Committee Monotonicity and Proportional Representation for Ranked Preferences

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Abstract

We study committee voting rules under ranked preferences, which map the voters' preference relations to a subset of the alternatives of predefined size. In this setting, the compatibility between proportional representation and committee monotonicity is a fundamental open problem that has been mentioned in several works. We address this research question by designing a new committee voting rule called the Solid Coalition Refinement (SCR) rule that simultaneously satisfies committee monotonicity and Dummett's PSC as well as one of its variants called inclusion PSC. This is the first rule known to satisfy both of these properties. Moreover, we show that this is effectively the best that we can hope for as other fairness notions adapted from approval voting are incompatible with committee monotonicity. Finally, we prove that, for truncated preferences, the SCR rule still satisfies PSC and a property called independence of losing voter blocs, thereby refuting a conjecture of Graham-Squire *et al.* (2024).

1 Introduction

A ubiquitous phenomenon in collective decision-making is the task of choosing a fixed-size subset of candidates based on the possibly conflicting preferences of multiple agents. For instance, this problem captures parliamentary elections and the shortlisting of finalists for competitions as well as technical applications such as recommender systems. Due to this wide range of applications, mechanisms for selecting the winning candidates based on the agents' preferences have recently garnered significant attention in the field of computational social choice (see, e.g., Faliszewski *et al.*, 2017; Lackner and Skowron, 2023). More specifically, such mechanisms are typically called *committee voting rules* and are formalized as functions that map the agents' preferences to a subset of the candidates with a predefined size.

We are interested in two properties of committee voting rules, namely proportional representation and committee monotonicity. Roughly, proportional representation formalizes that every group of voters with similar preferences should be represented proportionally to their size. Such proportionality notions have recently attracted significant attention in various settings (e.g., Lackner and Skowron, 2023; Rey and Maly, 2023; Aziz *et al.*, 2020; Ebadian and Micha, 2025). On the other hand, committee monotonicity is a basic consistency axiom which requires that when the target size of the committee is increased, the previously selected candidates remain selected. This axiom is especially desirable for online or sequential problems, where it is necessary to adapt a committee voting rule to select a ranking over the candidates rather than a subset (Skowron *et al.*, 2017; Israel and Brill, 2024).

In our work, we focus on the setting of (weak) ranked preferences. That is, voters submit ballots where they rank all candidates while possibly indicating indifference between candidates. For this setting, committee monotonicity has the additional interpretation of *rank aggregation*: voters submit rankings over candidates which then get aggregated into a single ranking. Perhaps surprisingly, while there are several proportional committee voting rules for ranked preferences (e.g., Aziz and Lee, 2020; Brill and Peters, 2023; Delemazure and Peters, 2024) and proportional rank aggregation methods (Lederer *et al.*, 2024), no rule is known to satisfy committee monotonicity together with strong notions of proportional representation. Thus, our main question is the following:

> *"To what extent are proportional representation and committee monotonicity compatible in committee voting with ranked preferences?"*

For proportional representation, we focus on variants of Dummett's *Proportionality for Solid Coalitions (PSC)* (Dummett, 1984) that has been referred to as "a sine qua non for a fair election rule" (Woodall, 1994). The compatibility between PSC and committee monotonicity has been mentioned as an open problem that also has a bearing on finding proportional rankings. For instance, Lederer *et al.* (2024, p. 18) write that "it is an open question whether axioms for proportional multi-winner rules (such as Proportionality for Solid Coalitions, PSC, Aziz and Lee, 2020) are compatible with committee monotonicity, which is necessary to adapt a multi-winner rule to output a ranking." Moreover, Lackner and Skowron (2023, p. 105) list the compatibility of committee monotonicity and proportional representation as one of the major problems of the field (although they focus on approval preferences).

Contributions. In this paper, we aim to design voting rules that satisfy both committee monotonicity and PSC. A graphical overview of our results is given in Figure 1.

In more detail, we first show that all committee voting rules known to satisfy PSC fail committee monotonicity. This makes it necessary to design new voting rules. For strict preferences, we show that there is a wide variety of rules satisfying our desired properties, by developing a simple scheme that turns any voting rule satisfying PSC into a committee monotone voting rules satisfying PSC (Theorem 1). This scheme operates by running the rule in a "reverse sequential" mode.

Unfortunately, this construction does not work for weak preferences. We hence design a new voting rule called the Solid Coalition Refinement (SCR) rule which repeatedly adds the candidates that represent the most underrepresented groups of voters to the winning committee. As we show, this rule satisfies both PSC (or, more precisely, a generalization of this axiom called inclusion PSC) and committee monotonicity, even if the voters report weak preferences (Theorem 2). To our knowledge, the SCR rule is the first committee voting rule that satisfies both of these properties at the same time.

Moreover, we use the SCR rule to refute a hypothesis by Graham-Squire *et al.* (2024). In more detail, these authors conjecture that for truncated preferences (i.e., voters do not need to rank all candidates), no voting rule satisfies both PSC and a property called independence of losing voter blocs. Roughly, this property requires that the outcome of a rule is not allowed to change if we delete voters who only rank unselected candidates. However, we show that the SCR rule satisfies both PSC and independence of losing voter blocs for truncated preferences (Theorem 3), thus disproving the conjecture.

Finally, we examine the compatibility of committee monotonicity with a family of proportionality notions due to Brill and Peters (2023) that adapt fairness axioms from approval-based committee voting to ranked preferences. However, it turns out that committee monotonicity is even incompatible with Rank-JR, the weakest such proportionality notion (Theorem 4). This suggests that it may be impossible to attain stronger proportionality conditions than PSC with committee monotone voting rules.

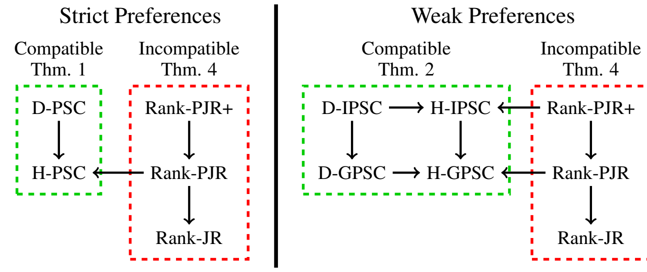

Figure 1: A summary of our results. Axioms in the green boxes are compatible with committee monotonicity, whereas the axioms in the red boxes are incompatible. An arrow from one axiom to another means that the first implies the second. "D-" is short for Droop and "H-" for Hare. Definitions for GPSC (Generalized PSC), Rank-PJR, and Rank-PJR+ as well as proofs of the relationships between the axioms can be found in the work of Brill and Peters (2023).

2 Related Work

We next discuss the prior works on proportional representation and committee monotonicity for ranked preferences.

Proportional Representation. The problem of finding representative committees has a long tradition. Already in the 19th century, a rule called *Single Transferable Vote (STV)* was proposed for finding such committees. We refer the reader to the article by Tideman (1995) for historical details on the development of this rule. STV is now widespread and used for elections in many countries such as Australia, Ireland, India, and Pakistan. Further, it has been shown that STV provides proportional representation since it satisfies PSC (Woodall, 1994; Tideman and Richardson, 2000). As a consequence of its widespread use, many works have focused on better understanding STV (e.g., Bartholdi, III and Orlin, 1991; Elkind *et al.*, 2017; Delemazure and Peters, 2024).

Moreover, numerous other committee voting rules have been suggested with the aim of finding representative outcomes. Examples include the Chamberlin-Courant rule (Chamberlin and Courant, 1983; Lu and Boutilier, 2011), Monroe's rule (Monroe, 1995), and various forms of positional scoring rules (Faliszewski *et al.*, 2019). However, none of these rules are known to satisfy PSC. Indeed, there are only two rules other than STV in the literature that satisfy this condition: the *Quota Borda System (QBS)* of Dummett (1984) and the *Expanding Approvals Rule (EAR)* of Aziz and Lee (2020). Both of these rules were defined specifically to attain PSC while addressing some flaws of STV. In particular, Dummett (1984) suggests QBS with the informal argument that it is less chaotic than STV, while Aziz and Lee (2020) motivate EAR by the observation that it satisfies monotonicity conditions that STV fails. Moreover, it was shown by Brill and Peters (2023) that EAR satisfies a fairness condition called Rank-PJR+ which is violated by STV. Finally, we note that Aziz and Lee (2022) give a characterization of committees satisfying PSC in terms of minimal demand rules; however, it is not clear how to derive appealing rules from this characterization.

Committee Monotonicity. Committee monotonicity (called *house monotonicity* in other settings) was identified early on as a desirable property of committee elections. For instance, in 1880, the chief clerk of the census office of the United States noticed that Alabama would be allocated 8 seats in a 299-seat parliament but only 7 seats in a 300-seat parliament (Balinski and Young, 2001). However, in contrast to proportional representation, committee monotonicity has attracted much less attention in the literature. It is known that committee monotonicity and proportional representation are compatible in the simpler setting of (approval-based) apportionment (Balinski and Young, 2001; Brill *et al.*, 2024), where voters vote for parties which can be assigned multiple seats. By contrast, the analysis of committee monotonicity for committee elections consists largely of counterexamples showing that specific classes of rules fail this axiom (Staring, 1986; Ratliff, 2003; Barberà and Coelho, 2008; Kamwa, 2013; Elkind *et al.*, 2017; McCune and Graham-Squire, 2024). For instance, Elkind *et al.* (2017) present a counterexample showing that STV fails committee monotonicity. On the other hand, Janson (2016) showed that several rules, such as Phragmén's Ordered Method and Thiele's Ordered Method, satisfy committee monotonicity. However, these rules fail PSC.

Committee monotonicity is also often considered when studying rules that return a ranking over the candidates rather than a set of winning candidates. In particular, Elkind *et al.* (2017) noted that committee monotonicity is necessary and sufficient to turn a committee voting rule into a ranking rule. This means that the study of fair rank aggregation rules (e.g., Skowron *et al.*, 2017; Lederer *et al.*, 2024) can also be interpreted as a study of fair and committee monotone voting rules for committee elections. However, these authors focus on fairness notions specific to rank aggregation and their results are hence not directly related to ours. Finally, also for approval-based committee elections, no voting rule is known to satisfy both committee monotonicity and strong proportionality guarantees (Lackner and Skowron, 2023).

3 Preliminaries

Let $N = \{1, \dots, n\}$ denote a set of n voters and let $C = \{c_1, \dots, c_m\}$ denote a set of m candidates. We assume that every voter $i \in N$ reports a *(weak) preference relation* $\succsim_i$ over the candidates, which is formally a complete and transitive binary relation on C. The notation $c \succsim_i c'$ denotes that voter i weakly prefers candidate c to c', whereas $c \succ_i c'$ indicates a strict preference. We call a preference relation *strict* if it is antisymmetric, i.e., there is no indifference between any two candidates. The set of all weak preference relations is denoted by $\mathcal{R}$ and the set of all strict preference relations by $\mathcal{L}$. A *preference profile* R is the collection of the voters' preferences, i.e., it is a function from N to $\mathcal{R}$. A preference profile is *strict* if all voters have strict preference relations. The set of all preference profiles is $\mathcal{R}^N$ and the set of all strict preference profiles is $\mathcal{L}^N$.

Given a preference profile, our goal is to select a *committee*, which is formally a subset of the candidates of a given size k. To this end, we use *committee voting rules*, which for every preference profile and committee size k return a committee of that size. More formally, a committee voting rule f for weak (resp. strict) preferences maps every profile $R \in \mathcal{R}^N$ (resp. $\mathcal{L}^N$) and target committee size $k \in \{1, \dots, m\}$ to a winning committee $W = f(R, k)$ with $|W| = k$. We emphasize that committee voting rules always choose a single committee.

We next introduce our central axioms, namely committee monotonicity and proportionality for solid coalitions.

3.1 Committee Monotonicity

The idea of committee monotonicity is that if some candidate is selected for a committee size k, then it should also be selected for a committee size $k' > k$. More formally, we define committee monotonicity as follows:

Definition 1 (Committee monotonicity)**.** *A committee voting rule f is* committee monotone *if $f(R,k) \subseteq f(R, k+1)$ for all preference profiles R and committee sizes $k \in \{1, \dots, m-1\}$.*

Faliszewski *et al.* (2017) deem committee monotonicity a necessity for excellence-based elections, where the goal is to choose the individually best candidates for the considered problem. Moreover, Elkind *et al.* (2017) have shown that every committee monotone rule can be transformed into a rule that returns rankings of the candidates instead of committees.

3.2 Proportionality for Solid Coalitions

Next, we introduce proportionality for solid coalitions (PSC). The rough idea of this axiom is that, if there is a sufficiently large set of voters $N' \subseteq N$ that all prefer the candidates in a subset $C' \subseteq C$ to the candidates in $C \setminus C'$, then this group should be represented by a number of candidates in C' that is proportional to the size of N'. Following Dummett (1984), we will first define this axiom for strict preferences before presenting a variant called inclusion PSC (IPSC) due to Aziz and Lee (2021) that can also accommodate weak preferences. Moreover, we will define these axioms as properties of committees; a committee voting rule satisfies PSC or IPSC if its selected committee always satisfies the given axiom.

To formalize PSC, we define a *solid coalition* for a set of candidates $C' \subseteq C$ as a group of voters $N' \subseteq N$ such that $c' \succ_i c$ for all voters $i \in N'$ and candidates $c' \in C'$, $c \in C \setminus C'$. In this case, we also say that the voters in N' *support* the candidates C'. We emphasize that the voters in N' do not have to agree on the order of the candidates in C' and that a voter can be part of multiple solid coalitions. Now, proportionality for solid coalitions postulates that each solid coalition N' of size $|N'| > \ell \cdot \frac{n}{k+1}$ (for some $\ell \in \mathbb{N}$) should be represented by at least ℓ candidates or the set C' if $|C'| < \ell$.

Definition 2 (Proportionality for Solid Coalitions (Dummett, 1984))**.** *A committee W satisfies* proportionality for solid coalitions (PSC) *for a preference profile R and a committee size k if for all integers $\ell \in \mathbb{N}$ and solid coalitions N' supporting a set C' with $|N'| > \ell \cdot \frac{n}{k+1}$, it holds that $C' \subseteq W$ or $|W \cap C'| \geq \ell$.*

We will next present a generalization of PSC to weak preferences due to Aziz and Lee (2021). To this end, we call a set of voters $N' \subseteq N$ a *generalized solid coalition* supporting a set of candidates $C' \subseteq C$ if for all voters $i \in N'$, we have $c' \succsim_i c$ for all candidates $c' \in C'$ and $c \in C \setminus C'$. That is, generalized solid coalitions only have to weakly prefer the candidates in C' to those in $C \setminus C'$. Furthermore, following Aziz and Lee (2021), we define the *periphery* $\overline{C'}(N') = \{c \in C :$ there exists $i \in N'$and $c' \in C'$ such that $c \succsim_i c'\}$ of C' with respect to N'. This is a "closure" of C', containing C' as well as all candidates that at least one member of N' weakly prefers to a member of C'. Then, IPSC demands that, for every generalized solid coalition N' supporting a set C', a number of candidates proportional to $|N'|$ needs to the chosen from the periphery $\overline{C}'(N')$.

Definition 3 (Inclusion PSC (Aziz and Lee, 2021))**.** *A committee W satisfies* inclusion PSC (IPSC) *for a profile R and committee size k if for all integers $\ell \in \mathbb{N}$ and generalized solid coalitions N' supporting a set C' with $|N'| > \ell \cdot \frac{n}{k+1}$, it holds that $C' \subseteq W$ or $|W \cap \overline{C'}(N')| \geq \ell$.*

We note that both PSC and IPSC are sometimes defined based on a parameter $q \in (n/(k+1), n/k]$ and the requirement that $|N'| \geq \ell \cdot q$ instead of $|N'| > \ell \cdot \frac{n}{k+1}$. Our variants of these axioms choose the minimal quota q in this interval, which results in the strongest proportionality notion. In particular, our proportionality notions are sometimes called Droop-PSC and Droop-IPSC (see, e.g., Aziz and Lee, 2020). Another commonly studied variant of these axioms is obtained for $q = \frac{n}{k}$, which are called Hare-PSC and Hare-IPSC.

4 PSC and Committee Monotonicity

We will now investigate the compatibility of PSC and committee monotonicity. To this end, we recall that there are only three rules known to satisfy PSC: STV, EAR, and QBS. In Appendix A, we define these rules and show that all of them fail committee monotonicity. We further show that Dummett's family of *Quota Preference Score* rules that generalize QBS to use any positional scoring rule for tie-breaking are also incompatible with committee monotonicity.

Since none of the known committee voting rules satisfy both PSC and committee monotonicity, we will design new voting rules to achieve both axioms simultaneously. As our first result we show that for strict preferences, there is a simple way to achieve PSC and committee monotonicity by modifying existing rules. To this end, we introduce the *reverse sequential rule* f^{RS} of a committee voting rule f. Roughly, these reverse sequential rules compute the winning committee by repeatedly using the original rule f to identify alternatives that should be removed from the winning committee. To make this more formal, we let $R|_X$ denote the restriction of a preference profile R to the set $X \subseteq C$, i.e., we derive $R|_X$ by deleting the alternatives $C \setminus X$ from R. Then, the reverse sequential rule f^{RS} of a committee voting rule f is defined recursively by $f^{RS}(R, m) = C$ and $f^{RS}(R, k) = f(R|_{f^{RS}(R,k+1)}, k)$ for all $k \in \{m-1, \dots, 1\}$.

We show next that for strict preferences, the reverse sequential rule f^{RS} satisfies committee monotonicity and PSC if the original rule f satisfies PSC. This means that, e.g., the reverse sequential rule of STV satisfies both of our desiderata.

Theorem 1. *For strict preferences, the reverse sequential rule f^{RS} of every committee voting rule f is committee monotone. Moreover, if f satisfies PSC, then f^{RS} satisfies PSC, too.*

Proof. It is straightforward that the reverse sequential rule f^{RS} of a committee voting rule f satisfies committee monotonicity because $f^{RS}(R, k) = f(R|_{f^{RS}(R,k+1)}, k) \subseteq f^{RS}(R, k+1)$ for all $k \in \{1, \dots, m-1\}$ and $R \in \mathcal{L}^N$.

Next, assume that f satisfies PSC and fix a profile $R \in \mathcal{L}^N$. We show via backwards induction on the committee size $k \in \{1, \dots, m\}$ that $f^{RS}(R, k)$ satisfies PSC, too. The induction basis $k = m$ is trivial since every rule satisfies PSC for the committee size m. Now, fix $k \in \{1, \dots, m-1\}$ and assume that the committee $W = f^{RS}(R, k+1)$ satisfies PSC for the profile R and committee size $k+1$. We will show that the committee $W' = f^{RS}(R, k)$ satisfies PSC, too.

For this, let $N' \subseteq N$ be a solid coalition such that $|N'| > \ell \frac{n}{k+1}$ for some $\ell \in \mathbb{N}$, and let $C' \subseteq C$ denote the set of candidates supported by N'. Since $\ell \frac{n}{k+1} > \ell \frac{n}{k+2}$ and W satisfies PSC on R by the induction hypothesis, we get that $|C' \cap W| \geq \min(\ell, |C'|)$. Next, W' satisfies PSC for the profile $R|_W$ and the committee size k because $f(R|_W, k) = W'$ and f satisfies PSC by definition. Since N' solidly supports the set $C' \cap W$ in $R|_W$, we derive that

$$|C' \cap W'| = |(C' \cap W) \cap W'| \geq \min(|C' \cap W|, \ell) \geq \min(\min(\ell, |C'|), \ell) = \min(\ell, |C'|).$$

Thus, W' satisfies PSC for the profile R, which proves the induction step. □

Unfortunately, Theorem 1 does not extend to weak preferences and IPSC. To see this, consider the proportional approval voting (PAV) rule which is defined for approval preferences, a special case of weak preferences, and satisfies IPSC (Brill and Peters, 2023). If Theorem 1 held for weak prefernces, then the reverse sequential version of PAV should also satisfy IPSC, and thus also the weaker axioms PJR and JR (Aziz and Lee, 2021). But reverse sequential PAV fails JR (Aziz, 2017).

5 The Solid Coalition Refinement Rule

A drawback of Theorem 1 is that the rules it produces (e.g., the reverse sequential rule of STV) seem unnatural. In this section, we design a more intuitive rulewhich we call the Solid Coalition Refinement rule (or the SCR rule or simply SCR). It has the additional advantage of working for weak preferences as well, for which it satisfies IPSC. The basic idea of this rule is to repeatedly identify the generalized solid coalition for which IPSC is currently violated the most and add one of the supported candidates of this solid coalition to the outcome.

To formally define the SCR rule, we need to introduce additional notation. First, we let $\rho(W, N', C') = \frac{|N'|}{|W \cap \overline{C'}(N')|+1}$ denote the *underrepresentation value* of a committee W for a generalized solid coalition N' supporting the set C'. In words, this value computes the ratio between the size of the generalized solid coalition N' and the number of selected candidates in $\overline{C'}(N')$ plus one. Hence, a large underrepresentation value means that the generalized solid coalition N' is far from being proportionally represented. Moreover, it can be shown that a committee W satisfies IPSC for a profile R and a committee size k if and only if $\rho(W, N', C') \leq \frac{n}{k+1}$ for all generalized solid coalitions N' that support a candidate set C' with $C' \not\subseteq W$. Finally, we denote by $\Phi(R, W, D) = \{(N', C') \colon C' \subsetneq D,\ C' \not\subseteq W,$ and N' is a generalized solid coalition supporting C' in $R\}$ the set of generalized solid coalitions N' supporting a set C' with $C' \subsetneq D$ and $C' \not\subseteq W$.

We are now ready to define the SCR rule. Starting from $W = \emptyset$, this rule computes the winning committee by repeatedly adding single candidates to W. The next candidate is chosen as follows: we first identify the generalized solid coalition N' supporting a set $C' \not\subseteq W$ such that (N', C') maximizes the underrepresentation value $\rho(W, N', C')$ among all generalized solid coalitions in $\Phi(R, W, C)$. The goal is then to select a candidate from $C' \setminus W$. Moreover, to decide which candidate from $C' \setminus W$ to choose, we identify the generalized solid coalition N'' supporting a candidate set C'' with $C'' \subsetneq C'$ and $C'' \not\subseteq W$ such that $\rho(W, N'', C'')$ is maximal among all generalized solid coalitions in $\Phi(R, W, C')$. By repeating this step, we will eventually arrive at a generalized solid coalition N^* supporting a set C^* such that $|C^* \setminus W| = 1$, and we add the single candidate in $C^* \setminus W$ to W. Put differently, starting from $D = C$, we repeatedly update D to be the set of candidates C^* corresponding to a generalized solid coalition N^* such that (N^*, C^*) maximizes $\rho(W, N^*, C^*)$ among all elements in $\Phi(R, W, D)$ until $D \setminus W = 1$. Then, we add the candidate in $D \setminus W$ to W. A pseudocode description of the SCR rule is given in Algorithm 1.

We note that multiple generalized solid coalitions in $\Phi(R, W, D)$ may have the same maximal underrepresenta-

Algorithm 1: The Solid Coalition Refinement Rule

Input : A preference profile R and committee size k
Output : A committee of k candidates
1 $W \leftarrow \emptyset$
2 **for** $i \in \{1, \dots, k\}$ **do**
3 $\quad D \leftarrow C$
4 $\quad$ **while** $|D \setminus W| > 1$ **do**
5 $\quad\quad \Phi(R, W, D) \leftarrow \{(N', C') : C' \subsetneq D, C' \not\subseteq W$, and N' is a generalized solid coalition supporting $C'\}$
6 $\quad\quad (N^*, C^*) \leftarrow \arg\max_{(N', C') \in \Phi(R,W,D)} \frac{|N'|}{|W \cap \overline{C'}(N')|+1}$
7 $\quad\quad D \leftarrow C^*$
8 $\quad W \leftarrow W \cup D$
9 **return** W

tion value in some step; in such cases, we assume that ties are broken by an arbitrary but fixed ranking $\triangleright$ over the sets of candidates $C' \subseteq C$. That is, if two generalized solid coalitions N' and N'' with candidate sets C' and C'' have the same maximal underrepersentation value in some step of the SCR rule, we choose (N', C') if $C' \triangleright C''$ and (N'', C'') otherwise.

We next present an example illustrating how SCR works.

Example 1. *Consider the following strict preference profile with $n = 5$ voters and $m = 4$ candidates:*

Voter 1*:* $c_1 \succ c_2 \succ c_3 \succ c_4$
Voter 2*:* $c_1 \succ c_3 \succ c_2 \succ c_4$
Voter 3*:* $c_3 \succ c_2 \succ c_1 \succ c_4$
Voter 4*:* $c_4 \succ c_1 \succ c_2 \succ c_3$
Voter 5*:* $c_4 \succ c_1 \succ c_2 \succ c_3$*.*

With a committee size $k = 3$, the SCR rule runs as follows.

1) Initially, $D = \{c_1, c_2, c_3, c_4\}$ and $W = \emptyset$. Then:
- *The solid coalition $N' = \{1, 2, 3\}$ supporting $C' = \{c_1, c_2, c_3\}$ has $\rho(W, N', C') = 3$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{c_1, c_2, c_3\}$.*
- *The solid coalition $N' = \{1, 2\}$ supporting $C' = \{c_1\}$ has $\rho(W, N', C') = 2$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{c_1\}$.*

Candidate c_1 is selected, so $W = \{c_1\}$.

2) D is reset to $\{c_1, c_2, c_3, c_4\}$. Then:
- *The solid coalition $N' = \{4, 5\}$ supporting $C' = \{c_4\}$ has $\rho(W, N', C') = 2$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{c_4\}$.*

Candidate c_4 is selected, so $W = \{c_1, c_4\}$.

3) D is reset to $\{c_1, c_2, c_3, c_4\}$. Then:
- *The solid coalition $N' = \{1, 2, 3\}$ supporting $C' = \{c_1, c_2, c_3\}$ has $\rho(W, N', C') = \frac{3}{2}$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{c_1, c_2, c_3\}$.*
- *The solid coalition $N' = \{3\}$ supporting $C' = \{c_3\}$ has $\rho(W, N', C') = 1$, which is maximal among all solid coalitions in $\Phi(R, W, D)$. SCR sets $D = \{c_3\}$.*

Candidate c_3 is selected, so the final committee is $W = \{c_1, c_3, c_4\}$.

We next show that SCR is well-defined and runs in polynomial time if voters have strict preferences. Due to space restrictions, we defer the proof of the subsequent proposition to Appendix B and discuss a proof sketch instead.

Proposition 1. *The SCR rule always terminates and produces a committee of the target size k. Furthermore, for strict preferences, it can be implemented to run in polynomial time.*

Proof Sketch. For showing that the SCR rule is well-defined, we note that each iteration of the for-loop (line 2) adds exactly one candidate to W. This is true because it holds for every pair (N', C') in $\Phi(R, W, D)$ that $|C' \setminus W| \geq 1$ and the while-loop (line 4) is only exited when $|D \setminus W| \leq 1$. Moreover, it can be shown that the set $\Phi(R, W, D)$ is always non-empty during the execution of SCR, so the rule indeed produces a committee of the desired committee size k. Next, the SCR rule runs for strict preferences in polynomial time because, in this case, the set of (voter-maximal) generalized solid coalitions can be efficiently computed and contains at most mn elements. We hence can solve the maximization problem in line 6 by iterating through all voter-maximal generalized solid coalitions, so SCR can be computed in polynomial time for strict preferences. $\square$

A natural follow-up question to Proposition 1 is whether the SCR rule can also be computed in polynomial time for weak preferences. Unfortunately, there is no clear way to compute line 6 in this case. Moreover, if we could solve this line for every profile R, committee W, and the set $\Phi(R, W, C)$, we could also decide whether there is a generalized coalition N' supporting a set C' such that $C' \not\subseteq W$ and $\rho(W, N', C') > \frac{n}{k+1}$. However, this is equivalent to deciding whether the committee W satisfies IPSC for the profile R, which is known to be a coNP-complete problem (Brill and Peters, 2023).

In the remainder of this section, we will show that the SCR rule satisfies committee monotonicity and IPSC.

Theorem 2. *Even for weak preferences, the SCR rule is committee monotone and satisfies IPSC.*

Proof. Since the SCR rule selects candidates sequentially and independently of the target committee size, it follows immediately that it satisfies committee monotonicity.

Next, to show that SCR satisfies IPSC, we fix a profile R and a committee size k. Moreover, for all $t \in \{1, \dots, k\}$, we let c_t denote the t-th candidate that SCR adds to the winning committee for R, and we define $W^t = \{c_1, \dots, c_t\}$ for all $t \in \{1, \dots, k\}$ and $W^0 = \emptyset$. For our proof, we assume that each voter $i \in N$ has a virtual budget $b_i = 1$ and that the candidates will be bought using these budgets for a price of $\frac{n}{k+1}$. We will next show that, for every $t \in \{1, \dots, k\}$, there is a payment scheme for W^t such that *(i)* the budgets of all voters are always non-negative and *(ii)* if a voter i is part of a generalized solid coalition N' supporting a set C' such that $C \not\subseteq W^t$ and $\rho(W^t, N', C') > \frac{n}{k+1}$, then voter i only spent his budget on candidates in $\overline{C'}(N')$. To derive such a scheme, we fix $t \in \{1, \dots, k\}$ and inductively assume that there is a payment scheme for the committee W^{t-1} that satisfies conditions *(i)* and *(ii)*. If $t = 1$, such a scheme exist for W^{t-1} as $W^0 = \emptyset$ and no money has been spent. We next explain how to extend the payment scheme for W^{t-1} to W^t.

For this, we proceed with a case distinction and first assume that $\rho(W^{t-1}, N', C') \leq \frac{n}{k+1}$ for all generalized solid coalitions N' that support a set C' with $C' \not\subseteq W^{t-1}$. In this case, we deduct the money for candidate c_t arbitrarily from the budgets of the voters while ensuring that no budget becomes negative. This is possible as the voters' total budget exceeds the necessary budget to pay for k candidates, i.e., $n > k\frac{n}{k+1}$. Moreover, it holds that $\rho(W^t, N', C') \leq \rho(W^{t-1}, N', C') \leq \frac{n}{k+1}$ for all generalized solid coalitions N' with set $C' \not\subseteq W^t$, so condition *(ii)* holds trivially.

For the second case, suppose that there is a generalized solid coalition N' supporting a set C' such that $C' \not\subseteq W^{t-1}$ and $\rho(W^{t-1}, N', C') > \frac{n}{k+1}$. In this case, let N^* and C^* denote the last generalized solid coalition and the corresponding set of candidates in the execution of the while-loop of SCR (line 4) that satisfies these conditions. By condition *(ii)*, the voters in N^* have only spent money on the candidates in $\overline{C^*}(N^*)$ so far. Hence, these voter have spent a total budget of at most $|W^{t-1} \cap \overline{C^*}(N^*)|\frac{n}{k+1}$. By rearranging the assumption that $\rho(W^{t-1}, N^*, C^*) = \frac{|N^*|}{|W^{t-1} \cap \overline{C^*}(N^*)|+1} > \frac{n}{k+1}$, we infer that $|N^*| > (|W^{t-1} \cap \overline{C^*}(N^*)| + 1)\frac{n}{k+1}$. Therefore, these voters have a total remaining budget of at least $\frac{n}{k+1}$, so they can pay for the candidate c_t without violating condition *(i)*.

It remains to show that this payment scheme for W^t satisfies condition *(ii)*. For this, let N'' denote a generalized solid coalition supporting a set C'' such that $C'' \not\subseteq W^t$ and $\rho(W^t, N'', C'') > \frac{n}{k+1}$. We assume for contradiction that there is a voter $i \in N''$ who spent money on candidates outside of $\overline{C''}(N'')$. Since $\rho(W^{t-1}, N'', C'') \geq \rho(W^t, N'', C'') > \frac{n}{k+1}$, we infer from the induction hypothesis that voter i has not spent money on candidates outside of $\overline{C''}(N'')$ during the first $t-1$ steps. Hence, voter i spent money on candidate c_t and $c_t \notin \overline{C''}(N'')$. This means that $i \in N^*$, so C^* forms a prefix of voter i's preference relation. Further, because $c_t \notin \overline{C''}(N'')$, it follows that $C'' \subsetneq C^*$. However, this means that after SCR selected the solid coalition N^* with set C^* in the while-loop, there is another iteration of the while-loop such $\frac{|N'|}{|W^{t-1} \cap \overline{C'}(N')|+1} \geq \frac{|N'|}{|W^t \cap \overline{C'}(N')|+1} > \frac{n}{k+1}$. This violates the definition of N^* and C^*, so the assumption that voter i spent some money on c_t is wrong and condition *(ii)* holds.

Finally, we will show that SCR satisfies IPSC. Assume for contradiction that the committee W^k fails IPSC for R and the committee size k. Thus, there is a generalized solid coalition N' supporting a set C' and an integer $\ell \in \mathbb{N}$ such that $|N'| > \frac{\ell n}{k+1}$, $C' \not\subseteq W^k$, and $|W^k \cap \overline{C'}(N')| < \ell$. This implies that $\rho(W^k, N', C') = \frac{|N'|}{|W^k \cap \overline{C'}(N')|+1} \geq \frac{|N'|}{\ell} > \frac{n}{k+1}$. In turn, condition *(ii)* of the payment scheme shows that the voters in N' only paid for candidates in $\overline{C'}(N')$. Hence, they spent a budget of at most $(\ell - 1)\frac{n}{k+1}$. Since the total initial budget of these voters is $|N'| > \frac{\ell n}{k+1}$, their remaining budget is more than $\frac{n}{k+1}$ in the end. However, the total remaining budget of all voters after k candidates have been bought is $n - \frac{kn}{k+1} = \frac{n}{k+1}$. Hence there must be a voter $i \notin N'$ who has a negative budget. This contradicts condition *(i)*, which means that SCR satisfies IPSC. □

6 PSC and Irrelevant Voter Blocks

We next use the SCR rule to answer an open question of Graham-Squire *et al.* (2024) by showing that a rule satisfying independence of losing voter blocs and PSC exists. In more detail, Graham-Squire *et al.* study the setting of truncated (strict) preferences, i.e., voters have strict preferences but it is no longer necessary to rank all alternatives. More formally, a preference relation $\succsim_i$ is *truncated* if it is a strict preference relation over a subset of the candidates.

First, the notion of solid coalitions, and thus also PSC as well as the SCR rule, can be easily extended to truncated preferences. Specifically, solid coalitions are defined just as before and, in particular, only form for sets of voters who rank all alternatives in the supported set of candidates. Then, PSC and the SCR rule can be adapted to truncated preferences by using this new definition of solid coalitions.

Furthermore, Graham-Squire *et al.* (2024) suggest two consistency notions regarding the behavior of committee voting rules when some voters are removed from the election. One of these notions, independence of losing voter blocs, requires that the outcome should not change when we remove voters who only rank unchosen candidates. To formalize this notion, we define by $X(\succsim_i)$ the set of alternatives that are not ranked in the truncated preference relation $\succsim_i$. Moreover, $R_{-N'}$ denotes the profile derived from another profile R by deleting the voters in $N' \subseteq N$ from R. Based on this notation, we now define independence of losing voter blocs.

Definition 4 (Independence of losing voter blocs)**.** *A committee voting rule f satisfies* independence of losing voter blocs *if $f(R, k) = f(R_{-N'}, k)$ for all truncated preference profiles R, committee sizes k, and sets of voters $N' \subsetneq N$ such that $f(R, k) \subseteq X(\succsim_i)$ for all $i \in N'$.*

Graham-Squire *et al.* (2024) show that no classical proportional rules such as STV and EAR satisfy this property. In fact, these authors even conjecture that no voting rule satisfies PSC and independence of losing voter blocs at the same time. We show that this conjecture is false as SCR (adapted to truncated rankings) satisfies both conditions

Theorem 3. *For truncated preferences, the SCR rule satisfies PSC and independence of losing voter blocs.*

Proof. First, an analogous argument as in the proof of Theorem 2 shows that SCR satisfies PSC for truncated preferences. Hence, we focus on independence of losing voter blocs. To this end, we fix a profile R, a committee size k, and a set of voters N' such that $W \subseteq X(\succsim_i)$ for all $i \in N'$ and the committee W chosen by SCR for the profile R and the committee size k. We need to show that SCR also chooses the committee W for the profile $R_{-N'}$ and the committee size k. For this, let W' denote the intermediate committee and D the current set of candidates of Algorithm 1 during some step of the execution of SCR for R such that $|D \setminus W'| > 1$. Moreover, let (N^*, C^*) be the solid coalition and the set of candidates that is chosen next at line 6. Since $W \subseteq X(\succsim_i)$ for all $i \in N'$, there must be for every voter $i \in N'$ a candidate $x \in C^* \cap X(\succsim_i)$. Otherwise, voter i ranks the candidate that will be selected next because $C^* \subseteq C \setminus X(\succsim_i)$. Since solid coalitions form only over sets of voters N'' and sets C'' such that all voters $i \in N''$

rank all candidates in C'', this implies that $N^* \cap N' = \emptyset$. In turn, it follows that $(N^*, C^*) \in \Phi(R_{-N'}, W', D)$. Further, it holds that $\Phi(R_{-N'}, W', D) \subseteq \Phi(R, W', D)$ as introducing new voters can only create more solid coalitions. Thus, if SCR agreed on W' and D for both R and $R_{N'}$ for the current step, it will still agree on these sets for the next step. Since it initially always holds that $W' = \emptyset$ and $D = C$, it now follows that SCR chooses W for both R and $R_{-N'}$. □

7 Committee Monotonicity and Rank-JR

In light of our positive results so far, it is a natural follow-up question whether the SCR rule—or, more generally, any committee monotone voting rule—also satisfies other forms of proportionality. Unfortunately, we will give a negative answer to this question by showing that no committee voting rule satisfies both committee monotonicity and a proportionality notion called Rank-JR due to Brill and Peters (2023).

In more detail, Brill and Peters suggested a whole family of proportionality notions inspired by fairness axioms from approval-based committee elections. To explain these axioms, we define the rank of a candidate c in a preference relation $\succsim_i$ as $\text{rank}(\succsim_i, x) = 1 + |\{y \in A\colon y \succ_i x\}|$. Now, Brill and Peters observe that for each $r \in \{1, \dots, m\}$, we can transform a preference profile R into an approval profile $A(R, r)$ by letting each voter approve the alternatives with a rank of at most r. Based on this, proportionality notions for approval-based committee elections can be transferred to ranked preferences by requiring that a committee satisfies the given proportionality axiom for the approval profiles $A(R, r)$ for all $r \in \{1, \dots, m\}$. Applying this approach to justified representation (JR), one of the weakest fairness notions in approval-based committee voting, results in the following axiom.

Definition 5 (Rank-JR). *A committee W satisfies Rank-JR for a preference profile R and committee size k if for all ranks $r \in \{1, \dots, m\}$ and groups of voters $N' \subseteq N$ such that $|N'| \geq \frac{n}{k}$ and $\bigcap_{i \in N'} \{x \in C\colon \text{rank}(\succsim_i, x) \leq r\} \neq \emptyset$, it holds that $W \cap \bigcup_{i \in N'} \{x \in C\colon \text{rank}(\succsim_i, x) \leq r\} \neq \emptyset$.*

We note that Brill and Peters also adopt several other approval-based fairness axioms to ranked preferences. However, since JR is the weakest approval-based fairness notion, Rank-JR is their weakest fairness notion for ranked preferences. Thus, showing that no committee monotone voting rule satisfies Rank-JR implies that, for ranked preferences, no committee monotone rule satisfies any of the proportionality conditions of Brill and Peters. Moreover, we note that some of these fairness notions are logically related to variants of PSC; we refer to Figure 1 for an overview of these relations.

We are now ready to prove our impossibility theorem.

Theorem 4. *Even for strict preference relations, no committee voting rule satisfies both committee monotonicity and Rank-JR if $m \geq 5$, $n \geq 4$, and n is divisible by 4.*

Proof. We will focus on the case that $m = 5$ and $n = 4$; to increase m, we can simply append new candidates at the bottom of the preference relations of all voters, and to increase n, we can duplicate the whole profile multiple times. Now, assume that f is a committee voting rule that satisfies Rank-JR and consider the following profile R.

Voter 1: $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5$
Voter 2: $c_3 \succ c_2 \succ c_1 \succ c_4 \succ c_5$
Voter 3: $c_4 \succ c_2 \succ c_1 \succ c_3 \succ c_5$
Voter 4: $c_5 \succ c_2 \succ c_1 \succ c_3 \succ c_4$

We will show that Rank-JR necessitates that $c_2 \in f(R, 2)$ and $f(R, 4) = \{c_1, c_3, c_4, c_5\}$. This implies that f fails committee monotonicity because $c_2 \in f(R, 2)$ but $c_2 \notin f(R, 4)$.

Claim 1: $c_2 \in f(R, 2)$**.** Assume for contradiction that $c_2 \notin f(R, 2)$. Since $k = 2$, at most two out of the four voters receive their top-ranked candidate in $f(R, 2)$. However, Rank-JR for $r = 2$ requires also that one of the top-ranked candidates of the other two voters is chosen as all voters rank c_2 second but $c_2 \notin f(R, 2)$. Hence, it holds that $c_2 \in f(R, 2)$.

Claim 2: $f(R, 4) = \{c_1, c_3, c_4, c_5\}$**.** When $k = 4$, Rank-JR for $r = 1$ requires that the top-ranked candidate of each voter is selected. Thus, $f(R, 4) = \{c_1, c_3, c_4, c_5\}$ has to hold. □

8 Conclusions

In this paper, we present the first committee voting rules for ranked preferences that satisfy committee monotonicity and proportionality for solid coalitions (PSC). Specifically, we first give a general scheme for defining such rules based on known rules that satisfy PSC. To extend our positive results to weak preferences, we further design a new committee voting rule called the Solid Coalition Refinement rule that simultaneously satisfies committee monotonicity and IPSC, a variant of PSC for weak preferences. Using this rule, we also disprove a conjecture by Graham-Squire *et al.* (2024) regarding the compatibility of PSC and notion called independence of losing voter blocs for truncated preferences. Finally, we show that committee monotonicity is not compatible with a recently suggested family of fairness concepts by Brill and Peters (2023).

A natural follow-up question is whether our positive results extend to the more general setting of participatory budgeting (PB). In this setting, each candidate has a cost and the task is to choose a set of candidates within a prescribed budget. For PB, committee monotonicity has been generalized to an axiom called *limit monotonicity* (Talmon and Faliszewski, 2019) and IPSC has been defined for this setting by Aziz and Lee (2021). However, a simple counterexample shows that limit monotonicity is incompatible with IPSC. First, we note that IPSC implies *exhaustiveness* (the left-over budget cannot be enough to purchase another candidate). Now, assume that there are two voters and three candidates c_1, c_2, c_3 where c_1 and c_2 have cost 2 and c_3 has cost 1. If c_1 and c_2 are each ranked first by one of the voters, these candidates must be chosen when the budget is 4. However, if the budget is 3, exhaustiveness requires c_3 to be chosen. This violates limit monotonicity, so weaker variants of proportionality or committee monotonicity are required to obtain positive results in PB.

Our results also give other directions for future work. In particular, we leave it open whether there is a polynomial-time computable rule for weak preferences that satisfies the axiomatic properties of SCR. Moreover, much remains unknown about the compatibility of committee monotonicity and proportionality under approval preferences (see also Lackner and Skowron, 2023).

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A Existing rules and committee monotonicity

In this appendix, we show that all existing rules known to satisfy PSC all fail committee monotonicity, even when voters have strict preferences. The three rules known to satisfy PSC are *Single Transferable Vote* (STV), the *Expanding Approvals Rule* (EAR) of Aziz and Lee (2020) and the *Quota Borda System* (QBS) of Dummett (1984). Because Elkind *et al.* (2017) prove that STV does not satisfy committee monotonicity, we only discuss to the other two rules.

A.1 EAR

EAR works as follows. First, each voter is given a budget of $b_i = 1$, which will be used to buy candidates at a cost $q \in (\frac{n}{k+1}, \frac{n}{k}]$ (our subsequent counterexample holds for any such q). Then, EAR works as follows until k candidates have been selected. For each rank r, starting at $r = 1$, the rule checks whether any candidates can be afforded by voters who all rank this candidate in their top r. If so, this candidate is added to the committee and the total budget of the corresponding voters is decreased by q. If no such candidate exists, we increase r by 1 and repeat this process. We note that the choice of the exact candidate and the way to decrease the voters' budgets needs, in principle, further clarification; however, these issues will not matter for our counterexample.

Now, we first note that Brill and Peters (2023) have shown that EAR satisfies Rank-PJR+ and thus also Rank-JR. Hence, it follows immediately from Theorem 4 that EAR fails committee monotonicity. Nevertheless, we additionally present a simple profile R with $n = 2$ voters and $m = 3$ candidates where EAR violates committee monotonicity:

Voter 1: $c_1 \succ c_3 \succ c_2$
Voter 2: $c_2 \succ c_3 \succ c_1$

For this profile, if $k = 1$, no candidate can be afforded if $r = 1$. Hence, the rank r is increased to 2, and c_3 is chosen. So, the selected committee is $\{c_3\}$. By contrast, if $k = 2$, candidates c_1 and c_2 are both elected at rank $r = 1$ and so the elected committee is $\{c_1, c_2\}$. This fails committee monotonicity since c_3 is elected when $k = 1$ but not when $k = 2$.

A.2 QBS

QBS was introduced by Dummett (1984) and later generalized into a class called *Minimal Demand* (MD) rules by Aziz and Lee (2022). For each rank r, starting at 1 and increasing one by one, these rules perform two steps:

1. Partition the voters into solid coalitions, where two voters are in the same solid coalition if their top r ranked candidates are the same, regardless of the ordering within the top r.
2. Consider each solid coalition N' supporting candidate set C' in the partition. If N' is entitled to more representation under PSC, then additional candidates from C' are elected until this entitlement is met.

The family of rules differ by the tie-breaking used in the second step. Dummett (1984) suggests that the Borda score[1] be used. Aziz and Lee (2022) showed that PSC is satisfied regardless of the tie-breaking method.

We show that MD rules fail committee monotonicity whenever positional scoring is used for tie-breaking. A positional scoring rule consists of a score vector $(s_1, s_2, \ldots, s_m)$, where $s_1 \geq s_2 \geq \ldots \geq s_m \geq 0$ and $s_1 > s_m$. A candidate earns s_r points if a voter ranks them in position r. The tie-breaking in step 2 selects the candidate with the highest total score across all n voters. Further tie-breaking may be needed if there are ties in the positional scoring, but our impossibility result holds regardless of the additional tie-breaking method.

Proposition 2. *Minimal Demand (MD) rules that break ties using positional scoring do not satisfy committee monotonicity, even for strict preferences.*

Proof. Consider an MD rule with a positional scoring vector $(s_1, s_2, \ldots, s_m)$. We assume $s_m = 0$: if it wasn't, then we could use a new vector $(s_1 - s_m, s_2 - s_m, \ldots, 0)$ without changing any tie-breaking decisions. We consider two cases depending on the scoring vector.

Case 1: $s_1 < 2s_2$. For this case, we construct a profile R with $n = 2$ voters and $m = 3$ candidates. To increase n, the entire profile can be duplicated multiple times. To increase m, extra candidates can be added to the preference relations of the voters as shown below.

Voter 1: $c_1 \succ c_3 \succ$ any extra candidates $\succ c_2$
Voter 2: $c_2 \succ c_3 \succ$ any extra candidates $\succ c_1$

For $k = 1$, a solid coalition must include both voters to be entitled to representation and so no candidates are elected until $r = m$. The tie-breaking will select candidate c_3 since its score of $2s_2$ exceeds the scores of the other candidates. For $k = 2$, voters 1 and 2 both separately form solid coalitions when $r = 1$, so candidates c_1 and c_2 are selected. This violates committee monotonicity because c_3 is elected when $k = 1$ but not when $k = 2$.

Case 2: $s_1 \geq 2s_2$. For this case, we construct a profile R with $n = 11$ voters and $m = 12$ candidates. To increase n, the entire profile can be duplicated ℓ times. To increase m, extra candidates can be added to the end of preference relations of each voter.

Voter 1: $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_9 \succ$ other candidates
Voter 2: $c_2 \succ c_3 \succ c_4 \succ c_1 \succ c_9 \succ$ other candidates
Voter 3: $c_3 \succ c_4 \succ c_1 \succ c_2 \succ c_9 \succ$ other candidates
Voter 4: $c_4 \succ c_1 \succ c_2 \succ c_3 \succ c_9 \succ$ other candidates
Voter 5: $c_5 \succ c_6 \succ c_7 \succ c_8 \succ c_9 \succ$ other candidates
Voter 6: $c_6 \succ c_7 \succ c_8 \succ c_5 \succ c_9 \succ$ other candidates
Voter 7: $c_7 \succ c_8 \succ c_5 \succ c_6 \succ c_9 \succ$ other candidates
Voter 8: $c_8 \succ c_5 \succ c_6 \succ c_7 \succ c_9 \succ$ other candidates
Voters 9 to 11: $c_9 \succ c_{10} \succ c_{11} \succ c_{12} \succ$ other candidates

[1] For each voter, their lowest-ranked candidate is given 0 points, their second lowest candidate 1 point, their third lowest 2 points, etc. The candidate with the highest total score is chosen.

Note that voters 1 to 4 form a solid coalition supporting $\{c_1, c_2, c_3, c_4\}$ and voters 5 to 8 form a solid coalition supporting $\{c_5, c_6, c_7, c_8\}$. For $k = 2$, the committee will consist of one candidate from $\{c_1, c_2, c_3, c_4\}$ and another from $\{c_5, c_6, c_7, c_8\}$. Now consider $k = 1$. A solid coalition needs at least 6 voters to be entitled to representation, and any such solid coalition will include c_9 in its supported candidates. We will show that c_9 has a higher score than all of c_1 through c_8, meaning that committee monotonicity will be violated:

- Candidates c_1 through c_8 are ranked first, second, third, and fourth by exactly one voter each. Therefore their scores are upper bounded by $s_1 + s_2 + s_3 + s_4 + 7s_5 \leq s_1 + 3s_2 + 7s_5$.
- Candidate c_9 has a score of $3s_1 + 8s_5$.

If $s_2 = 0$, then $s_i = 0$ for all $i \geq 2$ and c_9's score of $3s_1$ is higher than c_1 through c_8's score of s_1. Otherwise, assume that $s_2 > 0$:

$$\begin{aligned} 3s_1 + 8s_5 &\geq s_1 + 4s_2 + 8s_5 && \text{since } s_1 \geq 2s_2 \\ &> s_1 + 3s_2 + 8s_5 && \text{since } s_2 > 0 \\ &\geq s_1 + 3s_2 + 7s_5, \end{aligned}$$

and so c_9 has a higher score than c_1 through c_8. □

Since the Borda rule is a positional scoring rule, we have the following corollary.

Corollary 1. *QBS does not satisfy committee monotonicity.*

B Missing Proof of Proposition 1

We next present the proof of Proposition 1, which was omitted from the main body.

Proposition 1. *The SCR rule always terminates and produces a committee of the target size k. Furthermore, for strict preferences, it can be implemented to run in polynomial time.*

Proof. First, we will show that if the SCR rule terminates, it produces a committee of size k. To this end, we note that each iteration of the outer loop (line 2) adds exactly one candidate to W. The reason for this is that the set $\Phi(R, W, D)$ only contains pairs (N', C') such that $C' \not\subseteq W'$, which means that D will always contain at least one alternative not in W. On the other hand, we only reach line 8 after the end of the while-loop, which requires that $|D \setminus W| \leq 1$. In combination, this means that $|D \setminus W| = 1$, so each iteration of the outer loop adds a single candidate.

Next, we will show that SCR always terminates. To this end, we note that if $\Phi(R, W, D)$ is always non-empty, then each iteration of the while-loop (line 4) reduces the size of D by at least 1. This holds because C' is a strict subset of D for all $(N', C') \in \Phi(R, W, D)$. Hence, we only need to show that $\Phi(R, W, D)$ is always non-empty during the execution of the SCR rule. For this, consider the set D in some round during the execution of the SCR rule and suppose that $|D \setminus W| > 1$. We first note that D is supported by a generalized solid coalition N''. In more detail, if $D = C$, this holds as C is supported by the set of all voters N. On the other hand, if $D \neq C$, then D was chosen as the set of candidates supported by a generalized solid coalition N'' in the previous iteration of the while-loop. Next, let $i \in N''$, which means that $d \succsim_i c$ for all $d \in D$, $c \in C \setminus D$. Moreover, let d^* denote one of voter i's least favorite candidates in $D \setminus W$ and define $D' = \{c \in C \setminus \{d^*\} \colon c \succsim_i d^*\}$. By the definition of this set and d^*, it holds that $D' \subsetneq D$, $D' \not\subseteq W$, and that $\{i\}$ is a generalized solid coalition supporting D'. Hence,$(\{i\}, D') \in \Phi(R, W, D)$, which proves that $\Phi(R, W, D) \neq \emptyset$. Thus, the SCR rule is indeed well-defined.

Finally, we show that the SCR rule can be computed in polynomial time for strict preference profiles. Since line 4 is repeated at most mk times, we only need to show that the procedure in the while-loop can be done in polynomial time. For this, we note that for strict preferences, IPSC coincides with PSC and a generalized solid coalition is equivalent to a solid coalition. Furthermore, there are only nm candidate sets C' such that (N', C') can be in $\Phi(R, W, D)$: for each voter i and each rank r, we can obtain one such set by considering the r most preferred candidates of voter i. Finally, we can compute line 6 by evaluating $\rho(W, N', C')$ for all such sets C' and the maximal solid coalitions N' supporting C'. This works because for any two sets N', N'' supporting C', it holds that $\overline{C'}(N') = \overline{C'}(N'')$ and thus $\rho(W, N', C') \geq \rho(W, N'', C')$ if $|N'| \geq |N''|$. Moreover, we can compute the maximal solid coalition supporting a set C' by simply checking for each voter whether he supports C'. Since all of these steps can be done in polynomial time, it follow that the SCR rule can be computed in polynomial time for strict preferences. □