setminus \{b\}$. Thus, only the rank vector of $a$ and $b$ are modified, which is equalized by voter 4 who ranks $a$ first and $b$ third in $R^7$ and vice versa in $R^8$. As consequence, $r^*_=(R^7) = r^*_=(R^8)$. In contrast, $r^*_+(R^7, a) \neq r^*_+(R^8, a)$ and $r^*_r(R^7, a) \neq r^*_r(R^8, a)$ as voter 2 weakens $a$ from rank 1 to rank 2 while voter 4 strengthens $a$ from rank 3 to rank 1 with respect to these rank extensions. Furthermore, $r^*_-(R^7, a) \neq r^*_-(R^8, a)$ because voter 2 weakens $a$ from rank 1 to rank 4 and voter 4 strengthens $a$ from rank 3 to rank 1 with respect to $r_-$. Thus, we can finally derive that all rank extensions lead to independent definitions of rank-basedness in the weak domain.

As already discussed, every rank extension has its advantageous. However, a consequence of the independence of the different variants of rank-basedness is that we need for every rank extension a separate proof for a statement. This contradicts the intuition that it should not matter which rank extension is used for a theorem. Therefore, we develop another rather technical rank extension that refines the ones introduced in Definition 3.19. The main idea for this extension is that we do not require that the rank is a number for the definition of rank-based social choice functions. Instead, we use a tuple stating how many alternatives are strictly preferred to the considered alternative and how many alternatives are equally good. We refer to this idea as rank tuple and formalize it next.

**Definition 3.20 (Rank tuple of an alternative).** *The rank tuple $r_2$ of an alternative $a$ in the preference $R_i \in \mathcal{W}$ is defined as*

$$r_2(R_i, a) = (|\{x \in A \mid x \succ_i a\}|, |\{x \in A \setminus \{a\} \mid x \sim_i a\}|) \quad .$$

Observe that we still use the standard definition of the rank vector $r_2^*(R, a)$ and the rank matrix $r_2^*(R)$ in which we sort the rank tuples in lexicographical ascending order. As consequence, the only difference between the rank extensions presented in Definition 3.19 and the rank tuple is that the elements of the rank vector $r_2^*(R, x)$ and the rank matrix $r_2^*(R)$ are tuples instead of numbers.
On the one hand, this definition does not agree with the intuitive understanding of the rank anymore as it should be a number. However, we do not require this intuition for the definition of rank-based social choice functions. These functions only use the rank matrix for deciding the outcome of an election. However, it is completely unclear how they choose the winners based on this information. From this perspective, the rank tuple is more desirable than the previously discussed rank extensions as it provides more information. We formalize this observation in the following lemma by showing that we can recompute the rank extensions $r_+$, $r_-$ and $r_=$ from the rank tuple.

**Lemma 3.6.** *It holds for all preference profiles $R, R' \in \mathcal{W}^n$ with $r_2^*(R) = r_2^*(R')$ that $r_-^*(R) = r_-^*(R')$, $r_+^*(R) = r_+^*(R')$ and $r_=^*(R) = r_=^*(R')$.*

Proof: This lemma follows from the observation that the rank extensions $r_-$, $r_+$ and $r_=$ only depend on the rank tuple $r_2$. Therefore, it is possible to define a mapping from the rank tuple to these rank extensions. Explicitly, these mappings are defined for all alternatives $a \in A$ and preferences $R_i \in \mathcal{W}$ as follows:

- We obtain the rank extension $r_-(R_i, a)$ from the rank tuple $r_2(R_i, a) = (x_1, x_2)$ by applying $f_-(x_1, x_2) = 1 + x_1 + x_2$.
- We obtain the rank extension $r_+(R_i, a)$ from the rank tuple $r_2(R_i, a) = (x_1, x_2)$ by applying $f_+(x_1, x_2) = 1 + x_1$.
- We obtain the rank extension $r_=(R_i, a)$ from the rank tuple $r_2(R_i, a) = (x_1, x_2)$ by applying $f_=(x_1, x_2) = 1 + x_1 + 0.5x_2$.

$R^1$ :

| 1 | 1 |
|---|---|
| $a, b$ | $a$ |
| $d$ | $b$ |
| $c$ | $c$ |
| | $d$ |

$R^2$ :

| 1 | 1 |
|---|---|
| $a$ | $a, b$ |
| $b$ | $c$ |
| $d$ | $d$ |
| $c$ | |

Figure 3.11: Preference profiles showing the independence of $r_2$-basedness and $r_r$-basedness

As consequence of these mappings, we can derive that $r_2(R_i, a) = r_2(R'_i, a)$ implies that $r_-(R_i, a) = r_-(R'_i, a)$, $r_+(R_i, a) = r_+(R'_i, a)$ and $r_=(R_i, a) = r_=(R'_i, a)$ for all alternatives $a \in A$ and preferences $R_i \in \mathcal{W}$. The reason for this is that we can simply apply the respective mapping to both sides of the equation. Consequently, it follows that $r_2^*(R) = r_2^*(R')$ implies $r_-^*(R) = r_-^*(R')$, $r_+^*(R) = r_+^*(R')$ and $r_=^*(R) = r_=^*(R')$ for all preference profiles $R, R' \in \mathcal{W}^n$. □

Observe that this lemma is not true for the rank extension $r_r$. Consider the preference profiles $R^1$ and $R^2$ shown in Figure 3.11 for a counter example. It holds that $r_2^*(R^1) = r_2^*(R^2)$ as these profiles only differ in which voter prefers $a$ strictly to $b$ and which voter is indifferent between those alternatives. However, $r_r^*(R^1, c) \neq r_r^*(R^2, c)$ as $c$ appears in $R^1$ twice in the third row implying that $r_r^*(R^1, c) = (3, 3)$, whereas $r_r(R_1^2, c) = 4$. Thus, it follows that $r_r^*(R^1) \neq r_r^*(R^2)$.
The last two observations lead to the question on the difference between the rank extension $r_r$ and the other rank extensions. The answer to this is that $r_+(R_i, a)$, $r_-(R_i, a)$ and $r_=(R_i, a)$ only depend on the number of alternatives that are strictly preferred to $a$ in $R_i$ and the number of alternatives that are indifferent to $a$ in $R_i$. This dependency can be formalized by stating that these rank extensions only depend on the rank tuple. In contrast, $r_r(R_i, a)$ additionally depends on the preferences between alternatives that are strictly preferred to $a$. This is also the main problem in the example shown in Figure 3.11. However, in many cases this dependency is not desirable and therefore, we want to distinguish these two types of rank extension. Thus, we introduce the class on $r_2$-dependent rank extensions.

**Definition 3.21 ($r_2$-dependent rank extensions).** *A rank extension $r_x$ is $r_2$-dependent if there is a function $f$ such that $f(r_2(R_i, a)) = r_x(R, a)$ for all preferences $R_i \in \mathcal{W}$ and alternatives $a \in A$.*

It follows from Lemma 3.6 that $r_+$, $r_-$ and $r_=$ are $r_2$-dependent rank extensions. Another consequence of this definition is that we can discuss all $r_2$-dependent social choice functions in one theorem if we work with $r_2$-basedness in the proof. The reason for this is that if $r_2^*(R) = r_2^*(R')$ for two profiles $R, R' \in \mathcal{W}^n$, then $r_x^*(R) = r_x^*(R')$ for every $r_2$-dependent rank extension $r_x$ as we simply can apply the mapping from $r_2$ to $r_x$ as explained in the proof of Lemma 3.6. Thus, we have found a convenient way for working with rank extensions. However, note that $r_r$ is no $r_2$-dependent rank extensions. Thus, every result for $r_r$ requires a separate proof.

Finally, note that we cannot think of any other reasonable rank extension but $r_r$ that is not $r_2$-dependent. This means that we cover all reasonable rank extensions if we discuss $r_2$-dependent rank extensions and $r_r$.

### 3.4.2 Rank-basedness and $\tilde{P}$-strategyproofness in the Weak Domain

After discussing various rank extensions, we focus on properties of rank-based social choice functions in the weak domain. In particular, we are interested in $\tilde{P}$-strategyproof social choice functions that are rank-based and that try to choose small choice sets. We formalize the latter condition by using Pareto-optimality. Since there are not that many rank-based social choice functions that are both Pareto-optimal and $\tilde{P}$-strategyproof in the strict domain, it seems reasonable to conjecture that there are no such functions in the weak domain. We prove as main result of this section that this conjecture is true for all $r_x$-based social choice functions where $r_x$ denotes either $r_r$ or an arbitrary $r_2$-dependent rank extension.
However, note that this impossibility works only if there are at least 4 alternatives. Therefore, we discuss first the case that $m \leq 3$. In this situation, we can show that the Pareto-rule, i.e., the SCF that chooses all Pareto-optimal alternatives, is $r_2$-based. Thus, there is a $r_2$-based social choice function that is both Pareto-optimal and $\tilde{P}$-strategyproof if $m \leq 3$.

**Lemma 3.7.** *The Pareto-rule is $r_2$-based, Pareto-optimal and $\tilde{P}$-strategyproof if $m \leq 3$.*

Proof: First note that the Pareto-rule is known to be Pareto-optimal and $\tilde{P}$-strategyproof in the weak domain [BSS]. Thus, we only need to prove that it is possible to decide whether an alternative is Pareto-dominated in a profile $R$ based on $r_2^*(R)$. For showing this, we construct a preference profile for a given $r_2$-rank matrix $Q$. Note that we claim in the sequel often that some preferences are determined uniquely by $Q$. This means only that there is a voter with the discussed preference, but it is not clear which voter submits the preference. This is no problem as the Pareto-rule is anonymous.
Next, we discuss the entries of $Q$ in detail and distinguish the three possible values for $m$. First, it should be mentioned that $m = 1$ is trivial as the single alternative is always the unique winner. Even more, $Q$ uniquely specifies all preferences if $m = 2$ as $Q$ consists only of three different entries: $(0, 0)$ indicating that the corresponding alternative is uniquely first-ranked, $(1, 0)$ denoting that the corresponding alternative is uniquely second ranked, and $(0, 1)$ which means that a voter is indifferent between both available alternatives. Furthermore, it is straightforward that if an alternative has a $(0, 0)$-entry, then the other alternative has a $(1, 0)$-entry. Even more, if an alternative has a $(0, 1)$-entry, the other alternative has a $(0, 1)$-entry, too. These correspondences determine the preferences uniquely. Therefore, we can

recompute the original preference profile from $Q$ up to reordering the voters and calculate the Pareto-rule on this profile.
Finally, consider the case $m = 3$. We also discuss in this case the possible entries of the rank matrix $Q$. First note that if an alternative has the entry $(0, 2)$, a voter is indifferent between all three alternatives, which means that all other alternatives have such an entry. Thus, these entries determine the preference of a voter uniquely and therefore, we can dismiss them. Next, we consider the entries $(x, 1)$ for $x \in \{0, 1\}$. We focus without loss of generality on the case that $x = 0$ as $x = 1$ is symmetric. Our goal is to show these entries determine the preferences of some voters uniquely. Therefore, let $n_i$ denote the number of $(0, 1)$-entries in $Q$ corresponding to the $i$-th alternative, $i \in \{1, 2, 3\}$. Intuitively, $n_i$ denotes the number of voters that are indifferent between the $i$-th alternative and an arbitrary other alternative and prefer both of these alternatives strictly to the last one. This means that every voter contributes either to no $n_i$, $i \in \{1, 2, 3\}$, or he increments $n_i$ and $n_j$ by 1, $i, j \in \{1, 2, 3\}$ and $i \neq j$. We use this observation to construct a system of linear equations where we use the variables $v^{(1,2)}$, $v^{(1,3)}$ and $v^{(2,3)}$. These variables denote how many voters are indifferent between the alternatives denoted in the superscript, e.g., $v^{(1,2)}$ denotes how many voters are indifferent between the first and the second alternative and prefer both of them strictly to the third one. This leads to the following system of linear equations.

$$v^{(1,2)} + v^{(1,3)} = n_1 \qquad v^{(1,2)} + v^{(2,3)} = n_2 \qquad v^{(1,3)} + v^{(2,3)} = n_3$$

This equation system has a unique solution with $v^{(1,2)} = \frac{n_1+n_2-n_3}{2}$, $v^{(1,3)} = \frac{n_1+n_3-n_2}{2}$ and $v^{(2,3)} = \frac{n_2+n_3-n_1}{2}$. As this solution is unique, there is only one way to distribute ties involving two alternatives to the voters. Furthermore, if we know that a voter prefers two alternatives the most, it follows that the remaining alternative is his least preferred one. Thus, the $(x, 1)$-entries in $Q$, $x \in \{0, 1\}$, determine the preferences of a subset of voters uniquely. Therefore, we also ignore them from now on.
Consequently, we can focus on the $(0, 0)$-, $(1, 0)$- and $(2, 0)$-entries in $Q$. These entries correspond to strict preferences, which means that we can use results about the rank in the strict domain. Unfortunately, these entries do not determine the preferences uniquely. Instead, we prove that if $Q$ corresponds to a preference profile $R$ in which an alternative $a$ Pareto-dominates another alternative $b$, then $a$ Pareto-dominates $b$ in every preference profile $R'$ with $r_2^*(R') = Q$. As the voters who are indifferent between some alternatives are uniquely determined, we can focus on the voters with strict preferences. For these voters, we use a result proven in [Las96] stating that if $a$ Pareto-dominates $b$ in a preference profile $R \in \mathcal{S}^n$, then $r^*(R, a)_i < r^*(R, b)_i$ for all $i \in \{1, ..., n\}$. This means that if $a$ Pareto-dominates $b$, then every voter $i$ with $R_i \in \mathcal{S}$ gives $a$ a rank of 1 or 2 and $b$ a rank of 2 or 3. Thus, it follows immediately that $a$ Pareto-dominates $b$ regardless of how we place the alternatives in the strict part of $R$ as every rank of $a$ is as least as good as every rank of $b$ and ties are not

$$Q = \begin{pmatrix} (0,0) & (0,1) & (0,1) & (0,2) & (1,0) & (2,0) & (2,0) \\ (0,0) & (0,1) & (0,1) & (0,2) & (1,0) & (2,0) & (2,0) \\ (0,0) & (0,1) & (0,1) & (0,2) & (1,0) & (2,0) & (2,0) \end{pmatrix}$$

$R^1$ :

| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $a, b$ | $a, c$ | $b, c$ | $a, b, c$ |
| $b$ | $c$ | $a$ | $c$ | $b$ | $a$ | |
| $c$ | $a$ | $b$ | | | | |

Figure 3.12: $r_2$-majority matrix and preference profiles used for the explaining constructions in the proof of Lemma 3.7

allowed. Thus, it follows from these observations that we can indeed calculate the Pareto-rule based on an $r_2$-rank matrix if $m \leq 3$. □

For a better understanding, we illustrate the construction of the preference profile discussed in Lemma 3.7 for the case $m = 3$ with an example shown in Figure 3.12. In this figure an $r_2$-rank matrix $Q$ is shown and we construct the profile $R$ based on its entries. Thus, note that the $(0, 2)$-entries in $Q$ lead to the right most voter in $R$ who is indifferent between all alternatives. Furthermore, the $(0, 1)$-entries combined with some $(2, 0)$-entries determine the preferences of voters 4 to 6 uniquely. The remaining entries in $Q$ correspond to the strict preferences in $R$. Note that we can assign different preferences to the voters 1,2 and 3 to obtain these entries. For instance, it is also possible that $R_1 = a \succ c \succ b$, $R_2 = c \succ b \succ a$ and $R_3 = b \succ a \succ c$. Next, we focus on the situation that there are $m \geq 4$ alternatives. As we show in the sequel, no social choice function satisfies rank-basedness, Pareto-optimality and $\tilde{P}$-strategyproofness under this assumption. However, we want to focus in the proof on preference profiles with a small number of alternatives and voters. Therefore, we propose the following lemma which allows to generalize results for a fixed number of voters and alternatives to arbitrary larger values.

**Lemma 3.8.** *Consider a rank extension $r_x$ that is either $r_r$ or an $r_2$-dependent rank extension and assume that there is a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies $\tilde{P}$-strategyproofness, Pareto-optimality and $r_x$-basedness. It holds for all $n' \leq n$ and $A' \subseteq A$ that there is a social choice function $g : \mathcal{W}^{n'} \mapsto 2^{A'} \setminus \emptyset$ that satisfies all previously mentioned axioms.*

Proof: Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies all axioms stated in the lemma, an arbitrary number of voters $n' \leq n$ and a subset of alternatives $A' \subseteq A$. We prove the existence of a SCF $g : \mathcal{W}^{n'} \mapsto 2^{A'} \setminus \emptyset$ that satisfies all required axioms by providing induction steps with respect to $n$ and $m = |A|$. By repeatedly applying these induction steps, it follows that the statement holds indeed for all $n' \leq n$ and $m' = |A'| \leq |A| = m$.
We first construct a social choice function $g_1$ defined on $n$ voters and $m - 1$ alternatives. Given a preference profile $R$, the social choice function $g_1$ does the following:

For every voter $i \in N$, we add a new alternative $a$ as uniquely worst alternative to his preference. This leads to a preference profile $R'$ defined on $n$ voters and $m$ alternatives. Finally, we set $g_1(R) = f(R')$. Note that $a \notin f(R')$ as it is Pareto-dominated by every other alternative and therefore, $g_1$ is well-defined. Furthermore, it is obvious that $g_1$ inherits all required axioms from $f$. Thus, there is a $\tilde{P}$-strategyproof, $r_x$-based and Pareto-optimal social choice function that is defined on $n$ voters and $m-1$ alternatives.
Next, we discuss the induction step with respect to the voters. Hence, we construct a $r_x$-based, Pareto-optimal and $\tilde{P}$-strategyproof social choice function $g_2$ defined on $n-1$ voters and $m$ alternatives. Given a preference profile $R$, this social choice function adds a new voter to the profile that is indifferent between all alternatives. This leads to a new preference profile $R'$ defined on $n$ voters and $m$ alternatives. Finally, we set $g_2(R) = f(R')$. It follows that $g_2$ inherits the Pareto-optimality of $f$ as a voter who is indifferent between all alternatives does not change the set of Pareto-optimal alternatives. Furthermore, if $f$ is $r_x$-based, then $g_2$ is $r_x$-based, too. The reason for this that adding a voter that is indifferent between all alternatives adds the same entry to the rank vector of every alternative. Consequently, two profiles $R^1, R^2 \in \mathcal{W}^{n-1}$ with $r_x^*(R^1) = r_x(R^2)$ have also the same rank matrix after adding a completely indifferent voter. Thus, $g_2$ inherits also the $r_x$-basedness of $f$. Finally, it is obvious that $g_2$ inherits the $\tilde{P}$-strategyproofness of $f$ and therefore, $g_2$ satisfies all required axioms. Thus, we can inductively deduce this lemma. □

Observe that we use Lemma 3.8 in the inverse direction: We show that there is no social choice function $f$ that satisfies all axioms required by Lemma 3.8 for a small number of voters and alternatives and, as a consequence of this lemma, there is also no such social choice function if we increase the number of voters or alternatives. With the help of this observation, we prove that there is no rank-based social choice function that satisfies $\tilde{P}$-strategyproofness and Pareto-optimality in the weak domain if $m \geq 4$ and $n \geq 3$.

**Theorem 3.12.** *Consider an arbitrary rank extension $r_x$ that is either $r_2$-dependent or equal to $r_r$. There is no social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies Pareto-optimality, $\tilde{P}$-strategyproofness and $r_x$-basedness if $n \geq 3$ and $|A| = m \geq 4$.*

Proof: We prove the theorem for the case $n = 3$ and $m = 4$ as the remaining cases follow from Lemma 3.8. Thus, assume for contradiction that there is a rank extension $r_x$ that is either $r_2$-dependent or equal to $r_r$ and a SCF $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ satisfying Pareto-optimality, $\tilde{P}$-strategyproofness and $r_x$-basedness. It should be mentioned that we discuss in the sequel only $r_r$-basedness and $r_2$-basedness as $r_2^*(R) = r_2^*(R')$ implies $r_x^*(R) = r_x^*(R')$ for every $r_2$-dependent rank extension $r_x$. Thus, we can replace every application of $r_2$-basedness with an application with $r_x$-basedness, which means that it suffices to consider $r_2$ and $r_r$.
Before we go into the details of the proof, we first discuss a short outline of its key ideas because these ideas are also used in subsequent chapters. The first step

| | 1 | 1 | 1 |
|---|---|---|---|
| $R^1$ | $a \succ d \sim b \sim c$ | $a \sim b \succ c \sim d$ | $c \sim d \succ a \sim b$ |
| $R^2$ | $a \succ d \sim b \sim c$ | $a \sim c \succ b \sim d$ | $b \sim d \succ a \sim c$ |
| $R^3$ | $a \succ d \sim b \sim c$ | $a \sim d \succ b \sim c$ | $b \sim c \succ a \sim d$ |
| $R^4$ | $a \succ b \succ c \succ d$ | $a \sim d \succ b \sim c$ | $b \sim c \succ a \sim d$ |
| $R^5$ | $a \succ b \succ c \succ d$ | $a \succ b \succ c \succ d$ | $b \sim c \succ a \sim d$ |
| $R^6$ | $a \succ b \succ c \succ d$ | $a \succ b \succ c \succ d$ | $b \sim d \succ c \succ a$ |
| $R^7$ | $a \succ b \succ c \succ d$ | $a \sim c \succ b \succ d$ | $b \sim d \succ c \succ a$ |
| $R^8$ | $a \succ b \succ c \succ d$ | $a \sim b \succ c \succ d$ | $c \sim d \succ b \succ a$ |
| $R^9$ | $a \succ b \succ c \succ d$ | $a \succ b \succ c \succ d$ | $c \sim d \succ b \succ a$ |
| $R^{10}$ | $a \succ b \succ c \succ d$ | $a \sim b \succ c \succ d$ | $b \sim d \succ c \succ a$ |
| $R^{11}$ | $a \sim c \succ b \succ d$ | $a \sim b \succ c \succ d$ | $b \sim d \succ c \succ a$ |
| $R^{12}$ | $a \sim b \succ c \succ d$ | $a \sim b \succ c \succ d$ | $c \sim d \succ b \succ a$ |
| $R^{13}$ | $a \succ b \succ c \succ d$ | $a \sim b \succ c \succ d$ | $c \sim d \succ b \succ a$ |

Figure 3.13: Preference profiles used in the proof of Theorem 3.12

in the proof is to find a preference profile $R$ such that $f(R)$ does not contain any of the most preferred alternatives of a voter $i \in N$. After this, we use the combination of Pareto-optimality and $\tilde{P}$-strategyproofness to construct a new preference profile $R'$ which is even worse for voter $i$: The only winner in this profile is the least preferred alternative of voter $i$. This means that this voter can reorder his alternatives arbitrarily and the unique winner is still his worst alternative because of $\tilde{P}$-strategyproofness. Finally, we repeatedly use this observation combined with $r_x$-basedness to modify the preferences of the other voters without changing the choice set. In this step, another central idea is applied: We can use different paths to reach a preference profile, which means that the number of feasible choice sets decreases. In the sequel, we apply this technique to deduce the choice sets for the profiles $R^8$ and $R^{12}$ shown Figure 3.13. Finally, the contradiction is obtained by showing that there is no valid choice for a preference profile left.

These ideas are implemented with the preference profiles shown in Figure 3.13. As first step, we analyze the preference profiles $R^1$, $R^2$ and $R^3$. Thus, note that $r^*_x(R^1) = r^*_x(R^2) = r^*_x(R^3)$ for every rank extension allowed by this theorem and that $a$ Pareto-dominates $b$ in $R^1$, $c$ in $R^2$ and $d$ in $R^3$. Hence, $r_x$-basedness and Pareto-optimality imply that $f(R^1) = f(R^2) = f(R^3) = \{a\}$. This means that $f(R^3)$ does not contain any alternative that is among the most preferred ones of voter 3.

As next step, we consider the profiles $R^4$ and $R^5$. In these profiles, voter 1 and 2 manipulate one after another such that they prefer $a$ the most and $b$ second most. It follows from $\tilde{P}$-strategyproofness that $f(R^i) = \{a\}$ for $i \in \{4, 5\}$ as otherwise a voter can $\tilde{P}$-manipulate by switching from $R^i$ to $R^{i-1}$. Note that $b$ Pareto-dominates both $c$ and $d$ in $R^5$. This observation is also true for $R^6$ in which voter 3 ranks $a$ uniquely the worst. Thus, we can deduce that $f(R^6) \subseteq \{a, b\}$. Furthermore, if $b \in f(R^6)$, voter 3 can $\tilde{P}$-manipulate by switching from $R^5$ to $R^6$ as he prefers $b$ strictly to $a$. Hence, it follows from $\tilde{P}$-strategyproofness that $f(R^6) = \{a\}$ which means that the unique worst alternative of voter 3 is the only winner.

Subsequently, voter 2 manipulates such that he prefers $a$ and $c$ the most in the profile $R^7$. It follows from $\tilde{P}$-strategyproofness that $f(R^7) \subseteq \{a, c\}$; otherwise, voter 2 can $\tilde{P}$-manipulate by switching from $R^7$ to $R^6$. Thereafter, we apply $r_x$-basedness to derive $R^8$ from $R^7$: Voter 2 and 3 exchange their preferences over $b$ and $c$. It should be stressed that this modification does not change the rank matrix with respect to $r_r$ or $r_2$. Thus, it holds that $f(R^8) \subseteq \{a, c\}$. Next, observe that we can go from the profile $R^6$ also to the profile $R^9$ if voter 3 manipulates. It follows from the $\tilde{P}$-strategyproofness of $f$ that $f(R^9) = \{a\}$; otherwise, it is a $\tilde{P}$-manipulation for voter 3 to switch from $R^6$ to $R^9$. Furthermore, we derive the profile $R^8$ from $R^9$ by letting voter 2 push $b$ to the top. These observations imply that $f(R^8) \subseteq \{a, b\}$ as otherwise voter 2 can $\tilde{P}$-manipulate by switching back to $R^9$. Thus, we can deduce that $f(R^8) = \{a\}$.

In the profiles $R^{10}$ to $R^{13}$, we repeat the steps explained in the last paragraph with voter 1 and voter 3. First, voter 3 reorders his alternatives in the same way as in $R^6$, which leads to the profile $R^{10}$. It follows from the $\tilde{P}$-strategyproofness of $f$ that $f(R^{10}) = \{a\}$ as voter 3 can $\tilde{P}$-manipulate otherwise. Thereafter, voter 2 pushes $c$ to the top in $R^{11}$ implying that $f(R^{11}) \subseteq \{a, c\}$. As $r_x^*(R^{11}) = r_x^*(R^{12})$, it results from $r_x$-basedness that $f(R^{12}) = f(R^{11}) \subseteq \{a, c\}$. Finally, $R^{13}$ provides a second path from $R^9$ to $R^{12}$: We first let voter 3 reorder his alternatives to deduce profile $R^{13}$ and it follows from $\tilde{P}$-strategyproofness that $f(R^{13}) = \{a\}$. Subsequently, voter 1 pushes $b$ to the top to derive $R^{12}$, which implies that $f(R^{12}) \subseteq \{a, b\}$ because of $\tilde{P}$-strategyproofness. Thus, we can deduce that $f(R^{12}) = \{a\}$ is true. However, it is easy to see that $a$ is Pareto-dominated by $b$ in $R^{12}$ implying that $a \notin f(R^{12})$. This is a contradiction and therefore, no social choice function satisfying $\tilde{P}$-strategyproofness, Pareto-optimality and $r_x$-basedness exists if $m \geq 4$ and $n \geq 3$. □

As first remark, we discuss the independence of the axioms required for Theorem 3.12. Hence, note that there are various $r_2$-based social choice functions that are $\tilde{P}$-strategyproof but not Pareto-optimal, for instance the generalized OMNI-rule which contains all most preferred alternatives of every voter. Furthermore, the Pareto-rule is $\tilde{P}$-strategyproof and Pareto-optimal but not $r_2$-based if $m \geq 4$. Next, there are many Pareto-optimal and rank-based social choice functions that are not $\tilde{P}$-strategyproof, such as generalizations of dominance rules introduced in Definition 3.17 where we replace the rank with $r_+$. Even more, $m \geq 4$ is required as shown by Lemma 3.7. In contrast, we can show that $n \geq 3$ is not necessary. For proving this, consider the preference profiles $R^1$ and $R^2$ defined on $n = 2$ voters and $m = 5$ alternatives that are depicted in Figure 3.14 and an arbitrary rank extension $r_x$ that is either $r_2$-dependent or equal to $r_r$. Note that $r_x^*(R^1) = r_x^*(R^2)$, which implies that every $r_x$-based social choice function returns the same winner for both

$R^1$ :

| 1 | 1 |
|---|---|
| $b, d$ | $c, e$ |
| $a, e$ | $a$ |
| $c$ | $b$ |
| | $d$ |

$R^2$ :

| 1 | 1 |
|---|---|
| $c, d$ | $b, e$ |
| $a, e$ | $a$ |
| $b$ | $c$ |
| | $d$ |

Figure 3.14: Preference profiles used to show the independence of the bounds on $n$ and $m$ in Theorem 3.12

profiles. Furthermore, $e$ Pareto-dominates both $a$ and $c$, and $b$ Pareto-dominates $d$ in $R^1$. Similarly, $e$ Pareto-dominates $b$ in $R^2$, which implies that every $r_x$-based and Pareto-optimal social choice function $f$ satisfies $f(R^1) = f(R^2) = \{e\}$. Thus, it is easy to adapt the proof of Theorem 3.12 to start with the profiles $R^1$ and $R^2$ defined in the figure. In contrast, the Pareto-rule is also $r_2$-based if $n = 2$ and $m = 4$. We have derived this result by SAT-solving and omit the proof here. As consequence, Theorem 3.12 does not hold for $n = 2$ voters and $m = 4$ alternatives and therefore, we have a tight bound between possibility and impossibility results. Finally, it should be mentioned that for $n = 1$, the Pareto-rule is obviously $r_2$-based, which means that $n \geq 2$ is required.

A consequence of Theorem 3.12 is that no $r_+$-based, $r_-$-based, $r_=$-based and $r_r$-based SCF can be both Pareto-optimal and $\tilde{P}$-strategyproof if there is a sufficiently large number of voters and alternatives. We can strengthen this observation and state that we cannot think of any reasonable rank extension that avoids Theorem 3.12. This means that rank-based social choice functions are in the weak domain either $\tilde{P}$-manipulable or choose unreasonably large choice sets. Thus, it seems valid to say that there is no reasonable rank-based social choice function in the weak domain. Hence, even though many rank-based social choice functions seem advisable and are used in practice, we should be wary of possible disadvantages.

Chapter 4

# Social Choice Functions in C2

One of the most important hierarchies of social choice functions has been introduced in [Fis77]. This hierarchy distinguishes social choice functions with respect to the information required for computing them. One of the classes within this hierarchy is called C2 and it contains all social choice functions that only depend on the majorities between alternatives. This class contains a huge number of well-studied social choice functions such as Black's rule [BNM$^+$58], Nanson's rule [Cop51] and Kemeny's rule [Kem59, Lev75]. Furthermore, the class of C2-functions is a superset of the class of tournament solutions [BBH16] and therefore, it contains also many $\tilde{P}$-strategyproof social choice functions. However, the results with respect to $\tilde{P}$-strategyproofness only hold in the strict domain. Even more, the class of C2-functions is only rarely considered in the weak domain. Thus, we focus in this chapter on C2-functions in the weak domain and discuss them with respect to $\tilde{P}$-strategyproofness.

This chapter is organized as follows. First, we present a formal introduction to the concept of C2 in Section 4.1. Next, we introduce $\tilde{P}$-strategyproof C2-functions in the weak domain in Section 4.2. While there are many $\tilde{P}$-strategyproof C2-functions that violate Pareto-optimality, only few of them additionally satisfy this axiom. This observation leads to two necessary conditions for the $\tilde{P}$-strategyproofness of C2-functions which are discussed in Section 4.3. We even deduce a characterization of the Pareto-rule from these requirements.

## 4.1 Introduction to Social Choice Functions in C2

In this section, we formally introduce the class C2, discuss some known results and compare the concepts of rank-based social choice functions and C2-functions. This is necessary as we need a strong understanding of majorities and C2 in the following sections.

For introducing the class C2, we need to recall the concept of majorities of alternatives against each other discussed in Definition 1.7. This term refers to the number of voters that prefer an alternative $a$ strictly to an alternative $b$. For convenience, we present here the formal definition again.

**Definition 4.1 (Majority for $a$ against $b$).** *Consider a preference profile $R$ and two alternatives $a, b \in A$. The majority for $a$ against $b$ in the profile $R$ is denoted by $n_{ab} = |\{i \in N \mid a \succ_i b\}|$.*

With the help of this term, it is easy to define C2-functions. These social choice functions only rely on the majorities $n_{ab}$ for all alternatives $a, b \in A$ to compute the choice set. Thus, we may also refer to them as majority-based social choice functions. Formally, C2-functions are defined as follows.

**Definition 4.2 (C2-functions).** *A social choice function $f$ is a C2-function if $f(R) = f(R')$ for all preference profiles $R$, $R'$ with $n_{ab} = n'_{ab}$ for all pairs of alternatives $a, b \in A$.*

Note that we often write that a social choice function is in C2 or satisfies C2 instead of stating that it is a C2-function. Furthermore, it should be stressed that Definition 4.2 does not specify the domain or the number of voters used for defining $R$ and $R'$ as it does not depend on these details. Even more, C2-functions are in the weak domain always defined on a variable electorate as there can be profiles $R$ and $R'$ defined on electorates with varying sizes that have the same majorities. For instance, the profiles $R$ and $R' = (R, R_{i^*})$ in which voter $i^*$ is indifferent between all alternatives have the same majorities and therefore, it every C2-function $f$ is required to return the same choice sets for these profiles. Another argument for defining social choice functions in C2 on a variable electorate is that these functions are often defined on weighted and directed graphs where the alternatives are the nodes and there is an edge with weight $n_{ab}$ between all nodes $a, b \in A$. Therefore, we assume that C2-functions in the weak domain are always defined on a variable electorate.

Note that the class of C2-functions contains many well-known social choice functions. For instance, Kemeny's rule [Kem59, Lev75], the maximin rule [You77], Black's rule [BNM+58], Nanson's rule [Cop51], the ranked pairs method [Tid87] and the essential set [DL99] are in C2. Even Borda's rule discussed in the previous chapter can be defined only based on the majorities [Bra17]. However, most of these functions are only defined for the strict domain and therefore, they are not relevant for our results. The only well-known C2-function that is $\tilde{P}$-strategyproof in the weak domain is the Pareto-rule which picks all Pareto-optimal alternatives as winners [BSS]. This rule is formalized in C2 by choosing all alternatives $a \in A$ such that $n_{ab} > 0$ or $n_{ab} = n_{ba} = 0$ for all alternatives $b \in A \setminus \{a\}$. Even though this rule often chooses large choice sets, it is Pareto-optimal. Thus, it shows that there are $\tilde{P}$-strategyproof and Pareto-optimal C2-functions.

Because of Borda's rule, it follows that the set of C2-functions and the set of rank-based SCFs are not disjoint. Nevertheless, it is easy to see that they are conceptually rather different and their intersection does contain only very few interesting social choice functions. Intuitively, rank-basedness is a row-wise concept, i.e., we can reorder alternatives within a row in a preference profile, while we only have to ensure

$R^1$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $d$ |
| $c$ | $d$ | $b$ |
| $d$ | $a$ | $a$ |

$R^2$:

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $b$ | $c$ |
| $c$ | $a$ | $d$ |
| $d$ | $c$ | $a$ |
| $a$ | $d$ | $b$ |

$R^3$:

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $a$ | $c$ |
| $d$ | $c$ | $b$ |
| $c$ | $b$ | $d$ |
| $a$ | $d$ | $a$ |

Figure 4.1: Preference profiles used for comparing C2 and rank-basedness

that no alternative is placed twice in the preference of a voter. In contrast, C2 is a column-wise concept: A voter can move an alternative arbitrarily in his preference if other voters equalize the changes of the majorities. This intuition can be seen in the example in Figure 4.1. In this example, profile $R^2$ is deduced from $R^1$ by making $a$ the worst alternative in the preference of voter 1. Furthermore, the other two voters adapt the preferences such that $n^1_{xy} = n^2_{xy}$ for all pairs of alternatives $x, y \in A$. It follows that every C2-function $f_1$ satisfies $f_1(R^1) = f_1(R^2)$. In contrast, rank-basedness does not relate $R^1$ and $R^2$ as $r^*(R^1) \neq r^*(R^2)$. Moreover, the profile $R^3$ is deduced from profile $R^1$ by switching the positions of $a$ and $b$ in the first row and adapting the remaining rows to get a feasible preference profile. This means that every rank-based SCF $f_2$ satisfies $f_2(R^1) = f_2(R^3)$. Furthermore, note that $d$ is Pareto-dominated by $c$ in $R^1$ but not in $R^3$, which implies that a C2-function may choose different choice sets for $R^1$ and $R^3$.

## 4.2 $\tilde{P}$-strategyproof Social Choice Functions in C2

In this section, we focus on $\tilde{P}$-strategyproof social choice functions that are defined on the weak domain and that are in C2. While many social choice functions satisfying these axioms are known for the strict domain, this problem has been rarely considered in the weak domain. Even more, it turns out that it is very hard to find Pareto-optimal SCFs that satisfy $\tilde{P}$-strategyproofness and C2 and that are not equal to the Pareto-rule. Therefore, we first discuss C2-functions that are $\tilde{P}$-strategyproof but not Pareto-optimal in Section 4.2.1. After that, we focus on $\tilde{P}$-strategyproof C2-functions that additionally satisfy Pareto-optimal in Section 4.2.2.

### 4.2.1 Social Choice Functions Violating Pareto-optimality

In this section, we discuss social choice functions in C2 that are $\tilde{P}$-strategyproof and that do not need to satisfy any other axiom. These assumptions make it easy to find $\tilde{P}$-strategyproof social choice functions as we can return rather large choice sets. Even though one might criticize social choice functions with large choice sets

$R^1$ :

| 1 | 1 | 1 |
|---|---|---|
| $a, b$ | $a$ | $b$ |
| $c$ | $b$ | $a$ |
| $d$ | $c, d$ | $c, d$ |

$R^2$ :

| 1 | 1 | 1 |
|---|---|---|
| $a, b$ | $a$ | $b$ |
| $d$ | $b$ | $a$ |
| $c$ | $c, d$ | $c, d$ |

Figure 4.2: Preference profiles showing that remark 2 of [Bra15] is not true in the weak domain

as they are indecisive and fail important axioms such as Pareto-optimality, the existence of these social choice functions shows that the Pareto-rule is not the only $\tilde{P}$-strategyproof C2-function. For finding $\tilde{P}$-strategyproof C2-functions, we discuss a strong criterion that implies the $\tilde{P}$-strategyproofness of social choice functions. This condition states that many coarsenings of $\tilde{P}$-strategyproof social choice functions are also $\tilde{P}$-strategyproof. Note that we formalize this criterion as general as possible and therefore, it can also be applied for SCFs that are not in C2.
Our idea for finding $\tilde{P}$-strategyproof social choice functions in C2 is to consider coarsenings of the Pareto-rule. As this social choice function is known to be $\tilde{P}$-strategyproof, it seems reasonable that its coarsenings satisfy this axiom, too. For formally proving this intuition, we aim to generalize remark 2 in [Bra15] which states that a coarsening $g : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ of a $\tilde{P}$-strategyproof social choice function $f : \mathcal{S}^n \mapsto 2^A \setminus \emptyset$ is also $\tilde{P}$-strategyproof if $f(R) = g(R)$ for all preference profiles $R \in \mathcal{S}^n$ with $|f(R)| = 1$. Unfortunately, this result only holds in the strict domain. An example proving this claim can be constructed with the help of the profiles $R^1$ and $R^2$ shown in Figure 4.2. Assume that there is a social choice function $g$ with $g(R^1) = \{a, b, c\}$ and $g(R) = \text{PO}(R)$ otherwise. Clearly, voter 1 can $\tilde{P}$-manipulate $g$ by switching from $R^1$ to $R^2$ as $g(R^2) = \text{PO}(R^2) = \{a, b\}$. However, $g$ is a coarsening of the Pareto-rule which satisfies that $g(R) = \text{PO}(R)$ if $|\text{PO}(R)| = 1$. Thus, remark 2 of [Bra15] fails in the weak domain.
The problem in the last example is that voter 1 is indifferent between all alternatives in $\text{PO}(R^1) = \text{PO}(R^2)$. Furthermore, there are alternatives in $g(R^1)$ that are less preferred than those in $\text{PO}(R^1)$. Thus, a voter can $\tilde{P}$-manipulate by excluding the alternatives in $g(R^1) \setminus \text{PO}(R^1)$ as he is indifferent between all alternatives in $\text{PO}(R^1)$. This example shows that we have to be careful about the choice set of a coarsening $g$ of a $\tilde{P}$-strategyproof SCF $f$ if a voter is indifferent between all alternatives in $f(R)$. The reason for this is that $g$ might be $\tilde{P}$-manipulable in such a profile. Note that we can also observe this problem in the strict domain. The second remark of [Bra15] solves it by demanding that $f(R) = g(R)$ if $|f(R)| = 1$. However, it turns out that this assumption is often violated but $g$ is still $\tilde{P}$-strategyproof. For instance, we consider in Theorem 3.4 coarsenings $g$ of the OMNI-rule which are $\tilde{P}$-strategyproof even though they do not satisfy that $g(R) = \text{OMNI}(R)$ if $|\text{OMNI}(R)| = 1$. In contrast, if $\text{OMNI}(R) = \{a\}$ for a preference profile $R$, then there is an alternative $b \in g(R')$ such that $a \succ_i b$ for every profile $R'$ and voter $i \in N$ with $R_{-i} = R'_{-i}$. Thus, it is not necessary that $f(R) = g(R)$ if a voter is indifferent between all

alternatives in $f(R)$. Instead, it suffices that $g(R')$ contains for every profile $R'$ such that $R_{-i} = R'_{-i}$ for a voter $i \in N$ a less preferred alternatives to avoid $\tilde{P}$-manipulability. We combine the last two observations to derive a new theorem analyzing the $\tilde{P}$-strategyproofness of coarsenings of $\tilde{P}$-strategyproof social choice functions.

**Theorem 4.1.** *Let $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ and $g : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ denote social choice functions such that $f(R) \subseteq g(R)$ for all $R \in \mathcal{W}^n$. Furthermore, assume for all preference profiles $R, R' \in \mathcal{W}^n$ and all voters $i \in N$ such that $R_{-i} = R'_{-i}$ and $a \sim_i b$ for all $a, b \in f(R) \cup f(R')$ that either $a \sim_i b$ holds for all $a, b \in g(R) \cup g(R')$ or that there are alternatives $a \in g(R)$, $b \in g(R')$ with $a \succ_i b$. If $f$ is $\tilde{P}$-strategyproof, then $g$ is also $\tilde{P}$-strategyproof.*

Proof: Consider two social choice functions $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ and $g : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ as specified in the theorem and assume that $f$ is $\tilde{P}$-strategyproof. We have to show that $g$ is also $\tilde{P}$-strategyproof. Thus, assume for contradiction that $g$ is $\tilde{P}$-manipulable, i.e., there are preference profiles $R, R' \in \mathcal{W}^n$ and a voter $i \in N$ such that $R_{-i} = R'_{-i}$ and $g(R')\ \tilde{P}_i\ g(R)$. We derive a contradiction to this assumption with the help of a case distinction on $f(R)$ and $f(R')$.

First, assume that there are alternatives $a, b \in f(R)$ with $a \succ_i b$. In this case, let $a^*$ denote one of voter $i$'s best alternatives in $f(R)$, i.e., $a^* \succeq_i x$ for all $x \in f(R)$. It holds that every set $B \subseteq \{a \in A \mid a \succeq_i a^*\}$ is a $\tilde{P}$-improvement to $f(R)$ for voter $i$ as he prefers every alternative in $B$ strictly to the alternative $b \in f(R)$. Thus, it follows from the $\tilde{P}$-strategyproofness of $f$ that there is an alternative $c \in f(R')$ with $a^* \succ_i c$. As $f(R) \subseteq g(R)$ and $f(R') \subseteq g(R')$, it follows that $a^* \in g(R)$ and $c \in f(R')$ contradicting that $g(R')\ \tilde{P}_i\ g(R)$. Hence, there is no $\tilde{P}$-manipulation possible in this case.

Next, assume that $a \sim_i b$ for all alternatives $a, b \in f(R)$ and there are alternatives $c, d \in f(R')$ with $c \succ_i d$. Furthermore, let $a^*$ denote one of voter $i$'s worst alternatives in $f(R')$, i.e., $x \succeq_i a^*$ for all $x \in f(R')$. If $a^* \succeq_i a$ for an alternative $a \in f(R)$, voter $i$ prefers $a^*$ weakly to all alternatives in $f(R)$. Moreover, there is by assumption an alternative in $f(R')$ that is strictly preferred to $a^*$, which means that $f(R')\ \tilde{P}_i\ f(R)$. This contradicts the $\tilde{P}$-strategyproofness of $f$ and therefore, $a^*$ is strictly less preferred to the alternatives in $f(R)$. As $g$ is a coarsening of $f$, this implies that $g$ cannot be $\tilde{P}$-manipulated in this case either.

Finally, assume that $a \sim_i b$ for all alternatives $a, b \in f(R)$ and $c \sim_i d$ for all alternatives $c, d \in f(R')$. It follows directly from the $\tilde{P}$-strategyproofness of $f$ that $a \sim_i b$ for all alternatives $a, b \in f(R) \cup f(R')$. Otherwise, there are alternatives $a \in f(R)$, $c \in f(R')$ with $a \succ_i c$, which implies that voter $i$ can $\tilde{P}$-manipulate by switching from $R'$ to $R$, or with $c \succ_i a$, which implies that voter $i$ can $\tilde{P}$-manipulate by switching from $R$ to $R'$. Thus, the assumptions of the theorem imply that either $a \sim_i b$ for all $a, b \in g(R) \cup g(R')$ or there are alternatives $a \in g(R)$, $b \in g(R')$ with $a \succ_i b$. It is straightforward to deduce from these observations that voter $i$ cannot $\tilde{P}$-manipulate and therefore, $g$ is in all cases $\tilde{P}$-strategyproof. □

Note that the condition on $g$ in the case that $a \sim_i b$ for all $a, b \in f(R) \cup f(R')$ is rather technical. However, it allows to prove the $\tilde{P}$-strategyproofness of many coarsenings and therefore, it is a very useful condition. Furthermore, we can deduce many simplified variants of Theorem 4.1. For instance, if we require that $f(R) = g(R)$ for all preference profiles $R$ where a voter $i$ exists with $a \sim_i b$ for all $a, b \in f(R)$ and that $f(R) \subseteq g(R)$ for all other profiles, we can still use the theorem to prove the $\tilde{P}$-strategyproofness of $g$. Thus, our results imply remark 2 in [Bra15]. Furthermore, Theorem 4.1 is indeed more general than this remark as we can use it to prove Theorem 3.4 by observing that all functions discussed in this result are coarsenings of the OMNI-rule which satisfy the conditions of Theorem 4.1. Note that this example shows that this theorem can also be applied for social choice functions that are not in C2.

Furthermore, we can prove with the help of Theorem 4.1 that many C2-functions that coarsen the Pareto-rule are $\tilde{P}$-strategyproof. For example, consider the social choice function $f_1(R) = \{a \in A \mid \nexists b \in A : n_{ab} = 0 \wedge n_{ba} \geq 2\}$. Intuitively, this social choice function chooses an alternative $a$ if it is Pareto-optimal or if it is Pareto-dominated by an alternative $b$ but only a single voter prefers $b$ strictly to $a$. Therefore, it is obvious that $f_1$ is a coarsening of the Pareto-rule. We want to use Theorem 4.1 to prove the $\tilde{P}$-strategyproofness of $f_1$. Consequently, we focus on preference profiles $R, R' \in \mathcal{W}^n$ which contain a voter $i$ such that $R_{-i} = R'_{-i}$ and $a \sim_i b$ for all $a, b \in \text{PO}(R) \cup \text{PO}(R')$ as this is the only possibility to $\tilde{P}$-manipulate $f_1$. Because the Pareto-rule always contains at least one of the most preferred alternatives of every voter and $a \sim_i b$ for all $a, b \in \text{PO}(R) \cup \text{PO}(R')$, it follows that $\text{PO}(R)$ and $\text{PO}(R')$ are subsets of voter $i$'s most preferred alternatives. Thus, if $a \sim_i b$ for all $a, b \in f_1(R)$, we have no $\tilde{P}$-manipulation is possible as $f_1(R)$ is a subset of the most preferred alternatives of voter $i$. For this reason, assume that there is an alternative $b \in f_1(R)$ that is not among the most preferred ones of voter $i$. This means that for every alternative $a$ with $a \succ_i b$, there is either a voter $j$ with $b \succ_j a$ or every voter $j \in N \setminus \{i\}$ submits $a \sim_j b$. Both cases imply that there is no alternative $a$ with $a \succ_i b$ in $R$ such that $n'_{ba} = 0$ and $n'_{ab} \geq 2$ in $R'$ regardless of the preference $R'_i$. Thus, there is an alternative $c \in f(R')$ with $b \succeq_i c$ as the dominance relation implied by $f_1$ is transitive. This means that there is an alternative $c \in f(R')$ with $b \succeq_i c$. Therefore, the conditions of Theorem 4.1 hold and it follows that $f_1$ is $\tilde{P}$-strategyproof. As it follows from the definition of $f_1$ that this SCF is in C2, we have found another $\tilde{P}$-strategyproof C2-function but the Pareto-rule.

Another SCF for which we can apply Theorem 4.1 to show its $\tilde{P}$-strategyproofness is the weak Pareto-rule defined as $f_2(R) = \{a \in A \mid \nexists b \in A : n_{ab} = 0 \wedge n_{ba} = n\}$. This social choice function does only exclude an alternative $a$ from the choice set if there is another alternative $b$ such that $b \succ_i a$ for every voter $i \in N$. The arguments showing why we can use Theorem 4.1 to prove the $\tilde{P}$-strategyproofness of $f_2$ are the same as shown for $f_1$ and therefore, we omit them here. Furthermore, it should be mentioned that $f_2$ is no C2-function as profiles have the same majorities if we add a completely indifferent voter but the number of voters changes. Consequently, it

is possible that $f_2(R) = \{a\}$ and $f_2(R') = A$ where $R'$ results from $R$ by adding a completely indifferent voter. This contradicts that $f_2$ is in C2 as $R$ and $R'$ have the same majorities but not the same choice set. This example shows that Theorem 4.1 can also be used for social choice functions that are not in C2.
As consequence of Theorem 4.1, many coarsenings of the Pareto-rule are both in C2 and $\tilde{P}$-strategyproof, for instance $f_1$. Furthermore, it should be stressed that this theorem can also be used to derive the $\tilde{P}$-strategyproofness of social choice functions that are not in C2, e.g., $f_2$. However, all social choice functions derived with this theorem from the Pareto-rule violate Pareto-optimality in the weak domain. Thus, it does not seem promising to continue this line of work. Nevertheless, it shows that there are other $\tilde{P}$-strategyproof C2-functions but the Pareto-rule.

### 4.2.2 Social Choice Functions Satisfying Pareto-optimality

After discussing an approach for defining $\tilde{P}$-strategyproof C2-functions that violate Pareto-optimality, we focus next on the situation where this axiom is additionally required. The question about social choice functions satisfying all these properties is interesting as it is already known that no pairwise social choice function satisfies both Pareto-optimaity and $\tilde{P}$-strategyproofness. [BSS]. Recall that pairwiseness demands from a social choice function $f$ that $f(R) = f(R')$ for all preference profiles $R, R'$ with $n_{ab} - n_{ba} = n'_{ab} - n'_{ba}$ for all alternatives $a, b \in A$. Thus, this axiom is only slightly stronger than C2, which leads to the question whether there are any social choice functions in C2 that are Pareto-optimal and $\tilde{P}$-strategyproof. It turns out that it is easy to answer this question as the Pareto-rule satisfies all required axioms. However, it is not clear whether there are any other $\tilde{P}$-strategyproof and Pareto-optimal C2-functions. Thus, we discuss two approaches for defining refinements of the Pareto-rule which satisfy these properties.
Our first approach for deriving refinements of the Pareto-rule which are $\tilde{P}$-strategyproof and in C2 is to focus on the case where all voters are indifferent between some alternatives. Formally, we discuss sets of indistinguishable alternatives which are introduced in the sequel.

**Definition 4.3 (Sets of indistinguishable alternatives).** *We call two alternatives $a, b \in A$ indistinguishable in a preference profile $R \in \mathcal{W}^n$ if all voters are indifferent between $a$ and $b$. Furthermore, a set of indistinguishable alternatives $B$ is a set of alternatives such that $a \sim_i b$ for all alternatives $a, b \in B$ and voters $i \in N$.*

Intuitively, a set of indistinguishable alternatives consists only of clones of a single alternative. Clearly, every voter is indifferent between an alternative and its clones, which leads to a set of indistinguishable alternatives. It should be stressed that sets of indistinguishable alternatives are by definition not inclusion-maximal. This means that every singleton set is trivially also a set of indistinguishable alternatives. Nevertheless, there is always a unique inclusion-maximal set of indistinguish-

| | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $R:$ | $a, b$ | $c$ | $c, d$ | $d$ |
| | $c, d$ | $a, b$ | $a, b$ | $c$ |
| | | $d$ | | $a, b$ |

Figure 4.3: Preference profile used for explaining sets of indistinguishable alternatives

able alternatives that contains a specific alternative because indistinguishability is a transitive relation. Furthermore, we call a set of indistinguishable alternatives Pareto-optimal if it contains only Pareto-optimal alternatives.

An example for a preference profile yielding a set of indistinguishable alternatives with size larger than 1 can be seen in Figure 4.3. In this profile, every voter is indifferent between $a$ and $b$ and therefore, $\{a, b\}$ is a set of indistinguishable alternatives. Even more, as voter 1 prefers $a$ and $b$ the most, $\{a, b\}$ is a Pareto-optimal set of indistinguishable alternatives. Finally, no other alternatives are indistinguishable from each other and therefore, $\{c\}$ and $\{d\}$ are inclusion-maximal sets of indistinguishable alternatives.

We are interested in sets of indistinguishable alternatives for a simple reason: It is irrelevant for every voter with respect to the set extension $\tilde{P}$ whether the whole set is in the choice set or just a single representative. This observation leads to the social choice function $f_1 : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that is defined as follows: $f_1(R)$ contains for every Pareto-optimal and inclusion-maximal set of indistinguishable alternatives the lexicographic smallest alternative. For instance, $f(R) = \{a, c, d\}$ for the preference profile $R$ shown in Figure 4.3, whereas $\text{PO}(R) = \{a, b, c, d\}$. It is easy to see that $f_1$ is a refinement of the Pareto-rule. Thus, it remains to show that this social choice function is also $\tilde{P}$-strategyproof and in C2.

**Theorem 4.2.** *The social choice function $f_1$ is $\tilde{P}$-strategyproof and in C2.*

Proof: First, we focus on proving that $f_1$ is in C2. Note for this that for every pair of alternatives $a, b \in A$ and for all preference profiles $R \in \mathcal{W}^n$, it holds that if $a$ Pareto-dominates $b$ in $R$, then $n_{ab} > 0$ and $n_{ba} = 0$ and if $a$ is indistinguishable from $b$, then $n_{ab} = n_{ba} = 0$. Thus, we can detect all Pareto-optimal and inclusion-maximal sets of indistinguishable alternatives based on majorities, which implies that $f_1$ is indeed in C2.

Next, we show that the social choice function $f_1$ is also $\tilde{P}$-strategyproof. Thus, assume for contradiction that $f_1$ is $\tilde{P}$-manipulable, i.e., there is a voter $i \in N$ and preference profiles $R, R' \in \mathcal{W}^n$ such that $R_{-i} = R'_{-i}$ and $f_1(R') \tilde{P}_i f_1(R)$. This means that $x \succeq_i y$ for all alternatives $x \in f_1(R')$, $y \in f(R)$ and that this preference is strict for at least one pair of alternatives. Moreover, note that $f_1(R)$ differs from $\text{PO}(R)$ only if there are alternatives $a \in f_1(R)$ and $b \in \text{PO}(R) \setminus f(R)$ with $a \sim_j b$ for every voter $j \in N$. As this observation also holds for $f_1(R')$ and $\text{PO}(R')$, it follows from the transitivity of individual preferences that $x \succeq_i y$ for all alternatives $x \in \text{PO}(R')$,

$$R^* : \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline a, c & b, c & d \\ e & e & a, b, e \\ b, d & a, d & c \end{array}$$

Figure 4.4: Preference profile where $f_2$ refines the Pareto-rule

$y \in \text{PO}(R)$ and that this is also strict for at least one pair of alternatives. This means that $\text{PO}(R') \; \tilde{P}_i \; \text{PO}(R)$ which contradicts the $\tilde{P}$-strategyproofness of the Pareto-rule. Hence, our initial assumption is wrong and $f_1$ is also $\tilde{P}$-strategyproof. Thus, the social choice function $f_1$ satisfies all axioms required by this theorem. □

Note that there are many $\tilde{P}$-strategyproof social choice functions in C2 that are similar to $f_1$ as a representative of a set of indistinguishable alternatives can be chosen arbitrarily. Furthermore, we can also choose multiple representatives for a single set. The $\tilde{P}$-strategyproofness of such SCFs follows from the same argument as provided in the proof of Theorem 4.2. However, note that these social choice functions as well as $f_1$ are not neutral. Even more, it is often assumed in the literature that no alternatives are indistinguishable as it does usually not make a difference whether all voters are indifferent between some alternatives or we replace these alternatives with a single one.
As consequence of the last remarks, the question about the existence of social choice functions that are neutral, $\tilde{P}$-strategyproof and in C2 and that refine the Pareto-rule arises. As we show in the sequel, there is a social choice function that satisfies all required axioms. The main idea for defining this social choice function is best explained by considering the preference profile $R^*$ shown in Figure 4.4. In this profile every alternative is Pareto-optimal, i.e., $\text{PO}(R^*) = \{a, b, c, d, e\}$. Furthermore, alternative $e$ is Pareto-dominated in $R^*_{-i}$ for all $i \in N$. Thus, $e$ is almost Pareto-dominated and therefore, it might be possible to construct a social choice function $f_2$ such that $f_2(R^*) = \{a, b, c, d\}$ and that satisfies Pareto-optimality, neutrality, $\tilde{P}$-strategyproofness and C2. It turns out that this conjecture is true. For proving this claim, we first analyze the majorities of $R^*$ and their relation to the structure of this profile. In particular, we show that all profiles with the majorities displayed in Figure 4.5 consist of three voters that submit $R^*$ and $k - 1$ voters that submit $a \sim b \sim c \sim d \succ e$. Note that the majorities in Figure 4.5 are the majorities of $R^*$ if $k = 1$. Thus, the following lemma implies that $R^*$ is the only profile defined on three voters that has these majorities.

**Lemma 4.1.** *If a preference profile $R$ satisfies the majorities shown in Figure 4.5 for an arbitrary $k \geq 1$, then it consists of 3 voters who submit $R^*$, $k - 1$ voters who submit $a \sim b \sim c \sim d \succ e$ and an arbitrary number of voters that are indifferent between all alternatives.*

Proof: Consider a profile $R$ that satisfies the majorities shown in Figure 4.5 for an arbitrary $k \geq 1$. We prove this lemma by reconstructing $R$ from its majorities.

| $n^*_{xy}$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | – | 1 | 1 | 1 | $k$ |
| $b$ | 1 | – | 1 | 1 | $k$ |
| $c$ | 1 | 1 | – | 2 | $k+1$ |
| $d$ | 1 | 1 | 1 | – | $k$ |
| $e$ | 1 | 1 | 1 | 2 | – |

Figure 4.5: Majorities for Lemma 4.1

Therefore, we first focus on the majorities between the alternatives $a$, $b$, $c$ and $d$ as these are independent of $k$. Note that there are two voters $i_1$ and $i_2$ who submit $c \succ d$ as $n_{cd} = 2$ and a single voter $i_3$ who submits $d \succ c$ as $n_{dc} = 1$; all other voters are indifferent between $c$ and $d$. Furthermore, observe that $n_{ac}+n_{ad}+n_{ca}+n_{da} = 4$ and that every voter in $I = \{i_1, i_2, i_3\}$ contributes at least one to this sum. Moreover, every voter in $N\setminus I$ contributes either 0 or 2 to this sum as these voters are indifferent between $c$ and $d$. Consequently, every voter in $N \setminus I$ is indifferent between $a$, $c$ and $d$ as otherwise $n_{ac} + n_{ad} + n_{ca} + n_{da} > 4$. A symmetric argument for $b$ implies that every voter in $N \setminus I$ is indifferent between all alternatives in $A \setminus \{e\}$.
Therefore, every profile that satisfies the majorities shown in Figure 4.5 between alternatives in $A \setminus \{e\}$ is defined on three voters. We can find all these profiles by simply enumerating all profiles defined on four alternatives and three voters and compute their majorities. The result of this search is shown in Figure 4.6 where all profiles that satisfy the majorities $n^*_{xy}$ for all $x, y \in A \setminus \{e\}$ are displayed up to renaming the voters. Thus, every profile $R'$ that is defined on the alternatives $A \setminus \{e\}$ and that satisfies $n'_{xy} = n^*_{xy}$ for all $x, y \in A \setminus \{e\}$ consists of one of the profiles shown in Figure 4.6 and an arbitrary number of voters that are indifferent between all alternatives in $A \setminus \{e\}$.
Finally, we add alternative $e$ to these profiles such that the majorities shown in Figure 4.5 are satisfied. In this step, we use an induction on $k$ to prove the lemma. If $k = 1$ or $k = 2$, we can simply enumerate all possibilities to add $e$ to see that the lemma holds. This proves the induction basis. Next, consider an arbitrary integer $l \geq 2$ and assume that the lemma is true for all $k \leq l$. We prove under this assumption that the lemma also holds if $k = l + 1 > 2$ by showing that there is at least one voter $i$ who submits $a \sim b \sim c \sim d \succ e$. This means that we can remove voter $i$ from the preference profile, which reduces all majorities $n_{xe}$, $x \in A \setminus \{e\}$ by 1 and does not change the remaining majorities. Consequently, the induction hypothesis can be used, which means that profile $R_{-i}$ consists of profile $R^*$ and $l-1$ voters who submit $a \sim b \sim c \sim d \succ e$. Finally, the lemma follows by adding voter $i$ to the profile again.
Thus, it only remains to prove that there is a voter $i$ who submits $a \sim b \sim c \sim d \succ e$ if $k > 2$. It follows from $k > 2$ that there are at least $n_{ce} + n_{ec} = (k + 1) + 1 \geq 5$ voters who are not indifferent between all alternatives. Furthermore, we know that only three of these voters are not indifferent between all alternatives in $A \setminus \{e\}$.

| | 1 | 1 | 1 |
|---|---|---|---|
| $R_1$ | $a \succ c \succ b \sim d$ | $b \succ c \succ a \sim d$ | $d \succ a \sim b \sim c$ |
| $R_2$ | $a \succ b \sim c \succ d$ | $c \succ a \sim b \sim d$ | $d \succ b \succ a \sim c$ |
| $R_3$ | $b \succ a \sim c \succ d$ | $c \succ a \sim b \sim d$ | $d \succ a \succ b \sim c$ |
| $R_4$ | $c \succ a \sim d \succ b$ | $a \sim b \sim c \succ d$ | $b \sim d \succ a \succ c$ |
| $R_5$ | $c \succ b \sim d \succ a$ | $a \sim b \sim c \succ d$ | $a \sim d \succ b \succ c$ |
| $R_6$ | $a \sim c \succ d \succ b$ | $b \sim c \succ d \succ a$ | $a \sim b \sim d \succ c$ |
| $R_7$ | $a \sim c \succ b \sim d$ | $b \sim c \succ a \sim d$ | $d \succ a \sim b \succ c$ |

Figure 4.6: Preference profiles that agree with $R^*$ on all majorities defined on $\{a, b, c, d\}$

Furthermore, only a single voter can submit $e \succ a \sim b \sim c \sim d$ as $n_{ec} = 1$. Therefore, there is a voter who is not indifferent between all alternatives in $A$ but he is indifferent between the alternatives in $A \setminus \{e\}$ and he prefers $c$ weakly to $e$. This implies that this voter submits $a \sim b \sim c \sim d \succ e$, which proves the lemma. □

As consequence of Lemma 4.1, the profile $R^*$ yields a special structure that can be recognized only based on its majorities. This means that we can use this profile to design a social choice function $f_2$ that is in C2 and that refines the Pareto-rule: If the profile $R$ consists of 3 voters $i_1$, $i_2$ and $i_3$ such that $R^* = (R_{i_1}, R_{i_2}, R_{i_3})$ and all other voters submit $a \sim b \sim c \sim d \succeq e$, then $f_2(R) = \{a, b, c, d\}$. If the profile $R$ satisfies this condition after renaming the alternatives, then we simply rename the alternatives in $f_2(R^*)$ such that neutrality is satisfied. In all other cases, we set $f_2(R) = \text{PO}(R)$. Note that $f_2$ is only defined for the case that $m = 5$. Furthermore, it is easy to see that $f_2$ indeed refines the Pareto-rule. Therefore, it only remains to show that it is neutral, $\tilde{P}$-strategyproof and in C2.

**Theorem 4.3.** *The social choice function $f_2$ is $\tilde{P}$-strategyproof, Pareto-optimal and in C2.*

Proof: First note that $f_2(R) \subsetneq \text{PO}(R)$ only if there are three voters that submit a profile $R$ such that $R$ and $R^*$ are equal after renaming some alternatives and all other voters prefer the alternatives in $f_2(R)$ the most. It follows from Lemma 4.1 that this profile has unique majorities and therefore, we can detect in C2 when to refine the Pareto-rule. Thus, it is easy to see that $f_2$ is in C2. Furthermore, the SCF $f_2$ is also neutral as the set of profiles $R$ with $f_2(R) \subsetneq \text{PO}(R')$ is closed under renaming alternatives. More formally, if the profiles $R$ and $R'$ are equal after renaming alternatives, then $f_2(R) \subsetneq \text{PO}(R)$ if and only if $f_2(R') \subsetneq \text{PO}(R')$. Furthermore, as the social choice function $f_2$ also renames the choice set correctly, it follows that it is neutral.

Thus, it only remains to show that $f_2$ is also $\tilde{P}$-strategyproof. Therefore, assume for contradiction that $f_2$ is $\tilde{P}$-manipulable, i.e., there are preference profiles $R$, $R'$ and a voter $i$ such that $R_{-i} = R'_{-i}$ and $f_2(R')\ \tilde{P}_i\ f_2(R)$. We derive a contradiction to this assumption by making a case distinction distinguishing whether $f_2(R) \subsetneq \text{PO}(R)$ and $f_2(R') \subsetneq \text{PO}(R')$ or not. First, consider the case that $f_2$ agrees with the Pareto-rule on the choice sets for both $R$ and $R'$. In this case, $f_2$ cannot be $\tilde{P}$-manipulable as otherwise the Pareto-rule is $\tilde{P}$-manipulable, too. However, this SCF is known to be $\tilde{P}$-strategyproof and therefore, $f_2$ also satisfies this property if $f_2(R) = \text{PO}(R)$ and $f_2(R') = \text{PO}(R')$.

Next, consider the case that $f_2(R) \subsetneq \text{PO}(R)$. This means that $f_2(R) = \{a, b, c, d\}$ for some alternatives in $a, b, c, d \in A$. Furthermore, there are is a set of three voters $I = \{i_1, i_2, i_3\}$ such that $R^* = (R_{i_1}, R_{i_2}, R_{i_3})$ after renaming alternatives and all other voters prefer the alternatives in $\{a, b, c, d\}$ the most. Thus, no voter in $N \setminus I$ can $\tilde{P}$-manipulate as a subset of his most preferred alternatives is chosen. This means that the manipulator $i$ is a voter in $I$. Moreover, observe that $f_2(R)$ contains some of the most preferred alternatives of every voter in $I$. Thus, a voter can only $\tilde{P}$-manipulate if $f_2(R')$ is a subset of his most preferred alternatives. This implies that $|f_2(R')| \leq 2$ as the set of most preferred alternatives of every voter in $R^*$ has at most size 2. Consequently, $f_2(R') = \text{PO}(R')$ as $|f_2(R')| = 4$ if $f_2$ refines the Pareto-rule. We can derive from these observations that there are profiles $R$ and $R'$ such that $\text{PO}(R)$ contains alternatives that are not among the most preferred ones of voter $i$, $\text{PO}(R')$ is a subset of voter $i$'s most preferred alternatives and $R_{-i} = R'_{-i}$. Consequently, the Pareto-rule is $\tilde{P}$-manipulable which is known to be false. Thus, the assumption is wrong and $f_2$ is $\tilde{P}$-strategyproof if $f_2(R) \subsetneq \text{PO}(R)$.

Finally, consider the case that $f_2(R) = \text{PO}(R)$ and $f_2(R') \subsetneq \text{PO}(R')$. As the Pareto-rule contains at least one of the most preferred alternatives of every voter and $f_2(R) = \text{PO}(R)$, it follows that voter $i$ can only $\tilde{P}$-manipulate if $f_2(R')$ is a subset of his most preferred alternatives. Furthermore, as $f_2(R') \subsetneq \text{PO}(R')$, there are four alternatives $a, b, c, d \in A$ such that $f_2(R') = \{a, b, c, d\}$. Thus, voter $i$ prefers these alternatives the most in $R$. Even more, there are three voters $I = \{i_1, i_2, i_3\}$ in $R'$ such that $R^* = (R'_{i_1}, R'_{i_2}, R'_{i_3})$ after renaming alternatives and all other voters in $N \setminus I$ prefer the alternatives in $f_2(R')$ the most. If the manipulator is not among the voters in $I$, then $f_2(R) = \{a, b, c, d\} \subsetneq \text{PO}(R)$ as all voters in $N \setminus I$ prefer the alternatives in $f_2(R')$ the most in $R$. This contradicts that $f_2(R) = \text{PO}(R)$ and therefore, $i \in I$. Finally, note that $e$ is Pareto-dominated in $R'_{-j}$ for all $j \in I$. In particular, $e$ is Pareto-dominated in $R'_{-i} = R_{-i}$ and voter $i$ prefers all alternatives weakly to $e$ in $R_i$. This means that $e$ is also Pareto-dominated in $R$ and therefore, $f_2(R) = \text{PO}(R) \subseteq \{a, b, c, d\}$ is true. Thus, $f_2(R)$ is already a subset of voter $i$'s most preferred alternatives in $R$, which means that he cannot $\tilde{P}$-manipulate in this case either. Hence, $f_2$ is in all cases $\tilde{P}$-strategyproof, which proves the theorem. □

As consequence of the last theorem, it follows that there are also neutral and $\tilde{P}$-strategyproof C2-functions that refine the Pareto-rule. Thus, the Pareto-rule is not the only social choice function that satisfies $\tilde{P}$-strategyproofness, Pareto-optimality, neutrality and C2. Furthermore, it should be mentioned that the SCF $f_2$ is only defined for five alternatives. Nevertheless, it is easy to increase the number of alternatives by demanding that we only refine the Pareto-rule if there is a set of alternatives $B$ such that $|B| = 5$, $f_2(R|_B) \subsetneq \text{PO}(R|_B)$ and all alternatives in $A \setminus B$ are Pareto-dominated. It is easy to adapt the arguments in the proof of Theorem 4.3 to show that this approach leads to neutral and $\tilde{P}$-strategyproof refinements of the Pareto-rule in C2 that are defined on more than five alternatives.

## 4.3 Requirements for $\tilde{P}$-strategyproof C2-Functions

In this section, we focus on necessary conditions for the $\tilde{P}$-strategyproofness of C2-functions in the weak domain. As in the previous section, we try to formalize the criteria as general as possible and therefore, they sometimes do not require C2. Unfortunately, it is very hard to find strong criteria. Thus, we always use additional axioms in our results. More precisely, we discuss when a $\tilde{P}$-strategyproof and neutral C2-function is allowed to return a single winner in Section 4.3.1. Furthermore, we analyze the combination of Pareto-optimality, $\tilde{P}$-strategyproofness and C2 in Section 4.3.2. In this section, we can prove that the choice set of every social choice function satisfying these axioms contains at least one of the most preferred alternatives of every voter. This criterion has many important implications such as a characterization of the Pareto-rule discussed in Section 4.3.3.

### 4.3.1 $\tilde{P}$-strategyproofness and Neutrality

In this section, we focus on properties of $\tilde{P}$-strategyproof and neutral social choice functions in the weak domain. Note that neutrality is a very common property of social choice functions and therefore, we can derive many implications on the $\tilde{P}$-strategyproofness of SCFs. However, this also means that the required axioms are rather weak. As consequence, we cannot prove results that affect all preference profiles. Instead, we focus on preference profiles for which a neutral and $\tilde{P}$-strategyproof social choice function returns a single winner and discuss the properties of the winning alternative.

Therefore, we observe that many $\tilde{P}$-strategyproof social choice functions in the weak domain only return a single winner if this alternative is strongly preferred by many voters. For instance, the Pareto-rule and the OMNI-rule, which are known to be $\tilde{P}$-strategyproof in the weak domain, return only a single winner if an alternative is first-ranked by every voter. Even $\tilde{P}$-strategyproof social choice functions in the strict domain need to suffice strong requirements for picking a single winner. For instance, all $\tilde{P}$-strategyproof tournament solutions return only a single winner if it is the Condorcet winner.

Thus, we want to show that only strongly preferred alternatives are chosen as single winner by a $\tilde{P}$-strategyproof and neutral social choice function. We first prove a very weak variant of this claim: If a $\tilde{P}$-strategyproof and neutral social choice function returns a single winner, then this alternative is not Pareto-dominated.

**Theorem 4.4.** *Let $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ denote a neutral and $\tilde{P}$-strategyproof social choice function. If $f(R) = \{a\}$ for an arbitrary alternative $a \in A$ and preference profile $R \in \mathcal{W}^n$, then $a$ is Pareto-optimal in $R$.*

Proof: Consider an arbitrary social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies neutrality and $\tilde{P}$-strategyproofness and let $R \in \mathcal{W}^n$ denote a preference profile such that $f(R) = \{a\}$ for an arbitrary alternative $a \in A$. Furthermore, assume for contradiction that $a$ is Pareto-dominated by an alternative $b$ in $R$, i.e., $\forall i \in N : b \succeq_i a$ and $\exists i \in N : b \succ_i a$.
Next, we determine the set $N_{b \succ a} = \{i \in N \mid b \succ_i a\}$ which contains all voters who prefer $b$ strictly to $a$. We let every voter $i \in N_{b\succ a}$ manipulate one after another in the following way: First, we determine the sets $U_i = \{x \in A \mid x \succ_i a\}$ containing all alternatives strictly preferred to $a$ by voter $i$ and $L_i = \{x \in A \setminus \{a\} \mid a \succeq_i x\}$ containing all alternatives which are weakly less preferred than $a$. Thereafter, we change the preference of voter $i$ to $R'_i$ in which he is indifferent between all alternatives in $U_i \cup \{a\}$ and prefers all alternatives in $U_i \cup \{a\}$ strictly to all alternatives in $L_i$. Formally,

$$R'_i = \{(x, y) \mid x, y \in U_i \cup \{a\}\} \ \cup \ \{(x, y) \mid x \in U_i \cup \{a\}, y \in L_i\} \ \cup \ R|_{L_i} \quad .$$

This leads to a sequence of preference profiles $R^{i_0} = R$, $R^{i_1}$, ..., $R^{i_k} = R^*$, where $\{i_1, ..., i_k\} = N_{a\succ b}$ and $R^{i_j}$ differs from $R^{i_{j-1}}$ by replacing the current preference of voter $i_j$ with $R'_{i_j}$. This process leads to the preference profile $R^*$ which satisfies that $f(R^*) = \{a\}$; otherwise, there is a voter $i_j \in N_{a\succ b}$ such that $f(R^{i_j}) \neq \{a\}$ and $f(R^{i_{j-i}}) = \{a\}$. If an alternative $b \in L_{i_j}$ is in $f(R^{i_j})$, then voter $i_j$ can $\tilde{P}$-manipulate by switching from $R^{i_j}$ to $R^{i_{j-1}}$ as $a$ is one of his most preferred alternatives in $R^{i_j}$. Thus, $L_{i_j} \cap f(R^{i_j}) = \emptyset$. Furthermore, if an alternative $b \in U_{i_j}$ is in $f(R^{i_j})$, then voter $i_j$ can $\tilde{P}$-manipulate by switching from $R^{i_{j-1}}$ to $R^{i_j}$ as he prefers all alternatives in $f(R^{i_j}) \setminus \{a\}$ strictly to those in $f(R^{i_{j-1}}) = \{a\}$. Hence, this is also not possible because of the $\tilde{P}$-strategyproofness. Consequently, there cannot exists such a voter $i_j$, which implies that $f(R^*) = \{a\}$.
Finally, note that every voter is indifferent between $a$ and $b$ in the profile $R^*$ as every voter $i \in N_{b\succ a}$ changed his preference to $b \sim_i a$ and all other voters $j \in N \setminus N_{b\succ a}$ have already been indifferent between $a$ and $b$ in $R$. Thus, we can apply neutrality to rename $a$ to $b$ and vice versa and the profile does not change. However, the winner changes as $f(R^*) = \{a\}$ and after renaming $a$ to $b$, the winner is $b$ because of neutrality. This is a contradiction as $f(R^*) = \{a\}$ and $f(R^*) = \{b\}$ cannot be simultaneously true, which means that the initial assumption is wrong. Hence, if $f(R) = \{a\}$, then $a$ is Pareto-optimal in $R$. □

| | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|
| $R:$ | $a, b$ | $c$ | $c$ | $b$ | $b$ |
| | $c$ | $a, b$ | $b$ | $a, c$ | $a$ |
| | | | $a$ | | $c$ |

| | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|
| $R^*$ | $a, b$ | $c$ | $a, b, c$ | $b, a$ | $b, a$ |
| | $c$ | $a, b$ | | $c$ | $c$ |

Figure 4.7: Preference profiles illustrating the proof of Theorem 4.4

Next, we discuss an example of the construction used in the proof of Theorem 4.4 with the help of the profiles shown in Figure 4.7. We assume for this example that $f$ denotes a neutral and $\tilde{P}$-strategyproof SCF with $f(R) = \{a\}$. Furthermore, note that $b$ Pareto-dominates $a$ in $R$. According to the proof of Theorem 4.4, the first two voters do not change their preferences as they are indifferent between $a$ and $b$. In contrast, the other three voters manipulate such that they are indifferent between $a$ and all alternatives that are originally strictly preferred to $a$. As consequence of these modifications, every voter is indifferent between $a$ and $b$ in $R^*$. However, $f(R^*) = \{a\}$, which contradicts neutrality.

It should be stressed that Theorem 4.4 does not require C2. Even more, we do not need neutrality as it suffices that two alternative are treated equally if they are indistinguishable. Formally, this means that $a \in f(R)$ if and only if $b \in f(R)$ for all preference profiles $R$ and indistinguishable alternatives $a, b$ in $R$. Observe that this condition is very weak and arises naturally. Thus, almost every $\tilde{P}$-strategyproof social choice function is forbidden to choose a Pareto-dominated alternative as single winner.

Unfortunately, Theorem 4.4 is also a rather weak result and can only rarely be used to disprove the $\tilde{P}$-strategyproofness of a social choice function. Therefore, we strengthen this result next. However, it seems that we cannot improve the last theorem without assuming additional axioms. Therefore, we focus on neutral and $\tilde{P}$-strategyproof C2-functions and prove that these social choice functions only choose a single alternative if it is a Condorcet winner.

For proving this claim, we first discuss the combination of C2 and $\tilde{P}$-strategyproofness. More precisely, we prove two lemmas which state that these two axioms imply properties similar to set-monotonicity for a restricted set of preference profiles.

**Lemma 4.2.** *Let $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ denote a $\tilde{P}$-strategyproof C2-function and let $R \in \mathcal{W}^n$ denote an arbitrary preference profile such that $f(R) = \{a\}$ for an arbitrary alternative $a \in A$ and there is a voter $i \in N$ with $a \succ_i b$ for all alternatives $b \in A \setminus \{a\}$. Then, it holds that $f(R') = \{a\}$ for all preference profiles $R' \in \mathcal{W}^n$ such that $a \succeq_j b$ if and only if $a \succeq'_j b$ and $b \succeq_j a$ if and only if $b \succeq'_j a$ for all voters $j \in N$ and alternatives $b \in A \setminus \{a\}$.*

Proof: Consider an arbitrary social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that is $\tilde{P}$-strategyproof and in C2. Furthermore, let $R \in \mathcal{W}^n$ denote a preference profile such that $f(R) = \{a\}$ for an arbitrary alternative $a \in A$ and there is a voter $i \in N$ such that $a \succ_i b$ for all $b \in A \setminus \{a\}$. For proving the theorem, we show that every voter

can reorder the unchosen alternatives arbitrarily without changing the choice set of $f$ if these modifications do not affect preferences involving $a$. This means that $f$ satisfies a conditioned variant of independence of unchosen alternatives.
For proving this lemma, we focus on a single voter $j \in N$ and show that he can reorder the unchosen alternatives. As we can simply repeat this argument for every voter, the lemma follows. First, assume that $j = i$, i.e., we focus on a voter who prefers $a$ uniquely the most in $R$. It is easy to see that this voter can reorder all alternatives $b \in A \setminus \{a\}$ without affecting the choice set as long as $a$ remains his uniquely most preferred alternative. If this was not the case, then there is a profile $R' = (R_{-i}, R'_i)$ with $f(R') \neq \{a\}$ and voter $i$'s most preferred alternative in $R'$ is $a$. Thus, voter $i$ can $\tilde{P}$-manipulate by switching from $R'$ to $R$, which contradicts the $\tilde{P}$-strategyproofness of $f$. Consequently, $f(R') = \{a\}$ for all profiles $R' = (R_{-i}, R'_i)$ in which voter $i$ prefers $a$ strictly the most.
Next, consider a voter $j$ who does not prefer $a$ the most and a profile $R' = (R_{-j}, R'_j)$ where the preferences of voter $j$ that involve $a$ do not change. We show in the sequel that $f(R') = \{a\}$ by constructing a path of preference profiles from $R$ to $R'$ such that $a$ is the unique winner for every profile on this path. Therefore, consider the sets $U_j = \{x \in A \mid x \succ_j a\}$ containing all alternatives that voter $j$ prefers strictly to $a$, $L_j = \{x \in A \mid a \succ_j x\}$ containing all alternatives that voter $j$ prefers strictly less than $a$ and $I_j = \{x \in A \setminus \{a\} \mid x \sim_j a\}$ containing all alternatives that are indifferent to $a$ in $R_j$. With the help of these sets, we can represent the preference of voter $j$ as follows.

$$R_j = R_j|_{U_j} \cup R_j|_{I_j \cup \{a\}} \cup R_j|_{L_j} \cup \Big(U_j \times (L_j \cup I_j \cup \{a\})\Big) \cup \Big((I_j \cup \{a\}) \times L_j\Big)$$

We use in this definition the cross product $X \times Y$ between sets $X, Y$ as abbreviation indicating that all alternatives in $X$ are strictly preferred to all alternatives in $Y$. Furthermore, note that $U_j$, $L_j$ and $I_j$ do not change if we define these sets with respect to $R'_j$ as voter $j$ is only allowed to reorder alternatives in $U_j$ and $L_j$. Formally, this means that he can only modify a preference $x \succeq_j y$ if $x, y \in U_j$ or $x, y \in L_j$.
As $a \notin U_j \cup L_j$, we aim to derive $R'_j$ from $R_j$ by first letting voter $i$ reorder the alternatives in $U_j$ and $L_j$ in the same way as in $R'_j$ and then exchange the preferences of voter $i$ and $j$ with C2. In the first step, we let voter $i$ switch from his current preference $R_i$ to the preference $R^1_i$ in which he orders the alternatives in $U_j$ and $L_j$ according to $R'_j$. Formally, this is defined as follows.

$$R^1_i = \Big(\{a\} \times (A \setminus \{a\})\Big) \cup R'_j|_{U_j} \cup R'_j|_{I_j} \cup R'_j|_{L_j} \cup \Big(U_j \times (L_j \cup I_j)\Big) \cup \Big(I_j \times L_j\Big)$$

It should be stressed that it is irrelevant how we order the sets $U_j$, $L_j$ and $I_j$ in $R^1_i$ as only the preferences between alternatives in $U_j$ and $L_j$ are important. Furthermore, it follows from previous observations that voter $i$ can switch to this preference and the unique winner in the profile $R^1 = (R_{-i}, R^1_i)$ is still $a$. Next, we can use C2 and

$R$ :

| 1 | 1 |
|---|---|
| $a$ | $b$ |
| $d$ | $c$ |
| $e$ | $a$ |
| $b, c$ | $d, e$ |

$R^1$ :

| 1 | 1 |
|---|---|
| $a$ | $b$ |
| $c$ | $c$ |
| $b$ | $a$ |
| $d$ | $d, e$ |
| $e$ | |

$R^2$ :

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $b$ | $b$ |
| $c$ | $a$ |
| $d, e$ | $d$ |
| | $e$ |

$R'$ :

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $d$ | $b$ |
| $e$ | $a$ |
| $b, c$ | $d$ |
| | $e$ |

Figure 4.8: Preference profiles illustrating the proof of Lemma 4.2

let voter $i$ and $j$ exchange their preferences over $U_j$ and $L_j$. Formally, the preferences of voter $i$ and $j$ are now defined as follows.

$$R_i^2 = \Big(\{a\} \times (A \setminus \{a\})\Big) \cup R_j|_{U_j} \cup R_j|_{I_j} \cup R_j|_{L_j} \cup \Big(U_j \times (L_j \cup I_j)\Big) \cup \Big(I_j \times L_j\Big)$$

$$R_j^2 = R'_j|_{U_j} \cup R'_j|_{I_j \cup \{a\}} \cup R'_j|_{L_j} \cup \Big(U_j \times (L_j \cup I_j \cup \{a\})\Big) \cup \Big((I_j \cup \{a\}) \times L_j\Big)$$

It follows from C2 that this step results in a profile $R^2$ with $f(R^2) = f(R^1) = \{a\}$. Furthermore, note that $R_j^2 = R'_j$ and therefore, we only have to revert the preference of voter $i$ back to $R_i$ to arrive at $R'$. This is possible as $a$ is the unique winner in $R^2$ and voter $i$ prefers this alternative the most in $R$ and $R^2$. Thus, $a$ is still the unique winner after voter $i$ changes his preference back to $R_i$, i.e., $f(R') = \{a\}$. Otherwise, he can $\tilde{P}$-manipulate by switching from $R'$ to $R^2$ as $a$ is his uniquely most preferred alternative in $R'_i$. This means that every voter $j \in N$ can reorder the alternatives in $U_j$ and $L_j$ arbitrarily, which proves the lemma. □

Next, we discuss an example of the constructions in the proof of Lemma 4.2. Therefore, consider the preference profiles shown in Figure 4.8 and assume that $f$ denotes a $\tilde{P}$-strategyproof C2-function with $f(R) = \{a\}$. We want to change the preferences of voter 2 from $R_2 = b \succ c \succ a \succ d \sim e$ to $R'_2 = c \succ b \succ a \succ d \succ e$. Hence, we first calculate that $U_2 = \{b, c\}$, $L_2 = \{d, e\}$ and $I_2 = \emptyset$. Subsequently, we use the $\tilde{P}$-strategyproofness of $f$ to reorder the alternatives in $U_2$ and $L_2$ in the preference of the first voter according to $R'_2$. This leads to the profile $R^1$ and we know that $f(R^1) = \{a\}$ because of $\tilde{P}$-strategyproofness. Thereafter, we use C2 to exchange the preferences of voter 1 and 2 between the alternatives in $U_2 = \{b, c\}$ and $L_2 = \{d, e\}$, which implies that $f(R^2) = \{a\}$. Finally, we use again the $\tilde{P}$-strategyproofness of $f$ to let voter 1 switch back to his original preference. Thus, voter 2 has changed his preference to $R'_2$ and $a$ is still the unique winner.
As consequence of Lemma 4.2, every $\tilde{P}$-strategyproof C2-function $f$ satisfies independence of unchosen alternatives if $f(R) = \{a\}$ and a voter prefers $a$ uniquely the most. This is already a very powerful result because it allows us to reorder many alternatives arbitrarily. Even more, we can prove that voters can reinforce the unique winner if it is the uniquely most preferred alternative of a voter.

**Lemma 4.3.** *Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that is $\tilde{P}$-strategyproof and in C2. Furthermore, let $R \in \mathcal{W}^n$ denote an arbitrary preference profile such that $f(R) = \{a\}$ for an arbitrary alternative $a \in A$ and there is a voter $i \in N$ with $a \succ_i b$ for all $b \in A \setminus \{a\}$. Then, it holds that $f(R') = \{a\}$ for all voters $j \in N$ and preference profiles $R' = (R_{-j}, R'_j)$ such that $a \succeq_j b$ implies $a \succ'_j b$ for all $b \in A \setminus \{a\}$ and there is no alternative $b \in A \setminus \{a\}$ with $a \sim'_j b$.*

Proof: Consider a $\tilde{P}$-strategyproof C2-function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ and a preference profile $R \in \mathcal{W}^n$ such that $f(R) = \{a\}$ for an arbitrary alternative $a \in A$ and there is a voter $i \in N$ who prefers $a$ uniquely the most. We show that an arbitrary voter $j \in N$ can reinforce $a$ in his preference without changing the choice set if there are no ties involving $a$ in his new preference $R'_j$. Note that this lemma also allows for reordering unchosen alternatives. However, after we place $a$ correctly in the preference of voter $j$, we can use Lemma 4.2 to reorder the alternatives. Thus, it suffices to focus on how to reinforce $a$.
Therefore, consider an arbitrary voter $j \in N$ and a preference profile $R' \in \mathcal{W}^n$ as specified in the lemma. We prove that $f(R') = \{a\}$ by constructing a path of preference profiles from $R$ to $R'$ such that $a$ is the unique winner for every preference profile on this path. As this sequence is rather complicated, we discuss an example for the transformation after the proof. As in the proof of Lemma 4.2, we first focus on a voter $i$ who prefers $a$ uniquely the most in $R$. As in the last lemma, it holds that $f(R^+) = \{a\}$ for all preference profiles $R^+ = (R_{-i}, R_i^+)$ with $a \succ_i^+ b$ for all $b \in A \setminus \{a\}$. Hence, the lemma holds for this voter.
Next, consider a voter $j \neq i$ who does not prefer $i$ uniquely the most. Thus, we first determine the sets $U_j = \{x \in A \mid x \succ_j a\}$ containing all alternatives that are strictly preferred to $a$ in $R_j$, $L_j = \{x \in A \mid a \succ_j x\}$ containing all alternatives that are strictly less preferred than $a$ and $X_j = \{a \in A \mid x \succeq_j a \wedge a \succ'_j x\}$ containing the alternatives that are weakly preferred to $a$ in $R_j$ and that are strictly less preferred than $a$ in $R'_j$. Note that $X_j$ contains all alternatives $b \in A \setminus \{a\}$ with $a \sim_j b$ in $R$ as no alternative is allowed to be tied with $a$ in $R'_j$. With the help of these sets, we can formalize the required preference profiles. First, we let voter $j$ ensure that there are no alternatives $z \in A \setminus (X_j \cup \{a\})$, $y \in X_j$ such that $y \succeq_j z \succ_j a$. Formally, this leads to the following preference.

$$R_j^1 = R_j|_{U_j \setminus X_j} \ \cup \ R_j|_{L_j \cup X_j \cup \{a\}} \ \cup \ \Big((U_j \setminus X_j) \times (X_j \cup L_j \cup \{a\})\Big)$$

Note that we use the cross product $X \times Y$ between sets $X, Y$ to indicate that all alternatives in $X$ are strictly preferred to those in $Y$. The intuition in this step is to push the alternatives in $U_j \setminus X_j$ away from $a$ such that they are not affected by the following steps. The reason for this is that the alternatives $U_j \setminus X_j$ are still strictly preferred to $a$ in $R'_j$, whereas the alternatives in $X_j$ are less preferred than $a$ in $R'_j$. Furthermore, observe that the profile $R^1 = (R_{-j}, R_j^1)$ satisfies $f(R^1) = \{a\}$ because we derive it by an application of Lemma 4.2.

Subsequently, we let voter $i$ reorder his alternatives such that only alternative $a$ is preferred to the alternatives in $X_j$. Furthermore, we reorder the alternatives in $X_j$ according to $R_j$. Formally, this results in the following preference.

$$R_i^2 = \Big(\{a\} \times (A \setminus \{a\}\Big) \ \cup \ \Big(X_j \times (A \setminus (X_j \cup \{a\}))\Big) \ \cup \ R_j|_{X_j} \ \cup \ R_i|_{A\setminus(X_j\cup\{a\})}$$

It follows again from Lemma 4.2 that $f(R^2) = \{a\}$ where $R_2 = (R^1_{-i}, R_i^2)$. Thereafter, we use C2 to exchange the preferences of voter $i$ and $j$ over the alternatives in $X_j \cup \{a\}$. This means that voter $i$ prefers now the alternatives in $X_j$ the most and voter $j$ prefers $a$ strictly to the alternatives in $X_j$. Formally, this leads to the following preferences.

$$\begin{aligned} R_i^3 &= R_j^2|_{X_j\cup\{a\}} \ \cup R_i^2|_{A\setminus(X_j\cup\{a\})} \ \cup \ \Big((X_j \cup \{a\}) \times (A \setminus (X_j \cup \{a\}))\Big) \\ R_j^3 &= R_i^2|_{X_j\cup\{a\}} \ \cup \ \Big((U_j \setminus X_j) \times (X_j \cup L_j \cup \{a\})\Big) \ \cup \ \Big((X_j \cup \{a\}) \times L_j\Big) \end{aligned}$$

In the preference of voter $j$, only the preferences between $a$ and the alternatives in $X_j$ change because we have already ensured in the first step that the alternatives in $X_j$ are placed directly above $a$. Furthermore, as voter $i$ prefers only $a$ to the alternatives in $X_j$, it follows that $R^2$ and $R^3 = (R_{-\{i,j\}}, R_i^3, R_j^3)$ have the same majorities and therefore, $f(R^3) = f(R^2) = \{a\}$ as $f$ is in C2. Next, we let voter $i$ switch back to his original preference $R_i$, which leads to the profile $R^4 = (R^3_{-i}, R_i)$. It follows from the $\tilde{P}$-strategyproofness of $f$ that $f(R^4) = \{a\}$ as otherwise voter $i$ can $\tilde{P}$-manipulate by switching from $R^4$ back to $R^3$. Finally, note that the preferences involving $a$ in $R_j^4$ are equal to those in $R'_j$, which means that we can use Lemma 4.2 to derive $R'_j$ from $R_j^4$. This leads to the preference profile $R'$ and it holds that $f(R') = \{a\}$ as we deduce this profile with the help of Lemma 4.2. Thus, we have shown that an arbitrary voter can reinforce the unique winner $a$ of a $\tilde{P}$-strategyproof C2-function arbitrarily if there are no ties involving $a$ in his new preference. □

Note that even though Lemma 4.3 looks at first glance like a conditioned variant of set-monotonicity, it is not. While set-monotonicity also allows for going from $b \succ a$ to $b \sim a$, Lemma 4.3 does not allow such a modification. Even more, we strongly believe that there is no relation between the combination of $\tilde{P}$-strategyproofness and C2 and arbitrary variants of set-monotonicity.
As next point, we discuss an example for the proof of the last lemma with the help of the preference profiles in Figure 4.9. For this example, assume that $f$ denotes a $\tilde{P}$-strategyproof C2-function with $f(R) = \{a\}$. Furthermore, voter 2 wants to change his preference from $R_2 = d \succ c \succ a \sim b$ to $R'_2 = c \succ a \succ b \sim d$. Therefore, we first swap $d$ and $c$ in the preference of voter 2 to deduce $R^1$ because alternative $c$ is strictly preferred to $a$ in $R'_2$. Subsequently, voter 1 ensures that $b$ and $d$ are placed directly under $a$, which leads to $R^2$. It holds that $f(R^2) = f(R^1) = \{a\}$

$R$:

| 1 | 1 |
|---|---|
| $a$ | $d$ |
| $b$ | $c$ |
| $c$ | $a, b$ |
| $d$ | |

$R^1$:

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $b$ | $d$ |
| $c$ | $a, b$ |
| $d$ | |

$R^2$:

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $d$ | $d$ |
| $b$ | $a, b$ |
| $c$ | |

$R^3$:

| 1 | 1 |
|---|---|
| $d$ | $c$ |
| $b, a$ | $a$ |
| $c$ | $d$ |
| | $b$ |

$R^4$:

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $b$ | $a$ |
| $c$ | $d$ |
| $d$ | $b$ |

$R'$:

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $b$ | $a$ |
| $c$ | $b, d$ |
| $d$ | |

Figure 4.9: Preference profiles illustrating the proof of Lemma 4.3

because of Lemma 4.2. After this, we use C2 to exchange the preferences of voter 1 and 2 over $\{a, b, d\}$. Therefore, $f(R^3) = f(R^2) = \{a\}$. As fourth step, we use the $\tilde{P}$-strategyproofness of $f$ to let voter 1 switch back to his original preference $R_1$, which results in $R^4$. Finally, voter 2 applies Lemma 4.2 to order the unchosen alternatives according to $R'_2$ and consequently, we can deduce that $f(R') = \{a\}$. Thus, we have derived two powerful lemmas to manipulate preference profiles where a $\tilde{P}$-strategyproof C2-function returns a single winner and this winner is first-ranked by at least one voter. Note that the constraint that a voter must prefer the winner uniquely the most is very weak as $\tilde{P}$-strategyproofness allows to manipulate the preference of a voter such that he prefers the unique winner the most without changing the choice set. This is also one of the key ideas of the proof of the next theorem stating that a $\tilde{P}$-strategyproof and neutral C2-function only returns a single winner if it is a Condorcet winner.

**Theorem 4.5.** *Let $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ denote a social choice function that is neutral, $\tilde{P}$-strategyproof and in C2. If $f(R) = \{a\}$ for a preference profile $R \in \mathcal{W}^n$ and an alternative $a \in A$, then $a$ is the Condorcet winner in $R$.*

Proof: Let $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ denote a social choice function that is neutral, $\tilde{P}$-strategyproof and in C2. Furthermore, let $R \in \mathcal{W}^n$ denote a preference profile such that $f(R) = \{a\}$ for an arbitrary alternative $a \in A$ and assume for contradiction that $a$ is no Condorcet winner in $R$. We know already from Theorem 4.4 that $a$ is not Pareto-dominated by an alternative $b$. Furthermore, we can deduce from the neutrality of $f$ that there is no alternative $b$ such that every voter is indifferent between $a$ and $b$. Thus, there is an alternative $b$ with $n_{ba} \geq n_{ab} \geq 1$. This means that there is a voter $i \in N$ with $a \succ_i b$. We let this voter modify his preference by pushing $a$ to the top. This leads to a new profile $R^1$ which only differs from $R$ in the fact that $a$ is the uniquely preferred alternative of voter $i$. We can deduce that $f(R^1) = \{a\}$ as otherwise voter $i$ can $\tilde{P}$-manipulate by switching from $R^1$ to $R$. Furthermore, we can now use Lemma 4.2 and Lemma 4.3. Thus, we repeatedly use

$R$:

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $b$ | $b$ | $c$ | $c$ | $c$ |
| $a$ | $c$ | $a$ | $b$ | $a, b$ |
| $c$ | $a$ | $b$ | $a$ | |

$R^1$:

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $b$ | $b$ | $a$ | $c$ | $c$ |
| $a$ | $c$ | $c$ | $b$ | $a, b$ |
| $c$ | $a$ | $b$ | $a$ | |

$R^2$:

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $b$ | $b$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $b$ | $a$ | $a, b$ |
| $c$ | $c$ | $c$ | $c$ | |

$R^3$:

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $b$ | $a$ | $a, b$ |
| $c$ | $c$ | $c$ | $c$ | |

$R^4$:

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $b$ | $b$ | $b$ | $a$ | $c$ |
| $a$ | $a$ | $a$ | $b$ | $a, b$ |
| $c$ | $c$ | $c$ | $c$ | |

Figure 4.10: Preference profiles illustrating the proof of Theorem 4.5

Lemma 4.3 to go from $R^1$ to a profile $R^2$ in which every voter with $a \succ b$ prefers $a$ uniquely the most and $b$ uniquely the second most and every voter with $b \succ a$ prefers $b$ uniquely the most and $a$ uniquely second most; the preferences of voters with $a \sim b$ are not modified. As consequence of Lemma 4.3, it holds that $f(R^2) = \{a\}$.
Next, define the majority margin between $a$ and $b$ in $R^2$ as $k = n^2_{ba} - n^2_{ab} \geq 0$, i.e., there are $k$ more voters in $R^2$ who prefer $b$ strictly to $a$ than voters who prefer $a$ strictly to $b$. Subsequently, we use again Lemma 4.3 to let $k$ voters with $b \succ a$ in $R^2$ switch to $a \succ b$. This leads to a preference profile $R^3$ with $n^3_{ab} = n^2_{ba}$, $n^3_{ba} = n^2_{ab}$ and all other majorities stay the same as all voters either are indifferent between $a$ and $b$ or prefer both $a$ and $b$ strictly to all other alternatives. Furthermore, $f(R^3) = \{a\}$ as we derive this profile with the help of Lemma 4.3. Finally, we apply neutrality to rename $a$ to $b$ and vice versa, which leads to a preference profile $R^4$ with $f(R^4) = \{b\}$. As consequence of this renaming, we have that $n^4_{ab} = n^3_{ba} = n^2_{ab}$ and $n^4_{ba} = n^3_{ab} = n^2_{ba}$ and all other majorities are not affected. Thus, $R^4$ and $R^2$ have the same majorities but a different outcome, which contradicts that $f$ is in C2. Therefore, the initial assumption is wrong and $a$ is the Condorcet winner in $R$. □

As first remark, we discuss an example of the constructions used in the proof of the last theorem. Thus, consider the preference profiles shown in Figure 4.10 and assume that $f$ is a $\tilde{P}$-strategyproof and neutral C2-function with $f(R) = \{a\}$ even though $n_{ba} > n_{ab}$. As first step, we let the only voter with $a \succ b$, the third one, push $a$ uniquely to the top to derive $R^1$. It follows from $\tilde{P}$-strategyproofness that $f(R^1) = \{a\}$. Thereafter, we use Lemma 4.3 to deduce the profiles $R^2$ and $R^3$. Note that $n^2_{ba} - n^2_{ab} = 2$ and therefore, we let two voters in $R^2$ swap from $b \succ a$ to $a \succ b$. Finally, we apply neutrality to derive $R^4$ by renaming $a$ to $b$ and $b$ to $a$, which means that $f(R^4) = \{b\}$. Thus, $f(R^2) \neq f(R^4)$ even though the majorities of $R^4$ equals those of $R^2$. This contradicts that $f$ is in C2 and therefore, one of the initial assumptions is wrong.
Note that Theorem 4.5 provides a strong criterion on the preference profiles for which a $\tilde{P}$-strategyproof and neutral C2-function is allowed to return a single alternative. For instance, it follows from this theorem that the maximin rule and Borda's rule

are not $\tilde{P}$-strategyproof as they are neutral and in C2 and they can return a single winner that is no Condorcet winner. Even more, this result shows that there is no large gap between required conditions on $\tilde{P}$-strategyproof and neutral C2-functions in the weak domain and impossibility results. The reason for this is that the main result of [Bra11] states that every Condorcet extension defined on the weak domain is $\tilde{P}$-manipulable if there are sufficiently many voters. Thus, it is a requirement for $\tilde{P}$-strategyproofness that neutral C2-functions choose the Condorcet winner if they return a single winner. However, this SCF is not allowed to pick a Condorcet winner whenever it exists because of the result in [Bra11].

### 4.3.2 $\tilde{P}$-strategyproofness and Pareto-optimality

In this section, we discuss further requirements for the $\tilde{P}$-strategyproofness of social choice functions in C2. While the main focus of the last section is the combination of $\tilde{P}$-strategyproofness and neutrality, we analyze in this section the combination of $\tilde{P}$-strategyproofness and Pareto-optimality. We prove a strong condition that is necessary for $\tilde{P}$-strategyproof and Pareto-optimal C2-functions stating that every such function returns at least one of the most preferred alternatives of every voter for every preference profile. This condition strongly restricts $\tilde{P}$-strategyproof and Pareto-optimal C2-functions and has many important consequences. For instance, we can derive with the help of this theorem a simple proof for the impossibility of $\tilde{P}$-strategyproof, Pareto-optimal and pairwise SCFs discussed in [BSS].

As the proof of our main result is rather difficult, we break it down into multiple lemmas and give a short overview first. In the first two lemmas, we focus on the opposite setting assuming that there is a preference profile $R$ and a $\tilde{P}$-strategyproof and Pareto-optimal C2-function $f$ such that $f(R)$ does not contain any of the most preferred alternatives of voter $i$. We prove that this implies that there is a profile $R'$ such that $f(R') = \{a\}$ for an alternative $a \in f(R)$, a voter $i$ prefers $a$ uniquely the least and every other voter prefers $a$ uniquely the most. Thus, we know a lot about the structure of a profile for which a voter does not get any of his most preferred alternatives. This information is used in Lemma 4.6, Lemma 4.7 and Lemma 4.8 where we prove the with an induction on $n$ that every $\tilde{P}$-strategyproof and Pareto-optimal C2-function returns at least one of the most preferred alternatives of every voter if there are only three alternatives. Finally, we generalize this result from $m = 3$ alternatives inductively to an arbitrary larger number of alternatives as shown in Theorem 4.6. Note that not all lemmas need all axioms and therefore, the intermediate steps may also be used in different contexts.

It should be mentioned that our results require that $m \geq 3$ alternatives are available. The reason for this is that the majority rule satisfies C2, Pareto-optimality and $\tilde{P}$-strategyproofness if $m = 2$. This social choice function returns $\{a\}$ if $n_{ab} > n_{ba}$, $\{a, b\}$ if $n_{ab} = n_{ba}$, and $\{b\}$ otherwise. It is easy to see that the majority rule satisfies all required axioms and that its choice set may not include the most preferred alternative of a voter. Thus, $m \geq 3$ is indeed required for our main theorem. Furthermore, we assume in all lemmas that $n \geq 2$. The reason for this is that if there is only a single winner, then only his most preferred alternatives are Pareto-optimal, which means that every Pareto-optimal social choice function can only return a subset of his most preferred alternatives. Thus, our claim holds trivially in this case. Finally, it should be mentioned that we work in this section always with social choice function defined on a fixed electorate even though C2-functions in the weak domain are usually defined on a variable electorate. The reason for this is that a fixed electorate is easier to handle and still sufficiently strong. Furthermore, all results carry over to social choice functions defined on a variable electorate as these functions imply SCFs with fixed electorates.

As a first step of proving the main result of this section, we assume that there is a preference profile $R$ and a Pareto-optimal and $\tilde{P}$-strategyproof social choice function $f$ such that $f(R)$ does not contain any of voter $i$'s most preferred alternatives. We can deduce from this assumption that there is a preference profile $R'$ such that $f(R') = \{a\}$ and $a$ is not among the most preferred alternatives of voter $i$. The reason why we are interested in such a profile $R'$ is that $\tilde{P}$-strategyproofness becomes much more powerful if only a single winner is chosen.

**Lemma 4.4.** *Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that is defined on $m \geq 3$ alternatives and $n \geq 2$ voters and that satisfies Pareto-optimality and $\tilde{P}$-strategyproofness. Furthermore, assume that there is a preference profile $R \in \mathcal{W}^n$ and a voter $i \in N$ such that none of voter $i$'s most preferred alternatives are in $f(R)$. Then, there is a profile $R'$ such that $f(R') = \{a\}$ for an alternative $a \in f(R)$ and $a$ is not among the most preferred alternatives of voter $i$ in $R'$.*

Proof: Consider an arbitrary social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ and a preference profile $R \in \mathcal{W}^n$ as specified in the lemma. Furthermore, let $i$ denote a voter in $R$ such that none of his most preferred alternative are in $f(R)$. This means formally that the set $X = \{a \in A \mid \nexists b \in A : \ b \succ_i a\}$ containing all of voter $i$'s most preferred alternatives and $f(R)$ are disjoint, i.e., $f(R) \cap X = \emptyset$. In the sequel, we explain how to construct the profile $R'$ required by this lemma.

Therefore, we first iterate over all voters $j \in N \setminus \{i\}$ and let them manipulate one after another in the following way: Every voter $j \in N \setminus \{i\}$ switches from $R_j$ to a preference $R_j^1$ such that the alternatives in $f(R)$ are his most preferred alternatives, i.e., it holds in the new preference that $x \sim_j^1 y$ for all $x, y \in f(R)$ and $x \succ_j^1 y$ for all $x \in f(R), y \in A \setminus f(R)$. The alternatives in $A \setminus f(R)$ can be ordered arbitrarily. This leads to a new preference profile $R^1$ and it holds that $f(R^1) \subseteq f(R)$; otherwise, there is a voter $j \in N \setminus \{i\}$ such that a subset of $f(R)$ is chosen before his manipulations, but afterwards an alternative $c \in A \setminus f(R)$ is element of the choice set. This voter can $\tilde{P}$-manipulate by undoing the manipulation as he prefers the alternatives in $f(R)$ strictly to all other alternatives in $R_j^1$. Thus, $f(R^1) \subseteq f(R)$.

Furthermore, it follows from Pareto-optimality that only the alternatives of $f(R)$ that voter $i$ prefers the most can be in $f(R^1)$ as the remaining alternatives of $f(R)$ are Pareto-dominated. We use this fact to ensure that only a single alternative of $f(R)$ is chosen. Hence, let voter $i$ pick an arbitrary alternative $a \in f(R)$ and change his preference to $R_i^2$ in which he prefers the alternatives in $X$ the most, the alternative $a$ uniquely second most and the remaining alternatives are ordered arbitrarily. Formally, this preference is defined as follows.

$$R_i^2 = R_i|_X \cup R_i|_{A\setminus(X\cup\{a\})} \cup \{(x,y) \mid x \in X, y \in A\setminus X\} \cup \{(a,y) \mid y \in A\setminus(X\cup\{a\})\}$$

Note that we copy the preferences of $R_i$ on the alternatives in $A \setminus (X \cup \{a\})$ for simplicity as they are immaterial. This leads to a preference profile $R^2 = (R^1_{-i}, R_i^2)$ such that $f(R^2) = \{a\}$. The reason for this is the that every voter but $i$ prefers the alternatives in $f(R)$ the most and voter $i$ prefers $a \in f(R)$ strictly to all alternatives in $A \setminus (X \cup \{a\})$. Thus, $a$ Pareto-dominates all alternative in $A \setminus (X \cup \{a\})$ and therefore, $f(R^2) \subseteq X \cup \{a\}$. Furthermore, as $f(R^1) \subseteq f(R)$, it follows that no alternative in $X$ is chosen in $f(R^1)$. As the set $X$ contains the most preferred alternatives of voter $i$, it follows that none of these alternatives can be chosen for $R^2$ either; otherwise, voter $i$ can $\tilde{P}$-manipulate by switching from $R^1$ to $R^2$. Thus, it holds that $f(R^2) = \{a\}$, which means that we have found a preference profile that satisfies all requirements of the lemma. □

First, we illustrate the proof of Lemma 4.4 with the preference profiles shown in Figure 4.11. Note that the naming of the profiles in the example does not coincide with the naming in the proof of Lemma 4.4. Furthermore, assume that $f$ denotes a $\tilde{P}$-strategyproof and Pareto-optimal social choice function with $f(R^1) = \{a, d, e\}$. This implies that $\{b, c\} \cap f(R^1) = \emptyset$, which means that $f(R^1)$ contains no alternatives that are among the most preferred ones of voter 1. Thus, we can apply Lemma 4.4. The first step in the proof of this lemma is to let every voter except the first one manipulate such that they prefer the alternatives in $f(R^1) = \{a, d, e\}$ the most. This step results in the profile $R^2$ and it follows from $\tilde{P}$-strategyproofness and Pareto-optimality that $f(R^2) \subseteq \{a, e\}$. Next, we let voter 1 manipulate such that $a$ Pareto-dominates both $d$ and $e$ to derive $R^3$. Consequently, $f(R^3) = \{a\}$ as otherwise either a Pareto-dominated alternative is chosen or voter 1 can $\tilde{P}$-manipulate by switching from $R^2$ to $R^3$. Thus, we have now a single winner that is not among the most preferred alternatives of the first voter.
As consequence of Lemma 4.4, if a Pareto-optimal and $\tilde{P}$-strategyproof social choice function does not choose any of the most preferred alternatives of a voter, we can go to a profile with the same situation for this voter where only a single winner is chosen. This situation is desirable as $\tilde{P}$-strategyproofness becomes more powerful if a social choice function returns only a single winner for a profile. For instance, it becomes possible to use Lemma 4.2 and Lemma 4.3 if the social choice function additionally is in C2 and a voter prefers the unique winner the most.

| | 1 | 1 | 1 |
|---|---|---|---|
| | $b, c$ | $d$ | $a, d$ |
| $R^1$ : | $a, e$ | $a$ | $e$ |
| | $d$ | $e$ | $b, c$ |
| | | $b, c$ | |

| | 1 | 1 | 1 |
|---|---|---|---|
| | $b, c$ | $a, d, e$ | $a, d, e$ |
| $R^2$ : | $a, e$ | $b, c$ | $b, c$ |
| | $d$ | | |

| | 1 | 1 | 1 |
|---|---|---|---|
| | $b, c$ | $a, d, e$ | $a, d, e$ |
| $R^3$ : | $a$ | $b, c$ | $b, c$ |
| | $d$ | | |
| | $e$ | | |

Figure 4.11: Preference profiles illustrating the proof of Lemma 4.4

Next, we show that we can derive a profile $R''$ that is even worse for voter $i$: If a social choice function satisfies Pareto-optimality and $\tilde{P}$-strategyproofness, we can go from a profile with a unique winner that is not among the most preferred alternatives of voter $i$ to a profile where the unique winner is the worst alternative of voter $i$.

**Lemma 4.5.** *Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that is defined on $n \geq 2$ voters and $m \geq 3$ alternatives and that satisfies Pareto-optimality and $\tilde{P}$-strategyproofness. Furthermore, assume that there is a profile $R \in \mathcal{W}^n$ and a non-empty set of voters $I \subseteq N$ such that $f(R) = \{a\}$ even though there is an alternative $b \in A$ that is among the most preferred alternatives of every voter in $I$ and $b \succ_i a$ for all $i \in I$. Then, there is a profile $R' \in \mathcal{W}^n$ such that $f(R') = \{a\}$ and $a$ is the uniquely worst alternative of every voter in $I$ and the uniquely best alternative of every voter in $N \setminus I$.*

Proof: Consider an arbitrary SCF $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies Pareto-optimality and $\tilde{P}$-strategyproofness and a profile $R \in \mathcal{W}^n$ such that $f(R) = \{a\}$. Furthermore, assume that there is an alternative $b \in A$ and a non-empty set of voters $I \subseteq N$ such that $b \succ_i a$ for all $i \in I$ and $b$ is among the most preferred alternatives of every voter in $I$. Note that $I \neq N$ as otherwise $b$ Pareto-dominates $a$ and thus, $f(R) = \{a\}$ cannot be true. In the sequel we provide a sequence of profiles starting at $R$ and leading to a profile $R^2$ that satisfies all requirements of the lemma.
Thus, we first iterate over all voters $j \in N \setminus I$ and let them manipulate one after another in the following way: The currently considered voter $j$ modifies his preference such that $a$ is his uniquely most preferred alternative and $b$ is his uniquely second most preferred alternative in his new preference. The order of the remaining alternatives does not matter. After every voter $j \in N \setminus I$ manipulated this way, we derive a new preference profile $R^1$ for which $f(R^1) = \{a\}$ is true; otherwise, there is a voter $j$ such that $a$ is the unique winner before his modifications but not afterwards. This means that voter $j$ can $\tilde{P}$-manipulate by inverting his changes as he prefers $a$ uniquely the most after the modification.
Note that $b$ Pareto-dominates every alternative in $A \setminus \{a, b\}$ as every voter in $I$ prefers this alternative the most and every voter in $N \setminus I$ ranks $b$ strictly over every other alternative but $a$. We use this observation to derive the profile $R'$ by letting each voter $i \in I$ manipulate one after another in the following way: Every voter $i \in I$ switches from his current preference to a preference where he prefers $b$ uniquely the most and $a$ uniquely the least. The remaining alternatives can be

$R^3$ :

| 1 | 1 | 1 |
|---|---|---|
| $b, c$ | $a, d, e$ | $a, d, e$ |
| $a$ | $b, c$ | $b, c$ |
| $d$ | | |
| $e$ | | |

$R^4$ :

| 1 | 1 | 1 |
|---|---|---|
| $b, c$ | $a$ | $a$ |
| $a$ | $b$ | $b$ |
| $d$ | $d, e$ | $d, e$ |
| $e$ | $c$ | $c$ |

$R^5$ :

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $a$ | $a$ |
| $c$ | $b$ | $b$ |
| $d$ | $d, e$ | $d, e$ |
| $e$ | $c$ | $c$ |
| $a$ | | |

Figure 4.12: Preference profiles illustrating the proof of Lemma 4.5

ordered arbitrarily. This leads to a sequence of profiles that starts at $R^1$, ends at $R^2$ and two consecutive profiles differ only in the discussed modifications of a single voter. Note that the preference profile $R^2$ satisfies the required form of the lemma: Every voter in $I$ prefers $a$ uniquely the least and every voter in $N \setminus I$ prefers $a$ uniquely the most in this profile.

Thus, it only remains to show that $f(R^2) = \{a\}$. Assume for contradiction that this is not true. This means that there are two preference profiles $R^3$ and $R^4$ in the sequence of profiles starting at $R^1$ and ending at $R^2$ such that $f(R^3) = \{a\}$, $f(R^4) \neq \{a\}$ and $R^3_{-i} = R^4_{-i}$ for a voter $i \in I$. Note that every voter in $I$ prefers $b$ the most in $R^4$ and every voter in $N \setminus I$ prefers $b$ uniquely second most. This means that $f(R^4) \subseteq \{a, b\}$ because every other alternative is Pareto-dominated by $b$. Thus, $f(R^4) \neq \{a\}$ implies that $b \in f(R^4)$ and therefore, voter $i$ can $\tilde{P}$-manipulate as he prefers $b$ strictly to $a$. However, this contradicts that $f$ is $\tilde{P}$-strategyproof and therefore, the initial assumption is wrong. Hence, $f(R^4) = \{a\}$ and consequently, $f(R^2) = \{a\}$. This means that $R^2$ satisfies all requirements of the lemma. □

We illustrate the proof of this lemma by continuing the example in Figure 4.11. Thus, consider the profiles shown in Figure 4.12 and note that the naming of the profiles in the proof of Lemma 4.5 and the example differs. Furthermore, recall that $f$ denotes a Pareto-optimal and $\tilde{P}$-strategyproof social choice function and that $f(R^3) = \{a\}$. This means that none of the most preferred alternatives of the first voter are in $f(R^3)$. The first step of the proof of Lemma 4.5 is that every voter but the first one manipulates such that $a$ is his uniquely best alternative and $b$ his uniquely second best alternative. This leads to the profile $R^4$ for which $f(R^4) = \{a\}$ is true because of $\tilde{P}$-strategyproofness. Subsequently, voter 1 manipulates such that $b$ is his most preferred alternative and $a$ his least preferred one. This leads to the profile $R^5$ which satisfies that $f(R^5) = \{a\}$ because of Pareto-optimality and $\tilde{P}$-strategyproofness. Finally, we have derived a profile in which the unique winner is the unique worst alternative of a voter.

As consequence of Lemma 4.4 and Lemma 4.5, it follows that if a Pareto-optimal and $\tilde{P}$-strategyproof social choice function does not contain any of the most preferred alternatives of every voter for every preference profile $R$, then there is a profile $R'$ such that $f(R') = \{a\}$, a voter prefers $a$ the least and the remaining voters prefer $a$ the most. This is a very powerful setting as it follows from $\tilde{P}$-strategyproofness

that both the voters who prefer $a$ uniquely the most and the voter who prefers $a$ uniquely the least can modify their preferences without affecting the choice set.
We use this observation to derive a contradiction if the SCF $f$ additionally satisfies C2. However, it is difficult to find this contradiction if $n$ and $m$ are arbitrary and therefore, we focus on the simplified setting with $m = 3$ alternatives. Unfortunately, even with only three alternatives, we need auxiliary lemmas to derive the contradiction. Thus, we first discuss some properties of social choice functions that are $\tilde{P}$-strategyproof, Pareto-optimal and in C2. More precisely, we focus on the profiles that we derive by applying the previous lemmas, i.e., those where all voters but one agree on a uniquely most preferred alternative $a$, the remaining voter dislikes $a$ uniquely the most and $a$ is the unique winner. We show in the next lemma that if such a profile exists for a $\tilde{P}$-strategyproof and Pareto-optimal SCF, then it holds that $f(R) = \{x\}$ as soon as $n-1$ voters agree that $x$ is the uniquely best alternative.

**Lemma 4.6.** *Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \backslash \emptyset$ defined on $n \geq 2$ voters and $m = 3$ alternatives that satisfies Pareto-optimality and $\tilde{P}$-strategyproofness. If there is a preference profile $R \in \mathcal{W}^n$ such that $n-1$ voters agree on an alternative $a \in A$ as the uniquely best alternative, the last voter prefers $a$ uniquely the least and $f(R) = \{a\}$, then $f(R') = \{x\}$ for all alternatives $x \in A$ and preference profiles $R' \in \mathcal{W}^n$ in which $n-1$ voters agree that $x$ is the uniquely best alternative.*

Proof: Consider an arbitrary social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies all axioms required by the lemma. Furthermore, assume that there is a preference profile $R$ such that $f(R) = \{a\}$ for an arbitrary alternative $a \in A$, $n-1$ voters prefer $a$ uniquely the most and the last voter prefers $a$ uniquely the least. It follows from the $\tilde{P}$-strategyproofness of $f$ that $f(R') = \{a\}$ for all profiles $R' \in \mathcal{W}^n$ that result from $R$ if the voter who prefers $a$ the least changes his preference. Furthermore, all other voters can reorder their unchosen alternatives because of $\tilde{P}$-strategyproofness without affecting the choice set. Even more, we can rename the voters without changing the choice set because $f$ is in C2. This means that $f(R^*) = \{a\}$ for all profiles $R^*$ where $n-1$ voters agree that $a$ is the uniquely best alternative.
Next, we show that this result also holds if $n-1$ voters agree that another alternative $b \neq a$ is the uniquely best one. Note that it suffices to show that there is a profile $R' \in \mathcal{W}^n$ such that $f(R') = \{b\}$ and $b$ is the uniquely most preferred alternative of $n-1$ voters and the least preferred alternative of the last voter. If such a profile exists, we can simply apply the same arguments discussed in the last paragraph to show that $f(R^*) = \{b\}$ for all profiles $R^* \in \mathcal{W}^n$ where $n-1$ voters agree that $b$ is the uniquely best alternative. Thus, consider the profiles shown in Figure 4.13.
It follows from the explanations in the first paragraph that $f(R^1) = \{a\}$. Moreover, we let every voter but the first one manipulate one after another such that he prefers $a$ and $b$ the most. Note that $c$ cannot be chosen as consequence of these modifications; otherwise, there is a voter such that $c$ is not chosen before his modification, but it is afterwards. Hence, this voter can $\tilde{P}$-manipulate by undoing this modification, which contradicts the $\tilde{P}$-strategyproofness of $f$. Therefore, this process leads

| | 1 | $n-1$ |
|---|---|---|
| $R^1$ | $c \succ b \succ a$ | $a \succ b \succ c$ |
| $R^2$ | $c \succ b \succ a$ | $a \sim b \succ c$ |
| $R^3$ | $c \succ b \succ a$ | $b \succ c \succ a$ |
| $R^4$ | $c \succ a \succ b$ | $b \succ c \succ a$ |

Figure 4.13: Preference profiles used in the proof of Lemma 4.6

to the profile $R^2$ and $\tilde{P}$-strategyproofness implies that $f(R^2) \subseteq \{a, b\}$. Even more, $f(R^2) = \{b\}$ because $b$ Pareto-dominates $a$ in $R^2$.
Subsequently, we let every voter but the first one change his preference one after another from $a \sim b \succ c$ to $b \succ c \succ a$ in $R^2$. This leads to a preference profile $R^3$ and it follows from the $\tilde{P}$-strategyproofness of $f$ that $f(R^3) = \{b\}$. Furthermore, note that $c$ Pareto-dominates $a$ in $R^3$. We use this fact to derive the profile $R^4$ by letting voter 1 manipulate such that $b$ is his uniquely worst alternative. As $c$ still Pareto-dominates $a$ in $R^4$, it holds that $a \notin f(R^4)$ and it follows from $\tilde{P}$-strategyproofness that $c \notin f(R^4)$; otherwise, voter 1 can $\tilde{P}$-manipulate by switching from $R^3$ to $R^4$. Thus, $R^4$ satisfies all of our requirements: It holds that $f(R^4) = \{b\}$, $n-1$ voters prefer $b$ the most and the last voter prefers $b$ the least. □

Note that this lemma even holds if there are $n = 2$ voters. This implies that $f(R) \neq \{a\}$ for all $\tilde{P}$-strategyproof and Pareto-optimal C2-functions $f$ defined on two voters and preference profiles $R \in \mathcal{W}^2$ where one voter prefers $a$ the most and the second voter prefers $a$ the least. Otherwise, we can deduce from Lemma 4.6 that $f(R') = \{a\}$ and $f(R') = \{c\}$ are simultaneously true for all profiles $R'$ where $n-1 = 1$ voter prefers $a$ uniquely the most and $n-1 = 1$ voter prefers $c$ uniquely the most. This is a contradiction and therefore, it follows that the assumptions of Lemma 4.6 are never true for such social choice functions defined on $n = 2$ voters.
We want to derive a similar contradiction if there are more than two voters. Unfortunately, it becomes more difficult to deduce this result in this case. Therefore, we first provide another auxiliary lemma. We assume for this lemma that there is a social choice function $f$ defined on $n$ voters that satisfies $\tilde{P}$-strategyproofness, Pareto-optimality and C2 and for which there is a profile $R$ such that $f(R) = \{a\}$ even though at least two voters prefer another alternative $b$ the most. Based on this social choice function, we show how to construct a $\tilde{P}$-strategyproof and Pareto-optimal C2-function $g$ defined on $n-1$ voters for which a profile $R' \in \mathcal{W}^{n-1}$ exists such that $g(R') = \{a\}$ even though a voter prefers another alternative $b$ the most.

**Lemma 4.7.** *Consider a Pareto-optimal and $\tilde{P}$-strategyproof C2-function $f : \mathcal{W}^n \mapsto 2^A$ defined on $n \geq 3$ voters and $m = 3$ alternatives. Furthermore, assume that there is a profile $R \in \mathcal{W}^n$ such that $f(R) = \{a\}$ even though at least two voters prefer another alternative $b$ uniquely the most. Then, there is a $\tilde{P}$-strategyproof and Pareto-optimal C2-function $g : \mathcal{W}^{n-1} \mapsto 2^A \setminus \emptyset$ for which a preference profile $R' \in \mathcal{W}^{n-1}$ exists such that $b \notin g(R')$ even though a voters prefers $b$ uniquely the most.*

| | 1 | 1 | $n-2$ |
|---|---|---|---|
| $R^1$ | $b \succ c \succ a$ | $b \succ c \succ a$ | $a \succ c \succ b$ |
| $R^2$ | $b \sim c \sim a$ | $b \succ c \succ a$ | $a \succ c \succ b$ |
| $R^3$ | - | $b \succ c \succ a$ | $a \succ c \succ b$ |

Figure 4.14: Preference profiles used in the proof of Lemma 4.7

Proof: Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that is defined on $m = 3$ alternatives and an arbitrary number of voters $n \geq 3$ and that satisfies all axioms specified in the lemma. Furthermore, assume that there is a profile $R^* \in \mathcal{W}^n$ such that $f(R^*) = \{a\}$ even though there are at least two voters who prefer another alternative $b \neq a$ uniquely the most. We construct with the help of $f$ a social choice function $g : \mathcal{W}^{n-1} \mapsto 2^A \setminus \emptyset$ that satisfies the axioms required by the lemma. This SCF $g$ is defined as follows: Given a preference profile $R \in \mathcal{W}^{n-1}$, this SCF adds a new voter $i^*$ who is indifferent between all alternatives to obtain the profile $R' = (R, R_{i^*})$. Then, $g(R) = f(R')$. Clearly, $g$ is Pareto-optimal, $\tilde{P}$-strategyproof and in C2; otherwise, $f$ violates these properties, too.
Thus, it only remains to show that there is a profile $R' \in \mathcal{W}^{n-1}$ such that $b \notin g(R')$ even though a voter prefers $b$ uniquely the most. Therefore, recall that there is a profile $R \in \mathcal{W}^n$ such that $f(R) = \{a\}$ even though two voters prefer $b$ uniquely the most. It follows from Lemma 4.5 that there is a preference profile $R'' \in \mathcal{W}^n$ such that $f(R'') = \{a\}$ even though there are two voters who prefer $b$ uniquely the most and $a$ uniquely the least and every other voter prefers $a$ uniquely the most. Furthermore, we can also reorder unchosen alternatives with the help of Lemma 4.2 and voters because of C2 and therefore, we can deduce that $f(R^1) = \{a\}$ where $R^1$ is shown in Figure 4.14. Next, we let the first voter in $R^1$ manipulate such that he is indifferent between all alternatives. This leads to the profile $R^2$ and it holds that $f(R^2) = \{a\}$; otherwise, voter 1 can $\tilde{P}$-manipulate by switching from $R^1$ to $R^2$. Finally, consider the profile $R^3$ defined on $n-1$ voters and observe that $g(R^3) = f(R^2)$. Thus, it follows that $g(R^3) = \{a\}$ even though a voter prefers $b$ uniquely the most. Consequently, the SCF $g$ satisfies indeed all required axioms. □

Note that we use Lemma 4.7 is in the inverse direction, i.e., we show that there is no $\tilde{P}$-strategyproof and Pareto-optimal C2-function $g$ defined on $n-1$ voters for which a profile $R$ exists such that $f(R) = \{a\}$ if a voter prefers another alternative uniquely the most. It follows from this statement and Lemma 4.7 that there is no social choice function $f$ defined on $n$ voters that satisfies the same axioms as $g$ and that returns $a$ as unique winner even though two voters agree that another alternative is the uniquely best one. This allows us to deduce the contradiction inductively. We assume that no such social choice function $g$ defined on $n-1$ voters exists. Additionally, we prove that if there is a $\tilde{P}$-strategyproof and Pareto-optimal C2-function $f$ defined on $n$ voters and a profile $R$ such that such that $f(R) = \{a\}$

even though two voters prefer another alternative $b$ uniquely the most, there is a profile $R'$ such that $f(R') = \{b\}$ and a voter prefers another alternative $c$ uniquely the most. This contradicts the induction hypothesis because of Lemma 4.7. Even more, this approach is also the main idea for proving the following lemma.

**Lemma 4.8.** *Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ defined on $m = 3$ alternatives and $n \geq 2$ voters that satisfies $\tilde{P}$-strategyproofness, Pareto-optimality and is in C2. For every preference profile $R \in \mathcal{W}^n$ and every voter $i \in N$, it holds that $f(R)$ contains at least one of voter $i$'s most preferred alternatives.*

Proof: We prove this lemma by an induction on the number of voters $n$. Thus, we first focus on the base case $n = 2$, i.e., we consider an arbitrary $\tilde{P}$-strategyproof and Pareto-optimal C2-function $f : \mathcal{W}^2 \mapsto 2^A \setminus \emptyset$. Furthermore, assume for contradiction that there is a profile $R \in \mathcal{W}^2$ and a voter $i \in N$ such that $f(R)$ does not contain any of voter $i$'s most preferred alternatives. These assumptions imply that we can use Lemma 4.4 and Lemma 4.5 to construct a preference profile $R'$ such that $f(R') = \{a\}$ for an alternative $a \in f(R)$, voter $i$ prefers $a$ uniquely the least and the other voter prefers $a$ uniquely the most. Consequently, we can use Lemma 4.6 which implies that $f(R) = \{x\}$ for all preference profiles in which $n - 1 = 1$ voter prefers an arbitrary alternative $x \in A$ the most. However, this means that $f(R) = \{a\}$ and $f(R) = \{b\}$ are simultaneously true if the first voter prefers $a$ the most and the second voter prefers $b$ the most in $R$. This is a contradiction and therefore, it holds for all $\tilde{P}$-strategyproof and Pareto-optimal C2-functions $f$, all preference profiles $R \in \mathcal{W}^2$ and all voters $i \in N$ that $f(R)$ contains at least one of the most preferred alternatives in $R_i$.

Next, we focus on the induction step. Thus, assume that this lemma holds for every $\tilde{P}$-strategyproof and Pareto-optimal C2-function $g$ defined on $n-1$ voters and $m = 3$ alternatives. Furthermore, consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies the same axioms as $g$ and assume for contradiction that there is a profile $R \in \mathcal{W}^n$ such that $f(R)$ does not contain any of voter $i$'s most preferred alternatives. Thus, we can use Lemmas 4.4 to 4.6 to derive that $f(R^1) = \{a\}$ for an alternative $a \in f(R)$ where $R^1$ is shown in Figure 4.15. We derive from the profiles shown in this figure that $f(R^{10}) = \{b\}$ even though two voters prefer $c$ uniquely the most. This means that we can use Lemma 4.7 to construct a social choice function $g$ defined on $n - 1$ voters that satisfies the same axioms as $f$ and for which there is a profile $R' \in \mathcal{W}^{n-1}$ such that $f(R') = \{b\}$ even though a voter prefers $c$ uniquely the most. This contradicts the induction hypothesis and therefore, $f$ cannot exist.

It remains to prove that $f(R^{10}) = \{b\}$. For explaining this claim, we use the preference profiles shown in Figure 4.15. Note that this table only summarizes all required profiles as the second column is associated with an arbitrary number of voters. As already explained, it holds that $f(R^1) = \{a\}$. Next, we derive the profile $R^2$ by letting the last voter manipulate. As $a$ is his least preferred alternative in $R^1$, it follows that $f(R^2) = \{a\}$ because of $\tilde{P}$-strategyproofness. As consequence of this modification $a$ Pareto-dominates $b$ in $R^2$. Thereafter, we let voter $n - 1$ switch

| | 1 | $n-3$ | 1 | 1 |
|---|---|---|---|---|
| $R^1$ | $a \succ c \succ b$ | $a \succ b \succ c$ | $a \succ b \succ c$ | $c \succ b \succ a$ |
| $R^2$ | $a \succ c \succ b$ | $a \succ b \succ c$ | $a \succ b \succ c$ | $c \succ a \sim b$ |
| $R^3$ | $a \succ c \succ b$ | $a \succ b \succ c$ | $a \sim b \succ c$ | $c \succ a \sim b$ |
| $R^4$ | $a \succ c \succ b$ | $a \sim b \succ c$ | $a \sim b \succ c$ | $c \succ b \sim a$ |
| $R^5$ | $a \succ c \succ b$ | $a \sim b \succ c$ | $a \sim b \succ c$ | $c \succ b \succ a$ |
| $R^6$ | $a \succ c \succ b$ | $a \sim b \succ c$ | $b \succ a \succ c$ | $c \succ a \sim b$ |
| $R^7$ | $a \succ c \succ b$ | $a \sim b \succ c$ | $b \succ a \succ c$ | $c \succ b \succ a$ |
| $R^8$ | $a \succ c \succ b$ | $b \succ a \succ c$ | $b \succ a \succ c$ | $c \succ b \succ a$ |
| $R^9$ | $a \succ c \succ b$ | $b \succ a \succ c$ | $b \succ a \succ c$ | $b \succ c \succ a$ |
| $R^{10}$ | $c \succ a \succ b$ | $b \succ a \succ c$ | $b \succ a \succ c$ | $c \succ b \succ a$ |

Figure 4.15: Preference profiles used in the proof of Lemma 4.8

from $a \succ b \succ c$ to $a \sim b \succ c$ to derive the profile $R^3$. Observe that $a$ still Pareto-dominates $b$ in $R^3$ and therefore, $b \notin f(R^3)$. Furthermore, $c \notin f(R^3)$ as otherwise voter $n-1$ can $\tilde{P}$-manipulate by switching from $R^3$ to $R^2$. Thus, $f(R^3) = \{a\}$. Thereafter, we let the other $n-3$ voters that have the same preference as voter $n-1$ manipulate one after another in the same way. This leads to the profile $R^4$ and it follows from the same argument as for $R^3$ that neither $b$ nor $c$ can be chosen, i.e., $f(R^4) = \{a\}$.

After that, we let voter $n$ modify his preference by making $a$ his least preferred alternative. Note that we lose in this step the information about the exact choice set. Nevertheless, it follows from $\tilde{P}$-strategyproofness that $c \notin f(R^5)$ as voter $n$ can $\tilde{P}$-manipulate otherwise. Subsequently, we use C2 to exchange the preferences of voter $n$ and $n-1$ over $b$ and $c$ to derive the profile $R^6$. Consequently, $f(R^5) = f(R^6)$ which means that $c$ is still not chosen. Furthermore, as voter $n$ is indifferent between $a$ and $b$, and $c$ is his uniquely most preferred alternative in $R^6$, it follows again from $\tilde{P}$-strategyproofness that he can switch from $a \sim b$ to $b \succ a$ without making $c$ win. This leads to the profile $R^7$ which satisfies that $c \notin f(R^7)$. It is easy to see that we can repeat this process for all $n-3$ voters who are indifferent between $a$ and $b$ in $R^7$ without making $c$ win. This leads to the profile $R^8$ and to the fact that $c \notin f(R^8)$. Furthermore, note that $f(R^9) = \{b\}$ because of Lemma 4.6. This implies that $f(R^8) \notin \{\{a\}, \{a, b\}\}$ as otherwise voter $n$ can $\tilde{P}$-manipulate by switching from $R^8$ to $R^9$. Thus, we can conclude that $f(R^8) = \{b\}$. This observation leads to the profile $R^{10}$ by letting voter 1 switch from $a \succ c \succ b$ to $c \succ a \succ b$. As $b$ is his uniquely least preferred alternative in $R^8$, it follows from $\tilde{P}$-strategyproofness that $f(R^{10}) = \{b\}$. Hence, this proves the claim and therefore, we can derive with the help of Lemma 4.7 a contradiction to the induction hypothesis. Hence, the initial assumption is wrong and it holds for all $\tilde{P}$-strategyproof and Pareto-optimal C2-functions $f$, preference profiles $R \in \mathcal{W}^n$ and voters $i \in N$ that $f(R)$ contains at least one of the most preferred alternatives in $R_i$. $\square$

Note that this lemma provides a strong requirement for $\tilde{P}$-strategyproof and Pareto-optimal social choice functions in C2 if there are only $m = 3$ alternatives. However, only considering social choice functions defined on three alternatives is very restrictive. Therefore, we generalize this result to an arbitrary larger number of alternatives. We use an induction on $m$ for this task for which the base case is covered by Lemma 4.8. More precisely, we assume that there is a $\tilde{P}$-strategyproof and Pareto-optimal C2-function $f$ defined on $m > 3$ alternatives and $n$ voters for which a profile $R$ exists such that $b \notin f(R)$ even though a voter prefers $b$ the most. These assumptions imply that there is a social choice function $g$ defined on $m - 1$ alternatives that satisfies all these axioms, too. However, this is a contradiction to the induction hypothesis stating that no such function exists. This idea leads to our main theorem stating that the choice set of every $\tilde{P}$-strategyproof and Pareto-optimal C2-function contains for every voter and every preference profile at least one of his most preferred alternatives

**Theorem 4.6.** *Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ defined on $m \geq 3$ alternatives that satisfies $\tilde{P}$-strategyproofness, Pareto-optimality and C2. For every preference profile $R \in \mathcal{W}^n$ and every voter $i \in N$, it holds that $f(R)$ contains at least one of voter $i$'s most preferred alternatives.*

Proof: First note that if $n = 1$, the theorem is trivially true as only the most preferred alternatives of the single voter are Pareto-optimal. Thus, every Pareto-optimal social choice function chooses a subset of the most preferred alternatives of this voter, which proves the theorem in this case.
Next, we focus on the more complicated case that $n \geq 2$. In this case, we prove the theorem by an induction on the number of alternatives $m$. Note that the base case $m = 3$ is covered in Lemma 4.8 and therefore, we only focus on the induction step. Hence, assume that the theorem holds for all $\tilde{P}$-strategyproof and Pareto-optimal C2-functions $g$ that are defined on $n$ voters and $m - 1 \geq 3$ alternatives. Furthermore, consider an arbitrary $\tilde{P}$-strategyproof and Pareto-optimal C2-function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ defined on $n$ voters and $m$ alternatives. We prove that the intersection of $f(R)$ and the set of most preferred alternatives of voter $i$ in $R$ is non-empty for every voter $i \in N$ and every preference profile $R \in \mathcal{W}^n$.
For this reason, assume for contradiction that there is a profile $R$ such that $f(R)$ does not contain any of the most preferred alternatives of a voter. We derive from this assumption a contradiction to the induction hypothesis by constructing a social choice function $g : \mathcal{W}^n \mapsto 2^{A'}$ with $|A'| = m - 1$ that satisfies all required axioms and for which a preference profile $R^*$ exists such that $b \notin g(R^*)$ even though a voter prefers $b$ uniquely the most. This is a contradiction to the induction hypothesis because the theorem implies that an alternative is chosen if a voter prefers it uniquely the most. The social choice function $g$ is defined as follows: Given a preference profile $R$, it appends a new alternative $c \notin A'$ as the uniquely least preferred alternative to the preference of every voter. This leads to a new preference profile $R'$ defined on $m$ alternatives. Finally, we set $g(R) = f(R')$. Note that $g$ is well-defined as a

is Pareto-dominated in $R'$ by every alternative and therefore, it is not chosen by $f$. Furthermore, it is obvious that $g$ inherits $\tilde{P}$-strategyproofness, Pareto-optimality and C2 from $f$; otherwise, $f$ violates these properties itself.

Thus, the only point left to prove is that there is also a profile $R^*$ defined on $m-1$ alternatives such that $b \notin g(R^*)$ for an alternative $b \in A'$ that is uniquely most preferred by at least one voter. Therefore, we discuss how to construct a profile $R^+$ defined on $m$ alternatives such that $b \notin f(R^+)$ for an alternative $b \in A$, a voter prefers $b$ uniquely the most and there is an alternative $c \in A$ that is uniquely least preferred by every voter. Then, it follows that $R^* = R^+|_{A\setminus\{c\}}$ which implies that $g(R^*) = f(R^+)$. Thus, recall that, by assumption, there is a preference profile $R$ such that $b \notin f(R)$ and a voter prefers $b$ uniquely the most. We can apply Lemma 4.4 and Lemma 4.5 to this profile to deduce that there is a preference profile $R^1$ such that $f(R^1) = \{a\}$ for an alternative $a \in f(R)$, a single voter prefers $b$ uniquely the most and $a$ uniquely the least and every other voter prefers $a$ uniquely the most. After that, we use Lemma 4.3 to reinforce $a$ in the preference of the voter who prefers $b$ the most such that there is an alternative $c$ with $a \succ c$. Finally, we use Lemma 4.2 to reorder the unchosen alternatives in the preferences of the voters who prefer $a$ uniquely the most to ensure that every voter prefers $c$ uniquely the least. Note that the last two steps do not affect the choice set and therefore, we have found a profile $R^+$ such that $b \notin f(R^+) = g(R^*)$. Hence, if there is a $\tilde{P}$-strategyproof and Pareto-optimal and C2-function $f$ defined on $n$ voters and $m > 3$ alternatives for which there is a profile $R$ such that $f(R)$ does not contain any of the most preferred alternatives of a voter, then there is also such a social choice function $g$ defined on $n$ voters and $m-1$ alternatives. This contradicts the induction hypothesis and therefore, all $\tilde{P}$-strategyproof and Pareto-optimal C2-functions return for all preference profiles $R \in \mathcal{W}^n$ and all voters $i \in N$ an alternative that is among the most preferred ones in $R_i$. □

It should be stressed here that Theorem 4.6 provides a very strong restriction on $\tilde{P}$-strategyproof and Pareto-optimal C2-functions. A consequence of this theorem is that the choice sets of such functions are usually rather large. Even more, there is no large gap between satisfying this theorem and violating Pareto-optimality. Thus, Theorem 4.6 implies that there can only be few $\tilde{P}$-strategyproof and Pareto-optimal C2-functions. Furthermore, it should be stressed here that this theorem also holds if we allow social choice functions defined on variable electorates. The reason for this is that a SCF with variable electorate implies SCFs with fixed electorates for which Theorem 4.6 can be applied.

Note that the main theorem has many important consequences and is often used in subsequent sections. For instance, we can use it to reproof theorem 2 in [BSS] by hand. This theorem states that there is no $\tilde{P}$-strategyproof, Pareto-optimal and pairwise social choice function in the weak domain if $m \geq 3$ and $n \geq 3$. Recall that a social choice function $f$ is pairwise if $f(R) = f(R')$ for all preference profiles $R, R'$ with $n_{ab} - n_{ba} = n'_{ab} - n'_{ba}$ for all $a, b \in A$. It follows immediately that pair-

$R^1$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a, b$ | $a, b$ |
| $b$ | $c$ | $c$ |
| $c$ | | |

$R^2$ :

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $b$ | $a$ |
| $c$ | $c$ | $c$ |

Figure 4.16: Preference profiles used for re-proving theorem 2 of [BSS]

wise SCFs are in C2 and therefore, Theorem 4.6 can be used for $\tilde{P}$-strategyproof, Pareto-optimal and pairwise social choice functions. However, it is easy to construct a preference profile $R$ for such social choice functions such that the choice set does not contain any of the most preferred alternatives of a voter. For example, consider the profiles $R^1$ and $R^2$ shown in Figure 4.16. Note that $a$ Pareto-dominates $b$ in $R^1$ and therefore, it cannot be chosen by a Pareto-optimal social choice function. Furthermore, the majority margins $n_{ab} - n_{ba}$ are equal in $R^1$ and $R^2$ for all $a, b \in A$. Thus, $b$ is not chosen by a Pareto-optimal and pairwise SCF even though the last voter prefers this alternative the most. This contradicts Theorem 4.6 and therefore, no pairwise social choice function satisfies both Pareto-optimality and $\tilde{P}$-strategyproofness.

Finally, it should be mentioned that we can deduce a result similar to Theorem 4.6 from the proof of Theorem 3.12 for $\tilde{P}$-strategyproof, Pareto-optimal and rank-based social choice functions in the weak domain. Thus, it seems reasonable to conjecture that all $\tilde{P}$-strategyproof, Pareto-optimal and anonymous social choice functions return for every preference profile $R \in \mathcal{W}^n$ and voter $i \in N$ an alternative that is among the most preferred ones in $R_i$ if there are sufficiently many voters and alternatives. Unfortunately, we can disprove this conjecture. For showing this claim, we construct a social choice function that refines the Pareto-rule. Therefore, note that the Pareto-rule is defined by a dominance relation. In particular, we can define the function $t_1(R) = \{(a, b) \in A^2 \mid n_{ab} > 0 \wedge n_{ba} = 0\}$ that computes all Pareto-dominance relations for the input profile $R$ and define the Pareto-rule as $PO(R) = \{a \in A \mid \nexists b \in A : (b, a) \in t_1(R)\}$. Our goal is to find a stronger dominance relation than Pareto-dominance and use it to define a $\tilde{P}$-strategyproof social choice function. Therefore, consider the following function.

$$t_2(R) = \{(a, b) \in A^2 \mid n_{ab} = n - 1 \wedge n_{ba} = 1 \wedge a \text{ is first-ranked by } n - 1 \text{ voters}\}$$

This function computes a dominance relation for an input profile $R$ that prefers an alternative $a$ to an alternative $b$ if $n - 1$ voters prefer $a$ the most and $a$ strictly to $b$ and the last voter prefers $b$ strictly to $a$. Finally, we set $t(R) = t_1(R) \cup t_2(R)$ and define the SCF $f^*(R) = \{a \in A \mid \nexists b \in A : (b, a) \in t(R)\}$.

First of all, we prove that $f^*$ is well-defined, i.e., it returns always a non-empty choice set. Therefore, we show that $t(R)$ is for every preference profile $R' \in \mathcal{W}^n$ asymmetric and transitive, which implies that there are always maximal elements.

**Lemma 4.9.** *The dominance relation $t(R)$ is asymmetric and transitive for all preference profiles $R \in \mathcal{W}^n$ if $n \geq 3$.*

Proof: Consider an arbitrary preference profile $R \in \mathcal{W}^n$. First, we prove that $t(R)$ is asymmetric. Thus, we assume that $(a, b) \in t(R)$ and show that $(b, a) \notin t(R)$. First note that if $a = b$, then $(a, b) \notin t(R)$ as both $t_1(R)$ and $t_2(R)$ demand that $n_{ab} > 0$. Furthermore, it is easy to see that both $t_1(R)$ and $t_2(R)$ are asymmetric if $n \geq 3$. Thus, it only remains to show that if $(a, b) \in t_1(R)$, then $(b, a) \notin t_2(R)$ and if $(a, b) \in t_2(R)$, then $(b, a) \notin t_1(R)$. Therefore, assume first that $(a, b) \in t_1(R)$. This means that $n_{ba} = 0$, which contradicts that $n_{ba} = n - 1$ required for $(b, a) \in t_2(R)$. Furthermore, if $(a, b) \in t_2(R)$, then $n_{ab} = n - 1$ and $n_{ba} = 1$. This means that $b$ does not Pareto-dominate $a$ as $n_{ba} > 0$ and therefore, $(b, a) \notin t_1(R)$. Consequently, $t(R)$ is indeed asymmetric.

Next, we prove that $t(R)$ is also transitive. Thus, consider three arbitrary alternatives $a, b, c \in A$ with $(a, b) \in t(R)$ and $(b, c) \in t(R)$. We have to show that $(a, c) \in t(R)$ in order to prove the transitivity of $t(R)$. For this proof, we make a case distinction with respect to $(a, b) \in t_1(R)$ or $(a, b) \in t_2(R)$ and $(b, c) \in t_1(R)$ or $(b, c) \in t_2(R)$. First, assume that $(a, b) \in t_1(R)$ and $(b, c) \in t_1(R)$. This means that $a$ Pareto-dominates $b$ and $b$ Pareto-dominates $c$. As Pareto-dominance is transitive, it follows that $(a, c) \in t_1(R) \subseteq t(R)$. As second case, assume that $(a, b) \in t_1(R)$ and $(b, c) \in t_2(R)$. This means that there are $n-1$ voters who prefer $b$ the most and who submit $b \succ c$. As $a$ Pareto-dominates $b$, it follows that each of these $n - 1$ voters prefers $a$ also the most and that they submit $a \succ c$. Thus, either $(a, c) \in t_1(R)$ if the last voter prefers $a$ weakly to $c$, or $(a, c) \in t_2(R)$ if the last voter prefers $c$ strictly to $a$. Both cases imply that $(a, c) \in t(R)$. Next, assume that $(a, b) \in t_2(R)$ and $(b, c) \in t_1(R)$. Then, every voter prefers $b$ weakly to $c$ and there are $n-1$ voters who prefer $a$ strictly to $b$ and prefer $a$ the most. Consequently, these $n - 1$ voters prefer $a$ also strictly to $c$. This means that $(a, c) \in t_1(R)$ if the last voter prefers $a$ weakly to $c$, or $(a, c) \in t_2(R)$ if the last voter prefers $c$ strictly to $a$. Thus, it also holds in this case that $(a, c) \in t(R)$. Finally, assume that $(a, b) \in t_2(R)$ and $(b, c) \in t_2(R)$. This assumption is a contradiction as it means that $n - 1$ voters prefer $b$ the most and $n_{ab} = n - 1$, which cannot be simultaneously true if $n \geq 3$. Thus, this case is impossible and we can deduce for all other cases that $(a, c) \in t(R)$. This shows that $t(R)$ is indeed transitive. □

Note that Lemma 4.9 has significant consequences for the social choice function $f^*(R) = \{a \in A \mid \nexists b \in A : (b, a) \in t(R)\}$. Most importantly, it implies that $f^*$ is well-defined. Even more, it follows from the transitivity of $t(R)$ that if $a \notin f(R)$, then there is an alternative $b \in f(R)$ with $(b, a) \in t(R)$. This observation plays an important role when we prove the $\tilde{P}$-strategyproofness of $f^*$. However, before we discuss this axiom, it should be mentioned that $f^*$ is clearly anonymous, neutral and Pareto-optimal. Furthermore, $f^*(R)$ does not contain any of the most preferred alternatives of a voter if $n - 1$ voters agree that $a$ is the uniquely best alternative and a single voter prefers $b$ uniquely the most. Thus, it only remains to show that $f^*$ is indeed $\tilde{P}$-strategyproof to complete our counter example.

**Lemma 4.10.** *The social choice function $f^* : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ is $\tilde{P}$-strategyproof if $n \geq 3$ and $m \geq 3$.*

Proof: Assume for contradiction that $f^*$ is $\tilde{P}$-manipulable. Therefore, there are preference profiles $R, R' \in \mathcal{W}^n$ and a voter $i \in N$ such that $R_{-i} = R'_{-i}$ and $f^*(R')\ \tilde{P}_i\ f^*(R)$. For deriving a contradiction to this assumption, we make a case distinction with respect to $f^*(R)$. Therefore, let $X_i = \{a \in A \mid \forall b \in A : a \succeq_i b\}$ denote the set of voter $i$'s most preferred alternatives in $R$.
First, assume that $f^*(R) \cap X_i \neq \emptyset$, i.e., some of voter $i$'s most preferred alternatives are chosen. This means that voter $i$ can only $\tilde{P}$-manipulate if $f^*(R') \subseteq X_i$ and there is an alternative $a \in f^*(R) \setminus X_i$. As $a \in f^*(R)$, it follows that there is no $b \in X_i$ with $(b, a) \in t(R)$. Furthermore, as $t(R')$ is transitive and $a \notin f(R') \subseteq X_i$, we can deduce that there is a $c \in X_i$ such that $(c, a) \in t(R')$. However, it is straightforward that $(c, a) \notin t_1(R')$ as $c \succ_i a$ and $(c, a) \notin t_1(R)$. These observations imply that there is a voter $j \neq i$ with $a \succ_j c$ and thus, $c$ cannot Pareto-dominate $a$ in $R'$. We can similarly reason that $(c, a) \notin t_2(R')$: As $(c, a) \notin t_2(R)$ and $(c, a) \notin t_1(R)$, it follows that there are not $n-1$ voters in $R$ who rank $c$ first, $n_{ca} < n-1$ or $n_{ac} > 1$. None of these conditions can be fixed if voter $i$ reorders his alternatives because he already prefers $c$ the most and submits $c \succ_i a$. This implies that $(c, a) \notin t_2(R')$ and therefore, $(c, a) \notin t(R')$ for every $c \in X_i$. It follows from the transitivity of $t(R')$ that either $a \in f^*(R')$ or there is an alternative $b \in f^*(R')$ with $a \succeq_i b$. Consequently, $f^*(R') \subseteq X_i$ cannot be true, which means that $f^*$ is $\tilde{P}$-strategyproof in this case.
Next, assume that $f^*(R) \cap X_i = \emptyset$. This means that there is a tuple $(a, b) \in t_2(R)$ with $a \notin X_i$. Otherwise, there is a Pareto-optimal alternative $c \in X_i \cap f^*(R)$. Thus, the set $B = \{a \in A \mid \forall b \in A, j \in N \setminus \{i\} : a \succeq_j b\}$ containing all alternatives that are first-ranked by all voters in $N \setminus \{i\}$ is non-empty. Observe that for every alternative $y \in A \setminus B$, there is a voter $j \in N \setminus \{i\}$ with $x \succ_i y$ for all $x \in B$; otherwise, $y$ is in $B$, too. Furthermore, let $c$ denote an alternative in $B$ that voter $i$ prefers the most, i.e., $c \succeq_i x$ for all $x \in B$. It follows for all $x \in A$ that $(x, c) \notin t_2(R)$ as $n_{xc} \leq 1$ because $n-1$ voters prefer $c$ the most. Even more, $(x, c) \notin t_1(R)$ for all $x \in A$ because only alternatives in $B$ can Pareto-dominate $c$. However, all voters $j \in N \setminus \{i\}$ are indifferent between all alternatives in $B$ and $c \succeq_i x$ for all $x \in B$. This means that $c \in f^*(R)$. Even more, every alternative $x \in A$ with $c \succ_i x$ is Pareto-dominated by $c$ as every other voter submits $c \succeq x$. This implies that $c$ is among voter $i$'s least preferred alternatives in $f^*(R)$.
Furthermore, note that $(x, c) \notin t_2(R')$ for all $x \in A \setminus \{c\}$ as $c$ is among the most preferred alternatives of $n-1$ voters, which implies that $n_{xc} \leq 1$ regardless of the preference of voter $i$. Furthermore, $(x, c) \in t_1(R)$ is only possible for an alternative $x \in B$ since there is for every alternative $y \in A \setminus B$ a voter $j \in N \setminus \{i\}$ with $c \succ_j y$. This implies that if $(x, c) \in t(R')$, then $c \succeq_i x$. Because of the transitivity of $f(R')$, it follows that there is an alternative $x \in f^*(R')$ with $c \succeq_i x$. If there is an

alternative $b \in f^*(R)$ with $b \succ_i c$, then $b \succ_i x$ contradicting that $f^*(R') \; \tilde{P}_i \; f^*(R)$. As $c$ is among voter $i$'s least preferred alternative in $f^*(R)$, it follows that this voter is indifferent between all alternatives in $f^*(R)$. As consequence, every voter $j \in N$ satisfies $c \succeq_j x$ for all $x \in f^*(R)$. If this preference is strict for an alternative $x$ and a voter, then the alternative $x$ is Pareto-dominated. Thus, every voter is indifferent between all alternatives in $f^*(R)$, i.e., all chosen alternatives are indistinguishable. Combined with the transitivity of $t(R)$, this implies that $(b, c) \in t(R)$ for every alternative $b \in A$ with $b \succ_i c$. Even more, it follows from $b \succ_i c$ that $c$ does not Pareto-dominate $b$ and therefore, we can deduce that $(b, c) \in t_2(R)$. Thus, every voter $j \in N \setminus \{i\}$ prefers $c$ strictly to all alternatives $b$ with $b \succ_i c$ and $c$ is among his most preferred alternatives. Consequently, no alternative with $b \succ_i c$ can be chosen in $f^*(R')$ because $(c, b) \in t_2(R')$ if $b \succ'_i c$ and $(c, b) \in t_1(R)$ if $c \succeq'_i b$. Therefore, $c$ is among voter $i$'s most preferred alternatives in $f^*(R')$ contradicting that $f^*(R') \; \tilde{P}_i \; f^*(R)$. This implies that $f^*$ is $\tilde{P}$-strategyproof. □

As consequence of this lemma, it follows that Theorem 4.6 cannot be generalized to use anonymity and neutrality instead of C2. However, it is unclear whether it is possible that two voters agree on a uniquely most preferred alternative and it is not in the choice set of an anonymous, neutral, Pareto-optimal and $\tilde{P}$-strategyproof social choice function in the weak domain. Thus, it might still be possible to derive a weaker form of Theorem 4.6 for such social choice functions. However, it clearly is a lot harder to prove such a result.

### 4.3.3 A Characterization of the Pareto-rule

In this section, we prove that the Pareto-rule is the only reasonable $\tilde{P}$-strategyproof and Pareto-optimal social choice function in C2. Therefore, we provide two characterizations of this rule which show that every other C2-function violates an important axiom. In particular, we show that the Pareto-rule is the only social choice function that satisfies $\tilde{P}$-strategyproofness, Pareto-optimality, neutrality and a new axiom which we call C2-monotonicity if there are $m \geq 3$ alternatives. Even more, we prove that the Pareto-rule is the only $\tilde{P}$-strategyproof, Pareto-optimal and neutral C2-function if we restrict the number of alternatives to $3 \leq m \leq 4$. Thus, the Pareto-rule is the only reasonable social choice function in C2.

Our idea for proving these characterizations of the Pareto-rule is to use Theorem 4.6. Consequently, we have to show that every $\tilde{P}$-strategyproof refinement $f$ of the Pareto-rule which is in C2 has a profile $R$ such that $f(R)$ does not contain any of the most preferred alternatives of a voter. Unfortunately, this claim is in general not true as the social choice function $f_1$ defined in Section 4.2.2 satisfies C2, $\tilde{P}$-strategyproofness and Pareto-optimality and $f_1(R)$ contains at least one of the most preferred alternatives of every voter for every preference profile $R$. However, $f_1$ only refines the Pareto-rule if there are indistinguishable alternatives by non-neutrally

| | 1 | 1 | 1 |
|---|---|---|---|
| $R^*$ : | $a, c$ | $b, c$ | $d$ |
| | $e$ | $e$ | $a, b, e$ |
| | $b, d$ | $a, d$ | $c$ |

| | 1 | 1 | 1 |
|---|---|---|---|
| | $e$ | $b, c$ | $d$ |
| $R'$ : | $a, c$ | $a, d$ | $a, b$ |
| | $b, d$ | $e$ | $c$ |
| | | | $e$ |

| $n^*_{xy}$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | – | 1 | 1 | 1 | 1 |
| $b$ | 1 | – | 1 | 1 | 1 |
| $c$ | 1 | 1 | – | 2 | 2 |
| $d$ | 1 | 1 | 1 | – | 1 |
| $e$ | 1 | 1 | 1 | 2 | – |

| $n'_{xy}$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | – | 1 | 1 | 1 | 2 |
| $b$ | 1 | – | 1 | 1 | 2 |
| $c$ | 1 | 1 | – | 2 | 2 |
| $d$ | 1 | 1 | 1 | – | 2 |
| $e$ | 1 | 1 | 1 | 1 | – |

Figure 4.17: Preference profiles used for explaining C2-monotonicity

picking only a single alternative instead of the whole set of indistinguishable alternatives. Therefore, we additionally require neutrality for the characterization of the Pareto-rule. Unfortunately, the social choice function $f_2$ defined in Section 4.2.2 still satisfies all conditions. Thus, we have to strengthen or add an axiom for the characterization of the Pareto-rule such that $f_2$ violates it.
For finding an axiom that is satisfied by the Pareto-rule and violated by $f_2$, we analyze the profiles $R$ such that $f_2(R) \subsetneq \text{PO}(R)$. Thus, note that all these profiles contain three voters who submit the preference profile $R^*$ shown in Figure 4.17. In this profile $e$ is Pareto-optimal but $f_2$ does not choose it. In contrast, it holds for the profile $R'$ shown next to $R^*$ that $f_2(R') = A$. This is possible as neither C2 nor $\tilde{P}$-strategyproofness relates $R'$ and $R^*$. However, if we analyze the majorities of these profiles, we see that $n^*_{xe} \leq n_{xe}$ and $n^*_{ex} \geq n'_{xe}$ for all $x \in A \setminus \{e\}$ and the remaining majorities are equal in $R^*$ and $R'$. The detailed majorities are shown in Figure 4.17. Intuitively, this means that the position of $e$ weakens and the remaining alternatives are not affected when switching from $R^*$ to $R'$. Nevertheless, $e \notin f_2(R^*)$ but $e \in f(R')$. This contradicts the intuition of monotonicity stating that a chosen alternative should also be chosen after reinforcing it. However, it turns out that monotonicity is not sufficient to formalize this problem as it is defined on preference relations instead of majorities. Therefore, we introduce a new variant of monotonicity which we refer to as C2-monotonicity.

**Definition 4.4 (C2-monotonicity).** *A social choice function $f$ satisfies C2-monotonicity if for all preference profiles $R$, $R'$ and sets of indistinguishable alternatives $B$ such that $n_{ab} \geq n'_{ab}$, $n_{ba} \leq n'_{ba}$ for all $a \in A \setminus B$, $b \in B$ and $n_{ab} = n'_{ab}$ for all $a, b \in A \setminus B$ and $a, b \in B$, it holds that $b \in f(R) \cap B$ implies $b \in f(R')$.*

Intuitively, C2-monotonicity states that a chosen alternative is still chosen if its majorities increase and nothing else changes. Furthermore, it should be stressed that we discuss sets of indistinguishable alternatives instead of single alternatives in this definition. The reason for this is that we can always clone alternatives without violating an axiom. This leads to new social choice functions which still do not suffice the intuition of monotonicity, but a definition based on alternatives is not sufficient to formalize this behavior. Furthermore, note that the Pareto-rule and the social choice function $f_1$ discussed in Section 4.2.2 satisfy C2-monotonicity because Pareto-optimal alternatives stay Pareto-optimal if we reinforce their majorities. In contrast, the social choice $f_2$ discussed in Section 4.2.2 violates this axiom as the profiles in Figure 4.17 show.

Next, observe that every C2-monotonic social choice function is monotonic. This follows from the fact that if a voter reinforces an alternative, its majorities can only increase. Hence, we can consider this alternative as singleton set of indistinguishable alternatives and therefore, a chosen alternative is still in the choice set of a C2-monotonic social choice function after reinforcing it. Furthermore, every C2-monotonic social choice function is also in C2. The reason for this is that C2-monotonicity also allows to compare preference profiles with equal majorities. Thus, we can deduce from C2-monotonicity that $a \in f(R^1)$ if and only if $a \in f(R^2)$ for all alternatives $a \in A$ and preference profiles $R^1$ and $R^2$ with $n^1_{xy} = n^2_{xy}$ for all $x, y \in A$. This means that $f$ is in C2. However, the inverse is not true: The combination of monotonicity and C2 does not imply C2-monotonicity. For seeing this, consider the preference profiles shown in Figure 4.17. We know that if the alternative $e$ is chosen in $R'$ by a C2-monotonic social choice function, then it is also chosen in $R^*$. However, if we want to derive $R^*$ from $R'$ by applying monotonicity and C2, we can only use C2 to weaken $e$ in the preference of the second voter. This requires that we break the tie $a \sim_3 b \sim_3 e$ before we weaken $e$. Unfortunately, we cannot do this without losing the information whether $f$ is still chosen. Thus, C2-monotonicity is stronger than the combination of C2 and monotonicity.

Finally, we use this new axiom and Theorem 4.6 to prove the characterization of the Pareto-rule. Therefore, note that if a Pareto-optimal alternative $a$ is not chosen by a C2-monotonic social choice function $f$, then a voter can prefer $a$ uniquely the most and all other voters prefer it uniquely the least and $a$ is still not chosen as its majorities only worsen. As consequence of Theorem 4.6, we can deduce that $f$ cannot be additionally Pareto-optimal and $\tilde{P}$-strategyproof. This idea has also been used to derive the profile $R'$ from $R^*$ shown in Figure 4.17. Thus, we only have to formalize this approach to prove the characterization of the Pareto-rule.

**Theorem 4.7.** *The Pareto-rule is the only social choice function satisfying Pareto-optimality, neutrality, $\tilde{P}$-strategyproofness and C2-monotonicity if $m \geq 3$.*

Proof: First note that it is trivial that the Pareto-rule satisfies Pareto-optimality, neutrality, $\tilde{P}$-strategyproofness and C2-monotonicity. Thus, we focus on showing that no other social choice function satisfies these axioms if $m \geq 3$. Therefore, assume for contradiction that there is another social choice function $f$ that satisfies Pareto-optimality, neutrality, $\tilde{P}$-strategyproofness and C2-monotonicity and that is defined on $m \geq 3$ alternatives. As $f$ is Pareto-optimal but not equal to the Pareto-rule, there is a profile $R$ such that a Pareto-optimal alternative $b$ is not chosen, i.e., $b \notin f(R)$. By neutrality, it follows that all alternatives that are indistinguishable to $b$ are not in $f(R)$ either. Thus, there is a Pareto-optimal and inclusion-maximal set of indistinguishable alternatives $B$ such that $f(R) \cap B = \emptyset$. If $n = 1$, this means that $f(R)$ is empty or not Pareto-optimal as there is only a single Pareto-optimal and inclusion-maximal set of indistinguishable alternatives. Hence, the characterization holds in this case trivially.

Therefore, assume that $n \geq 2$. As $B$ is a Pareto-optimal and inclusion-maximal set of indistinguishable alternatives, it follows that $n_{ca} \geq 1$ for all $c \in B$, $a \in A \setminus B$. Thus, we can use the C2-monotonicity of $f$ to modify the preference profile $R$ in the following way: The first voter prefers the alternatives in $B$ the most and every other voter prefers the alternatives in $B$ the least; the preferences between the remaining alternatives are not affected. This leads to a new profile $R'$ and the majorities involving alternatives in $B$ only worsen by going from $R$ to $R'$. In particular, it holds that $n'_{ca} = 1$ and $n'_{ac} = n - 1$ for all $c \in B$, $a \in A \setminus B$ and $n_{xy} = n'_{xy}$ for all other majorities. Consequently, C2-monotonicity implies that $B \cap f(R') = \emptyset$ because $B \cap f(R) = \emptyset$. This means that we have derived a profile $R'$ such that $f(R')$ does not contain any of the most preferred alternatives of the first voter. Thus, we can use Theorem 4.6 to derive that $f$ violates $\tilde{P}$-strategyproofness, Pareto-optimality or C2. This implies that there is no other $\tilde{P}$-strategyproof, Pareto-optimal, neutral and C2-monotonic social choice function but the Pareto-rule if $m \geq 3$. □

First, we discuss the independence of the axioms used in the characterization of the Pareto-rule. Therefore, note that the social choice function $f_1$ discussed in Theorem 4.2 satisfies all axioms of the characterization but neutrality. Furthermore, many ideas for neutral, $\tilde{P}$-strategyproof and C2-monotonic social choice functions that violate Pareto-optimality arise from the results in Section 4.2.1. It is also straightforward that there are neutral, Pareto-optimal and C2-monotonic social choice functions that are not $\tilde{P}$-strategyproof such as the intersection of the Pareto-rule and the Condorcet-rule. Even more, C2-monotonicity is independent from the remaining axioms of the characterization as the social choice function $f_2$ discussed in Section 4.2.2 only violates this axiom. Finally, even the condition $m \geq 3$ is required as the majority rule satisfies all axioms if $m = 2$.

Next, note that we can weaken neutrality and still derive Theorem 4.7. It suffices that all indistinguishable alternatives are treated equally, i.e., if $a$ and $b$ are indistinguishable in a preference profile $R$, then $a \in f(R)$ if and only if $b \in f(R)$. Even more, if we assume that no alternatives are indistinguishable in the preference profile for which a social choice function refines the Pareto-rule, then we can completely omit neutrality in the characterization. However, it is not possible to remove all profiles with indistinguishable alternatives from the weak domain as the proof of Lemma 4.4 may lead to a profile with indistinguishable alternatives. Furthermore, the social choice functions discussed in Section 4.2.1 also imply that Pareto-optimality cannot be weakened without making the characterization of the

Pareto-rule invalid. The reason for this is that social choice functions such as the SCF $f(R) = \{a \in A \mid \nexists b \in A : n_{ab} = 0 \wedge n_{ba} > 1\}$ satisfy all axioms of the characterization but Pareto-optimality. Finally, the social choice function $f_2$ shows that we cannot weaken C2-monotonicity without violating Theorem 4.7.
However, observe that the idea for defining the social choice function $f_2$ requires at least five alternatives. Thus, it might be possible to derive a stronger characterization of the Pareto-rule if we restrict the number of alternatives to $m \leq 4$. As we show in the sequel, this conjecture is true and we can characterize the Pareto-rule as the only $\tilde{P}$-strategyproof, Pareto-optimal and neutral C2-function if $3 \leq m \leq 4$. As in the proof of Theorem 4.7, we use Theorem 4.6 to show that no refinement of the Pareto-rule can be simultaneously $\tilde{P}$-strategyproof, neutral and in C2. Therefore, we prove that every refinement $f$ of the Pareto-rule which satisfies all these axioms has a profile $R$ such that $f(R)$ does not contain any of the most preferred alternatives of a voter. However, as we do not require C2-monotonicity anymore, it is more complicated to derive such a profile. Therefore, we discuss in the next lemma a sufficient condition on Pareto-optimal and $\tilde{P}$-strategyproof C2-functions that allows to make an unchosen and Pareto-optimal alternative to the uniquely most preferred alternative of a voter.

**Lemma 4.11.** *Consider a Pareto-optimal and $\tilde{P}$-strategyproof C2-function $f$ and a profile $R$ such that $c \notin f(R)$ even though $c$ is Pareto-optimal. Furthermore, assume that there are disjoint sets $X_i$ for all voters $i \in N$ such that $x \succ_i y$ for all alternatives $x \in A \setminus X_i$, $y \in X_i$ and $\bigcup_{i \in N} X_i = A \setminus \{c\}$. Then, there is a profile $R'$ such that $c \notin f(R')$ even though a voter prefers $c$ uniquely the most.*

Proof: Consider an arbitrary Pareto-optimal and $\tilde{P}$-strategyproof C2-function $f$ and a profile $R$ such that a Pareto-optimal alternative $c$ is not in $f(R)$. Furthermore, assume that there are disjoint sets $X_i$ for all voters $i \in N$ such that $x \succ_i y$ for all alternatives $x \in A \setminus X_i$, $y \in X_i$ and $\bigcup_{i \in N} X_i = A \setminus \{c\}$. In the sequel, we explain how to modify the preference profile $R$ to derive a profile $R'$ such that $c \notin f(R')$ even though $c$ is the uniquely most preferred alternative of a voter. As a first step, we add a new voter $i^*$ who is indifferent between all alternatives to the profile $R$. This leads to a new preference profile $R^1 = (R, R_{i^*})$ that satisfies that $f(R^1) = f(R)$ because $f$ is in C2 and a completely indifferent voter does not affect the majorities. Next, we let voter $i^*$ change his preference such that he prefers $c$ uniquely the least and is indifferent between all other alternatives. This leads to a new preference profile $R^2$ and $c \notin f(R^2)$ holds because of $\tilde{P}$-strategyproofness. Otherwise, voter $i^*$ can $\tilde{P}$-manipulate by switching from $R^1$ to $R^2$ as $c \notin f(R^1)$. Finally, we let every voter $i \in N \setminus \{i^*\}$ change his preference such that he prefers the alternatives in $X_i$ strictly to all alternatives in $A \setminus X_i$, which leads to the preference profile $R^3$. Formally, the preference of voter $i$ in $R^3$ is defined as follows.

$$R_i^3 = R|_{X_i} \cup R|_{A \setminus X_i} \cup \{(x,y) \mid x \in X_i, y \in A \setminus X_i\}$$

$R$:

| 1 | 1 |
|---|---|
| $b$ | $a$ |
| $c$ | $d$ |
| $a$ | $c$ |
| $d$ | $b$ |

$R^1$:

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $a$ | $a, b, c, d$ |
| $c$ | $d$ | |
| $a$ | $c$ | |
| $d$ | $b$ | |

$R^2$:

| 1 | 1 | 1 |
|---|---|---|
| $b$ | $a$ | $a, b, d$ |
| $c$ | $d$ | $c$ |
| $a$ | $c$ | |
| $d$ | $b$ | |

$R'$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $c$ |
| $d$ | $a$ | $a, b, d$ |
| $b$ | $d$ | |
| $c$ | $c$ | |

Figure 4.18: Preference profiles illustrating the proof of Lemma 4.11

As voter $i$ initially prefers every alternative in $A \setminus X_i$ strictly to every alternative in $X_i$, this change increases the majorities $n_{xy}$ by 1 and decreases the majorities $n_{yx}$ by 1 for all $x \in X_i$, $y \in A \setminus X_i$. This holds for every voter $i \in N \setminus \{i^*\}$ and as $\bigcup_{i \in N \setminus \{i^*\}} X_i = A \setminus \{c\}$ and $X_i \cap X_j = \emptyset$ for all $i, j \in N \setminus \{i^*\}$, $i \neq j$, it follows that this step does not affect the majorities between alternatives in $A \setminus \{c\}$. Furthermore, $n^3_{xc} = n^2_{xc} + 1$ and $n^3_{cx} = n^2_{cx} - 1$ for all $x \in A \setminus \{c\}$. Thus, if voter $i^*$ changes his preference such that $c$ is his uniquely most preferred alternative, then we arrive at a profile $R^4$ with the same majorities as $R^2$. Consequently, $f(R^4) = f(R^2)$ because of C2, which means that $c \notin f(R^4)$ even though $c$ is the uniquely most preferred alternative of voter $i^*$. Hence, we have found the required profile. □

First, we illustrate the construction used in the proof of Lemma 4.11 with an example. Therefore, consider the preference profiles shown in Figure 4.18 and let $f$ denote a Pareto-optimal and $\tilde{P}$-strategyproof C2-function with $f(R) = \{a, b\}$. In the first step, we add a new voter who is indifferent between all alternatives to derive $R^1$. This implies that $f(R^1) = f(R) = \{a, b\}$ as $f$ is in C2. Next, we let the new voter manipulate such that his preference is $a \sim b \succ c$. This leads to the profile $R^2$ and it follows from $\tilde{P}$-strategyproofness that $c \notin f(R^2)$ as otherwise the new voter can $\tilde{P}$-manipulate by switching from $R^2$ to $R^1$. Finally, we compute that $X_1 = \{a, d\}$ and $X_2 = \{b\}$, and change the preferences of voter 1 and 2 such that they prefer these sets the most. Furthermore, we let the third voter switch from $a \sim b \succ c$ to $c \succ a \sim b$ to derive the profile $R'$. It can be easily checked that $R'$ and $R^2$ have the same majorities and therefore, $f(R') = f(R^2)$ which means that $c$ is not chosen even though it is the most preferred alternative of a voter.

It should be mentioned that Lemma 4.11 can be used for an arbitrary number of alternatives. However, there are preference profiles that do not yield suitable sets $X_i$ and therefore, this lemma is not always applicable. For instance, the profile $R^*$ shown in Figure 4.17 does not yield suitable sets if alternative $e$ is not chosen and we want to make it to the uniquely most preferred alternative of a voter. Therefore, this lemma does not affect the social choice function $f_2$ that shows that C2-monotonicity is required for the characterization of the Pareto-rule if there are arbitrarily many alternatives. Nevertheless, Lemma 4.11 is rather powerful if there are only few alternatives as it becomes easy to find the sets $X_i$ in this situation. Furthermore, if we can apply this lemma, then Theorem 4.6 implies that the corresponding social choice function violates either Pareto-optimality, $\tilde{P}$-strategyproofness or C2. This is the main idea for characterizing the Pareto-rule as the only $\tilde{P}$-strategyproof, Pareto-optimal and neutral C2-function if $3 \leq m \leq 4$.

**Theorem 4.8.** *The Pareto-rule is the only $\tilde{P}$-strategyproof, Pareto-optimal and neutral C2-function if* $3 \leq m \leq 4$.

Proof: First note that the Pareto-rule obviously satisfies all required axiom. Therefore, we focus on proving that there is no other $\tilde{P}$-strategyproof, Pareto-optimal and neutral C2-function if $3 \leq m \leq 4$. Thus, assume for contradiction that there is another social choice function $f$ but the Pareto-rule that satisfies all required axioms and that is defined on 3 or 4 alternatives. As $f$ is Pareto-optimal but no equal to the Pareto-rule, there is a profile $R$ such that $f(R)$ does not contain a Pareto-optimal alternative $c$. Furthermore, because of the neutrality of $f$, every alternative that is indistinguishable from $c$ is also not in $f(R)$. Thus, there are at least two inclusion-maximal sets of indistinguishable alternatives as the alternatives in $f(R)$ and $c$ are distinguishable. Even more, if there are exactly two inclusion-maximal sets of indistinguishable alternatives in $R$, then these sets are $f(R)$ and $A \setminus f(R)$ as these sets are non-empty and neutrality implies that either all alternatives in a set of indistinguishable alternatives or none of them are chosen. Furthermore, there is a voter who prefers all alternatives in $A \setminus f(R)$ strictly to all alternatives in $f(R)$ as otherwise the alternatives in $A \setminus f(R)$ are not Pareto-optimal. This contradicts our assumptions as $c \in A \setminus f(R)$ and $c$ is Pareto-optimal. Thus, we can directly apply Theorem 4.6 as none of the most preferred alternatives of a voter are chosen and deduce that $f$ violates $\tilde{P}$-strategyproof, Pareto-optimality or C2. Consequently, we assume that there are 3 or 4 inclusion-maximal sets of indistinguishable alternatives in $R$ and we treat every set as single alternative, i.e., we assume in the sequel that all alternatives are distinguishable from each other. This is possible because the SCF $f$ satisfies neutrality and therefore, it treats indistinguishable alternatives equally.
Next, we choose a set of voters $I \subseteq N$ such that $n_{cx} \geq 1$ for all $x \in A \setminus c$ in the profile $R_{-I}$ and $|I|$ is maximal. Note that the set $N \setminus I$ contains at most $m-1$ voters because there must be for every voter $i \in N \setminus I$ an alternative $a \in A$ such that $c \succ_i a$ and $a \succeq_j c$ for every other voter $j \in N \setminus I$. Otherwise, $n_{cx} \geq 1$ also holds for all $x \in A \setminus c$ in the profile $R_{-I \cup \{i\}}$ contradicting that $I$ is among the largest sets that satisfy this property. Furthermore, if $|N \setminus I| = 1$, then a single voter prefers $c$ strictly to all other alternatives. This means that $c$ is his uniquely most preferred alternative and therefore, we can use Theorem 4.6 to derive that $f$ violates Pareto-optimality, $\tilde{P}$-strategyproofness or C2. Moreover, if $|N \setminus I| = m-1$, there is a unique alternative $b_i \in A \setminus \{c\}$ for every voter $i \in N \setminus I$ such that $c \succ_i b_i$. If this is not true, there is a voter who prefers $c$ strictly to two or more alternatives. This implies that we only need at most $m-2$ voters such that $n_{cx} \geq 1$ for all $x \in A \setminus \{c\}$ as there is for every voter $i \in N \setminus I$ an alternative $b \in A \setminus \{c\}$ such that $c \succ_i b$ and $b \succeq_j c$ for all other voters $j \in A \setminus \{I\}$. Thus, every voter $i \in N \setminus I$ prefers $c$ strictly to a single alternative $b_i$ if $|N \setminus I| = m-1$. This implies that $b_i$ is the unique

worst alternative of voter $i$. Hence, we can use Lemma 4.11 as the sets $X_i = \{b_i\}$ for $i \in N \setminus I$ and $X_i = \emptyset$ for $i \in I$ are disjoint and their union is $A \setminus \{c\}$. This means that there is a profile $R'$ such that $c$ is the most preferred alternative of a voter and $c \notin f(R')$ if $|N \setminus I| = m - 1$. Thus, Theorem 4.6 implies that $f$ violates one of the axioms of this characterization.

As consequence of these observations, there is no refinement of the Pareto-rule that satisfies $\tilde{P}$-strategyproofness, neutrality and C2 if $m = 3$. The reason for this is that $1 \leq |N \setminus I| \leq m - 1 = 2$, which means that only the two previously discussed situations are possible. Even more, if $m = 4$, the only case that remains open is $|N \setminus I| = 2$. Therefore, we assume that $|N \setminus I| = 2$ and analyze the preference of the voters $i_1, i_2 \in N \setminus I$ in detail. Thus, observe that there are alternatives $a, b \in A$ such that $b \succsim_{i_1} c \succ_{i_1} a$ and $a \succsim_{i_2} c \succ_{i_2} b$. Furthermore, there is a fourth alternative $d \in A$ and at least one voter in $\{i_1, i_2\}$ submits $c \succ d$. Next, we make a case distinction with respect to the preferences of voter $i_1$ and $i_2$ over $d$. First, assume that $d$ is not among the least preferred alternatives of voter $i_1$. This means that $a$ is his least preferred alternative as $b \succsim_{i_1} c \succ_{i_1} a$. Furthermore, if voter $i_2$ prefers $c$ strictly to both $b$ and $d$, then we can set $X_{i_1} = \{a\}$, $X_{i_2} = \{b, d\}$ and $X_j = \emptyset$ for all voters in $N \setminus I$ to apply Lemma 4.11. If voter $i_2$ prefer $c$ not strictly to both $d$ and $b$, then he submits $d \succsim_{i_2} c$ as $c \succ_{i_2} b$ is true by our assumptions. This implies that voter $i_1$ prefers $c$ strictly to $d$ as otherwise $n_{cd} = 0$ in $R_{-I}$. Hence, we set $X_{i_1} = \{a, d\}$, $X_{i_2} = \{b\}$ and $X_j = \emptyset$ for all $j \in I$ to apply Lemma 4.11. This means that we can derive a profile $R'$ such that $c$ is the most preferred alternative of a voter but $c \notin f(R')$ if voter $i_1$ does not prefer $d$ the least. Therefore, it follows from Theorem 4.6 that $f$ violates $\tilde{P}$-strategyproofness, Pareto-optimality or C2. Even more, we can deduce from a symmetric argument that $f$ also violates one of these axioms if $d$ is not among the least preferred alternatives of voter $i_2$.

Thus, it only remains to consider the case where $d$ is the least preferred alternative of both voter $i_1$ and $i_2$. This means that $b \succsim_{i_1} c \succ_{i_1} a \succsim_{i_1} d$ and $a \succsim_{i_1} c \succ_{i_1} b \succsim_{i_1} d$. If voter $i_1$ prefers $a$ strictly to $d$, we can exchange the preferences of voter $i_1$ and $i_2$ on the alternatives $a$ and $c$ without changing the majorities. This leads to a new profile $R'$ in which voter $i_2$ prefers $c$ uniquely the most. However, $c \notin f(R')$ because we derive the profile $R'$ with C2. Thus, we can use again Theorem 4.6 to show that $f$ violates an axiom of this characterization. Even more, a symmetric argument can be applied if voter $i_2$ prefers $b$ strictly to $d$.

Thus, the only case that is left open is that voter $i_1$ is indifferent between $a$ and $d$ and voter $i_2$ is indifferent between $b$ and $d$. In this case, we let the voters in $I$ manipulate their preferences one after another in the following way: Every voter $i \in I$ switches to a preference in which he prefers $c$ uniquely the least and is indifferent between all other alternatives. As consequence of these changes, $c$ cannot be chosen as otherwise there is a voter such that $c$ is not chosen before his modification, but it is afterwards. Thus, this voter can $\tilde{P}$-manipulate by undoing the modification. This process results in a profile $R^1$ after every voter $i \in I$ manipulated his preference for which $c \notin f(R^1)$ is true. Furthermore, $d$ is Pareto-dominated in $R^1$ and therefore,

$f(R^1) \subseteq \{a, b\}$. Next, we add a new voter $i_3$ to the profile $R^1$ who is indifferent between all alternatives. This step leads to the profile $R^2 = (R^1, R_{i_3})$ which satisfies that $f(R^2) = f(R^1)$ as voter $i_3$ does not affect the majorities. Thereafter, voter $i_3$ changes his preference to $a \sim b \succ d \succ c$ and the choice set of the resulting profile $R^3$ does still not contain $c$ and $d$. Otherwise, voter $i_3$ can $\tilde{P}$-manipulate by switching back to $R^2$. Finally, we change the profile such that voter $i_3$ is indifferent between $a$, $b$ and $d$, voter $i_2$ prefers $b$ strictly to $d$ and voter $i_1$ prefers $a$ strictly to $d$. We call the resulting preference profile $R^4$. This step does not affect the majorities and therefore, $f(R^4) = f(R^3)$, i.e., $c$ is still not chosen. Furthermore, voter $i_1$ prefers $a$ strictly to $d$ and therefore, we can use the steps explained in the last paragraph to deduce a contradiction. Thus, we have derived a contradiction in all cases, which means that there is no other $\tilde{P}$-strategyproof, Pareto-optimal and neutral C2-function but the Pareto-rule if $3 \leq m \leq 4$. □

Note that all axioms in this characterization of the Pareto-rule are independent. Most importantly, the bounds on $m$ are tight, i.e., if $m = 2$, the majority rule satisfies all axioms of the characterization and if $m = 5$, the SCF $f_2$ discussed in Section 4.2.2 satisfies all axioms. Furthermore, C2 is independent of the remaining axioms as the social choice function discussed in Lemma 4.10 is Pareto-optimal, $\tilde{P}$-strategyproof and neutral but not in C2. The independence of the remaining axioms follows from the same examples discussed in the remarks after Theorem 4.7 because C2-monotonicity implies C2.
As consequence of Theorem 4.8, there is no other reasonable $\tilde{P}$-strategyproof and Pareto-optimal C2-function but the Pareto-rule if $m = 3$ or $m = 4$. Therefore, this rule is uncontroversial if we require a C2-function that is defined on three or four alternatives in the weak domain. Even more, if $m \geq 5$, the Pareto-rule is the most preferable C2-function because of Theorem 4.7. This result requires C2-monotonicity instead of C2, which restricts its strength only slightly. The reason for this is that C2-monotonicity is a very desirable axiom that only translates monotonicity into the C2-setting. Therefore, we deem the Pareto-rule as the most desirable C2-function in the weak domain.

# Chapter 5

# Responsively Efficient Social Choice Functions

As we have seen in the last chapters, there are many strong results on $\tilde{P}$-strategyproof and Pareto-optimal social choice functions in the weak domain. For instance, it is known that no such function can be additionally pairwise [BSS] and we have proven in Theorem 3.12 that no $\tilde{P}$-strategyproof and Pareto-optimal SCF can be rank-based. However, there are $\tilde{P}$-strategyproof and Pareto-optimal social choice functions, such as the Pareto-rule. This rule satisfies many other axioms such as monotonicity, neutrality and anonymity. Hence, we need a stronger notion of efficiency than Pareto-optimality in order to find an impossibility result based on $\tilde{P}$-strategyproofness and anonymity. Therefore, we discuss a new axiom called responsive efficiency in this chapter. This axiom generalizes Pareto-optimality from single alternatives to sets of alternatives and therefore, it is more restrictive. Thus, it seems reasonable to conjecture that there is no social choice function that satisfies $\tilde{P}$-strategyproofness, responsive efficiency and anonymity.

For answering this conjecture, we formally introduce the concept of responsive efficiency in Section 5.1. As the pure technical introduction leaves many questions about this axiom open, we thoroughly analyze the axiom itself in Section 5.2. Finally, we discuss social choice functions that are $\tilde{P}$-strategyproof and responsively efficient in Section 5.3. We can show that there is a $\tilde{P}$-strategyproof social choice function that satisfies responsive efficiency but violates anonymity. Furthermore, we prove two impossibility results that rule out that simple social choice functions satisfy $\tilde{P}$-strategyproofness, responsive efficiency and anonymity. However, we cannot completely disprove the existence of such social choice functions.

## 5.1 Introduction to Responsive Efficiency

In this section, we formally introduce a new axiom called responsive efficiency. Similar to Pareto-optimality, this axiom tries to formalize the idea that many choice sets are not reasonable because they contain alternatives such that every voter agrees that there are better options. However, Pareto-optimality allows only to compare individual alternatives, which does not seem sufficient in the context of set-valued social choice functions. For instance, consider the profile shown in Figure 5.1. It

| | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $R:$ | $a$ | $a$ | $b$ | $b$ |
| | $c$ | $d$ | $c$ | $d$ |
| | $b$ | $b$ | $a$ | $a$ |
| | $d$ | $c$ | $d$ | $c$ |

Figure 5.1: Preference profile used for explaining responsive efficiency

can be easily seen that every alternative is Pareto-optimal in $R$ and therefore, a Pareto-optimal social choice function might return $\{a, b, c, d\}$. However, it is not reasonable to return this set because every voter seems to prefer the set $\{a, b\}$ to the set $\{c, d\}$. Thus, a more appropriate choice set for $R$ is $\{a, b\}$.
The intuition of responsive efficiency is to formalize the behavior seen in the last example: A set of alternatives can be more desirable than another set for every voter and therefore, the latter set should not be a subset of the choice set. However, this intuition leaves it open when we consider a set of alternatives preferable to another one. We solve this problem in the following definition.

**Definition 5.1 (Responsive dominance).** *A set of alternatives $X \subseteq A$ responsively dominates another set $Y \subseteq A$ in a preference profile $R$ if $|X| = |Y|$ and for every voter $i \in N$, there is a bijection $\pi_i : X \mapsto Y$ such that $x \succeq_i \pi_i(x)$ for all $x \in X$ and there is an alternative $x^* \in X$ and a voter $j \in N$ such that $x^* \succ_j \pi_j(x^*)$.*

This definition only allows to compare sets that have the same size as it requires bijections between the sets. The intuitive meaning of these bijections is that for every voter $i \in N$ and alternative $x \in X$, there is a unique alternative $y \in Y$ that is less preferred than $x$ by voter $i$. Thus, if every alternative in $Y$ is chosen, we can replace $Y$ with $X$ in the choice set and every voter favors the new outcome. Furthermore, we demand that at least one preference with respect to $\pi_i$ is strict for at least one voter $i \in N$ to ensure that a set does not responsively dominate itself.
Moreover, note that if there are sets $X = \{x\}$ and $Y = \{y\}$ such that $X$ responsively dominates $Y$, then alternative $x$ Pareto-dominates alternative $y$. Even more, the inverse is also true, i.e., if $x$ Pareto-dominates $y$, then $\{x\}$ responsively dominates $\{y\}$. Thus, we see that responsive dominance is indeed stronger than Pareto-dominance.
It should be mentioned that it is rather restrictive that we only allow to compare sets of the same size in Definition 5.1. However, if we allow sets with different size, we cannot use a bijection for comparing them anymore. This leads to difficult workarounds and less intuitive definitions. Furthermore, by using a restrictive definition, we strengthen the impossibility results that are discussed in subsequent sections. Nevertheless, we want to mention a reasonable method to generalize responsive dominance to sets with different size. For instance, if we allow sets $X$, $Y$ with $|X| \leq |Y|$, we may require that there is for every voter $i \in N$ a surjection $\pi_i : Y \mapsto X$ such that $\pi_i(y) \succeq_i y$ for all $y \in Y$ and this preference is strict for at least one voter and one alternative. This leads to a variant of responsive dominance that

| | 1 | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|---|
| | $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
| | $c$ | $d$ | $e$ | $c$ | $d$ | $e$ |
| $R'$ : | $d$ | $e$ | $c$ | $d$ | $e$ | $c$ |
| | $b$ | $b$ | $b$ | $a$ | $a$ | $a$ |
| | $e$ | $c$ | $d$ | $e$ | $c$ | $d$ |

Figure 5.2: Preference profile used to show the differences between variants of responsive dominance

is reasonable and even stronger than Definition 5.1. For instance, consider the preference profile $R'$ shown in Figure 5.2. It seems that every voter prefers the set $\{a, b\}$ to the set $\{c, d, e\}$. Even more, it can be easily checked that the strengthened variant of responsive dominance implies that $\{a, b\}$ dominates $\{c, d, e\}$. In contrast, no sets are responsively dominated with respect to Definition 5.1 in the profile $R'$. The reason for this is that for every subset $\{x, y\} \subsetneq \{c, d, e\}$ there is a voter who prefers this subset strictly to $a$ or $b$. Thus, the variant suggested in this paragraph is indeed stronger that Definition 5.1. Nevertheless, we discuss in the sequel the weaker version of responsive efficiency based on bijections.
Even though the definition of responsive dominance is rather interesting and allows for many variations, we want to analyze social choice functions. Therefore, we introduce an axiom called responsive efficiency that is defined similarly to Pareto-optimality. Thus, recall that a SCF is Pareto-optimal if its choice set never contains a Pareto-dominated alternative. It is straightforward to generalize this approach to responsive dominance by demanding that a responsively efficient SCF never contains a responsively dominated subset.

**Definition 5.2 (Responsive efficiency).** *We call a set of alternatives $X \subseteq A$ responsively efficient if it does not contain a responsively dominated subset. Furthermore, we call a social choice function $f$ responsively efficient if $f(R)$ is for every preference profile $R$ responsively efficient.*

As consequence of this definition, every responsively efficient social choice function $f$ satisfies that $\{c, d\} \nsubseteq f(R)$ for the profile $R$ shown in Figure 5.1 because $\{a, b\}$ responsively dominates $\{c, d\}$ in this profile. For instance, $f(R) = \{a, b, c\}$ is a valid choice. Note that it is no problem that $c$ is chosen as responsive efficiency only requires that not both $c$ and $d$ are chosen. As a consequence of this observation, it is not possible to define a social choice function that returns the maximal responsively efficient set as this set is not unique. For instance, both $\{a, b, c\}$ and $\{a, b, d\}$ are maximal and responsively efficient choice sets for the profile $R$ shown in Figure 5.1. Hence, there is no responsively efficient social choice function that can be compared to the Pareto-rule. Due to this observation, it is also rather hard to check whether a social choice function is responsively efficient. This makes it difficult to discuss this axiom and therefore, we analyze it in detail in the next section.

## 5.2 Analysis of Responsive Efficiency

As we have already explained in the last section, responsively efficient social choice functions may behave rather unexpected and it is hard to check whether a social choice function satisfies responsive efficiency. Therefore, we discuss in this section the responsive dominance relation itself with respect to two goals: Firstly, we want to improve the understanding of this relation and therefore, we propose some lemmas about responsive dominance. Secondly, we aim to introduce approaches for constructing responsively efficient social choice functions. We are able to prove that various social choice functions, for instance Borda's rule, are responsively efficient. However, all social choice functions introduced in this section are $\tilde{P}$-manipulable.

Our first idea for defining a responsively efficient social choice function is to simply remove all responsively dominated sets from the choice set. However, it is easy to see that this approach does not lead to a well-defined social choice function as often all alternatives are part of a responsively dominated set. For instance, consider the profile $R^1$ shown in Figure 5.3. In this profile, $c$ is Pareto-dominated by $b$ and $d$ is Pareto-dominated by $a$. Thus, the sets $\{c\}$ and $\{d\}$ are responsively dominated. These observations imply that $\{a, b\}$ responsively dominates $\{a, c\}$ and that $\{a, b\}$ responsively dominates $\{b, d\}$. Therefore, every alternative is indeed element of a responsively dominated set.

The problem in this example is that we compare intersecting sets with respect to responsive dominance. This means that we only have to find two sets $X$ and $Y$ such that $X$ responsively dominates $Y$. Then, we can add an alternative $z \notin X \cup Y$ to both sets and it holds that $X \cup \{z\}$ responsively dominates $Y \cup \{z\}$. Therefore, $z$ is also part of a responsively dominated set. Thus, it follows that all alternatives can be element of a responsively dominated set.

As a consequence of this observation, it seems reasonable to discuss only disjoint sets with respect to responsive dominance. However, even if we only consider disjoint sets, all alternatives may still be element of a responsively dominated set. For instance, consider the profile $R^2$ shown in Figure 5.3. In this profile, every voter prefers $d$ the least and is indifferent between the three remaining alternatives $a$, $b$ and $c$. Consequently, $d$ is Pareto-dominated by every alternative. Furthermore, it is easy to see that every subset of size 2 of $\{a, b, c\}$ responsively dominates the set consisting of $d$ and the remaining alternative. Thus, every alternative is element of a responsively dominated set, even if we only discuss disjoint sets. Hence, we do not obtain a well-defined social choice function by deleting all responsively dominated sets, even if we only discuss disjoint sets with respect to responsive dominance.

Note that both previously explained examples rely on the fact that sets containing a Pareto-dominated alternative are often also responsively dominated. However, we actually do not have to worry about these sets for constructing a responsively efficient social choice function if we simply remove the Pareto-dominated alternatives from the choice set. For instance, a responsively efficient social choice function $f$ is only required to satisfy that $f(R^1) \cap \{c, d\} = \emptyset$ for the profile $R^1$ shown in Figure 5.3.

$$R^1: \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline a & b & b \\ d & c & c \\ b & a & a \\ c & d & d \end{array} \qquad R^2: \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline a,b,c, & a,b,c, & a,b,c, \\ d & d & d \end{array}$$

Figure 5.3: Preference profiles used for explaining problems with responsive dominance

This observation leads to the notion of minimal responsive dominance, i.e., we are only interested in the fact that $X$ responsively dominates $Y$ if no set $X' \subsetneq X$ responsively dominates a subset $Y' \subsetneq Y$. The reason for this is that the choice set of a responsively efficient social choice function is no superset of $Y'$ and therefore, it is also no superset of $Y$. This intuition is formalized in the following definition.

**Definition 5.3 (Minimal responsive dominance).** *A set of alternatives $X \subseteq A$ minimally responsively dominates another set $Y \subseteq A$ in a preference profile $R$ if $X$ responsively dominates $Y$ in $R$ and no set $X' \subsetneq X$ responsively dominates a set $Y' \subsetneq Y$.*

For observing the difference between minimal responsive dominance and responsive dominance, consider the profile $R^1$ shown in Figure 5.3. In this profile, $\{a\}$ minimally responsively dominates $\{d\}$, but $\{a, b\}$ only responsively dominates $\{b, d\}$. Thus, it follows that removing all alternatives that are element of a minimally responsively dominated set leads to a non-empty choice set. Even more, it is enough to consider minimal responsive dominance to derive a responsively efficient social choice function. Therefore, we investigate minimal responsive dominance in more detail. Unfortunately, this dominance relation is rather difficult to understand and to analyze. One reason for this is that it is not straightforward to decide whether a set is minimally responsively dominated or only responsively dominated. Furthermore, minimal responsive dominance can behave rather unexpected if we modify a preference profile. For instance, a minimally responsively dominated set might grow larger if we swap to alternatives in a preference profile.
Therefore, we first discuss the minimal responsive dominance relation in more detail and analyze its properties. As a first step, we prove that if a set minimally responsively dominates another set, then these sets are disjoint.

**Lemma 5.1.** *If a set $X$ responsively dominates another set $Y$ in a profile $R \in \mathcal{W}^n$, then $X \setminus Y$ responsively dominates $Y \setminus X$ in $R$.*

Proof: We prove this lemma by an induction on size of the intersection $X \cap Y$. Thus, consider an arbitrary preference profile $R \in \mathcal{W}^n$ and two sets $X$ and $Y$ such that $X$ responsively dominates $Y$ in $R$. The base case $|X \cap Y| = 0$ is trivially true as this assumption implies that $X \setminus Y = X$ and $Y \setminus X = Y$.

Next, we focus on the induction step. Thus, assume that $|X \cap Y| = k$ for some $k \in \{1, ..., m\}$ and that the lemma is already proven for all sets $X'$, $Y'$ such that $|X' \cap Y'| = k - 1$ and $X'$ responsively dominates $Y'$ in $R$. In the sequel, we explain how to remove an alternative $a \in X \cap Y$ from $X$ and $Y$ while maintaining that $X \setminus \{a\}$ responsively dominates $Y \setminus \{a\}$. Then, the lemma follows from the induction hypothesis and this claim. Thus, consider an arbitrary alternative $a \in X \cap Y$ and a voter $i \in N$ and let $\pi_i : X \mapsto Y$ denote the bijection of voter $i$ required for showing that $X$ responsively dominates $Y$. As $a \in Y$, there is an alternative $b \in X$ such that $\pi_i(b) = a$. Furthermore, there is also an alternative $c \in Y$ with $\pi_i(a) = c$ as $a \in X$. By the definition of responsive dominance, it follows that $b \succeq_i a \succeq_i c$. Thus, we can define a new bijection $\pi_i' : X \setminus \{a\} \mapsto Y \setminus \{a\}$ that is required to show that $X \setminus \{a\}$ responsively dominates $Y \setminus \{a\}$. This bijection $\pi_i'$ maps $b$ to $c$ and is equal to $\pi_i$ otherwise. As we can apply this construction for every voter, it follows that $X \setminus \{a\}$ responsively dominates $Y \setminus \{a\}$. Finally, note that the intersection of these two sets has size $k - 1$ and therefore, the lemma follows from the induction hypothesis. □

As a consequence of this lemma, a set can only minimally responsively dominate another set if they are disjoint. Otherwise, we can simply remove the intersection from both sets and still have a responsive dominance relation between the sets, which contradicts the minimality assumption. Furthermore, note that Lemma 5.1 also holds if $a \in X$ and $b \in Y$ and $a$ is indistinguishable from $b$. The reason for this is that we can exchange $a$ with $b$ in $X$ and then apply the lemma. Moreover, as a consequence of Lemma 5.1, it suffices to consider responsive dominance only between sets $X$, $Y$ with $|X| = |Y| \leq m/2$ as all other dominance relation cannot be minimal. This means that if $m \leq 3$, responsive dominance and Pareto-dominance are equal. Consequently, the Pareto-rule is responsively efficient if $m \leq 3$. This observation shows also why it is hard to work with responsively efficient social choice functions: We need many alternatives such that responsive efficiency becomes restrictive. However, this makes it hard to find interesting profiles.
Note that Lemma 5.1 provides a simple criterion for deciding whether a responsive dominance relation is minimal. However, it is clearly not enough to characterize all minimally responsively dominated sets. Therefore, we propose another lemma on minimal responsive dominance that focuses on the situation when two responsive dominance relations are defined on the same set of alternatives.

**Lemma 5.2.** *Consider an arbitrary preference profile $R \in \mathcal{W}^n$ and sets of alternatives $X_1$, $Y_1$, $X_2$ and $Y_2$ such that $X_1$ responsively dominates $Y_1$, $X_2$ responsively dominates $Y_2$, $X_1 \cap Y_1 = X_2 \cap Y_2 = \emptyset$ and $X_1 \cup Y_1 = X_2 \cup Y_2$. Then $X_1 \cap X_2$ responsively dominates $Y_1 \cap Y_2$ in $R$.*

Proof: Consider an arbitrary preference profile $R \in \mathcal{W}^n$ and sets $X_1$, $Y_1$, $X_2$ and $Y_2$ as specified in the lemma, i.e., $X_1 \cap Y_1 = X_2 \cap Y_2 = \emptyset$, $X_1 \cup Y_1 = X_2 \cup Y_2$ and $X_i$ responsively dominates $Y_i$, $i \in \{1, 2\}$. First note that if $X_1 = X_2$, then it holds that $Y_1 = Y_2$ and therefore, the lemma is trivially true. Therefore, we assume in the sequel that $X_1 \cap X_2 \neq \emptyset$. Furthermore, note that $|X_1| = |X_2| = |Y_1| = |Y_2|$ and $|X_1 \cap X_2| = |Y_1 \cap Y_2|$ follows from our assumptions.

As $X_j$ responsively dominates $Y_j$, $j \in \{1, 2\}$, there are bijections $\pi_i^j : X_j \mapsto Y_j$ for every voter $i \in N$ and $j \in \{1, 2\}$ such that $x \succeq_i \pi_i^j(x)$ for all alternatives $x \in X_j$. We use these bijections to construct a new bijection $\pi_i^3 : X_1 \cap X_2 \mapsto Y_1 \cap Y_2$ for every voter $i \in N$ that shows that $X_1 \cap X_2$ responsively dominates $Y_1 \cap Y_2$. Thus, consider an arbitrary alternative $a \in X_1 \cap X_2$ and voter $i \in N$. If $\pi_i^1(a) \in Y_1 \cap Y_2$, we set $\pi_i^3(a) = \pi_i^1(a)$. Otherwise, it follows that $\pi_i^1(a) \notin Y_2$ as $\pi_i^1(a) \in Y_1$ because of the definition of $\pi_i^1$. This implies that $\pi_i^1(a) \in X_2$ as $X_1 \cup Y_1 = X_2 \cup Y_2$ and therefore, $\pi_i^2(\pi_i^1(a))$ is well-defined. If this alternative is in $Y_1 \cap Y_2$, then we set $\pi_i^3(a) = \pi_i^2(\pi_i^1(a))$. Otherwise, we can repeat this argument with $X_1$ and $Y_1$, i.e., we apply $\pi_i^1$ again. Hence, we alternate between applications of $\pi_i^1$ and $\pi_i^2$ until $\pi_i^j(\pi_i^k(...\pi_i^1(a))) = y \in Y_1 \cap Y_2$ and set $\pi_i^3(a) = y$. Note that we eventually arrive at an element in $Y_1 \cap Y_2$ as $\pi_i^1$ and $\pi_i^2$ are bijections and $X_1 \cap Y_1 = X_2 \cap Y_2 = \emptyset$. This means that every alternative in $Y_1 \setminus Y_2 = Y_1 \cap X_2$ and $Y_2 \setminus Y_1 = Y_2 \cap X_1$ has a unique successor and a unique predecessor with respect to $\pi_i^1$ and $\pi_i^2$. Thus, no alternative can appear twice in this sequence for deriving $y$, which means that we eventually reach an element in $Y_1 \cap Y_2$. Furthermore, for $a, b \in X_1 \cap X_2$, $a \neq b$, it holds that $\pi_i^3(a) \neq \pi_i^3(b)$ as otherwise there are alternatives $c, d \in X_j$, $c \neq d$, and $j \in \{1, 2\}$ such that $\pi_i^j(c) = \pi_i^j(d)$, which contradicts that $\pi_i^j$ is a bijection. Thus, it follows that $\pi_i^3$ defines indeed a bijection from $X_1 \cap X_2$ to $Y_1 \cap Y_2$ as these two sets have the same size. Finally, note that $x \succeq_i \pi_i^1(x) \succeq_i \pi_i^2(\pi_i^1(x)) \succeq_i ... \succeq_i \pi_3(x)$ for all $x \in X_1 \cap X_2$ and that this preference is strict for at least one voter and one alternative. As we can construct the bijection $\pi_i^3$ for every voter $i \in N$, it shows that $X_1 \cap X_2$ responsively dominates $Y_1 \cap Y_2$ in $R$. □

As consequence of this lemma, there is at most one minimal dominance relation defined on $X \cup Y$. For instance, this implies that there can only be a single minimal responsive dominance relation involving sets of size 2 if $m = 4$. Thus, responsive efficiency is not very restrictive in this case.

Unfortunately, neither Lemma 5.1 nor Lemma 5.2 suffice to provide strong restrictions on minimal responsive dominance. Thus, many problems with respect to this relation remain unsolved. For instance, it is not clear if there is a preference profile such that every alternative is element of a minimally responsively dominated set. One of the main problems for rejecting this conjecture is that many simple ideas focusing on single voters fail. For instance, it is possible that all most preferred alternative of a single voter are element of minimally responsively dominated sets. An example of such a profile is shown in Figure 5.4. In this profile, $\{a, c\}$ minimally responsively dominates $\{b, d\}$ and $\{b, e\}$ minimally responsively dominates $\{a, f\}$. Thus, both most preferred alternatives of voter 1 are element of a minimally responsively dominated set. Such observations are not intuitive and explain the difficulties in proving that not all alternatives can be element of a minimally responsively dominated set. Nevertheless, we strongly believe that this conjecture is true. However,

| | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|
| $R:$ | $a, b$ | $c$ | $e$ | $a, e, d$ | $b, c, f$ |
| | $c, d, e, f$ | $a, b, d, e, f$ | $a, b, c, d, f$ | $b, c, f$ | $a, e, d$ |

Figure 5.4: Preference profile used to show the unexpected behavior of minimal responsive dominance

we do not investigate it any further because Theorem 5.5 shows that the social choice function that removes all alternatives contained in a minimally responsively dominated set from the choice set is $\tilde{P}$-manipulable.
Instead, we focus on other approaches for designing responsively efficient social choice functions. Therefore, we note that responsive efficiency is related to the ranks of alternatives. Recall for this that if a set $X$ responsively dominates the set $Y$, then there is a bijection $\pi_i : X \mapsto Y$ for every voter $i \in N$ such that $x \succeq_i \pi(x)$. This implies that the rank of $x$ is at most as large as the rank of $\pi_i(x)$. Consequently, the sum $\sum_{x \in X} r_+(R_i, x)$ is not larger than the sum $\sum_{y \in Y} r_+(R_i, y)$. Even more, this inequality is strict for at least one voter as there must be a voter $j$ and an alternatives $x \in X$ such that $x \succ_j \pi_j(x)$. Hence, the sum of the ranks of all alternatives in $X$ over all voters is strictly less than the same sum for $Y$. This means that a set which minimizes this sum cannot be responsively dominated.
For formalizing this approach, we first introduce the rank sum of a set of alternatives. This term is discussed in the next definition where we use the rank extension $r_+$ introduced in Definition 3.19. However, the results hold also for all other rank extensions and we use $r_+$ only because of its simplicity.

**Definition 5.4 (Rank sum of a set of alternatives).** *The rank sum of a set $X \subseteq A$ in the preference profile $R$ is defined as $rank(R, X) = \sum_{x \in X} \sum_{i \in N} r_+(R_i, x)$.*

Thus, the previous mentioned intuition is that if $X$ responsively dominates $Y$ in $R$, then $rank(R, X) < rank(R, Y)$. We formalize this intuition to derive a criterion for designing responsively efficient social choice functions.

**Theorem 5.1.** *Consider a preference profile $R \in \mathcal{W}^n$ and a set of alternatives $X \subseteq A$. If $X$ is not responsively efficient in $R$, then there are sets $Y \subseteq X$ and $Z \subseteq A$ such that $|Y| = |Z|$ and $rank((X \setminus Y) \cup Z) < rank(X)$.*

Proof: Consider an arbitrary preference profile $R$ and a set of alternatives $X$ that is not responsively efficient in $R$. Thus, there is a subset $Y \subseteq X$ such that $Y$ is responsively dominated by another set $Z \subseteq A$. This means that there is a bijection $\pi_i : Z \mapsto Y$ for every voter $i \in N$ such that $z \succeq_i \pi_i(z)$ for all $z \in Z$ and for at least one voter and one alternative this is strict. Consequently, we can deduce that $rank(Z) < rank(Y)$ and therefore, the following inequality is true.

$$rank(X) = rank(X \setminus Y) + rank(Y) > rank(X \setminus Y) + rank(Z) \geq rank((X \setminus Y) \cup Z)$$

The strict inequality in this equation follows from the fact that $rank(Z) < rank(Y)$ and the second inequality follows as $X \setminus Y$ and $Z$ may have a non-empty intersection. Thus, the theorem is proven. □

Note that Theorem 5.1 is very helpful for designing responsively efficient social choice functions. For instance, it follows from this result that Borda's rule is responsively efficient as it always chooses a set that contains only alternatives $a$ that maximize $rank(\{a\})$. Similarly, we can develop many other ideas for designing responsively efficient social choice functions with the help of Theorem 5.1. Two of these ideas are presented in the sequel.

The first approach directly uses the intuition of Theorem 5.1: If a set is not responsively efficient, then we can simply replace a responsively dominated subset with its dominator. Hence, we need to choose an initial set and apply this replacing procedure until we arrive at a responsively efficient choice set. As the rank sum of the set strictly decreases after replacing a responsively dominated set and the rank sum of a set is bounded, it follows that this process terminates eventually. Thus, this approach is indeed well-defined. A drawback of this approach is that it cannot be neutral as we always have to decide on a set to replace. For instance, assume that $\{e, f\}$ responsively dominates $\{a, b\}$ and $\{d, e\}$ responsively dominates $\{a, c\}$. Then, we have to choose whether to remove $\{a, b\}$ or $\{a, c\}$ from the choice set. This is one of the key problems of this approach which leads to the fact that the social choice functions based on this idea are $\tilde{P}$-manipulable.

Next, we discuss a social choice function implementing this replacement routine. This SCF starts with a set of alternatives that contains for every voter one his most preferred alternatives and that has minimal size. For instance, consider the profile $R$ shown in Figure 5.5 and assume that we start at the lexicographic smallest set that contains one of the most preferred alternatives of every voter and that we apply the replacement routine in lexicographic order. Thus, we start for $R$ with the set $\{a, b, c\}$. Subsequently, the SCF notes that this set is not responsively efficient and therefore, $\{a, b\}$ is replaced with $\{e, f\}$. This leads to the responsively efficient set $\{c, e, f\}$ which is also the choice set. The beauty of this social choice function is that it leads to a rather small choice set that still contains one of the most preferred alternatives of every voter. The reason for this is that if an alternative $x$ is element of a responsively dominated set $X$ even though it is among the most preferred alternatives of voter $i$, then the dominating set $Y$ contains another one of voter $i$'s most preferred alternatives. Otherwise, there is no alternative in $Y$ that is weakly preferred to $x$ by voter $i$. This shows that a responsively efficient social choice function can contain one of the most preferred alternatives of every voter. Even though this property is often an indicator for $\tilde{P}$-strategyproofness, it is easy to show that this social choice function is $\tilde{P}$-manipulable by exploiting its non-neutrality.

The second approach for defining social choice functions with the help of Theorem 5.1 is to choose sets that minimize the rank sum. Without any further restrictions, this leads directly to a variant of Borda's rule based on $r_+$. However, note that we can

| | 1 | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|---|
| | $a, d, e, i$ | $a, d, f, j$ | $b, f, g$ | $b, e, h$ | $c, g, i$ | $c, h, j$ |
| $R:$ | $b, f, g$ | $b, e, h$ | $c, h, j$ | $c, g, i$ | $a, d, f, j$ | $a, d, e, i$ |
| | $h, j$ | $g, i$ | $d, e, i$ | $a, f, j$ | $e, h$ | $f, g$ |
| | $c$ | $c$ | $a$ | $d$ | $b$ | $b$ |

Figure 5.5: Preference profiles used for explaining responsive efficiency

also try to find larger social choice functions by adding constraints to the choice set. One idea for this is to require that the choice set contains for every voter at least one alternative that is among his most preferred ones. This condition gives a lower bound on the size of the choice set and therefore, it leads to larger choice sets. Furthermore, the choice set of this SCF contains for every voter at least one of his most preferred alternatives. For an application of this social choice function, consider again the profile $R$ shown in Figure 5.5. We have to calculate the rank sum of every set that contains for every voter one of his most preferred alternatives. For instance, we compute that $rank(\{c, e, f\}) = 84$ and $rank(\{a, g, h\}) = 82$. As 82 is minimal, the choice set for this profile is $\{a, g, h\}$. Finally, note that this approach is not neutral as there can be multiple choice sets with the same score in the end, which means that we have to select a set non-neutrally. Furthermore, it is also not $\tilde{P}$-strategyproof as a voter can manipulate the tie-breaking between sets with minimal rank sum in his favor by worsening alternatives.

## 5.3 $\tilde{P}$-strategyproofness and Responsive Efficiency

We have seen several responsively efficient social choice functions in the last section. However, all of them failed $\tilde{P}$-strategyproofness, which leads to the conjecture that there are no or only very few social choice functions that are both responsively efficient and $\tilde{P}$-strategyproof. For answering this conjecture, we consider the combination of these two axioms in detail. First, we prove in Section 5.3.1 that there are some $\tilde{P}$-strategyproof and responsively efficient social choice functions in rather restricted settings. After that, we consider impossibility results based on $\tilde{P}$-strategyproofness and responsive efficiency in Section 5.3.2. We can show in this section that there is no simple social choice function that satisfies $\tilde{P}$-strategyproofness, responsive efficiency and anonymity. However, we cannot completely rule out the existence of these social choice functions.

### 5.3.1 $\tilde{P}$-Strategyproof and Responsively Efficient Social Choice Functions

In this section, we discuss $\tilde{P}$-strategyproof and responsively efficient social choice functions in the weak domain. Unfortunately, it is very hard to find such functions and therefore, we either have to restrict the number of voters and alternatives or to violate other important axioms such as anonymity. Nevertheless, the results in this section show that there are $\tilde{P}$-strategyproof and responsively efficient social choice functions and they give lower bounds for the number of voters and alternatives required for impossibility results.
First, we focus on the setting with a restricted number of alternatives and voters. The easiest way to design a responsively efficient social choice function under these assumptions is to show that responsive dominance is equivalent to Pareto-dominance. Thus, we prove in the next lemma that the Pareto-rule, i.e., the SCF that picks all Pareto-optimal alternatives, is responsively efficient if $n \leq 2$ or $m \leq 3$. This implies that there is a $\tilde{P}$-strategyproof and responsively efficient social choice function if there are very few alternatives or voters as the Pareto-rule satisfies these axioms.

**Lemma 5.3.** *The Pareto-rule is responsively efficient in the weak domain if $m \leq 3$ or $n \leq 2$.*

Proof: We first consider the case that there are $m \leq 3$ alternatives and an arbitrary number of voters. In this case, Lemma 5.1 implies that every minimal responsive dominance relation is defined on sets of size at most $m/2$. This means that minimal responsive dominance is equivalent to Pareto-dominance and therefore, the Pareto-rule is responsively efficient if $m \leq 3$.
Next, consider the case that there are $n \leq 2$ voters and an arbitrary number of alternatives. As in the first case, we are only interested in minimal responsive dominance as this relation suffices to discuss responsive efficiency. We prove again that every minimal responsive dominance relation is defined on sets of size 1, which means that it is equivalent to Pareto-dominance. First, we focus consider the case that $n = 1$. This assumption means that only the most preferred alternatives of the single voter are Pareto-optimal. Furthermore, it is impossible that a subset of his most preferred alternatives is responsively dominated as there cannot be a strict preference. Hence, Pareto-dominance and minimal responsive dominance are in this case equivalent, which means that the Pareto-rule is $\tilde{P}$-strategyproof.
Finally, we analyze the situation with $n = 2$ voters. Thus, assume for contradiction that there are sets $X, Y \subseteq A$ and a profile $R \in \mathcal{W}^2$ such that $|X| = |Y| > 1$ and $X$ minimally responsively dominates $Y$. Furthermore, let $a$ denote one of the most preferred alternatives of voter 1 in $X$, i.e., $a \succeq_1 b$ for all $b \in X$. As $X$ responsively dominates $Y$, it follows that for every alternative $x \in X$ there is an alternative $y \in Y$ such that $x \succeq_1 y$. Because of the transitivity of individual preferences, it follows that

voter 1 prefers $a$ weakly to all alternatives in $Y$. Moreover, there is an alternative $b \in Y$ such that voter 2 prefers $a$ weakly to $b$ as otherwise $X$ cannot responsively dominate $Y$. Thus, there is an alternative $b \in Y$ such that $a \succeq_i b$ for all voters $i \in N$. If one of the voters prefers $a$ strictly to $b$, then $a$ Pareto-dominates $b$ contradicting that $X$ minimally responsively dominates $Y$. Otherwise, both voters are indifferent between $a$ and $b$, which means that these alternatives are indistinguishable. This implies that $X$ responsively dominates $(Y \setminus \{b\}) \cup \{a\}$ and by Lemma 5.1, it follows that $X \setminus \{a\}$ responsively dominates $Y \setminus \{a\}$. This contradicts again that $X$ minimally responsively dominates $Y$ and therefore, the initial assumption is wrong. Thus, Pareto-dominance and minimal responsive dominance are equivalent if $n \leq 2$, which implies that the Pareto-rule is responsively efficient in this case. □

As consequence of Lemma 5.3, it follows that every impossibility result relying on responsive efficiency, $\tilde{P}$-strategyproofness, neutrality and anonymity requires at least 4 alternatives and 3 voters. Otherwise, the Pareto-rule satisfies all these axioms. Moreover, we can push the boundaries even further if we restrict both the number of voters and alternatives. Therefore, we show that the Pareto-rule is still responsively efficient if $m \leq 5$ and $n \leq 3$.

**Lemma 5.4.** *The Pareto-rule is responsively efficient in the weak domain if $m \leq 5$ and $n \leq 3$.*

Proof: First note that the cases that $n \leq 2$ and that $m \leq 3$ are covered by Lemma 5.3. Thus, we assume that there are $n = 3$ voters and $4 \leq m \leq 5$ alternatives. As in the proof of Lemma 5.3, we derive from these assumptions that Pareto-dominance is equal to minimal responsive dominance. Thus, assume for contradiction that the previous claim is wrong, i.e., there is a profile $R$ and sets $X, Y \subseteq A$ such that $|X| = |Y| > 1$ and $X$ minimally responsively dominates $Y$. As consequence of Lemma 5.1, this means that $|X| = |Y| = 2$, i.e., $X = \{a, b\}$ and $Y = \{c, d\}$ for some alternatives $a, b, c, d \in A$. We prove in the sequel that there are always alternatives $x \in X$, $y \in Y$ such that every voter prefers $x$ weakly to $y$. This contradicts the minimality of the responsive dominance relation as either $x$ Pareto-dominates $y$ if the preference of at least one voter is strict, or $x$ and $y$ are indistinguishable in $R$. The latter case means that $X$ responsively dominates $(Y \setminus \{y\}) \cup \{x\}$ and therefore, $X \setminus \{x\}$ responsively dominates $Y \setminus \{x\}$ because of Lemma 5.1. This contradicts again that $X$ minimally responsively dominates $Y$.
Thus, it only remains to find alternatives $x \in X$, $y \in Y$ such that $x \succeq_i y$ for all $i \in N$. Therefore, we consider the preference between the alternatives $a, b \in X$. The reason for this is that if a voter prefers $a$ weakly to $b$, he prefers $a$ weakly to every alternative in $Y$. Otherwise, there is an alternative $y \in Y$ that is strictly preferred to $a$ and therefore, it is strictly preferred to every alternative in $X$. However, this contradicts that $X$ responsively dominates $Y$. Clearly, a symmetric argument also holds for voters who prefer $b$ weakly to $a$. Hence, let $X_a = \{i \in N \mid a \succeq: ib\}$ denote the set of voters who prefer $a$ weakly to $b$ and $X_b = \{i \in N \mid b \succeq_i a\}$

$R^1$:

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ |
| $c$ | $d$ | $c$ | $d$ |
| $b$ | $b$ | $a$ | $a$ |
| $d$ | $c$ | $d$ | $c$ |

$R^2$:

| 1 | 1 | 1 |
|---|---|---|
| $a$ | $b$ | $c$ |
| $d$ | $e$ | $f$ |
| $b$ | $c$ | $a$ |
| $f$ | $d$ | $e$ |
| $c$ | $a$ | $b$ |
| $e$ | $f$ | $d$ |

Figure 5.6: Preference profiles used to show that the bounds in Lemma 5.4 are tight

denote the set of voters who prefer $b$ weakly to $a$. As $n = 3$, it is clear that at least one of the sets contains at least two voters. We assume in the sequel that $|X_a| \geq 2$ as the case $|X_b| \geq 2$ is symmetric. This assumption means that there are two voters who prefer $a$ at least weakly to every alternative in $Y$. The remaining voter also prefers $a$ weakly to at least one alternative in $Y$ as otherwise $X$ does not responsively dominate $Y$. Thus, $a$ is weakly preferred to an alternative in $Y$ by every voter. As previously explained, this contradicts that $X$ minimally responsively dominates $Y$ and therefore, our initial assumption is wrong. Hence, there cannot be a preference profile $R$ and sets $X, Y \subseteq A$ such that $X$ responsively dominates $Y$ and $|X| = |Y| > 1$. This implies that minimal responsive dominance is equivalent to Pareto-dominance if $m \leq 5$ and $n \leq 3$. Therefore, it follows that the Pareto-rule is responsively efficient. □

It should be mentioned that the bound on both the number of alternatives and voters presented in the last lemma is tight. To see this, consider the preference profiles shown in Figure 5.6. In the profile $R^1$, $\{a, b\}$ minimally responsively dominates $\{c, d\}$, which means that Lemma 5.4 is not true if there are $n = 4$ voters and $m = 4$ alternatives. Furthermore, $\{a, b, c\}$ minimally responsively dominates $\{d, e, f\}$ in $R^2$. Thus, the Pareto-rule is not responsively efficient either if $m = 6$ and $n = 3$.

Even though the Pareto-rule is not responsively efficient anymore if $m = n = 4$, this does not mean that there cannot be a $\tilde{P}$-strategyproof and responsively efficient social choice function. It can indeed be shown by SAT-solving that there is a social choice function that satisfies these axioms and is even neutral and anonymous if there are $m = 4$ alternatives and $n = 4$ voters. However, this SCF is rather complicated and it is not clear how to represent it in a formal and succinct way. Therefore, we do not discuss it in detail. Next, it should be mentioned that, while it is appealing to use SAT-solving to find responsively efficient and $\tilde{P}$-strategyproof social choice functions, we cannot use this method to derive social choice functions that satisfy these axioms and are defined on more than four voters and alternatives due to the exponential growth of the SAT-formula. Thus, we have to leave it as an open problem whether there is a social choice function that satisfies $\tilde{P}$-strategyproofness, responsive efficiency, neutrality and anonymity for larger numbers of alternatives and voters than discussed in the previous results. However, if there is an impossibility

| | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| | $a, b, c$ | $d$ | $a$ | $e, c$ |
| $R$ : | $d$ | $a, b$ | $c$ | $d, b$ |
| | $e$ | $c, e$ | $b$ | $a$ |
| | | | $d, e$ | |

Figure 5.7: Preference profile used for explaining sequential dictator

result based on these axioms, it seems likely that it requires at least 6 alternatives. The reason for this is that responsive efficiency becomes more powerful if there are more alternatives and it is not that restrictive if there are $m < 6$ alternatives. Even more, we know that there is a social choice function that satisfies $\tilde{P}$-strategyproofness and responsive efficiency if $n = m = 4$. Thus, it seems rather likely that there is such a function for $m = 4$ and $n > 4$ as Lemma 5.2 implies that responsive efficiency does not become more restrictive if there are more voters and $m = 4$.
As consequence of the previous results, the question arises whether we can find a social choice function that satisfies $\tilde{P}$-strategyproofness and responsive efficiency for an unbounded number of alternatives. It turns out that there is such a function which we call sequential dictator and abbreviate with SD. This social choice function fixes an order over the voters, i.e., it considers them in increasing order one by one. For each voter, it removes all alternatives from the choice set but those that he prefers the most. For instance, consider the preference profile $R$ shown in Figure 5.7 and assume that sequential dictator considers the voters from left to right. Thus, before we consider the first voter, the choice set contains all alternatives. In the next step, only $\{a, b, c\}$ remains as these are the most preferred alternatives of the first voter. Subsequently, voter 2 removes $c$ from this set as he is indifferent between $a$ and $b$ but prefers both alternatives strictly to $c$. Note that the alternatives that have already been removed from the choice set are ignored. Furthermore, the choice set reduces to $\{a\}$ after considering the third voter. Thus, the last voter does not have a choice anymore, which means that $\text{SD}(R) = \{a\}$.
Note that sequential dictator is clearly not anonymous as it depends on the order in which we iterate over the voters. However, as we prove in the sequel, it is both $\tilde{P}$-strategyproof and responsively efficient in the weak domain for arbitrary numbers of voters and alternatives.

**Theorem 5.2.** *Sequential dictator is $\tilde{P}$-strategyproof and responsively in the weak domain.*

Proof: First, we prove that sequential dictator is responsively efficient. Thus, assume for contradiction that there is a profile $R \in \mathcal{W}^n$ such that $\text{SD}(R)$ is not responsively efficient. This means that there is a set of alternatives $X \subseteq \text{SD}(R)$ that is minimally responsively dominated by another set $B$ in $R$. Note that every voter is indifferent between all alternatives in $\text{SD}(R)$. Otherwise, there is a voter $i \in N$ who restricts the choice set even further. Therefore, every voter prefers every alternative in $B$

weakly to every alternative in $\mathrm{SD}(R)$. As this preference is strict for at least one voter and one alternative $b \in B$, it follows that $b$ Pareto-dominates every alternative in $\mathrm{SD}(R)$. Hence, we can deduce that there is a voter $i \in N$ during the computation of SD such that he prefers $b$ strictly to all alternatives in $\mathrm{SD}(R)$ and all voters considered before voter $i$ are indifferent between $b$ and the alternatives in $\mathrm{SD}(R)$. However, this means that the choice set contains both $b$ and $\mathrm{SD}(R)$ until sequential dictator considers voter $i$. Furthermore, as this voter prefers $b$ strictly to $\mathrm{SD}(R)$, it follows that $\mathrm{SD}(R)$ is not the correct choice set of sequential dictator. This is a contradiction, which means that the initial assumption is wrong. Hence, sequential dictator is indeed responsively efficient.
Next, we focus on the $\tilde{P}$-strategyproofness of sequential dictator. Thus, consider an arbitrary voter $i \in N$ and let $X$ denote the choice set before sequential dictator considers voter $i$. It follows from the definition of sequential dictator that no alternative in $A \setminus X$ is chosen regardless of the preference of voter $i$. Furthermore, SD returns a subset of voter $i$'s most preferred alternatives in $R|_X$. Thus, it is straightforward that voter $i$ cannot $\tilde{P}$-manipulate, which means that sequential dictator is $\tilde{P}$-strategyproof. □

As consequence of Theorem 5.2, there is a $\tilde{P}$-strategyproof, responsively efficient and even neutral social choice function. Furthermore, SD satisfies many other desirable axioms such as monotonicity. Hence, every impossibility result based on $\tilde{P}$-strategyproofness and responsive efficiency requires additionally anonymity as sequential dictator violates this axiom.

### 5.3.2 Impossibility Results based on $\tilde{P}$-strategyproofness and Responsive Efficiency

As we have seen in the last section, it is very hard to find meaningful social choice functions that satisfy responsive efficiency and $\tilde{P}$-strategyproofness in the weak domain. Furthermore, we have already discussed many impossibility results involving Pareto-optimality such as Theorem 3.12 or theorem 2 in [BSS]. Thus, it seems reasonable to conjecture that there are also impossibility results based on responsive efficiency. We prove two such results: Firstly, we show in Theorem 5.3 that there is no $\tilde{P}$-strategyproof and responsively efficient C2-function in the weak domain. This result can be easily derived from Theorem 4.6. Furthermore, this result can be viewed as a first step of answering the initial question of this chapter asking about the existence of anonymous, $\tilde{P}$-strategyproof and responsively efficient social choice functions. Since it follows from Theorem 3.12 and Theorem 5.3 that there is neither a rank-based social choice function nor a C2-function that satisfies $\tilde{P}$-strategyproofness and responsive efficiency, we can deduce that no commonly considered approach leads to an anonymous social choice function that satisfies these two axioms. However, the existence of more complicated social choice functions

that satisfy anonymity, responsive efficiency and $\tilde{P}$-strategyproofness is still possible. Thus, we use a strong technical assumption and neutrality to prove that there is no such social choice function. Note that the technical assumption is easily seen to be true if we use another axiom. However, we cannot prove it in general and therefore, we can neither prove nor disprove the existence of $\tilde{P}$-strategyproof, responsively efficient and anonymous social choice functions.
First, we focus on a variation of theorem 2 in [BSS] stating that there is no $\tilde{P}$-strategyproof, Pareto-optimal and pairwise social choice function if $m \geq 3$ and $n \geq 3$. The question that arises naturally is whether we can weaken pairwiseness if we require responsive efficiency instead of Pareto-optimality. As we prove in the sequel with the help of Theorem 4.6, we can weaken pairwiseness to C2 and show that there is no $\tilde{P}$-strategyproof and responsively efficient social choice function in C2.
As in previously discussed impossibility results, we want to focus on a small number of alternatives and voters. This means that we need a lemma that allows us to generalize an impossibility result for a fixed number of voters and alternatives to arbitrary larger values. This issue is addressed with the next lemma.

**Lemma 5.5.** *Assume there is a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies $\tilde{P}$-strategyproofness, responsive efficiency and C2. Then, there is a social choice function $g : \mathcal{W}^{n'} \mapsto 2^{A'} \setminus \emptyset$ that satisfies $\tilde{P}$-strategyproofness, responsive efficiency and C2 for all $n' \leq n$ and $A' \subseteq A$.*

<u>Proof:</u> Consider a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies all axioms in the lemma, a subset of alternatives $A' \subseteq A$ and a number of voters $n' \leq n$. We prove in the following that there is a social choice function $g : \mathcal{W}^{n'} \mapsto 2^{A'} \setminus \emptyset$ that is also $\tilde{P}$-strategyproof, responsively efficient and in C2 by an induction on $n'$ and $m' = |A'|$. By repeatedly applying the induction steps, we can deduce the existence of $g$.
First, we focus on the number of voters $n'$. Therefore, we assume that there is a SCF $f$ that satisfies all requirements of the lemma and show that there is a $\tilde{P}$-strategyproof and responsively efficient C2-function $g_1 : \mathcal{W}^{n-1} \mapsto 2^A \setminus \emptyset$. We construct $g_1$ as follows: Given an input profile $R$ defined on $n-1$ voters, we add a new voter that is indifferent between all alternatives. This leads to a new profile $R'$ defined on $n$ voters. Finally, we set $g_1(R) = f(R')$. Clearly, $g_1$ inherits the $\tilde{P}$-strategyproofness from $f$ and is also in C2. Furthermore, adding a voter that is indifferent between all voters does not affect any responsive dominance relation. Thus, $g_1$ is also responsively efficient because $f$ satisfies this axiom, which means that $g_1$ satisfies all required properties.
Next, we discuss the induction step with respect to the number of alternatives $m$. Thus, assume that there is a social choice function $f$ defined on $m = |A|$ alternatives and $n$ voters that satisfies $\tilde{P}$-strategyproofness, responsive efficiency and C2. We construct in the sequel a social choice function $g_2 : \mathcal{W}^n \mapsto 2^{A'} \setminus \emptyset$ with $A' \subsetneq A$ and $m' = |A'| = m - 1$ that satisfies the same axioms as $f$. This SCF is defined

as follows: Given an input profile $R$ defined on $m-1$ alternatives, it adds a new alternative $x \in A \setminus A'$ as the least preferred alternative of every voter. This leads to a new preference profile $R'$ defined on $m$ alternatives. Finally, we set $g_2(R) = f(R')$. It is easy to see that $g_2$ satisfies $\tilde{P}$-strategyproofness, responsive efficiency and C2 as $f$ also satisfies these axioms. Furthermore, this social choice function is well-defined because the new alternative $x$ cannot be chosen as it is Pareto-dominated by every other alternative. Thus, we have shown the induction steps on $n$ and $m$ and by repeatedly applying them, we can deduce that the lemma holds for all $n' \leq n$ and $A' \subseteq A$. □

Note that this lemma uses a standard construction that we have already seen in Lemma 3.8. We even use this result in the same way: We show that there is no social choice function $f$ defined for a small fixed number of voters and alternatives that satisfies $\tilde{P}$-strategyproofness, responsive efficiency and C2. From the contraposition of Lemma 5.5, it follows that there is no such social choice function defined on more voters or alternatives either. Thus, we focus on proving the impossibility for $m = 4$ alternatives and $n = 4$ voters.

**Theorem 5.3.** *There is no social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ in C2 that satisfies $\tilde{P}$-strategyproofness and responsive efficiency if $m \geq 4$ and $n \geq 4$.*

Proof: We focus in this proof on the case that $n = 4$ and $m = 4$ and show that there is no social choice function that satisfies all axioms required by the theorem. It follows from Lemma 5.5 that there cannot be such a social choice function for a larger number of voters and alternatives. Thus, assume for contradiction that there is a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that is defined on $n = 4$ voters and $m = 4$ alternatives and that satisfies $\tilde{P}$-strategyproofness, responsive efficiency and C2. Note that responsive efficiency implies Pareto-optimality and therefore, we can use Theorem 4.6. Therefore, we deduce a contradiction by showing that there is a profile $R$ such that $f(R)$ does not contain any of the most preferred alternatives of a voter.
Hence, consider the profile $R^1$ shown in Figure 5.8. It is easy to see that no alternative is Pareto-dominated and that the set $\{a, b\}$ responsively dominates the set $\{c, d\}$ in $R^1$. This means that $c$ or $d$ or both alternatives are not in $f(R^1)$. Furthermore, we use C2 to derive the profile $R^2$: Voter 1 and 3 exchange their preferences over $b$ and $c$, and voter 2 and 4 exchange the preferences over $b$ and $d$. As consequence, voter 3 prefers $c$ uniquely the most and voter 4 prefers $d$ uniquely the most. Even more, it follows from C2 that $f(R^2) = f(R^1)$ as $R^1$ and $R^2$ have the same majorities. This means that the most preferred alternative of the third or the fourth voter is not in $f(R^2)$, which contradicts Theorem 4.6. Thus, it follows that the initial assumption is wrong, which means that there is no social choice function $f$ defined on $n \geq 4$ voters and $m \geq 4$ alternatives that satisfies $\tilde{P}$-strategyproofness, Pareto-optimality and C2. □

| | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $R^1$ | $a \succ c \succ b \succ d$ | $a \succ d \succ b \succ c$ | $b \succ c \succ a \succ d$ | $b \succ d \succ a \succ c$ |
| $R^2$ | $a \succ b \succ c \succ d$ | $a \succ b \succ d \succ c$ | $c \succ b \succ a \succ d$ | $d \succ b \succ a \succ c$ |

Figure 5.8: Preference profiles used in the proof of Theorem 5.3

First, we discuss the independence of the axioms used in the Theorem 5.3. Thus, note that the social choice function sequential dictator discussed in Theorem 5.2 is responsively efficient and $\tilde{P}$-strategyproof but not in C2. Furthermore, the Pareto-rule is $\tilde{P}$-strategyproof and in C2 but not responsively efficient. This observation combined with Lemma 5.3 and Lemma 5.4 implies that $m \geq 4$ is required and that $n \geq 4$ is required unless we use more alternatives. Hence, even the conditions on the number of voters and alternatives are necessary. Finally, there are responsively efficient C2-functions that are not $\tilde{P}$-strategyproof. For instance, consider the SCF that always chooses non-neutrally a single Pareto-optimal alternative. This shows that all axioms used in the theorem are independent from each other.
It should be mentioned that Theorem 5.3 shows another interesting application of Theorem 4.6. However, it is easy to see that the general idea of this impossibility cannot be used to prove stronger results. The reason for this is that the proof is based on the fact that C2 and responsive efficiency imply that the uniquely most preferred alternative of a voter is not chosen. However, we have already discussed responsively efficient social choice functions that choose for every voter at least one of his most preferred alternatives in Section 5.2. Thus, we additionally require neutrality in the sequel and show that every neutral, anonymous, responsively efficient and $\tilde{P}$-strategyproof social choice function has a profile in which none of the most preferred alternatives of a voter are chosen. Consequently, it only remains to prove a variation of Theorem 4.6 using responsive efficiency instead of Pareto-optimality, and anonymity and neutrality instead of C2. Unfortunately, we cannot prove this claim. Therefore, we assume it as given for the next theorem and leave it as open problem whether this assumption is true.

**Theorem 5.4.** *Assume that every responsively efficient, neutral and anonymous social choice function $f$ is $\tilde{P}$-manipulable if there is a preference profile where $f$ does not choose any of the most preferred alternatives of a voter. Then, there is no social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies anonymity, neutrality, $\tilde{P}$-strategyproofness and responsive efficiency if $m \geq 4$ and $n \geq 4$.*

Proof: First note that we focus on the case that there are $m = 4$ alternatives and $n = 4$ voters. The reason for this is that the constructions discussed in Lemma 5.5 also maintain neutrality and anonymity. Thus, it suffices to discuss the theorem for fixed values of $n$ and $m$ as we can generalize it to arbitrary larger values. Furthermore, we assume for contradiction that there is a neutral, anonymous, $\tilde{P}$-strategyproof and responsively efficient social choice function $f$ defined on $n = 4$ voters and $m = 4$ alternatives. We show in the sequel with the profiles in Figure 5.9 that

| | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $R^1$ | $a \sim b \succ c \sim d$ | $a \sim d \succ b \sim c$ | $c \sim d \succ a \succ b$ | $b \sim c \succ a \succ d$ |
| $R^2$ | $a \sim c \succ b \sim d$ | $a \sim d \succ b \sim c$ | $b \sim d \succ a \succ c$ | $b \sim c \succ a \succ d$ |
| $R^3$ | $a \sim b \succ c \sim d$ | $a \sim d \succ b \sim c$ | $a \sim c \succ b \sim d$ | $b \sim c \succ a \succ d$ |

Figure 5.9: Preference profiles used in the proof of Theorem 5.4

$f(R^3) = \{a\}$, which means that none of the most preferred alternatives of the fourth voter are chosen. Thus, we can derive from the assumptions of the theorem that $f$ is $\tilde{P}$-manipulable, which is a contradiction.

It only remains to prove the claim on $f(R^3)$. Therefore, consider the profile $R^1$ and note that $\{a, c\}$ responsively dominates $\{b, d\}$ in this profile. This means that $\{b, d\} \not\subseteq f(R^1)$ by responsive efficiency. Moreover, if we rename $b$ to $d$ and vice versa and reorder the voters in $R^1$, we are still in the same profile. Thus, it follows from neutrality, anonymity and responsive efficiency that $b, d \notin f(R^1)$. Even more, observe that $R^2$ only differs from $R^1$ by renaming $b$ to $c$ and vice versa and therefore, we can derive from neutrality that $c, d \notin f(R^2)$.

Finally, note that we can derive $R^3$ from both $R^1$ and $R^2$ by letting the third voter manipulate and applying anonymity to reorder the voters. For going from $R^1$ to $R^3$, we replace the original preference of voter 3 with $a \sim c \succ b \sim d$. Since $f$ is $\tilde{P}$-strategyproof and $f(R^1) \subseteq \{a, c\}$, we can deduce that $f(R^3) \subseteq \{a, c\}$; otherwise, voter 3 can $\tilde{P}$-manipulate by switching from $R^3$ to $R^1$. Similarly, for going from $R^2$ to $R^3$, we let voter 3 replace his current preference $R^2_3$ with $a \sim b \succ c \sim d$ and exchange the preferences of voter 1 and 3. It follows from $\tilde{P}$-strategyproofness and anonymity that $f(R^3) \subseteq \{a, b\}$ as otherwise voter 3 can $\tilde{P}$-manipulate. These two observations imply that $f(R^3) = \{a\}$, which means that none of the most preferred alternatives of voter 4 are chosen. Thus, it follows from the assumptions of the theorem that $f$ is $\tilde{P}$-manipulable, which is a contradiction. □

It should be mentioned that this theorem assumes a variant of Theorem 4.6. Thus, it only remains to prove this variant using responsive efficiency, neutrality and anonymity. However, we are not sure whether this is possible at all as we have already discussed a neutral, anonymous, $\tilde{P}$-strategyproof and Pareto-optimal social choice function that does not choose any of the most preferred alternatives of a voter in Lemma 4.10.

As we cannot prove the assumptions of Theorem 5.4, we discuss one of the implications of this result instead. Hence, we use an additional axiom which we call dependence on responsive dominance. The intuition of this axiom is that a social choice function satisfying it returns the same choice set for all profiles with the same responsive dominance relations. While this axiom has not much meaning in general, it is often satisfied by simple approaches for defining responsively efficient social choice functions. For instance, dependence on responsive dominance is satisfied by the SCF that removes all alternatives contained in a minimally responsively dominated set from the choice set. Formally, this axiom is defined as follows.

**Definition 5.5 (Dependence on responsive dominance).** *A social choice function $f$ depends on responsive dominance if $f(R) = f(R')$ for all preference profiles $R$, $R'$ such that $X$ responsively dominates $Y$ in $R$ if and only if $X$ responsively dominates $Y$ in $R'$ for all set of alternatives $X, Y \subseteq A$.*

Note that dependence on responsive dominance implies anonymity as reordering the voters does not affect the responsive dominance relation. Furthermore, dependence on responsive dominance is a rather strong axiom, i.e., a social choice function satisfying it returns for many profiles the same choice set. Finally, it should be mentioned that dependence on responsive dominance is independent from C2. For seeing this, consider the profiles shown in the proof of Theorem 5.3. First, observe that a C2-function returns the same choice set for the profiles $R^1$ and $R^2$ shown in this proof. In contrast, this is not implied by dependence on responsive dominance as no set is responsively dominated in $R^2$ but $\{a, b\}$ responsively dominated $\{c, d\}$ in $R^1$. Furthermore, we can deduce that a social choice function satisfying dependence on responsive dominance is not allowed to change the choice set if we replace the preference of voter 1 in $R^1$ with $a \sim c \succ b \sim d$. However, this modification changes the majorities and therefore, C2 allows for different choice sets.
Our main intention in introducing this new axiom is to show that there is no neutral, responsively efficient and $\tilde{P}$-strategyproof social choice function that additionally satisfies dependence on responsive dominance. We use Theorem 5.4 to prove this claim as shown in the next theorem.

**Theorem 5.5.** *There is no responsively efficient social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies neutrality, $\tilde{P}$-strategyproofness and dependence on responsive dominance if $m \geq 4$ and $n \geq 4$.*

<u>Proof:</u> Assume for contradiction that there is a social choice function $f : \mathcal{W}^n \mapsto 2^A \setminus \emptyset$ that satisfies neutrality, $\tilde{P}$-strategyproofness, responsive efficiency and dominance on responsive dominance and that is defined on $n \geq 4$ voters and $m \geq 4$ alternatives. As we want to use Theorem 5.4 to derive a contradiction, we have to show that a voter can $\tilde{P}$-manipulate the social choice function $f$ if $f(R)$ does not contain any of the most preferred alternatives of a voter. Thus, assume that there is a profile $R^1 \in \mathcal{W}^n$ such that $f(R^1)$ does not contain any of voter $i$'s most preferred alternatives. As $f$ satisfies $\tilde{P}$-strategyproofness and responsive efficiency which implies Pareto-optimality, we can use Lemma 4.4 and Lemma 4.5. Thus, we can derive a profile $R^2$ in which all voters in $N \setminus \{i\}$ prefer an alternative $a \in f(R^1)$ uniquely the most, voter $i$ prefers $a$ uniquely the least and $f(R^2) = \{a\}$. Next, we let every voter $j \in N \setminus \{i\}$ manipulate one after another such that all voters in $N \setminus \{i\}$ have the same strict preference $R^3_j \in \mathcal{S}$ and prefer $a$ uniquely the most. It follows from the $\tilde{P}$-strategyproofness of $f$ that $a$ is still the unique winner in the resulting profile $R^3$

$R^1$ :

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a, b$ | $a, c$ | $a, d$ | $b, c$ |
| $c, d$ | $b, d$ | $b, c$ | $a$ |
| | | | $d$ |

$R^2$ :

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a$ | $a$ | $a$ | $b, c$ |
| $b$ | $c$ | $d$ | $d$ |
| $c, d$ | $b, d$ | $b, c$ | $a$ |

$R^3$ :

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a$ | $a$ | $a$ | $b, c$ |
| $b$ | $b$ | $b$ | $d$ |
| $c$ | $c$ | $c$ | $a$ |
| $d$ | $d$ | $d$ | |

$R^4$ :

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a$ | $a$ | $a$ | $d$ |
| $b$ | $b$ | $b$ | $c$ |
| $c$ | $c$ | $c$ | $b$ |
| $d$ | $d$ | $d$ | $a$ |

$R^5$ :

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $a, b, c, d$ | $a, b, c, d$ | $a, b, c, d$ | $a, b, c, d$ |

Figure 5.10: Preference profiles illustrating the proof of Theorem 5.5

as otherwise a voter can $\tilde{P}$-manipulate by reverting this modification. Finally, we let voter $i$ change his preference such that he orders all alternatives inversely, i.e., if $a \succ_j b$ for a voter $j \in N \setminus \{i\}$ and alternatives $a, b \in A$, then $b \succ_i a$. This leads to the profile $R^4$ and because of $\tilde{P}$-strategyproofness, it holds that $f(R^4) = \{a\}$.
Furthermore, observe that every set is responsively efficient in $R^4$ because all alternatives are strict and the worst alternative of a set $B \subseteq A$ in $R^4_j$, $j \in N \setminus \{i\}$, is the best one in $R^4_i$. Finally, consider the profile $R^5$ where every voter is indifferent between all alternatives. It follows from neutrality that $f(R^5) = A$ as we can rename alternatives arbitrarily without changing the preference profile. Furthermore, every set is responsively efficient in $R^5$ and therefore, it must hold that $f(R^4) = f(R^5)$ because $f$ depends on responsive dominance. However, this contradicts that $f(R^4) = \{a\}$, which implies that the initial assumption is wrong. Hence, we can use Theorem 5.4 to deduce that there cannot be a neutral, $\tilde{P}$-strategyproof and responsively efficient social choice function that depends on responsive dominance if $m \geq 4$ and $n \geq 4$. □

As a first remark, we provide an example illustrating the constructions in the proof of Theorem 5.5. Thus, consider the profiles shown in Figure 5.10 and assume that $f$ denotes a $\tilde{P}$-strategyproof, responsively efficient and neutral social choice function that depends on responsive dominance. Note that $f(R^1) = \{a\}$ holds because of Theorem 5.4. Next, we use Lemma 4.4 and Lemma 4.5 to deduce that $f(R^2) = \{a\}$. After that, we ensure that the preferences of the voters preferring $a$ the most are strict to derive the profile $R^3$. As consequence of the $\tilde{P}$-strategyproofness of $f$, it follows that $f(R^3) = \{a\}$. Even more, we use again this axiom to reorder the preference of the fourth voter in the inverse order of the remaining voters. This leads to the profile $R^4$ which satisfies that $f(R^4) = \{a\}$. Finally, dependence on responsive dominance implies that $f(R^4) = f(R^5)$, which contradicts neutrality.
Next, we discuss the independence of the axioms used in Theorem 5.5. First note that the Pareto-rule satisfies all axioms if $m \leq 3$, or $m \leq 5$ and $n \leq 3$. Thus, we need

at least 4 alternatives for this result and we need at least 4 voters unless we use more alternatives. Furthermore, if we assume that the social choice function that removes all alternatives that are element of a minimally responsively dominated set from the choice set is well-defined, then this SCF satisfies all axioms but $\tilde{P}$-strategyproofness. Next, the social choice function sequential dictator discussed in Theorem 5.2 satisfies all axioms of Theorem 5.5 but dependence of responsive dominance. Furthermore, the trivial social choice function that always returns all choice sets satisfies all axioms but responsive efficiency. Finally, it should be mentioned that we do not know whether neutrality is required for Theorem 5.5.

As consequence of Theorem 5.5, many straightforward ideas for designing $\tilde{P}$-strategyproof, neutral, anonymous and responsively efficient social choice functions are doomed to fail as these approaches often depend on responsive dominance. Even more, note that many of the coarsest responsively efficient social choice functions depend on responsive dominance. Thus, it follows from Theorem 4.1 and Theorem 5.5 that also many responsively efficient social choice functions cannot be simultaneously neutral, anonymous and $\tilde{P}$-strategyproof as they refine a SCF that additionally depends on responsive dominance.

Hence, the results in this chapter indicate that there is no anonymous, neutral, $\tilde{P}$-strategyproof and responsively efficient social choice function if there are sufficiently many voters and alternatives. The reason for this is that most social choice functions considered in the literature are either in C2 or rank-based, which cannot lead to a $\tilde{P}$-strategyproof and responsively efficient social choice function as shown in Theorem 5.3 and Theorem 3.12. Thus, the results in this section provide strong evidence for the impossibility of such a social choice function even though we cannot prove it.

Chapter 6

# Conclusion

In this thesis, we discuss election schemes with respect to manipulability. As it has already been shown in [Gib73, Sat75] that all reasonable single-valued social choice functions are manipulable, we focus on social choice functions that are allowed to return multiple winners. This leads to the problem of comparing sets of alternatives with each other because the preferences of the voters are only defined on individual alternatives. Thus, we use the set extension $\tilde{P}$ introduced in [Kel77] to compare sets of alternatives and discuss social choice functions with respect to $\tilde{P}$-strategyproofness.

First, we discuss sufficient conditions that imply $\tilde{P}$-strategyproofness. In particular, we analyze which variants of monotonicity imply variants of $\tilde{P}$-strategyproofness in the weak domain. These results generalize the results in [Bra15] and prove some of the remarks of this paper. For instance, we can show that set-monotonicity implies a weakened variant of $\tilde{P}$-group-strategyproofness even if preferences are allowed to be intransitive.

Thereafter, we take a closer look at a large class of social choice functions called rank-based social choice functions. We examine an important subclass of these functions which we call independent rank-based social choice functions. Furthermore, we prove that $\tilde{P}$-strategyproof and independent rank-based social choice functions pick rather large choice sets, which leads to a characterization of the OMNI-rule. Similarly, we also discuss other subclasses of rank-based social choice functions with respect to $\tilde{P}$-strategyproofness. For instance, we show that no non-trivial scoring rule is $\tilde{P}$-strategyproof. Furthermore, we generalize the concept of rank-basedness to the weak domain by introducing various rank extensions. Even though there are multiple reasonable rank extensions, we can provide an impossibility result stating that there is no $\tilde{P}$-strategyproof, rank-based and Pareto-optimal social choice function in the weak domain.

Moreover, we discuss in a similar manner another large class of social choice functions called C2-functions. This class contains many interesting social choice functions and even some that are known to be $\tilde{P}$-strategyproof in the strict domain. However, C2-functions are only rarely considered in the weak domain. Therefore, we analyze these social choice functions with respect to $\tilde{P}$-strategyproofness in the weak domain where only the Pareto-rule is known to be $\tilde{P}$-strategyproof and Pareto-optimal. We first discuss some ideas for defining $\tilde{P}$-strategyproof social choice functions in C2 which even lead to refinements of the Pareto-rule. Furthermore, we also see that these social choice functions must satisfy restrictive criteria. Thus, we discuss necessary conditions for $\tilde{P}$-strategyproof C2-functions. The strongest one states that every $\tilde{P}$-

strategyproof and Pareto-optimal C2-function must choose at least one of the most preferred alternatives of every voter if preferences may contain ties. This result has many important implications. For instance, we derive a characterization of the Pareto-rule from this condition.

Finally, we consider social choice functions that satisfy an axiom called responsive efficiency. This axiom is a generalization of Pareto-optimality that allows to compare sets of alternatives with each other. First, we analyze this axiom in detail in order to improve its interpretability. This leads to ideas for defining responsively efficient social choice functions. Unfortunately, all these approaches are $\tilde{P}$-manipulable. Consequently, we provide evidence suggesting that there are no reasonable social choice functions that satisfy $\tilde{P}$-strategyproofness and responsive efficiency. In particular, we show that there is no social choice function in C2 that satisfies $\tilde{P}$-strategyproofness and responsive efficiency in the weak domain. Combined with a previously mentioned result stating that there is no rank-based SCF that satisfies $\tilde{P}$-strategyproofness and Pareto-optimality, it follows that no commonly considered social choice function satisfies $\tilde{P}$-strategyproofness, responsive efficiency and anonymity. As consequence of these results, it seems likely that there is no $\tilde{P}$-strategyproof, anonymous and responsively efficient social choice function.

Unfortunately, some questions about $\tilde{P}$-strategyproofness remain unsolved. It is unclear to us whether there is a simple axiom that implies $\tilde{P}$-strategyproofness in the weak domain similar to set-monotonicity in the strict domain. Such a criterion would enhance the understanding of $\tilde{P}$-strategyproofness significantly as it allows to find social choice functions satisfying this axiom more easily. Furthermore, some conjectures about impossibility results are left open. For instance, it remains unclear whether there are anonymous, neutral, $\tilde{P}$-strategyproof and responsively efficient social choice functions. Even though we provide strong evidence suggesting that this conjecture is false, a definite answer has not been found yet.

Chapter 6

# Bibliography


[Ban82] Taradas Bandyopadhyay. Threats, counter-threats and strategic manipulation for non-binary group decision rules. *Mathematical Social Sciences*, 2(2):145–155, 1982.

[Ban83] Taradas Bandyopadhyay. Manipulation of non-imposed, non-oligarchic, non-binary group decision rules. *Economics Letters*, 11(1-2):69–73, 1983.

[Bar77a] Salvador Barberà. The manipulation of social choice mechanisms that do not leave "too much" to chance. *Econometrica*, 45(7):1573–1588, 1977.

[Bar77b] Salvador Barberà. Manipulation of social decision functions. *Journal of Economic Theory*, 15(2):266–278, 1977.

[BB11] Felix Brandt and Markus Brill. Necessary and sufficient conditions for the strategyproofness of irresolute social choice functions. In *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge*, pages 136–142. ACM press, 2011.

[BBGH15] Florian Brandl, Felix Brandt, Christian Geist, and Johannes Hofbauer. Strategic abstention based on preference extensions: Positive results and computer-generated impossibilities. In *24th International Joint Conference on Artificial Intelligence*, pages 18–24. AAAI press, 2015.

[BBH16] Felix Brandt, Markus Brill, and Bernhard Harrenstein. Tournament solutions. In *Handbook of Computational Social Choice*, pages 57–84. Cambridge University Press, 2016.

[BBH18] Felix Brandt, Markus Brill, and Paul Harrenstein. Extending tournament solutions. *Social Choice and Welfare*, 51(2):193–222, 2018.

[BBP04] Salvador Barberà, Walter Bossert, and Prasanta K. Pattanaik. Ranking sets of objects. In *Handbook of Utility Theory*, pages 893–977. Springer, 2004.

[BG16] Felix Brandt and Christian Geist. Finding strategyproof social choice functions via SAT solving. *Journal of Artificial Intelligence Research*, 55:565–602, 2016.

[BNM+58] Duncan Black, Robert Albert Newing, Iain McLean, Alistair McMillan, and Burt L. Monroe. *The theory of committees and elections*. Springer, 1958.

[Bor81] Jean-Charles de Borda. Mémoire sur les élections au scrutin. *Histoire de l'Académie Royale des Sciences pour*, 1781.

[Bor76] Georges Bordes. Consistency, rationality and collective choice. *The Review of Economic Studies*, 43(3):451–457, 1976.

[Bra11] Felix Brandt. Group-strategyproof irresolute social choice functions. In *22nd International Joint Conference on Artificial Intelligence*, pages 79–84, 2011.

[Bra15] Felix Brandt. Set-monotonicity implies kelly-strategyproofness. *Social Choice and Welfare*, 45(4):793–804, 2015.

[Bra17] Felix Brandt. Rolling the dice: Recent results in probabilistic social choice. *Trends in Computational Social Choice*, pages 3–26, 2017.

[BSS] Felix Brandt, Christian Saile, and Christian Stricker. Strategyproof social choice when preferences and outcomes may contain ties. Forthcoming.

[Chi96] Stephen Ching. A simple characterization of plurality rule. *Journal of Economic Theory*, 71(1):298–302, 1996.

[Cop51] Arthur H. Copeland. A reasonable social welfare function. Technical report, Mimeo, 1951. University of Michigan, 1951.

[CS98] Pavel Yu Chebotarev and Elena Shamis. Characterizations of scoring methods for preference aggregation. *Annals of Operations Research*, 80:299–332, 1998.

[DL99] Bhaskar Dutta and Jean-Francois Laslier. Comparison functions and choice correspondences. *Social Choice and Welfare*, 16(4):513–532, 1999.

[Dut88] Bhaskar Dutta. Covering sets and a new Condorcet choice correspondence. *Journal of Economic Theory*, 44(1):63–80, 1988.

[Fel79] Allan Feldman. Manipulation and the Pareto rule. *Journal of Economic Theory*, 21(3):473–482, 1979.

[Fis72] Peter C. Fishburn. Even-chance lotteries in social choice theory. *Theory and Decision*, 3(1):18–40, 1972.

[Fis77] Peter C. Fishburn. Condorcet social choice functions. *SIAM Journal on Applied Mathematics*, 33(3):469–489, 1977.

[Gär76] Peter Gärdenfors. Manipulation of social choice functions. *Journal of Economic Theory*, 13(2):217–228, 1976.

[Gär79] Peter Gärdenfors. On definitions of manipulation of social choice functions. *Aggregation and Revelation of Preferences*, 1979.

[Gib73] Allan Gibbard. Manipulation of voting schemes. *Econometrica*, 41(4):587–601, 1973.

[Gib77] Allan Gibbard. Manipulation of schemes that mix voting with chance. *Econometrica*, 45(3):665–681, 1977.

[Kel77] Jerry S. Kelly. Strategy-proofness and social choice functions without single-valuedness. *Econometrica*, pages 439–446, 1977.

[Kem59] John G. Kemeny. Mathematics without numbers. *Daedalus*, 88(4):577–591, 1959.

[Las96] Jean-François Laslier. Rank-based choice correspondences. *Economics Letters*, 52(3):279–286, 1996.

[Lep92] Dominique Lepelley. Une caractérisation du vote à la majorité simple. *RAIRO Operations Research*, 26(4):361–365, 1992.

[Lev75] Arthur Levenglick. Fair and reasonable election systems. *Behavioral Science*, 20(1):34–46, 1975.

[LLLB93] Gilbert Laffond, Jean-Francois Laslier, and Michel Le Breton. The bipartisan set of a tournament game. *Games and Economic Behavior*, 5(1):182–201, 1993.

[Mer03] Vincent Merlin. The axiomatic characterizations of majority voting and scoring rules. *Mathématiques et Sciences Humaines. Mathematics and Social Sciences*, 163, 2003.

[Mil80] Nicholas R. Miller. A new solution set for tournaments and majority voting: Further graph-theoretical approaches to the theory of voting. *American Journal of Political Science*, 24(1):68–96, 1980.

[MP81] Ian MacIntyre and Prasanta K. Pattanaik. Strategic voting under minimally binary group decision functions. *Journal of Economic Theory*, 25(3):338–352, 1981.

[Neh00] Klaus Nehring. Monotonicity implies generalized strategy-proofness for correspondences. *Social Choice and Welfare*, 17(2):367–375, 2000.

[NR81] Shmuel Nitzan and Ariel Rubinstein. A further characterization of Borda ranking method. *Public Choice*, 36(1):153–158, 1981.

[Saa] Donald G. Saari. Geometry of voting. 1994.

[Sat75] Mark Allen Satterthwaite. Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.

[Smi73] John H. Smith. Aggregation of preferences with variable electorate. *Econometrica*, pages 1027–1041, 1973.

[Tid87] Nicolaus T. Tideman. Independence of clones as a criterion for voting rules. *Social Choice and Welfare*, 4(3):185–206, 1987.

[You75] Hobart Peyton Young. Social choice scoring functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.

[You77] Hobart Peyton Young. Extending Condorcet's rule. *Journal of Economic Theory*, 16(2):335–353, 1977.