

# The Impossibility of Strategyproof Rank Aggregation

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## ABSTRACT

In rank aggregation, the goal is to combine multiple input rankings into a single output ranking. In this paper, we analyze rank aggregation methods, so-called social welfare functions (SWFs), with respect to strategyproofness, which requires that no agent can misreport his ranking to obtain an output ranking that is closer to his true ranking in terms of the Kemeny distance. As our main result, we show that no anonymous SWF satisfies unanimity and strategyproofness if there are at least four alternatives. This result is proven by SAT solving, a computer-aided theorem proving technique, and verified by Isabelle, a highly trustworthy interactive proof assistant. Moreover, we show by hand that strategyproofness is incompatible with majority consistency, a variant of Condorcet-consistency for SWFs. Lastly, we demonstrate for two large classes of SWFs that all SWFs within these classes have a high incentive ratio and are thus severely manipulable.

## KEYWORDS

Rank aggregation, Social welfare functions, Social choice theory, Strategyproofness, Computer-aided theorem proving

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## 1 INTRODUCTION

An important problem for multi-agent systems is rank aggregation: multiple input rankings need to be aggregated into a single output ranking. For instance, this task arises when a hiring committee is asked to produce a ranking of the applicants based on the preferences of the committee members [e.g., 24, 49], when aggregating the outputs of multiple ranking algorithms in ensemble learning [e.g., 56, 63], or when recommender systems infer an output ranking based on the preferences of multiple users [e.g., 1, 62]. Moreover, rank aggregation finds applications in computational biology [48, 55], engineering [38], and meta-search [36, 64]. Motivated by this wide range of applications, we will investigate rank aggregation in this paper through the lens of social choice theory. In this field, rank aggregation is formalized via *social welfare functions (SWF)*, which map every profile of (complete and strict) input rankings to a single output ranking.

Specifically, we are interested in the question of whether there are SWFs that incentivize voters to report their rankings truthfully—a property that is commonly known as *strategyproofness*. We believe strategyproofness to be important for many applications of rank aggregation: without it, voters may try to game the mechanism in order to obtain a better outcome from their individual perspective. For instance, if we use an SWF that violates strategyproofness to aggregate the preferences of the members of a hiring committee, a committee member may misreport his preferences to ensure that his preferred candidates are more likely to get the job. Similarly, in recommender systems, a user may try to manipulate the output ranking to ensure that the final recommendations are closer to his preferences. Lastly, even in technical applications such as ensemble learning, strategyproofness may be desirable as it offers resistance against malicious behavior from the individual algorithms.

However, while both SWFs and strategyproofness are generally well understood [see, e.g., 3, 8, 72], the study of strategyproof SWFs has only recently gained attention [e.g., 4–6, 14]. One possible reason for this is that it is challenging to define strategyproofness for SWFs because it is unclear how voters compare different output rankings. For instance, if a voter’s true ranking is  $a \succ b \succ c$ , does he prefer the ranking  $b \succ a \succ c$  or the ranking  $c \succ a \succ b$ ? Following the recent literature [e.g., 4, 6, 53], we will address this issue by using the Kemeny distance to define the voters’ preferences over rankings. This distance counts the number of pairs of alternatives on which two rankings disagree, and we suppose that voters prefer rankings that have a smaller Kemeny distance to their true ranking. Less formally, this means that voters want the output ranking to align as closely as possible with their true ranking. Lastly, (*Kemeny*)-strategyproofness requires that, by misreporting their true ranking, voters cannot obtain an output ranking that is closer to their true ranking than the one that is chosen when voting honestly.

It is known that appealing SWFs, such as the Kemeny rule, satisfy strategyproofness when there are  $m \leq 3$  alternatives, but all known SWFs fail this property if  $m \geq 4$  [4, 6]. The central question of this paper is thus whether strategyproofness allows for the design of desirable SWFs or whether an impossibility theorem similar to the Gibbard-Satterthwaite theorem [41, 67] holds for rank aggregation.

**Contribution.** As our main result, we show that no reasonable SWF satisfies strategyproofness, thereby establishing an analogue of the Gibbard-Satterthwaite theorem for SWFs. In more detail, we prove that no SWF simultaneously satisfies anonymity, unanimity, and strategyproofness when there are  $m \geq 5$  alternatives and an even number of voters  $n$ , or when there are  $m = 4$  alternatives and  $n$  is a multiple of 4 (Theorem 2). We note here that anonymity and unanimity are very basic properties—anonymity requires that all voters are treated equally and unanimity that the output ranking ranks one alternative  $x$  ahead of another alternative  $y$  if all voters

## Negative results

- ⊖ No majority consistent and strategyproof SWF if  $m \geq 4$  (Thm. 1)
- ⊖ No unanimous, anonymous, and strategyproof SWF if  $m \geq 4$  (Thm. 2)
- ⊖ The Kemeny rule, all distance scoring rules, and all positional scoring rules have an incentive ratio of at least  $\binom{m}{2} - m$  (Thm. 3)

**Table 1: Summary of our results and comparison to related work.** All negative results have been proven in this paper. While there have been impossibility results for strategyproof SWFs before our work [e.g., 6, 15], Theorem 2 supersedes all of them.

## Positive results

- ⊕ The Kemeny rule is strategyproof, unanimous, anonymous, and majority consistent if  $m \leq 3$  [4, 6]
- ⊕ There are non-dictatorial, unanimous, and strategyproof SWFs [6]
- ⊕ The Kemeny rule is betweenness strategyproof for all  $m$  [14]

prefer  $x$  to  $y$ . Hence, this result shows that no voting rule that seems acceptable in practice can satisfy strategyproofness.

The proof of our main theorem is obtained via SAT solving, a computer-aided theorem proving technique. Specifically, we encode the problem of deciding whether an SWF satisfies anonymity, unanimity, and strategyproofness in a logical formula and show with the help of a computer that this formula is unsatisfiable when there are  $m = 5$  alternatives and  $n = 2$  voters, and when there are  $m = 4$  alternatives and  $n = 4$  voters. This proves two base cases for our impossibility theorem, which we then lift to our final statement by applying inductive arguments. Following standard practices, we also extract a proof of one of our base cases in a human-readable format. However, since this proof spans over 20 pages, we additionally verify our main theorem with Isabelle [59], a highly trustworthy computer program designed to verify mathematical proofs.

Furthermore, we manually prove that no strategyproof SWF satisfies a form of Condorcet-consistency we call majority consistency (Theorem 1). To introduce this axiom, we define the majority relation of a profile as the binary relation that prefers an alternative  $x$  to another alternative  $y$  if a majority of the voters prefers  $x$  to  $y$ . Then, majority consistency requires that, when the majority relation is transitive and antisymmetric (and thus a ranking), an SWF needs to choose the corresponding ranking. Hence, this impossibility theorem can be seen as a counterpart to the observation that no Condorcet-consistent social choice function (which return single alternatives instead of rankings) satisfies strategyproofness.

Lastly, we investigate the incentive ratio of several SWFs to measure how manipulable they are. Roughly, the incentive ratio of an SWF quantifies the worst-case ratio between the utility of a voter when manipulating and when voting honestly. This notion has been successfully applied for private goods settings [e.g., 29, 54, 73] to show that manipulable rules still limit the manipulation gain of agents, which may suffice to disincentivize strategic behavior in practice. Unfortunately, all SWFs that we consider have a very high incentive ratio. Specifically, we show that the incentive ratio of the Kemeny rule and all distance scoring rules (where voters assign scores to the rankings depending on their Kemeny distance to the ranking and the ranking with minimal total score is chosen) is roughly  $\binom{m}{2}$  when there are  $m$  alternatives. Moreover, we prove that positional scoring rules have an unbounded incentive ratio for all  $m \geq 3$  as voters with utility 0 can manipulate these SWFs.

**Related Work.** As mentioned before, both SWFs and strategyproofness have been studied for decades. We hence refer to the textbook by Arrow et al. [3] and the survey of Barberà [8] for introductions to these topics. In more detail, SWFs are studied since Arrow's foundational work of social choice theory [2]. To date,

there is a large range of SWFs, including the Kemeny rule [46, 47], various types of scoring rules [e.g., 30, 52, 70, 75], sequential scoring rules [e.g., 12, 70], and Condorcet-style rules [e.g., 31, 68, 69]. These SWFs are primarily studied with respect to consistency properties (such as monotonicity, population consistency, or independence axioms), which have lead to several influential characterizations of, e.g., the Kemeny rule [26, 76, 77] or the Borda rule [60, 74].

Similarly, strategyproofness in voting has attracted significant attention, although the results in this line of work are more negative. In particular, Gibbard and Satterthwaite [41, 67] have independently shown that no reasonable deterministic voting rule that always selects a single winner satisfies strategyproofness. Motivated by this result, significant efforts have been spent to circumvent this impossibility theorem, for instance by allowing for randomized or set-valued outcomes [e.g., 18, 21, 42, 45], restricting the feasible input rankings [e.g., 23, 28, 58], or studying weakenings of strategyproofness [e.g., 25, 50, 65]. Except for domain restrictions, these approaches have mostly lead to strengthened impossibility theorems. We note that our paper can also be interpreted in this line of work: since no reasonable single-winner voting rule is strategyproof, one may attempt to escape this negative result by considering rankings as output. As our results show, this approach does not allow to circumvent the Gibbard-Satterthwaite theorem.

More directly related, there are a number of works that study (Kemeny-)strategyproofness for SWFs. To our knowledge, Bossert and Storcken [15] were the first to study this strategyproofness notion. These authors focus on group Kemeny-strategyproofness, which allows for groups of voters to jointly deviate, and show that this condition leads to an impossibility when requiring a technical auxiliary property called weak extrema independence. Moreover, Athanasoglou [4] and Athanasoglou et al. [6] investigate Kemeny-strategyproofness and prove that, for  $m \leq 3$  alternatives, the Kemeny rule (with suitable tie-breaking) and other SWFs are strategyproof, whereas these positive results break when  $m \geq 4$ . Moreover, Athanasoglou et al. [6] show that no anonymous SWF satisfies Kemeny-strategyproofness and a technical property called preference selection. This condition requires an SWF to always return a ranking that is present in the input profile, a property that is violated by all commonly studied SWFs.

Lastly, different strategyproofness notions have been explored for SWFs. For instance, Bossert and Sprumont [14] study betweenness strategyproofness, which requires that, by manipulating, voters cannot obtain a ranking that lies on a single-crossing sequence of rankings from the manipulator's true ranking to the output ranking chosen when voting truthfully. This notion is strictly weaker than Kemeny-strategyproofness and Bossert and Sprumont [14]

show that, e.g., the Kemeny rule always satisfies it. Moreover, Sato [66] uses betweenness strategyproofness and strong side conditions to obtain an impossibility theorem, whereas Harless [44] and Athanasoglou [5] use this notion to characterize a family of SWFs called status-quo rules. Further strategyproofness notions for SWFs have been studied by Bonkoungou [13] and Dindar et al. [35], which are, however, unrelated to Kemeny-strategyproofness.

## 2 PRELIMINARIES

Let  $A = \{a, b, c, \dots\}$  be a set of  $m$  alternatives and  $N = \{1, \dots, n\}$  be a set of  $n$  voters. Every voter  $i \in N$  reports a *ranking*  $\succ_i$  over the alternatives to indicate his preferences. Formally, a ranking  $\succ_i$  is a transitive, antisymmetric, and complete binary relation on  $A$ . The set of all rankings is denoted by  $\mathcal{R}$ . A (*ranking*) *profile*  $R = (\succ_1, \dots, \succ_n)$  is the collection of the rankings of all voters in  $N$ , and the set of all profiles is  $\mathcal{R}^N$ . We will write rankings as sequences of alternatives and indicate the voter submitting a ranking directly before it. For example,  $3 : abc$  means that voter 3 prefers  $a$  to  $b$  to  $c$ .

The study object of this paper are *social welfare functions* (SWFs) which map every ranking profile to a single output ranking. More formally, a social welfare function is a function  $f$  of the type  $\mathcal{R}^N \rightarrow \mathcal{R}$ . To clearly distinguish between input and output rankings, we will denote the former by  $\succ$  and the latter by  $\triangleright$ .

### 2.1 Classes of SWFs

We will next introduce several natural classes of SWFs. Since all of the following rules may return multiple winning rankings, we assume that such ties are broken based on an external ranking  $\succ$  over the alternatives that is lexicographically extended to rankings. Specifically, given two rankings  $\triangleright_1 = x_1 \dots x_m$  and  $\triangleright_2 = y_1 \dots y_m$ , it holds that  $\triangleright_1 > \triangleright_2$  if and only if there is  $\ell \in \{1, \dots, m\}$  such that  $x_\ell > y_\ell$  and  $x_i = y_i$  for all  $i \in \{1, \dots, \ell - 1\}$ . To fully specify our SWFs, we always choose the most preferred ranking with respect to  $\succ$  that is winning for the considered SWF. We note, however, that all our results are independent of this tie-breaking convention.

**Kemeny rule.** The Kemeny rule was first suggested by Kemeny [46] and is maybe the most prominent method in rank aggregation. To introduce this rule, we define the *Kemeny distance* (which is also known as swap distance or Kendall-tau distance) between two rankings  $\succ$  and  $\triangleright$  by  $\Delta(\succ, \triangleright) = |\{(x, y) \in A^2 : x \succ y \wedge y \triangleright x\}|$ . Less formally,  $\Delta(\succ, \triangleright)$  is the number of pairs of alternatives on which  $\succ$  and  $\triangleright$  disagree. The *Kemeny rule* chooses the (lexicographically most preferred) ranking that minimizes the total Kemeny distance to the input rankings, i.e.,  $f_{\text{Kemeny}}(R) = \arg \min_{\triangleright \in \mathcal{R}} \sum_{i \in N} \Delta(\succ_i, \triangleright)$ .

**Distance scoring rules.** In distance scoring rules, every voter  $i$  assigns a score to every ranking  $\triangleright$  depending on the Kemeny distance between his input ranking  $\succ_i$  and  $\triangleright$ , and we choose the ranking with the minimal total score. More formally, these rules are defined based on *distance scoring functions*  $s : \{0, \dots, \binom{m}{2}\} \rightarrow \mathbb{R}$ , and a voter with a Kemeny distance of  $x$  to a ranking assigns a score of  $s(x)$  to this ranking. Throughout the paper, we will require of distance scoring functions  $s$  that  $s(x) > s(x - 1)$  for all  $x \in \{1, \dots, \binom{m}{2}\}$ , and  $s(x) - s(x - 1) \geq s(x - 1) - s(x - 2)$  for all  $x \in \{2, \dots, \binom{m}{2}\}$ . These conditions formalize that rankings that are further away from a voter's ranking get a higher score and that  $s$

is convex. Finally, a *distance scoring rule*  $f$  is defined by a distance scoring function  $s$  and chooses the (lexicographically most preferred) ranking that minimizes  $\sum_{i \in N} s(\Delta(\succ_i, \triangleright))$ . For example, the Kemeny rule is defined by the distance scoring function  $s(x) = x$ , and the Squared Kemeny rule of Lederer et al. [53] by  $s(x) = x^2$ .

**Positional scoring rules.** Another prominent class of SWFs are positional scoring rules. For these rules, the voters assign points to the alternatives depending on their positions in the input ranking, and the output ranking orders the alternatives in decreasing order of their total score. To formalize this, we define the *rank* of an alternative  $x$  in a ranking  $\succ$  by  $r(\succ, x) = 1 + |\{y \in A \setminus \{x\} : y \succ x\}|$ . Then, positional scoring rules are defined by *positional scoring functions*  $p : \{1, \dots, m\} \rightarrow \mathbb{R}$  and a voter who places an alternative at rank  $k$  assigns a score of  $p(k)$  to this alternative. We will require of positional scoring functions that  $p(1) \geq p(2) \geq \dots \geq p(m)$  and  $p(1) > p(m)$ , i.e., voters give more points to higher-ranked alternatives and do not assign the same score to all alternatives. Finally, an SWF  $f$  is a *positional scoring rule* if there is a positional scoring function  $p$  such that  $f$  returns for every profile  $R$  the (lexicographically most preferred) ranking  $\triangleright$  such that  $x \triangleright y$  implies  $\sum_{i \in N} p(r(\succ_i, x)) \geq \sum_{i \in N} p(r(\succ_i, y))$ . For instance, the Borda rule is induced by the positional scoring function  $p(x) = m - x$ .

### 2.2 Strategyproofness

The central axiom in our analysis is strategyproofness, which requires that voters cannot benefit by lying about their true ranking. Following the literature [e.g., 4, 6, 15], we will define this axiom by assuming that the voters' preferences over rankings are induced by the Kemeny distance  $\Delta(\succ, \triangleright) = |\{(x, y) \in A^2 : x \succ y \wedge y \triangleright x\}|$ : a voter with ranking  $\succ_i$  prefers a ranking  $\triangleright$  to another ranking  $\triangleright'$  if  $\Delta(\succ_i, \triangleright) < \Delta(\succ_i, \triangleright')$ . This formalizes that voters prefer rankings that are more similar to their true ranking. Based on this assumption, we can define strategyproofness as usual by requiring that voters cannot obtain a more preferred ranking by voting strategically.

**Definition 1** (Strategyproofness). An SWF  $f$  is *strategyproof* if  $\Delta(\succ_i, f(R)) \leq \Delta(\succ_i, f(R'))$  for all voters  $i \in N$  and profiles  $R, R' \in \mathcal{R}^N$  such that  $\succ_j = \succ'_j$  for all  $j \in N \setminus \{i\}$ .

This definition of strategyproofness is motivated by the fact that the Kemeny distance is by far the most common distance over rankings in rank aggregation [e.g., 24, 36, 48, 56]. In particular, many papers propose to minimize the total Kemeny distance to find good output rankings, which implicitly assumes that voters prefer rankings that have a closer Kemeny distance to their input ranking. Moreover, the Kemeny distance is also theoretically well-understood and allows for appealing characterizations [27, 34, 46]. Nevertheless, we acknowledge that one can define alternative strategyproofness notions by, e.g., using different distances on rankings or even approaches that are not based on any distance measure, which may lead to different results.

To further illustrate strategyproofness for SWFs, we will next discuss an example showing that the Kemeny rule is manipulable.

**Example 1.** Consider the following profile  $R$  with  $m = 4$  alternatives  $A = \{a, b, c, d\}$  and  $n = 5$  voters  $N = \{1, \dots, 5\}$ .

$R: 1: abcd \quad 2: cdab \quad 3: dbac \quad 4: bcda \quad 5: adcb$

For this profile, the Kemeny rule chooses the ranking  $\triangleright = dabc$ , which has a total Kemeny distance of  $\sum_{i \in N} \Delta(\succ_i, \triangleright) = 3+3+1+4+2 = 13$ . By contrast, if voter 4 misreports his true ranking by swapping  $b$  and  $c$ , i.e., if  $\succ'_4 = cbda$ , the Kemeny rule chooses the ranking  $\triangleright' = cdab$ . Further, it holds that  $\Delta(\succ_4, \triangleright) = \Delta(bcda, dabc) = 4 > 3 = \Delta(bcda, cdab) = \Delta(\succ_4, \triangleright')$ . This shows that voter 4 prefers the ranking  $\triangleright'$  to the ranking  $\triangleright$  selected for  $R$ , so the Kemeny rule fails strategyproofness when  $m = 4$  and  $n = 5$ .

### 2.3 Further Axioms

Additionally to strategyproofness, we will consider three further axioms, namely anonymity, unanimity, and majority consistency.

**Anonymity.** Intuitively, anonymity requires that the identities of voters should not matter for the outcome. Formally, we say an SWF  $f$  is *anonymous* if  $f(R) = f(\pi(R))$  for all profiles  $R \in \mathcal{R}^N$  and permutations  $\pi : N \rightarrow N$ . Here,  $R' = \pi(R)$  is the profile given by  $\succ'_i = \succ_{\pi(i)}$  for all  $i \in N$ . When assuming anonymity, we may interpret profiles as multisets of rankings because we only need to know how often each ranking is reported to compute the outcome.

**Unanimity.** Unanimity is a minimal efficiency notion which requires that if all voters unanimously prefer one alternative  $x$  to another alternative  $y$ , then the output ranking should also rank  $x$  ahead of  $y$ . More formally, an SWF  $f$  is *unanimous* if, for all profiles  $R \in \mathcal{R}^N$  and alternatives  $x, y \in A$ , it holds for the output ranking  $\triangleright = f(R)$  that  $x \triangleright y$  if  $x \succ_i y$  for all voters  $i \in N$ .

**Majority consistency.** One of the dominant notions in social choice theory is the Condorcet principle: if an alternative is favored to another alternative by a majority of the voters, then the former is typically seen as socially more desirable than the latter. To formalize this idea, we define the *majority relation*  $\succ_R$  of a profile  $R$  by  $x \succ_R y$  if and only if  $|\{i \in N : x \succ_i y\}| \geq |\{i \in N : y \succ_i x\}|$ , i.e.,  $x \succ_R y$  if and only if a majority of voters prefer  $x$  to  $y$ . Put differently,  $\succ_R$  encodes the majority preferences. Finally, we say an SWF  $f$  is *majority consistent* if returns the majority relation whenever this relation is a ranking, i.e.,  $f(R) = \succ_R$  for all profiles  $R$  such that the majority relation  $\succ_R$  is transitive and antisymmetric. We note that the majority relation does not necessarily form a ranking, and majority consistency permits any outcome in such situations.

## 3 MAJORITY CONSISTENCY AND STRATEGYPROOFNESS

As our first contribution, we will show that no majority consistent SWF satisfies strategyproofness if there are sufficiently many voters and at least four alternatives. While we include this theorem primarily to showcase a proof for an impossibility theorem based on strategyproofness, it is also one of the most general impossibility results in rank aggregation.

**Theorem 1.** *No strategyproof SWF satisfies majority consistency if  $m \geq 4$ ,  $n \geq 9$ , and  $n \notin \{10, 12, 14, 16\}$ .*

**PROOF.** We will first focus on the case that  $m = 4$  and  $n = 9$  and later on explain how to generalize the result to larger values of  $n$  and  $m$ . Hence, assume for contradiction that there is a strategyproof and

majority consistent SWF  $f$  for 4 alternatives and 9 voters. We will focus on the following two profiles  $\bar{R}$  and  $\hat{R}$  to derive a contradiction.

$\bar{R}$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbac</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\hat{R}$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbca</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	

We will show that  $f$  has to choose  $\triangleright = adcb$  for  $\bar{R}$  and  $\hat{\triangleright} = dcba$  for  $\hat{R}$ . Since the profiles  $\bar{R}$  and  $\hat{R}$  only differ in the ranking of voter 3, this means that  $f$  is manipulable because  $\Delta(\succ_3, \triangleright) = 3 > 2 = \Delta(\succ_3, \hat{\triangleright})$ , i.e., voter 3 prefers  $\hat{\triangleright}$  to  $\triangleright$ . It remains to show that  $f$  indeed needs to choose *adcb* and *dcba* for  $\bar{R}$  and  $\hat{R}$ , respectively.

**Claim 1:**  $f(\bar{R}) = adcb$ . For proving this claim, we consider the following five profiles  $\bar{R}^1, \dots, \bar{R}^5$ , all of which differ from  $\bar{R}$  in the ranking of a single voter. This ranking is highlighted in blue.

$\bar{R}^1$	1: <i>cdab</i>	2: <i>badc</i>	3: <i>dbac</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\bar{R}^2$	1: <i>cdba</i>	2: <i>abdc</i>	3: <i>dbac</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\bar{R}^3$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dabc</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\bar{R}^4$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbac</i>	4: <i>cabd</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\bar{R}^5$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbac</i>	4: <i>cbad</i>	5: <i>dabc</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	

In all five of these profiles, the majority relation is transitive and antisymmetric and therefore a ranking. In more detail, in  $\bar{R}^1$  to  $\bar{R}^4$ , all of which arise from  $\bar{R}$  by swapping  $a$  and  $b$  in the ranking of a single voter, the majority relation corresponds to the ranking *adcb*. Further, in  $\bar{R}^5$ , the majority relation is given by *dbac*. Hence, majority consistency requires that  $f(\bar{R}^1) = \dots = f(\bar{R}^4) = adcb$  and  $f(\bar{R}^5) = dbac$ . Consequently, strategyproofness implies the following constraints for the ranking  $\triangleright$  chosen for  $\bar{R}$ .

- (1) Strategyproofness from  $\bar{R}$  to  $\bar{R}^1$  requires that  $\Delta(cdba, \triangleright) \leq \Delta(cdba, f(\bar{R}^1)) = \Delta(cdba, adcb) = 4$ .
- (2) Strategyproofness from  $\bar{R}$  to  $\bar{R}^2$  requires that  $\Delta(badc, \triangleright) \leq \Delta(badc, f(\bar{R}^2)) = \Delta(badc, adcb) = 3$ .
- (3) Strategyproofness from  $\bar{R}$  to  $\bar{R}^3$  requires that  $\Delta(dbac, \triangleright) \leq \Delta(dbac, f(\bar{R}^3)) = \Delta(dbac, adcb) = 3$ .
- (4) Strategyproofness from  $\bar{R}$  to  $\bar{R}^4$  requires that  $\Delta(cbad, \triangleright) \leq \Delta(cbad, f(\bar{R}^4)) = \Delta(cbad, adcb) = 4$ .
- (5) Strategyproofness from  $\bar{R}$  to  $\bar{R}^5$  requires that  $\Delta(adcb, \triangleright) \leq \Delta(adcb, f(\bar{R}^5)) = \Delta(adcb, dbac) = 3$ .

We claim that only the ranking *adcb* satisfies these constraints. To prove this, we will consider several cases. First,  $d$  cannot be bottom-ranked in  $\triangleright$  because  $\Delta(dbac, \triangleright) \geq 4$  for all rankings  $\triangleright$  that rank  $d$  last and are not equal to *bacd*. Hence, all these rankings fail Condition (3). Further, it holds that  $\Delta(adcb, bacd) = 4$ , so the ranking *bacd* fails Condition (5), thus showing that  $d$  is not ranked last in  $\triangleright$ . Next,  $\triangleright$  cannot bottom-rank  $a$ : every ranking  $\triangleright$  that places  $a$  last and is not equal to *dcba* violates Condition (5) since  $\Delta(adcb, \triangleright) \geq 4$ , and the ranking *dcba* fails Condition (2) since  $\Delta(badc, dcba) = 4$ . Thirdly, we show that alternative  $c$  cannot be

bottom-ranked by  $\hat{\triangleright}$ . For this, we observe that Condition (1) implies that  $\hat{\triangleright} \notin \{abdc, badc, adbc\}$  because all these rankings have a swap distance of at least 5 to  $cdba$ . Further, Condition (4) requires that  $\hat{\triangleright} \notin \{dabc, dbac\}$  because these rankings have a swap distance of at least 5 to  $cbad$ . Lastly, Condition (5) shows that  $\hat{\triangleright} \neq bdac$  because  $\Delta(adcb, bdac) = 4$ . We now conclude that  $c$  is not bottom-ranked by  $\hat{\triangleright}$ . Because all other options have been ruled out,  $b$  must be bottom-ranked by  $\hat{\triangleright}$ . This means that  $\Delta(badc, \hat{\triangleright}) \geq 3$ . Moreover, this inequality needs to be tight due to Condition (2), which is only true if  $f(\hat{R}) = \hat{\triangleright} = adcb$ .

**Claim 2:**  $f(\hat{R}) = dcba$ . We will next show that  $f(\hat{R}) = dcba$ , for which we consider the following five profiles. All of these profiles only differ in the highlighted ranking from  $\hat{R}$ .

$\hat{R}^1$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbca</i>	4: <i>cbda</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\hat{R}^2$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbca</i>	4: <i>cbad</i>	5: <i>dacb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\hat{R}^3$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbca</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cdab</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\hat{R}^4$	1: <i>cdba</i>	2: <i>bdac</i>	3: <i>dbca</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>dcba</i>	8: <i>dabc</i>	9: <i>abcd</i>	
$\hat{R}^5$	1: <i>cdba</i>	2: <i>badc</i>	3: <i>dbca</i>	4: <i>cbad</i>	5: <i>adcb</i>
	6: <i>cadb</i>	7: <i>cdab</i>	8: <i>dabc</i>	9: <i>abcd</i>	

The profiles  $\hat{R}^1, \dots, \hat{R}^4$  only differ from  $\hat{R}$  in the fact that a voter swapped  $d$  and  $a$ . As a consequence, the majority relation of these profiles corresponds to the ranking  $dcba$ . On the other hand, in  $\hat{R}^5$ , the majority relation is given by the ranking  $cadb$ . Hence, majority consistency requires that  $f(\hat{R}^1) = \dots = f(\hat{R}^4) = dcba$  and  $f(\hat{R}^5) = cadb$ . In turn, strategyproofness between  $\hat{R}$  and our five profiles requires the following constraints for the ranking  $\hat{\triangleright}$  chosen for  $\hat{R}$ .

- (1) Strategyproofness from  $\hat{R}$  to  $\hat{R}^1$  requires that  $\Delta(cbad, \hat{\triangleright}) \leq \Delta(cbad, f(\hat{R}^1)) = \Delta(cbad, dcba) = 3$ .
- (2) Strategyproofness from  $\hat{R}$  to  $\hat{R}^2$  requires that  $\Delta(adcb, \hat{\triangleright}) \leq \Delta(adcb, f(\hat{R}^2)) = \Delta(adcb, dcba) = 3$ .
- (3) Strategyproofness from  $\hat{R}$  to  $\hat{R}^3$  requires that  $\Delta(cadb, \hat{\triangleright}) \leq \Delta(cadb, f(\hat{R}^3)) = \Delta(cadb, dcba) = 3$ .
- (4) Strategyproofness from  $\hat{R}$  to  $\hat{R}^4$  requires that  $\Delta(badc, \hat{\triangleright}) \leq \Delta(badc, f(\hat{R}^4)) = \Delta(badc, dcba) = 4$ .
- (5) Strategyproofness from  $\hat{R}$  to  $\hat{R}^5$  requires that  $\Delta(dcba, \hat{\triangleright}) \leq \Delta(dcba, f(\hat{R}^5)) = \Delta(dcba, cadb) = 3$ .

Analogously to the last claim, we will show that these constraints entail that  $\hat{\triangleright} = dcba$ . To this end, we first note that Conditions (2) and (5) show that  $d$  cannot be bottom-ranked by  $\hat{\triangleright}$ . In more detail, every ranking  $\triangleright$  other than  $cbad$  that bottom-ranks  $d$  violates Condition (5) since  $\Delta(dcba, \triangleright) \geq 4$ . On the other hand, the ranking  $cbad$  violates Condition (2) as  $\Delta(adcb, cbad) = 4$ . Next, Conditions (2) and (3) show that  $c$  cannot be bottom-ranked by  $\hat{\triangleright}$ : the only ranking  $\triangleright$  that bottom-ranks  $c$  and satisfies that  $\Delta(cbad, \triangleright) \leq 3$  (Condition (2)) is  $badc$ , but this ranking fails Condition (3) since  $\Delta(badc, cadb) = 5$ . Thirdly,  $\hat{\triangleright}$  cannot bottom-rank  $b$ . Specifically, Condition (4) rules out that  $\hat{\triangleright} \in \{cdab, dcab, cadb\}$  since all these rankings have a distance of at least 5 to  $badc$ . Condition (1) shows that  $\hat{\triangleright} \notin \{adcb, dacb\}$  since these rankings have a Kemeny distance

of at least 4 to  $cbad$ , and Condition (5) shows that  $\hat{\triangleright} \neq acdb$  since  $\Delta(dcba, acdb) = 4$ . As a consequence of the analysis so far, we conclude that  $a$  must be bottom-ranked in  $\hat{\triangleright}$ . In turn, we infer from Condition (2) that  $\hat{\triangleright}$  must be  $dcba$  since every other ranking that bottom-ranks  $a$  satisfies that  $\Delta(adcb, \triangleright) \geq 4$ .

**Extension to larger values of  $m$  and  $n$ .** Lastly, we explain how to generalize our result to larger numbers of voters  $n$  and alternatives  $m$ . First, to increase  $m$ , we can add new alternatives in the same order at the bottom of the rankings of all voters. After this extension, the majority relation is still transitive and antisymmetric for all profiles  $\hat{R}^i$  and  $\tilde{R}^i$  with  $i \in \{1, \dots, 5\}$ . Using majority consistency, the inequalities (1) to (5) thus remain intact for both cases. Finally, these inequalities still imply that we need to choose rankings for  $\hat{R}$  and  $\tilde{R}$  that permit a manipulation for voter 3.

To extend our construction to larger numbers of voters, we apply two different techniques. Firstly, we can generalize our impossibility to every odd  $n > 9$  by adding pairs of voters with inverse rankings. These voters cancel each other out with respect to the majority relation and therefore do not affect our analysis. Secondly, to extend our impossibility to an even number of voters, we can double all voters in our profiles. While this requires intermediate profiles  $\tilde{R}$  to go from, e.g.,  $\hat{R}$  to  $\hat{R}^1$ , we can still infer the same inequalities by chaining the strategyproofness conditions. For instance, for  $\hat{R}^1$ , strategyproofness implies that  $\Delta(cdba, f(\hat{R})) \leq \Delta(cdba, f(\tilde{R}^1)) \leq \Delta(cdba, f(\hat{R}^1))$ , where  $\tilde{R}^1$  denotes the intermediate profile. Hence, our analysis remains intact after this extension. Lastly, for any even  $n > 18$ , we can again add pairs of voters with inverse rankings.  $\square$

**Remark 1.** When  $m \leq 3$ , Theorem 1 ceases to hold as the Kemeny rule (with suitable tie-breaking) is strategyproof and majority consistent in this case [6, 77]. Moreover, under mild additional conditions, namely anonymity, cancellation (i.e., adding pairs of voters with inverse rankings does not affect the outcome), and a weak form of neutrality, it can be shown that strategyproofness and majority consistency require to choose a ranking that minimizes the total Kemeny distance to the input rankings when there are 3 alternatives. We refer to Appendix A for details.

**Remark 2.** We did not minimize the number of voters for Theorem 1 as we aimed for a simple proof. However, with the help of a computer, we showed that this impossibility already holds when there are  $m = 4$  alternatives and  $n \in \{3, 4\}$  voters. Based on our inductive arguments for  $n$ , it thus follows that no majority consistent SWF is strategyproof if there are  $m = 4$  alternatives and  $n \geq 3$  voters. By contrast, our inductive argument for  $m$  is specific to the profiles in Theorem 1, so it is unclear whether the computer proof extends to more alternatives. Further, we verified the correctness of the computer proof and our human-readable proof by Isabelle/HOL, a highly trustworthy interactive theorem prover [59].

## 4 STRATEGYPROOFNESS AND UNANIMITY

We will now turn to our main theorem: there is no anonymous SWF that satisfies strategyproofness and unanimity if there are  $m \geq 5$  alternatives and an even number of voters  $n$ , or when there are  $m = 4$  alternatives and the number of voters  $n$  is a multiple of 4. Put differently, this result shows that every reasonable SWF is

manipulable and it can thus be seen as an analog of the Gibbard-Satterthwaite theorem for rank aggregation.

**Theorem 2.** *No anonymous SWF satisfies strategyproofness and unanimity if  $m \geq 5$  and  $n$  is even, or  $m = 4$  and  $n$  is a multiple of 4.*

We note that we have shown Theorem 2 based on a computer-aided theorem proving technique called SAT solving. In the context of social choice theory, such computer-aided techniques have been pioneered by Tang and Lin [71] and have since then been used to show a large number of results [e.g., 16, 17, 19, 21, 33, 37, 39, 61]. We refer to the survey of Geist and Peters [40] for an introduction to these techniques. In the following three sections, we outline how we apply SAT solving to obtain Theorem 2 (cf. Sections 4.1 and 4.2) and how we verified our result (cf. Section 4.3).

**Remark 3.** All axioms of Theorem 2 are necessary for the impossibility. Specifically, constant SWFs, which always return a fixed ranking, satisfy strategyproofness and anonymity but violate unanimity. Dictatorships, which return the ranking of a fixed voter, satisfy unanimity and strategyproofness but violate anonymity. It is also not possible to weaken anonymity to non-dictatorship as Athanassoglou et al. [6] design non-dictatorial (and non-anonymous) SWFs that are strategyproof and unanimous. Thirdly, e.g., the Kemeny rule satisfies unanimity and anonymity but violates strategyproofness. Further, we cannot significantly weaken strategyproofness as Bossert and Sprumont [14] shows that the Kemeny rule satisfies betweenness strategyproofness, which is only slightly weaker than our Kemeny-strategyproofness. Finally, when  $m \leq 3$ , Theorem 2 ceases to hold as the Kemeny rule is strategyproof in this case [6].

**Remark 4.** A drawback of Theorem 2 is that we cannot extend this result to an odd number of voters. The primary reason for this is technical: we could not find an inductive argument that generalizes our theorem from an even number of voters to an odd one. Moreover, based on our SAT approach, we showed that there are SWFs that satisfy all axioms of Theorem 2 when  $m = 4$  and  $n \in \{3, 5\}$ , which indicates that such an argument may not exist. Similar problems are common for impossibility theorems in social choice theory [e.g., 19, 20, 22, 33, 61], as it is often challenging to generalize such results from a fixed number of voters  $n$  to arbitrary values of  $n$ . We note, however, that we can extend Theorem 2 to odd  $n$  when strengthening unanimity. Specifically, based on Theorem 2, one can show to that no anonymous SWF satisfies strategyproofness and a property called near unanimity when  $m \geq 5$  and  $n \geq 3$  is odd. This latter condition requires that the output ranking puts  $x$  ahead of  $y$  whenever all but one voter prefer  $x$  to  $y$  [10, 51]. In particular, if we had an SWF that satisfies anonymity, strategyproofness, and near unanimity for odd  $n \geq 3$ , we could construct an SWF that satisfies anonymity, strategyproofness, and unanimity for  $n - 1$  voters by fixing the ranking of a single voter. We further observe that near unanimity becomes less demanding as  $n$  increases, and all common SWFs satisfy this condition when  $n$  is sufficiently larger than  $m$ .

## 4.1 SAT Solving

To show Theorem 2, we rely on SAT solving, a computer-aided theorem proving technique. The central idea of this approach is that, for fixed numbers of voters  $n$  and alternatives  $m$ , there is a large

but finite number of ranking profiles and possible outcomes. For instance, when  $n = 2$  and  $m = 5$ , there are  $(5!)^2 = 14,400$  ranking profiles, for each of which one of  $5! = 120$  rankings must be chosen. Based on this observation, it is possible to write a large logical formula that is satisfiable if and only if there is an anonymous SWF that satisfies unanimity and strategyproofness for the given values of  $n$  and  $m$ . We then prove two base cases of our theorem by letting a computer program, a so-called SAT solver, show that our formula is unsatisfiable when there are  $n = 2$  voters and  $m = 5$  alternatives, and when there  $n = 4$  voters and  $m = 4$  alternatives.

In our logical formula, we follow the standard encoding of voting rules. Specifically, our formula will use variables  $x_{R,\triangleright}$  for all profiles  $R$  and rankings  $\triangleright$ , which will encode whether the ranking  $\triangleright$  is chosen for the profile  $R$ . Moreover, since we focus on anonymous SWFs, we will treat profiles as multisets of rankings. Formally, this means that the variable  $x_{R,\triangleright}$  will state whether the ranking  $\triangleright$  is chosen for all non-anonymous profiles that can be obtained by assigning the rankings in the multiset  $R$  to the voters in  $N$ . This is possible as anonymity necessitates that we need to choose the same ranking for all such profiles. Furthermore, by representing profiles as multisets, we reduce the number of variables in our formula as a single multiset corresponds up to  $n!$  non-anonymous ranking profiles. Also, due to this representation, anonymity is implicitly encoded, so we do not need to add constraints for this axiom.

Next, we have to ensure that our variables  $x_{R,\triangleright}$  indeed encode an (anonymous) SWF. This necessitates us to formalize that for every profile  $R$ , exactly one ranking  $\triangleright$  is chosen. To this end, we will require for each profile  $R$  that exactly one variable  $x_{R,\triangleright}$  is true. Moreover, to further simplify our formula, we encode unanimity in this step by only introducing variables  $x_{R,\triangleright}$  for profiles  $R$  and rankings  $\triangleright$  such that  $\triangleright$  satisfies unanimity for  $R$ . To make this more formal, let  $X(R) = \{(x, y) \in A^2 : \forall i \in N : x \succ_i y\}$  denote the pairs of alternatives  $(x, y)$  such that all voters prefer  $x$  to  $y$  in  $R$ . Then,  $U(R) = \{\triangleright \in \mathcal{R} : \forall (x, y) \in X(R) : x \triangleright y\}$  is the set of rankings that satisfy unanimity for  $R$ . We will only introduce variables  $x_{R,\triangleright}$  for each profile  $R$  and ranking  $\triangleright \in U(R)$  as rankings outside of  $U(R)$  are not allowed to be chosen by unanimity. We can now enforce that our variables encode an unanimous SWF by adding the following constraints for every profile  $R$ , which respectively state that at least and at most one ranking in  $U(R)$  must be chosen for  $R$ .

$$\bigvee_{\triangleright \in U(R)} x_{R,\triangleright} \quad \text{and} \quad \bigwedge_{\triangleright, \triangleright' \in U(R) : \triangleright \neq \triangleright'} (\neg x_{R,\triangleright} \vee \neg x_{R,\triangleright'})$$

Lastly, we need to encode strategyproofness. We recall here that this axiom requires that, for every profile  $R$  and voter  $i$ , it is not possible for the voter to deviate such that the ranking chosen when lying has a smaller Kemeny distance to his truthful ranking than the one chosen when reporting his truthful ranking. Put differently, if  $R$  and  $R'$  only differ in the ranking of voter  $i$ , we cannot choose rankings  $\triangleright$  and  $\triangleright'$  for these profiles such that  $\Delta(\triangleright, \succ_i) > \Delta(\triangleright', \succ_i)$ . To make this more formal, we define by  $D(R, \succ)$  the set of profiles that can be derived from  $R$  by letting a voter with ranking  $\succ$  deviate to an arbitrary other ranking. We note that  $D(R, \succ) = \emptyset$  if no voter in  $R$  reports  $\succ$ . Further, given two rankings  $\succ$  and  $\triangleright$ , we let  $B(\succ, \triangleright) = \{\triangleright' \in \mathcal{R} : \Delta(\succ, \triangleright') < \Delta(\succ, \triangleright)\}$  denote the set of rankings that have a smaller Kemeny distance to  $\succ$  than  $\triangleright$ . Based on this notation, strategyproofness can be formalized via implications:

if we choose a ranking  $\triangleright$  for  $R$ , we cannot choose a ranking  $\triangleright' \in B(\succ, \triangleright) \cap U(R')$  for all profiles  $R' \in D(R, \succ)$  and rankings  $\succ \in \mathcal{R}$ . Using our variables  $x_{R, \triangleright}$ , this results in the following constraints.

$$\bigwedge_{R \in \mathcal{R}^N} \bigwedge_{\succ \in \mathcal{R}} \bigwedge_{R' \in D(R, \succ)} \bigwedge_{\triangleright \in U(R)} \bigwedge_{\triangleright' \in B(\succ, \triangleright) \cap U(R')} (\neg x_{R, \triangleright} \vee \neg x_{R', \triangleright'})$$

We can now write a computer program that generates this logical formula for given values of  $m$  and  $n$ . For instance, for  $m = 5$  and  $n = 2$ , our program produces a formula with 227,880 variables and 59,445,060 clauses. We then hand this formula for both  $m = 5$  and  $n = 2$ , and  $m = n = 4$  to a SAT solver (e.g., Glucose [7] or Cadical [11]), which proves both formulas unsatisfiable in less than a minute. We hence derive the following result.

**Proposition 1.** *No anonymous SWF satisfies both strategyproofness and unanimity if  $m = 5$  and  $n = 2$  or  $m = 4$  and  $n = 4$ .*

## 4.2 Inductive Arguments

Proposition 1 shows that no anonymous SWF satisfies strategyproofness and unanimity for only two cases, namely when there are  $m = 5$  alternatives and  $n = 2$  voters or  $m = 4$  alternatives and  $n = 4$  voters. By contrast, Theorem 2 claims that the impossibility holds for a large range of combinations of  $m$  and  $n$ . To close this gap, we will next present a lemma that generalizes our impossibility theorem from fixed numbers of voters and alternatives to a large range. In combination with Proposition 1, this lemma proves Theorem 2. The full proof of Lemma 1 can be found in Appendix B.

**Lemma 1.** *Assume there is no anonymous SWF that satisfies strategyproofness and unanimity for  $m$  alternatives and  $n$  voters. The following claims hold:*

- (1) *For every  $m' > m$ , there is no anonymous SWF that satisfies strategyproofness and unanimity for  $m'$  alternatives and  $n$  voters.*
- (2) *For every  $\ell \in \mathbb{N}$ , there is no anonymous SWF that satisfies strategyproofness and unanimity for  $m$  alternatives and  $\ell n$  voters.*

**PROOF SKETCH.** For both claims, we will show the contrapositive: we assume that there is an anonymous, unanimous, and strategyproof SWF for the larger numbers of alternatives and voters, and show that this implies that there is also an SWF that satisfies our properties for  $m$  alternatives and  $n$  voters. In more detail, to show Claim (1), we suppose that there is an SWF  $f$  for  $m' > m$  alternatives and  $n$  voters that satisfies our axioms. We then define an SWF  $g$  for  $m$  alternatives and  $n$  voters as follows: given a profile for these parameters, we add  $m' - m$  dummy alternatives in the same order at the bottom of the rankings of all voters, apply  $f$  to compute a ranking on these  $m'$  alternatives, and then delete the  $m' - m$  dummy alternatives from this ranking to infer the final output ranking. Since  $f$  is unanimous, the dummy alternatives must appear at the bottom of the intermediate output ranking. Based on this insight, it can be shown that  $g$  inherits strategyproofness, anonymity, and unanimity from  $f$ , which contradicts that no SWF satisfies these axioms for  $m$  alternatives and  $n$  voters.

For Claim (2), we assume that there is a unanimous, anonymous, and strategyproof SWF  $f$  for  $m$  alternatives and  $\ell n$  voters. We then define an SWF  $g$  for  $m$  alternatives and  $n$  voters as follows: we clone each voter's ranking  $\ell$  times and then apply  $f$  to compute

the output ranking. It can be verified that  $g$  is anonymous, unanimous, and strategyproof as  $f$  satisfies these conditions, which again contradicts the assumption of this lemma.  $\square$

## 4.3 Verification

Because Proposition 1 has been shown by SAT solving, it is not immediately clear how to verify the correctness of this result. We have chosen a threefold approach to address this issue.

Firstly, following prior works [e.g., 16, 19, 21, 61], we provide in Appendix D a proof of Proposition 1 for the case that  $m = 5$  and  $n = 2$  in a human-readable format. This proof was obtained by analyzing minimal unsatisfiable subsets (MUSes) of the original formula, i.e., inclusion-minimal subsets of the formula that are unsatisfiable. More intuitively, such MUSes can be seen as the reason why a formula is unsatisfiable and they tend to be much smaller than the original formula. For instance, for their Theorem 2, Brandt et al. [21] ended up with a MUS that only reasons about 13 profiles, which allows to give a compact human-readable proof. Unfortunately, our MUSes, which we obtained by using the program MUSer2 [9], are much bigger: even after several optimizations, the smallest MUS we found requires roughly 200 profiles and uses rather intricate reasoning. Although we were able to extract a human-readable proof by using a custom computer program that translates MUSes into a readable format, the resulting proof spans over 20 pages and requires the verification of more than 3000 strategyproofness applications. As such, the proof allows readers to build confidence in the correctness of our SAT-based approach by inspecting intermediate steps, but manually verifying the full argument would be extremely tedious. We moreover note that we cannot provide a human-readable proof for the case that  $m = n = 4$  as the computer reasons about thousands of profiles in this case.

Because of these issues, we offer two further means of verification. Firstly, alongside the paper, we submitted the necessary code for creating the formula described in Section 4.1 and we plan to make this code publicly available upon acceptance. This enables researchers to directly check that our code correctly constructs the formula described before, which means we only need to trust the correctness of the SAT solvers. We note here also that our implementation is rather standard and we expect a researcher familiar with the literature to be able to verify our code in less than a day.

Lastly, following more recent works [e.g., 16, 21, 33], we have fully verified Theorem 2—including both base cases and the inductive arguments—with the interactive theorem prover Isabelle/HOL [59]. Such interactive theorem provers are computer programs designed to verify the correctness of mathematical proofs and thus have a high degree of trustworthiness. In more detail, Isabelle/HOL offers a rich mathematical logic which makes it simple to formalize our setting. Thus, our Isabelle verification directly derives Proposition 1 as well as Lemma 1 from our axioms. Moreover, this formal verification releases us from the need to check the intermediate steps because Isabelle verifies the correctness of each deduction step based on a small and highly trustworthy set of logical operations. Consequently, to trust our results, one only needs to trust the implementation of our axioms in Isabelle. We further note that experts in verification see such formal proofs as the “gold standard” for increasing the trustworthiness of mathematical results [43].

Upon acceptance, we will publish our Isabelle proof development in the *Archive of Formal Proof*, the standard venue for such proofs.

## 5 APPROXIMATE STRATEGYPROOFNESS

As our last contribution, we analyze how manipulable particular SWFs are. Indeed, while Theorem 2 shows that all reasonable SWFs must be manipulable, it may be the case that voters can only gain a small amount of utility by lying about their true ranking. In practice, this may be enough to disincentivize voters from manipulating as casting strategic votes requires effort to, e.g., learn the rankings of the other voters and to compute a successful manipulation.

Unfortunately, all the SWFs discussed in Section 2.1 are severely manipulable. To formalize this, we will make several changes in our assumptions. Firstly, we will consider a variable electorate setting and use as many voters as necessary for our counterexamples. Secondly, to make our result better comparable to the existing literature, we will analyze the potential utility gain rather than the decrease in cost. To this end, we define the utility of a voter  $i$  with ranking  $\succ_i$  for another ranking  $\triangleright$  by  $u(\succ_i, \triangleright) = \binom{m}{2} - \Delta(\succ_i, \triangleright) = |\{(x, y) \in A^2 : x \succ_i y \wedge x \triangleright y\}|$ . We note that  $u$  is minimal if  $\succ$  and  $\triangleright$  are inverse to each other (yielding  $u(\succ, \triangleright) = 0$ ) and maximal if  $\succ = \triangleright$  (yielding  $u(\succ, \triangleright) = \binom{m}{2}$ ). When using this utility function, strategyproofness demands that voters cannot increase their utility by lying about their ranking, which is equivalent to Definition 1.

Finally, we will use the incentive ratio to measure the manipulability of SWFs. Intuitively, this ratio quantifies the worst-case ratio between a voters' utility when lying and when voting truthfully. Formally, the *incentive ratio* of an SWF  $f$  for  $m$  alternatives is defined by  $\gamma_m(f) = \sup_{R, i, \succ'_i} \frac{u_i(f(\succ'_i, R_{-i}))}{u_i(f(R))}$ , where we only consider profiles on  $m$  alternatives and  $(\succ'_i, R_{-i})$  denotes the profile obtained from  $R$  by letting voter  $i$  deviate to  $\succ'_i$ . Since  $u_i(f(R))$  can be 0, we use the conventions that  $\frac{x}{0} = \infty$  for all  $x > 0$  and  $\frac{0}{0} = 1$ . We observe that  $\gamma_m(f) = 1$  if  $f$  is strategyproof for  $m$  alternatives and  $\gamma_m(f) > 1$  otherwise. Moreover, an SWF has an incentive ratio of  $\gamma_m(f) = \infty$  if a voter with utility 0 can manipulate and it holds that  $\gamma_m(f) \leq \binom{m}{2}$  otherwise. The incentive ratio has been successfully used for private good settings to show that several manipulable mechanisms are still close to strategyproof [e.g., 29, 54, 73].

We will next prove that all SWFs discussed in Section 2.1 have a large incentive ratio, which demonstrates that these SWFs are severely manipulable. Specifically, we will show that, while distance scoring rules (including the Kemeny rule) cannot be manipulated by voters with utility 0, voters with a utility of 1 can gain almost their maximal utility when lying. Moreover, for positional scoring rules, we show that voters with utility 0 can manipulate, which means that their incentive ratio is unbounded.

**Theorem 3.** *The following statements are true:*

- (1) *For all  $m \geq 4$ , the incentive ratio of the Kemeny rule  $f_{\text{Kemeny}}$  satisfies  $\binom{m}{2} - m \leq \gamma_m(f_{\text{Kemeny}}) \leq \binom{m}{2}$*
- (2) *For all  $m \geq 3$ , the incentive ratio of every distance scoring rule  $f_{\text{dist}}$  other than  $f_{\text{Kemeny}}$  satisfies  $\binom{m}{2} - 1 \leq \gamma_m(f_{\text{dist}}) \leq \binom{m}{2}$ .*
- (3) *For all  $m \geq 3$ , the incentive ratio of every positional scoring rule  $f_{\text{pos}}$  is  $\gamma_m(f_{\text{pos}}) = \infty$ .*

**PROOF SKETCH.** We will only prove here that  $\gamma_m(f) \leq \binom{m}{2}$  for all distance scoring rules  $f$  and defer the proofs of our lower bounds

to Appendix C. Equivalently, this upper bound means that voters with utility 0 cannot manipulate distance scoring rules. To prove this claim, we fix such a rule  $f$  and its distance scoring function  $s$ , a profile  $R$ , a voter  $i$  with ranking  $\succ_i$ , and let  $\triangleright = f(R)$ . Moreover, we suppose that  $u(\succ_i, \triangleright) = 0$ , which means that  $\succ_i$  is inverse to  $\triangleright$  and  $\Delta(\succ_i, \triangleright) = \binom{m}{2}$ . Lastly, let  $R'$  denote a profile derived from  $R$  by assigning an arbitrary ranking  $\succ'_i \neq \succ_i$  to voter  $i$ .

As the first step, we consider another ranking  $\triangleright' \in \mathcal{R} \setminus \{\triangleright\}$  and show that  $s(\Delta(\succ_i, \triangleright)) - s(\Delta(\succ'_i, \triangleright)) \geq s(\Delta(\succ_i, \triangleright')) - s(\Delta(\succ'_i, \triangleright'))$ . To this end, we observe that  $\Delta(\succ_i, \triangleright) - \Delta(\succ'_i, \triangleright) = \Delta(\succ_i, \succ'_i)$  since  $\triangleright$  and  $\succ_i$  are inverse to each other. Next, let  $z = \Delta(\succ_i, \triangleright') - \Delta(\succ'_i, \triangleright')$  and note that  $z \leq \Delta(\succ_i, \succ'_i)$  since  $\Delta$  is a metric. If  $z \leq 0$ , our inequality holds since  $s$  is non-decreasing,  $\Delta(\succ_i, \triangleright) \geq \Delta(\succ'_i, \triangleright)$ ,  $\Delta(\succ_i, \triangleright') \leq \Delta(\succ'_i, \triangleright')$ . In particular, these insights imply that  $s(\Delta(\succ_i, \triangleright)) - s(\Delta(\succ'_i, \triangleright)) \geq 0 \geq s(\Delta(\succ_i, \triangleright')) - s(\Delta(\succ'_i, \triangleright'))$ . Next, suppose that  $z > 0$ . In this case, we recall that, by definition,  $s(x) - s(x-1) \geq s(x-1) - s(x-2)$  for all  $x \in \{2, \dots, \binom{m}{2}\}$ , which implies that  $s(\binom{m}{2}) - s(\binom{m}{2} - z) \geq s(\Delta(\succ_i, \triangleright')) - s(\Delta(\succ_i, \triangleright) - z)$ . Since  $s$  is non-decreasing,  $\Delta(\succ_i, \triangleright) = \binom{m}{2}$ , and  $z \leq \Delta(\succ_i, \succ'_i)$ , we can again infer our target inequality as

$$\begin{aligned} s(\Delta(\succ_i, \triangleright)) - s(\Delta(\succ'_i, \triangleright)) &= s(\Delta(\succ_i, \triangleright)) - s(\Delta(\succ_i, \triangleright) - \Delta(\succ_i, \succ'_i)) \\ &\geq s(\Delta(\succ_i, \triangleright)) - s(\Delta(\succ_i, \triangleright) - z) \\ &\geq s(\Delta(\succ_i, \triangleright')) - s(\Delta(\succ_i, \triangleright') - z) \\ &= s(\Delta(\succ_i, \triangleright')) - s(\Delta(\succ'_i, \triangleright')). \end{aligned}$$

Next, it holds that  $\sum_{j \in N} s(\Delta(\succ_j, \triangleright)) \leq \sum_{j \in N} s(\Delta(\succ_j, \triangleright'))$  for all  $\triangleright' \in \mathcal{R}$  since  $f(R) = \triangleright$ . Hence, we derive for all  $\triangleright' \in \mathcal{R}$  that

$$\begin{aligned} \sum_{j \in N} s(\Delta(\succ'_j, \triangleright)) &= \sum_{j \in N \setminus \{i\}} s(\Delta(\succ_j, \triangleright)) - (s(\Delta(\succ_i, \triangleright) - \Delta(\succ'_i, \triangleright))) \\ &\leq \sum_{j \in N \setminus \{i\}} \Delta(\succ_j, \triangleright') - (s(\Delta(\succ_i, \triangleright') - s(\Delta(\succ'_i, \triangleright')))) \\ &= \sum_{j \in N} \Delta(\succ'_j, \triangleright'). \end{aligned}$$

This proves that  $\triangleright$  minimizes the total score in  $R'$ . Moreover, if this inequality is tight for some ranking  $\triangleright'$ , then  $\sum_{j \in N} s(\Delta(\succ_j, \triangleright)) = \sum_{j \in N} s(\Delta(\succ_j, \triangleright'))$  and  $\triangleright$  is lexicographically preferred to  $\triangleright'$  as  $f(R) = \triangleright$ . We hence conclude that  $f(R') = \triangleright$ , which shows that voters with utility 0 cannot manipulate  $f$ .  $\square$

**Remark 5.** A natural follow-up question to Theorem 3 is whether there are appealing SWFs that have a significantly better incentive ratio than our considered SWFs. Motivated by this question, we discuss in Appendix C the *minimal compromise rule*  $f_{MC}$ , which has an incentive ratio of  $m - 2$  when  $m \geq 4$ . To define this rule, we denote the min score of an alternative  $x$  in a profile  $R$  by  $s_{\min}(R, x) = \min_{i \in N} m - r(\succ_i, x)$ . Then,  $f_{MC}$  orders the alternatives in decreasing order of their min scores, with ties broken lexicographically.

## 6 CONCLUSION

In this paper, we study social welfare functions (SWFs) with respect to (Kemeny-)strategyproofness, which requires that voters cannot obtain a ranking that is closer to their true ranking in terms of the Kemeny distance by voting strategically. As our main result, we show a sweeping impossibility theorem, demonstrating that

no anonymous and unanimous SWF satisfies strategyproofness if there are  $m \geq 4$  alternatives. Moreover, we prove that no majority consistent SWF is strategyproof when  $m \geq 4$  and that many natural SWFs are severely manipulable as they have a high incentive ratio.

Perhaps the most natural follow-up question to our work is how we can circumvent our impossibility theorems. Possible directions to this end are the study of randomized or set-valued SWFs, a more detailed analysis of the incentive ratio of SWFs, or the study of alternative strategyproofness notions. For instance, one could analyze strategyproofness based on other distances between rankings in the hope of more positive results.

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## A MAJORITY CONSISTENCY AND STRATEGYPROOFNESS FOR $m = 3$

As mentioned in Remark 1, when  $m = 3$ , the Kemeny rule satisfies both majority consistency and strategyproofness. In this appendix, we will show that, under mild additional assumptions, these two axioms even requires us to choose a ranking minimizing the total Kemeny distance to the input rankings. To make this more formal, we first extend the domain of SWFs from profiles for a fixed electorate  $\mathcal{R}^N$  to the set of all profiles  $\mathcal{R}^* = \bigcup_{N \subseteq \mathcal{R}}: N \text{ is non-empty and finite } \mathcal{R}^N$  that are defined for any non-empty and finite electorate. Furthermore, we define the *majority margin* between two alternatives  $x, y \in A$  in a profile  $R$  by  $g_{xy}(R) = |\{i \in N: x \succ_i y\}| - |\{i \in N: y \succ_i x\}|$ , i.e., the majority margin between two alternatives counts how many more voters prefer  $x$  to  $y$  than vice versa. Then, we say an SWF  $f: \mathcal{R}^* \rightarrow \mathcal{R}$  satisfies

- *quasi-neutrality* if  $f(\tau(R)) = \tau(f(R))$  for all profiles  $R$  and permutations  $\tau: A \rightarrow A$  such that (i)  $g_{xy}(R) \neq g_{vw}(R)$  for all  $v, w, x, y \in A$  with  $v \neq w, x \neq y$ , and  $\{x, y\} \neq \{v, w\}$  and (ii) the alternatives can be labeled such that  $x \succ_R y \succ_R z \succ_R x$ .
- *cancellation* if  $f(R) = f(R')$  for all profiles  $R$  and  $R'$  such that  $R'$  arises from  $R$  by adding two voters with inverse preferences.

Less formally, quasi-neutrality enforces a mild degree of neutrality for profiles where the majority relation is cyclic and all majority margins are unique. On the other hand, cancellation requires that adding pairs of voters with inverse preferences does not affect the outcome. As we show next, every SWF on  $\mathcal{R}^*$  that satisfies strategyproofness, majority consistency, anonymity, quasi-neutrality, and cancellation must always choose a Kemeny ranking, i.e., a ranking from the set  $K(R) = \arg \max_{\succ \in \mathcal{R}} \sum_{i \in N_R} \Delta(\succ_i, \succ)$  (where  $N_R$  denotes the voters in  $R$ ).

**Proposition 2.** *Assume  $m = 3$ . If an SWF  $f$  on  $\mathcal{R}^*$  satisfies strategyproofness, majority consistency, anonymity, quasi-neutrality, and cancellation, then  $f(R) \in K(R)$  for all profiles  $R \in \mathcal{R}^*$ .*

**PROOF.** Let  $m = 3$  and  $f$  be a SWF on  $\mathcal{R}^*$  that satisfies all given axioms. We will show in multiple steps that  $f(R) \in K(R)$  for all profiles  $R \in \mathcal{R}^*$ .

**Step 1:** First, we will show that  $f$  only depends on the majority margins: it holds that  $f(R) = f(R')$  for all profiles  $R, R' \in \mathcal{R}^*$  such that  $g_{xy}(R) = g_{xy}(R')$  for all alternatives  $x, y \in A$ . To prove this claim, we denote the six possible input rankings by

$$\begin{array}{lll} \succ_1 = abc & \succ_2 = bca & \succ_3 = cab \\ \succ_4 = cba & \succ_5 = acb & \succ_6 = bac. \end{array}$$

Furthermore, given a profile  $R$ , we define by  $n_i^R$  for  $i \in \{1, \dots, 6\}$  the number of voters that report the ranking  $\succ_i$ . By anonymity, we can compute  $f$  only based on these six numbers. We further observe that  $\succ_i$  is inverse to  $\succ_{i+3}$  for  $i \in \{1, 2, 3\}$ . We hence define  $\delta_i^R = n_i^R - n_{i+3}^R$  for all  $i \in \{1, 2, 3\}$  to count how many more voters report  $\succ_i$  than  $\succ_{i+3}$  (or vice versa). By cancellation, these values are sufficient to compute  $f$  as we can add and remove pairs of voters with inverse preferences without affecting the outcome. To prove this step, we will next show that if  $g_{xy}(R) = g_{xy}(R')$  for all  $x, y \in A$ , then also  $\delta_i^R = \delta_i^{R'}$ . This implies that  $f(R) = f(R')$  by our previous insights, so  $f$  can indeed only be computed based on the pairwise majority margins. Now, to prove our claim, we observe that

$$g_{ab}(R) = \delta_1^R - \delta_2^R + \delta_3^R, \quad g_{bc}(R) = \delta_1^R + \delta_2^R - \delta_3^R, \quad g_{ca}(R) = -\delta_1^R + \delta_2^R + \delta_3^R.$$

By rearranging these equations, we infer that

$$\delta_1^R = \frac{g_{ab}(R) + g_{ab}(R)}{2}, \quad \delta_2^R = \frac{g_{bc}(R) + g_{ca}(R)}{2}, \quad \delta_3^R = \frac{g_{ca}(R) + g_{ab}(R)}{2}.$$

Hence, the majority margins fully determine the values of  $\delta_i^R$ . In particular, if  $g_{xy}(R) = g_{xy}(R')$  for all  $x, y \in A$ , then it also holds that  $\delta_i^R = \delta_i^{R'}$  for all  $i \in \{1, 2, 3\}$ . This completes the proof of this step.

**Step 2:** We will next show that  $f(R) \in K(R)$  for all profiles  $R \in \mathcal{R}^*$ . To this end, we first note that if the majority relation  $\succ_M$  of a profile  $R$  is a ranking, then  $K(R) = \{\succ_M\}$  and our inclusion holds by majority consistency. Furthermore, if  $g_{xy}(R) = 0$  for all  $x, y \in A$ , then  $K(R) = \mathcal{R}$  and it holds trivially that  $f(R) \in K(R)$ . We next proceed with a case distinction regarding the structure of the considered profile.

**Case 2.1:** Let  $R$  denote a profile such that  $g_{ab}(R) = g_{bc}(R) = g_{ca}(R) > 0$ . For this profile, it holds that  $K(R) = \{abc, bca, cab\}$  and we assume for contradiction that  $f(R) \notin \{abc, bca, cab\}$ . Without loss of generality, we can make this more precise by letting  $f(R) = cba$ . Using cancellation, we next add  $g_{ab}(R)$  pairs of voters such that one voter reports  $abc$  and the other voter reports  $cba$ . This leads to a new profile  $R'$  with  $f(R') = cba$  due to cancellation. Finally, we let the newly added voters who report  $abc$  deviate to  $bac$ . Since these voters completely disagree with  $f(R')$ , strategyproofness requires that the outcome is not allowed to change at any step. However, in the resulting profile  $R''$ , we have that  $g_{ba}(R'') = g_{ab}(R') > 0$ ,  $g_{bc}(R'') > 0$ , and  $g_{ca}(R'') > 0$ , i.e., the majority relation is transitive. Hence, we need to choose the ranking  $bca$  for  $R''$ , which contradicts strategyproofness. This proves that the assumption that  $f(R) \notin \{abc, bca, cab\}$  is wrong.

**Case 2.2:** We will next turn to profiles such that the majority relation is cyclic and all majority margins are unique and non-zero. More specifically, we analyze profiles  $R$  such that  $g_{ab}(R) > 0$ ,  $g_{bc}(R) > 0$ ,  $g_{ca}(R) > 0$ , and  $g_{xy}(R) \neq g_{yz}(R)$  for all distinct  $x, y, z \in A$ . We start by again considering a profile  $R^*$  such that  $g_{ab}(R^*) = g_{bc}(R^*) = g_{ca}(R^*) = \lambda$  for some  $\lambda > 0$ . By Case 2.1, it holds that  $f(R^*) \in \{abc, bca, cab\}$  and we suppose without loss of generality that  $f(R^*) = abc$ . Using cancellation, we next add pairs of voters to  $R^*$  such that one reports  $abc$  and the other  $cba$ . By strategyproofness, the voters who report  $cba$  cannot change the outcome by deviating as any other outcome is better for them. In particular, this means that, when swapping  $b$  and  $c$ , or  $a$  and  $b$ , the outcome does not change. By this argument and Step 1, it follows that  $f(R') = abc$  for all profiles  $R'$  such that  $g_{ab}(R') \geq g_{ab}(R)$ ,  $g_{bc}(R') \geq g_{bc}(R)$ , and  $g_{ca}(R') = \lambda$ . In particular, this

holds for when  $g_{ab}(R') \neq g_{bc}(R')$ ,  $g_{ab}(R') > g_{ca}(R')$ , and  $g_{bc}(R') > g_{ca}(R')$ , i.e., when all majority margins are different and  $g_{ca}(R')$  has the smallest weight. Put differently, for all such profiles, we “cut” the edge in the majority graph with the least weight, which means that  $f(R') \in K(R') = \{abc\}$ . Further, since all majority margins are unique in  $R'$ , we can generalize this insight to all profiles whose majority margins can be obtained from  $R'$  by permuting the alternatives using quasi-neutrality. This shows that  $f(R) \in K(R)$  for all profiles such that the minimal majority margin has weight  $\lambda$  and all majority margins are different. Finally, we note that we can apply this argument for all values of  $\lambda > 0$ . Hence,  $f(R) \in K(R)$  for all profiles with unique non-zero majority margins and a cyclic majority relation.

*Case 2.3:* Thirdly, we consider the case that the majority relation is cyclic and the majority margins have two different non-zero values. To this end, let  $R$  denote a profile and  $\lambda_1, \lambda_2 \in \mathbb{N}$  denote two non-zero integers such that  $g_{ab}(R) = \lambda_1$ ,  $g_{bc}(R) = \lambda_1$ ,  $g_{ca}(R) = \lambda_2$ . First, suppose that  $\lambda_1 > \lambda_2$ , in which case  $K(R) = \{abc\}$ . Suppose for contradiction that  $f(R) \neq abc$ . We first note that it can be shown analogously to Case 2.1 that  $f(R) \notin \{cba, bac, acb\}$  as we can deviate to a profile with transitive majority relation otherwise. It thus holds that  $f(R) \in \{bca, cab\}$ . Suppose that  $f(R) = bca$ . In this case, we add a pair of voters with preference relations  $bca$  and  $acb$  and cancellation shows that we still choose  $bca$ . Next, let the voter reporting  $acb$  swap  $b$  and  $c$ , resulting in a profile  $R'$  with  $g_{ab}(R') = \lambda_1$ ,  $g_{bc}(R') = \lambda_1 + 2$ , and  $g_{ca}(R') = \lambda_2$ . Moreover, strategyproofness requires that  $f(R') = bca$  as any other outcome constitutes a manipulation. However, this contradicts Case 2.2 as all majority margins are unique, non-zero, and  $g_{ca}(R')$  is minimal. On the other hand, if  $f(R) = cab$ , we can apply an analogous argument by adding a pair of voters with preferences  $cab$  and  $bac$  and reinforcing  $a$  against  $b$ .

As the second subcase, suppose that  $\lambda_1 < \lambda_2$ , which means that  $K(R) = \{bca, cab\}$ . Using again the argument of Case 2.1, it follows that  $f(R) \in \{abc, bca, cab\}$ . So, we assume for contradiction that  $f(R) = abc$ . In this case, we can add pairs of voters with preference relations  $abc$  and  $cba$ , which does not affect the outcome due to cancellation. Moreover, strategyproofness requires that the voters reporting  $cba$  cannot change the outcome by deviating. By letting these voters swap  $b$  and  $c$ , we can increase the majority margin  $g_{bc}(R)$  arbitrarily, so we now infer that  $f(R') = abc$  for all profiles  $R'$  with  $g_{ab}(R') = \lambda_1$ ,  $g_{bc}(R') \geq \lambda_1$ , and  $g_{ca}(R') = \lambda_2$ . However, once  $g_{bc}(R') > \lambda_2$ , this conflict with Case 2.2: in this case, the minimal edge is  $g_{ab}(R')$ , so  $f(R')$  must be  $bca$ .

*Case 2.4:* In our fourth case, we assume that exactly one majority margin has value 0. Without loss of generality, we suppose that  $g_{ac}(R) = 0$  and we consider three subcases. Moreover, we will assume that every preference relation is reported by at least one voter; this is without loss of generality due to cancellation. Now, as the first subcase, suppose  $g_{bc}(R) > 0$  and  $g_{ba}(R) > 0$ , which implies that  $K(R) = \{bca, bac\}$ . Assume without for contradiction that  $f(R) \notin K(R)$ . If  $f(R) = acb$ , let  $R'$  denote the profile derived from  $R$  by letting a voter change his preference relation from  $bca$  to  $bac$ . Strategyproofness requires that the outcome does not change. However, the majority relation now agrees with the ranking  $bca$ , so majority consistency requires that  $f(R) = bca$ , a contradiction. If  $f(R) = abc$ , let  $R'$  denote the profile derived from  $R$  by letting a voter change his preference relation from  $cba$  to  $bac$ . Strategyproofness requires that  $f(R') = abc$ , but majority consistency requires that  $f(R') = bac$ , hence yielding a contradiction. Finally, the cases that  $f(R) = cab$  and  $f(R) = cba$  are symmetric to  $f(R) = acb$  and  $f(R) = abc$ , respectively, so we have a contradiction in every case.

As the second subcase, suppose that  $g_{bc}(R) < 0$  and  $g_{ba}(R) < 0$ , so  $K(R) = \{acb, cab\}$ . If  $f(R) = bca$ , a voter can again benefit by manipulating from the preference relation  $acb$  to  $cab$ . In particular, after this deviation, the majority relation agrees with the ranking  $cab$ , so this ranking must be chosen, but  $\Delta(acb, bca) = 3 > 1 = \Delta(acb, cab)$ . On the other hand, if  $f(R) = cba$ , a voter can manipulate by deviating from the preference relation  $abc$  to  $cab$ . After this step, the outcome must be  $cab$  by majority consistency, which again decreases the Kemeny distance of the manipulator. The case that  $f(R) = bac$  and  $f(R) = abc$  are again symmetric, so it follows that  $f(R) \in K(R)$  as every other outcome means that  $f$  fails strategyproofness.

As the third case, we suppose that  $g_{ab}(R) > 0$  and  $g_{bc}(R) > 0$  and note that the case  $g_{ab}(R) < 0$  and  $g_{bc}(R) < 0$  is symmetric. For this case, let  $R^*$  denote a profile with  $g_{xy}(R^*) = 0$  for all  $x, y \in A$  and suppose that  $f(R^*) = bca$  (this choice will not matter due to quasi-neutrality). Using the same argument as in Case 2.2, it holds for every profile  $R'$  with  $g_{bc}(R') \geq 0$ ,  $g_{ca}(R') \geq 0$ , and  $g_{ab}(R) = 0$  that  $f(R') = bca$ , too. Based on quasi-neutrality, this insight generalizes to all rankings such that one majority margin is 0, all majority margins are distinct and the majority relation is cyclic. Hence, if  $g_{ab}(R) \neq g_{bc}(R)$ , it follows for our profile  $R$  that  $f(R) = abc$  and thus  $f(R) \in K(R)$ . So, suppose that  $g_{ab}(R) = g_{bc}(R)$  and assume that  $f(R) \neq abc$ . If  $f(R) = cba$ , we repeatedly let voters with preference relation  $abc$  change their preference relation to  $acb$ . (Note that we may use cancellation to add these voters). This eventually results in a profile  $R'$  with  $g_{ab}(R') > 0$ ,  $g_{cb}(R') > 0$ , and  $g_{ac}(R') = 0$ , so  $f(R') \in \{acb, cab\}$  by the last subcase. However, this contradicts strategyproofness as the voters with preference relation  $acb$  have  $\Delta(acb, bca) = 3$  and any other outcome is therefore a manipulation. Next, if  $f(R) = bac$ , we let a voter with preference relation  $cab$  swap  $a$  and  $c$ . For the resulting profile  $R'$ , majority consistency requires that  $f(R') = abc$ , which constitutes a manipulation. A similar construction also works if  $f(R) = acb$ . Lastly, if  $f(R) = bca$  or  $f(R) = cab$ , we can infer a manipulation analogously to Case 2.3 by ensuring that all majority margins are unique.

*Case 2.5:* As the last case, suppose that exactly two majority margins are zero. To make this more precise, we assume that  $g_{ab}(R) > 0$  and  $g_{bc}(R) = g_{ca}(R) = 0$ , so  $K(R) = \{abc, acb, cab\}$ . Suppose for contradiction that  $f(R) \notin K(R)$ . If  $f(R) = bac$ , let  $R'$  denote the profile derived from  $R$  by letting a voter deviate from  $cab$  to  $acb$ . Consequently,  $g_{ab}(R') > 0$ ,  $g_{ac}(R') > 0$ , and  $g_{bc}(R') = 0$ , so  $f(R) \in \{abc, acb\}$  by Case 2.4. Regardless of the exact outcome this constitutes a manipulation. If  $f(R) = bca$ , we can use an analogous construction to infer a contradiction. Finally, if  $f(R) = cba$ , we let a voter deviate from  $abc$  to  $acb$ . This results in a profile  $R'$  with  $g_{ab}(R') > 0$ ,  $g_{cb}(R') > 0$ , and  $g_{ac}(R') = 0$ . Hence,  $f(R) \in \{acb, cab\}$  by Claim 2.4, which means that a voter can again manipulate. Hence, it indeed holds that  $f(R) \in K(R)$  for all cases.  $\square$

## B PROOF OF LEMMA 1

We next turn to the inductive arguments, which are necessary to generalize Proposition 1 to Theorem 2.

**Lemma 1.** *Assume there is no anonymous SWF that satisfies strategyproofness and unanimity for  $m$  alternatives and  $n$  voters. The following claims hold:*

- (1) *For every  $m' > m$ , there is no anonymous SWF that satisfies strategyproofness and unanimity for  $m'$  alternatives and  $n$  voters.*
- (2) *For every  $\ell \in \mathbb{N}$ , there is no anonymous SWF that satisfies strategyproofness and unanimity for  $m$  alternatives and  $\ell n$  voters.*

**PROOF.** Fix some values  $m$  and  $n$  and assume that there is no anonymous SWF satisfying strategyproofness and unanimity for  $m$  alternatives and  $n$  voters. For both claims of our lemma, we will show that if there was an SWF satisfying our axioms for the given parameters, then there would also be one for  $m$  alternatives and  $n$  voters, which contradicts our previous assumption.

**Claim (1):** Assume for contradiction that there is an integer  $m' > m$  such that there is an anonymous SWF  $f$  for  $m'$  alternatives and  $n$  voters that satisfies strategyproofness and unanimity. Moreover, let  $A'$  denote the set of  $m'$  alternatives for which  $f$  is defined, and let  $A \subseteq A'$  be a subset of this set with  $|A| = m$ . We will next construct an SWF  $g$  for  $n$  voters and the alternatives  $A$  that satisfies all our axioms. To this end, given a profile  $R$  on the alternatives  $A$ , we define by  $R^{A \rightarrow A'}$  the profile obtained from  $R$  by adding the alternatives in  $A' \setminus A$  in a fixed order at the bottom of the rankings of all voters. Moreover, given a ranking  $\triangleright$  on  $A'$ , we define by  $\triangleright|_A$  the ranking obtained by deleting the alternatives in  $A' \setminus A$  from  $\triangleright$ . Then, given a profile  $R$  on the alternatives  $A$ , our new SWF  $g$  first constructs the profile  $R^{A \rightarrow A'}$ , computes the output ranking  $\triangleright$  of  $f$  on this profile, and finally returns the restriction of  $\triangleright$  to  $A$ . Or, more compactly,  $g$  is given by  $g(R) = f(R^{A \rightarrow A'})|_A$ .

We next will show that  $g$  satisfies anonymity, unanimity, and strategyproofness. For anonymity, we note that  $\pi(R^{A \rightarrow A'}) = \pi(R)^{A \rightarrow A'}$  for every permutation  $\pi : N \rightarrow N$  since we extend the ranking of every voter in the same way, regardless of his reported ranking. Since  $f$  is by assumption anonymous, it hence holds that  $g(\pi(R)) = f(\pi(R)^{A \rightarrow A'})|_A = f(\pi(R^{A \rightarrow A'}))|_A = f(R^{A \rightarrow A'})|_A = g(R)$  for all permutations  $\pi : N \rightarrow N$ , thus showing that  $g$  is unanimous. Next, for unanimity, we observe that, if  $x \succ_i y$  for all voters  $i$  in some profile  $R$  on  $A$ , then the same holds for the profile  $R^{A \rightarrow A'}$ . Since  $f$  is unanimous, it therefore follows that the output ranking  $g(R) = f(R^{A \rightarrow A'})|_A$  ranks  $x$  ahead of  $y$ .

Lastly, for strategyproofness, we note that if  $g$  is manipulable, then so must be  $f$ . To make this more formal, assume that there are profiles  $R$  and  $\hat{R}$  over the set of alternatives  $A$  and a voter  $i$  in  $N$  such that  $R$  and  $\hat{R}$  only differ in the ranking of voter  $i$  and  $\Delta(\succ_i, g(R)) > \Delta(\succ_i, g(\hat{R}))$ . Further, let  $\tilde{\succ}_i$  denote voter  $i$ 's extended ranking in  $R^{A \rightarrow A'}$ . Since  $f$  is unanimous and all voters in  $R^{A \rightarrow A'}$  (resp.  $\hat{R}^{A \rightarrow A'}$ ) rank all alternatives in  $A$  ahead of those in  $A' \setminus A$  and agree on the order of the alternatives in  $A' \setminus A$ , the same must be true for the output ranking  $f(R^{A \rightarrow A'})$  (resp.  $f(\hat{R}^{A \rightarrow A'})$ ). This implies that  $\Delta(\tilde{\succ}_i, f(R^{A \rightarrow A'})) = \Delta(\succ_i, g(R)) > \Delta(\succ_i, g(\hat{R})) = \Delta(\tilde{\succ}_i, f(\hat{R}^{A \rightarrow A'}))$ . Since  $R^{A \rightarrow A'}$  and  $\hat{R}^{A \rightarrow A'}$  only disagree in the ranking of voter  $i$ , we conclude that  $f$  is manipulable, contradicting our assumptions. Hence, if the SWF  $f$  exists, we could also construct an SWF  $g$  for  $m$  alternatives and  $n$  voters that satisfies anonymity, unanimity, and strategyproofness, contradicting the premise of this lemma.

**Claim (2):** For the second case, we fix some integer  $\ell \in \mathbb{N}$  and suppose that there is an anonymous SWF  $f$  that satisfies strategyproofness and unanimity for  $m$  voters and  $\ell n$  voters. This time, we define the following transformation: given a profile  $R$  on  $n$  voters and  $m$  alternatives, we define by  $\ell R$  the profile that contains  $\ell$  copies of each voter in  $R$ . Then, we define an SWF  $g$  for  $n$  voters and  $m$  alternatives by  $g(R) = f(\ell R)$ . We first note that it is again easy to see that  $g$  inherits both anonymity and unanimity from  $f$ . In more detail, permuting our input profile  $R$  corresponds to permuting our input profile  $\ell R$  accordingly, which suffices to prove that  $g$  inherits anonymity. Similarly, if all voters in  $R$  prefer  $x$  to  $y$ , the same holds for  $\ell R$ . Hence, the output ranking  $g(R) = f(\ell R)$  also has to rank  $x$  ahead of  $y$  by the unanimity of  $f$ .

Lastly, for strategyproofness, consider two profile  $R$  and  $R'$  (on  $n$  voters) such that  $R$  and  $R'$  only differ in the ranking of voter  $i$ . Consequently,  $\ell R$  and  $\ell R'$  only differ in the  $\ell$  clones of voter  $i$ . Now, consider the sequence of profile  $R^0 = \ell R, R^1, \dots, R^\ell = \ell R'$  derived by letting the clones of voter  $i$  one after another change their ranking from  $\succ_i$  to  $\succ'_i$ . By the strategyproofness of  $f$ , it holds that  $\Delta(\succ_i, f(R^j)) \leq \Delta(\succ_i, f(R^{j+1}))$  for all  $j \in \{0, \dots, \ell - 1\}$ . By chaining these inequalities, we get that  $\Delta(\succ_i, g(R)) = \Delta(\succ_i, f(\ell R)) \leq \Delta(\succ_i, f(\ell R')) = \Delta(\succ_i, g(R'))$ . This proves that  $g$  is strategyproof if  $f$  satisfies this condition. Hence, if  $f$  satisfies anonymity, unanimity, and strategyproofness, so does  $g$ , which contradicts our assumption that no SWF for  $m$  alternatives and  $n$  voters simultaneously satisfies all three conditions.  $\square$

## C PROOF OF THEOREM 3

We next turn to the proof Theorem 3. Since we have already shown that every distance scoring function has an incentive ratio of at most  $\gamma_m(f) \leq \binom{m}{2}$ , we will only focus on the lower bounds. Further, to improve legibility, we show each lower bound as a separate proposition.

**Proposition 3.** *For all  $m \geq 4$ , the incentive ratio of the Kemeny rule  $f_{\text{Kemeny}}$  satisfies  $\binom{m}{2} - m \leq \gamma_m(f_{\text{Kemeny}})$ .*

**PROOF.** To prove this proposition, we first note that the Kemeny rule can be computed only based on the majority margins  $g_{xy}(R) = |\{i \in N : x \succ_i y\}| - |\{i \in N : y \succ_i x\}|$  for all alternatives  $x, y \in A$  in a profile  $R$ . In more detail, a ranking  $\triangleright$  minimizes the total Kemeny distance  $\sum_{i \in N} \Delta(\succ_i, \triangleright)$  if and only if it maximizes  $\sum_{x, y \in A} \mathbb{I}[x \triangleright y] g_{xy}(R)$ . This holds because  $\sum_{x, y \in A} \mathbb{I}[x \triangleright y] g_{xy}(R) = \binom{m}{2}n - 2 \sum_{x, y \in A} \mathbb{I}[x \triangleright y] |\{i \in N : y \succ_i x\}| = \binom{m}{2}n - 2 \sum_{x, y \in A} \sum_{i \in N} \mathbb{I}[x \triangleright y \wedge y \succ_i x] = \binom{m}{2}n - 2 \sum_{i \in N} \Delta(\succ_i, \triangleright)$ . Here,  $\mathbb{I}[x \triangleright y \wedge y \succ_i x]$  is an indicator function that takes value 1 if  $x \triangleright y$  and  $y \succ_i x$  and 0 otherwise. For an easier notation, we define the Kemeny score of an ranking  $\triangleright$  in a profile  $R$  by  $s(R, \triangleright) = \sum_{x, y \in A} \mathbb{I}[x \triangleright y] g_{xy}(R)$ . The key insight for our proof is that there is a profile  $R^*$  for which two rankings  $\triangleright_1$  and  $\triangleright_2$  with  $\Delta(\triangleright_1, \triangleright_2) = \binom{m}{2} - (m - 1)$  maximize the Kemeny score, i.e.,  $s(R^*, \triangleright_1) = s(R^*, \triangleright_2) > s(R^*, \triangleright)$  for all  $\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}$ . We will construct this profile  $R^*$  later on.

Based on this profile  $R^*$ , we will construct another profile  $\hat{R}$  to prove that  $\gamma_m(f_{\text{Kemeny}}) \geq \binom{m}{2} - m$ . To this end, let  $\succ_1, \dots, \succ_k$  (with  $k = \Delta(\succ_1, \succ_2) + 1$ ) denote a sequence of rankings such that  $\succ_1 = \succ_1$ ,  $\succ_k = \succ_2$ , and for all  $i \in \{1, \dots, k-1\}$ , the ranking  $\succ_{i+1}$  is derived from  $\succ_i$  by swapping one adjacent pair of alternatives in  $\succ_i$ . Put differently, this sequence transforms  $\succ_1$  and  $\succ_2$  by repeatedly swapping pairs of alternatives and each pair of alternatives is swapped at most once. Hence, it holds for  $\succ_2$  and  $\succ_{k-1}$  that  $\Delta(\succ_2, \succ_1) = \Delta(\succ_{k-1}, \succ_2) = 1$  and  $\Delta(\succ_{k-1}, \succ_2) = \Delta(\succ_2, \succ_1) = \Delta(\succ_1, \succ_2) - 1$ . Moreover, this means for the inverse rankings of  $\succ_2$  and  $\succ_{k-1}$ , denoted by  $\tilde{\succ}_2$  and  $\tilde{\succ}_{k-1}$ , that  $\Delta(\tilde{\succ}_2, \succ_1) = \Delta(\tilde{\succ}_{k-1}, \succ_2) = \binom{m}{2} - 1$  and  $\Delta(\tilde{\succ}_2, \succ_2) = \Delta(\tilde{\succ}_{k-1}, \succ_1) = \binom{m}{2} - (\Delta(\succ_1, \succ_2) - 1)$ . Finally, the profile  $\hat{R}$  is the profile derived from  $R^*$  by adding for each ranking  $\succ \in \{\succ_2, \succ_{k-1}, \tilde{\succ}_2, \tilde{\succ}_{k-1}\}$  one voter who reports  $\succ$ .

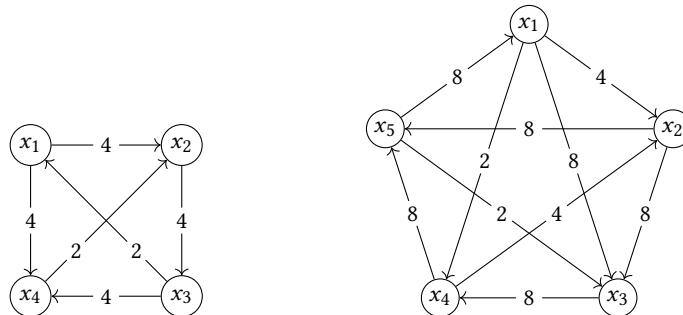
We first note that  $s(\hat{R}, \succ) = s(R^*, \succ)$  for all rankings  $\succ$  since  $\succ_2$  and  $\tilde{\succ}_2$  as well as  $\succ_{k-1}$  and  $\tilde{\succ}_{k-1}$  are inverse to each other. Thus, the corresponding voters cancel each other out with respect to the majority margins. This means that the Kemeny rule has to choose either  $\succ_1$  or  $\succ_2$  for  $\hat{R}$  and we suppose without loss of generality that  $\succ_1$  is selected. Now, let  $\hat{R}'$  denote the profile derived from  $\hat{R}$  by letting the voter who reports  $\tilde{\succ}_2$  deviate to the ranking that is completely inverse to  $\succ_1$ , denoted by  $\tilde{\succ}_1$ . Since  $\Delta(\tilde{\succ}_1, \succ_1) = \binom{m}{2} > \Delta(\tilde{\succ}_2, \succ_1)$  and  $\Delta(\tilde{\succ}_1, \succ_2) = \binom{m}{2} - \Delta(\succ_1, \succ_2) < \binom{m}{2} - (\Delta(\succ_1, \succ_2) - 1) = \Delta(\tilde{\succ}_2, \succ_2)$ , it holds that  $s(\hat{R}', \succ_1) < s(\hat{R}', \succ_2)$ . Further, it still holds that  $s(\hat{R}', \succ) < s(\hat{R}', \succ_2)$  for all  $\succ \in \mathcal{R} \setminus \{\succ_1, \succ_2\}$ . To see this, we note that  $\Delta(\tilde{\succ}_2, \tilde{\succ}_1) = 1$  because  $\Delta(\succ_2, \succ_1) = 1$ , so there is a single pair of alternatives  $x, y$  such that  $x \succ_2 y$  and  $y \succ_1 x$ . Consequently, it holds for every ranking  $\succ$  that  $s(\hat{R}', \succ) - s(\hat{R}, \succ) = 2$  if  $y \succ x$  and  $s(\hat{R}', \succ) - s(\hat{R}, \succ) = -2$  if  $x \succ y$ . Moreover, since the score of  $\succ_2$  increases when going from  $\tilde{\succ}_2$  to  $\tilde{\succ}_1$ , we have that  $s(\hat{R}', \succ_2) = s(\hat{R}, \succ_2) + 2 > s(\hat{R}, \succ_2) + 2 \geq s(\hat{R}', \succ)$  for all rankings  $\succ \in \mathcal{R} \setminus \{\succ_1, \succ_2\}$ . This proves that Kemeny's rule chooses  $\succ_2$  for  $\hat{R}'$ . Finally, we note that the deviator has a utility of  $u(\tilde{\succ}_2, \succ_1) = \binom{m}{2} - \Delta(\tilde{\succ}_2, \succ_1) = 1$  in  $\hat{R}$  and a utility of  $u(\tilde{\succ}_2, \succ_2) = \binom{m}{2} - \Delta(\tilde{\succ}_2, \succ_2) = \binom{m}{2} - m$  in  $\hat{R}'$ . Hence, the incentive ratio of the Kemeny rule is at least  $\binom{m}{2} - m$ .

It remains to show that there is indeed a profile  $R^*$  and two rankings  $\succ_1$  and  $\succ_2$  such that  $s(R^*, \succ_1) = s(R^*, \succ_2) > s(R^*, \succ)$  for all rankings  $\succ \in \mathcal{R} \setminus \{\succ_1, \succ_2\}$  and  $\Delta(\succ_1, \succ_2) = \binom{m}{2} - (m - 1)$ . To this end, we note that it suffices to specify a matrix containing all majority matrices because the Kemeny rule can be computed only based on these. Moreover, McGarvey's construction shows that every majority margin matrix can be induced by a ranking profile if all majority margins have the same parity [32, 57], so it is without loss of generality to focus on these matrices. To improve legibility, we will represent such majority margin matrices via weighted tournaments  $T = (A, E, w)$  on the alternatives, where  $(x, y) \in E$  if and only if  $g_{xy}(R) > 0$  and  $w(x, y) = g_{xy}(R)$ .

Furthermore, we also extend the Kemeny rule and the Kemeny score to such weighted tournaments. Specifically, given a weighted tournament  $T = (A, E, w)$ , we define the Kemeny score by  $s(T, \succ) = \sum_{(x, y) \in E: x \succ y} w(u, v)$  and the Kemeny rule chooses the ranking that maximizes this score. Given a profile  $R$  that induces the weighted tournament  $T$ , maximizing  $s(T, \succ)$  is equivalent to maximizing  $s(R, \succ) = \sum_{x, y \in A: x \succ y} g_{xy}(R)$ . To see this, let  $T = (V, E, w)$  denote the weighted tournament induced by  $R$  and let  $C = \sum_{e \in E} w(e) = \sum_{x, y \in A: g_{xy}(R) > 0} g_{xy}(R)$  denote the sum of all positive majority margins. Since  $g_{xy}(R) = -g_{yx}(R)$  for all  $x, y \in A$ , we have that

$$\begin{aligned} \sum_{x, y \in A: x \succ y} g_{xy}(R) &= \sum_{x, y \in A: x \succ y \wedge g_{xy}(R) > 0} g_{xy}(R) + \sum_{x, y \in A: x \succ y \wedge g_{xy}(R) < 0} g_{xy}(R) \\ &= \sum_{x, y \in A: x \succ y \wedge g_{xy}(R) > 0} g_{xy}(R) - \sum_{x, y \in A: x \succ y \wedge g_{yx}(R) > 0} g_{yx}(R) \\ &= -C + 2 \sum_{x, y \in A: x \succ y \wedge g_{xy}(R) > 0} g_{xy}(R) \\ &= -C + 2 \sum_{(x, y) \in E: x \succ y} w(x, y). \end{aligned}$$

Now, to prove the existence of  $R^*$ , we will proceed inductively on the number of alternatives and consider  $m = 4$  and  $m = 5$  as base cases. For these cases, the following weighted tournaments prove our claim.



For the weighted tournament  $T_4$  on  $m = 4$  alternatives on the left, it can be checked that precisely  $\succ_1 = x_1x_2x_3x_4$  and  $\succ_2 = x_3x_1x_4x_2$  maximize the Kemeny score. To see this, we note that both of these rankings only need to reverse edges with a total weight of 4, so

$s(T_4, \triangleright_1) = s(T_4, \triangleright_2) = 16$ , whereas all other rankings need to reverse edges with higher total weight and have thus less score. Further, it holds that  $\Delta(\triangleright_1, \triangleright_2) = 3 = \binom{m}{2} - (m - 1)$ , so our claim holds in this case. Similarly, when  $m = 5$ , the Kemeny score is maximized by the rankings  $\triangleright_1 = x_1 x_2 x_3 x_4 x_5$  and  $\triangleright_2 = x_4 x_2 x_5 x_1 x_3$  in the weighted tournament  $T_5$  on the right. Both of these rankings require us to revert edges with a total weight of 14, so there total score is  $s(T_5, \triangleright_2) = 46$ . By using a case distinction with respect to which edge of the cycle  $(x_1, x_3, x_4, x_5)$  is reversed in  $T_5$ , we can further conclude that every other ranking  $\triangleright$  has strictly less score, so  $\triangleright_1$  and  $\triangleright_2$  are indeed the only possible winning rankings. Further, it holds again that  $\Delta(\triangleright_1, \triangleright_2) = 6 = \binom{m}{2} - (m - 1)$ .

For our induction step, we assume that there is a weighted tournament  $T_m = (A, E, w)$  for  $m \geq 4$  alternatives  $A = \{x_1, \dots, x_m\}$  for which two rankings  $\triangleright_1$  and  $\triangleright_2$  with  $\Delta(\triangleright_1, \triangleright) = \binom{m}{2} - (m - 1)$  maximize the Kemeny score, i.e.,  $s(T_m, \triangleright_1) = s(T_m, \triangleright_2) > s(T_m, \triangleright)$  for all  $\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}$ . Additionally, we will suppose that (i) each edge weight  $w(e)$  in  $T_m$  is non-zero and even, (ii)  $\triangleright_1 = x_1 \dots x_m$ , and (iii)  $\triangleright_2$  top-ranks  $x_{m-1}$  and second-ranks  $x_{m-3}$ . It is straightforward to verify that these assumptions holds for our base cases, and it will become clear that they are preserved in the induction step. Given the weighted tournament  $T_m$ , we will construct another weighted tournament  $T_{m'}$  on  $m' = m + 2$  alternatives  $A' = \{x_1, \dots, x_{m+2}\}$ . In particular, for this weighted tournament  $T_{m'}$ , precisely the following two rankings  $\triangleright'_1$  and  $\triangleright'_2$  will maximize the Kemeny score:  $\triangleright'_1 = x_1 \dots x_m x_{m+1} x_{m+2}$  and  $\triangleright'_2$  agrees with  $\triangleright_2$  on all alternatives in  $A$ , places  $x_{m+1}$  first,  $x_{m-1} = x_{m'-3}$  second, and  $x_{m+2}$  third. For instance,  $\triangleright'_2 = x_5 x_3 x_6 x_1 x_4 x_2$  when  $m = 6$  and  $\triangleright'_2 = x_7 x_5 x_8 x_3 x_6 x_1 x_4 x_2$  when  $m = 8$ . Since  $\triangleright'_1$  agrees with  $\triangleright_1$  and  $\triangleright'_2$  agrees with  $\triangleright_2$  when restricted to  $A$ , it holds that  $\Delta(\triangleright'_1, \triangleright'_2) = \Delta(\triangleright_1, \triangleright_2) + m + (m - 1) = \binom{m}{2} + m = \binom{m+2}{2} - (m + 1)$ , so these rankings satisfy our distance condition. Further, it is easy to check that they satisfy the conditions (ii) and (iii) for the induction.

Given  $T_m = (A, E, w)$ , we construct the weighted tournament  $T_{m'} = (A', E', w')$  as follows:

- For all  $x_i, x_j \in \{x_1, \dots, x_m\}$ , we have  $(x_i, x_j) \in E'$  if and only if  $(x_i, x_j) \in E$ . Further, for all these edges except for  $(y_1, y_2) = (x_2, x_3)$  if  $m$  is even and  $(y_1, y_2) = (x_3, x_4)$  if  $m$  is odd, we set  $w'(e) = c \cdot w(e)$ , where  $c$  is a large constant that will be specified later. Further, we set  $w'(y_1, y_2) = c \cdot w(y_1, y_2) + 2$ .
- For all  $x_i \in \{x_1, \dots, x_{m-4}, x_{m-2}\}$ , we add the edges  $(x_{m+2}, x_i)$  and  $(x_i, x_{m+1})$  with weight  $w'(x_{m+2}, x_i) = w'(x_i, x_{m+1}) = 2$ .
- We add the edges  $(x_m, x_{m+1})$ ,  $(x_{m+2}, x_m)$ ,  $(x_{m+2}, x_{m-3})$ , and  $(x_{m-3}, x_{m+1})$  with weights  $w'(x_m, x_{m+1}) = w'(x_{m+2}, x_{m-3}) = 4m$  and  $w'(x_{m+2}, x_m) = w'(x_{m-3}, x_{m+1}) = 2$ .
- Lastly, we add the edges  $(x_{m-1}, x_{m+2})$ ,  $(x_{m+1}, x_{m-1})$ , and  $(x_{m+1}, x_{m+2})$  with weights  $w'(x_{m-1}, x_{m+2}) = w'(x_{m+1}, x_{m+2}) = 14m$  and  $w'(x_{m+1}, x_{m-1}) = 2$ .

Now, we will choose the constant  $c$  so large that the ranking chosen for  $T_{m'}$  has to agree either with  $\triangleright_1$  or  $\triangleright_2$  when restricted to  $A = \{x_1, \dots, x_m\}$ . Specifically, we set  $c = 4 + \sum_{(x, y) \in E' \setminus \{x, y \in A\}} w(x, y)$  and we denote by  $\triangleright|_A$  the restriction of a ranking  $\triangleright$  on  $A' = \{x_1, \dots, x_{m+2}\}$  to the alternatives in  $A$ . Now, let  $(y_1, y_2) = (x_2, x_3)$  if  $m$  is even and  $(y_1, y_2) = (x_3, x_4)$  if  $m$  is odd. By construction, it holds that  $s(T_{m'}, \triangleright) = c \cdot s(T_m, \triangleright|_A) + 2 + \sum_{(x, y) \in E' \setminus \{x, y \in A\}} w'(x, y)$  if  $y_1 \triangleright y_2$  and  $s(T_{m'}, \triangleright) = c \cdot s(T_m, \triangleright|_A) + \sum_{(x, y) \in E' \setminus \{x, y \in A\}} w'(x, y)$  otherwise. Furthermore, it holds by the induction hypothesis that  $s(T_m, \triangleright_1) = s(T_m, \triangleright_2) > s(T_m, \triangleright)$  for all other rankings  $\triangleright$  and so  $s(T_m, \triangleright_1) = s(T_m, \triangleright_2) \geq 1 + s(T_m, \triangleright)$  because the Kemeny score is always an integer. Hence, if  $\triangleright'$  and  $\triangleright''$  are two rankings on  $A'$  such that  $\triangleright'|_A \in \{\triangleright_1, \triangleright_2\}$  and  $\triangleright''|_A \notin \{\triangleright_1, \triangleright_2\}$ , it holds that  $s(T_{m'}, \triangleright') > s(T_{m'}, \triangleright'')$  because

$$\begin{aligned} s(T_{m'}, \triangleright') - s(T_{m'}, \triangleright'') &\geq c \cdot (s(T_m, \triangleright'|_A) - s(T_m, \triangleright''|_A)) - 2 - \sum_{(x, y) \in E \setminus \{x, y \in A\}} w'(x, y) \\ &\geq c - 2 - \sum_{(x, y) \in E \setminus \{x, y \in A\}} w'(x, y) \\ &= 2. \end{aligned}$$

Next, we will derive the rankings on  $A'$  that agree with  $\triangleright_1$  when restricted to  $A$  and maximizes the score  $s(T_m, \triangleright)$ . To this end, we first compute the score of  $\triangleright'_1 = x_1 \dots x_{m+2}$  as

$$\begin{aligned} s(T_{m'}, \triangleright'_1) &= c \cdot s(T_m, \triangleright_1) + 2 + w'(x_{m+1}, x_{m+2}) + w'(x_{m-1}, x_{m+2}) + w'(x_m, x_{m+1}) + w'(x_{m-3}, x_{m+1}) + \sum_{x_i \in \{x_1, \dots, x_{m-4}, x_2\}} w'(x_i, x_{m+1}) \\ &= c \cdot s(T_m, \triangleright_1) + 2 + 14m + 14m + 4m + 2 + 2(m - 3) \\ &= c \cdot s(T_m, \triangleright_1) + 34m - 2 \end{aligned}$$

Note here that the first +2 is due to the fact that  $x_2 \triangleright_1 x_3 \triangleright_1 x_4$  and that we increased the weight of  $(x_2, x_3)$  (if  $m$  is even) or  $(x_3, x_4)$  (if  $m$  is odd) by 2. Next, to show that  $\triangleright'_1$  maximizes the Kemeny scores among all rankings  $\triangleright$  with  $\triangleright|_A = \triangleright_1$ , we fix one such ranking  $\triangleright$  and show that  $s(T_{m'}, \triangleright'_1) > s(T_{m'}, \triangleright)$ . To this end, we observe that the total weight of the edges  $(x_i, x_j)$  with  $\{x_i, x_j\} \not\subseteq A$  is  $\sum_{(x_i, x_j) \in E' \setminus \{x, y \in A\}} w'(x, y) = 2 \cdot 14m + 2 \cdot 4m + (2m - 3) \cdot 2 = 40m - 6$ , because there are  $2m + 1$  edges that are not contained in  $A$  and only four of these edges have a value other than 2. As a consequence of this, it holds for  $\triangleright$  that  $s(T_{m'}, \triangleright) = c \cdot s(T_m, \triangleright_1) + 2 + 40m - 6 - \sum_{(x, y) \in E \setminus \{x, y \in A\}} w'(x, y)$ . If  $x_{m+1} \triangleright x_{m+2}$  or  $x_{m+2} \triangleright x_{m-1}$ , our observation implies that  $s(T_{m'}, \triangleright) \leq c \cdot s(T_m, \triangleright_1) + 2 + 40m - 6 - 14m < s(T_m, \triangleright'_1)$ . On the other hand, if  $x_{m-1} \triangleright x_{m+2}$ , we have by transitivity that  $x_{m-3} \triangleright x_{m+2}$  because  $x_{m-3} \triangleright_1 x_{m-1}$ . If additionally  $x_{m+1} \triangleright x_m$ , it holds that  $s(T_{m'}, \triangleright) \leq c \cdot s(T_m, \triangleright_1) + 2 + 40m - 6 - 8m < s(T_m, \triangleright'_1)$ . Hence, we must have that  $x_m \triangleright x_{m+1} \triangleright x_{m+2}$ , which implies that  $\triangleright = \triangleright'_1$ . We hence conclude that  $\triangleright'_1$  indeed uniquely maximizes  $s(T_{m'}, \triangleright)$  among all rankings  $\triangleright$  with  $\triangleright|_A = \triangleright_1$ .

We next repeat the exercise for the rankings  $\triangleright$  on  $A'$  that agree with  $\triangleright_2$  when restricted to  $A$ . To this end, we let  $X = \{x_1, \dots, x_{m-4}, x_{m-2}\}$  and first compute the score of the ranking  $\triangleright'_2$ , which ranks  $x_{m+1}$  first,  $x_{m-1}$  second, and  $x_{m+2}$  third.

$$\begin{aligned} s(T_{m'}, \triangleright'_2) &= c \cdot s(T_m, \triangleright_2) + w'(x_{m+1}, x_{m-1}) + w'(x_{m+1}, x_{m+2}) + w'(x_{m-1}, x_{m+2}) + w'(x_{m+2}, x_{m-3}) + w'(x_{m+2}, x_m) + \sum_{x_i \in X} w'(x_{m+2}, x_i) \\ &= c \cdot s(T_m, \triangleright_2) + 2 + 14m + 14m + 4m + 2 + 2(m-3) \\ &= c \cdot s(T_m, \triangleright_2) + 34m - 2 \end{aligned}$$

Note here that  $x_3 \triangleright_2 x_2$  if  $m$  is even and  $x_4 \triangleright_2 x_3$  if  $m$  is even, so we do not have the "+2" from the edge in  $(x_2, x_3)$  (resp.  $(x_3, x_4)$ ). Just as for  $\triangleright'_1$ , we next fix a ranking  $\triangleright$  on  $A'$  with  $\triangleright|_A = \triangleright_2$  and show that  $s(T_{m'}, \triangleright'_2) > s(T_{m'}, \triangleright)$ . Analogous to the analysis of  $\triangleright'_1$ , we observe that  $s(T_{m'}, \triangleright) = c \cdot s(T_m, \triangleright_2) + 40m - 6 - \sum_{(x, y) \in E: y \triangleright x \wedge \{x, y\} \notin A} w'(x, y)$ . Hence, if  $x_{m+2} \triangleright x_{m+1}$  or  $x_{m+2} \triangleright x_{m-1}$ , we can immediately conclude our inequality. So, we assume that  $x_{m+1} \triangleright x_{m+2}$  and  $x_{m-1} \triangleright x_{m+2}$ . Next, if  $x_{m+2} \triangleright x_{m-3}$ ,  $\triangleright$  must be either  $\triangleright'_2$  or the ranking that places  $x_{m-1}$  first,  $x_{m+1}$  second, and  $x_{m+2}$  third, because  $x_{m-3}$  is the second-best alternative in  $\triangleright_2$ . Further, if  $x_{m-1}$  is first ranked, it can be computed that  $s(T_{m'}, \triangleright) = s(T_{m'}, \triangleright'_2) - w'(x_{m+1}, x_{m-1}) = c \cdot s(T_m, \triangleright_2) + 34m - 4$ , so it holds that  $s(T_{m'}, \triangleright) < s(T_{m'}, \triangleright'_2)$ . Hence, we suppose next that  $x_{m-3} \triangleright x_{m+2}$ . If additionally  $x_{m+1} \triangleright x_m$ , it holds that  $s(T_{m'}, \triangleright) \leq c \cdot s(T_m, \triangleright_2) + 40m - 6 - 8m < s(T_{m'}, \triangleright'_2)$ . Hence, we now assume that  $x_m \triangleright x_{m+1}$ , which implies that  $x_{m-1} \triangleright x_{m+1}$  and  $x_{m-3} \triangleright x_{m+1}$  because  $x_{m-1} \triangleright_2 x_{m-3} \triangleright_2 x_m$ . Lastly, recall that  $X = \{x_1, \dots, x_{m-4}, x_{m-2}\}$ , and define  $\ell_1 = |\{x_i \in X: x_i \triangleright x_{m+1}\}|$  and  $\ell_2 = |\{x_i \in X: x_{m+2} \triangleright x_i\}|$ . Since  $x_{m+1} \triangleright x_{m+2}$ , it holds that  $\ell_1 + \ell_2 \geq m - 3$ . Therefore, we conclude for this case that

$$\begin{aligned} s(T_{m'}, \triangleright) &= c \cdot s(T_m, \triangleright_2) + 40m - 6 - w'(x_m, x_{m+1}) - w'(x_{m+1}, x_{m-1}) - w'(x_{m+1}, x_{m-1}) \\ &\quad - \sum_{x_i \in X: x_i \triangleright x_{m+1}} w'(x_i, x_{m+1}) - \sum_{x_i \in X: x_{m+2} \triangleright x_i} w'(x_{m+2}, x_i) \\ &\leq c \cdot s(T_m, \triangleright_2) + 40m - 6 - 4m - 2 - 2 - 2(m-3) \\ &= c \cdot s(T_m, \triangleright_2) + 32m - 4. \end{aligned}$$

Hence, it holds in every case that  $s(T_{m'}, \triangleright'_2) > s(T_{m'}, \triangleright)$ , thereby proving that  $\triangleright'_2$  uniquely maximizes the Kemeny score among all rankings  $\triangleright$  with  $\triangleright|_A = \triangleright_2$ . Since our computations also show that  $s(T_{m'}, \triangleright'_1) = s(T_{m'}, \triangleright'_2)$  as  $s(T_m, \triangleright_1) = s(T_m, \triangleright_2)$ , the weighted tournament  $T_{m'}$  satisfies all our requirements. This completes the proof of the induction step and thus of this proposition.  $\square$

Next, we turn to our lower bound for distance scoring rules.

**Proposition 4.** *Assume  $m \geq 3$ . The incentive ratio of every distance scoring rule  $f_{\text{dist}}$  other than  $f_{\text{Kemeny}}$  satisfies  $\binom{m}{2} - 1 \leq \gamma_m(f_{\text{dist}})$ .*

**PROOF.** Fix a distance scoring rule  $f$  for  $m$  alternatives other than the Kemeny rule and let  $s$  denote its distance scoring rule. To show this proposition, we will follow the same approach as for the Kemeny rule and construct a profile  $R^*$  such that that two rankings  $\triangleright_1$  and  $\triangleright_2$  with  $\Delta(\triangleright_1, \triangleright_2) = \binom{m}{2}$  minimize the total score, i.e.,  $\sum_{i \in N} s(\Delta(\triangleright_i, \triangleright_1)) = \sum_{i \in N} s(\Delta(\triangleright_i, \triangleright_2)) < \sum_{i \in N} s(\Delta(\triangleright_i, \triangleright))$  for all  $\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}$ . For simplicity, we subsequently define the score of a ranking  $\triangleright$  in a profile  $R$  by  $s(R, \triangleright) = \sum_{i \in N} s(\Delta(\triangleright_i, \triangleright))$ . Based on the profile  $R^*$ , we construct another profile  $\hat{R}$  as follows. First, we let  $\lambda$  denote an integer such that  $\lambda \cdot \min_{\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}} s(R^*, \triangleright) - s(R^*, \triangleright_1) > 2(s(\binom{m}{2}) - s(0))$ . Further, we let  $\triangleright_1$  and  $\triangleright_2$  denote two rankings with  $\Delta(\triangleright_1, \triangleright_1) = 1$  and  $\Delta(\triangleright_2, \triangleright_2) = 1$ , respectively. Since  $\Delta(\triangleright_1, \triangleright_2) = \binom{m}{2}$ , this also means  $s(\triangleright_1, \triangleright_2) = s(\triangleright_2, \triangleright_1) = \binom{m}{2} - 1$ .

Now, let  $R$  denote the profile that consists of  $\lambda$  copies of  $R^*$ , one voter reporting  $\triangleright_1$ , and another voter reporting  $\triangleright_2$ . First, we observe that

$$\begin{aligned} s(R, \triangleright_1) &= \lambda \cdot s(R^*, \triangleright_1) + s(\Delta(\triangleright_1, \triangleright_1)) + s(\Delta(\triangleright_2, \triangleright_1)) \\ &= \lambda \cdot s(R^*, \triangleright_2) + s(1) + s\left(\binom{m}{2} - 1\right) \\ &= \lambda \cdot s(R^*, \triangleright_2) + s(\Delta(\triangleright_2, \triangleright_2)) + s(\Delta(\triangleright_1, \triangleright_2)) \\ &= s(R, \triangleright_2). \end{aligned}$$

Further, it holds for all  $\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}$  that

$$\begin{aligned} s(R, \triangleright) - s(R, \triangleright_1) &= \lambda \cdot (s(R^*, \triangleright) - s(R^*, \triangleright_1)) + s(\Delta(\triangleright_1, \triangleright)) - s(\triangleright_1, \triangleright_1) + s(\Delta(\triangleright_2, \triangleright)) - s(\triangleright_2, \triangleright_1) \\ &> 2(s\left(\binom{m}{2}\right) - s(0)) + s(\Delta(\triangleright_1, \triangleright)) - s(\triangleright_1, \triangleright_1) + s(\Delta(\triangleright_2, \triangleright)) - s(\triangleright_2, \triangleright_1) \\ &\geq 0. \end{aligned}$$

Here, the first inequality follows from the choice of  $\lambda$  and the second one by the fact that  $s(x) < s(x+1)$  for all  $x \in \{0, \dots, \binom{m}{2} - 1\}$ , which means that  $s(\binom{m}{2}) - s(0) \geq s(y) - s(z)$  for all  $y, z \in \{0, \dots, \binom{m}{2}\}$ .

By this analysis, it holds that precisely  $\triangleright_1$  and  $\triangleright_2$  minimize the total score in  $R$ . Without loss of generality, we may thus assume that  $\triangleright_1$  is chosen, which leaves the voter who reports  $\triangleright_2$  with a utility of  $u(\triangleright_2, \triangleright_1) = \binom{m}{2} - \Delta(\triangleright_2, \triangleright_1) = 1$ . Now, assume that this voter deviates to report  $\triangleright_2$  instead. For the resulting profile  $R'$ , it holds that  $s(R', \triangleright_2) - s(R, \triangleright_2) = s(0) - s(1) < 0 < s(\binom{m}{2}) - s(\binom{m}{2} - 1) = s(R', \triangleright_1) - s(R, \triangleright_1)$ .

Since  $s(R, \triangleright_1) = s(R, \triangleright_2)$ , this means that  $s(R', \triangleright_2) < s(R', \triangleright_1)$ . Further, an analogous argument as for  $R$  shows that  $s(R', \triangleright_2) < s(R, \triangleright)$  for all  $\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}$ . Hence,  $f$  needs to choose the ranking  $\triangleright_2$  for  $R$ . This means that the utility of our voter after the manipulation is  $u(\succ_2, \triangleright_2) = \binom{m}{2} - 1$ , which implies that  $\gamma_m(f) \geq \binom{m}{2} - 1$ .

It remains to construct the profile  $R^*$ . To this end, we first consider the profile  $R^\succ$ , where one voter reports  $\succ$  and another voter reports the inverse ranking  $\succ$ . It holds for every ranking  $\triangleright$  that  $s(R^\succ, \triangleright) = s(\Delta(\succ, \triangleright)) + s(\Delta(\succ, \triangleright)) = s(\Delta(\succ, \triangleright)) + s(\binom{m}{2} - \Delta(\succ, \triangleright))$ . Furthermore, we recall that distance scoring functions satisfy that  $s(x+2) - s(x+1) \geq s(x+1) - s(x)$  for all  $x \in \{0, \dots, \binom{m}{2}\}$ . By chaining these inequalities, it holds that  $s(y+1) - s(y) \geq s(x+1) - s(x)$  for all  $x, y \in \{0, \dots, \binom{m}{2}\} - 1$  with  $x < y$ . When letting  $x \in \{1, \dots, \lfloor \binom{m}{2}/2 \rfloor\}$ , this implies that  $s(\binom{m}{2} - x+1) - s(\binom{m}{2} - x) \geq s(x) - s(x-1)$  and thus  $s(\binom{m}{2} - x+1) + s(x-1) \geq s(x) + s(\binom{m}{2} - x)$ . For our profile  $R^\succ$ , this means for all  $\triangleright, \triangleright' \in \mathcal{R}$  that  $s(R^\succ, \triangleright) \leq s(R^\succ, \triangleright')$  when  $|\Delta(\succ, \triangleright) - \binom{m}{2}/2| \leq |\Delta(\succ, \triangleright') - \binom{m}{2}/2|$ . Further, since  $f$  is not the Kemeny rule, there must be an index  $i \in \{1, \dots, \binom{m}{2} - 1\}$  such that  $s(x+1) - s(x) > s(x) - s(x-1)$ , which implies that  $s(\binom{m}{2}) + s(0) > s(\lceil \binom{m}{2}/2 \rceil) + s(\lfloor \binom{m}{2}/2 \rfloor)$ . Since  $s(x) + s(\binom{m}{2} - x) \leq s(y) - s(\binom{m}{2} - y)$  for all  $x, y \in \{0, \dots, \lfloor \binom{m}{2} \rfloor\}$  with  $x \geq y$ , we thus conclude that there is an index  $\ell \in \{0, \dots, \lfloor \binom{m}{2}/2 \rfloor - 1\}$  such that

- (1)  $s(\binom{m}{2} - x) + s(x) = s(\binom{m}{2} - y) + s(y)$  for all  $x, y \in \{\lfloor \binom{m}{2}/2 \rfloor - \ell, \dots, \lfloor \binom{m}{2}/2 \rfloor\}$  and
- (2)  $s(\binom{m}{2} - x) + s(x) < s(\binom{m}{2} - y) + s(y)$  for all  $x \in \{\lfloor \binom{m}{2}/2 \rfloor - \ell, \dots, \lfloor \binom{m}{2}/2 \rfloor\}$  and  $y \in \{0, \dots, \lfloor \binom{m}{2}/2 \rfloor - \ell - 1\}$ .

Now, let  $X = \{\lfloor \binom{m}{2}/2 \rfloor - \ell, \dots, \lceil \binom{m}{2}/2 \rceil + \ell\}$ . From our insights on  $s$ , it follows that  $s(R^\succ, \triangleright) = s(R^\succ, \triangleright') < s(R^\succ, \triangleright'')$  for all  $\triangleright, \triangleright', \triangleright'' \in \mathcal{R}$  such that  $\Delta(\succ, \triangleright) \in X$ ,  $\Delta(\succ, \triangleright') \in X$ , and  $\Delta(\succ, \triangleright'') \notin X$ .

Next, we fix our desired output ranking  $\triangleright_1$  and we define  $D(\triangleright_1) = \{\succ \in \mathcal{R} : \Delta(\succ, \triangleright_1) \in X\}$ . Then, we let  $R^*$  denote the profile that concatenates the profiles  $R^\succ$  for all  $\succ \in D(\triangleright_1)$ . Now, we first note that  $s(R^*, \triangleright_1) = \sum_{\succ \in D(\triangleright_1)} s(R^\succ, \triangleright_1) \leq \sum_{\succ \in D(\triangleright_1)} s(R^\succ, \triangleright) = s(R^*, \triangleright)$  for all rankings  $\triangleright \in \mathcal{R}$ , because  $\Delta(\succ, \triangleright_1) \in X$  for all  $\succ \in D(\triangleright_1)$ . In particular, by the insights of the previous paragraph, this means that  $\triangleright_1$  is a minimizer of  $s(R^*, \triangleright)$ . Secondly, we note for the inverse ranking of  $\triangleright_1$ , denoted by  $\triangleright_2$  that  $s(R^*, \triangleright_2) = s(R^*, \triangleright_1)$  because  $s(R^*, \triangleright_1) = s(R^*, \triangleright_2)$  for all  $\succ \in \mathcal{R}$ . The latter observation is true as  $\Delta(\succ, \triangleright_1) = \Delta(\succ, \triangleright_2)$  and  $\Delta(\succ, \triangleright_1) = \Delta(\succ, \triangleright_2)$  (where  $\succ$  is the inverse ranking of  $\succ$ ), where we use that, if two rankings disagree on a pair of alternatives, the inverse rankings will also disagree on this pair.

Lastly, it remains to show that  $s(R^*, \triangleright_1) < s(R^*, \triangleright)$  for all rankings  $\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}$ . To this end, we first recall that  $s(R^\succ, \triangleright_1) \leq s(R^\succ, \triangleright)$  for all  $\succ \in D(\triangleright_1)$  and  $\triangleright \in \mathcal{R}$ . Hence, it suffices to identify a single ranking  $\succ \in D(\triangleright_1)$  to prove our claim. If  $\triangleright \in D(\triangleright_1)$ , we can simply pick  $\triangleright$  for this. Indeed, since  $\Delta(\triangleright, \triangleright) = 0 \notin X$ , it holds that  $s(R^\triangleright, \triangleright_1) < s(R^\triangleright, \triangleright)$ . Hence, assume that  $\triangleright \notin D(\triangleright_1)$ , which means that  $\Delta(\triangleright, \triangleright_1) < \lfloor \binom{m}{2}/2 \rfloor - \ell$  or  $\Delta(\triangleright, \triangleright_1) > \lceil \binom{m}{2}/2 \rceil + \ell$ . We focus on the first case, i.e.,  $\Delta(\triangleright, \triangleright_1) < \lfloor \binom{m}{2}/2 \rfloor - \ell$  because the other case is symmetric when exchanging the role of  $\triangleright_1$  and  $\triangleright_2$ . Now, let  $\succ_0, \dots, \succ_{\binom{m}{2}}$  denote a sequence of rankings from  $\triangleright_1$  to  $\triangleright_2$  through  $\triangleright$ . More formally, these rankings satisfy that  $\succ_0 = \triangleright_1$ ,  $\succ_{\binom{m}{2}} = \triangleright_2$ , there is  $k \in \{1, \dots, \binom{m}{2} - 1\}$  such that  $\succ_k = \triangleright$ , and for all  $i \in \{0, \dots, \binom{m}{2} - 1\}$ ,  $\succ_{i+1}$  emerges from  $\succ_i$  by swapping one pair of alternatives. By the last condition, we know that  $k = \Delta(\triangleright, \triangleright_1)$  because each swap must move the ranking further away from  $\triangleright_1$  and towards  $\triangleright_2$ . Now, let  $\succ = \succ_j$  denote the ranking for the index  $j = \lfloor \binom{m}{2}/2 \rfloor - \ell$ . Since our sequence transforms  $\triangleright_1$  one after another to  $\triangleright_2$  and  $0 < k < j$ , it holds that  $\Delta(\triangleright_1, \succ) = \lfloor \binom{m}{2}/2 \rfloor - \ell > \Delta(\triangleright, \succ)$ . Hence, we have that  $s(R^\succ, \triangleright_1) < s(R^\succ, \triangleright)$ . Moreover, it holds by definition that  $\succ \in D(\triangleright_1)$  because  $\Delta(\triangleright_1, \succ) \in X$ . Since our two cases are exhaustive, we conclude that  $s(R^*, \triangleright_1) < s(R^*, \triangleright)$  for all rankings  $\triangleright \in \mathcal{R} \setminus \{\triangleright_1, \triangleright_2\}$ , which completes the proof.  $\square$

As the third point of this section, we turn to positional scoring rules.

**Proposition 5.** *For all  $m \geq 3$ , the incentive ratio of every positional scoring rule  $f_{\text{pos}}$  is  $\gamma_m(f_{\text{pos}}) = \infty$ .*

**PROOF.** For proving this claim, we fix a positional scoring rule  $f$  and let  $p$  denote its positional scoring function. To show that  $\gamma_m(f_{\text{pos}}) = \infty$ , it suffices to give two profiles  $R$  and  $R'$  on  $m$  alternatives and a voter  $i$  such that  $R$  and  $R'$  only differ in the ranking of voter  $i$ , this voter obtains a utility of 0 from the ranking  $\triangleright = f(R)$ , and a non-zero utility from the ranking  $\triangleright' = f(R')$ . To construct such profiles, we let  $R^x$  denote a profile on  $m - 1$  voters such that (i) alternative  $x$  is top-ranked by all voters and (ii) for every alternative  $y \in A \setminus \{x\}$  and each rank  $k \in \{2, \dots, m\}$ , there is one voter such that  $r(\succ_j, y) = k$ . For these profiles, the total score of  $x$ , denoted by  $p(R^x, x)$ , is  $p(R^x, x) = (m-1)p(1)$  and the total score of all other alternatives  $y \in A \setminus \{x\}$  is  $p(R^x, y) = \sum_{k=2}^m p(k)$ . Since  $p(1) \geq p(2) \geq \dots \geq p(m)$  and  $p(1) > p(m)$ , this means that  $p(R^x, x) > p(R^x, y)$  for all  $y \in A \setminus \{x\}$ . Next, we define  $c = p(R^x, y)$  and  $\delta = p(R^x, x) - p(R^x, y)$  for some pair of alternatives  $x, y \in A$  with  $x \neq y$  and note that  $c$  and  $\delta$  are independent of the choice of  $x$  and  $y$ .

We proceed with a case distinction and first suppose that  $p(1) > p(m-1)$ . In this case, we enumerate the alternatives by  $x_1, \dots, x_m$  and consider the profile  $\hat{R}$  that consists of  $i$  copies of  $R^{x_i}$  for all  $i \in \{1, \dots, m-1\}$  and  $m-1$  copies of  $R^{x_m}$ . For instance, if  $m = 3$ ,  $\hat{R}$  consists one copy of  $R^{x_1}$ , two copies of  $R^{x_2}$  and two copies of  $R^{x_3}$ . In this profile, every alternative  $x_i \in A \setminus \{x_m\}$  has a score of  $p(\hat{R}, x_i) = (\frac{m(m+1)}{2} - 1) \cdot c + i \cdot \delta$  and alternative  $x_m$  has a score of  $p(\hat{R}, x_m) = (\frac{m(m+1)}{2} - 1) \cdot c + (m-1) \cdot \delta$ . Hence, it holds that  $p(\hat{R}, x_m) = p(\hat{R}, x_{m-1}) > p(\hat{R}, x_{m-2}) > \dots > p(\hat{R}, x_1)$ . Next, let  $\lambda \in \mathbb{N}$  denote an integer such that  $\delta \cdot \lambda > 2(p(1) - p(m))$  and let  $R$  denote the profile that consists of  $\lambda$  copies of  $\hat{R}$ , one voter reporting  $\succ_1 = x_1 \dots x_m$ , and one voter reporting  $\succ_2 = x_1 \dots x_{m-2} x_m x_{m-1}$ . We first note that  $p(R, x_m) = \lambda p(\hat{R}, x_m) + p(m-1) + p(m) = \lambda p(\hat{R}, x_{m-1}) + p(m-1) + p(m) = p(R, x_{m-1})$ . Further, by the choice of  $\lambda$ , it holds that  $p(\hat{R}, x_{m-1}) > p(\hat{R}, x_{m-2}) > \dots > p(\hat{R}, x_1)$  because  $p(\hat{R}, x_i) - p(\hat{R}, x_{i-1}) = \lambda \cdot \delta + p(r(\succ_1, x_i)) - p(r(\succ_1, x_{i-1})) + p(r(\succ_2, x_i)) - p(r(\succ_2, x_{i-1})) \geq \lambda \cdot \delta - 2(p(1) - p(m)) > 0$  for all  $i \in \{2, \dots, m-1\}$ . Lastly, we suppose without loss of generality that  $f$  chooses the ranking  $\triangleright = x_m \dots x_1$ , which means that the voter reporting  $\succ_1$  obtains a utility of 0. On the other hand, if this voter deviates to report

$\succ'_1 = x_{m-1}x_1 \dots x_{m-2}x_m$ , it holds for the resulting profile  $R'$  that  $p(R', x_{m-1}) > p(R, x_{m-1}) = p(R, x_m) = p(R', x_m)$  because  $p(1) > p(m-1)$ . This means that  $x_{m-1} \succ' x_m$  for the ranking chosen for  $R'$  and therefore  $u(\succ_1, f(R')) > 0$ . This completes our proof in this case.

Secondly, we suppose that  $p(1) = p(m-1)$ . Since  $p(1) \geq p(2) \geq \dots \geq p(m-1)$ , this means that  $p(1) = p(2) = \dots = p(m-1)$ . Moreover, because  $p(1) > p(m)$ , we conclude that  $p(m-1) > p(m)$ . In this case, we consider the profile  $\hat{R}$  that consists of one copy of  $R^{x_1}$  and  $i-1$  copies of  $R^{x_i}$  for all  $i \in \{2, \dots, m\}$ . Hence, it holds in this profile that  $p(\hat{R}, x_1) = (\frac{m(m-1)}{2} + 1)c + \delta$  and  $p(\hat{R}, x_i) = (\frac{m(m-1)}{2} + 1)c + (i-1)\delta$  for all  $i \in \{2, \dots, m\}$ . This shows that  $p(\hat{R}, x_m) > \dots > p(\hat{R}, x_2) = p(\hat{R}, x_1)$ . Further, we let  $\lambda$  again denote an integer such that  $\delta \cdot \lambda > 2(p(1) - p(m))$  and define  $R$  as the profile that consists of  $\lambda$  copies of  $\hat{R}$ , one voter reporting  $\succ_1 = x_1 \dots x_m$ , and another voter reporting  $\succ_2 = x_2x_1x_3 \dots x_m$ . Just as in the last case, it can still be shown that  $p(R, x_m) > \dots > p(R, x_2) = p(R, x_1)$ . Without loss of generality, we suppose that  $f(R) = \succ = x_m \dots x_1$ , which means that the voter reporting  $\succ_1$  obtains a utility of 0. Lastly, let  $R'$  denote the profile where this voter deviates to the ranking  $\succ'_1 = x_1x_3 \dots x_mx_2$ . It holds for this profile that  $p(R', x_1) = p(R, x_1)$  as the manipulator does not change the position of this alternative and  $p(R', x_2) < p(R, x_2)$  because  $p(m) < p(2)$ . Hence, we have now that  $p(R', x_1) > p(R', x_2)$  which implies that  $x_1 \succ' x_2$  for the ranking  $\succ' = f(R')$ . This means that  $u(\succ_1, \succ') > 0$  and thus  $u(\succ_1, \succ')/u(\succ_1, \succ) = \infty$ .  $\square$

As the last point of this appendix, we turn to the minimal compromise rule defined in Remark 5. To this end, we recall that the min score of an alternative  $x$  in a profile  $R$  is  $s_{\min}(R, x) = \min_{i \in N} m - r(\succ_i, x)$ . Then, the minimal compromise rule sorts the alternatives in decreasing order of their min scores, with ties broken lexicographically.

**Proposition 6.** *For all  $m \geq 4$ , the incentive ratio of the minimal compromise rule  $f_{\min}$  is  $\gamma_m(f_{\min}) = m - 2$ .*

**PROOF.** To prove this proposition, we will show that  $\gamma_m(f_{\min}) \geq m - 2$  and  $\gamma_m(f_{\min}) \leq m - 2$  when  $m \geq 4$ .

**Claim 1:**  $\gamma_m(f_{\min}) \geq m - 2$ . First, to show our lower bound, we construct a profile  $R$  such that a voter  $i$  obtains a utility of 1 when voting truthfully and of  $m - 2$  when voting dishonestly. To this end, we suppose that the tie-breaking order  $>$  is given by  $x_1 > x_2 > \dots > x_m$ . Now, the ranking of our manipulator will be  $\succ_i = x_{m-1} \dots x_2x_mx_1$ . Further, for every alternative  $x_i \in A \setminus \{x_m\}$ , there is one voter in  $R$  who reports a ranking where  $x_i$  is bottom-ranked. By this definition, it follows that  $s_{\min}(R, x_i) = 0$  for all  $x_i \in A \setminus \{x_m\}$  because each such alternative is bottom-ranked by one voter. On the other hand,  $s_{\min}(R, x_m) = 1$  because  $r(\succ_j, x_m) \leq m - 1$  for all voters  $j \in N$  and  $r(\succ_i, x_m) = m - 1$ . Using our tie-breaking, it thus follows that the minimal compromise rule returns the ranking  $\succ = x_mx_1 \dots x_{m-1}$ . On the other hand, if voter  $i$  bottom-ranks  $x_m$ , every alternative has a min score of 0, because every alternative is bottom-ranked by a voter. Hence, in the corresponding profile  $R'$ , the minimal compromise rule picks the ranking  $\succ' = x_1 \dots x_m$  by our tie-breaking assumption. Finally, we note that  $\frac{u(\succ_i, \succ')}{u(\succ_i, \succ)} = \frac{m-2}{1} = m - 2$ , thus showing our lower bound.

**Claim 2:**  $\gamma_m(f_{\min}) \leq m - 2$ . To prove our upper bound, we will proceed in multiple steps. Firstly, we will show that we can restrict our analysis to deviations such that the min scores of all alternatives weakly decrease, because the maximal utility gain is attained with such a deviation. Secondly, we prove that the sum of the min scores is a lower bound of the utility of every voter. Based on these insights, we will prove this claim in the last step.

**Step 1:** First, we will show that it suffices to focus on deviations that weakly reduce the min scores of all alternatives. To prove this claim, let  $R$  and  $R'$  denote two profiles and  $i$  a voter such that  $R$  and  $R'$  only differ in the ranking of voter  $i$ . We will next show that there are two other profiles  $\hat{R}$  and  $\hat{R}'$  such that (i)  $\hat{\succ}_i = \succ_i$  and  $\hat{\succ}'_i = \succ'_i$  (i.e., voter  $i$  reports the same ranking in  $\hat{R}$  and  $\hat{R}'$  as in  $R$  and  $R'$ , respectively), (ii)  $\hat{R}$  differs from  $\hat{R}'$  only in the ranking of voter  $i$ , (iii)  $s_{\min}(\hat{R}', x) \leq s_{\min}(\hat{R}, x)$  for all  $x \in A$ , and (iv)  $f_{\min}(R) = f_{\min}(\hat{R})$  and  $u(\succ_i, f_{\min}(\hat{R}')) \geq u(\succ_i, f_{\min}(R'))$ . This means that the incentive ratio of  $f_{\min}$  is maximized when voters only decrease the min scores. Now, to prove this claim, we define  $\hat{R}$  and  $\hat{R}'$  as the profiles derived from  $R$  and  $R'$  by adding a new voter who reports  $\succ_i$ . Clearly, the resulting profiles  $\hat{R}$  and  $\hat{R}'$  satisfy our conditions (i) and (ii) by construction. Further, it holds for all  $x \in A$  that  $s_{\min}(\hat{R}', x) = \min(m - r(\succ_i, x), m - r(\succ'_i, x), \min_{j \in N \setminus \{i\}} m - r(\succ_j, x)) \leq \min(m - r(\succ_i, x), \min_{j \in N \setminus \{i\}} m - r(\succ_j, x)) = s_{\min}(\hat{R}, x)$ , which proves condition (iii). Moreover, since cloning rankings does not affect min scores, we have that  $s_{\min}(R, x) = s_{\min}(\hat{R}, x)$  for all  $x \in A$  and thus  $f_{\min}(R) = f_{\min}(\hat{R})$ .

As the last point, we need to show that  $u(\succ_i, f_{\min}(\hat{R}')) \geq u(\succ_i, f_{\min}(R'))$ . To ease notation, we set  $\hat{\succ} = f_{\min}(\hat{R}')$  and  $\succ = f_{\min}(R')$  for the rest of this step. Furthermore, we define for every alternative  $x \in A$  the set  $S(x) = \{y \in A: x \succ y \wedge y \hat{\succ} x\}$  and note that  $\Delta(\succ, \hat{\succ}) = \sum_{x \in A} |S(x)|$ . Now, we fix an alternative  $x \in A$  and analyze the set  $S(x)$ . To this end, we first observe that  $s_{\min}(\hat{R}', y) \leq s_{\min}(R', y)$  for all  $y \in A$  because  $\hat{R}'$  arises from  $R'$  by adding a new voter. Hence, if  $s_{\min}(\hat{R}', x) = s_{\min}(R', x)$ , it holds that  $S(x) = \emptyset$  because  $s_{\min}(\hat{R}', x) = s_{\min}(R', x) \geq s_{\min}(R', y) \geq s(\hat{R}', y)$  for all  $y$  with  $x \succ y$ . Moreover, if this inequality is tight, then  $s_{\min}(R', x) = s_{\min}(R', y)$  and  $x \succ y$  implies that  $x$  is favored lexicographically to  $y$ . Next, we suppose that  $s_{\min}(\hat{R}', x) < s_{\min}(R', x)$ , which means that  $s_{\min}(\hat{R}', x) = m - r(\succ_i, x)$ . By definition of  $S(x)$ , it holds that  $y \hat{\succ} x$  and thus  $s_{\min}(\hat{R}', y) \geq s_{\min}(\hat{R}', x)$  for all  $y \in S(x)$ . This inequality implies that  $y \succ_i x$  for all  $y \in S(x)$  because  $s_{\min}(\hat{R}', x) = m - r(\succ_i, x)$ . In particular, if  $x \succ_i y$ , it would hold that  $s_{\min}(\hat{R}', y) \leq m - r(\succ_i, y) < m - r(\succ_i, x) = s_{\min}(\hat{R}', x)$ , which contradicts that  $y \hat{\succ} x$ . By combining our two cases, we infer for all alternatives  $x, y \in A$  with  $x \succ y$  and  $y \hat{\succ} x$  that  $y \succ_i x$ . This means that  $\Delta(\succ_i, \hat{\succ}) = \Delta(\succ_i, \succ) - \Delta(\succ, \hat{\succ}) \leq \Delta(\succ_i, \succ)$  and thus  $u(\succ_i, \hat{\succ}) \geq u(\succ_i, \succ)$ .

**Step 2:** Next, we will show that the sum of all min scores is a lower bound for the utility of every voter in the considered profile, i.e.,  $u(\succ_i, f_{\min}(R)) \geq \sum_{x \in A} s_{\min}(R, x)$ . To see this, fix a profile  $R$  and a voter  $i$ , and let  $\succ = f_{\min}(R)$  denote the ranking chosen by the minimal compromise rule for  $R$ . We further define the utility of an alternative  $x \in A$  by  $u(x, \succ_i, \succ) = |\{y \in A: x \succ_i y \wedge x \succ y\}|$  and note that

$u(\succ_i, \triangleright) = \binom{m}{2} - \Delta(\succ_i, \triangleright) = |\{(x, y) \in A^2 : x \succ_i y \wedge x \triangleright y\}| = \sum_{x \in A} u(x, \succ_i, \triangleright)$ . Finally, fix an alternative  $x$ ; we will show that  $u(x, \succ_i, \triangleright) \geq s_{\min}(R, x)$ . If  $s_{\min}(R, x) = 0$ , this holds trivially, so we suppose that  $s_{\min}(R, x) > 0$ . In this case, let  $Y = \{y \in A : m - r(\succ_i, y) < s_{\min}(R, y)\}$  and note that  $|Y| = s_{\min}(R, x)$ . Further, it holds for all  $y \in Y$  that  $x \succ_i y$  and  $x \triangleright y$  because  $s_{\min}(R, y) \leq m - r(\succ_i, y) < s_{\min}(R, x) \leq m - r(\succ_i, x)$ . This entails that  $u_i(x, \succ_i, \triangleright) \geq |Y| = s_{\min}(R, x)$ , thus proving this step.

*Step 3:* Lastly, we will prove our upper bound. To this end, let  $R$  and  $R'$  denote two profiles on  $m$  alternatives that differ only in the ranking of a single voter  $i$  and suppose that  $R$  and  $R'$  maximize the incentive ratio of the minimal compromise rule. Further, we let  $\triangleright = f_{\min}(R)$  and  $\triangleright' = f_{\min}(R')$ . By Step 1, we may assume that  $s_{\min}(R', x) \leq s_{\min}(R, x)$  for all  $x \in A$ . In turn, this implies that  $\frac{u(\succ_i, \triangleright')}{u(\succ_i, \triangleright)} \neq \infty$ . In particular, if  $u(\succ_i, \triangleright) = 0$ , then  $\triangleright$  has to top-rank the bottom-ranked alternative  $x^*$  of  $\succ_i$ . However, it holds that  $s_{\min}(R, x^*) = m - r(\succ_i, x^*) = 0$  and thus also that  $s_{\min}(R, x) = 0$  for all  $x \in A$  because  $f_{\min}$  sorts the alternatives in decreasing order of their min scores. Finally, because  $s_{\min}(R', x) \leq s_{\min}(R, x)$  for all  $x \in A$ , we conclude that  $s_{\min}(R', x) = 0 = s_{\min}(R, x)$  for all  $x \in A$ . This implies that  $\triangleright = \triangleright'$ , so a voter with utility 0 cannot manipulate.

By Step 1, we can also conclude that no voter can manipulate if all min scores are 0. Hence, suppose that  $s_{\min}(R, x) > 0$  for at least one alternative  $x$ . Similar to Step 1, we define the sets  $S^+(x) = \{y \in A : x \triangleright y \wedge y \triangleright' x \wedge y \succ_i x\}$  and  $S^-(x) = \{y \in A : x \triangleright y \wedge y \triangleright' x \wedge x \succ_i y\}$ . We note for these sets that  $\{(x, y) \in A^2 : x \triangleright y \wedge y \triangleright' x\} = \{(x, y) \in A^2 : y \in S^+(x)\} \cup \{(x, y) \in A^2 : y \in S^-(x)\}$ , so  $\Delta(\triangleright, \triangleright') = \sum_{x \in A} |S^+(x)| + |S^-(x)|$ . Further, the set  $\{(x, y) \in A^2 : y \in S^+(x)\}$  contains all pairs of alternatives on which  $\triangleright$  and  $\triangleright'$  disagree and that move  $\triangleright$  closer to  $\succ_i$ , whereas  $\{(x, y) \in A^2 : y \in S^-(x)\}$  contains the pairs of alternatives on which  $\triangleright$  and  $\triangleright'$  disagree and that move  $\triangleright$  further away from  $\succ_i$ . In particular, this means that  $\Delta(\succ_i, \triangleright') = \Delta(\succ_i, \triangleright) + \sum_{x \in A} |S^-(x)| - |S^+(x)|$  and thus  $u(\succ_i, \triangleright') = u(\succ_i, \triangleright) + \sum_{x \in A} |S^+(x)| - |S^-(x)|$ .

We will next aim to bound the value  $|S^+(x)| - |S^-(x)|$  for every alternative  $x$ . To this end, we first note that if  $s_{\min}(R, x) = s_{\min}(R', x)$  for some alternative  $x$ , then  $S^+(x) = S^-(x) = \emptyset$ . The reason for this is that if  $s_{\min}(R, x) = s_{\min}(R', x)$ , then  $s_{\min}(R', x) = s_{\min}(R, x) \geq s_{\min}(R, y) \geq s_{\min}(R', y)$  for all  $y$  with  $x \triangleright y$ . Hence, we can infer that  $x \triangleright y$  for all such  $y$ , which means that there are no alternatives  $x, y$  such that  $x \triangleright y$  and  $y \triangleright' x$ . This implies that  $|S^+(x)| - |S^-(x)| = 0$  for all  $x \in A$  with  $s_{\min}(R, x) = 0$ .

Next, we turn to alternatives  $x$  with  $s_{\min}(R, x) > 0$ . In this case, let  $x^*$  again denote the bottom-ranked alternative in  $\succ_i$  and note that  $x \triangleright x^*$  as  $s_{\min}(R, x) > 0$ . Now, if  $x^* \triangleright' x$ , it holds that  $|S^+(x)| \leq m - 2$  (as  $x^* \notin S^+(x)$  and  $x \notin S^+(x)$ ) and  $|S^-(x)| \geq 1$  (as  $x^* \in S^-(x)$ ) and thus  $|S^+(x)| - |S^-(x)| \leq m - 3$ . Similarly, if  $x \triangleright' x^*$  and there is another alternative  $y$  with  $x^* \triangleright y$ , it holds that  $|S^+(x)| - |S^-(x)| \leq m - 3$  since  $x \notin S^+(x)$ ,  $x^* \notin S^+(x)$ , and  $y \notin S^+(x)$ . To conclude, if  $x^*$  is not bottom-ranked in  $\triangleright'$ , then  $|S^+(x)| - |S^-(x)| \leq m - 3$  for all  $x \in A$  with  $s_{\min}(R, x) > 0$ . Hence, when letting  $\ell$  denote the number of alternatives with  $s_{\min}(R, x) > 0$ , we derive that

$$\frac{u(\succ_i, \triangleright')}{u(\succ_i, \triangleright)} = \frac{u(\succ_i, \triangleright) + \sum_{x \in A} |S^+(x)| - |S^-(x)|}{u(\succ_i, \triangleright)} \leq \frac{u(\succ_i, \triangleright) + \ell(m - 3)}{u(\succ_i, \triangleright)} \leq \frac{\ell + \ell(m - 3)}{\ell} = m - 2.$$

Here, the first equality uses the definition of  $S^+(x)$  and  $S^-(x)$  and the second uses our insights regarding  $|S^+(x)| - |S^-(x)|$ . The third step holds because  $\ell \leq \sum_{x \in A} s_{\min}(R, x) \leq u(\succ_i, \triangleright)$ , where the last inequality follows by Step 2. Hence, if  $\triangleright'$  does not bottom-rank  $x^*$ , our upper bound on the incentive ratio holds.

As the second case, assume that  $x^*$  is bottom-ranked in  $\triangleright'$ . Since  $s_{\min}(R', x) \leq s_{\min}(R, x)$  for all  $x \in A$ , it holds that  $\{x \in A : s_{\min}(R, x) = 0\} \subseteq \{x \in A : s_{\min}(R', x) = 0\}$ . By our lexicographic tie-breaking, this means that  $x^*$  is also bottom-ranked in  $\triangleright$  and so  $S^+(x^*) = S^-(x^*) = \emptyset$ . Further, let  $y_i$  denote the  $i$ -th best alternative in  $\triangleright'$ , i.e.,  $r(y_i, \triangleright') = i$ . By definition of the sets  $S^+(x)$  and  $S^-(x)$ , it holds for alternatives  $y_i$  with  $i \in \{1, \dots, m - 1\}$  that  $|S^+(y_i)| - |S^-(y_i)| \leq |S^+(y_i)| \leq (i - 1)$  because there are only  $i - 1$  alternatives  $x$  with  $x \triangleright' y_i$ . Lastly, since  $x^*$  is bottom-ranked in  $\succ_i$  and  $\triangleright$ , we conclude that  $u(\succ_i, \triangleright) \geq m - 1$  and thus

$$\frac{u(\succ_i, \triangleright')}{u(\succ_i, \triangleright)} = \frac{u(\succ_i, \triangleright) + \sum_{x \in A} |S^+(x)| - |S^-(x)|}{u(\succ_i, \triangleright)} \leq \frac{u(\succ_i, \triangleright) + \sum_{i=1}^{m-1} (i - 1)}{u(\succ_i, \triangleright)} \leq \frac{(m - 1) + (m - 2)(m - 1)/2}{m - 1} = \frac{m}{2} \leq m - 2.$$

This chain of (in)equalities follows analogous to the last case, except for the last step where we use that  $m \geq 4$  implies  $\frac{m}{2} \leq m - 2$ .  $\square$

## D PROOF OF PROPOSITION 1

In this section, we will prove one of the base cases of our main impossibility theorem: if there are  $n = 2$  voters and  $m = 5$  alternatives, no anonymous SWF satisfies both unanimity and strategyproofness. We will prove this statement by contradiction and hence assume that there is an SWF  $f$  for the given numbers of voters and alternatives that satisfies our axioms. To derive a contradiction, we will subsequently reason about numerous profiles and show that, regardless of which outcomes we choose at certain profiles, strategyproofness must be violated. We start by considering the profile  $R^*$  shown below.

$$R^*: \quad abcde \quad acbed$$

We first note that for this profile, it holds that  $f(R^*) \in \{abcde, abced, acbde, acbed\}$  because of unanimity. Moreover, an analogous statement holds for all profiles  $R' = \pi(R^*)$  that are derived by permuting the alternatives in  $R^*$  according to a bijection  $\pi : A \rightarrow A$ . In more detail, it is easy to see that, for each permutation  $\pi : A \rightarrow A$ , it holds that  $f(\pi(R^*)) \in \{\pi(abcde), \pi(abced), \pi(acbde), \pi(acbed)\}$  because unanimity does not depend on the names of the alternatives.

As the first step of our proof, we will show that there is a permutation  $\pi : A \rightarrow A$  such that  $f(\pi(R^*)) \in \{\pi(abcde), \pi(acbed)\}$ . Assume for contradiction that this is not true, which means that  $f(\pi(R^*)) \in \{\pi(abced), \pi(acbde)\}$  for all permutations  $\pi : A \rightarrow A$ . In particular, this means that  $f(R^*) \in \{abced, acbde\}$ . We further note that  $b$  and  $c$  as well as  $c$  and  $d$  are symmetric in  $R^*$ , i.e., it holds that  $\pi^*(R^*) = R^*$  for

the permutation  $\pi^*$  given by  $\pi^*(a) = a$ ,  $\pi^*(b) = c$ ,  $\pi^*(c) = b$ ,  $\pi^*(d) = e$ , and  $\pi^*(e) = d$ . Consequently, we can assume that  $f(R^*) = acbde$ ; the case that  $f(R^*) = abced$  follows by simply permuting all profiles and rankings in the subsequent argument with respect to  $\pi^*$ .

In the following table, we show that our assumptions are in conflict with each other as no feasible outcome remains for the profile  $R^4$ . Note that we display this simple derivation in the highly compressed form that is used throughout this section. In particular, we will present all proofs in tabular forms, where each row consists of a profile marker (left most column), the two rankings that make up the profile (second and third column), and all feasible outcomes. We note that the feasible outcomes are either determined by our assumptions (as, e.g., for  $R^*$  in the following table) or correspond to the set of rankings that satisfy unanimity for the given profile (as, e.g., for  $R^2$  in the following table). Moreover, profiles may appear multiple times in our derivations as we may infer additional information about the possible outcomes.

Based on the assumptions, we show that strategyproofness rules out specific rankings at given profiles. For instance, in the following derivation, the fact that  $f(R^*) = acbde$  entails that  $f(R^2) \neq acbed$  as the first voter can otherwise manipulate by deviating from  $R^2$  to  $R^*$ . Hence, we know that  $acbde$  must be chosen for  $R^2$ , and we will use this fact in further derivations. More generally, all grayed out rankings in the right most column satisfy unanimity, but violate strategyproofness due to the outcome that has been inferred for the profile indicated in brackets. We note here that strategyproofness may apply in either directions (i.e., either a voter manipulated from the considered profile to the one in the brackets or vice versa). Moreover, it is possible that multiple rankings are feasible outcomes for the profile indicated in brackets; in this case, strategyproofness rules out that the indicated ranking is chosen, regardless of the exact outcome for a given profile. Note that strategyproofness may be applied in either of the two directions for each of the possible outcomes at the manipulated profile. The second possibility that a ranking is grayed out is that we explicitly assume that this outcome is not chosen (e.g., for  $R^4$ , our assumption that  $f(\pi(R^4)) \in \{\pi(abced), \pi(acbde)\}$  for all permutations  $\pi : A \rightarrow A$  rules out that  $f(R^4) = abced$  or  $f(R^4) = acbde$ ). Lastly, all our proofs end in a profile where no valid outcome remains, thereby showing that our axioms the assumptions are incompatible with each other.

$R^*$	$abcde$	$acbed$	$acbde$ (A)
$R^2$	$abcde$	$acbed$	$acbde$ $acbed$ ( $R^*$ )
$R^3$	$abcde$	$acbde$	$abcde$ ( $R^*$ ) $acbed$
$R^4$	$abced$	$acbde$	$abcde$ ( $R^3$ ) $acbed$ ( $R^2$ ) $abced$ (A) $acbde$ (A)

For this simple derivation, it is straightforward to translate our tabular form into natural language. We assume that  $f(R^*) = acbde$ . This implies that  $f(R^2) = f(R^3) = acbde$  as these profiles are derived by letting one of the voters in  $R^*$  deviate to  $acbed$ . Lastly, for  $R^4$ , we have by assumption that  $f(R^4) \neq abced$  and  $f(R^4) \neq acbde$ . However, if we choose  $f(R^4) = abcde$ , voter 1 can manipulate by deviating from  $R^3$  to  $R^4$ . Similarly, if  $f(R^4) = acbed$ , voter 2 can manipulate by deviating from  $R^2$  to  $R^4$ . Hence, no ranking that satisfies unanimity remains for  $R^4$ , thereby showing that our assumptions are in conflict.

By this derivation, we know that  $f(\pi(R^*)) \in \{\pi(abcde), \pi(acbed)\}$  for some permutation  $\pi : A \rightarrow A$ . We will subsequently assume that  $\pi$  is given by the identity, i.e., that  $f(R^*) \in \{abcde, acbed\}$ . Our proof applies to any other permutation  $\pi$  by simply renaming the alternatives in all proofs and outcomes accordingly as both unanimity and strategyproofness are independent of the names of alternatives. Further, by the fact that  $b$  and  $c$ , as well as  $d$  and  $e$  are symmetric in  $R^*$ , we can assume without loss of generality that  $f(R^*) = abcde$ ; if  $f(R^*) = acbed$ , we can just exchange the roles of  $b$  and  $c$  as well as  $d$  and  $e$  in the subsequent proofs.

Now, from here on, we proceed with a case distinction with respect to the profile  $R^+$  shown below.

$$R^+: \quad eabcd \quad ecabd$$

We note that only three outcomes satisfy unanimity for this profile:  $f(R^+) = eabcd$ ,  $f(R^+) = eacbd$ , or  $f(R^+) = ecabd$ . We will next show that all of these three cases result in a contradiction. To give further structure to our proof, we will discuss each of these cases as a separate lemma. In particular, we show in the next three lemmas that none of these outcomes is compatible with the fact that  $f$  is anonymous, strategyproof, unanimous, and satisfies that  $f(R^*) = abcde$ . Note that each case itself breaks down in several subcases and steps. Since no valid outcome remains for  $R^+$ , we conclude that our basic assumptions are in conflict, so no SWF satisfies anonymity, unanimity, and strategyproofness if  $m = 5$  and  $n = 2$ .

**Lemma 2.**  $f(R^+) \neq ecabd$ .

**PROOF.** To prove his lemma, we assume for contradiction that  $f(R^+) = ecabd$ . We then proceed in five steps to specify the outcomes for further profiles, which ultimately results in a contradiction. We note that, except for  $R^*$  and  $R^+$ , we will reset the profile markers for each step as the corresponding derivations are fully self-contained.

**Step 1:** Our first goal is to show that  $f(R) = aecbd$  for the profile  $R$  where one voter reports  $aebcd$  and the other  $aeabd$ . We hence assume for contradiction that  $f(R) \neq aecbd$ . The subsequent derivation shows that this assumption is invalid as no feasible outcome remains for profile  $R^{28}$ . In this table, the profile  $R$  appears in Steps 1 and 6 (i.e.,  $R = R^1$  and  $R = R^6$ ).

$R^*$	$abcde$	$acbed$	$abcde$ (A)
$R^+$	$eabcd$	$ecabd$	$ecabd$ (A)
$R^1$	$aebdc$	$aecbd$	$aebdc$ $aecbd$ (A)

$R^2$	$eabdc$	$ecabd$	$eabcd$ ( $R^+$ )	$eabdc$ ( $R^+$ )	$eacbd$ ( $R^+$ )	$ecabd$				
$R^3$	$aebdc$	$ecabd$	$aebcd$ ( $R^+$ )	$aebdc$ ( $R^2$ )	$aecbd$ ( $R^1$ )	$eabcd$ ( $R^+$ )	$eabdc$ ( $R^+$ )	$eacbd$ ( $R^+$ )	$ecabd$	
$R^4$	$aebdc$	$caebd$	$acebd$	$aebcd$ ( $R^3$ )	$aebdc$ ( $R^3$ )	$aecbd$ ( $R^1$ )	$eacbd$ ( $R^1$ )	$caebd$		
$R^5$	$acebd$	$aebdc$	$acebd$	$aebcd$ ( $R^4$ )	$aebdc$ ( $R^4$ )	$aecbd$ ( $R^1$ )	$eacbd$ ( $R^1$ )			
$R^6$	$aebdc$	$aecbd$	$aebcd$	$aebdc$ ( $R^5$ )	$aecbd$ ( $A$ )					
$R^7$	$abecd$	$aecbd$	$abecd$	$aebcd$	$aecbd$ ( $R^6$ )					
$R^8$	$abecd$	$ecabd$	$abecd$ ( $R^3$ )	$aebcd$ ( $R^+$ )	$aecbd$ ( $R^3$ )	$eabcd$ ( $R^+$ )	$eacbd$ ( $R^+$ )	$ecabd$		
$R^9$	$abecd$	$eacbd$	$abecd$ ( $R^8$ )	$aebcd$ ( $R^8$ )	$aecbd$ ( $R^7$ )	$eabcd$	$eacbd$			
$R^{10}$	$abecd$	$eabcd$	$abecd$ ( $R^9$ )	$aebcd$	$eabcd$					
$R^{11}$	$eabdc$	$eacbd$	$eabcd$	$eabdc$ ( $R^2$ )	$eacbd$					
$R^{12}$	$abecd$	$aecbd$	$abecd$	$abecd$	$aebcd$	$aebdc$ ( $R^6$ )	$aecbd$ ( $R^1$ )			
$R^{13}$	$abcde$	$acebd$	$abcde$	$abced$ ( $R^*$ )	$acdbe$ ( $R^*$ )	$acbed$ ( $R^*$ )	$acebd$ ( $R^*$ )			
$R^{14}$	$abcde$	$aecbd$	$abcde$	$abced$ ( $R^*$ )	$abecd$	$acbde$ ( $R^*$ )	$acbed$ ( $R^*$ )	$acebd$ ( $R^*$ )	$aebcd$ ( $R^{13}$ )	$aecbd$ ( $R^{13}$ )
$R^{15}$	$abdec$	$aecbd$	$abdec$	$abecd$	$abedc$	$aebcd$ ( $R^{14}$ )	$aebdc$ ( $R^6$ )	$aecbd$ ( $R^1$ )		
$R^{16}$	$abecd$	$aecbd$	$abecd$	$abedc$	$aebcd$ ( $R^{15}$ )	$aebdc$ ( $R^6$ )	$aecbd$ ( $R^1$ )			
$R^{17}$	$abedc$	$eacbd$	$abecd$ ( $R^9$ )	$abedc$ ( $R^9$ )	$aebcd$ ( $R^9$ )	$aebdc$	$aecbd$ ( $R^{12}$ )	$eabcd$	$eabdc$ ( $R^{11}$ )	$eacbd$ ( $R^{16}$ )
$R^{18}$	$abecd$	$eacbd$	$abecd$ ( $R^8$ )	$aebcd$ ( $R^8$ )	$aecbd$ ( $R^7$ )	$eabcd$	$eacbd$ ( $R^{17}$ )			
$R^{19}$	$abecd$	$eabcd$	$abecd$ ( $R^9$ )	$aebcd$ ( $R^{18}$ )	$eabcd$					
$R^{20}$	$abecd$	$aecbd$	$abecd$	$abedc$ ( $R^{17}$ )	$aebcd$ ( $R^{15}$ )	$aebdc$ ( $R^6$ )	$aecbd$ ( $R^1$ )			
$R^{21}$	$abecd$	$aecbd$	$abecd$	$aebcd$ ( $R^{20}$ )	$aecbd$ ( $R^6$ )					
$R^{22}$	$abced$	$eabcd$	$abced$ ( $R^{19}$ )	$abecd$ ( $R^{10}$ )	$aebcd$ ( $R^{19}$ )	$eabcd$				
$R^{23}$	$abecd$	$aecdb$	$abecd$	$aebcd$ ( $R^{21}$ )	$aecbd$ ( $R^7$ )	$aecdb$ ( $R^{21}$ )				
$R^{24}$	$acebd$	$eabcd$	$acebd$ ( $R^{22}$ )	$aebcd$ ( $R^5$ )	$aecbd$ ( $R^5$ )	$eabcd$ ( $R^5$ )	$eacbd$			
$R^{25}$	$abecd$	$eacdb$	$abecd$ ( $R^8$ )	$aebcd$ ( $R^8$ )	$aecbd$ ( $R^7$ )	$aecdb$ ( $R^{21}$ )	$eabcd$	$eacbd$ ( $R^{18}$ )	$eacdb$ ( $R^{23}$ )	
$R^{26}$	$eabcd$	$eacdb$	$eabcd$ ( $R^{24}$ )	$eacbd$ ( $R^{25}$ )	$eacdb$ ( $R^{25}$ )					

**Step 2:** Next, we will show that  $f(R) = aedcb$  for the profile  $R$  where one voters reports  $aecd$  and the other  $aedcb$ . We hence assume that  $f(R) \neq aedcb$  and infer a contradiction as shown below. The profile  $R$  corresponds to  $R^2$  and  $R^{13}$  in the subsequent derivation.

$R^{19}$	$aebdc$	$aedcb$	$aebdc$	$aedbc (R^{18})$	$aedcb$
$R^{20}$	$adceb$	$aedcb$	$adceb$	$aedcb (R^{18})$	$aedcb$
$R^{21}$	$adebc$	$aebcd$	$adebc (R^{15})$	$aebcd$	$aebdc$
$R^{22}$	$adebc$	$aebdc$	$adebc (R^{21})$	$aebdc$	$aedbc$
$R^{23}$	$abecd$	$aedcb$	$abecd$	$abecd (R^{18})$	$aebcd (R^3)$
$R^{24}$	$abecd$	$aedcb$	$abecd (R^{18})$	$aebdc$	$aedbc (R^{23})$
$R^{25}$	$aebdc$	$aedcb$	$aebdc$	$aedbc (R^{18})$	$aedcb (R^{24})$
$R^{26}$	$aedcb$	$aebdc$	$aedbc (R^{22})$	$aedcb (R^{14})$	$aebdc$
$R^{27}$	$adceb$	$aebdc$	$adceb (R^{26})$	$adceb (R^{22})$	$aebdc$
$R^{28}$	$acedb$	$aebdc$	$acebd (R^{27})$	$acedb (R^{27})$	$aebcd (R^1)$
$R^{29}$	$acedb$	$aecbd$	$acebd (R^{28})$	$acedb (R^{28})$	$aecbd$
$R^{30}$	$adceb$	$aecbd$	$acdeb (R^{14})$	$acebd (R^{29})$	$acdeb (R^{29})$
$R^{31}$	$acebd$	$adceb$	$acdeb (R^{30})$	$acebd$	$acdeb (R^{30})$
$R^{32}$	$acedb$	$adceb$	$acdeb$	$acedb$	$adceb (R^{31})$
$R^{33}$	$adbce$	$aedcb$	$adbce (R^{18})$	$adbec (R^{18})$	$adcbe (R^{18})$
$R^{34}$	$abcde$	$acedb$	$abcde$	$abced (R^*)$	$acbed (R^*)$
$R^{35}$	$acedb$	$adbce$	$acdbe$	$acdeb (R^{34})$	$acdeb (R^{34})$
$R^{36}$	$acdbe$	$acedb$	$acdbe$	$acdeb (R^{35})$	$acdeb (R^{35})$
$R^{37}$	$acdbe$	$aecdb$	$acdbe$	$acdeb (R^{36})$	$acedb (R^{36})$
$R^{38}$	$acdbe$	$aedcb$	$acdbe$	$acdeb (R^{36})$	$acdbe (R^{18})$
$R^{39}$	$adbce$	$aedcb$	$adbce (R^{18})$	$adbec (R^{18})$	$adcbe (R^{18})$
$R^{40}$	$adceb$	$aedcb$	$adceb$	$adceb (R^{18})$	$adecb (R^{18})$
$R^{41}$	$acdbe$	$adbec$	$acdbe$	$adbec (R^{35})$	$adcb (R^{35})$
$R^{42}$	$adceb$	$aedbc$	$adceb (R^{27})$	$adebc$	$adecb (R^{20})$
$R^{43}$	$adceb$	$adecb$	$adceb (R^{42})$	$adecb$	$adecb (R^{40})$
$R^{44}$	$acdbe$	$adbec$	$acdbe$	$adbec (R^{41})$	$adcb (R^{41})$
$R^{45}$	$acdeb$	$adebc$	$acdeb (R^{43})$	$adebc$	$adecb (R^{43})$
$R^{46}$	$acdeb$	$adbec$	$acdeb (R^{45})$	$adbec (R^{44})$	$adbec (R^{17})$
					$adcb (R^{44})$
					$adecb (R^{44})$
					$adecb (R^{44})$

**Step 3:** As our third step, we will show that  $f(R) = ecabd$  for the profile  $R$  where one voter reports  $abecd$  and the other reports  $ecabd$ . As usual, we assume that  $f(R)$  is not our desired outcome, i.e.,  $f(R) \neq ecabd$ , and derive a contradiction. We use our assumption on  $R$  at profile  $R^3$  and  $R^{19}$ .

$R^*$	$abcde$	$acbed$	$abcde (A)$
$R^+$	$eabcd$	$ecabd$	$ecabd (A)$
$R^1$	$aebdc$	$aecbd$	$aecbd (A, Step 1)$
$R^2$	$aecbd$	$aedcb$	$aedcb (A, Step 2)$
$R^3$	$abecd$	$ecabd$	$abecd$
			$aebcd (R^+)$
			$aecbd$
			$eabcd (R^+)$
			$ecabd$
$R^4$	$eacbd$	$ecabd$	$eacbd (R^+)$
			$ecabd$
$R^5$	$aecbd$	$ecabd$	$aecbd$
			$eacbd (R^+)$
			$ecabd$
$R^6$	$aebdc$	$ecabd$	$aebdc (R^1)$
			$aebdc (R^1)$
			$aecbd$
			$eabcd (R^+)$
$R^7$	$aecbd$	$ecabd$	$aecbd$
			$eacbd (R^+)$
			$ecabd (R^6)$
$R^8$	$aecbd$	$ecbad$	$aecbd$
			$eacbd (R^5)$
			$ecabd (R^7)$
			$ecbad (R^7)$
$R^9$	$eacbd$	$ecbad$	$eacbd (R^4)$
			$ecabd$
			$ecbad (R^8)$
$R^{10}$	$ecabd$	$ecbad$	$ecabd$
			$ecbad (R^9)$
$R^{11}$	$aebcd$	$aecbd$	$aebcd (R^1)$
			$aecbd$
$R^{12}$	$abecd$	$aecbd$	$abecd$
			$aebcd (R^1)$
$R^{13}$	$abcde$	$acebd$	$abcde$
			$abced (R^*)$
			$acbed (R^*)$
$R^{14}$	$abcde$	$aecbd$	$abcde$
			$abced (R^*)$
			$abecd$
			$acbed (R^*)$
			$acbed (R^*)$
			$acebd (R^*)$
			$adecb (R^1)$
			$aecbd (R^{13})$

$R^{15}$	$abdec$	$aecbd$	$abdec$	$abecd$	$abedc(R^1)$	$aebcd(R^1)$	$aebcd(R^1)$	$aecbd(R^{14})$
$R^{16}$	$abedc$	$aecbd$	$abecd$	$abedc(R^1)$	$aebcd(R^1)$	$aebdc(R^1)$	$aecbd(R^{15})$	
$R^{17}$	$abecd$	$aecbd$	$abecd$	$aebcd(R^1)$	$aecbd(R^{16})$			
$R^{18}$	$eabdc$	$ecabd$	$eabcd(R^+)$	$eabdc(R^+)$	$eacbd(R^+)$	$ecabd$		
$R^{19}$	$abecd$	$ecabd$	$abecd$	$abecd(R^+)$	$aecbd(R^{17})$	$eabcd(R^+)$	$eacbd(R^+)$	$ecabd(A)$
$R^{20}$	$beacd$	$ecabd$	$beacd$	$becad$	$eabcd(R^+)$	$eacbd(R^+)$	$ebacd(R^+)$	$ebcad$
$R^{21}$	$ebadc$	$ecabd$	$eabcd(R^+)$	$eabdc(R^+)$	$eacbd(R^+)$	$ebacd(R^+)$	$ebadc(R^{18})$	$ecabd(R^{19})$
$R^{22}$	$ebcad$	$ecabd$	$ebcad$	$ecabd(R^{21})$	$ebcad(R^{10})$			$ecabd(R^{20})$
$R^{23}$	$ebadc$	$ebcad$	$ebacd(R^{21})$	$ebadc(R^{21})$	$ebcad$			
$R^{24}$	$aecbd$	$ebcad$	$aebcd(R^1)$	$aecbd$	$eabcd$	$eacbd(R^5)$	$ebacd(R^8)$	$ebcad(R^8)$
$R^{25}$	$eacdb$	$ebcad$	$eabcd$	$eacdb$		$ebacd(R^{24})$	$ebcad(R^{24})$	$ecabd(R^{22})$
$R^{26}$	$eadbc$	$ebcad$	$eabcd$	$eabdc(R^{23})$	$eadbc$	$ebacd(R^{23})$	$ebadc(R^{23})$	$ebcad(R^{25})$
$R^{27}$	$eabcd$	$eadbc$	$eabcd$	$eabdc(R^{26})$	$eadbc$			
$R^{28}$	$acedb$	$aedcb$	$acedb$	$aecdb(R^2)$	$aedcb$			
$R^{29}$	$abecd$	$aecdb$	$abecd$	$aebcd(R^{12})$	$aecbd(R^{17})$	$aecdb(R^{17})$		
$R^{30}$	$aebcd$	$aecdb$	$aebcd(R^{11})$	$aecbd$	$aecdb(R^{29})$			
$R^{31}$	$aecbd$	$aecdb$	$aecbd$		$aecdb(R^{30})$			
$R^{32}$	$abecd$	$aedcb$	$abecd$	$abecd$	$aebcd(R^2)$	$aebdc$	$aecbd(R^2)$	$aedbc(R^{29})$
$R^{33}$	$abedc$	$aedcb$	$abedc$	$aebdc$	$aedbc(R^{32})$	$aedcb(R^{32})$		
$R^{34}$	$aebdc$	$aedcb$	$aebdc$	$aedbc$	$aedcb(R^{33})$			
$R^{35}$	$abecd$	$ebcad$	$abecd$	$aebcd(R^{17})$	$baecd$	$beacd$	$becad(R^{24})$	$eabcd(R^{19})$
$R^{36}$	$baecd$	$ebcad$	$baecd$	$beacd$	$becad(R^{35})$	$ebacd(R^{23})$	$ebcad(R^{35})$	
$R^{37}$	$beacd$	$ebcad$	$beacd$	$becad$	$ebacd(R^{23})$	$ebcad(R^{36})$		
$R^{38}$	$abecd$	$eacdb$	$abecd$	$aebcd(R^{12})$	$aecbd(R^{17})$	$aecdb(R^{17})$	$eabcd(R^{19})$	$eacbd(R^{19})$
$R^{39}$	$beacd$	$eacdb$	$beacd$	$eabcd$	$eacbd(R^{38})$	$eacdb(R^{38})$	$ebacd(R^{25})$	
$R^{40}$	$beacd$	$ecabd$	$beacd$	$becad$	$eabcd(R^+)$	$ecabd(R^+)$	$ebacd(R^+)$	$ecabd(R^{37})$
$R^{41}$	$beacd$	$eacdb$	$beacd$	$eabcd(R^{40})$	$ecabd(R^{38})$	$eacdb(R^{38})$	$ebacd(R^{25})$	
$R^{42}$	$beacd$	$eadbc$	$beacd$	$beadc$	$eabcd(R^{40})$	$eabdc(R^{26})$	$eadbc(R^{41})$	$ebacd(R^{39})$
$R^{43}$	$eadbc$	$ebcad$	$eabcd$	$eabdc(R^{23})$	$eadbc(R^{42})$	$ebacd(R^{23})$	$ebadc(R^{23})$	$ebcad(R^{25})$
$R^{44}$	$eabcd$	$eadbc$	$eabcd$	$eabdc(R^{26})$	$eadbc(R^{43})$			
$R^{45}$	$eabcd$	$edabc$	$eabcd$	$eabdc(R^{27})$	$eadbc(R^{44})$	$edabc(R^{44})$		
$R^{46}$	$deabc$	$eabcd$	$deabc(R^{45})$	$eabcd$	$eabdc(R^{27})$	$eadbc(R^{44})$	$edabc(R^{44})$	
$R^{47}$	$adebc$	$aecbd$	$adebc$	$adecb$	$aebcd(R^2)$	$aebdc(R^1)$	$aecbd(R^2)$	$aedbc(R^1)$
$R^{48}$	$adebc$	$eabcd$	$adebc(R^{46})$	$aebcd(R^{47})$	$aebdc$	$aedbc(R^{44})$	$eabcd$	$eabdc(R^{27})$
$R^{49}$	$adebc$	$aebcd$	$adebc(R^{48})$	$aebcd(R^{47})$	$aebdc$	$aedbc(R^{48})$		
$R^{50}$	$adebc$	$aebdc$	$adebc(R^{49})$	$aebdc$	$aedbc(R^{49})$			
$R^{51}$	$adecb$	$aebdc$	$adecb(R^{50})$	$aebdc$	$aedbc(R^{50})$	$aedcb(R^{34})$		
$R^{52}$	$adceb$	$aebdc$	$adceb(R^{51})$	$adebc(R^{50})$	$adecb(R^{50})$	$aedbc(R^{50})$	$aedcb(R^{34})$	
$R^{53}$	$acedb$	$aebdc$	$acebd(R^{52})$	$acedb(R^{52})$	$aebcd(R^1)$	$aebdc(R^1)$	$aecbd$	$aecdb(R^{28})$
$R^{54}$	$acedb$	$aecbd$	$acebd(R^{53})$	$acedb(R^{53})$	$aecbd$	$aecdb(R^{31})$		
$R^{55}$	$adebc$	$aecbd$	$adebc(R^{48})$	$adecb(R^{48})$	$aebcd(R^2)$	$aebdc(R^1)$	$aecbd(R^2)$	$aedbc(R^1)$
$R^{56}$	$adebc$	$aecdb$	$adebc(R^{55})$	$adecb(R^{55})$	$aecdcb(R^{47})$	$aedbc(R^{55})$	$aedcb$	
$R^{57}$	$adbec$	$aecdb$	$adbec(R^{56})$	$adebc(R^{56})$	$adecb(R^{56})$	$aecdcb(R^{29})$	$aedbc(R^{56})$	$aedcb$
$R^{58}$	$adbec$	$aedcb$	$adbec(R^{57})$	$adebc(R^{57})$	$adecb(R^{57})$	$aedbc(R^{57})$	$aedcb$	
$R^{59}$	$adceb$	$aecbd$	$adceb(R^{52})$	$acebd(R^2)$	$acedb(R^{54})$	$adceb(R^{52})$	$adecb(R^{55})$	$aecbd(R^2)$
$R^{60}$	$adceb$	$aedcb$	$adceb(R^{59})$	$adecb(R^{58})$	$aedcb$			
$R^{61}$	$adbce$	$aedcb$	$adbce(R^{58})$	$adbec(R^{58})$	$adcbe(R^{58})$	$adceb(R^{60})$	$adebc(R^{58})$	$aedbc(R^{58})$

$R^{62}$	$acdbe$	$aedcb$	$acdbe (R^{61})$	$acdeb (R^{60})$	$acedb$	$adcbe (R^{58})$	$adceb (R^{60})$	$adecb (R^{58})$	$aecdb (R^2)$	$aedcb$
$R^{63}$	$acdbe$	$aecdb$	$acdbe (R^{62})$	$acdeb (R^{62})$	$acedb$	$aecdb$				
$R^{64}$	$acdbe$	$acedb$	$acdbe (R^{63})$	$acdeb$	$acedb$					
$R^{65}$	$abcde$	$acedb$	$abcde$	$abced (R^*)$	$acbde (R^*)$	$acbed (R^*)$	$acdbe$		$acdeb (R^{13})$	$acebd (R^*)$
$R^{66}$	$acedb$	$adbce$	$acdbe (R^{64})$	$acdeb (R^{65})$	$acedb (R^{65})$	$adbce (R^{61})$	$adcbe (R^{61})$	$adceb$		
$R^{67}$	$adbce$	$adceb$	$adbce (R^{66})$	$adcbe (R^{66})$	$adceb$					
$R^{68}$	$acedb$	$adceb$	$acdeb (R^{66})$	$acedb (R^{66})$	$adceb$					
$R^{69}$	$acbed$	$aedcb$	$acbed$	$acebd (R^2)$	$acedb$	$aecbd (R^2)$	$aecdb (R^2)$	$aedcb$		
$R^{70}$	$acebd$	$adceb$	$acdeb (R^{68})$	$acebd (R^{59})$	$acedb (R^{68})$	$adceb$				
$R^{71}$	$abcde$	$acedb$	$abcde$	$abced (R^*)$	$acbde (R^*)$	$acbed (R^*)$	$acdbe (R^{64})$	$acdeb (R^{13})$	$acebd (R^*)$	$acedb (R^{13})$
$R^{72}$	$abcde$	$adceb$	$abcde$	$abdce (R^{67})$	$acbde (R^*)$	$acdbe (R^{71})$	$acdeb (R^{13})$	$adbce (R^{67})$	$adcbe (R^{71})$	$adceb (R^{71})$
$R^{73}$	$acbed$	$adceb$	$acdbe (R^{70})$	$acbed (R^{70})$	$acdbe$	$acdeb (R^{68})$	$acebd (R^{69})$	$acedb (R^{68})$	$adcbe (R^{67})$	$adceb (R^{72})$
$R^{74}$	$acdbe$	$adceb$	$acdbe$	$acdeb (R^{68})$	$adcbe (R^{67})$	$adceb (R^{73})$				
$R^{75}$	$acdbe$	$acedb$	$acdbe (R^{63})$	$acdeb (R^{74})$	$acedb$					
$R^{76}$	$acbed$	$acdbe$	$acdbe (R^{73})$	$acbed (R^{73})$	$acdbe$					
$R^{77}$	$acdbe$	$acedb$	$acdbe (R^{75})$	$acbed$	$acdbe (R^{64})$	$acdeb (R^{65})$	$acebd (R^{65})$	$acedb (R^{65})$		
$R^{78}$	$acbed$	$acedb$	$acbed$	$acebd (R^{77})$	$acedb (R^{77})$					
$R^{79}$	$acdeb$	$aecbd$	$acdeb (R^{54})$	$acebd (R^{54})$	$acedb (R^{54})$	$aecbd$	$aecdb (R^{31})$			
$R^{80}$	$acdbe$	$aecbd$	$acdbe (R^{79})$	$acbed (R^{76})$	$acdbe (R^{79})$	$acdeb (R^{54})$	$acebd (R^{54})$	$acedb (R^{54})$	$aecbd$	$aecdb (R^{31})$
$R^{81}$	$acbed$	$aecbd$	$acbed (R^{80})$	$acebd (R^{54})$	$aecbd$					
$R^{82}$	$acdbe$	$adecb$	$acdbe$	$acdeb (R^{74})$	$adcbe (R^{74})$	$adceb (R^{74})$	$adecb (R^{74})$			
$R^{83}$	$acdbe$	$aedcb$	$acdbe (R^{61})$	$acdeb (R^{60})$	$acedb$	$adcbe (R^{58})$	$adceb (R^{60})$	$adecb (R^{58})$	$aecdb (R^2)$	$aedcb (R^{82})$
$R^{84}$	$acbed$	$aedcb$	$acbed (R^{81})$	$acebd (R^2)$	$acedb (R^{78})$	$acebd (R^2)$	$aecdb (R^2)$	$aedcb (R^{83})$		

**Step 4:** Fourthly, we will show that  $f(R) = adcbe$ . Once again, we assume this to be wrong, which means that  $f(R) \neq adcbe$ . The subsequent derivation shows that this is impossible. We note that our assumption that  $f(R) \neq adcbe$  is only used at profile  $R^{40}$ , i.e.,  $R = R^{40}$ .

$R^*$	$abcde$	$acbed$	$abcde (A)$							
$R^1$	$aebdc$	$aecbd$	$aecbd (A, Step 1)$							
$R^2$	$aecbd$	$aedcb$	$aedcb (A, Step 2)$							
$R^3$	$abecd$	$ecabd$	$ecabd (A, Step 3)$							
$R^4$	$abecd$	$aecbd$	$abecd$	$aebcd (R^1)$	$aecbd$					
$R^5$	$aebcd$	$aecbd$	$aebcd (R^1)$	$aecbd$						
$R^6$	$abcde$	$acebd$	$abcde$	$abced (R^*)$	$acbde (R^*)$	$acbed (R^*)$	$acebd (R^*)$			
$R^7$	$abcde$	$aecbd$	$abcde$	$abced (R^*)$	$abecd$	$acbde (R^*)$	$acbed (R^*)$	$acebd (R^*)$	$aebcd (R^1)$	$aecbd (R^6)$
$R^8$	$abdec$	$aecbd$	$abdec$	$abecd$	$abedc (R^1)$	$aebcd (R^1)$	$aebdc (R^1)$	$aecbd (R^7)$		
$R^9$	$abedc$	$aecbd$	$abecd$	$abedc (R^1)$	$aebed (R^1)$	$aebdc (R^1)$	$aebdc (R^1)$	$aecbd (R^8)$		
$R^{10}$	$abecd$	$aecbd$	$abecd$	$aebcd (R^1)$	$acebd (R^9)$					
$R^{11}$	$abecd$	$aecdb$	$abecd$	$abecd (R^4)$	$acebd (R^{10})$	$aecdb (R^{10})$				
$R^{12}$	$abecd$	$caebd$	$abced (R^3)$	$abecd (R^3)$	$acbed$	$acebd (R^{10})$	$aebcd (R^3)$	$aecbd (R^{10})$	$cabed$	$caebd$
$R^{13}$	$abecd$	$acedb$	$abced$	$abecd (R^{12})$	$acbed$	$acebd (R^{10})$	$acedb (R^{11})$	$aebcd (R^4)$	$aecbd (R^{10})$	$aecdb (R^{10})$
$R^{14}$	$acbed$	$acedb$	$acbed$	$acebd$	$acedb (R^{13})$					
$R^{15}$	$abcde$	$abecd$	$abcede$	$abced (R^*)$	$abecd$					
$R^{16}$	$abecd$	$aedcb$	$abecd$	$abedc$	$aebcd (R^2)$	$aebdc$	$aecbd (R^2)$	$aecdb (R^2)$	$aedbc (R^{11})$	$aedcb (R^{11})$
$R^{17}$	$abedc$	$aedcb$	$abedc$	$aebdc$	$aedbc (R^{16})$	$aedcb (R^{16})$				
$R^{18}$	$aebdc$	$aedcb$	$aebdc$	$aedbc$	$aedcb (R^{17})$					
$R^{19}$	$acedb$	$aedcb$	$acedb$	$acedb (R^2)$	$aedcb$					
$R^{20}$	$aebcd$	$aecdb$	$aebcd (R^5)$	$aecbd$	$aecdb (R^{11})$					

**Step 5:** Finally, we will show that the information we have inferred so far is contradictory.

Since Step 5 only uses the assumption on  $R^*$  and the insights proven in the previous steps, this proves our lemma.

1

**Lemma 3.**  $f(R^+) \neq eacbd.$

PROOF. We again assume for contradiction that  $f(R^+) = eacbd$  and derive a contradiction in multiple steps.

**Step 1:** First, we will show that  $f(R) = eacbd$  for the profile  $R$  where one voter reports  $ceabd$  and the other reports  $eacbd$ . The following derivation shows that we get an impossibility if  $f(R) \neq eacbd$ , thus proving our claim. Our assumption that  $f(R) \neq eacbd$  is used at profile  $R^1$  and  $R^{41}$ .

$R^{17}$	$eacbd$	$ebacd$	$eabcd (R^5)$	$eacbd$	$ebacd$				
$R^{18}$	$abdec$	$aecbd$	$abdec$	$abecd$	$abecd$	$aebcd (R^7)$	$aebdc$	$aecbd (R^7)$	
$R^{19}$	$abedc$	$aecbd$	$abecd$	$abecd$	$aebcd$	$aebcd (R^{18})$	$aebdc$	$aecbd (R^{18})$	
$R^{20}$	$abedc$	$eacbd$	$abecd$	$abecd$	$aebcd (R^{19})$	$aebdc (R^5)$	$aecbd (R^{19})$	$eabcd (R^5)$	$eabdc (R^5)$
$R^{21}$	$beadc$	$eacbd$	$beacd$	$beadc$	$eabcd (R^5)$	$eabdc (R^5)$	$eacbd (R^{20})$	$ebacd$	$ebadc (R^5)$
$R^{22}$	$beacd$	$eacbd$	$beacd$	$eabcd (R^5)$	$eacbd (R^{21})$	$ebacd$			
$R^{23}$	$eacbd$	$ebacd$	$eabcd (R^5)$	$eacbd (R^{22})$	$ebacd$				
$R^{24}$	$acebd$	$eabcd$	$acebd$	$aebcd (R^{15})$	$aecbd$	$eabcd (R^+)$	$eacbd (R^{16})$		
$R^{25}$	$acebd$	$ebacd$	$acebd$	$aebcd (R^{15})$	$aecbd$	$eabcd (R^{17})$	$eacbd (R^{16})$	$ebacd (R^{24})$	
$R^{26}$	$aecbd$	$ebacd$	$aebcd$	$aecbd (R^{23})$	$eabcd (R^{17})$	$eacbd (R^{23})$	$ebacd (R^{25})$		
$R^{27}$	$aebcd$	$aecbd$	$aebcd$	$aecbd (R^{26})$					
$R^{28}$	$ceabd$	$ebacd$	$ceabd$	$cebad$	$eabcd (R^2)$	$eacbd (R^1)$	$ebacd (R^2)$	$ebcad (R^2)$	$ecabd (R^2)$
$R^{29}$	$ebacd$	$ecabd$	$eabcd (R^+)$	$eacbd (R^{23})$	$ebacd (R^{28})$	$ebcad$	$ecabd (R^+)$	$ecbad$	
$R^{30}$	$ebacd$	$ecbad$	$ebacd (R^{29})$	$ebcad$	$ecbad$				
$R^{31}$	$acbde$	$acebd$	$acbde$	$acbed$	$acebd (R^6)$				
$R^{32}$	$abcde$	$acbde$	$abcde$	$acbde (R^*)$					
$R^{33}$	$acebd$	$ebacd$	$acebd$	$aebcd (R^{15})$	$aecbd (R^{23})$	$eabcd (R^{17})$	$eacbd (R^{16})$	$ebacd (R^{24})$	
$R^{34}$	$acebd$	$baecd$	$abced$	$abecd (R^{33})$	$acbed$	$acebd (R^6)$	$aebcd (R^{15})$	$aecbd (R^{33})$	$baced (R^{33})$
$R^{35}$	$abced$	$baecd$	$abced$	$abecd$	$baced$	$baecd (R^{34})$			
$R^{36}$	$acebd$	$eabdc$	$acebd$	$aebcd$	$aebdc (R^3)$	$aecbd$	$eabcd (R^3)$	$eabdc (R^3)$	$eacbd (R^{16})$
$R^{37}$	$acebd$	$eabdc$	$acebd$	$aebcd (R^{15})$	$aebdc (R^3)$	$aecbd$	$eabcd (R^3)$	$eabdc (R^3)$	$eacbd (R^{16})$
$R^{38}$	$abedc$	$acebd$	$abced$	$abecd$	$abedc (R^{37})$	$acbed$	$acebd (R^6)$	$aebcd (R^{15})$	$aebdc (R^{36})$
$R^{39}$	$abced$	$abedc$	$abced$	$abecd$	$abedc (R^{38})$				
$R^{40}$	$aebcd$	$ecabd$	$aebcd$	$aecbd$	$eabcd (R^+)$	$eacbd$	$ecabd (R^+)$		
$R^{41}$	$ceabd$	$ebacd$	$ceabd$	$eacbd (A)$	$ecabd (R^2)$				
$R^{42}$	$aebcd$	$ecabd$	$aebcd$	$aecbd (R^{27})$	$eabcd (R^+)$	$eacbd (R^{12})$	$ecabd (R^+)$		
$R^{43}$	$eabdc$	$ecbad$	$eabcd (R^3)$	$eabdc (R^3)$	$eachd$	$ebacd$	$ebadc (R^3)$	$ebcad$	$ecabd (R^4)$
$R^{44}$	$aebcd$	$ecbad$	$aebcd$	$aecbd (R^{27})$	$eabcd (R^{43})$	$eacbd (R^{13})$	$ebacd$	$ebcad (R^{42})$	$ecabd (R^{40})$
$R^{45}$	$eabdc$	$ecbad$	$eabcd (R^3)$	$eabdc (R^3)$	$eachd$	$ebacd (R^{30})$	$ebadc (R^3)$	$ebcad (R^{44})$	$ecabd (R^4)$
$R^{46}$	$eacbd$	$ecbad$	$eacbd$	$ecabd (R^{45})$	$ecbad (R^{45})$				
$R^{47}$	$eacbd$	$ecbda$	$eacbd$	$ecabd (R^{46})$	$ecbad (R^{46})$	$ecbda (R^{46})$			
$R^{48}$	$cebda$	$ebacd$	$ceabd$	$cebad (R^{46})$	$cebda (R^{47})$	$ecabd (R^1)$	$ecabd (R^{41})$	$ecbad (R^{46})$	$ecbda (R^{46})$
$R^{49}$	$ceabd$	$cebda$	$ceabd$	$cebad (R^{48})$	$cebda (R^{48})$				
$R^{50}$	$ebadc$	$ecbad$	$ebacd (R^{30})$	$ebadc (R^{43})$	$ebcad$	$ecbad$			
$R^{51}$	$ceabd$	$ebacd$	$ceabd$	$cebad (R^{49})$	$eabcd (R^2)$	$eacbd (R^1)$	$ebacd (R^2)$	$ecabd (R^2)$	$ecbad$
$R^{52}$	$ebacd$	$ecabd$	$eabcd (R^+)$	$eacbd (R^{23})$	$ebacd (R^{28})$	$ebcad (R^{51})$	$ecabd (R^+)$	$ecbad$	
$R^{53}$	$ebacd$	$ecbad$	$ebacd (R^{29})$	$ebcad (R^{52})$	$ecbad$				
$R^{54}$	$baecd$	$ecbad$	$baecd$	$beacd (R^{53})$	$becad$	$ebacd (R^{30})$	$ebcad (R^{44})$	$ecbad (R^{44})$	
$R^{55}$	$beadc$	$ecbad$	$beacd (R^{53})$	$beadc (R^{50})$	$becad$	$ebacd (R^{30})$	$ebadc (R^{43})$	$ebcad (R^{53})$	$ecbad (R^{54})$
$R^{56}$	$beadc$	$becad$	$beacd (R^{55})$	$beadc (R^{55})$	$becad$				
$R^{57}$	$ceabd$	$ebcad$	$ceabd$	$cebad$	$ebcad (R^2)$	$ecabd (R^2)$	$ecbad$		
$R^{58}$	$ebacd$	$ecabd$	$eabcd (R^+)$	$eacbd$	$ebacd (R^{28})$	$ebcad$	$ecabd (R^+)$	$ecbad$	
$R^{59}$	$acebd$	$ecbad$	$acebd$	$aecbd (R^{33})$	$caebd$	$ceabd$	$cebad (R^{33})$	$eacbd (R^{16})$	$ecabd (R^{33})$
$R^{60}$	$caebd$	$ecbad$	$caebd$	$ceabd$	$cebad (R^{59})$	$ecabd (R^{59})$	$ecbad (R^{59})$		
$R^{61}$	$ceabd$	$ebcad$	$ceabd$	$cebad (R^{49})$	$ebcad (R^2)$	$ecabd (R^2)$	$ecbad$		
$R^{62}$	$beacd$	$ecbad$	$beacd (R^{53})$	$becad$	$ebacd (R^{30})$	$ebcad (R^{53})$	$ecbad (R^{54})$		
$R^{63}$	$beacd$	$ecabd$	$beacd (R^{52})$	$becad$	$eabcd (R^+)$	$eacbd (R^{22})$	$ebcad (R^{58})$	$ecabd (R^{52})$	$ecbad (R^{62})$

$R^{64}$	<b>becad</b>	<b>ecabd</b>	<b>becad</b>	<i>ebcad (R<sup>52</sup>)</i>	<i>ecabd (R<sup>29</sup>)</i>	<i>ecbad (R<sup>63</sup>)</i>				
$R^{65}$	<b>eacbd</b>	<b>ebcad</b>	<i>eabcd (R<sup>5</sup>)</i>	<i>eacbd (R<sup>22</sup>)</i>	<b>ebacd</b>	<i>ebcad (R<sup>46</sup>)</i>	<i>ecabd (R<sup>23</sup>)</i>	<i>ecbad (R<sup>46</sup>)</i>		
$R^{66}$	<b>caebd</b>	<b>ebcad</b>	<b>caebd</b>	<b>ceabd</b>	<i>cebad (R<sup>61</sup>)</i>	<i>ebcad (R<sup>57</sup>)</i>	<i>ecabd (R<sup>65</sup>)</i>	<i>ecbad (R<sup>60</sup>)</i>		
$R^{67}$	<b>ceabd</b>	<b>ebcad</b>	<b>ceabd</b>	<i>cebad (R<sup>49</sup>)</i>	<i>ebcad (R<sup>2</sup>)</i>	<i>ecabd (R<sup>2</sup>)</i>	<i>ecbad (R<sup>66</sup>)</i>			
$R^{68}$	<b>becad</b>	<b>ceabd</b>	<i>bcead (R<sup>67</sup>)</i>	<i>becad (R<sup>67</sup>)</i>	<b>cebad</b>	<i>ceabd (R<sup>64</sup>)</i>	<i>cebad (R<sup>49</sup>)</i>	<i>ebcad (R<sup>2</sup>)</i>	<i>ecabd (R<sup>2</sup>)</i>	<i>ecbad (R<sup>64</sup>)</i>
$R^{69}$	<b>cbead</b>	<b>ceabd</b>	<b>cbead</b>	<i>ceabd (R<sup>68</sup>)</i>	<i>cebad (R<sup>49</sup>)</i>					
$R^{70}$	<b>becad</b>	<b>ebacd</b>	<i>beacd (R<sup>56</sup>)</i>	<b>becad</b>	<b>ebacd</b>	<i>ebcad (R<sup>64</sup>)</i>				
$R^{71}$	<b>ebadc</b>	<b>ecbad</b>	<i>ebacd (R<sup>30</sup>)</i>	<i>ebadc (R<sup>43</sup>)</i>	<i>ebcad (R<sup>53</sup>)</i>	<b>ecbad</b>				
$R^{72}$	<b>ceabd</b>	<b>ebcda</b>	<b>ceabd</b>	<i>cebad (R<sup>49</sup>)</i>	<i>cebda (R<sup>49</sup>)</i>	<i>ebcad (R<sup>2</sup>)</i>	<i>ebcda (R<sup>67</sup>)</i>	<i>ecabd (R<sup>2</sup>)</i>	<i>ecbad (R<sup>67</sup>)</i>	<i>ecbda (R<sup>49</sup>)</i>
$R^{73}$	<b>ebcda</b>	<b>ecabd</b>	<i>ebcad (R<sup>52</sup>)</i>	<i>ebcda (R<sup>72</sup>)</i>	<i>ecabd (R<sup>29</sup>)</i>	<b>ecbad</b>		<i>ecbda (R<sup>72</sup>)</i>		
$R^{74}$	<b>ebcda</b>	<b>ecbad</b>	<i>ebcad (R<sup>53</sup>)</i>	<i>ebcda (R<sup>73</sup>)</i>	<b>ecbad</b>	<i>ebcda (R<sup>73</sup>)</i>				
$R^{75}$	<b>ebdca</b>	<b>ecbad</b>	<i>ebcad (R<sup>53</sup>)</i>	<i>ebcda (R<sup>74</sup>)</i>	<i>ebdca (R<sup>71</sup>)</i>	<b>ecbad</b>		<i>ecbda (R<sup>74</sup>)</i>		
$R^{76}$	<b>bedca</b>	<b>ecbad</b>	<b>becad</b>	<i>becda (R<sup>74</sup>)</i>	<i>bedca (R<sup>75</sup>)</i>	<i>ebcad (R<sup>53</sup>)</i>		<i>ebcda (R<sup>74</sup>)</i>	<i>ebdca (R<sup>71</sup>)</i>	<i>ecbad (R<sup>62</sup>)</i>
$R^{77}$	<b>becad</b>	<b>bedca</b>	<b>becad</b>	<i>becda (R<sup>76</sup>)</i>	<i>bedca (R<sup>76</sup>)</i>					
$R^{78}$	<b>acebd</b>	<b>aebcd</b>	<b>acebd</b>	<i>aebcd (R<sup>14</sup>)</i>	<i>acebd (R<sup>33</sup>)</i>					
$R^{79}$	<b>abcde</b>	<b>aebcd</b>	<b>abcde</b>	<i>abced (R<sup>*</sup>)</i>	<b>abecd</b>	<i>aebcd (R<sup>6</sup>)</i>				
$R^{80}$	<b>cbead</b>	<b>eacbd</b>	<b>cbead</b>	<i>ceabd (R<sup>69</sup>)</i>	<i>cebad (R<sup>46</sup>)</i>	<i>ebacd (R<sup>1</sup>)</i>	<i>ecabd (R<sup>41</sup>)</i>	<i>ecbad (R<sup>46</sup>)</i>		
$R^{81}$	<b>cbead</b>	<b>ebacd</b>	<b>bcead</b>	<i>beacd (R<sup>28</sup>)</i>	<b>becad</b>	<b>cbead</b>	<i>cebad (R<sup>69</sup>)</i>	<i>ebacd (R<sup>28</sup>)</i>	<i>ebcad (R<sup>28</sup>)</i>	<i>ecbad (R<sup>80</sup>)</i>
$R^{82}$	<b>bcead</b>	<b>ebacd</b>	<b>bcead</b>	<i>beacd (R<sup>70</sup>)</i>	<b>becad</b>	<i>ebacd (R<sup>81</sup>)</i>	<i>ebcad (R<sup>70</sup>)</i>			
$R^{83}$	<b>becad</b>	<b>ebacd</b>	<i>beacd (R<sup>56</sup>)</i>	<b>becad</b>	<i>ebacd (R<sup>82</sup>)</i>	<i>ebcad (R<sup>64</sup>)</i>				
$R^{84}$	<b>becad</b>	<b>eabcd</b>	<i>beacd (R<sup>56</sup>)</i>	<b>becad</b>	<i>eabcd (R<sup>64</sup>)</i>	<i>ebacd (R<sup>83</sup>)</i>	<b>ebcad (R<sup>64</sup>)</b>			
$R^{85}$	<b>aebcd</b>	<b>ecbad</b>	<b>aebcd</b>	<i>aecbd (R<sup>27</sup>)</i>	<i>eabcd (R<sup>43</sup>)</i>	<i>ebabd (R<sup>13</sup>)</i>	<i>ebacd (R<sup>30</sup>)</i>	<i>ebcad (R<sup>42</sup>)</i>	<i>ecabd (R<sup>40</sup>)</i>	<i>ecbad (R<sup>13</sup>)</i>
$R^{86}$	<b>aebcd</b>	<b>becad</b>	<i>abecd (R<sup>84</sup>)</i>	<i>aebcd (R<sup>84</sup>)</i>	<b>baecd</b>	<i>beacd (R<sup>85</sup>)</i>	<i>becad (R<sup>85</sup>)</i>	<i>eabcd (R<sup>42</sup>)</i>	<i>ebacd (R<sup>85</sup>)</i>	<i>ebcad (R<sup>42</sup>)</i>
$R^{87}$	<b>baecd</b>	<b>becad</b>	<b>baecd</b>	<i>beacd (R<sup>54</sup>)</i>	<i>becad (R<sup>86</sup>)</i>					
$R^{88}$	<b>aebcd</b>	<b>baecd</b>	<i>abecd (R<sup>86</sup>)</i>	<i>aebcd (R<sup>86</sup>)</i>	<b>baecd</b>					
$R^{89}$	<b>baecd</b>	<b>becda</b>	<b>baecd</b>	<i>beacd (R<sup>54</sup>)</i>	<i>becad (R<sup>87</sup>)</i>	<i>becda (R<sup>87</sup>)</i>				
$R^{90}$	<b>becad</b>	<b>bedac</b>	<i>beacd (R<sup>56</sup>)</i>	<b>beacd</b>	<i>beacd (R<sup>56</sup>)</i>	<i>becda (R<sup>77</sup>)</i>	<i>bedac (R<sup>56</sup>)</i>	<i>bedca (R<sup>77</sup>)</i>		
$R^{91}$	<b>bdeac</b>	<b>becad</b>	<i>bdeac (R<sup>90</sup>)</i>	<i>bdeca (R<sup>77</sup>)</i>	<i>beacd (R<sup>56</sup>)</i>	<i>beadc (R<sup>56</sup>)</i>	<b>becad</b>	<i>becda (R<sup>77</sup>)</i>	<i>bedac (R<sup>56</sup>)</i>	<i>bedca (R<sup>77</sup>)</i>
$R^{92}$	<b>bdeac</b>	<b>becda</b>	<i>bdeac (R<sup>91</sup>)</i>	<i>bdeca (R<sup>91</sup>)</i>	<b>becda</b>	<i>bedac (R<sup>91</sup>)</i>	<b>bedca</b>			
$R^{93}$	<b>bdaec</b>	<b>becda</b>	<i>bdaec (R<sup>92</sup>)</i>	<i>bdeac (R<sup>92</sup>)</i>	<i>bdeca (R<sup>92</sup>)</i>	<b>becda</b>		<i>bedac (R<sup>92</sup>)</i>	<b>bedca</b>	
$R^{94}$	<b>bdaec</b>	<b>becda</b>	<i>bdaec (R<sup>92</sup>)</i>	<i>bdeac (R<sup>92</sup>)</i>	<i>bdeca (R<sup>92</sup>)</i>	<i>becda (R<sup>89</sup>)</i>	<i>bedac (R<sup>92</sup>)</i>	<b>bedca</b>		
$R^{95}$	<b>bdaec</b>	<b>bedca</b>	<i>bdaec (R<sup>93</sup>)</i>	<i>bdeac (R<sup>93</sup>)</i>	<i>bdeca (R<sup>94</sup>)</i>	<i>bedac (R<sup>94</sup>)</i>	<b>bedca</b>			
$R^{96}$	<b>baecd</b>	<b>bedca</b>	<b>baecd</b>	<i>baecd (R<sup>95</sup>)</i>	<i>beacd (R<sup>54</sup>)</i>	<b>beadc</b>	<i>becad (R<sup>87</sup>)</i>	<i>becda (R<sup>76</sup>)</i>	<i>bedac (R<sup>76</sup>)</i>	<i>bedca (R<sup>76</sup>)</i>
$R^{97}$	<b>badec</b>	<b>bedca</b>	<i>badec (R<sup>95</sup>)</i>	<i>baecd (R<sup>95</sup>)</i>	<i>bdaec (R<sup>95</sup>)</i>	<i>bdeac (R<sup>95</sup>)</i>	<i>badec (R<sup>95</sup>)</i>	<b>beadc</b>	<i>bedac (R<sup>95</sup>)</i>	<i>bedca (R<sup>96</sup>)</i>
$R^{98}$	<b>badec</b>	<b>beadc</b>	<i>badec (R<sup>97</sup>)</i>	<i>baecd (R<sup>97</sup>)</i>	<b>beadc</b>					
$R^{99}$	<b>baecd</b>	<b>ecbad</b>	<b>baecd</b>	<i>beacd (R<sup>53</sup>)</i>	<i>becad (R<sup>87</sup>)</i>	<i>ebacd (R<sup>30</sup>)</i>	<i>ebcad (R<sup>44</sup>)</i>	<i>ecbad (R<sup>44</sup>)</i>		
$R^{100}$	<b>baecd</b>	<b>eacbd</b>	<i>abecd (R<sup>88</sup>)</i>	<i>aebcd (R<sup>88</sup>)</i>	<i>aecbd (R<sup>20</sup>)</i>	<b>baecd</b>	<i>beacd (R<sup>87</sup>)</i>	<i>eabcd (R<sup>5</sup>)</i>	<i>eachd (R<sup>9</sup>)</i>	<i>ebacd (R<sup>99</sup>)</i>
$R^{101}$	<b>acebd</b>	<b>baecd</b>	<b>abced</b>	<i>abecd (R<sup>33</sup>)</i>	<i>acbed (R<sup>100</sup>)</i>	<i>acebd (R<sup>6</sup>)</i>	<i>aebcd (R<sup>15</sup>)</i>	<i>aecbd (R<sup>33</sup>)</i>	<i>baced (R<sup>33</sup>)</i>	<i>baecd (R<sup>25</sup>)</i>
$R^{102}$	<b>abced</b>	<b>baecd</b>	<b>abced</b>	<i>abecd (R<sup>88</sup>)</i>	<i>baced (R<sup>101</sup>)</i>	<i>baecd (R<sup>34</sup>)</i>				
$R^{103}$	<b>badce</b>	<b>beadc</b>	<i>badce (R<sup>98</sup>)</i>	<i>badec (R<sup>98</sup>)</i>	<b>baecd</b>					
$R^{104}$	<b>abced</b>	<b>beacd</b>	<b>abced</b>	<i>abecd (R<sup>102</sup>)</i>	<i>baced (R<sup>102</sup>)</i>	<i>baecd (R<sup>35</sup>)</i>	<i>beacd (R<sup>102</sup>)</i>			
$R^{105}$	<b>acbde</b>	<b>aebcd</b>	<b>abcde</b>	<i>abced (R<sup>79</sup>)</i>	<i>abecd (R<sup>78</sup>)</i>	<i>acbde (R<sup>32</sup>)</i>	<b>acbed</b>	<i>acebd (R<sup>31</sup>)</i>	<i>aebcd (R<sup>79</sup>)</i>	<i>aecbd (R<sup>78</sup>)</i>
$R^{106}$	<b>acbde</b>	<b>baecd</b>	<b>abcde</b>	<i>abced (R<sup>105</sup>)</i>	<i>abecd (R<sup>34</sup>)</i>	<i>acbde (R<sup>32</sup>)</i>	<i>acbed (R<sup>100</sup>)</i>	<b>bacde</b>	<i>baced (R<sup>34</sup>)</i>	<i>baecd (R<sup>34</sup>)</i>
$R^{107}$	<b>abced</b>	<b>beadc</b>	<b>abced</b>	<i>abecd (R<sup>102</sup>)</i>	<i>abedc (R<sup>39</sup>)</i>	<i>baced (R<sup>102</sup>)</i>	<i>baecd (R<sup>35</sup>)</i>	<i>baedc (R<sup>102</sup>)</i>	<i>beacd (R<sup>102</sup>)</i>	<i>beadc (R<sup>104</sup>)</i>
$R^{108}$	<b>bacde</b>	<b>beadc</b>	<i>bacde (R<sup>103</sup>)</i>	<i>baced (R<sup>103</sup>)</i>	<i>badce (R<sup>98</sup>)</i>	<i>badec (R<sup>98</sup>)</i>	<b>baecd</b>	<i>baedc (R<sup>107</sup>)</i>	<i>beacd (R<sup>107</sup>)</i>	<i>beadc (R<sup>107</sup>)</i>
$R^{109}$	<b>bacde</b>	<b>baecd</b>	<i>bacde (R<sup>108</sup>)</i>	<i>baced (R<sup>108</sup>)</i>	<i>baecd (R<sup>106</sup>)</i>					

**Step 2:** Our next goal is to show that  $f(R) = aebcd$  for the profile  $R$  where one voter reports  $aebcd$  and the other reports  $ecabd$ . We hence assume that  $f(R) \neq aebcd$  and derive a contradiction, as shown in the following table. The profile  $R$  appears at Steps 2 and 11 (i.e.,  $R^2$  and  $R^{11}$ ) of our derivation.

$R^{43}$	$eacbd$	$ebcad$	$eabcd (R^{31})$	$eacbd$	$ebacd (R^{23})$	$ebcad (R^{24})$	$ecabd (R^1)$	$ecbad (R^{42})$
$R^{44}$	$abced$	$eacbd$	$abced (R^{41})$	$abecd (R^{32})$	$acbed$	$acebd (R^6)$	$aebcd (R^6)$	$aecbd (R^6)$
$R^{45}$	$cbead$	$eacbd$	$cbead$	$ceabd (R^1)$	$cebad (R^1)$	$eacbd (R^{44})$	$ecabd (R^1)$	$ecbad (R^{43})$
$R^{46}$	$cebad$	$eacbd$	$ceabd (R^1)$	$cebad (R^1)$	$eacbd (R^{45})$	$ecabd (R^1)$	$ecbad (R^{43})$	

**Step 3:** As the next step, we will show that  $f(R) = aebcd$  for the profile  $R$  where one voter reports  $acebd$  and the other reports  $aebcd$ . Once again, we assume for contradiction that this is not true and derive a contradiction. The profile  $R$  is called  $R^3$  in the subsequent derivation.

$R^*$	$abcde$	$acbed$	$abcde (A)$					
$R^+$	$eabcd$	$ecabd$	$eacbd (A)$					
$R^1$	$ceabd$	$eacbd$	$eacbd (A, Step 1)$					
$R^2$	$aebcd$	$ecabd$	$aebcd (A, Step 2)$					
$R^3$	$acebd$	$aebcd$	$acebd$	$aebcd (A)$	$acebd (R^2)$			
$R^4$	$abcde$	$acebd$	$abcde$	$abced (R^*)$	$acbed (R^*)$	$acbed (R^*)$	$acebd (R^*)$	
$R^5$	$acbde$	$acebd$	$acbde$	$acbed$	$acebd (R^4)$			
$R^6$	$abcde$	$acbde$	$abcde$	$acbed (R^*)$				
$R^7$	$acebd$	$eacbd$	$acebd$	$acebd (R^3)$	$eacbd$			
$R^8$	$eacbd$	$ecbad$	$eacbd$	$ecabd (R^1)$	$ecbad$			
$R^9$	$eabdc$	$eacbd$	$eabcd (R^+)$	$eabdc$	$ecabd$	$ecabd (R^2)$		
$R^{10}$	$eabdc$	$eacbd$	$eabcd (R^9)$	$eabdc$	$ecabd$			
$R^{11}$	$aebcd$	$aecbd$	$aebcd$	$acebd (R^2)$				
$R^{12}$	$abcde$	$aebcd$	$abcde$	$abced (R^*)$	$abecd$	$aebcd (R^4)$		
$R^{13}$	$acbde$	$aebcd$	$abcde$	$abced (R^{12})$	$abecd (R^3)$	$acbed (R^6)$	$acbed (R^5)$	$aebcd (R^3)$
$R^{14}$	$acbde$	$eacbd$	$acbde$	$acbed$	$acebd (R^5)$	$acbed (R^7)$	$ecabd (R^{13})$	
$R^{15}$	$acbed$	$eacbd$	$acbed$	$acebd$	$acebd (R^7)$	$ecabd (R^{14})$		
$R^{16}$	$acebd$	$eacbd$	$acebd$	$acebd (R^3)$	$ecabd (R^{15})$			
$R^{17}$	$acebd$	$eabcd$	$acebd$	$aebcd (R^3)$	$ecabd (R^+)$	$ecabd (R^{16})$		
$R^{18}$	$acebd$	$ebacd$	$acebd$	$aebcd (R^3)$	$ebacd (R^3)$	$ecabd (R^{16})$	$ebacd (R^{17})$	
$R^{19}$	$acebd$	$ecbad$	$acebd$	$acebd (R^3)$	$caebd$	$ceabd$	$cebad (R^{18})$	$ecabd (R^8)$
$R^{20}$	$caebd$	$ecbad$	$caebd$	$ceabd$	$cebad (R^{19})$	$ecabd (R^{19})$	$ecbad (R^{19})$	$ecbad (R^{18})$
$R^{21}$	$ceabd$	$eabcd$	$ceabd (R^1)$	$eabcd (R^+)$	$eacbd$	$ecabd (R^1)$		
$R^{22}$	$caebd$	$ebcad$	$caebd$	$ceabd$	$cebad (R^{20})$	$ebcad (R^{20})$	$ecabd (R^{20})$	$ecbad (R^{20})$
$R^{23}$	$ceabd$	$ebacd$	$ceabd (R^1)$	$cebad (R^1)$	$eabed (R^{21})$	$eacbd$	$ebacd (R^{18})$	$ebcad (R^{18})$
$R^{24}$	$ceabd$	$eabcd$	$ceabd$	$cebad$	$ebcad (R^{23})$	$ecabd$	$ecbad (R^{22})$	
$R^{25}$	$acebd$	$eabdc$	$acebd$	$aebcd (R^3)$	$aebcd (R^3)$	$ecabd (R^3)$	$eabdc (R^{17})$	$ecabd (R^{16})$
$R^{26}$	$ceabd$	$eabdc$	$ceabd (R^1)$	$eabed (R^{21})$	$eabdc (R^{25})$	$ecabd$	$ecabd (R^1)$	
$R^{27}$	$eabdc$	$eacbd$	$eabcd (R^9)$	$eabdc (R^{26})$	$eacbd$			
$R^{28}$	$ceabd$	$eabcd$	$ceabd (R^1)$	$cebad (R^1)$	$eabed (R^{21})$	$ecabd$	$ebacd (R^{18})$	$ecabd (R^1)$
$R^{29}$	$eacbd$	$ebacd$	$eabcd (R^{27})$	$eacbd$	$ebacd (R^{28})$			
$R^{30}$	$abecd$	$ecabd$	$abecd$	$aebcd$	$aebcd (R^2)$	$eabed (R^2)$	$ecabd (R^2)$	
$R^{31}$	$abecd$	$eacbd$	$abecd$	$aebcd$	$aebcd (R^{30})$	$ebcad (R^{30})$	$ecabd (R^{30})$	
$R^{32}$	$beacd$	$eacbd$	$beacd (R^{29})$	$eabcd (R^{27})$	$eacbd$	$ebacd (R^{29})$		
$R^{33}$	$beadc$	$eacbd$	$beadc (R^{29})$	$beadc (R^{32})$	$eabed (R^{10})$	$ebacd (R^{27})$	$eacbd$	$ebacd (R^{29})$
$R^{34}$	$abedc$	$eacbd$	$abedc (R^{32})$	$abedc (R^{33})$	$aebcd$	$aebcd (R^{27})$	$aebcd (R^{31})$	$eabdc (R^{27})$
$R^{35}$	$abedc$	$acebd$	$abedc (R^{34})$	$abedc (R^{34})$	$aebcd$	$aebcd (R^{34})$	$aebcd (R^{11})$	
$R^{36}$	$abcde$	$acebd$	$abcde$	$abced (R^*)$	$abecd$	$acbed (R^*)$	$acebd (R^*)$	$aebcd (R^4)$
$R^{37}$	$abdec$	$aecbd$	$abdec (R^{35})$	$abecd (R^{35})$	$abedc (R^{35})$	$aebcd (R^{36})$	$aebcd (R^{35})$	$aebcd (R^{11})$

**Step 4:** Fourthly, we will show that  $f(R) = abecd$  for the profile  $R$  where one voter reports  $acdbe$  and the other reports  $aebcd$ . In more detail, the following derivation shows that  $f(R) \neq abecd$  results in a contradiction. The profile  $R$  is called  $R^9$  in the following table.

**Step 5:** In our fifth step, we will prove that  $f(R) = acbed$  for the profile  $R$  where one voter reports  $acbed$  and the other reports  $eacbd$ . To this end, we assume that  $f(R) \neq acbed$  and derive a contradiction. Our assumption that  $f(R) \neq acbed$  appears at profile  $R^{12}$  and  $R^{27}$ .

$R^{30}$	$acbed$	$ecabd$	$acbed (R^{12})$	$acebd (R^{27})$	$aecbd$	$cabed (R^{27})$	$caebd$	$ceabd (R^{29})$	$eacbd (R^{12})$	$ecabd (R^2)$
$R^{31}$	$acebd$	$ecabd$	$acebd (R^{20})$	$aecbd$	$caebd$	$ceabd (R^{30})$	$eacbd (R^{20})$	$ecabd (R^2)$		
$R^{32}$	$caebd$	$ebcad$	$caebd (R^{24})$	$ceabd (R^{24})$	$ceabd (R^{24})$	$ebcad$	$ecabd (R^{24})$	$ecbad$		
$R^{33}$	$acebd$	$ecbad$	$acebd (R^{20})$	$aecbd$	$caebd$	$ceabd (R^{31})$	$ceabd (R^{31})$	$eacbd (R^{20})$	$ecabd (R^{18})$	$ecbad (R^{31})$
$R^{34}$	$caebd$	$ecbad$	$caebd (R^{32})$	$ceabd (R^{32})$	$ceabd (R^{32})$	$ebcad$	$ecabd (R^{33})$			
$R^{35}$	$caebd$	$ebcad$	$caebd (R^{24})$	$ceabd (R^{24})$	$ceabd (R^{24})$	$ebcad$	$ecabd (R^{24})$	$ecbad (R^{34})$		
$R^{36}$	$aecbd$	$ebcad$	$aecbd (R^{35})$	$acebd (R^5)$	$eabed (R^{26})$	$eacbd (R^{35})$	$ebacd$	$ebcad$	$ecabd (R^{24})$	$ecbad (R^{35})$
$R^{37}$	$aecbd$	$ebacd$	$aebcd (R^{36})$	$aecbd (R^5)$	$eabed (R^{25})$	$eacbd (R^{17})$	$ebacd$			
$R^{38}$	$acebd$	$ebacd$	$acebd (R^3)$	$aebcd (R^{37})$	$acebd (R^3)$	$eabed (R^{22})$	$eacbd (R^{20})$	$ebacd$		
$R^{39}$	$acebd$	$ecbad$	$acebd (R^{20})$	$acebd (R^{38})$	$caebd (R^{34})$	$ceabd (R^{31})$	$ceabd (R^{31})$	$eacbd (R^{20})$	$ecabd (R^{18})$	$ecbad (R^{31})$
$R^{40}$	$cebad$	$ecbad$	$cebad$		$ecbad (R^{39})$					
$R^{41}$	$eacbd$	$ecbad$	$eacbd (R^{39})$		$ecabd (R^1)$	$ecbad$				
$R^{42}$	$cebad$	$eacbd$	$cebad (R^1)$		$eacbd (R^{41})$	$ecabd (R^1)$	$ecbad (R^{40})$			

**Step 6:** Finally, we derive a contradiction by showing that the insights of our previous steps are incompatible with each other.

$R^+$	$eabcd$	$ecabd$	$eacbd (A)$							
$R^1$	$ceabd$	$eacbd$	$eacbd (A, Step 1)$							
$R^2$	$aebcd$	$ecabd$	$aebcd (A, Step 2)$							
$R^3$	$acebd$	$aebcd$	$aebcd (A, Step 3)$							
$R^4$	$acbde$	$aebcd$	$abecd (A, Step 4)$							
$R^5$	$acbed$	$eacbd$	$acbed (A, Step 5)$							
$R^6$	$aebcd$	$cabed$	$abced (R^4)$	$abecd$	$acbed (R^4)$	$acebd (R^4)$	$aebcd (R^4)$	$aecbd (R^3)$	$cabed (R^4)$	$caebd (R^3)$
$R^7$	$cabed$	$eacbd$	$acbed$	$acebd (R^5)$	$aecbd (R^5)$	$cabed (R^6)$	$caebd (R^1)$	$ceabd (R^5)$	$eacbd (R^5)$	$ecabd (R^5)$
$R^8$	$acbed$	$cabed$	$acbed$	$cabed (R^7)$						
$R^9$	$eacbd$	$ecbad$	$eacbd$	$ecabd (R^1)$	$ecbad$					
$R^{10}$	$acebd$	$eacbd$	$acebd$	$aecbd$	$ecabd (R^5)$					
$R^{11}$	$cbead$	$eacbd$	$cbead$	$ceabd (R^1)$	$cebad (R^1)$	$eacbd (R^5)$	$ecabd (R^1)$	$ecbad$		
$R^{12}$	$cebad$	$eacbd$	$ceabd (R^1)$	$cebad (R^1)$	$eacbd (R^{11})$	$ecabd (R^1)$	$ecbad$			
$R^{13}$	$cebad$	$ecbad$	$cebad (R^{12})$	$ecbad$						
$R^{14}$	$eacbd$	$ecbad$	$eacbd (R^{12})$	$ecabd (R^1)$	$ecbad$					
$R^{15}$	$aebcd$	$ecbad$	$aebcd$	$aecbd (R^3)$	$eabed (R^2)$	$eacbd (R^2)$	$ebacd$	$ebcad (R^2)$	$ecabd (R^2)$	$ecbad (R^2)$
$R^{16}$	$eabdc$	$ecbad$	$eabcd (R^{14})$	$eabdc$	$eacbd (R^{14})$	$ebacd$	$ebadc$	$ebcad (R^{15})$	$ecabd (R^{15})$	$ecbad (R^{15})$
$R^{17}$	$ebadc$	$ecbad$	$ebacd$	$ebadc$	$ebcad (R^{16})$	$ecbad (R^{16})$				
$R^{18}$	$ebacd$	$ecbad$	$ebacd$	$ebcad$	$ecbad (R^{17})$					
$R^{19}$	$aebcd$	$baecd$	$abecd$	$aebcd (R^4)$	$baecd$					
$R^{20}$	$caebd$	$ecbad$	$caebd$	$ceabd$	$cebad (R^{13})$	$ecabd (R^{14})$	$ecbad$			
$R^{21}$	$acbed$	$eacbd$	$acbed$	$acebd (R^5)$	$aecbd (R^5)$	$cabed (R^8)$	$caebd$	$ceabd (R^5)$	$eacbd (R^5)$	$ecabd (R^5)$
$R^{22}$	$acebd$	$ecabd$	$acebd$	$aecbd$	$caebd$	$ceabd (R^{21})$	$eacbd (R^{10})$	$ecabd (R^2)$		
$R^{23}$	$acebd$	$ecbad$	$acebd$	$aecbd (R^{14})$	$caebd$	$ceabd (R^{22})$	$cebad (R^{13})$	$eacbd (R^{10})$	$ecabd (R^9)$	$ecbad (R^{22})$
$R^{24}$	$caebd$	$ecbad$	$caebd$	$ceabd$	$cebad (R^{13})$	$ecabd (R^{14})$	$ecbad (R^{23})$			
$R^{25}$	$eacbd$	$ecbad$	$eacbd (R^{12})$	$eacbd (R^{12})$	$ebacd$	$ebcad$	$ecabd (R^1)$	$ecbad$		
$R^{26}$	$caebd$	$ecbad$	$caebd$	$ceabd$	$cebad (R^{20})$	$ebcad (R^{24})$	$ecabd (R^{25})$	$ecbad (R^{24})$		
$R^{27}$	$ceabd$	$ebcad$	$ceabd$	$cebad$	$ebcad (R^{26})$	$ecabd (R^{25})$	$ecbad (R^{26})$			
$R^{28}$	$ceabd$	$eabcd$	$ceabd (R^1)$	$eabcd (R^+)$	$eacbd$	$ecabd (R^1)$				
$R^{29}$	$ceabd$	$ebacd$	$ceabd (R^1)$	$cebad (R^1)$	$eacbd (R^{28})$	$ecabd$	$ebacd (R^{27})$	$ebcad (R^{27})$	$ecabd (R^1)$	$ecbad (R^{18})$
$R^{30}$	$eacbd$	$ebacd$	$eabcd (R^{29})$	$eacbd$	$ebacd (R^{29})$					
$R^{31}$	$abecd$	$ecabd$	$abecd$	$aebcd$	$aecbd (R^2)$	$eabed (R^2)$	$eacbd (R^2)$	$ecabd (R^2)$		

$R^{32}$	$abecd$	$eacbd$	$abecd$	$aebcd$	$aecbd (R^{31})$	$eabcd (R^{31})$	$eacbd (R^5)$
$R^{33}$	$beacd$	$eacbd$	$beacd (R^{30})$	$eabcd (R^{30})$	$eacbd$	$ebacd (R^{30})$	
$R^{34}$	$baecd$	$eacbd$	$abecd (R^{33})$	$aebcd (R^{19})$	$aecbd (R^{32})$	$baecd (R^{33})$	$beacd (R^{30})$

This concludes the proof of this lemma.  $\square$

**Lemma 4.**  $f(R^+) \neq eabcd$ .

PROOF. As usual, we will assume for contradiction that  $f(R^+) = eabcde$ . However, in contrast to the previous two lemmas, we will derive a contradiction by means of a further case distinction. Specifically, we will consider the outcomes for the profile  $R^1$  shown below.

$$R^1: caebd \quad cebad$$

For this profile, only three ranings satisfy unanimity:  $f(R^1) = caebd$ ,  $f(R^1) = ceabd$ , and  $f(R^1) = cebad$ . We will show that none of these outcomes is compatible with our other assumptions on  $f$ .

**Case 1:** We first assume that  $f(R^1) = ceabd$ . In this case, we immediately can derive a contradiction as witnessed by the following deduction.

$R^*$	$abcde$	$acbed$	$abcde (A)$				
$R^+$	$eabcd$	$ecabd$	$eabcd (A)$				
$R^1$	$caebd$	$cebad$	$ceabd (A)$				
$R^2$	$ceabd$	$eabcd$	$ceabd (R^+)$	$eabcd$	$eacbd (R^+)$	$ecabd (R^+)$	
$R^3$	$ceabd$	$ebcad$	$ceabd (R^2)$	$cebad$	$ebcad$	$ecabd$	$ecbad$
$R^4$	$abcde$	$cabed$	$abcde$	$abced (R^*)$	$acbed (R^*)$	$acbed (R^*)$	$cabde$
$R^5$	$acbde$	$cabed$	$acbde$	$acbed$	$cabde$	$cabed (R^4)$	
$R^6$	$abcde$	$acebd$	$abcde$	$abced (R^*)$	$acbed (R^*)$	$acbed (R^*)$	$acebd (R^*)$
$R^7$	$acbde$	$acebd$	$acbde$	$acbed$	$acebd (R^6)$		
$R^8$	$caebd$	$ceabd$	$caebd (R^1)$	$ceabd$			
$R^9$	$caebd$	$ecabd$	$caebd (R^1)$	$ceabd$	$ecabd$		
$R^{10}$	$ceabd$	$eacbd$	$ceabd (R^2)$	$eacbd$	$ecabd$		
$R^{11}$	$cabed$	$ceabd$	$cabed$	$caebd (R^8)$	$ceabd$		
$R^{12}$	$ceabd$	$ecbad$	$ceabd (R^3)$	$cebad$	$ecabd$	$ecbad$	
$R^{13}$	$abced$	$ceabd$	$abced$	$acbed$	$cabed$	$caebd (R^8)$	$ceabd (R^2)$
$R^{14}$	$acbde$	$ceabd$	$acbde$	$acbed$	$cabde$	$cabed (R^5)$	$caebd (R^8)$
$R^{15}$	$cabde$	$ceabd$	$cabde$	$cabed$	$caebd (R^8)$	$ceabd (R^{14})$	
$R^{16}$	$cabed$	$ceabd$	$cabed$	$caebd (R^8)$	$ceabd (R^{15})$		
$R^{17}$	$cabed$	$ecabd$	$cabed$	$caebd (R^9)$	$ceabd (R^{16})$	$ecabd (R^{16})$	
$R^{18}$	$caebd$	$ecabd$	$caebd (R^1)$	$ceabd$	$ecabd (R^{17})$		
$R^{19}$	$caebd$	$ecbad$	$caebd (R^1)$	$ceabd (R^{12})$	$cebad (R^1)$	$ecabd$	$ecbad$
$R^{20}$	$caebd$	$ecbad$	$caebd (R^1)$	$ceabd (R^{12})$	$cebad (R^1)$	$ecabd (R^{18})$	$ecbad$
$R^{21}$	$cabed$	$ecbad$	$cabed (R^{19})$	$caebd (R^{11})$	$cbaed (R^{20})$	$cbead$	$ceabd (R^{12})$
$R^{22}$	$cabed$	$cbead$	$cabed (R^{21})$	$cbaed (R^{21})$	$cbead$	$ceabd (R^{16})$	$ecabd (R^{16})$
$R^{23}$	$acbed$	$cbead$	$acbed (R^{22})$	$cabed (R^{22})$	$cbead$	$ceabd (R^{17})$	$ecbad (R^{17})$
$R^{24}$	$acbed$	$cebad$	$acbed (R^{23})$	$acebd (R^{23})$	$cabed (R^{23})$	$caebd (R^1)$	$cbaed (R^{23})$
$R^{25}$	$acebd$	$cebad$	$acebd (R^{24})$	$caebd (R^1)$	$ceabd$	$cabed (R^1)$	
$R^{26}$	$acebd$	$ceabd$	$acebd (R^{25})$	$caebd (R^8)$	$ceabd$		
$R^{27}$	$acdeb$	$ceabd$	$acdeb$	$acebd (R^{26})$	$acedb (R^{26})$	$cadeb$	$caebd (R^8)$
$R^{28}$	$cadeb$	$ceabd$	$cadeb$	$caebd (R^8)$	$caedb$	$ceabd (R^{27})$	$ceadb$
$R^{29}$	$caedb$	$cebad$	$caedb (R^1)$	$caedb$	$ceabd$	$ceadb$	$cebad (R^1)$
$R^{30}$	$caebd$	$ceabd$	$caebd (R^8)$	$caedb$	$ceabd (R^{28})$	$ceadb$	
$R^{31}$	$cbead$	$ecbad$	$cbead$	$cebad (R^{20})$	$ecbad (R^{21})$		

$R^{32}$	<b>caedb</b>	<b>cebad</b>	caebd ( $R^1$ )	<b>caedb</b>	ceabd ( $R^{30}$ )	<b>ceadb</b>	cebad ( $R^1$ )			
$R^{33}$	<b>caedb</b>	<b>cebad</b>	cabed ( $R^{22}$ )	caebd ( $R^{22}$ )	<b>caedb</b>	cbaed ( $R^{22}$ )	cbread ( $R^{32}$ )	ceabd ( $R^{30}$ )	<b>ceadb</b>	cebad ( $R^{29}$ )
$R^{34}$	<b>cbead</b>	<b>ecadb</b>	cbead ( $R^{33}$ )	<b>ceadb</b>	<b>ceadb</b>	cebad ( $R^{31}$ )	ecabd ( $R^{31}$ )	<b>ecadb</b>	ecbad ( $R^{31}$ )	
$R^{35}$	<b>cbead</b>	<b>eacbd</b>	cbead ( $R^{34}$ )	ceabd ( $R^{10}$ )	cebad ( $R^{31}$ )	<b>eacbd</b>	ecabd ( $R^{31}$ )	ecbad ( $R^{31}$ )		
$R^{36}$	<b>eacbd</b>	<b>ecbad</b>	<b>eacbd</b>	ecabd ( $R^{20}$ )	ecbad ( $R^{35}$ )					
$R^{37}$	<b>acebd</b>	<b>cebad</b>	acebd	ceabd ( $R^1$ )	<b>ceabd</b>	cebad ( $R^1$ )				
$R^{38}$	<b>cabed</b>	<b>eacbd</b>	cabed ( $R^{35}$ )	<b>acebd</b>	aecbd ( $R^{17}$ )	cabed ( $R^{35}$ )	caebd ( $R^{11}$ )	ceabd ( $R^{10}$ )	eacbd ( $R^{17}$ )	ecabd ( $R^{16}$ )
$R^{39}$	<b>acebd</b>	<b>ecbad</b>	acebd ( $R^{19}$ )	aecbd ( $R^{20}$ )	caebd ( $R^{37}$ )	ceabd ( $R^{12}$ )	cebad ( $R^{37}$ )	<b>eacbd</b>	ecabd ( $R^{20}$ )	ecbad ( $R^{36}$ )
$R^{40}$	<b>acebd</b>	<b>eacbd</b>	acebd ( $R^{39}$ )	aecbd ( $R^{38}$ )	ecabd ( $R^{38}$ )					

**Case 2:** As the second case, we assume  $f(R^1) = \text{caebd}$ . In this case, we proceed with a third tier case distinction with respect to the profile  $R^2$  where one voter reports  $\text{caebd}$  and the other reports  $\text{ecabd}$ .

**Case 2.1:** First, we assume that  $f(R^2) \neq \text{caebd}$ . Then, our assumption become incompatible as shown by the following derivation.

$R^*$	<b>abcde</b>	<b>acbed</b>	abcde (A)							
$R^1$	<b>caebd</b>	<b>cebad</b>	caebd (A)							
$R^2$	<b>caebd</b>	<b>ecabd</b>	caebd (A)	ceabd ( $R^1$ )	<b>ecabd</b>					
$R^3$	<b>cabed</b>	<b>eacbd</b>	cabed ( $R^2$ )	caebd ( $R^2$ )	ceabd ( $R^2$ )	<b>ecabd</b>				
$R^4$	<b>cabde</b>	<b>ecabd</b>	cabde ( $R^3$ )	cabed ( $R^2$ )	caebd ( $R^2$ )	ceabd ( $R^2$ )	<b>ecabd</b>			
$R^5$	<b>cabde</b>	<b>ceabd</b>	cabde ( $R^4$ )	cabed ( $R^4$ )	<b>caebd</b>	ceabd				
$R^6$	<b>cabde</b>	<b>cebad</b>	cabde ( $R^4$ )	cabed ( $R^4$ )	<b>caebd</b>	cbade ( $R^4$ )	<b>cbaed</b>	cbead ( $R^1$ )	ceabd ( $R^1$ )	cebad ( $R^1$ )
$R^7$	<b>cabde</b>	<b>ceabd</b>	cabde ( $R^4$ )	cabed ( $R^4$ )	<b>caebd</b>	ceabd ( $R^6$ )				
$R^8$	<b>cabde</b>	<b>caebd</b>	cabde ( $R^5$ )	cabed ( $R^7$ )	<b>caebd</b>					
$R^9$	<b>caebd</b>	<b>cebd</b>	caebd	ceabd ( $R^1$ )	cebad ( $R^1$ )	cebda ( $R^1$ )				
$R^{10}$	<b>caebd</b>	<b>cbeda</b>	cabed ( $R^8$ )	<b>caebd</b>	<b>cbaed</b>	cbead ( $R^1$ )	cbeda ( $R^9$ )	ceabd ( $R^1$ )	cebad ( $R^1$ )	cebda ( $R^1$ )
$R^{11}$	<b>cabed</b>	<b>cbeda</b>	cabed	<b>cbaed</b>	cbead ( $R^{10}$ )	cbeda ( $R^{10}$ )				
$R^{12}$	<b>cbaed</b>	<b>cbeda</b>	<b>cbaed</b>	<b>cbead</b>	cebda ( $R^{11}$ )					
$R^{13}$	<b>caebd</b>	<b>cbaed</b>	cabed ( $R^8$ )	<b>caebd</b>	<b>cbaed</b>					
$R^{14}$	<b>abcde</b>	<b>acebd</b>	abcde	abced ( $R^*$ )	acbde ( $R^*$ )	acbed ( $R^*$ )	acebd ( $R^*$ )			
$R^{15}$	<b>abcde</b>	<b>caebd</b>	abcde	abced ( $R^*$ )	acbed ( $R^*$ )	acbed ( $R^*$ )	acebd ( $R^*$ )	<b>cabde</b>	cabed ( $R^*$ )	caebd ( $R^{14}$ )
$R^{16}$	<b>acbde</b>	<b>caebd</b>	acbde ( $R^8$ )	<b>acbed</b>	acebd ( $R^{15}$ )	cabde ( $R^8$ )	cabed ( $R^{15}$ )	caebd ( $R^{15}$ )		
$R^{17}$	<b>acbed</b>	<b>caebd</b>	acbed	acebd ( $R^{16}$ )	cabed ( $R^8$ )	caebd ( $R^{16}$ )				
$R^{18}$	<b>acbed</b>	<b>acedb</b>	acbed	acebd ( $R^{17}$ )	<b>acedb</b>					
$R^{19}$	<b>bcade</b>	<b>caebd</b>	<b>bcade</b>	<b>bcaed</b>	cabde ( $R^8$ )	cabed ( $R^{15}$ )	caebd ( $R^{15}$ )	<b>cbade</b>	( $R^8$ )	<b>cbaed</b>
$R^{20}$	<b>caebd</b>	<b>cbdae</b>	cabde ( $R^8$ )	cabed ( $R^8$ )	caebd ( $R^{19}$ )	cbade ( $R^8$ )	<b>cbaed</b>	<b>cbdae</b>		
$R^{21}$	<b>caebd</b>	<b>cbeda</b>	cabed ( $R^8$ )	<b>caebd</b>	<b>cbaed</b>	cbead ( $R^1$ )	cbeda ( $R^9$ )	ceabd ( $R^1$ )	cebad ( $R^1$ )	cebda ( $R^1$ )
$R^{22}$	<b>caebd</b>	<b>cbaed</b>	cabed ( $R^8$ )	caebd ( $R^{21}$ )	<b>cbaed</b>					
$R^{23}$	<b>cbaed</b>	<b>cbeda</b>	<b>cbaed</b>	cbead ( $R^{21}$ )	cebda ( $R^{11}$ )					
$R^{24}$	<b>caebd</b>	<b>cbaed</b>	cabed ( $R^{13}$ )	caebd ( $R^{22}$ )	caebd ( $R^{22}$ )	<b>cbaed</b>				
$R^{25}$	<b>cbaed</b>	<b>ecabd</b>	cabed ( $R^2$ )	caebd ( $R^2$ )	cbaed ( $R^3$ )	cbead ( $R^3$ )	ceabd ( $R^2$ )	<b>cebad</b>	<b>ecabd</b>	ecbad
$R^{26}$	<b>cbaed</b>	<b>ceadb</b>	cabed ( $R^{13}$ )	caebd ( $R^{22}$ )	caebd ( $R^{22}$ )	cbaed ( $R^{25}$ )	cbead ( $R^{23}$ )	ceabd ( $R^{22}$ )	ceadb ( $R^{24}$ )	<b>cebad</b>
$R^{27}$	<b>cbaed</b>	<b>cebad</b>	cabed ( $R^{26}$ )	cbead ( $R^{23}$ )	<b>cebad</b>					
$R^{28}$	<b>acbed</b>	<b>cebad</b>	acbed	acebd ( $R^{17}$ )	cabed ( $R^6$ )	caebd ( $R^{17}$ )	cbaed ( $R^{27}$ )	cebade ( $R^1$ )	ceabd ( $R^1$ )	cebad ( $R^1$ )
$R^{29}$	<b>acbed</b>	<b>cbaed</b>	acbed	cabed ( $R^{17}$ )	<b>cbaed</b>	cabed ( $R^{28}$ )				
$R^{30}$	<b>acedb</b>	<b>cbaed</b>	acbed	acebd ( $R^{18}$ )	acedb ( $R^{24}$ )	cabed ( $R^{13}$ )	caebd ( $R^{22}$ )	caebd ( $R^{22}$ )	cbaed ( $R^{29}$ )	
$R^{31}$	<b>acbed</b>	<b>acedb</b>	acbed	acebd ( $R^{17}$ )	acedb ( $R^{30}$ )					
$R^{32}$	<b>ceadb</b>	<b>cebad</b>	ceabd ( $R^{26}$ )	ceabd ( $R^{26}$ )	<b>cebad</b>					
$R^{33}$	<b>acbed</b>	<b>ceadb</b>	acbed	acebd ( $R^{17}$ )	acedb ( $R^{31}$ )	cabed ( $R^{17}$ )	caebd ( $R^{17}$ )	caebd ( $R^{17}$ )	ceadb ( $R^{28}$ )	

$R^{34}$	<b>cabed</b>	<b>ceadb</b>	<b>cabed</b> ( $R^3$ )	<b>caebd</b>	<b>caedb</b> ( $R^{33}$ )	<b>ceabd</b> ( $R^{33}$ )	<b>ceadb</b> ( $R^{33}$ )
$R^{35}$	<b>caebd</b>	<b>ceadb</b>	<b>caebd</b>	<b>caedb</b> ( $R^{34}$ )	<b>ceabd</b> ( $R^1$ )	<b>ceadb</b> ( $R^{34}$ )	
$R^{36}$	<b>caebd</b>	<b>cedba</b>	<b>caebd</b>	<b>caedb</b> ( $R^{35}$ )	<b>ceabd</b> ( $R^1$ )	<b>ceadb</b> ( $R^{35}$ )	<b>cebad</b> ( $R^1$ )
$R^{37}$	<b>caebd</b>	<b>cedba</b>	<b>cbaed</b> ( $R^{26}$ )	<b>cbead</b> ( $R^{23}$ )	<b>cbeda</b> ( $R^{12}$ )	<b>cebad</b>	<b>cebda</b> ( $R^{23}$ )
$R^{38}$	<b>cebad</b>	<b>cedba</b>	<b>cebad</b>	<b>cebda</b> ( $R^{37}$ )	<b>cedba</b> ( $R^{37}$ )		<b>cedba</b> ( $R^{36}$ )
$R^{39}$	<b>cdeba</b>	<b>cebad</b>	<b>cdeba</b> ( $R^{38}$ )	<b>cebad</b>	<b>cebda</b> ( $R^{38}$ )	<b>cedba</b> ( $R^{38}$ )	
$R^{40}$	<b>cdeab</b>	<b>cebad</b>	<b>cdeab</b> ( $R^{39}$ )	<b>cdeba</b> ( $R^{38}$ )	<b>ceabd</b> ( $R^{32}$ )	<b>ceabd</b> ( $R^{32}$ )	<b>cebad</b> ( $R^{38}$ )
$R^{41}$	<b>caebd</b>	<b>cebad</b>	<b>caebd</b> ( $R^{40}$ )	<b>caebd</b>	<b>caedb</b> ( $R^{32}$ )	<b>ceabd</b> ( $R^1$ )	<b>ceadb</b> ( $R^{32}$ )
$R^{42}$	<b>caebd</b>	<b>caebd</b>	<b>caebd</b> ( $R^{41}$ )	<b>caebd</b>	<b>caedb</b> ( $R^{35}$ )		<b>cebad</b> ( $R^1$ )
$R^{43}$	<b>cebad</b>	<b>dceba</b>	<b>cdeba</b> ( $R^{38}$ )	<b>cebad</b>	<b>cebda</b> ( $R^{38}$ )	<b>cedba</b> ( $R^{38}$ )	<b>dceba</b> ( $R^{39}$ )
$R^{44}$	<b>cebda</b>	<b>dceba</b>	<b>cdeba</b> ( $R^{43}$ )	<b>cebda</b>	<b>cedba</b>		<b>dceba</b> ( $R^{43}$ )
$R^{45}$	<b>cdeab</b>	<b>cebda</b>	<b>cdeab</b> ( $R^{40}$ )	<b>cdeba</b> ( $R^{40}$ )	<b>cebd</b>	<b>cedab</b> ( $R^{40}$ )	<b>cedba</b>
$R^{46}$	<b>caebd</b>	<b>cdaeb</b>	<b>caebd</b> ( $R^{42}$ )	<b>caebd</b>	<b>caedb</b> ( $R^{35}$ )	<b>cdaeb</b> ( $R^{36}$ )	
$R^{47}$	<b>cdaeb</b>	<b>cebda</b>	<b>cdaeb</b> ( $R^{46}$ )	<b>cdeab</b> ( $R^{45}$ )	<b>cdeba</b> ( $R^{44}$ )	<b>cebda</b> ( $R^9$ )	<b>cedab</b> ( $R^{45}$ )
$R^{48}$	<b>cdaeb</b>	<b>cebda</b>	<b>cdaeb</b> ( $R^{47}$ )	<b>cdeab</b> ( $R^{47}$ )	<b>cedba</b> ( $R^{47}$ )	<b>cedab</b> ( $R^{47}$ )	<b>cedba</b>
$R^{49}$	<b>caebd</b>	<b>cedba</b>	<b>caebd</b>	<b>caedb</b>	<b>ceabd</b> ( $R^1$ )	<b>ceabd</b>	<b>cebad</b> ( $R^1$ )
$R^{50}$	<b>caebd</b>	<b>cedba</b>	<b>caebd</b> ( $R^{48}$ )	<b>caedb</b> ( $R^{48}$ )	<b>cdaeb</b> ( $R^{48}$ )	<b>cdeab</b> ( $R^{48}$ )	<b>ceadb</b>
$R^{51}$	<b>caebd</b>	<b>ceadb</b>	<b>caebd</b> ( $R^{50}$ )	<b>caebd</b> ( $R^{50}$ )	<b>ceadb</b>		
$R^{52}$	<b>cadbe</b>	<b>ceadb</b>	<b>cadbe</b> ( $R^{51}$ )	<b>caebd</b> ( $R^{51}$ )	<b>ceadb</b>		
$R^{53}$	<b>acdbe</b>	<b>ceadb</b>	<b>acdbe</b> ( $R^{52}$ )	<b>acdeb</b> ( $R^{51}$ )	<b>acedb</b>	<b>cadbe</b> ( $R^{51}$ )	<b>caedb</b> ( $R^{33}$ )
$R^{54}$	<b>acdbe</b>	<b>acedb</b>	<b>acdbe</b> ( $R^{53}$ )	<b>acdeb</b> ( $R^{53}$ )	<b>acedb</b>		
$R^{55}$	<b>abcde</b>	<b>acedb</b>	<b>abcde</b>	<b>abced</b> ( $R^*$ )	<b>acbde</b> ( $R^*$ )	<b>acbed</b> ( $R^*$ )	<b>acdbe</b>
$R^{56}$	<b>acedb</b>	<b>adbe</b>	<b>adbe</b> ( $R^{54}$ )	<b>acdeb</b> ( $R^{55}$ )	<b>acebd</b> ( $R^{55}$ )	<b>adbce</b>	<b>adcbe</b> ( $R^{54}$ )
$R^{57}$	<b>acedb</b>	<b>adcb</b>	<b>adbe</b> ( $R^{54}$ )	<b>acdeb</b> ( $R^{54}$ )	<b>acedb</b> ( $R^{56}$ )	<b>adcbe</b> ( $R^{54}$ )	<b>adceb</b>
$R^{58}$	<b>acedb</b>	<b>adceb</b>	<b>acdeb</b> ( $R^{54}$ )	<b>acedb</b> ( $R^{57}$ )	<b>adceb</b>		
$R^{59}$	<b>caebd</b>	<b>dcaeb</b>	<b>caebd</b> ( $R^{42}$ )	<b>caebd</b>	<b>caedb</b> ( $R^{35}$ )	<b>cdaeb</b> ( $R^{36}$ )	<b>dcaeb</b> ( $R^{46}$ )
$R^{60}$	<b>adceb</b>	<b>ceadb</b>	<b>acdeb</b> ( $R^{51}$ )	<b>acedb</b> ( $R^{58}$ )	<b>adceb</b>	<b>caebd</b> ( $R^{51}$ )	<b>ceadb</b> ( $R^{53}$ )
$R^{61}$	<b>adceb</b>	<b>caebd</b>	<b>acdeb</b> ( $R^{42}$ )	<b>acebd</b> ( $R^{58}$ )	<b>acedb</b> ( $R^{58}$ )	<b>adceb</b> ( $R^{59}$ )	<b>caebd</b> ( $R^{42}$ )
						<b>caebd</b> ( $R^{60}$ )	<b>caedb</b> ( $R^{35}$ )

**Case 2.2:** As our second subcase, we suppose that  $f(R^2) = \text{caebd}$ . We again derive a contradiction for this case, as shown in the followign. By combining our two subcases, it follows that  $f(R^1) = \text{caebd}$  must be wrong.

$R^*$	<b>abcde</b>	<b>acbed</b>	<b>abcde</b> (A)
$R^+$	<b>eabcd</b>	<b>ecabd</b>	<b>eabcd</b> (A)
$R^1$	<b>caebd</b>	<b>ceadb</b>	<b>caebd</b> (A)
$R^2$	<b>caebd</b>	<b>ecabd</b>	<b>caebd</b> (A)
$R^3$	<b>ceabd</b>	<b>eabcd</b>	<b>ceabd</b> ( $R^+$ )
$R^4$	<b>cebad</b>	<b>eabcd</b>	<b>ceabd</b> ( $R^+$ )
$R^5$	<b>cebad</b>	<b>ebacd</b>	<b>cebad</b> ( $R^4$ )
$R^6$	<b>cebad</b>	<b>ebcad</b>	<b>cebad</b> ( $R^5$ )
$R^7$	<b>caebd</b>	<b>cebda</b>	<b>ceabd</b> ( $R^2$ )
$R^8$	<b>caebd</b>	<b>ecbda</b>	<b>ceabd</b> ( $R^2$ )
$R^9$	<b>caebd</b>	<b>ebcad</b>	<b>ceabd</b> ( $R^2$ )
$R^{10}$	<b>ceabd</b>	<b>ebcad</b>	<b>ceabd</b> ( $R^3$ )
$R^{11}$	<b>ceabd</b>	<b>ecabd</b>	<b>ceabd</b> ( $R^{10}$ )
$R^{12}$	<b>caebd</b>	<b>cbeda</b>	<b>cabed</b>
$R^{13}$	<b>cabed</b>	<b>cbeda</b>	<b>cbaed</b>
$R^{14}$	<b>cbaed</b>	<b>cbeda</b>	<b>cbead</b>

$R^{15}$	$ebcad$	$ecabd$	$ebcad (R^{10})$	$ecabd$	$ecbad (R^{10})$					
$R^{16}$	$caebd$	$eabcd$	$acebd (R^3)$	$aebcd (R^9)$	$aecbd$	$caebd (R^3)$	$ceabd (R^2)$	$eabcd (R^9)$	$eacbd (R^2)$	$ecabd (R^2)$
$R^{17}$	$aecbd$	$caebd$	$acebd (R^{16})$	$aecbd$	$caebd (R^{16})$					
$R^{18}$	$cebad$	$ecabd$	$ceabd (R^{11})$	$cebad$	$ecabd$	$ecbad (R^{15})$				
$R^{19}$	$aecbd$	$cebad$	$acebd (R^{17})$	$aecbd$	$caebd (R^{17})$	$ceabd (R^1)$	$cebad (R^1)$	$eacbd$	$ecabd$	$ecbad (R^1)$
$R^{20}$	$cebad$	$eacbd$	$ceabd (R^{18})$	$cebad (R^{19})$	$eacbd$	$ecabd$	$ecbad (R^{18})$			
$R^{21}$	$cebad$	$ecabd$	$ceabd (R^{11})$	$cebad (R^{20})$	$ecabd$	$ecbad (R^{15})$				
$R^{22}$	$cebad$	$ecabd$	$cebad (R^{21})$	$ceabd (R^{11})$	$cebad (R^{21})$	$ecabd$	$ecbad (R^{15})$			
$R^{23}$	$cbaed$	$ecabd$	$cabed (R^{22})$	$caebd$	$cbaed (R^{22})$	$cbead (R^{21})$	$ceabd (R^2)$	$cebad (R^{21})$	$ecabd (R^2)$	$ecbad (R^2)$
$R^{24}$	$caebd$	$cbaed$	$cabed (R^{23})$	$caebd$	$cbaed (R^{23})$					
$R^{25}$	$cabde$	$ecabd$	$cabde$	$cabed$	$caebd$	$ceabd (R^2)$	$ecabd (R^2)$			
$R^{26}$	$abcde$	$acebd$	$abcde$	$abced (R^*)$	$acbde (R^*)$	$acbed (R^*)$	$acebd (R^*)$			
$R^{27}$	$abcde$	$caebd$	$abcde$	$abced (R^*)$	$acbbe (R^*)$	$acbed (R^*)$	$acebd (R^*)$	$cabde$	$cabed (R^*)$	$caebd (R^{26})$
$R^{28}$	$bcade$	$caebd$	$bcade$	$bcaed (R^9)$	$cabde$	$cabed (R^{27})$	$caebd (R^{27})$	$cbade (R^{24})$	$cbaed (R^{24})$	
$R^{29}$	$caebd$	$cbeda$	$cabed (R^{24})$	$caebd$	$cbaed (R^{24})$	$cbead (R^1)$	$cbeda (R^7)$	$ceabd (R^2)$	$cebad (R^1)$	$cebda (R^1)$
$R^{30}$	$caebd$	$cbdae$	$cabde$	$cabed (R^{24})$	$caebd (R^{28})$	$cbade (R^{24})$	$caebd (R^{24})$	$cbdae (R^{29})$		
$R^{31}$	$cabde$	$caebd$	$cabde$	$cabed (R^{24})$	$caebd (R^{30})$					
$R^{32}$	$cabde$	$ecabd$	$cabde$	$cabed (R^{22})$	$caebd$	$ceabd (R^2)$	$ecabd (R^2)$			
$R^{33}$	$cabde$	$ceabd$	$cabde$	$cabed (R^{32})$	$caebd (R^{31})$	$ceabd (R^{25})$				
$R^{34}$	$cbeda$	$ceabd$	$cbead (R^{29})$	$cbeda (R^{29})$	$ceabd$	$cebad$	$cebda (R^{29})$			
$R^{35}$	$cbdea$	$ceabd$	$cbdea (R^{34})$	$cbead (R^{34})$	$cbeda (R^{34})$	$ceabd (R^{33})$	$cebad$	$cebda (R^{34})$		
$R^{36}$	$cbdea$	$ecbda$	$cbdea (R^{35})$	$cbeda$	$cebda$	$ecbda$				
$R^{37}$	$cbdea$	$cebad$	$cbdea (R^{35})$	$cbead (R^{35})$	$cbeda (R^{35})$	$cebad$	$cebda (R^{35})$			
$R^{38}$	$cbaed$	$cebad$	$cbaed$	$cbead (R^{37})$	$cebad$					
$R^{39}$	$cabde$	$cebad$	$cabde$	$cabed (R^{32})$	$caebd (R^{31})$	$cbade (R^{37})$	$cbaed$	$cbead (R^1)$	$ceabd (R^1)$	$cebad (R^1)$
$R^{40}$	$cbdae$	$cebad$	$cbade (R^{37})$	$cbaed$	$cbdae (R^{37})$	$cbdea (R^{37})$	$cbead (R^{37})$	$cbeda (R^{37})$	$cebad (R^{39})$	$cebda (R^{37})$
$R^{41}$	$cbaed$	$cebad$	$cbaed$	$cbead (R^{37})$	$cebad (R^{40})$					
$R^{42}$	$cbaed$	$ecbda$	$cbaed$	$cbead (R^{38})$	$cbeda (R^{14})$	$cebad (R^{41})$	$cebda (R^8)$	$ecbad (R^8)$	$ecbda (R^8)$	
$R^{43}$	$cbdae$	$ecbda$	$cbdae$	$cbdea (R^{36})$	$cbeda$	$cebda (R^{40})$	$ecbda (R^{42})$			
$R^{44}$	$cbdea$	$ecbda$	$cbdea (R^{35})$	$cbeda$	$cebda (R^{43})$	$ecbda (R^{43})$				
$R^{45}$	$cbeda$	$ecbda$	$cbeda$	$cebda (R^{44})$	$ecbda (R^{44})$					
$R^{46}$	$cbeda$	$ecabd$	$cbead (R^{21})$	$cbeda (R^{22})$	$ceabd (R^{11})$	$cebad (R^{21})$	$cebda (R^{21})$	$ecabd$	$ecbad (R^{15})$	$ecbda$
$R^{47}$	$cbeda$	$ecdab$	$cbeda (R^{46})$	$cebda (R^{46})$	$cedab$	$cedba$	$ecbda (R^{45})$	$ecdab$	$ecdba (R^{45})$	
$R^{48}$	$cbeda$	$ecdba$	$cbeda (R^{47})$	$cebda (R^{45})$	$cedba$	$ecbda (R^{45})$	$ecdba (R^{45})$			
$R^{49}$	$cbeda$	$cedba$	$cbeda (R^{48})$	$cebda (R^{45})$	$cedba$					
$R^{50}$	$cebda$	$cedba$	$cebda (R^{49})$	$cedba$						
$R^{51}$	$bcead$	$cebad$	$bcead$	$cbead (R^{37})$	$cebad$					
$R^{52}$	$cebad$	$cedba$	$cebad$	$cebda (R^{37})$	$cedba$					
$R^{53}$	$cbeda$	$cedba$	$cbeda$	$cebda (R^{45})$	$cedba$					
$R^{54}$	$bceda$	$cedba$	$bceda (R^{49})$	$cbeda (R^{49})$	$cebda (R^{53})$	$cedba$				
$R^{55}$	$cbaed$	$cedba$	$cbaed$	$cbead (R^{38})$	$cbeda (R^{14})$	$cebad (R^{41})$	$cebda (R^{52})$	$cedba (R^{42})$		
$R^{56}$	$bcead$	$cedba$	$bcead (R^{54})$	$bceda (R^{49})$	$cbead (R^{51})$	$cbeda (R^{49})$	$cebad$	$cebda (R^{52})$	$cedba (R^{55})$	
$R^{57}$	$cebad$	$cedba$	$cebad$	$cebda (R^{37})$	$cedba (R^{56})$					
$R^{58}$	$cebad$	$dceba$	$cdeba (R^{37})$	$cebad$	$cebda (R^{37})$	$cedba$	$dceba$			
$R^{59}$	$cdeba$	$cebad$	$cdeba (R^{37})$	$cebad$	$cebda (R^{37})$	$cedba$				
$R^{60}$	$bcead$	$cebad$	$bcead (R^{56})$	$cbead (R^{37})$	$cebad$					
$R^{61}$	$cdeba$	$cebad$	$cdeba (R^{37})$	$cebad$	$cebda (R^{37})$	$cedba$	$cebda (R^{57})$			

$R^{62}$	<i>cebad</i>	<i>dceba</i>	<i>cdeba</i> ( $R^{37}$ )	<i>cebad</i>	<i>cebda</i> ( $R^{37}$ )	<i>cedba</i> ( $R^{57}$ )	<i>dceba</i> ( $R^{61}$ )				
$R^{63}$	<i>ceba</i>	<i>dceba</i>	<i>cdeba</i> ( $R^{58}$ )	<i>ceba</i> ( $R^{50}$ )	<i>cedba</i>		<i>dceba</i> ( $R^{62}$ )				
$R^{64}$	<i>cedba</i>	<i>dceba</i>	<i>cdeba</i> ( $R^{63}$ )	<i>cedba</i>		<i>dceba</i> ( $R^{63}$ )					
$R^{65}$	<i>cedba</i>	<i>ebcad</i>	<i>cebad</i> ( $R^6$ )	<i>cebda</i> ( $R^{52}$ )	<i>cedba</i> ( $R^{56}$ )	<i>ebcad</i>		<i>ebcda</i> ( $R^{54}$ )	<i>ecbad</i>		<i>ecbda</i>
$R^{66}$	<i>cedba</i>	<i>ebcda</i>	<i>cebda</i> ( $R^{49}$ )	<i>cedba</i> ( $R^{65}$ )	<i>ebcda</i> ( $R^{54}$ )	<i>ecbda</i>		<i>ecbda</i>			<i>ecbda</i> ( $R^{56}$ )
$R^{67}$	<i>cedba</i>	<i>ecbda</i>	<i>cebda</i> ( $R^{49}$ )	<i>cedba</i> ( $R^{66}$ )	<i>ecbda</i>	<i>ecbda</i>					
$R^{68}$	<i>cdeba</i>	<i>ecbda</i>	<i>cdeba</i> ( $R^{59}$ )	<i>cebad</i>	<i>cebda</i> ( $R^{67}$ )	<i>ecbda</i>		<i>ecbda</i>			
$R^{69}$	<i>bcdea</i>	<i>cebad</i>	<i>bcdea</i> ( $R^{37}$ )	<i>bcead</i> ( $R^{60}$ )	<i>bceda</i> ( $R^{37}$ )	<i>cbdea</i> ( $R^{37}$ )	<i>cbead</i> ( $R^{37}$ )	<i>cbeda</i> ( $R^{37}$ )	<i>cebad</i>		<i>cebda</i> ( $R^{37}$ )
$R^{70}$	<i>bcdea</i>	<i>cebd</i>	<i>bcdea</i> ( $R^{69}$ )	<i>bceda</i> ( $R^{69}$ )	<i>cbdea</i> ( $R^{69}$ )	<i>cbeda</i>		<i>cebda</i>			
$R^{71}$	<i>cbdae</i>	<i>cedba</i>	<i>cbdae</i>	<i>cbdea</i> ( $R^{43}$ )	<i>cbeda</i> ( $R^{49}$ )	<i>cdbae</i>		<i>cdbea</i>	<i>cdeba</i> ( $R^{55}$ )	<i>cebda</i> ( $R^{40}$ )	<i>cedba</i> ( $R^{55}$ )
$R^{72}$	<i>bcdea</i>	<i>cedba</i>	<i>bcdea</i> ( $R^{54}$ )	<i>bceda</i> ( $R^{49}$ )	<i>cbdea</i> ( $R^{49}$ )	<i>cbeda</i> ( $R^{49}$ )	<i>cdbea</i>		<i>cdeba</i> ( $R^{64}$ )	<i>cebda</i> ( $R^{53}$ )	<i>cedba</i> ( $R^{71}$ )
$R^{73}$	<i>bcdea</i>	<i>cebd</i>	<i>bcdea</i> ( $R^{69}$ )	<i>bceda</i> ( $R^{69}$ )	<i>cbdea</i> ( $R^{69}$ )	<i>cbeda</i>		<i>cebda</i> ( $R^{72}$ )			
$R^{74}$	<i>bdcea</i>	<i>cebd</i>	<i>bcdea</i> ( $R^{70}$ )	<i>bceda</i> ( $R^{70}$ )	<i>bdcea</i> ( $R^{70}$ )	<i>cbdea</i> ( $R^{70}$ )	<i>cbeda</i>		<i>cebda</i>		
$R^{75}$	<i>bdcea</i>	<i>cebd</i>	<i>bcdea</i> ( $R^{70}$ )	<i>bceda</i> ( $R^{70}$ )	<i>bdcea</i> ( $R^{70}$ )	<i>cbdea</i> ( $R^{70}$ )	<i>cbeda</i>		<i>cebda</i> ( $R^{73}$ )		
$R^{76}$	<i>bdcea</i>	<i>cbeda</i>	<i>bcdea</i> ( $R^{74}$ )	<i>bceda</i> ( $R^{75}$ )	<i>bdcea</i> ( $R^{74}$ )	<i>cbdea</i> ( $R^{75}$ )	<i>cbeda</i>				
$R^{77}$	<i>cedba</i>	<i>dbcea</i>	<i>cdbea</i>	<i>cdeba</i> ( $R^{64}$ )	<i>cedba</i> ( $R^{72}$ )	<i>dbcea</i>		<i>dcbea</i> ( $R^{64}$ )	<i>dceba</i> ( $R^{64}$ )		
$R^{78}$	<i>cbeda</i>	<i>dbcea</i>	<i>bcdea</i> ( $R^{76}$ )	<i>bceda</i> ( $R^{76}$ )	<i>bdcea</i> ( $R^{76}$ )	<i>cbdea</i> ( $R^{76}$ )	<i>cbeda</i>		<i>cdbea</i>	<i>dbcea</i> ( $R^{76}$ )	<i>dcbea</i> ( $R^{76}$ )
$R^{79}$	<i>cedba</i>	<i>dbcea</i>	<i>cdbea</i>	<i>cdeba</i> ( $R^{64}$ )	<i>cedba</i> ( $R^{72}$ )	<i>dbcea</i> ( $R^{78}$ )	<i>dcbea</i> ( $R^{64}$ )	<i>dceba</i> ( $R^{64}$ )			
$R^{80}$	<i>cdbea</i>	<i>dbcea</i>	<i>cdbea</i>		<i>dbcea</i> ( $R^{79}$ )	<i>dcbea</i> ( $R^{77}$ )					
$R^{81}$	<i>bcdea</i>	<i>cdbea</i>	<i>bcdea</i> ( $R^{72}$ )	<i>cbdea</i> ( $R^{72}$ )	<i>cdbea</i>						
$R^{82}$	<i>cdbea</i>	<i>dbeca</i>	<i>cdbea</i>		<i>dbcea</i> ( $R^{80}$ )	<i>dbeca</i> ( $R^{80}$ )	<i>dcbea</i> ( $R^{80}$ )				
$R^{83}$	<i>cdbea</i>	<i>ecbda</i>	<i>cbdea</i> ( $R^{36}$ )	<i>cbeda</i>		<i>cdbea</i> ( $R^{68}$ )	<i>cdeba</i> ( $R^{68}$ )	<i>cebda</i> ( $R^{43}$ )	<i>cedba</i> ( $R^{67}$ )	<i>ecbda</i> ( $R^{43}$ )	<i>ecbda</i> ( $R^{43}$ )
$R^{84}$	<i>bdeca</i>	<i>cdbea</i>	<i>bdeca</i> ( $R^{81}$ )	<i>bdcea</i> ( $R^{81}$ )	<i>bdeca</i> ( $R^{82}$ )	<i>cbdea</i> ( $R^{81}$ )	<i>cdbea</i>		<i>dbcea</i> ( $R^{80}$ )	<i>dbeca</i> ( $R^{80}$ )	<i>dcbea</i> ( $R^{80}$ )
$R^{85}$	<i>becda</i>	<i>cdbea</i>	<i>bcdea</i> ( $R^{81}$ )	<i>bceda</i> ( $R^{81}$ )	<i>becda</i> ( $R^{84}$ )	<i>cbdea</i> ( $R^{81}$ )	<i>cbeda</i>		<i>cdbea</i> ( $R^{83}$ )		
$R^{86}$	<i>becda</i>	<i>cbeda</i>	<i>bcdea</i> ( $R^{76}$ )	<i>becda</i> ( $R^{85}$ )	<i>cbeda</i>						
$R^{87}$	<i>bedca</i>	<i>cbeda</i>	<i>bcdea</i> ( $R^{76}$ )	<i>becda</i> ( $R^{86}$ )	<i>bedca</i> ( $R^{76}$ )	<i>cbeda</i>					
$R^{88}$	<i>bedac</i>	<i>cbeda</i>	<i>bcdea</i> ( $R^{76}$ )	<i>becda</i> ( $R^{86}$ )	<i>bedac</i> ( $R^{87}$ )	<i>bedca</i> ( $R^{76}$ )	<i>cbeda</i>				
$R^{89}$	<i>beacd</i>	<i>cbeda</i>	<i>bcead</i>		<i>bceda</i> ( $R^{76}$ )	<i>beacd</i> ( $R^{88}$ )	<i>becad</i> ( $R^{86}$ )	<i>becda</i> ( $R^{86}$ )	<i>cbead</i>		<i>cbeda</i>
$R^{90}$	<i>beacd</i>	<i>cbead</i>	<i>bcead</i>		<i>beacd</i> ( $R^{89}$ )	<i>becad</i> ( $R^{89}$ )	<i>cbead</i>				
$R^{91}$	<i>cbaed</i>	<i>ebcad</i>	<i>bcaed</i>		<i>bcead</i>	<i>becad</i> ( $R^9$ )	<i>cbaed</i>		<i>cbead</i> ( $R^{38}$ )	<i>cebad</i> ( $R^6$ )	<i>ebcad</i> ( $R^9$ )
$R^{92}$	<i>cbead</i>	<i>ebcad</i>	<i>bcead</i>		<i>becad</i>	<i>cbead</i> ( $R^{22}$ )	<i>cebad</i> ( $R^6$ )		<i>ebcad</i> ( $R^{91}$ )	<i>ecbad</i>	
$R^{93}$	<i>cbead</i>	<i>ebacd</i>	<i>bcead</i>		<i>beacd</i> ( $R^{90}$ )	<i>becad</i> ( $R^{90}$ )	<i>cbead</i> ( $R^5$ )	<i>cebad</i> ( $R^5$ )	<i>ebacd</i> ( $R^{90}$ )	<i>ebcad</i> ( $R^{92}$ )	<i>ecbad</i> ( $R^5$ )
$R^{94}$	<i>bcead</i>	<i>ebacd</i>	<i>bcead</i>		<i>beacd</i> ( $R^{93}$ )	<i>becad</i> ( $R^{93}$ )	<i>ebacd</i> ( $R^{93}$ )	<i>ebcad</i> ( $R^{93}$ )			
$R^{95}$	<i>bcead</i>	<i>ecabd</i>	<i>bcead</i> ( $R^{22}$ )	<i>becad</i> ( $R^{15}$ )	<i>cbead</i> ( $R^{21}$ )	<i>ceabd</i> ( $R^{11}$ )	<i>cebad</i> ( $R^{21}$ )	<i>ebcad</i> ( $R^{15}$ )	<i>ecabd</i>		<i>ecbad</i> ( $R^{15}$ )
$R^{96}$	<i>bcead</i>	<i>eabcd</i>	<i>bcead</i> ( $R^{95}$ )	<i>beacd</i> ( $R^{95}$ )	<i>ebacd</i> ( $R^{94}$ )	<i>ebacd</i> ( $R^{94}$ )	<i>ebcad</i> ( $R^{94}$ )	<i>ebcad</i> ( $R^{94}$ )			

**Case 3:** Lastly, we suppose that  $f(R^1) = \text{cebad}$ . To derive a contradiction in this case, we will use a further case distinction with respect to the profile  $R^2$  where one voter reports *aecbd* and the other reports *cabed*. It can be checked that there are 5 possible outcomes for this profile  $f(R^2) \in \{\text{acbed}, \text{acebd}, \text{aecbd}, \text{cabed}, \text{caebd}\}$ . We will subsequently show that none of these outcomes is feasible by considering four separate cases.

**Case 3.1:** As the first case, we suppose that  $f(R^2) \notin \{\text{acbed}, \text{acebd}, \text{aecbd}\}$ . In this case, the following derivation shows that our assumptions are incompatible. Note that the profile  $R^2$  appears again at  $R^{11}$  as we can conclude that  $f(R^2) = f(R^{11}) = \text{cabed}$  at this point.

$R^*$	<i>abcde</i>	<i>acbed</i>	<i>abcde</i> (A)								
$R^+$	<i>eabcd</i>	<i>ecabd</i>	<i>eabcd</i> (A)								
$R^1$	<i>caebd</i>	<i>cebad</i>	<i>cebad</i> (A)								
$R^2$	<i>aecbd</i>	<i>cabed</i>	<i>acbed</i> (A)	<i>acebd</i> (A)	<i>aecbd</i> (A)	<i>cabed</i>		<i>caebd</i>			
$R^3$	<i>aecbd</i>	<i>caebd</i>	<i>acebd</i>		<i>acebd</i> ( $R^2$ )	<i>caebd</i>					
$R^4$	<i>acebd</i>	<i>cebad</i>	<i>acebd</i> ( $R^1$ )	<i>caebd</i> ( $R^1$ )	<i>ceabd</i> ( $R^1$ )	<i>cebad</i>					

$R^5$	$acebd$	$ceabd$	$acebd (R^4)$	$caebd$	$ceabd$						
$R^6$	$ceabd$	$eabcd$	$ceabd (R^+)$	$eabcd$	$eacbd (R^+)$	$ecabd (R^+)$					
$R^7$	$caebd$	$eabcd$	$acebd (R^6)$	$aebcd$	$aecbd (R^3)$	$caebd (R^6)$	$ceabd (R^+)$	$eabcd$	$eacbd (R^+)$	$ecabd (R^+)$	
$R^8$	$aebcd$	$caebd$	$acebd (R^7)$	$aebcd$	$aecbd (R^3)$	$caebd (R^7)$					
$R^9$	$aebcd$	$aecdb$	$aebcd$	$aecbd (R^8)$	$aecdb$						
$R^{10}$	$aecbd$	$caebd$	$acebd$	$aecbd (R^2)$	$caebd (R^8)$						
$R^{11}$	$aecbd$	$cabed$	$acbed (A)$	$acebd (A)$	$aecbd (A)$	$cabed$	$caebd (R^{10})$				
$R^{12}$	$aebcd$	$cabed$	$abced$	$abecd (R^{11})$	$acbed (R^8)$	$acebd (R^2)$	$aebcd (R^2)$	$aecbd (R^2)$	$cabed (R^8)$	$caebd (R^8)$	
$R^{13}$	$abced$	$cabed$	$abced$	$abced (R^{11})$	$cabed (R^{12})$						
$R^{14}$	$abcde$	$acebd$	$abcde$	$abced (R^*)$	$acbde (R^*)$	$acbed (R^*)$	$acebd (R^*)$				
$R^{15}$	$abcde$	$aebcd$	$abcde$	$abced (R^*)$	$abecd (R^{12})$	$aebcd (R^{14})$					
$R^{16}$	$acdeb$	$aebcd$	$acdeb$	$acebd (R^8)$	$acedb (R^8)$	$aebcd (R^{15})$	$aecbd (R^8)$	$aecdb$			
$R^{17}$	$acebd$	$aebcd$	$acebd (R^8)$	$acedb (R^8)$	$aebcd (R^{16})$	$aecbd (R^8)$	$aecdb$				
$R^{18}$	$aebcd$	$aecdb$	$abced (R^{17})$	$acebd (R^8)$	$aecdb$						
$R^{19}$	$abced$	$ceabd$	$abced$	$abced (R^{13})$	$acebd (R^5)$	$cabed (R^{13})$	$caebd (R^{13})$	$ceabd (R^6)$			
$R^{20}$	$abced$	$acedb$	$abced$	$abced (R^{13})$	$acebd (R^{19})$	$acedb (R^{19})$					
$R^{21}$	$abecd$	$acedb$	$abced$	$abecd (R^{17})$	$acbed (R^{20})$	$acebd (R^{20})$	$acedb (R^{20})$	$aebcd (R^{17})$	$aecbd (R^{20})$	$aecdb (R^{20})$	
$R^{22}$	$abecd$	$aecdb$	$abecd (R^{18})$	$aebcd (R^{18})$	$acebd (R^9)$	$aecbd (R^{21})$					

**Case 3.2:** Next, we assume that  $f(R^2) = acbed$  and derive a contradiction as shown below.

$R^+$	$eabcd$	$ecabd$	$eabcd (A)$								
$R^1$	$caebd$	$cebad$	$cebad (A)$								
$R^2$	$aecbd$	$cabed$	$acbed (A)$								
$R^3$	$ceabd$	$eabcd$	$ceabd (R^+)$	$eabcd$	$eacbd (R^+)$	$ecabd (R^+)$					
$R^4$	$caebd$	$eabcd$	$acebd (R^3)$	$aebcd$	$aecbd$	$caebd (R^3)$	$ceabd (R^+)$	$eabcd$	$eacbd (R^+)$	$ecabd (R^+)$	
$R^5$	$aebcd$	$ceabd$	$acebd (R^4)$	$abecd$	$aecbd$	$caebd (R^4)$					
$R^6$	$aecbd$	$caebd$	$acebd$	$aecbd$	$caebd (R^5)$						
$R^7$	$acebd$	$cebad$	$acebd (R^1)$	$caebd (R^1)$	$ceabd (R^1)$	$cebad$					
$R^8$	$acebd$	$ceabd$	$acebd (R^7)$	$caebd$	$ceabd$						
$R^9$	$aecbd$	$cebad$	$acebd (R^1)$	$aecbd (R^7)$	$caebd (R^1)$	$ceabd (R^1)$	$cebad$	$eacbd (R^7)$	$ecabd$	$ecbad$	
$R^{10}$	$aecbd$	$ceabd$	$acebd (R^8)$	$aecbd (R^9)$	$caebd (R^6)$	$ceabd (R^3)$	$eacbd$	$ecabd$			
$R^{11}$	$aecbd$	$ecabd$	$acebd (R^{10})$	$eacbd$	$ecabd$						
$R^{12}$	$ceabd$	$eabcd$	$ceabd (R^+)$	$ceabd (R^3)$	$eabcd$	$eacbd (R^+)$	$ecabd$	$ecabd (R^+)$	$ecabd (R^+)$		
$R^{13}$	$ceabd$	$eadbc$	$ceabd (R^{12})$	$eacdb$	$eadbc$	$eadcb$	$ecabd (R^{12})$				
$R^{14}$	$ceabd$	$eadcb$	$ceabd (R^{13})$	$eacdb$	$eadcb$	$ecabd (R^{13})$					
$R^{15}$	$eadcb$	$ecadb$	$eacdb$	$eadcb$	$ecadb (R^{14})$						
$R^{16}$	$abced$	$cabed$	$abced$	$abced$	$cabed (R^2)$						
$R^{17}$	$acbed$	$cabed$	$acbed$	$cabed (R^2)$							
$R^{18}$	$acbde$	$aecbd$	$acbde$	$abced$	$acebd (R^2)$	$aecbd (R^2)$					
$R^{19}$	$acbde$	$ecabd$	$acbde$	$abced$	$acebd (R^{18})$	$aecbd (R^{18})$	$eacbd (R^{18})$				
$R^{20}$	$acbed$	$ecabd$	$acbed$	$acebd$	$aecbd (R^{19})$	$eacbd (R^{19})$					
$R^{21}$	$acbed$	$ceabd$	$acbed$	$acebd (R^8)$	$cabed (R^{17})$	$caebd$	$ceabd$				
$R^{22}$	$abced$	$ceabd$	$abced$	$abced$	$acebd (R^8)$	$cabed (R^{16})$	$caebd$	$ceabd (R^3)$			
$R^{23}$	$acbed$	$ceabd$	$acbed$	$acebd (R^8)$	$cabed (R^{17})$	$caebd$	$ceabd (R^{22})$				
$R^{24}$	$acebd$	$ceabd$	$acebd (R^7)$	$eacbd$	$ceabd (R^{23})$						
$R^{25}$	$acbed$	$ecabd$	$acbed$	$acebd (R^{21})$	$aecbd (R^{20})$	$cabed (R^{17})$	$caebd$	$ceabd (R^{23})$	$eacbd (R^{20})$	$ecabd (R^{20})$	
$R^{26}$	$acebd$	$ecabd$	$acebd (R^7)$	$aecbd (R^{11})$	$caebd$	$ceabd (R^{24})$	$eacbd (R^{25})$	$ecabd (R^{25})$			

$R^{27}$	<b>acebd</b>	<b>ecbad</b>	$acebd(R^7)$	$aecbd(R^7)$	$caebd(R^7)$	$ceabd(R^7)$	<b>cebad</b>		$eacbd(R^7)$	$ecabd(R^{26})$	$ecbad(R^{26})$
$R^{28}$	<b>cebad</b>	<b>ecbad</b>	<b>cebad</b>	$ecbad(R^{27})$							
$R^{29}$	<b>ceadb</b>	<b>eacdb</b>	$ceadb(R^{14})$	<b>eacdb</b>	<b>ecadb</b>						
$R^{30}$	<b>acbed</b>	<b>cebad</b>	$acbed(R^7)$	$acebd(R^1)$	$cabed(R^1)$	$caebd(R^1)$	<b>cbaed</b>		<b>cbead</b>	$ceabd(R^1)$	<b>cebad</b>
$R^{31}$	<b>acbed</b>	<b>cbead</b>	$acbed(R^{30})$	$cabed(R^{17})$	<b>cbaed</b>		<b>cbead</b>				
$R^{32}$	<b>acbed</b>	<b>cbaed</b>	$acbed(R^{31})$	$cabed(R^{17})$	<b>cbaed</b>						
$R^{33}$	<b>cebad</b>	<b>ecabd</b>	$ceabd(R^1)$	<b>cebad</b>	<b>ecabd</b>		$ecbad(R^{28})$				
$R^{34}$	<b>aecbd</b>	<b>cebad</b>	$acebd(R^1)$	$aecbd(R^7)$	$caebd(R^1)$	$ceabd(R^1)$	<b>cebad</b>		$eacbd(R^7)$	<b>ecabd</b>	$ecbad(R^{28})$
$R^{35}$	<b>aecbd</b>	<b>ceabd</b>	$acebd(R^8)$	$aecbd(R^9)$	$caebd(R^6)$	$ceabd(R^3)$	$eacbd(R^{34})$	<b>ecabd</b>			
$R^{36}$	<b>aecbd</b>	<b>ecabd</b>	$acebd(R^{10})$	$eacbd(R^{35})$	<b>ecabd</b>						
$R^{37}$	<b>aecbd</b>	<b>cbaed</b>	$acbed(R^{32})$	$acebd(R^2)$	$aecbd(R^2)$	$cabed(R^2)$	$caebd(R^6)$	<b>cbaed</b>			
$R^{38}$	<b>cbaed</b>	<b>ecabd</b>	$cabed(R^{32})$	$caebd(R^{37})$	<b>cbaed</b>		<b>cbead</b>	$ceabd(R^{37})$	<b>cebad</b>	$ecabd(R^{37})$	$ecbad(R^{37})$
$R^{39}$	<b>cbead</b>	<b>ecabd</b>	<b>cbead</b>	$ceabd(R^{33})$	<b>cebad</b>		$ecabd(R^{38})$	$ecbad(R^{33})$			
$R^{40}$	<b>cebad</b>	<b>ecabd</b>	$ceabd(R^1)$	<b>cebad</b>	$ecabd(R^{39})$	$ecbad(R^{28})$					
$R^{41}$	<b>acebd</b>	<b>ecbad</b>	$acebd(R^7)$	$aecbd(R^7)$	$caebd(R^7)$	$ceabd(R^7)$	<b>cebad</b>		$eacbd(R^7)$	<b>ecabd</b>	$ecbad$
$R^{42}$	<b>aecbd</b>	<b>cebad</b>	$acebd(R^1)$	$aecbd(R^7)$	$caebd(R^1)$	$ceabd(R^1)$	<b>cebad</b>		$eacbd(R^7)$	$ecabd(R^{40})$	$ecbad(R^{28})$
$R^{43}$	<b>cebad</b>	<b>eacdb</b>	$ceabd(R^1)$	$ceadb(R^{29})$	<b>cebad</b>		$ecabd(R^7)$	$eacdb(R^{42})$	$ecabd(R^{40})$	$ecadb(R^{40})$	$ecbad(R^{28})$
$R^{44}$	<b>eacdb</b>	<b>ecbad</b>	$acebd(R^{41})$	$eacbd(R^{43})$	<b>ecabd</b>		$ecabd(R^{43})$	<b>ecbad</b>			
$R^{45}$	<b>eadcb</b>	<b>ecbad</b>	$eadcb(R^{41})$	$eadcb(R^{44})$	$eadcb(R^{44})$	<b>ecabd</b>		$ecadb(R^{14})$	<b>ecbad</b>		
$R^{46}$	<b>eadcb</b>	<b>ecabd</b>	$acebd(R^{36})$	$eacbd(R^{45})$	$eadcb(R^{45})$	<b>ecabd</b>		$ecadb(R^{14})$			
$R^{47}$	<b>eadcb</b>	<b>ecadb</b>	<b>eadcb</b>	$eadcb(R^{46})$	$ecabd(R^{14})$						
$R^{48}$	<b>aedcb</b>	<b>ecabd</b>	$acebd(R^{11})$	$acebd(R^{36})$	$aedcb(R^{46})$	$ecabd(R^{36})$	$ecadb(R^{46})$	$eadcb(R^{46})$	<b>ecabd</b>		$ecadb(R^{46})$
$R^{49}$	<b>aedcb</b>	<b>ecadb</b>	$acebd(R^{48})$	$aedcb(R^{48})$	<b>ecabd</b>		$eadcb(R^{47})$	$ecadb(R^{15})$			
$R^{50}$	<b>aedcb</b>	<b>ecadb</b>	$acebd(R^{49})$	$aedcb(R^{49})$	<b>ecabd</b>		$eadcb(R^{49})$				
$R^{51}$	<b>acedb</b>	<b>eacdb</b>	<b>acedb</b>		$acebd(R^{50})$	<b>ecabd</b>					
$R^{52}$	<b>ceadb</b>	<b>eadcb</b>	$ceadb(R^{13})$	<b>ecabd</b>		$eadcb(R^{46})$	$ecadb(R^{13})$				
$R^{53}$	<b>ceadb</b>	<b>eacdb</b>	$ceadb(R^{14})$	<b>ecabd</b>		$ecadb(R^{52})$					
$R^{54}$	<b>cedba</b>	<b>eacdb</b>	$ceadb(R^{29})$	$cedab(R^{53})$	<b>cedba</b>		$ecadb(R^{43})$	$ecadb(R^{43})$	<b>ecdab</b>		<b>ecdba</b>
$R^{55}$	<b>ecadb</b>	<b>ecdba</b>	$acebd(R^{54})$	$ecadb(R^{53})$	<b>ecdab</b>		<b>ecdba</b>				
$R^{56}$	<b>caebd</b>	<b>ecabd</b>	<b>caebd</b>		$ceabd(R^{26})$	$ecabd(R^{26})$					
$R^{57}$	<b>cebad</b>	<b>eabcd</b>	$ceabd(R^1)$	$cebad(R^3)$	$eabcd(R^{42})$	$eachd(R^+)$	$ebacd(R^{42})$	<b>ebcad</b>		$ecabd(R^+)$	$ecbad(R^+)$
$R^{58}$	<b>cebad</b>	<b>eabcd</b>	$cebad(R^{57})$	<b>ebcad</b>		$ecbad(R^{28})$					
$R^{59}$	<b>eacdb</b>	<b>ecdab</b>	$acebd(R^{55})$	$ecadb(R^{53})$	<b>ecdab</b>						
$R^{60}$	<b>caebd</b>	<b>ebcad</b>	$caebd(R^1)$	$ceabd(R^1)$	$cebad(R^{58})$	<b>ebcad</b>		$ecabd(R^{56})$	$ecbad(R^{56})$		
$R^{61}$	<b>aecbd</b>	<b>ebcad</b>	$aebcd(R^9)$	$acebd(R^9)$	<b>eabcd</b>		$eachd(R^9)$	<b>ebacd</b>		$ecabd(R^{60})$	$ecbad(R^{58})$
$R^{62}$	<b>aecbd</b>	<b>ebacd</b>	$aebcd(R^{61})$	$acebd(R^{61})$	<b>eabcd</b>		$eachd(R^{61})$	<b>ebacd</b>			
$R^{63}$	<b>aecbd</b>	<b>eabcd</b>	$aebcd$		$acebd(R^{62})$	<b>eabcd</b>		$eachd(R^+)$			
$R^{64}$	<b>acbed</b>	<b>eacdb</b>	<b>acbed</b>	<b>acebd</b>	<b>acedb</b>		$acebd(R^{20})$	$acebd(R^{50})$	$ecabd(R^{20})$		
$R^{65}$	<b>adecb</b>	<b>eacdb</b>	$adecb(R^{50})$	$acebd(R^{50})$	$adecb(R^{50})$	<b>ecabd</b>		$eadcb(R^{50})$			
$R^{66}$	<b>acdbe</b>	<b>eacdb</b>	<b>acdbe</b>		$acdeb(R^{65})$	<b>acedb</b>		$acebd(R^{50})$	$ecadb(R^{64})$		
$R^{67}$	<b>adceb</b>	<b>eacdb</b>	$acdeb(R^{65})$	<b>acedb</b>		$adceb(R^{65})$	$adecb(R^{50})$	$acecd(R^{50})$	$adecb(R^{50})$	$ecadb(R^{66})$	$eadcb(R^{50})$
$R^{68}$	<b>acedb</b>	<b>eacdb</b>	<b>acedb</b>		$acebd(R^{50})$	$ecadb(R^{67})$					
$R^{69}$	<b>eabcd</b>	<b>ecdab</b>	<b>eabcd</b>		$ecabd(R^+)$	<b>ecadb</b>		$ecabd(R^+)$	$ecadb(R^+)$	<b>ecdab</b>	
$R^{70}$	<b>ceadb</b>	<b>eadbc</b>	$ceadb(R^{12})$	<b>ecadb</b>		$eadbc$		$eadcb(R^{52})$	$ecadb(R^{12})$		
$R^{71}$	<b>caebd</b>	<b>eabcd</b>	$acebd(R^3)$	<b>aebcd</b>		$acebd(R^{63})$	$caebd(R^3)$	$ceabd(R^+)$	<b>eabcd</b>		$eacbd(R^+)$
$R^{72}$	<b>acedb</b>	<b>eabcd</b>	$acebd(R^3)$	$acedb(R^{71})$	<b>aebcd</b>		$acebd(R^{63})$	$acecd(R^{51})$	<b>eabcd</b>		$ecabd(R^+)$
$R^{73}$	<b>acedb</b>	<b>eadbc</b>	$acedb(R^{72})$	$acebd(R^{51})$	<b>aedbc</b>		$aedcb$	$ecadb(R^{68})$	<b>eadbc</b>		$eadcb(R^{70})$

$R^{74}$	<b>acedb</b>	<b>edacb</b>	acedb ( $R^{73}$ )	aecdb ( $R^{51}$ )	<b>aedcb</b>	eacdb ( $R^{68}$ )	eadcb ( $R^{68}$ )	<b>edacb</b>
$R^{75}$	<b>ceadb</b>	<b>eabcd</b>	ceadb ( $R^+$ )	ceadb ( $R^3$ )	<b>eabcd</b>	eacbd ( $R^+$ )	eacdb ( $R^{72}$ )	<b>ecabd</b> ( $R^+$ )
$R^{76}$	<b>eabcd</b>	<b>ecdab</b>	<b>eabcd</b>	eacbd ( $R^+$ )	eacdb ( $R^{59}$ )	ecabd ( $R^+$ )	ecadb ( $R^+$ )	<b>ecdab</b> ( $R^{75}$ )
$R^{77}$	<b>eadbc</b>	<b>ecdab</b>	eadcb ( $R^{59}$ )	<b>eadbc</b>	eadcb ( $R^{70}$ )	ecadb ( $R^{69}$ )	ecdab ( $R^{76}$ )	<b>edabc</b>
$R^{78}$	<b>aedcb</b>	<b>ecdab</b>	aecdb ( $R^{48}$ )	aedcb ( $R^{48}$ )	eacdb ( $R^{59}$ )	eadcb ( $R^{48}$ )	eacdb ( $R^{49}$ )	ecdab ( $R^{77}$ )
$R^{79}$	<b>aedcb</b>	<b>edacb</b>	acedb ( $R^{78}$ )	eadcb ( $R^{74}$ )	<b>edacb</b>			
$R^{80}$	<b>acedb</b>	<b>edacb</b>	acedb ( $R^{73}$ )	aecdb ( $R^{51}$ )	aedcb ( $R^{79}$ )	eacdb ( $R^{68}$ )	eadcb ( $R^{68}$ )	<b>edacb</b>
$R^{81}$	<b>ecabd</b>	<b>edacb</b>	ecabd ( $R^{36}$ )	eacdb ( $R^{46}$ )	eadcb ( $R^{46}$ )	<b>ecabd</b>	eacdb ( $R^{46}$ )	<b>ecdab</b>
$R^{82}$	<b>ecadb</b>	<b>edacb</b>	ecadb ( $R^{80}$ )	eadcb ( $R^{47}$ )	ecadb ( $R^{15}$ )	<b>ecdab</b>	edacb ( $R^{81}$ )	edcab ( $R^{81}$ )
$R^{83}$	<b>ecdab</b>	<b>edacb</b>	ecdab ( $R^{78}$ )	edacb ( $R^{82}$ )	edcab ( $R^{82}$ )			

**Case 3.3:** Thirdly, we show that  $f(R^2) = acebd$  is not possible. As usual, we assume the opposite, i.e., that  $f(R^2) = acebd$  and infer a contradiction with the following table.

$R^*$	<b>abcde</b>	<b>acbed</b>	abcde (A)
$R^+$	<b>eabcd</b>	<b>ecabd</b>	eabcd (A)
$R^1$	<b>caebd</b>	<b>cebad</b>	cebad (A)
$R^2$	<b>aecbd</b>	<b>cabed</b>	acebd (A)
$R^3$	<b>aecbd</b>	<b>cbaed</b>	acbed ( $R^2$ )
$R^4$	<b>acbed</b>	<b>cbaed</b>	acbed
$R^5$	<b>acebd</b>	<b>cebad</b>	acebd ( $R^1$ )
$R^6$	<b>acbed</b>	<b>cebad</b>	acebd ( $R^5$ )
$R^7$	<b>acbed</b>	<b>cbead</b>	acbed ( $R^6$ )
$R^8$	<b>acbed</b>	<b>cbaed</b>	acbed ( $R^7$ )
$R^9$	<b>acbed</b>	<b>cabed</b>	acbed ( $R^8$ )
$R^{10}$	<b>acebd</b>	<b>ceabd</b>	acebd ( $R^5$ )
$R^{11}$	<b>aecbd</b>	<b>caebd</b>	acebd
$R^{12}$	<b>ceabd</b>	<b>eabcd</b>	ceabd ( $R^+$ )
$R^{13}$	<b>caebd</b>	<b>eabcd</b>	acebd ( $R^{12}$ )
$R^{14}$	<b>aebcd</b>	<b>caebd</b>	acebd ( $R^{13}$ )
$R^{15}$	<b>aebcd</b>	<b>aecbd</b>	acbed
$R^{16}$	<b>acbde</b>	<b>aecbd</b>	acbed ( $R^2$ )
$R^{17}$	<b>abcde</b>	<b>acebd</b>	abced ( $R^*$ )
$R^{18}$	<b>acbde</b>	<b>acebd</b>	abced ( $R^{16}$ )
$R^{19}$	<b>acbde</b>	<b>aecbd</b>	abced
$R^{20}$	<b>abcde</b>	<b>aecbd</b>	abced ( $R^*$ )
$R^{21}$	<b>abcde</b>	<b>aebcd</b>	abced ( $R^*$ )
$R^{22}$	<b>aebcd</b>	<b>cabed</b>	abced
$R^{23}$	<b>abced</b>	<b>cabed</b>	abced ( $R^9$ )
$R^{24}$	<b>aebcd</b>	<b>caebd</b>	abced
$R^{25}$	<b>acdeb</b>	<b>aebcd</b>	acebd ( $R^{14}$ )
$R^{26}$	<b>acedb</b>	<b>aebcd</b>	acebd ( $R^{14}$ )
$R^{27}$	<b>aebcd</b>	<b>aecbd</b>	acebd ( $R^{26}$ )
$R^{28}$	<b>abced</b>	<b>ceabd</b>	abced
$R^{29}$	<b>abced</b>	<b>acedb</b>	abced
$R^{30}$	<b>abced</b>	<b>acedb</b>	abced
$R^{31}$	<b>abecd</b>	<b>aecdb</b>	abecd ( $R^{27}$ )

**Case 3.4:** Lastly, we assume that  $f(R^2) = aecbd$  and derive again a contradiction, as demonstrated with the following table.

Since no valid ranking remains for  $R^2$ , this means that  $f(R^1) = cebad$  is not possible either. Since we exhausted all cases for  $f(R^1)$ , this proves the lemma.  $\square$

$\bar{R}$	$badc$	$cbad$	$dcba$
$\bar{R}^1$	$cbad$	$cbad$	$dcba$
$\bar{R}^2$	$badc$	$dcba$	$dcba$
$\bar{R}^3$	$badc$	$cbad$	$cbad$

**Step 1:** For our first step, we will analyze the profile  $\bar{R}$ , where voter 1 reports  $abcd$ , voter 2 reports  $cbda$ , and voter 3 reports  $dbac$ . Specifically, we will show for this profile that  $f(\bar{R}) \in \{bacd, bcda, bdac\}$ . To prove this claim, we consider the following profiles.

$\bar{R}$	$abcd$	$cbda$	$dbac$
$\bar{R}^1$	$abdc$	$cbda$	$dbac$
$\bar{R}^2$	$abcd$	$cbad$	$dbac$
$\bar{R}^3$	$abcd$	$cbda$	$dbca$

We first note that the majority relation for  $\bar{R}^1$ ,  $\bar{R}^2$ , and  $\bar{R}^3$  all form a ranking. Specifically, the majority relation for  $\bar{R}^1$ ,  $\bar{R}^2$ , and  $\bar{R}^3$  correspond to the rankings  $bdac$ ,  $bacd$ , and  $bcda$ . Hence, majority consistency requires that  $f(\bar{R}^1) = bdac$ ,  $f(\bar{R}^2) = bacd$ , and  $f(\bar{R}^3) = bcda$ . In turn, strategyproofness from  $\bar{R}$  to these three profiles results in the following inequalities, where  $\preceq$  is the ranking chosen for  $\bar{R}$ .

- (1) Strategyproofness from  $\bar{R}$  to  $\bar{R}^1$  requires that  $\Delta(abcd, \bar{\triangleright}) \leq \Delta(abcd, bdac) = 3$ .
  - (2) Strategyproofness from  $\bar{R}$  to  $\bar{R}^2$  requires that  $\Delta(cbda, \bar{\triangleright}) \leq \Delta(cbda, bacd) = 3$ .
  - (3) Strategyproofness from  $\bar{R}$  to  $\bar{R}^3$  requires that  $\Delta(dbac, \bar{\triangleright}) \leq \Delta(dbac, bcda) = 3$ .

We claim that these inequalities imply that  $\tilde{\succ} \in \{bacd, bcda, bdac\}$ . To this end, we first show that  $b$  must be top-ranked by  $\tilde{\succ}$ . Assume that this is not the case. If  $a$  is top-ranked by  $\tilde{\succ}$ , then either Condition (2) or Condition (3) is violated. In more detail, if  $\tilde{\succ}$  is top-ranked by  $a$  and is not  $acbd$ , then  $\Delta(cbda, \tilde{\succ}) > 3$ , which violates Condition (2). On the other hand, if  $\tilde{\succ} = acbd$ , then Condition (3) is violated because  $\Delta(dbac, \tilde{\succ}) = 4$ . Next, if  $c$  is top-ranked by  $\tilde{\succ}$ , then we either Condition (1) or (3) is violated. In more detail, any ranking  $\succ$  that top-ranks  $c$  other than  $cdba$  fails Condition (3) since  $\Delta(dbac, \succ) > 3$ . On the other hand, the ranking  $cdba$  fails Condition (1) because  $\Delta(abcd, cdba) = 4$ . Lastly, a symmetric argument based on Conditions (1) and (2) shows that  $d$  cannot be top-ranked by  $\tilde{\succ}$ . By our analysis so far, we have that  $b$

is top-ranked by  $\preceq$ . Our claim now follows by observing that the rankings  $badc$ ,  $bcad$ , and  $bdca$  respectively fail Conditions (2), (3), and (1) since  $\Delta(cbda, badc) = 4$ ,  $\Delta(dbac, bcad) = 4$ , and  $\Delta(abcd, bdca) = 4$ . Hence, it follows that  $f(\bar{R}) \in \{bacd, bdac, bcda\}$ .

Finally, we note that we can without loss of generality assume that  $f(\bar{R}) = bdac$ .

**Step 2:** For our second step, we consider the profile  $\hat{R}$ , where voter 1 reports  $abdc$ ,

$$\hat{R} \quad abcd \quad cdba \quad dbac$$

$$\hat{R}^1 \quad abcd \quad cdba \quad bdca$$

$$\hat{R}^2 \quad cdba \quad cdba \quad dbac$$

$$\bar{R} \quad abcd \quad cbda \quad dbac$$

We first note that the majority relation for  $\hat{R}^1$  corresponds to the ranking  $bcda$ . Hence, majority consistency postulates that  $f(\hat{R}^1) = bcda$ . Similarly, it is easy see that the majority relation in  $\hat{R}^2$  is  $cdba$ , so  $f(\hat{R}^2) = cdba$ . Finally, we have by our previous analysis that  $f(\bar{R}) =$