What Is A Skeptical Proof?

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Abstract. We investigate the task of skeptically reasoning in extension-based, nonmonotonic logics by concentrating on general argumentation theories. The restricted applicability of Dung's notion of skeptical provability in his well-known argumentation framework is illustrated, and a new approach based on the notion of a *claim* associated with each argument is proposed. We provide a formal definition of a skeptical proof in our framework. As a concrete formalism, default logic in case of normal default theories is embedded in the general framework. We prove a formal correspondence between the two notions of skeptical provability, which enables us to adopt the general concept of a skeptical proof into default logic.

1 Introduction

This paper deals with a specific way of drawing conclusions in nonmonotonic frameworks, namely, skeptical reasoning. The term nonmonotonicity stems from the observation that the addition of information to an incomplete knowledge base may change the set of drawable conclusions. The treatment of incomplete information constitutes one of the central problems for complex information systems.

A variety of topical nonmonotonic frameworks, such as default logic [6], THEORIST [4], or Autoepistemic Logic [3], are based on the notion of so-called *extensions*. This term describes the process of creating a reasonable set of beliefs by extending one's certain knowledge through the application of default or uncertain rules. An extension is then obtained by applying such rules as long as this is possible without getting tangled up.

Consider, as an example, the well-known *Nixon Diamond*: It consists of the fact that Nixon is known to be both Quaker and republican. Furthermore, we have the two vague rules that normally Quakers are pacifists and normally republicans are not (see Fig. 1(a)). Here it is possible to extend the certain knowledge about Nixon's membership of two organizations by either the default conclusion that he is a pacifist (as he is Quaker) or else the default conclusion that he is a non-pacifist (as he is republican). Note that one should not apply the respective instances of both default rules together as this would yield to a contradictory, hence unreasonable, set of beliefs. Therefore we obtain two different extensions in this example.

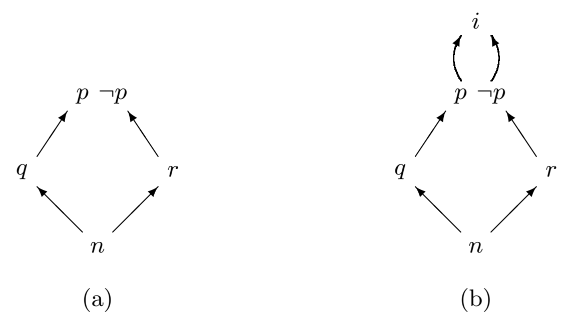

Fig. 1. (a) The basic Nixon Diamond, stating that Nixon (n) is known to be Quaker (q) as well as republican (r) and that Quakers normally are pacifists (p) while republicans normally are non-pacifists ($\neg p$). (b) The extended diamond where it is additionally stated that pacifists normally are interested in politics (i), as are non-pacifists.

The Nixon Diamond illustrates that a theory formalized in a nonmonotonic framework admits, in general, more than just one extension. This observation has led to two entirely different kinds of reasoning in such frameworks. A *credulous* reasoner proves a formula by showing it to be consequence of at least one extension. In contrast, a *skeptical* reasoner proves a formula only if it is a consequence of all possible extensions. Here, we focus on the latter proof paradigm.

Regarding our example, we can prove, say, Nixon be Quaker both credulously as well as skeptically. In contrast, Nixon being a pacifist is only credulously entailed as there exists an extension that does not include this information.

In [1], a general approach to nonmonotonic frameworks has been presented which is based on the concept of argumentations: Given a set of *arguments* along with a binary relation describing which argument *attacks* another argument, one can formally construct reasonable sets of beliefs (extensions) in the way we have informally described above. This framework can be taken as a general semantics for nonmonotonic logics such as, for instance, default logic, by interpreting reasoning in default theories as argumentation [1].

In order to illustrate Dung's method, recall our example. It can be adequately modeled in the argumentation framework by introducing these four arguments

$$\begin{array}{l} q\,, \\ r\,, \\ p\_\mathtt{since}\_q\,,\ \neg p\_\mathtt{since}\_r \end{array}$$

Furthermore, let the attacking relation be defined by

$$\begin{array}{rcl} p\_\mathtt{since}\_q & \mathit{attacks} & \neg p\_\mathtt{since}\_r\,, \\ \neg p\_\mathtt{since}\_r & \mathit{attacks} & p\_\mathtt{since}\_q \end{array}$$

Now, a so-called *preferred extension* [1] is defined as a maximal set of arguments which is conflict-free (i.e., there are no attacks within this set) and which defends itself (i.e., each argument attacking some element in this set is itself attacked by some element of this set). It is easy to verify that we obtain two preferred extensions in our example, viz. $\{q, r, p\_\texttt{since}\_q\}$ and $\{q, r, \neg p\_\texttt{since}\_r\}$, as intended. This is so because the critical arguments $p\_\texttt{since}\_q$ and $\neg p\_\texttt{since}\_r$ attack each other, hence defend themselves.

This concept of an extension supports credulous reasoning by defining an argument to be credulously entailed if it belongs to at least one preferred extension. In order to prove skeptically, the notion of a *grounded* extension has additionally been introduced [1]. This unique set of arguments is obtained by starting with all arguments that are not attacked by any other argument; then successively all arguments are added that are defended by those which are already included in the set being constructed. In our example, we obtain $\{q, r\}$ as the grounded extension because both q and r cannot be attacked and, moreover, we cannot add $p\_\texttt{since}\_q$ nor $\neg p\_\texttt{since}\_r$ as none of them is defended by q or r. Hence, we again obtain the desired result, namely, Nixon being Quaker as well as republican is skeptically provable but nothing can be skeptically determined about him being pacifist or not.

However, by slightly extending our example it is possible to illustrate the restrictive applicability of this concept: Consider two additional default rules stating, respectively, that pacifists normally are interested in politics and that non-pacifists normally are interested in politics as well (see Fig. 1(b)). Again, one would expect two extensions, one where Nixon is assumed to be a pacifist and interested in politics (aside from being Quaker and republican) and one extension where Nixon is expected to be a non-pacifist and interested in politics. In particular, we would like to conclude i also in case of reasoning skeptically since this fact is known to hold in both extensions.

Regarding our argumentation-based formalization, the additional two default rules of Fig. 1(b) bring about two new arguments, viz.

$$i\_\texttt{since}\_p\_\texttt{since}\_q\,,$$
$$i\_\texttt{since}\_\neg p\_\texttt{since}\_r$$

claiming that Nixon is interested in politics because he is a pacifist (as he is Quaker) or because he is a non-pacifist (as he is republican), respectively. The attacking relation is extended as follows:

$$\begin{array}{rcl} p\_\texttt{since}\_q & \textit{attacks} & i\_\texttt{since}\_\neg p\_\texttt{since}\_r\,, \\ \neg p\_\texttt{since}\_r & \textit{attacks} & i\_\texttt{since}\_p\_\texttt{since}\_q \end{array}$$

Again, we obtain two preferred extensions: $\{q, r, p\_\texttt{since}\_q, i\_\texttt{since}\_p\_\texttt{since}\_q\}$ and $\{q, r, \neg p\_\texttt{since}\_r, i\_\texttt{since}\_\neg p\_\texttt{since}\_r\}$. The argument $i\_\texttt{since}\_p\_\texttt{since}\_q$, say, is defended by $p\_\texttt{since}\_q$ against the attack $\neg p\_\texttt{since}\_r$.

Now, let us see what happens in case of skeptical reasoning. The computation of the grounded extension starts with the two arguments p and q that are not attacked by any other argument. However, there are no other arguments that are defended by these two; hence, the grounded extension is just $\{p, q\}$. This extension does not allow to conclude i, in contrast to what one expects for the extended Nixon Diamond.

In general, using the grounded extension is too restrictive with regard to the intuition behind skeptical reasoning. More concretely, for instance, in [1] it is shown that and how default logic can be embedded in the argumentation framework. By employing an adequate transformation, the set of preferred extensions coincides with the set of extensions of a default theory. On the other hand, the notions of skeptical reasoning do not coincide, as can be shown by our extended example, say. This unexpected difference is caused by the fact that the grounded extension is built up from—in a certain sense—general, irrefutable arguments only. It does not take into account the possibility that different arguments may have been designed to support the same proposition: Although our extended example does not include a non-disputable argument stating that Nixon is interested in politics, each possible set of beliefs includes an argument claiming this.

From this perspective, we suggest a more elaborated notion of skeptical reasoning in argumentation theories. To this end, we introduce the concept of a *claim* associated with each argument. Based on this conceptual extension, we define a claim to be skeptically entailed if each possible set of beliefs contains at least one argument claiming it. We will illustrate that, furthermore, this view leads us to the concept of a *skeptical proof* which, informally spoken, consists of a set of arguments each supporting the claim under consideration such that each reasonable set of beliefs is—in a certain sense—covered by this set of arguments. For instance, if we define both arguments $i\_\mathtt{since}\_p\_\mathtt{since}\_q$ and $i\_\mathtt{since}\_\neg p\_\mathtt{since}\_r$ to claim i then it is reasonable to consider these two arguments as a skeptical proof for i as each preferred extension includes one of them. We will illustrate the adequateness of our notion of skeptical provability as regards the common understanding of this notion. More concretely, we prove it to coincide with the definition of skeptical entailment in default logic when that formalism is embedded in the general framework.

The paper is organized as follows. In the next Section 2, we present a modified version of the general argumentation theory proposed in [1]. Aside from introducing the notion of a claim, we use a slightly different interpretation of an argument: Instead of defining an underlying attacking relation, we employ a *conflict* relation to describe collections of arguments which should not be accepted together. In order to make these differences visible, we call our modified approach *disputation* framework. Thereafter, we provide a formal definition of a skeptical proof following the example above. We show that our definition is reasonable in so far as it models the usual intention behind skeptical provability, namely, the formula at hand being supposed to belong to every preferred extension. In Section 3, we illustrate how Reiter's default logic [6], restricted to normal default theories, can be embedded in our general disputation framework. We prove that the respective concept of an extension in both frameworks coincide. Hence, we can adopt our notion of a skeptical proof into default logic. Finally, the significance of our approach is discussed and future work is outlined in Section 4.

2 Arguments and Claims

With the following fundamental definition we introduce our so-called disputation framework. It includes the notion of a *claim* associated with each argument, as argued in the preceding section. Moreover, it differs from the argumentation theoretic approach of [1] in so far as the arguments interact by means of a *conflict* relation. This relation is used to state which sets of arguments shall not occur together in a single, reasonable set of beliefs.

Definition 1. A *disputation framework* DF is a triple $\langle AR, CL, \mathit{conflict} \rangle$ where AR is a set of *arguments*, CL is a set of *claims* such that each argument $a \in AR$ is associated with a particular claim, written $\mathit{claims}(a) \in CL$, and $\mathit{conflict} \subseteq 2^{AR}$ such that $\emptyset \notin \mathit{conflict}$ and $\mathit{conflict} \cap \{\{a\} \mid a \in AR\} = \emptyset$.

The two restrictions placed on the *conflict* relation require, in words, that neither the empty set of arguments nor a set containing only a single argument is conflicting.

As an example, consider these four arguments, stated along with their claims:

$$\mathit{claims}(q) = q \tag{1}$$

$$\mathit{claims}(r) = r \tag{2}$$

$$\mathit{claims}(p\_\mathtt{since}\_q) = p \tag{3}$$

$$\mathit{claims}(\neg p\_\mathtt{since}\_r) = \neg p \tag{4}$$

Furthermore, let the conflict relation consist of just this single set of argument:

$$\{\, p\_\mathtt{since}\_q \,,\ \neg p\_\mathtt{since}\_r \,\} \tag{5}$$

The resulting disputation framework, $\langle (1) - (4), \{q, r, p, \neg p\}, \{(5)\} \rangle$, is an encoding of our introductory example, the basic Nixon Diamond (see Fig. 1(a)).

Next, we define the notion of a maximal acceptable set of beliefs, here called a *preferred extension* on the analogy of [1].

Definition 2. Let $\langle AR, CL, \mathit{conflict} \rangle$ be a disputation framework. A set of arguments $A \subseteq AR$ is *conflict-free* iff there is no $B \subseteq A$ such that $B \in \mathit{conflict}$. A *preferred extension* is a maximal (wrt set inclusion) conflict-free set.

In other words, a preferred extension is a maximal set of arguments that does not include any conflicting subset. For instance, $\{(1), (2), (4)\}$ is conflict-free whereas $\{(3), (4)\}$ is not; the former is also one of the two preferred extensions in our example while $\{(1), (2), (3)\}$ is the second one. Hence, we obtain the expected result.

Given a disputation framework, we interpret each argument as a credulous proof of its claim. This coincides with the usual interpretation of credulously provable, namely, being included in at least one (preferred) extension.[1] We now

[1] Note that each argument belongs to at least one preferred extension following Definition 2.

turn to the problem of skeptical reasoning in our approach. As argued in the introduction, it is too restrictive, hence not adequate, to search for arguments belonging to each preferred extension (i.e., which are not member of any conflicting set). Instead, we intend to verify that the claim under consideration is supported by at least one argument in every preferred extension. Consider, for example, our extended Nixon Diamond depicted in Fig. 1(b). In order to formalize it as a disputation framework, we employ, in addition to (1)–(4), the following two arguments, again stated along with their claims:

$$\mathit{claims}(i\_\mathtt{since}\_p\_\mathtt{since}\_q) = i \tag{6}$$

$$\mathit{claims}(i\_\mathtt{since}\_\neg p\_\mathtt{since}\_r) = i \tag{7}$$

I.e., we now have $CL = \{q, r, p, \neg p, i\}$. Henceforth, let *conflict* consist of (5) along with

$$\begin{array}{l} \{\, p\_\mathtt{since}\_q \,,\, i\_\mathtt{since}\_\neg p\_\mathtt{since}\_r \,\}\,, \\ \{\, \neg p\_\mathtt{since}\_r \,,\, i\_\mathtt{since}\_p\_\mathtt{since}\_q \,\}\,, \\ \{\, i\_\mathtt{since}\_p\_\mathtt{since}\_q \,,\, i\_\mathtt{since}\_\neg p\_\mathtt{since}\_r \,\} \end{array}$$

The reader is invited to verify that we obtain two preferred extensions again, viz. $\{(1),(2),(3),(6)\}$ and $\{(1),(2),(4),(7)\}$, which is the intended result according to our discussion in Section 1.

Clearly, one expects both q and r to be skeptically provable since the respective arguments (1) and (2) are contained in both preferred extension. Each of these two arguments can thus be taken as a skeptical proof for this claim. But, as argued in the introduction, we also expect i to be skeptically provable since each preferred extension includes an argument claiming this formula (namely, (6) or (7)). As, however, there is no unique argument supporting i that is included in every preferred extension, we cannot simply take one of these arguments as a skeptical proof. For this reason, we define a more general notion of a skeptical proof by allowing sets of arguments, each of them claiming the formula under consideration. This formula is skeptically provable as soon as such a collection of arguments takes into account every possible argumentation:

Definition 3. Let $\langle AR, CL, \mathit{conflict}\rangle$ be a disputation framework and $g \in CL$. A set $P \subseteq AR$ of arguments, each of them claiming g, is a *skeptical proof* for g iff for each conflict-free set of arguments $A \subseteq AR$ there is some $p \in P$ such that $A \cup \{p\}$ is conflict-free.

In other words, a skeptical proof contains, for each reasonable collection of arguments, an argument that can be added without getting tangled up. For instance, we can take $\{i\_\mathtt{since}\_p\_\mathtt{since}\_q, i\_\mathtt{since}\_\neg p\_\mathtt{since}\_r\}$ as a skeptical proof for i in our exemplary, extended disputation framework—while it is impossible to find a skeptical proof for, say, p.

It is easy to verify that the notion of a skeptical proof coincides with our intention to ensure the formula under consideration be contained in every preferred extension:

Proposition 4. *Let* $\langle AR, CL, \mathit{conflict}\rangle$ *be a disputation framework. For each* $g \in CL$ *, there is a skeptical proof for* g *iff* g *is claimed in every preferred extension by some element.*

Proof.

"$\Rightarrow$": Let P be a skeptical proof for g and let A be an arbitrary preferred extension. Since A is conflict-free, we can find some $p \in P$ such that $A \cup \{p\}$ is conflict-free. But A is a maximal conflict-free set, hence it must include p. Due to $\mathit{claims}(p) = g$ it follows that g is claimed by some member of A.

"$\Leftarrow$": Assume g be claimed in every preferred extension by some element. Let $\{A_i \subseteq AR \mid i \in \mathcal{I}\}$ be the set of all preferred extensions (for some index set $\mathcal{I}$). Then, for each $i \in \mathcal{I}$, let $a_i \in A_i$ denote an argument claiming g. We can find, for each conflict-free set of arguments $A \subseteq AR$, some preferred extension A_i containing A, which implies $A \cup \{a_i\}$ be conflict-free. Hence, defining $P = \{a_i \mid i \in \mathcal{I}\}$ provides us with a skeptical proof for g.

Furthermore, we can show that our notion of skeptical provability generalizes Dung's concept of a grounded extension. The latter concept is adapted to our approach in the following proposition:

Proposition 5. *Let* $DF = \langle AR, CL, \mathit{conflict}\rangle$ *be a disputation framework. If we define the* grounded *extension of* DF *to be the set of arguments*

$$GE \;=\; \{\, a \mid \forall A \subseteq AR.\ A \cup \{a\} \in \mathit{conflict} \;\Rightarrow\; A \in \mathit{conflict}\,\}$$

then, for each $a \in GE$ *,* $\mathit{claims}(a)$ *has a skeptical proof.*

Proof. It is obvious that $P = \{a\}$ is skeptical proof for $\mathit{claims}(a)$ since a can be added to each conflict-free set of arguments without causing a conflict.

3 Credulous and Skeptical Default Proofs

In what follows, we illustrate how a concrete nonmonotonic logic, namely, default logic, can be adequately formalized as disputation framework. The underlying intention is to interpret the set of (credulous) default proofs as the set of arguments in the corresponding framework. As will be shown at the end of this section, this allows us to adopt our notion of a skeptical proof developed above. To begin with, let us recapitulate the definition of (normal) default theories and their extensions [6]:

Definition 6. A *default theory* (D, W) consists of a set W, called *world knowledge*, of closed formulas and a set D of *defaults* of the form $\delta = \frac{\alpha\,:\,\beta}{\omega}$, where $\alpha = \mathit{Prereq}(\delta)$, called *prerequisite*, $\beta = \mathit{Justif}(\delta)$, called *justification*, and $\omega = \mathit{Conseq}(\delta)$, called *consequence*, are formulas. If these three components contain free variables then the default is taken as a representative of all its ground instances. The three selection functions Prereq, Justif and Conseq extend to sets of defaults in the obvious way. A default of the form $\frac{\alpha:\omega}{\omega}$ is called *normal*, as is a default theory containing only normal defaults.

Given a default theory $\Delta = (D, W)$, let, for any set of formulas S, the set $\Gamma(S)$ be the smallest set of formulas S' such that

1. $W \subseteq S'$;
2. $Th(S') = S'$;[2] and
3. for any $\frac{\alpha:\beta}{\omega} \in D$, if $\alpha \in S'$ and $\neg\beta \notin S$ then $\omega \in S'$.

A set of formulas E is called an *extension* of Δ iff $\Gamma(E) = E$. A closed formula g is *credulously* (resp. *skeptically*) *entailed* from Δ iff it belongs to some (resp. every) extension.

The basic version of our running example (Fig. 1(a)) can be formalized in default logic as follows. Let $W = \{q, r\}$ and $D = \{\frac{q:p}{p}, \frac{r:\neg p}{\neg p}\}$. Following the definition above, this theory admits two extensions, namely, $Th(\{q, r, p\})$ and $Th(\{q, r, \neg p\})$, respectively. In order to model the extended Nixon Diamond (Fig. 1(b)), we additionally use the two defaults $\frac{p:i}{i}$ and $\frac{\neg p:i}{i}$. This extended theory admits two extensions as well, viz. $Th(\{q, r, p, i\})$ and $Th(\{q, r, \neg p, i\})$, respectively.

For our purpose, the following alternative, (pseudo-)iterative characterization of an extension, stated and proved in [6], is useful.

Proposition 7. *Let $\Delta = (D, W)$ be a default theory and E be a set of formulas. Furthermore, let*

1. $E_0 := W$ *and*
2. $E_i := Th(E_{i-1}) \cup \{\omega \mid \frac{\alpha:\omega}{\omega} \in D \ \&\ \alpha \in E_{i-1} \ \&\ \neg\omega \notin E\}$ *for* $i = 1, 2, \ldots$

then E is an extension of Δ iff $E = \bigcup_{i=0}^{\infty} E_i$.

We now turn to the question how default logic can be embedded in our disputation framework. With the next definition we follow Dung's basic idea [1] by calling a set of justifications to *support* a closed formula if, informally spoken, this formula is entailed after having applied defaults whose justifications are contained in its support. Yet our definition differs from Dung's in so far as we do not only require *groundedness* (see, e.g., [8]) but also consistency of a support with the world knowledge. We use this additional criterion in view of identifying a formula—along with some support—with an argument.

Definition 8. Let (D, W) be a normal default theory. A *support* for a closed formula g is a set of formulas Jus such that $W \cup Jus \not\models \bot$ and there exists a sequence $e_0, e_1, \ldots, e_n$ of formulas with $e_n = g$ and, for each $0 \leq k \leq n$, either $e_k \in W$, or e_k is a logical consequence of $\{e_0, \ldots, e_{k-1}\}$, or there exists some $\frac{p:e_k}{e_k} \in D$ such that p is a logical consequence of $\{e_0, \ldots, e_{k-1}\}$ and $e_k \in Jus$.

[2] For any set F of formulas, $Th(F)$ denotes the deductive closure of F.

Recall, for instance, our exemplary (extended) default theory. Both $Jus_1 = \{p, i\}$ and $Jus_2 = \{q, i\}$ support i; the former, say, is justified to be support by the sequence $e_0 = q, e_1 = p, e_2 = i$ (based on $\frac{q:p}{p}$ and $\frac{p:i}{i}$).

The notion of support resembles the way extensions are constructed via Proposition 7. In order to prove a formal correspondence between these two concepts, we need the following notion [6]: If E is an extension of a default theory then E is said to be *based* on a sequence of defaults $\langle \delta_i \rangle_{i \in \mathcal{I}}$ (for some index set $\mathcal{I}$) if this sequence contains exactly those defaults which have been applied during the generation process given by Proposition 7. We then obtain the following relationship between extensions and the notion of support:

Proposition 9. *Let $\Delta = (D, W)$ be a normal default theory, E an extension of Δ based on $\langle \delta_i \rangle_{i \in \mathcal{I}}$, and g a closed formula.*

1. *If some $Jus \subseteq Justif(\langle \delta_i \rangle_{i \in \mathcal{I}})$ is support for g then $g \in E$.*
2. *If $g \in E$ then $Jus = Justif(\langle \delta_i \rangle_{i \in \mathcal{I}})$ is support for g.*

Proof.

1. Since Jus is support for g, we can find a sequence $e_0, e_1, \ldots, e_n = g$ of formulas satisfying the conditions of Definition 8. For each $k = 0, \ldots, n$ we show $e_k \in E$ provided $e_0, \ldots, e_{k-1} \in E$:
   - If $e_k \in W$ then $e_k \in E$.
   - If e_k is a logical consequence of $\{e_0, \ldots, e_{k-1}\}$ and $\{e_0, \ldots, e_{k-1}\} \subseteq E$ then $e_k \in E$ due to $E = Th(E)$.
   - If $\frac{p:e_k}{e_k} \in D$ such that $e_k \in Jus$ and $\{e_0, \ldots, e_{k-1}\} \models p$ then $p \in E$ and $e_k \in Justif(\langle \delta_i \rangle_{i \in \mathcal{I}})$, hence $e_k \in E$.
2. Following a result stated in [6], $g \in E$ implies the existence of a finite sequence $\delta_1, \ldots, \delta_m \in \langle \delta_i \rangle_{i \in \mathcal{I}}$ such that
   - $W \cup Conseq(\{\delta_1, \ldots, \delta_m\}) \models g$,
   - $W \cup Conseq(\{\delta_1, \ldots, \delta_{j-1}\}) \models Prereq(\delta_j)$, for $1 \leq j \leq m$, and
   - $W \cup Justif(\{\delta_1, \ldots, \delta_m\}) \not\models \bot$.

   Due to $Justif(\{\delta_1, \ldots, \delta_m\}) \subseteq Jus$ this implies Jus be support for g.

In order to model reasoning in normal default theories as disputation framework, we identify with an argument each pair consisting of a formula along with a minimal support. Clearly, such an argument shall claim exactly the formula which is supported. A set of arguments belongs to the conflict relation if the respective supports are contradictory (wrt the world knowledge):

Definition 10. Let $\Delta = (D, W)$ be a normal default theory. The *corresponding* disputation framework $DF_\Delta = \langle AR, CL, conflict \rangle$ is defined as follows:

- CL is the set of all closed formulas.
- $AR = \{ (g, Jus) \mid g \in CL,\ Jus$ minimal (wrt set inclusion) support for $g \}$, and if $(g, Jus) \in AR$ then $claims((g, Jus)) = g$.
- For each $A \subseteq AR$, $A \in conflict$ iff $W \cup \bigcup_{(g, Jus) \in A} Jus \models \bot$.[3]

[3] The reader should note that this definition of *conflict* meets the conditions of Definition 1.

The following main theorem of this section shows that the notions of an extension in a default theory and a preferred extension in the corresponding disputation framework coincide.

Theorem 11. *Let* $\Delta = (D, W)$ *be a normal default theory with corresponding disputation framework* $DF_\Delta = \langle AR, CL, \mathit{conflict}\rangle$ *.*

1. *For each extension* E *of* Δ *there is a preferred extension* A *of* DF *such that* $\{g \mid (g, Jus) \in A\} = E$ *.*
2. *For each preferred extension* A *of* DF_Δ *,* $E = \{g \mid (g, Jus) \in A\}$ *is an extension of* Δ *.*

Proof.

1. Let $\langle\delta_i\rangle_{i\in\mathcal{I}}$ denote the basis of E . Then, let $J = \mathit{Justif}(\langle\delta_i\rangle_{i\in\mathcal{I}})$ and define $A = \{\, (g, Jus) \mid g \in CL \;\&\; Jus \subseteq J$ minimal support for $g\,\}$. From Proposition 9 (1.), we conclude $\{g \mid (g, Jus) \in A\} \subseteq E$, and Proposition 9 (2.) implies $E \subseteq \{g \mid (g, Jus) \in A\}$. We show that A is preferred extension of DF_Δ :
   (a) If A were not conflict-free, we could find some subset $B \subseteq A$ such that $B \in \mathit{conflict}$, i.e., $W \cup \bigcup_{(g,Jus)\in B} Jus \models \bot$. As $J \supseteq \bigcup_{(g,Jus)\in A} Jus \supseteq \bigcup_{(g,Jus)\in B} Jus$, this would imply $W \cup J \models \bot$, which is a contradiction to $\langle\delta_i\rangle_{i\in\mathcal{I}}$ being the basis of E and $J = \mathit{Justif}(\langle\delta_i\rangle_{i\in\mathcal{I}})$.
   (b) To prove A be maximal, let $(g, Jus) \in AR \setminus A$, i.e., we can find a sequence $e_0, e_1, \ldots, e_n = g$ satisfying the conditions of Definition 8. We show the existence of some $a \in A$ such that $\{a, (g, Jus)\} \in \mathit{conflict}$. The fact that $(g, Jus) \notin A$ implies the existence of some $k \in \{0, \ldots, n\}$ such that $\frac{p\,:\,e_k}{e_k} \in D$ for some p , $\{e_0, \ldots, e_{k-1}\} \models p$, and $e_k \in Jus$ but $e_k \notin J$. Now, assume k be minimal wrt this property and define $Jus' = Jus \cap \{e_0, \ldots, e_{k-1}\}$.[4] Minimality of k implies $Jus' \subseteq J$. Hence, $p \in E$ following Proposition 9(1.); on the other hand, from $e_k \notin J = \mathit{Justif}(\langle\delta_i\rangle_{i\in\mathcal{I}})$ we learn that $\frac{p\,:\,e_k}{e_k}$ has not been applied to obtain E , hence $\neg e_k \in E$. From Proposition 9(2.) we then conclude J be support for $\neg e_k$. Let $Jus'' \subseteq J$ be minimal wrt this property then $(\neg e_k, Jus'') \in A$. From Definition 8 we conclude $W \cup Jus'' \models \neg e_k$; on the other hand, $W \cup Jus \models e_k$; hence, $W \cup Jus'' \cup Jus \models \bot$, i.e., $\{(\neg e_k, Jus''), (g, Jus)\} \in \mathit{conflict}$.
2. We prove $E = \bigcup_{i=0}^{\infty} E_i$ according to Proposition 7.
   (a) We first show $\bigcup_{i=0}^{\infty} E_i \subseteq E$ by induction on i .
   In the base case, $E_0 = W$, we know for each $g \in W$ that $(g, \{\}) \in AR$ according to Definition 8 and Definition 10. Hence, as this argument can be added to any conflict-free set without causing conflicts and since A is maximal, $(g, \{\}) \in A$, which implies $g \in E$.
   In case $i > 0$ let $E_i = Th(E_{i-1}) \cup \{\omega \mid \frac{\alpha\,:\,\omega}{\omega} \in D \;\&\; \alpha \in E_{i-1} \;\&\; \neg\omega \notin E\}$ and assume $E_{i-1} \subseteq E$ as induction hypothesis.

[4] I.e., Jus' contains the justifications of exactly those defaults whose successive application allows to derive p .

- If $g \in Th(E_{i-1})$ then the compactness theorem (see, e.g., [2]) implies the existence of a finite set of formulas $e_0, \ldots, e_n \in E_{i-1}$ entailing g. From the induction hypothesis and the construction of E we conclude the existence of elements $(e_0, Jus_0), \ldots, (e_n, Jus_n) \in A$; hence, there is some $(g, Jus) \in AR$ such that $Jus \subseteq \bigcup_{0 \leq j \leq n} Jus_j$ according to Definition 8 and Definition 10. Maximality of A implies $(g, Jus) \in A$, i.e., $g \in E$.
- If $\omega \in \{\omega \mid \frac{\alpha : \omega}{\omega} \in D \,\&\, \alpha \in E_{i-1} \,\&\, \neg\omega \notin E\}$ then there exists some $(\alpha, Jus) \in A$ according to the induction hypothesis. Hence, we can find some $(\omega, Jus') \in AR$ such that $Jus' \subseteq Jus \cup \{\omega\}$ according to Definition 8 and Definition 10. Now, assume $(\omega, Jus') \notin A$ then maximality of A implies the existence of some $B \subseteq A$ such that $W \cup Jus' \cup \bigcup_{(g, Jus'') \in B} Jus'' \models \bot$. This and the fact that $(\alpha, Jus) \in A$ imply both $W \cup Jus \cup \bigcup_{(g, Jus'') \in B} Jus'' \models \neg\omega$ and $W \cup Jus \cup \bigcup_{(g, Jus'') \in B} Jus'' \not\models \bot$. But from this and maximality of A, we conclude the existence of some $(\neg\omega, \overline{Jus}) \in A$ according to Definition 8 and Definition 10, which is a contradiction to $\neg\omega \notin E$. Hence, $(\omega, Jus') \in A$, i.e., $\omega \in E$.

(b) To show $E \subseteq \bigcup_{i=0}^{\infty} E_i$, let $g \in E$ be an arbitrary formula, i.e., there exists some $(g, Jus) \in A$. This implies the existence of a sequence $e_0, e_1, \ldots, e_n = g$ satisfying the conditions of Definition 8 wrt Jus. Let $\{\delta_j \in D \mid 1 \leq j \leq k\}$ be the set of defaults applied in this sequence ($k \geq 0$). From $(g, Jus) \in A$ and maximality of A, we conclude the existence of elements $(Justif(\delta_1), Jus_1), \ldots, (Justif(\delta_k), Jus_k) \in A$, hence $Justif(\{\delta_j \mid 1 \leq j \leq k\}) \subset E$. Therefore, all these defaults can also be applied according to Proposition 7, i.e., there exists some $l \in \mathbb{N}$ such that $g \in E_l$, which implies $g \in \bigcup_{i=0}^{\infty} E_i$.

This correspondence between default logic and our general disputation framework enables us to adopt the general notion of a skeptical proof into default logic:

Corollary 12. *Let $\Delta = (D, W)$ be a normal default theory with corresponding disputation framework $DF_\Delta = \langle AR, CL, conflict \rangle$. A closed formula g is skeptically entailed from Δ iff there is a skeptical proof for g from DF_Δ.*

Recall our exemplary, extended default theory. The corresponding disputation framework contains the two arguments $(i, \{p, i\})$ and $(i, \{\neg p, i\})$, both claiming i. To each conflict-free set of arguments in this disputation framework, one of these two arguments can be added without causing a conflict. Hence, both together form a skeptical proof for i. In terms of default logic, we can, analogously, regard the set $\{\{\frac{q : p}{p}, \frac{p : i}{i}\}, \{\frac{r : \neg p}{\neg p}, \frac{\neg p : i}{i}\}\}$ as a skeptical proof for i.

4 Discussion and Outlook

We have proposed a modified general, argumentation-based framework with the underlying concept of a claim supported by each argument. This development serves the purpose to overcome a weakness of Dung's argumentation-based approach [1] in case of skeptical reasoning. The concept of a skeptical proof has been defined, and we have illustrated that this notion coincides with the intention of being contained in every preferred extension. Furthermore, we have embedded normal default theories in our general disputation framework, and it has been proved that the respective concepts of an extension in these two methods coincide.

The most interesting and promising purpose of the framework presented in this paper shall be the development of a general algorithm to automate skeptical reasoning in nonmonotonic logics. Such an algorithm, based on the alternating generation of arguments and counterarguments, has been proposed for default logics in [10]. There, we have adapted the basic principles of a similar algorithm, originally formulated in [5] and extended in [9], for reasoning skeptically in THEORIST [4]. In [7], we have shown how to extract a proof from the algorithm for default logic in case of success. This notion resembles the concept of a skeptical proof developed in this paper. If the underlying principle of these algorithms are applied to the general disputation framework, the result should be a general algorithm for skeptical reasoning which we anticipate to be applicable to a variety of extension-based, nonmonotonic frameworks. Finally, we intend to relate this work to philosophical foundations of argumentation theory (see, e.g., [11]), which resembles our logical formalization of arguing in disputations.

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