Fair Course Allocation with Additive Utilities and a Conflict Graph

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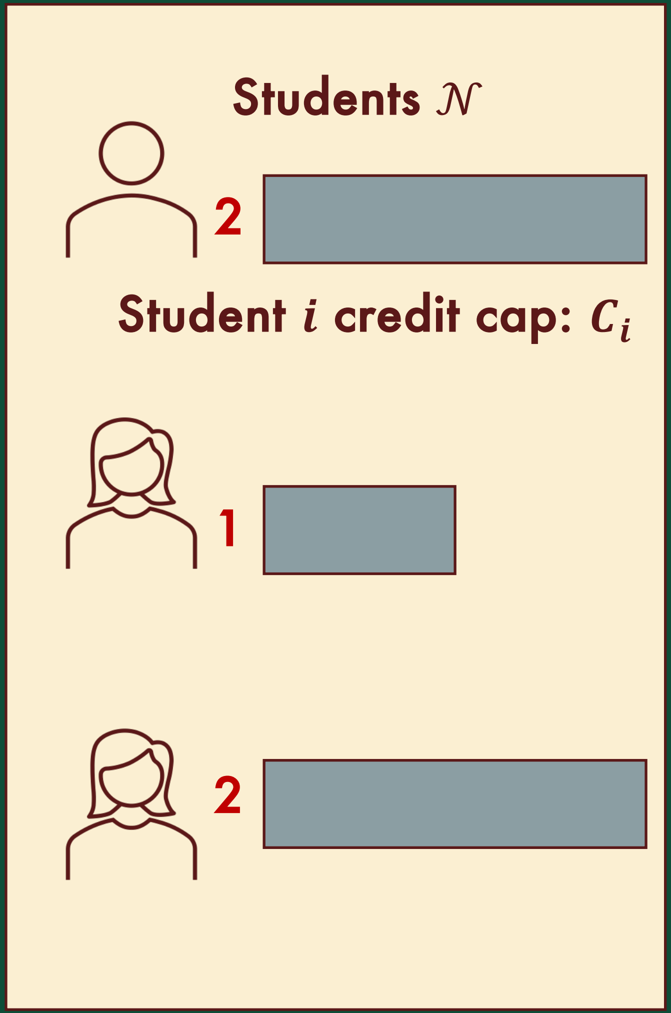
Motivation: a problem all universities face

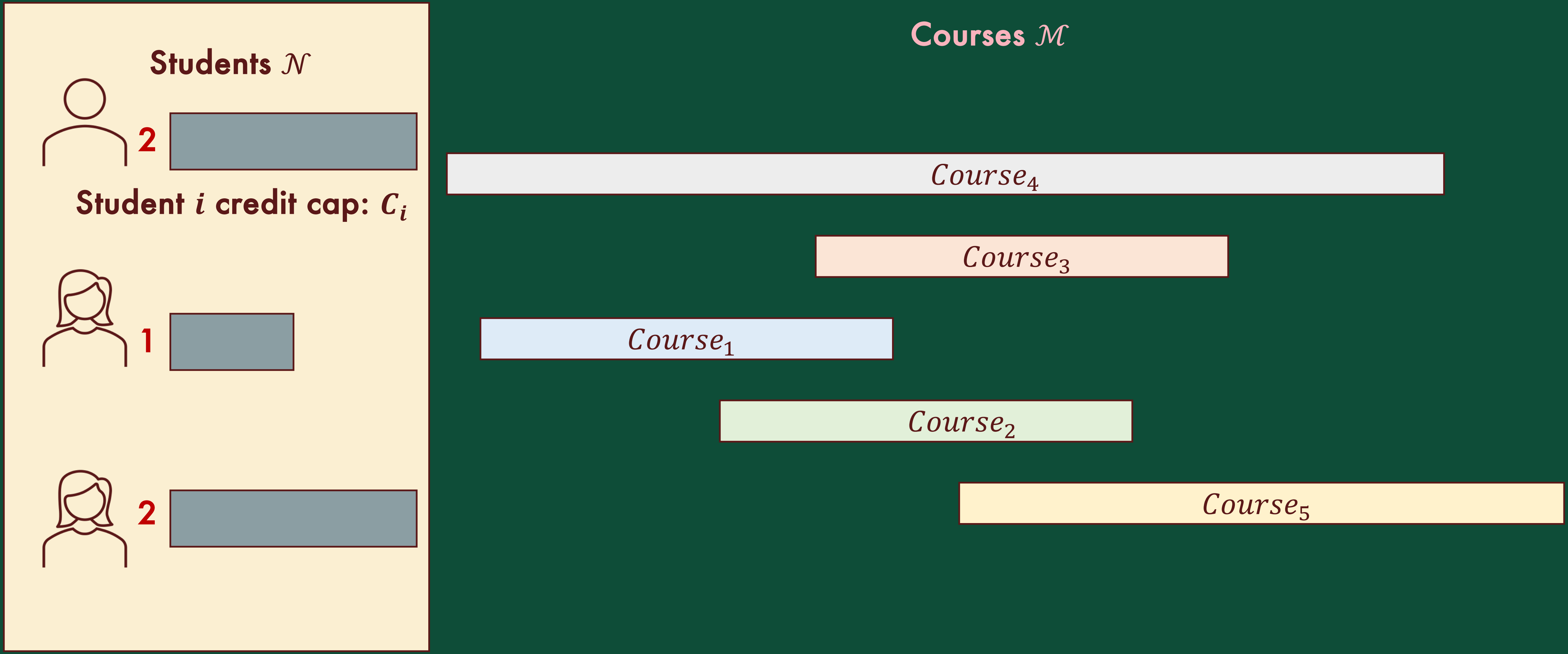
- Students want to take different courses
- Not enough seats in courses
- Some courses conflict with others
- How to allocate seats "fairly"?
- How it is now: whoever get online first
- Enrollment in CS courses has gone up significantly in recent years, for example

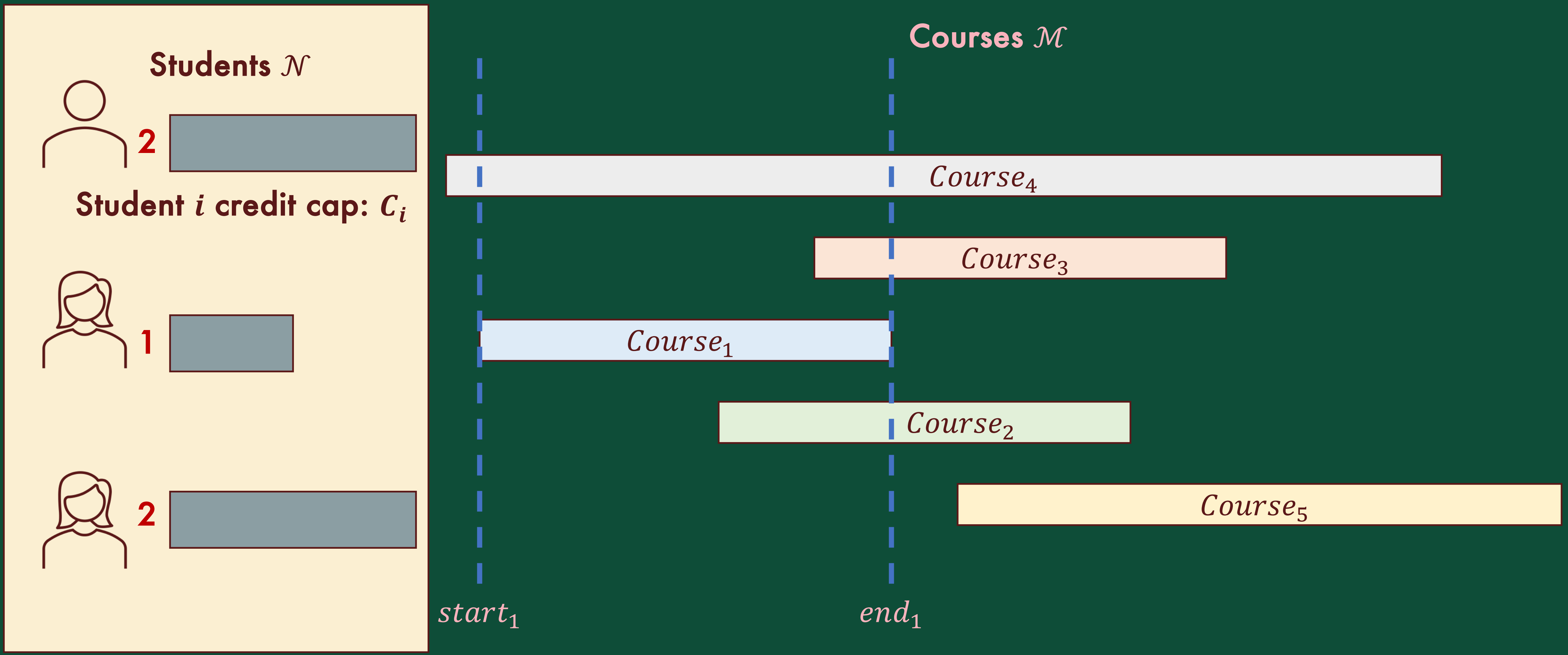
Problem setup, before we formally define fairness notions

Students $\mathcal{N}$

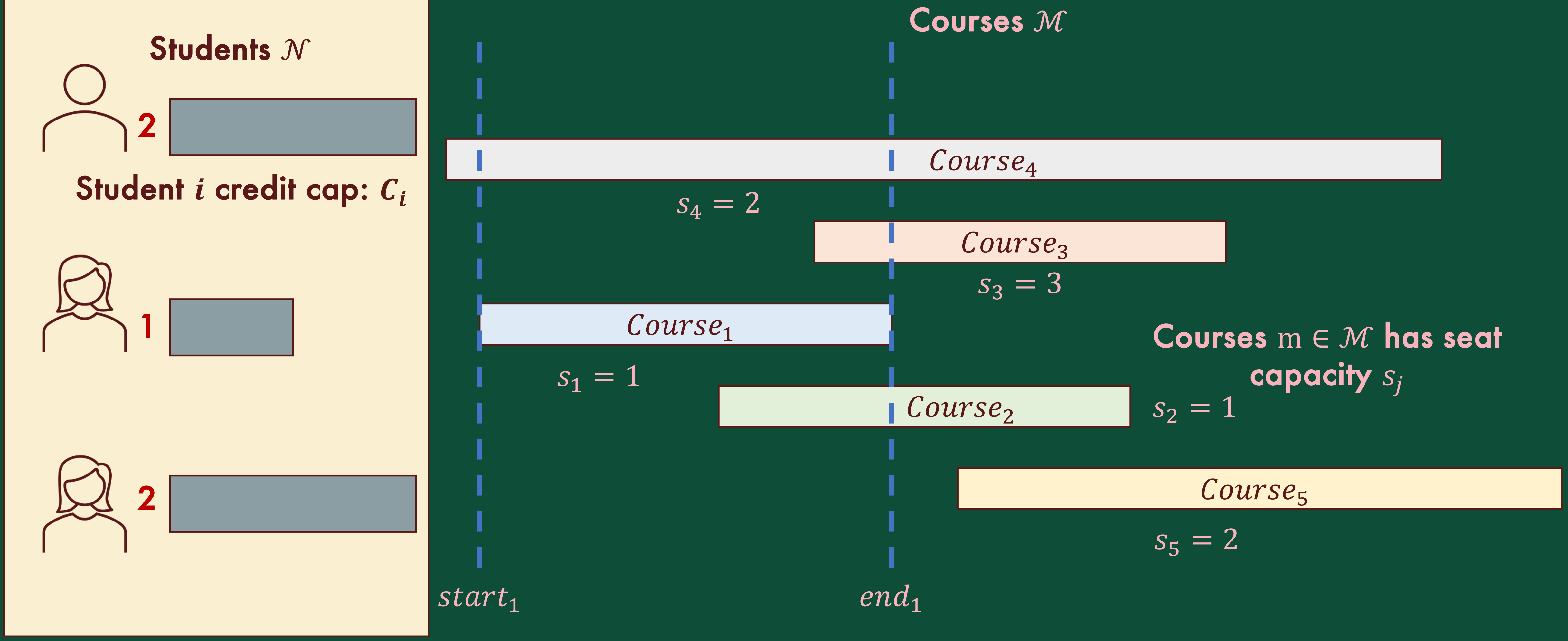


Students $\mathcal{N}$
2
Student i credit cap: C_i
1
2

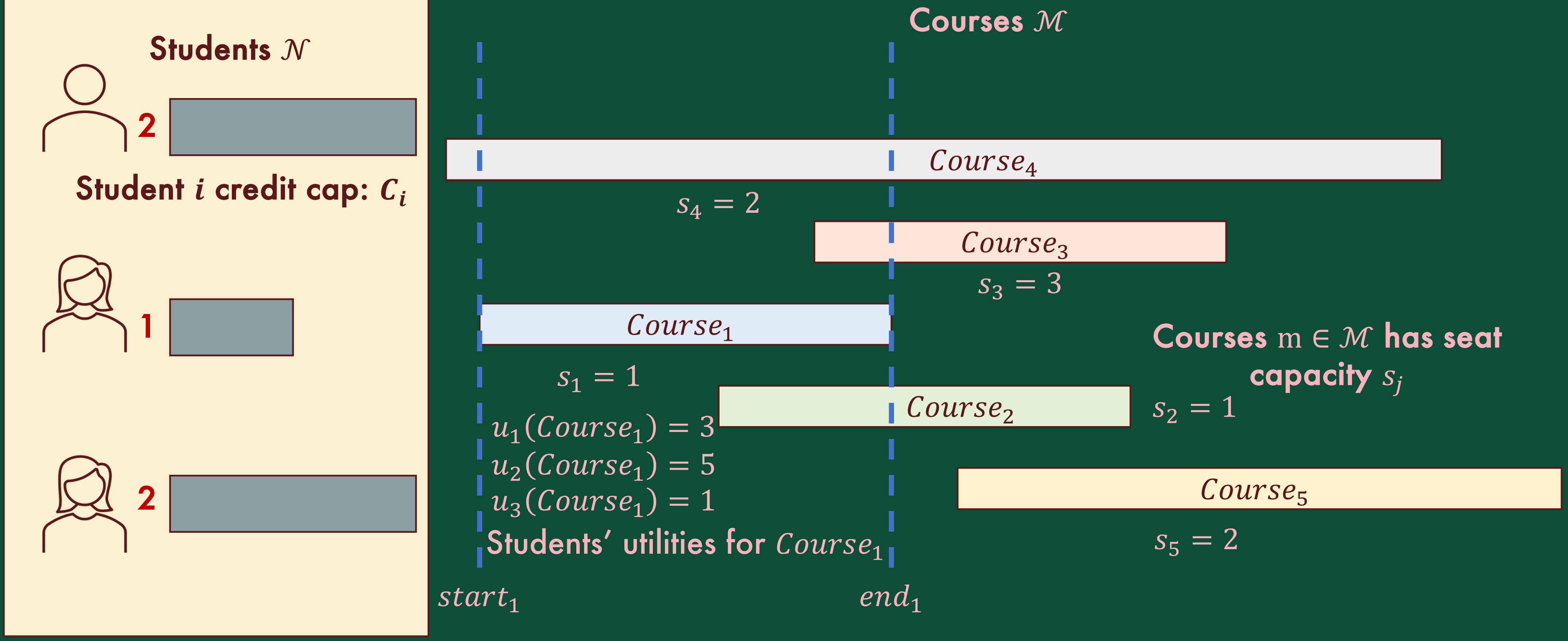

Students $\mathcal{N}$
2
Student i credit cap: C_i
1
2
Courses $\mathcal{M}$
$Course_4$
$Course_3$
$Course_1$
$Course_2$
$Course_5$



Students $\mathcal{N}$
2
Student i credit cap: C_i
1
2
Courses $\mathcal{M}$
$Course_4$
$Course_3$
$Course_1$
$Course_2$
$Course_5$
$start_1$
end_1



Students $\mathcal{N}$
2
Student i credit cap: C_i
1
2
Courses $\mathcal{M}$
$Course_4$
$s_4 = 2$
$Course_3$
$s_3 = 3$
$Course_1$
$s_1 = 1$
Courses $m \in \mathcal{M}$ has seat capacity s_j
$Course_2$
$s_2 = 1$
$Course_5$
$s_5 = 2$
$start_1$
end_1


Students $\mathcal{N}$
2
Student i credit cap: C_i
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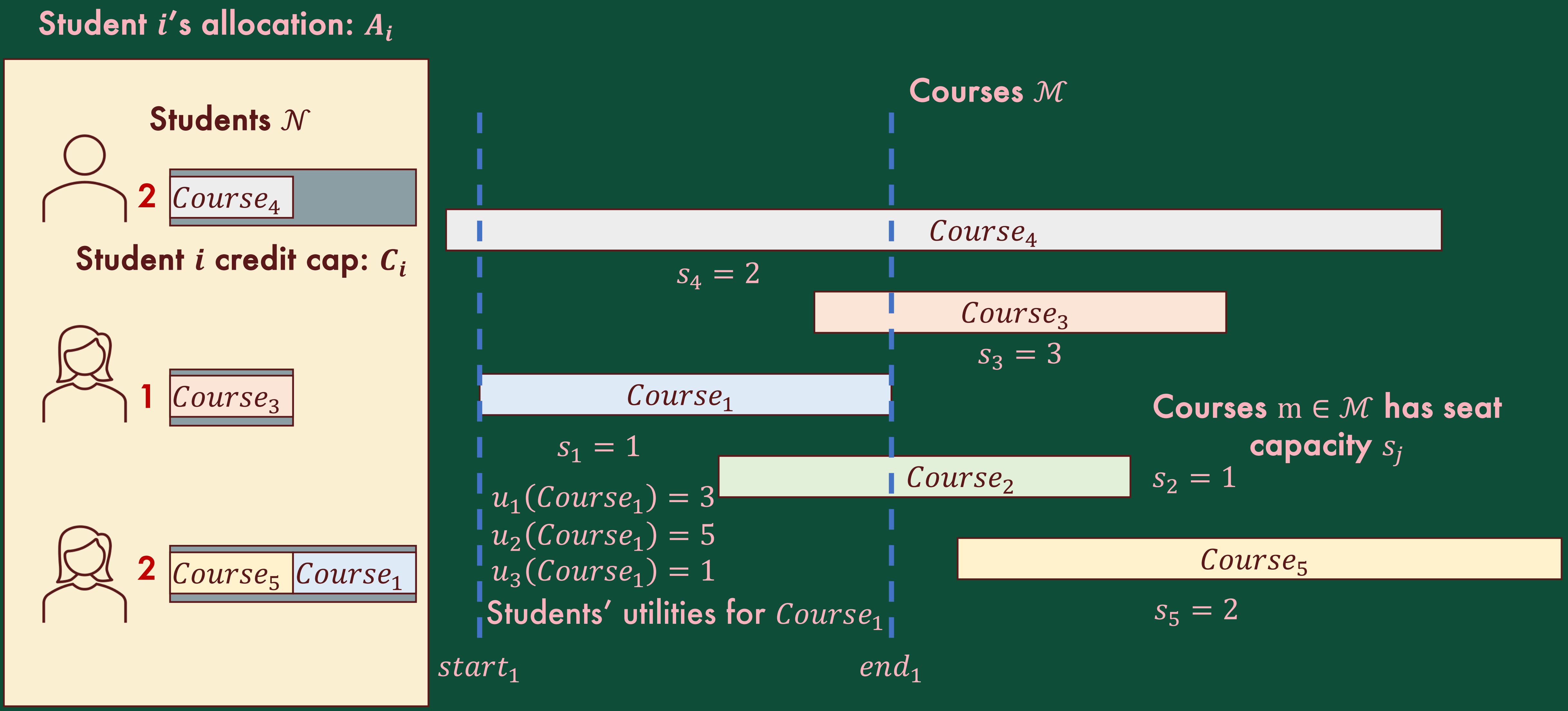
Courses $\mathcal{M}$
$Course_4$
$s_4 = 2$
$Course_3$
$s_3 = 3$
$Course_1$
$s_1 = 1$
$Course_2$
$s_2 = 1$
$Course_5$
$s_5 = 2$
Courses $m \in \mathcal{M}$ has seat capacity s_j
$u_1(Course_1) = 3$
$u_2(Course_1) = 5$
$u_3(Course_1) = 1$
Students' utilities for $Course_1$
$start_1$
end_1


Student i's allocation: A_i
Students $\mathcal{N}$
2 $Course_4$
Student i credit cap: C_i
1 $Course_3$
2 $Course_5$ $Course_1$
Courses $\mathcal{M}$
$Course_4$
$s_4 = 2$
$Course_3$
$s_3 = 3$
$Course_1$
$s_1 = 1$
$Course_2$
$s_2 = 1$
Courses $m \in \mathcal{M}$ has seat capacity s_j
$Course_5$
$s_5 = 2$
$u_1(Course_1) = 3$
$u_2(Course_1) = 5$
$u_3(Course_1) = 1$
Students' utilities for $Course_1$
$start_1$
end_1

What is “fair”?

What is “fair”?

1) envy-freeness

*Gamow, George; Stern, Marvin (1958). Puzzle-math. Viking Press. ISBN 0670583359.

What is “fair”?

1) envy-freeness*

What is “fair”?

1) envy-freeness
2) Max-min objective

Envy-freeness

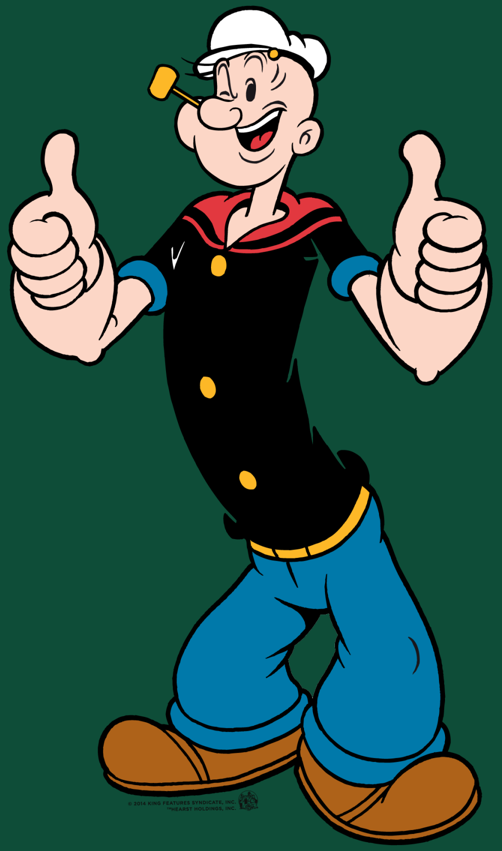
when no student envies
another student's allocation

For all students $i \in \mathcal{N}$, there does not exist a student $i' \in \mathcal{N}$ such that $u_i(A_{i'}) > u_i(A_i)$

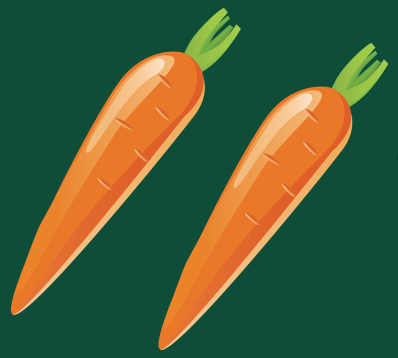
Recall: $u_i(A_{i'})$ is the utility of student i for $A_{i'}$

Bunny

Popeye

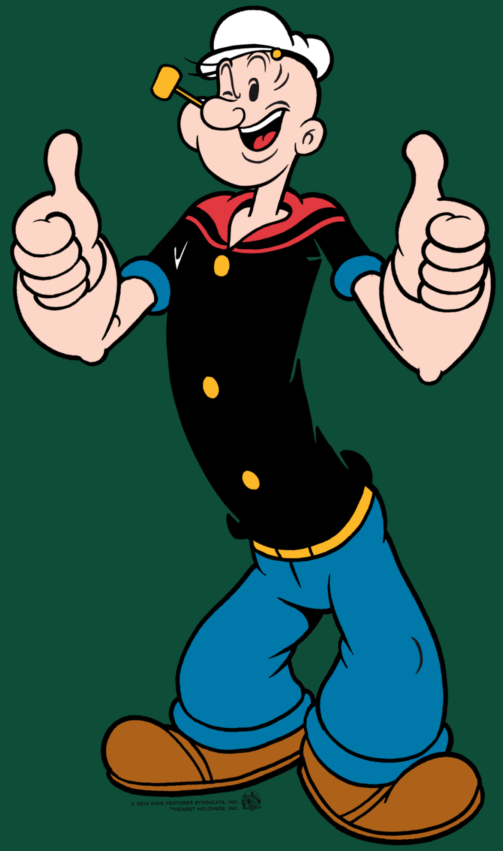


Bunny



carrot>spinach>Ø

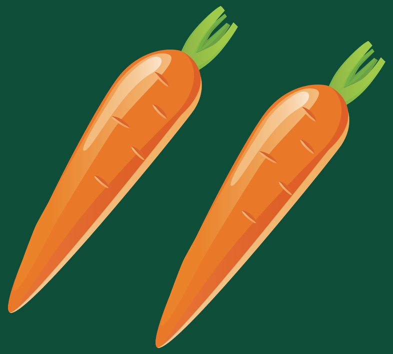
Popeye



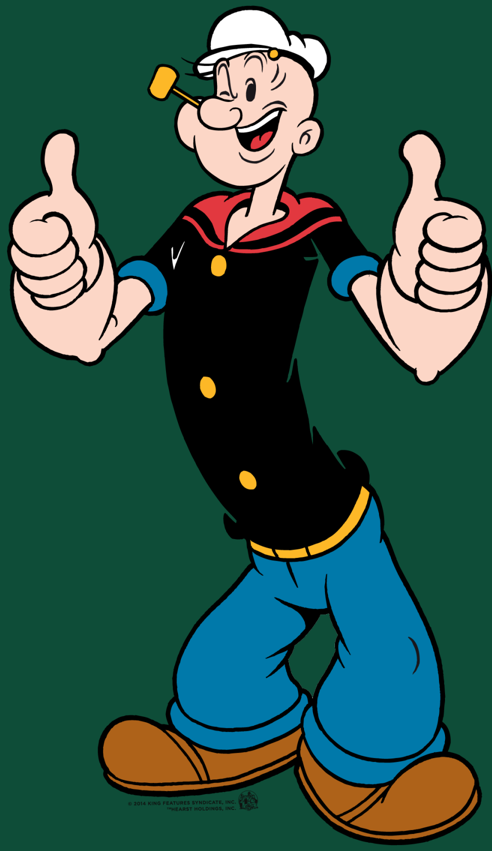
spinach>carrot>Ø

Bunny

Popeye



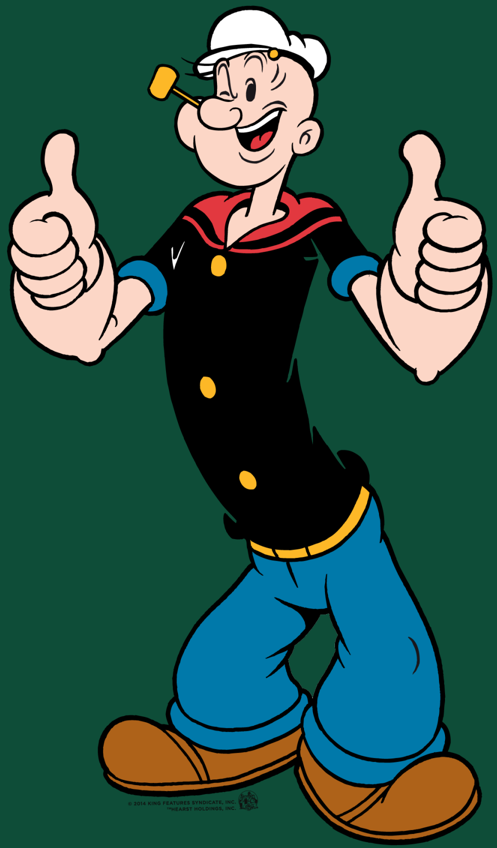
No envy!
Because of their preferences



But envy-freeness is a bit strict...

Bunny

Popeye



Envy-freeness, but relaxed:

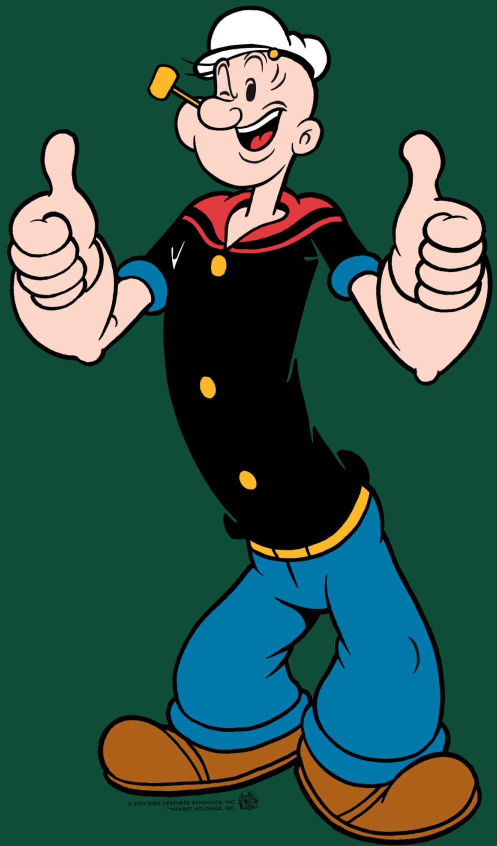
envy-freeness up to any good (EFX)

For all students $i \in \mathcal{N}$, if there is a student $i' \in \mathcal{N}$ such that $u_i(A_{i'}) > u_i(A_i)$, for any course $x \in A_{i'}$ it is true that $u_i(A_{i'} \setminus x) \leq u_i(A_i)$

Bunny

Still not EFX.
Why?

Popeye

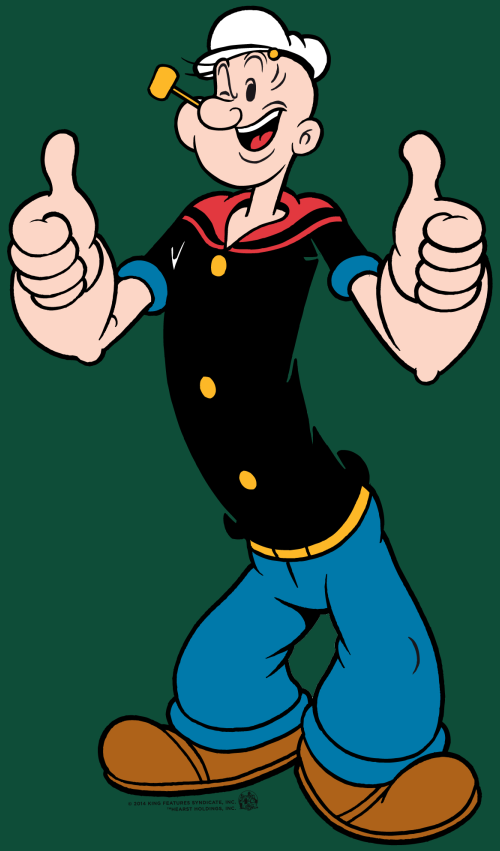


Bunny



Still not EFX.
Why?

Popeye

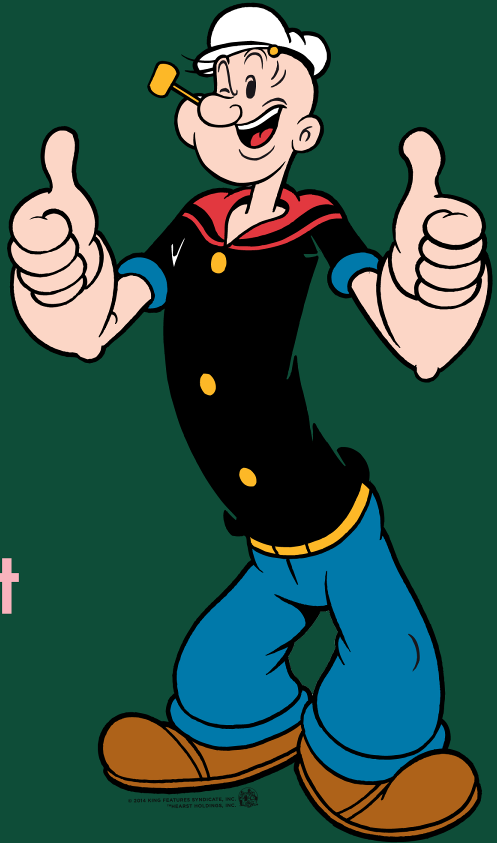



spinach>carrot>∅

Bunny

Popeye

Still not EFX.
Why?
Because taking away the carrot is not enough



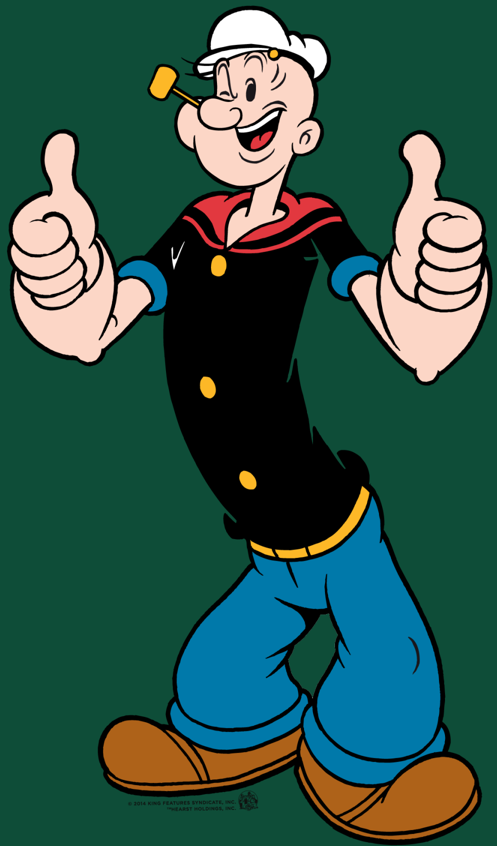
spinach>carrot>∅

Bunny

But if Popeye instead had a spinach, then it would be EFX.

Why?

Popeye



spinach>carrot>∅

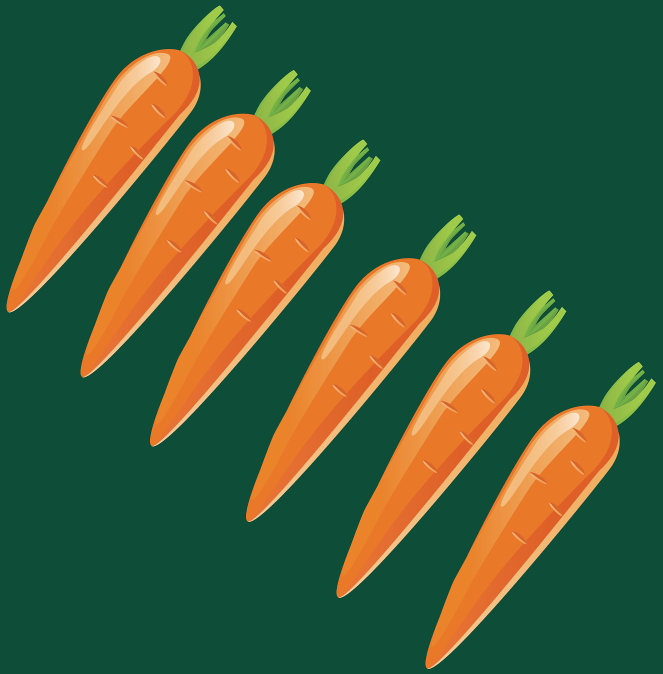
Max-min fairness

an allocation of courses to students $\mathcal{A} = (A_1, \ldots, A_n)$ is **max-min fair** if it maximizes the minimum utility of any student. Namely, we seek to satisfy the following objective: $\max_{\mathcal{A}} \left(\min_{i \in \mathcal{N}} (u_i(A_i)) \right)$

Bunny A

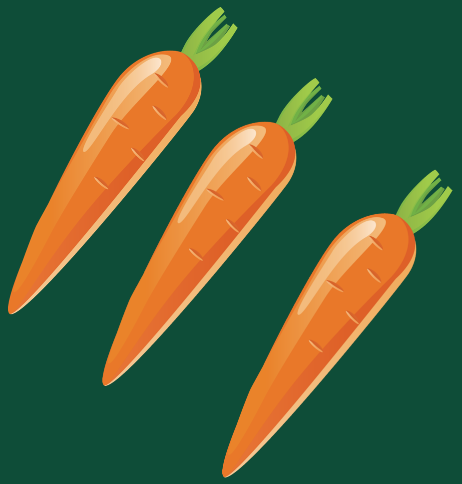
Bunny B

Bunny C



Not very fair!!

Bunny A



Bunny B

Bunny C

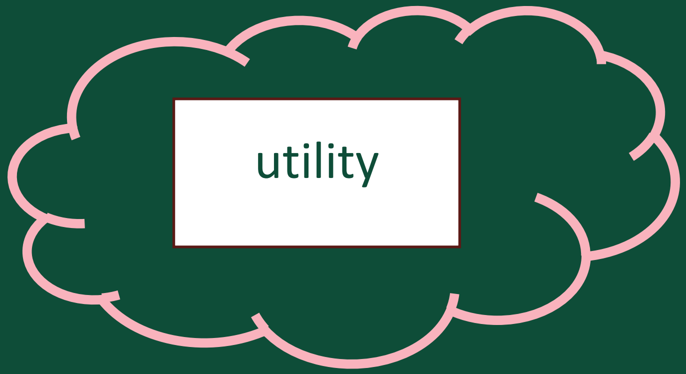
Fairer!

Uniform means $\forall i \in N, j \in M, u_i(j) = 1$

Binary means $\forall i \in N, j \in M, u_i(j) \in \{0,1\}$

Identical means $j \in M, u_i(j) = u_{i'}(j)\ \forall i, i' \in N$

Additive means a student's utility for a set S of items is the sum of his utilities for each item in the maximum weighted independent set of S: set: $u_i(S) = \sum_{j \in MWIS_i(S)} u_i(j)$



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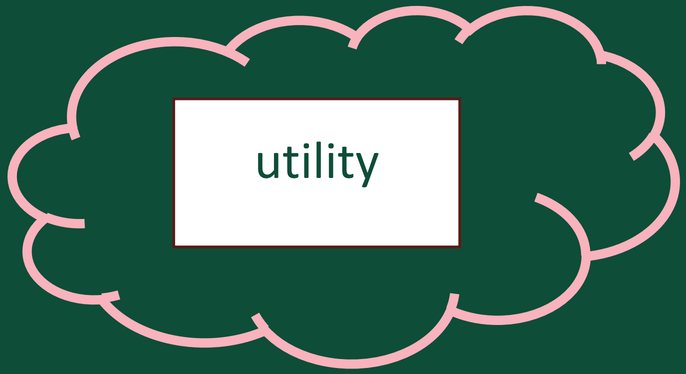
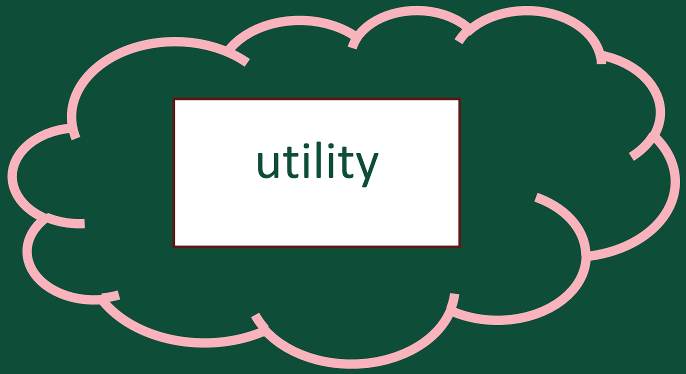


In previous work: Arpita Biswas, Yiduo Ke, Samir Khuller, and Quanquan C. Liu. An algorithmic approach to address course enrollment challenges. In Kunal Talwar, editor, 4th Symp. on Foundations of Responsible Computing, FORC 2023



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This talk. EFX and 1/2 max-min

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One more thing about envy...

Due to courses having conflicts with each other, it is not always possible to allocate every course seat to a student

For example:

$m = 6$
$n = 1$

The Charity

The set of all unallocated courses

The Charity

No student shall envy the Charity.

$$u_i(A_i) \geq u_i(\text{Charity}) \; \forall i \in N$$

The Charity

No student shall envy the Charity.

$$u_1(\text{Charity}) = MWIS_1(\text{Charity})$$

Theorem

Under identical additive utility for n students, there exists an EFX and ½-approx. with additive factor maxmin allocation

EFX algorithm

Exists for monotonic non-decreasing utility

EFX algorithm

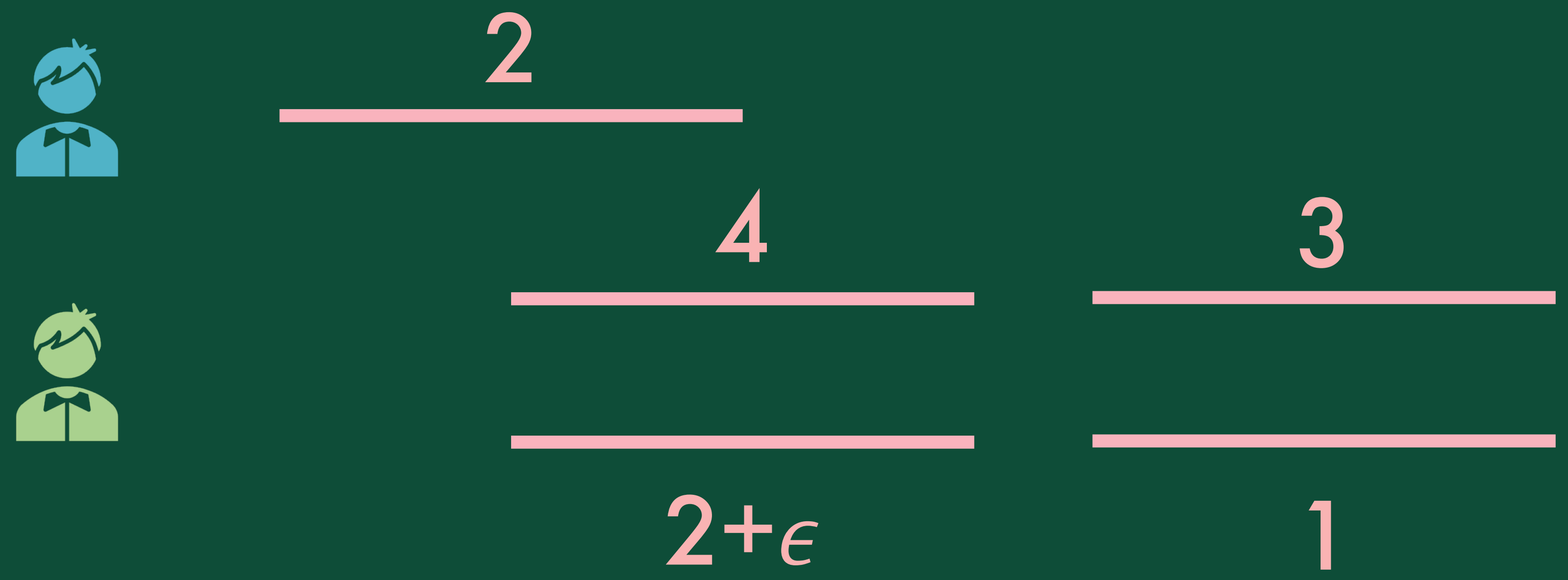
Exists for monotonic non-decreasing utility*

*Chaudhury et. al. A *Little Charity Guarantees Almost Envy-Freeness* SODA 2020

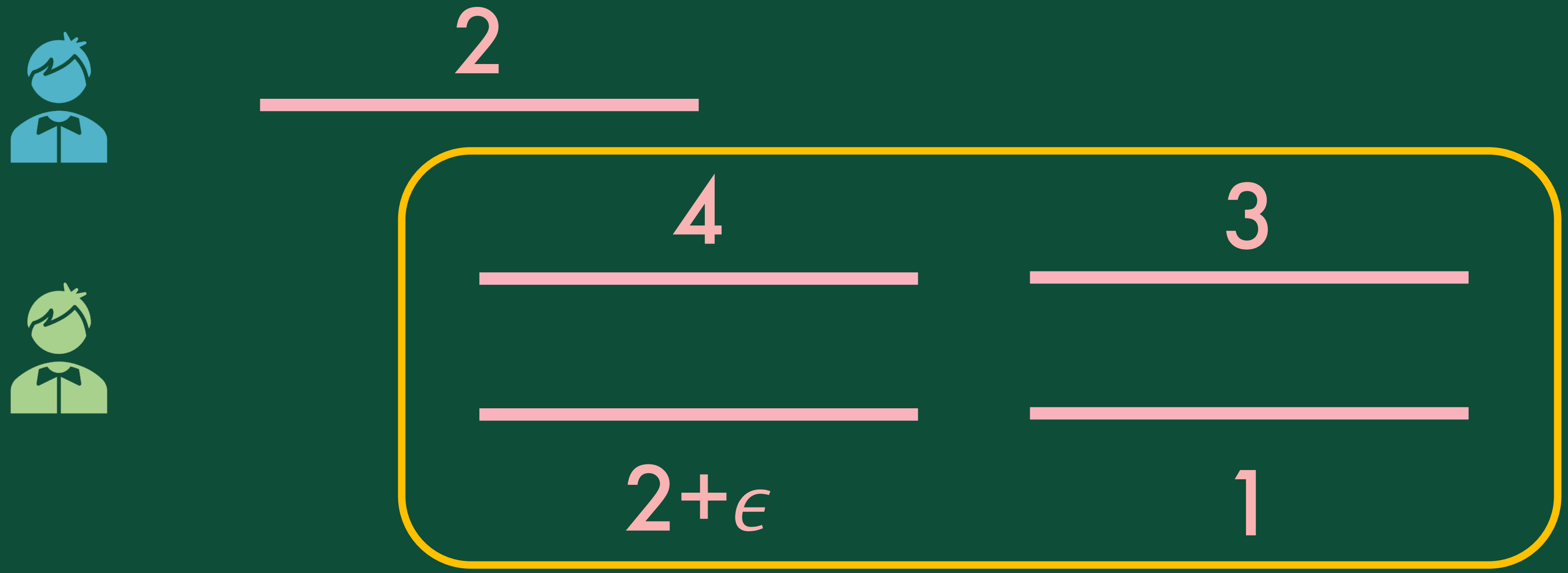
Our algorithm, roughly

- Find the n-largest colorable subgraph G′ of the interval graph that describes the course conflicts
- Find an EFX allocation on G′
- Continue finding an EFX allocation on the whole graph, G

An example

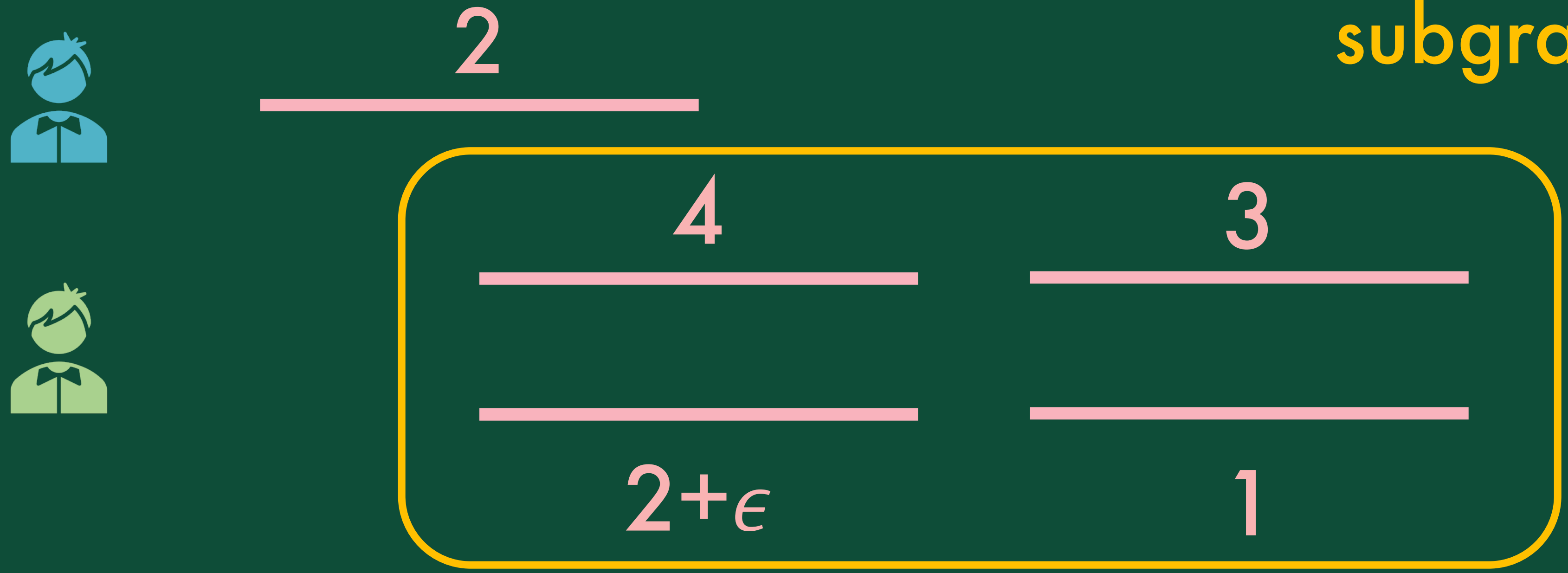


An example



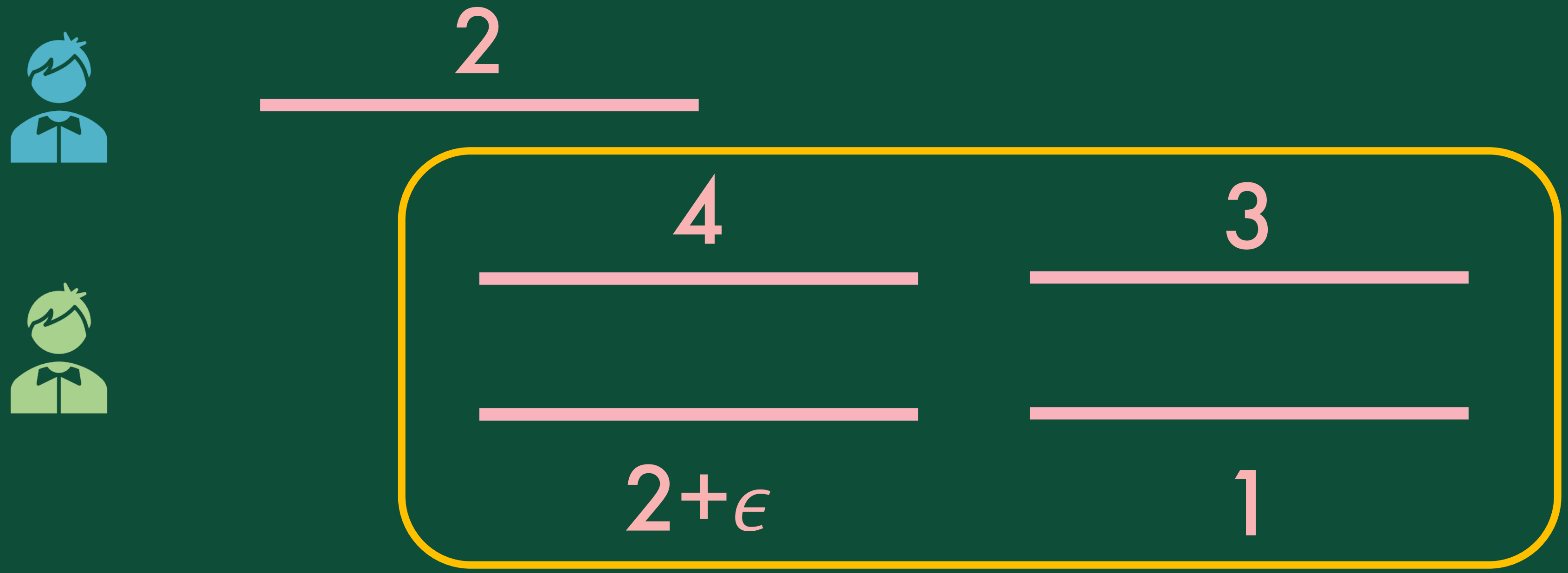
An example

Largest 2-colorable subgraph G'



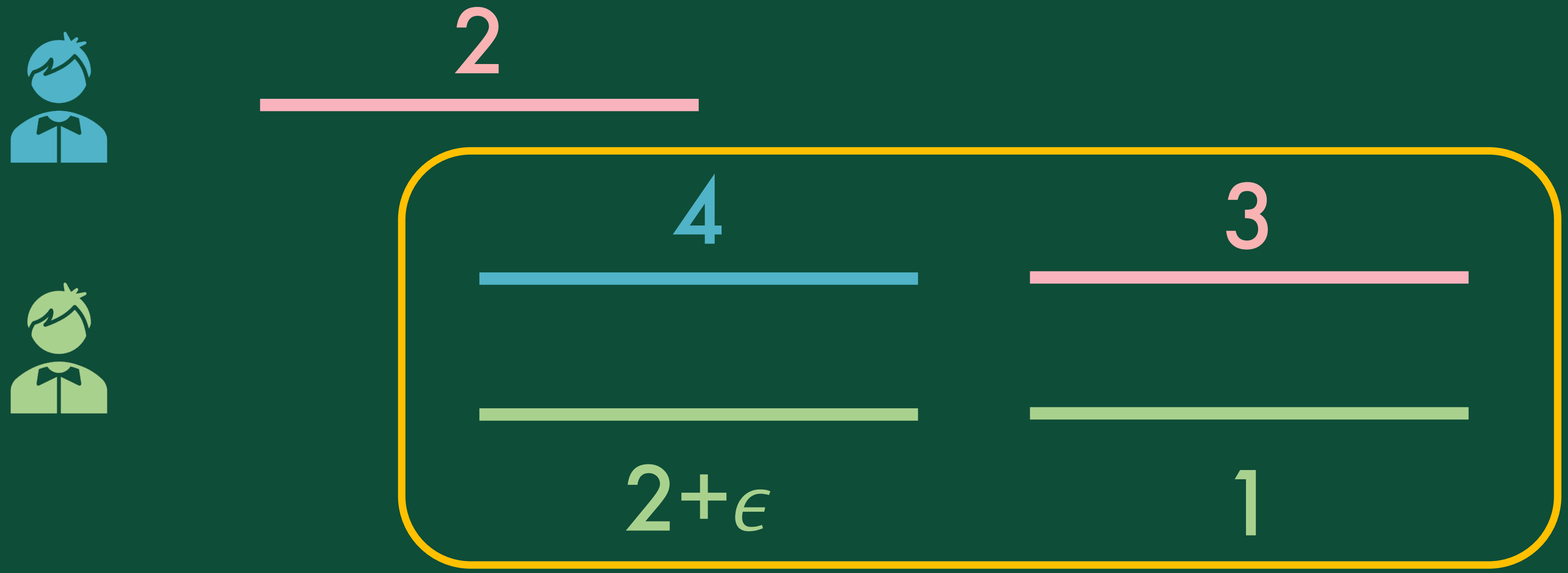
An example

Find an EFX allocation on G′



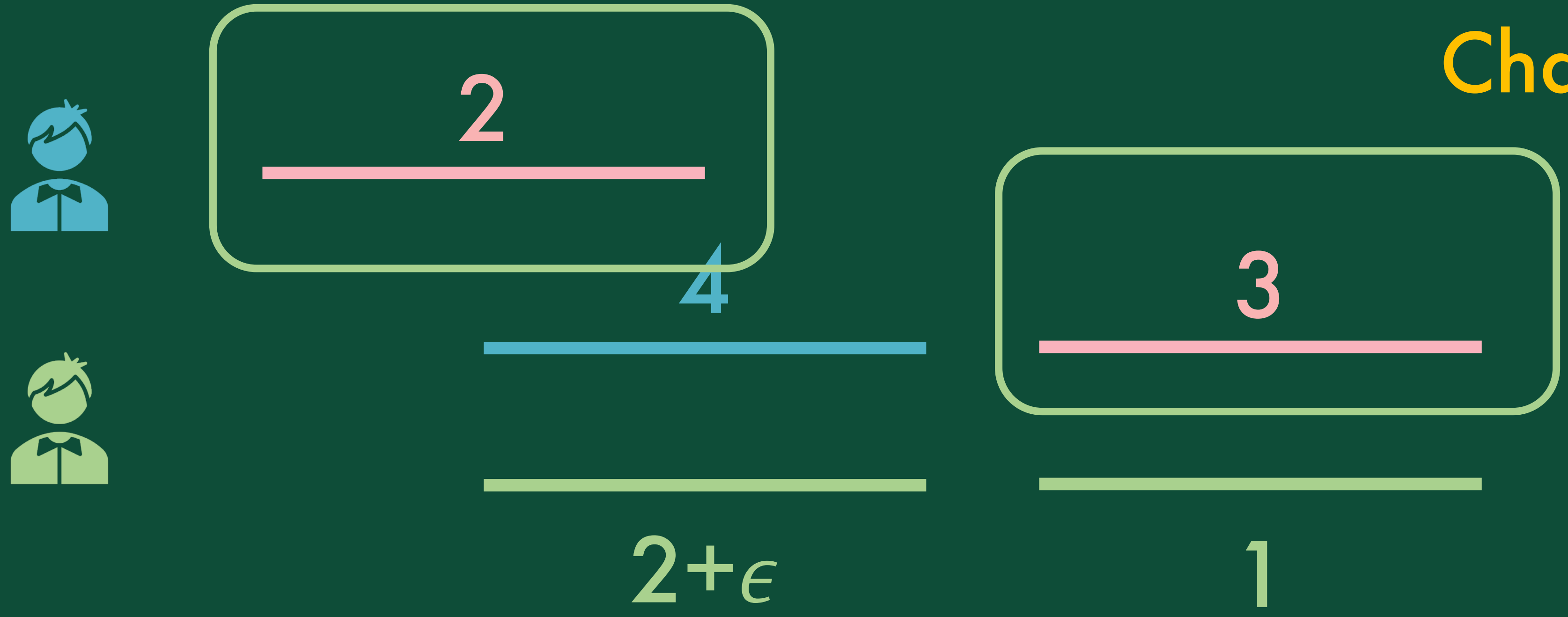
An example

Find an EFX allocation on G′



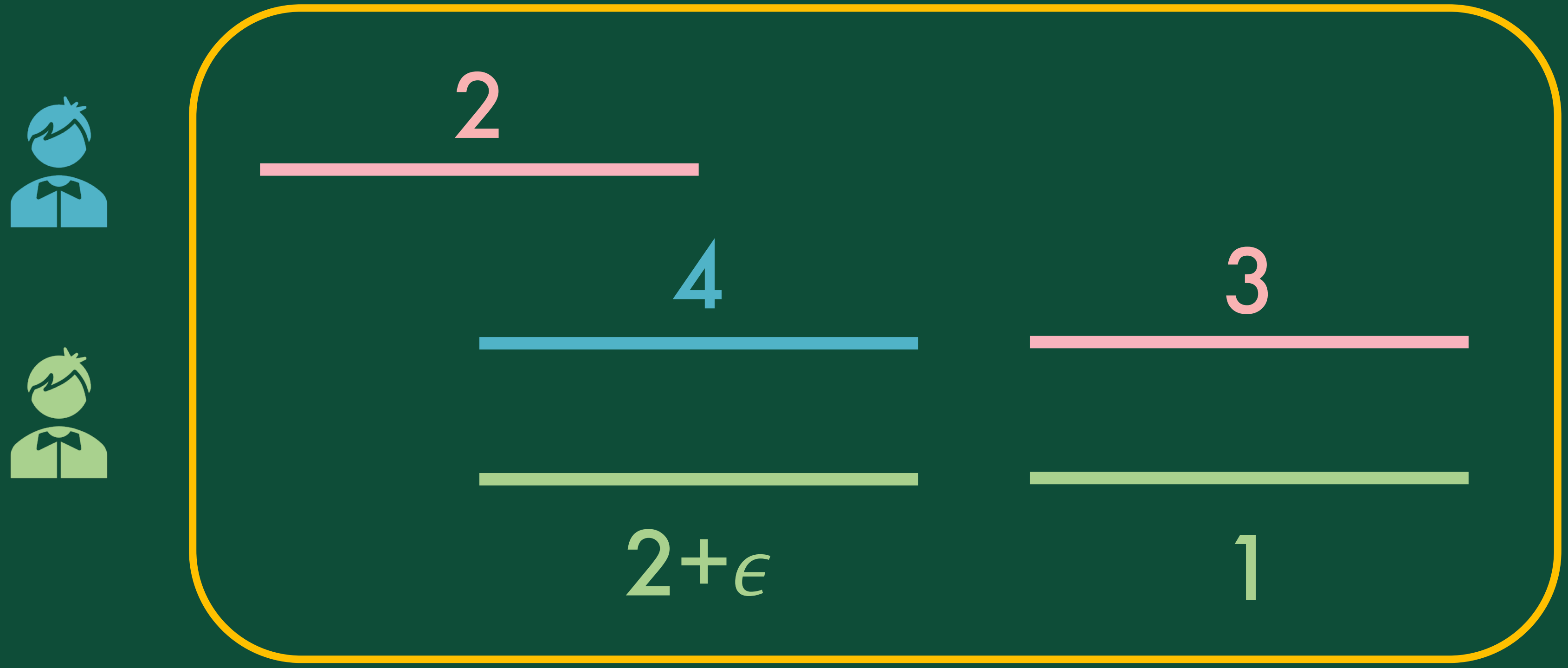
An example

Notice the green student envies the Charity



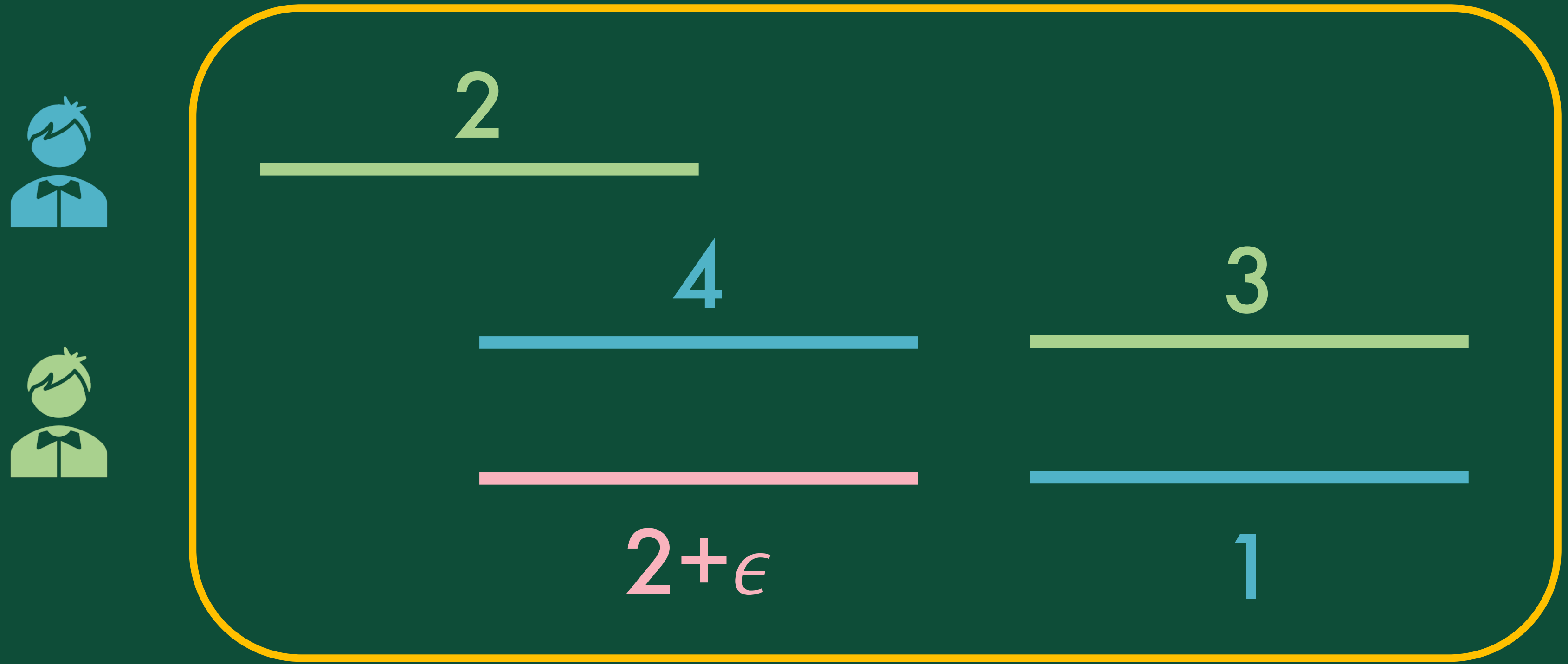
An example

Now, find an EFX allocation on the whole graph G



An example

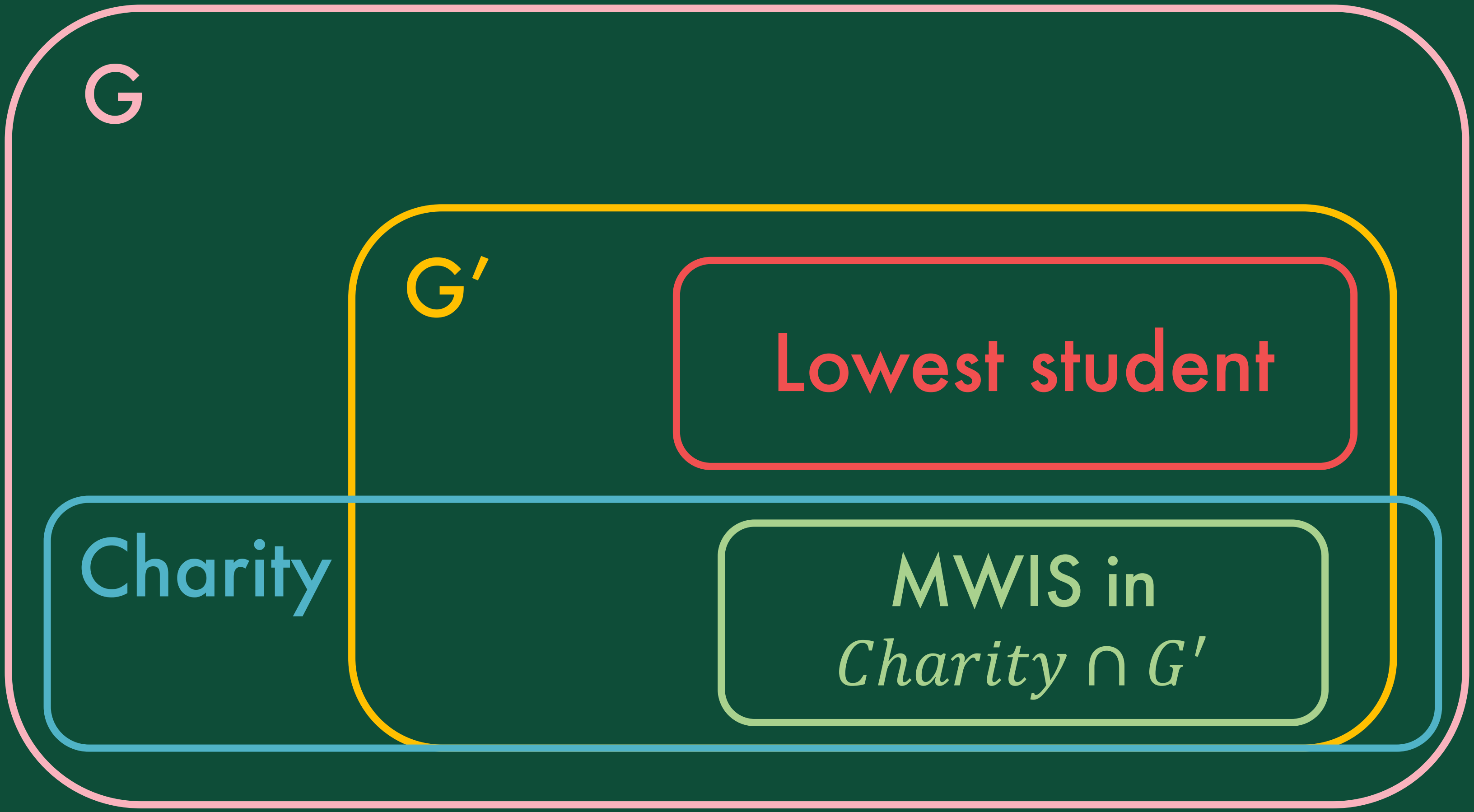
Now, find an EFX allocation on the whole graph G



Proof Sketch

- Allocation is EFX throughout the algorithm
- Optimal maxmin value $OPT \leq T/n$, where T is the sum of utilities of courses in the largest n-colorable subgraph G′
- Lowest student in G′ has a better allocation than the MWIS of $G' \cap \text{Charity}$

Proof Sketch



Related works

- *Fair Allocation with Interval Scheduling Constraints.* (Li et. al. 2021, NeurIPS)
- *Fair packing of independent sets.* (Chiarelli et. al. 2020)
- *Fair allocation of conflicting items.* (Hummel and Hetland, 2021)
- *Fair allocation of indivisible goods: Improvements and generalizations.* (Ghodsi et. al. 2018)

Future work?

- Max-min approximation without finding largest –colorable subgraph
- Multiple meeting days for each class
- Different credit counts for each course
- Other utility types such as non-identical, submodular, subadditive, etc.

Thank you for attending!