Proportional Allocations of Indivisible Resources: Insights via Matchings.

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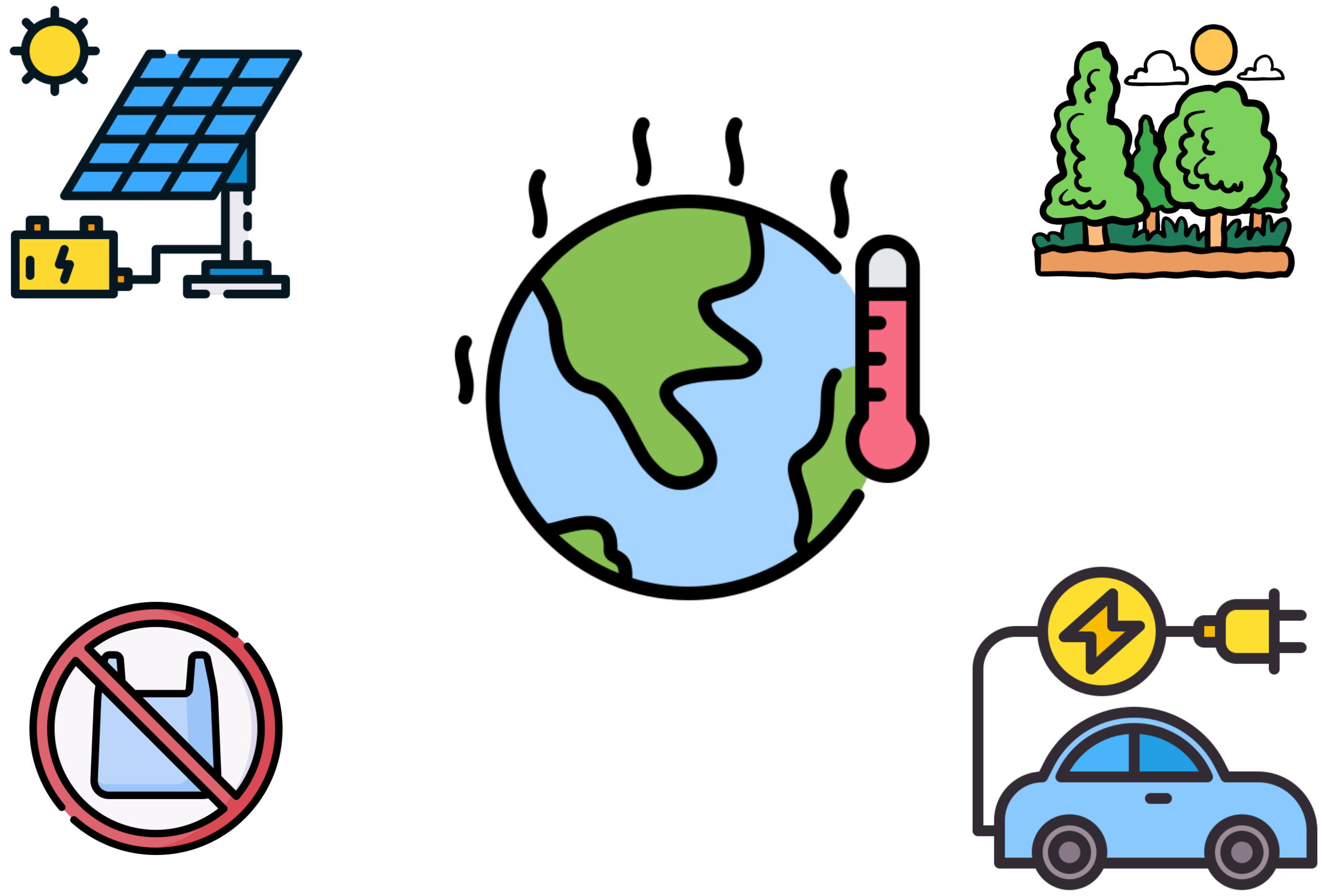
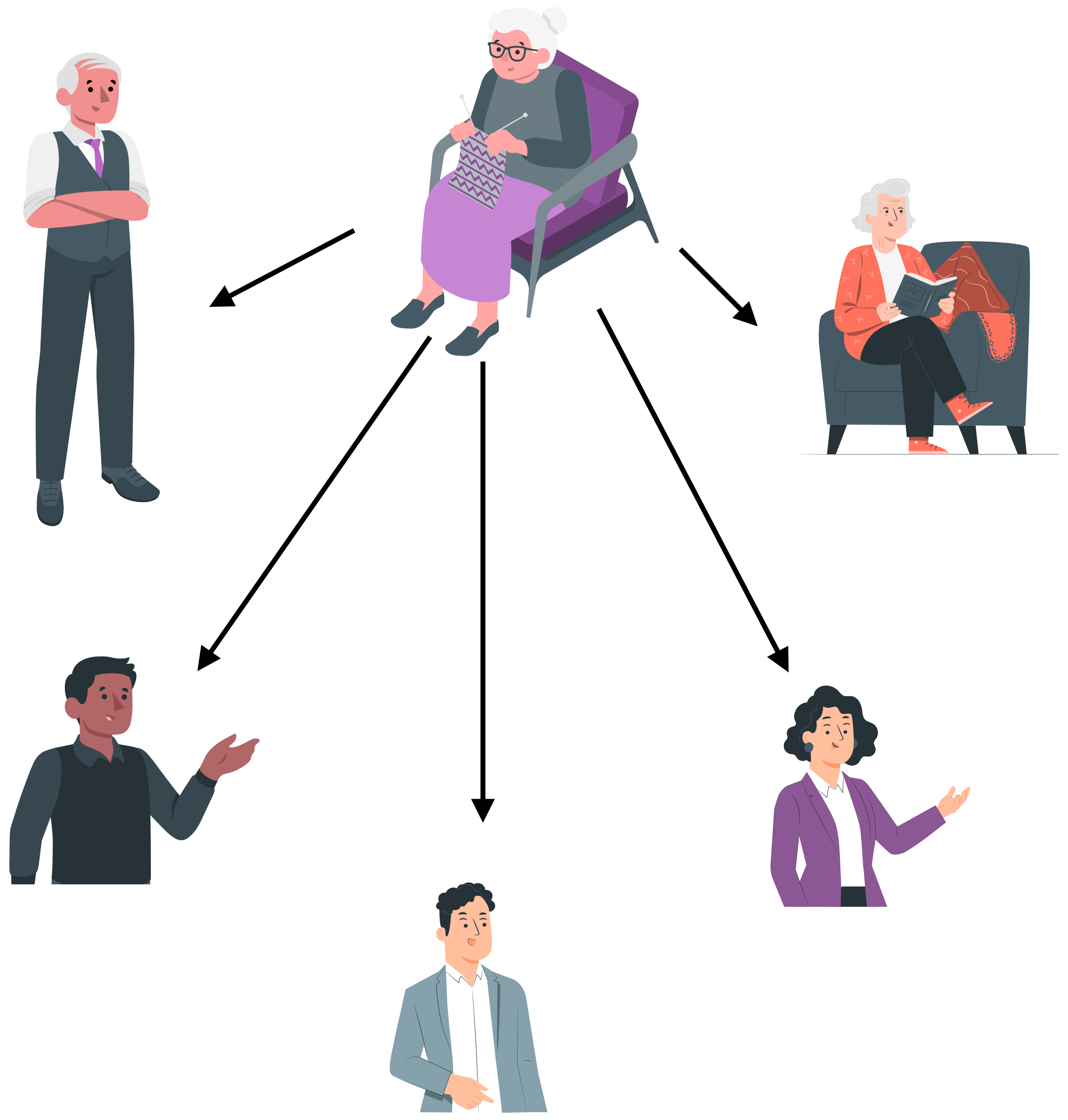
Fair Division: Examples

- Allocation of house chores among roommates
- Dividing assets between divorcing couples
- Fair allocation of responsibilities among countries
- Inheritance allocations

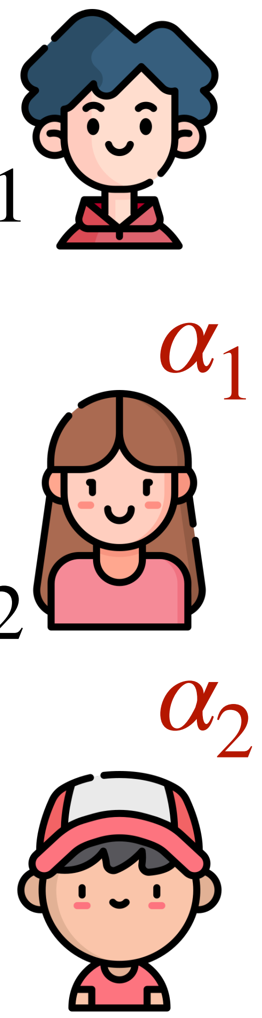




A Fair Allocation Instance

- Set of agents:
  $A = \{a_1, a_2, \dots, a_n\}$
- Set of items:
  $B = \{b_1, b_2, \dots, b_m\}$
- $\forall i \in [n],\ v_i : 2^B \rightarrow \mathbb{R}_{\geq 0}$ is the valuation function of a_i
  - Additive valuations:
  $$\forall S \subseteq B, v_i(S) = \sum_{b \in S} v_i(b)$$
- Each agent a_i has an entitlement $\alpha_i \in [0,1]$,
  $$\sum_{i \in [n]} \alpha_i = 1$$

A



a_1 (α_1)	10	20	50	30	15	45
a_2 (α_2)	40	50	10	60	20	15
a_3 (α_3)	40	50	20	10	60	70

B

A Fair Allocation Instance

- Set of agents:
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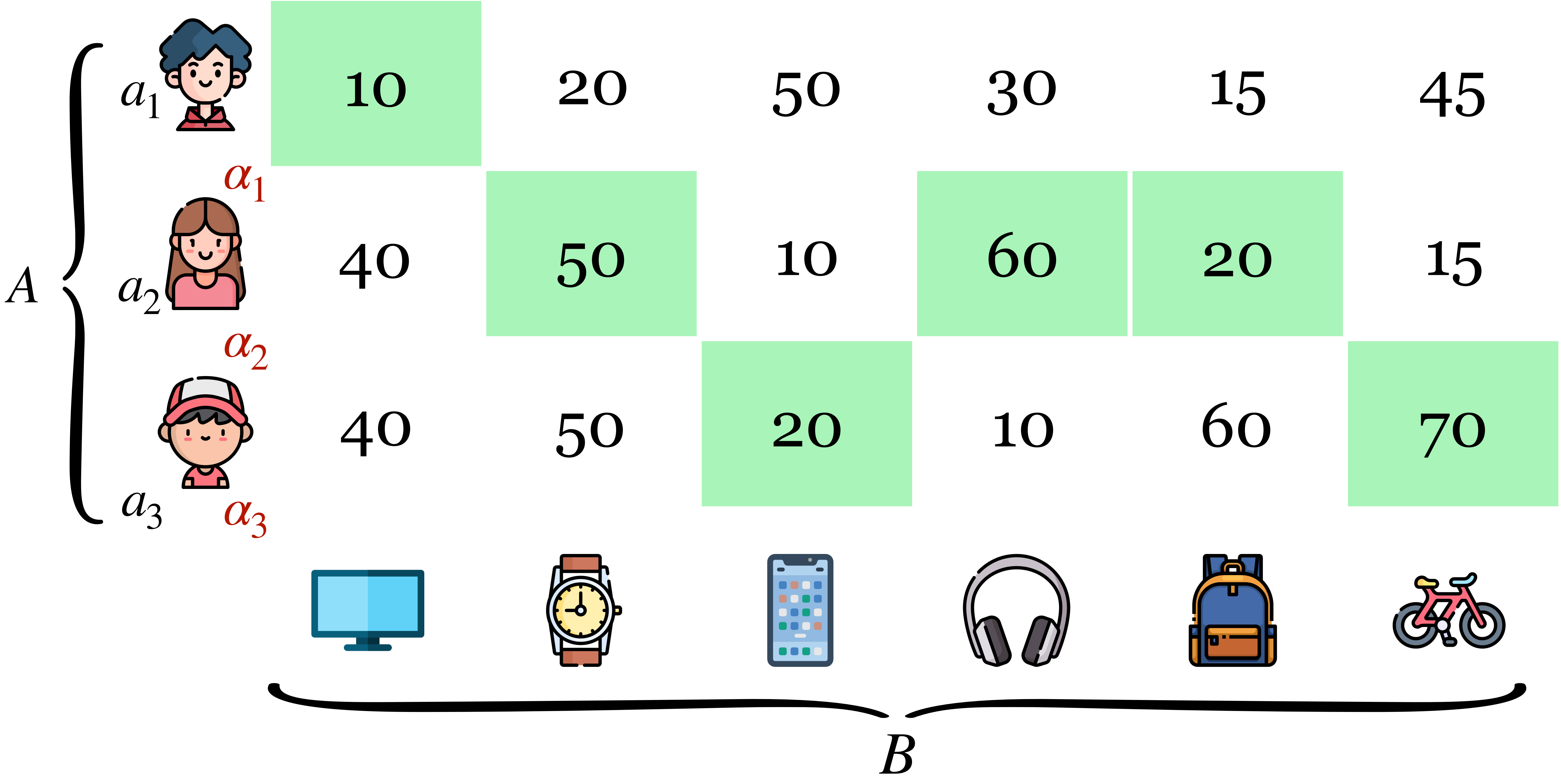


a_1 (α_1)	**10**	20	50	30	15	45
a_2 (α_2)	40	**50**	10	**60**	**20**	15
a_3 (α_3)	40	50	**20**	10	60	**70**

- An allocation $X = \langle X_1, X_2, \ldots X_n \rangle$ is a partition of B such that agent a_i gets the part, *bundle* X_i

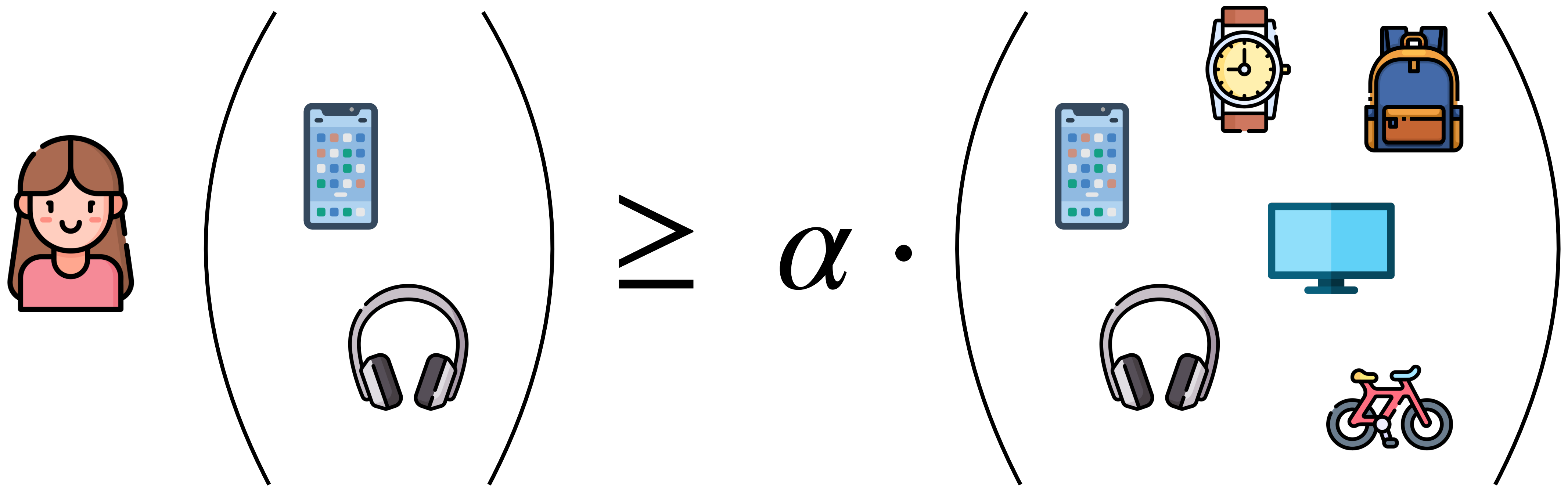
Proportional Item Allocation:

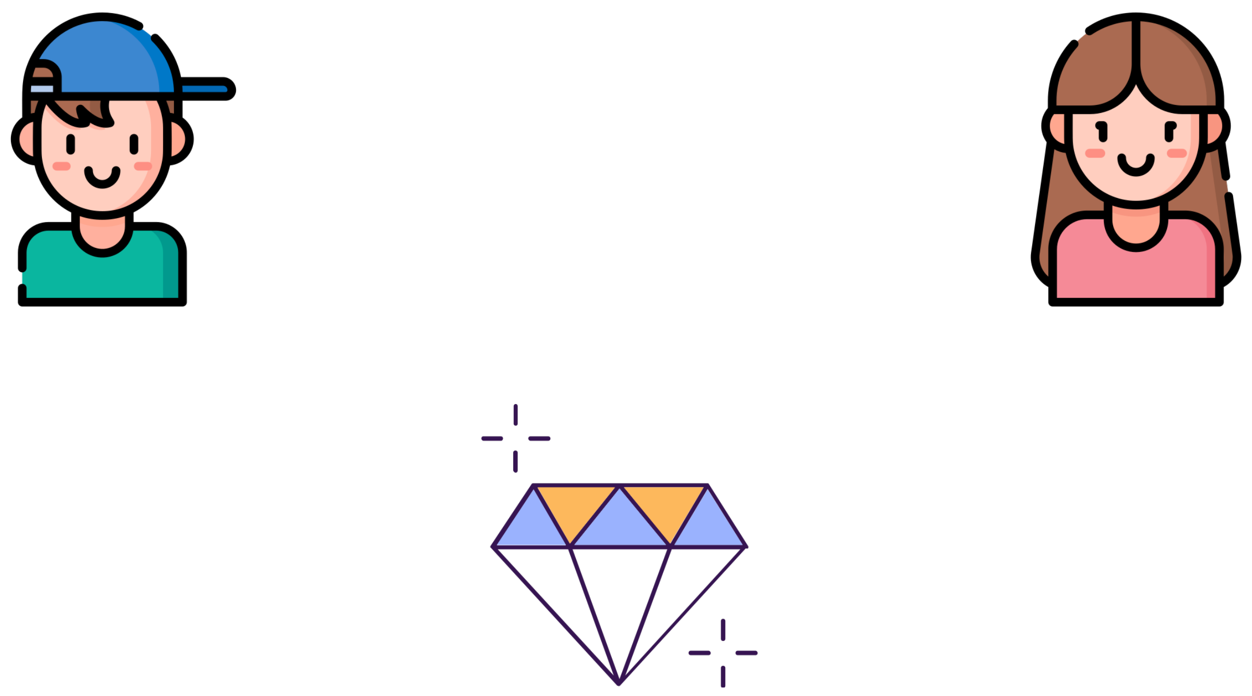
- An allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is weighted proportional (**WPROP**) if

$\forall_{i\in[n]} \quad v_i(X_i) \geq \alpha_i \cdot v_i(B)$ - When B is a set of goods

$\forall_{i\in[n]} \quad v_i(X_i) \leq \alpha_i \cdot v_i(B)$ - When B is a set of chores

  - We say X_i is a **WPROP** bundle for agent a_i





Does not always exist

Almost Proportional Allocations

- An allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is weighted proportional up to one item (**WPROP1**) if

  $\forall_{i \in [n]}, \exists b \in B \backslash X_i \quad v_i(X_i \cup b) \geq \alpha_i \cdot v_i(B)$ - When B is a set of goods

  $\forall_{i \in [n]}, \exists b \in X_i \quad v_i(X_i \backslash b) \leq \alpha_i \cdot v_i(B)$ - When B is a set of chores

  - We say X_i is a **WPROP1** *bundle* for agent a_i

- Does it always exist?

YES [Aziz, Moulin, Sandomirskiy; Oper. Res. Lett 2020]

Envy vs Proportionality

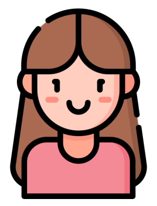
- Envy Free (EF): $\forall i, j \in [n]$ $v_i(X_i) \geq v_i(X_j)$
- Envy Free up to One Item (EF1): $\forall i, j \in [n]$ $\exists h \in X_j$, $v_i(X_i) \geq v_i(X_j \backslash h)$
- Appropriate generalisation to the weighted setting (WEF, WEF1) [Chakraborty, Igarashi, Suksompong, Zick; AAMAS 2020]

- EF $\implies$ PROP WEF $\implies$ WPROP
- EF1 $\implies$ PROP1

WEF1 $\not\Rightarrow$ WPROP1 [Chakraborty et al, AAMAS 2020]

Necessarily **Fair Allocation**

- Agents rank the items: $b_3 \succ_i b_1 \succ_i \cdots \succ_i b_n$
- Agents have private cardinal valuations that respects their ranking.

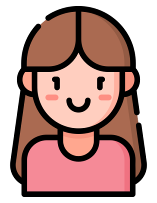


	$\cdots$	b_1	$\succ$	b_2	$\cdots$
		10		9	
		20		20	
		1		2	

- An allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is *necessarily* **WPROP1** (**WSD-PROP1**) if $\forall a_i \in A$, bundle X_i is **WPROP1** under all valuations that respects the agent rankings.

Necessarily Fair Allocation

An ordinal Instance of Fair Allocation: $I = \langle A, B, \Pi, \overrightarrow{\alpha} \rangle$

		10	9	2	1
0.3		$b_1 \succ$	$b_2 \succ$	$b_4 \succ$	b_3
		13	9	5	5
0.2		$b_2 \succ$	$b_4 \succ$	$b_1 \succ$	b_3
		7	7	7	0
0.5		$b_1 \succ$	$b_4 \succ$	$b_3 \succ$	b_2

WPROP1 ?

[b_1] [b_2 b_3] [b_4]

A Matching Approach

"Divorcing Made Easy"
[Pruhs and Woeginger; FUN 2012]

WSD-PROP - strict ordering

"Fair assignment of indivisible objects under ordinal preferences"
[Aziz, Gaspers, Mackenzie and Walsh; AAMAS 2014]

WSD-PROP - weak ordering

Our Contribution: A matching approach to find WSD-PROP1 allocations

Existence of WSD-PROP1 Allocation

Do WSD-PROP1 allocations always exist?

- Goods: Yes [Aziz et.al and Hoefer et.al AAMAS 2023]

  Approach: Eating Algorithm.

- Chores: Yes [Wu et.al EC 2023]

  Approach: Weighted Reverse Round Robin.

Matching approach:

- Works for both Goods and Chores. (Alternate proof of existence using Hall's Theorem)
- Gives an integral polytope of all WSD-PROP1 allocations.
- Also gives economic efficiency guarantees.
- Best of Both World fairness notions.
- Is Parallelizable. That is, WSD-PROP1 is in **RNC, Quasi-NC**
- Brings along notions from Matching Theory Literature - Popularity, Matchings with quotas...

What Makes a WSD-PROP1 Bundle?

Building intuition with an example

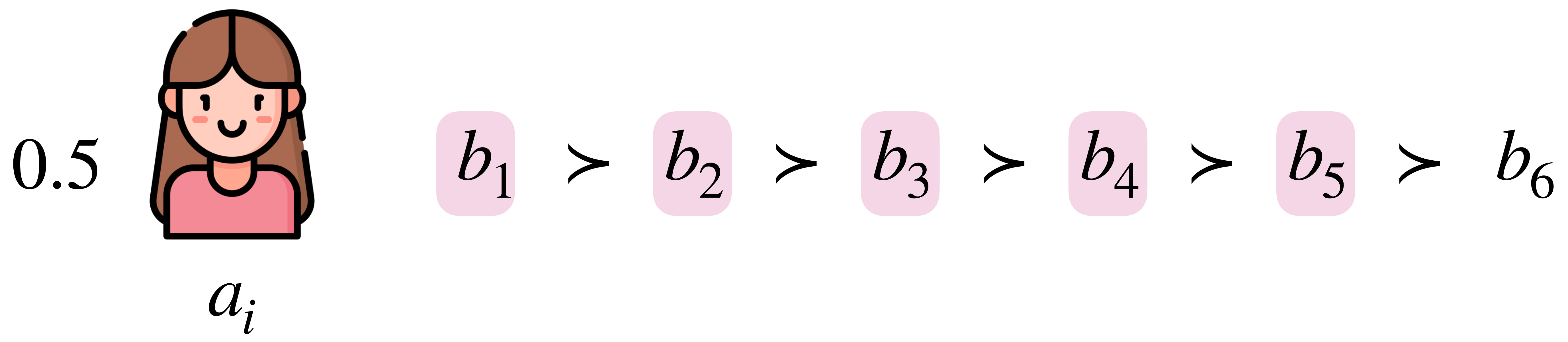
Recall - chore allocation set up:

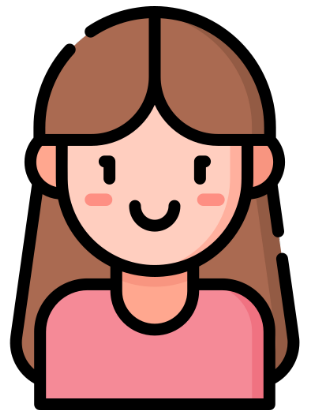
Lightest Chore
Most Favourite

9 5 5 2 0

$$a_i : b_1 \succ b_2 \succ b_3 \succ b_4 \cdots \succ b_m$$

Heaviest Chore
Least Favourite


0.5
a_i
$b_1 \succ b_2 \succ b_3 \succ b_4 \succ b_5 \succ b_6$

0.5 

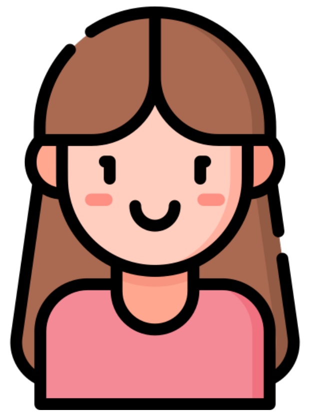
a_i

$1 \succ 1 \succ 1 \succ 1 \succ 1 \succ 1$

Total: 6 Entitled Share: 3

Bundle value after removal of one chore: 4

NOT WPROP1

0.5 

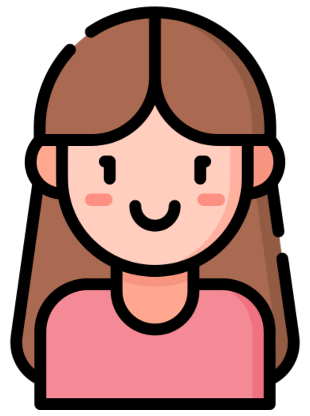
a_i

$1 \succ 1 \succ 1 \succ 1 \succ 1 \succ 1$

Total: 6 Entitled Share: 3

Bundle value after removal of one chore: 3

WPROP1 ✓

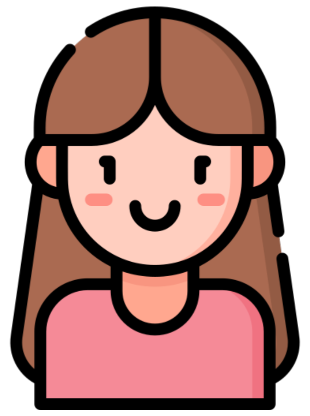
0.5  $1 \succ 1 \succ 1 \succ 1 \succ 1 \succ 0$

a_i

Total: 5 Entitled Share: 2.5

Bundle value after removal of one chore: 3

NOT WPROP1

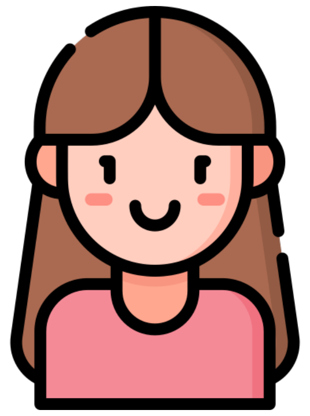
0.5  a_i

$1 \succ 1 \succ 1 \succ 1 \succ 1 \succ 0$

Total: 5   Entitled Share: 2.5

Bundle value after removal of one chore: 2

WPROP1 ✓

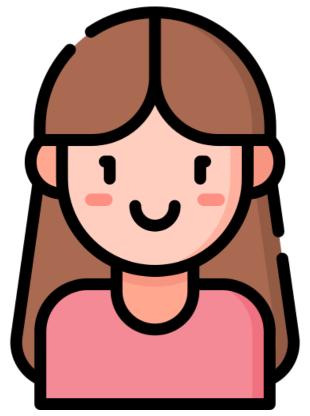
0.5  a_i

$1 \succ 1 \succ 1 \succ 0 \succ 0 \succ 0$

Total: 3 Entitled Share: 1.5

Bundle value after removal of one chore: 2

NOT WPROP1

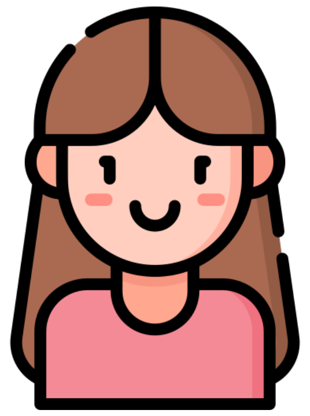
0.5  a_i

$1 \succ 1 \succ 1 \succ 0 \succ 0 \succ 0$

Total: 3 Entitled Share: 1.5

Bundle value after removal of one chore: 1

WPROP1 ✓

0.5 

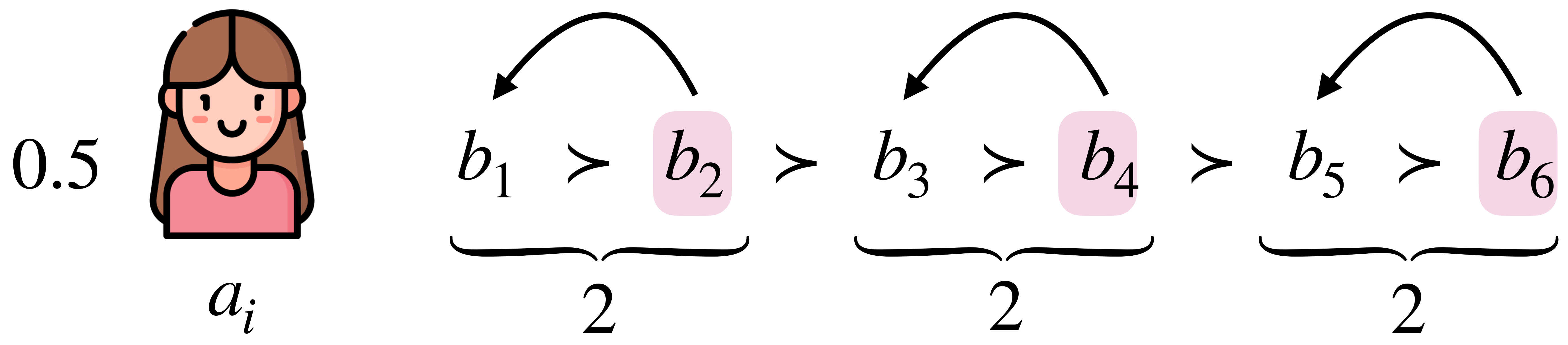
a_i

$b_1 \succ b_2 \succ b_3 \succ b_4 \succ b_5 \succ b_6$

Total: 3

Bundle value after removal of one chore: 1

WSD-PROP1

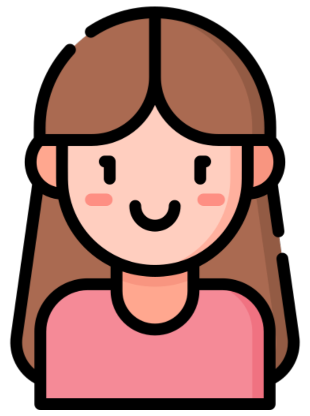


Total: 3

Bundle value after removal of one chore: 1

WSD-PROP

For an agent a_i , a bundle X_i is WSD-PROP1 if it has at most 1 chore per every $\frac{1}{\alpha_i}$ chore in the sorted order.

0.5 

a_i

$$\underbrace{b_1 \succ b_2} \succ \underbrace{b_3 \succ b_4} \succ \underbrace{b_5 \succ b_6}$$

Total: 3

Bundle value after removal of one chore: 1

WSD-PROP1

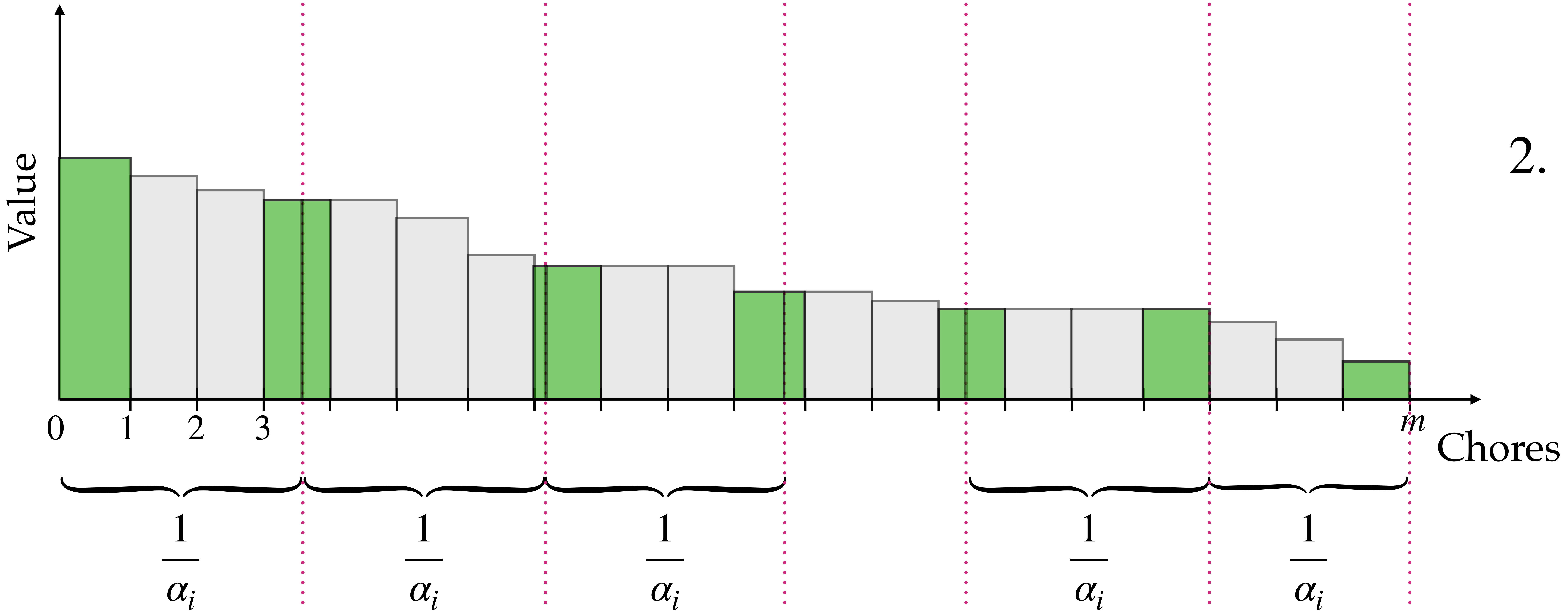
Characterizing WSD-PROP1 Bundles

A bundle is WSD-PROP1 for an agent a_i if and only if

- It has at most $\lfloor m\alpha_i \rfloor + 1$ chores (at least $\lceil m\alpha_i \rceil - 1$ many goods)
- The ℓ^{th} item in the bundle (sorted) is later than or equal to $\left\lceil \frac{\ell - 1}{\alpha_i} \right\rceil^{\text{th}}$ chore according to a_i. (or within the first $\left\lfloor \frac{\ell}{\alpha_i} \right\rfloor + 1$ goods)

Characterizing WSD-PROP1 Bundles

Proof sketch: (Sufficient)

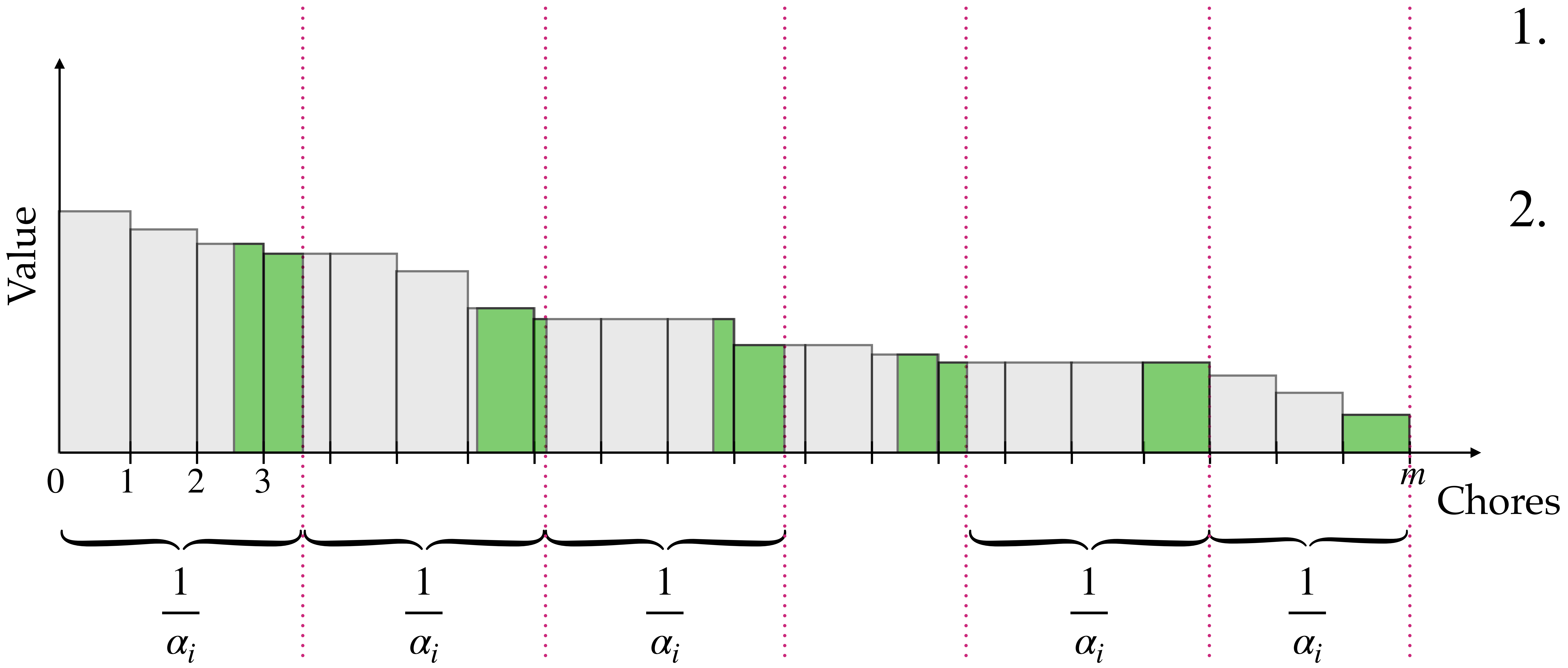


1. $m_i = \lfloor m\alpha_i \rfloor + 1$

2. $r_\ell = \left\lceil \frac{\ell - 1}{\alpha_i} \right\rceil$

Characterizing WSD-PROP1 Bundles

Proof sketch: (Sufficient)



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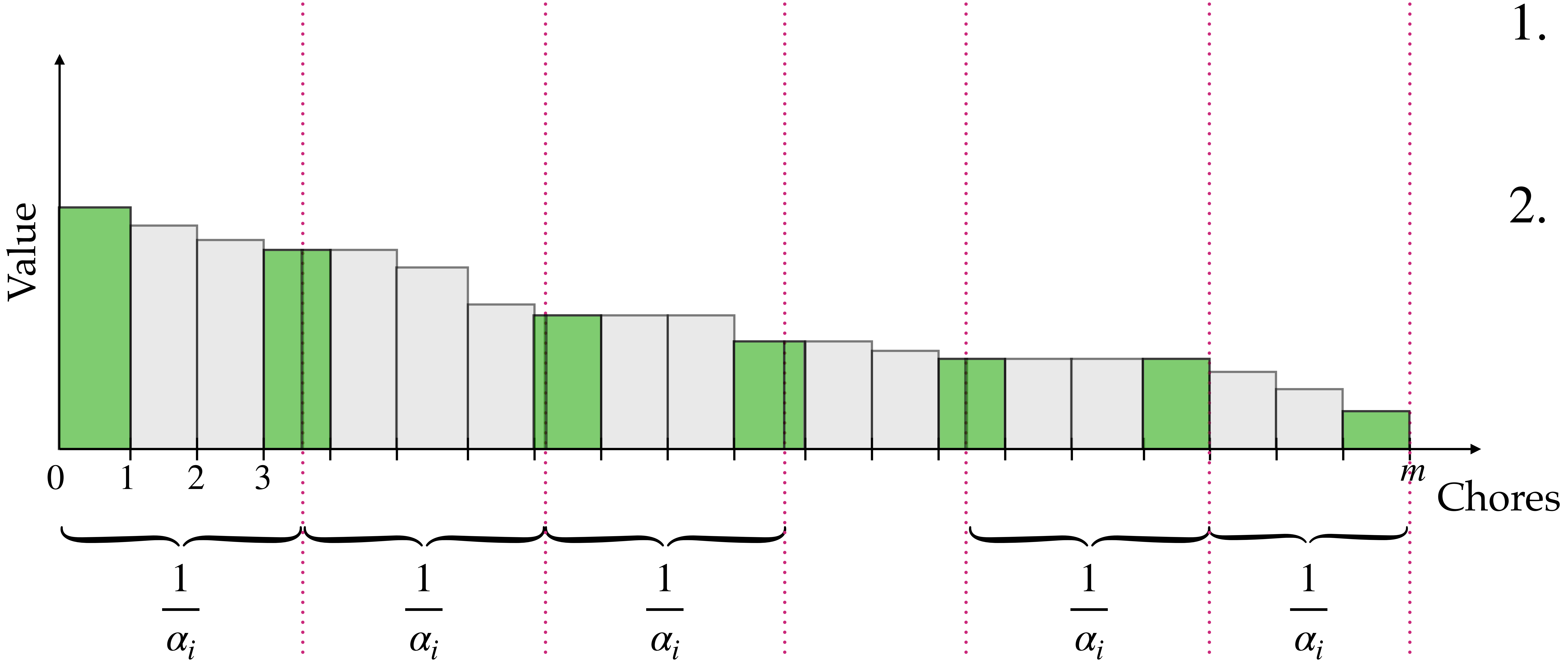


1. $m_i = \lfloor m\alpha_i \rfloor + 1$
2. $r_\ell = \left\lceil \frac{\ell - 1}{\alpha_i} \right\rceil$

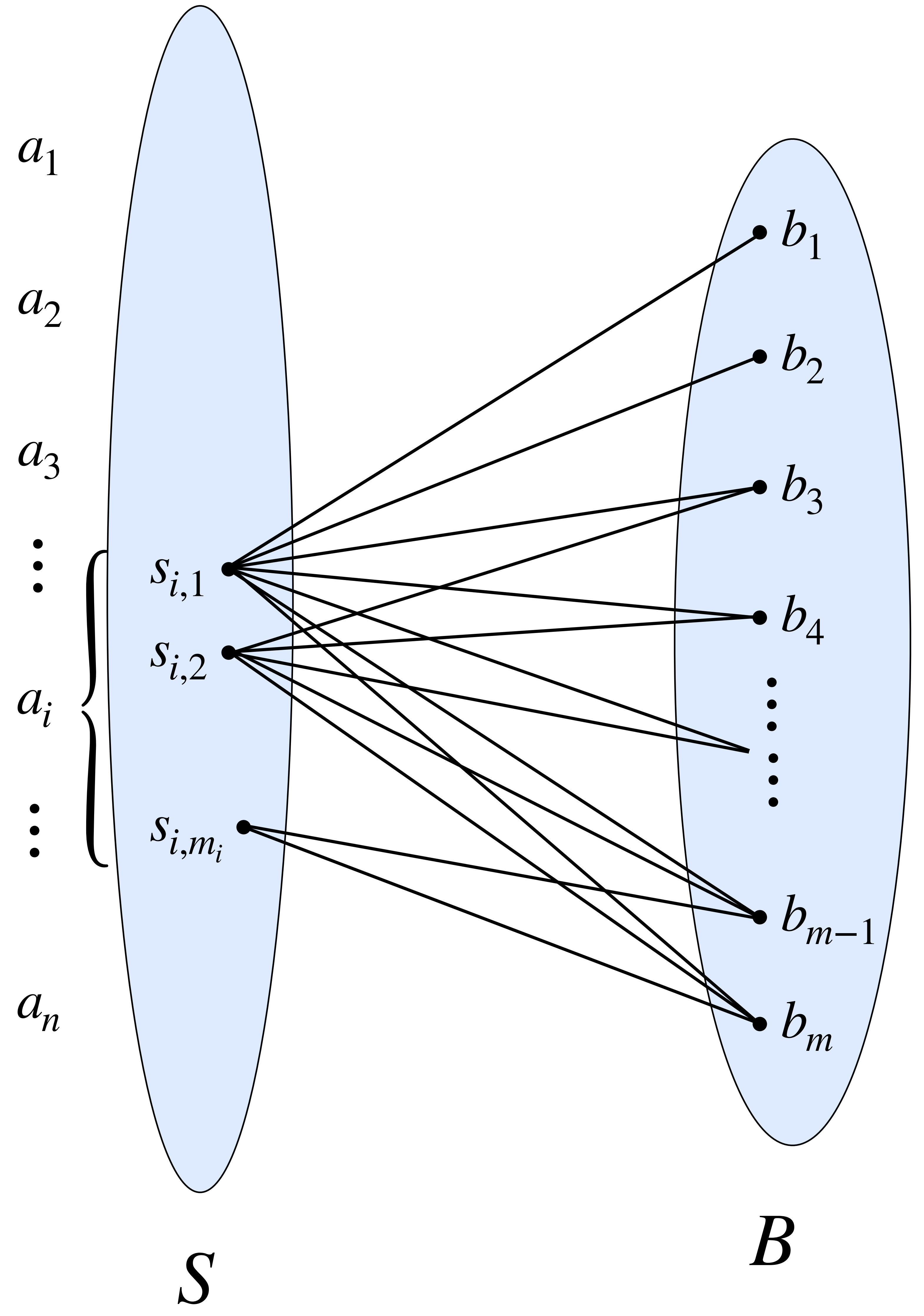
∎

Matching Items to *Slots*

Allocation Graph : $G_c = (S \cup B, E)$

$m_i = \lfloor m\alpha_i \rfloor + 1$

- at most $\lfloor m\alpha_i \rfloor + 1$ chores
- later than or equal to $\left\lceil \frac{\ell - 1}{\alpha_i} \right\rceil^{\text{th}}$ chore

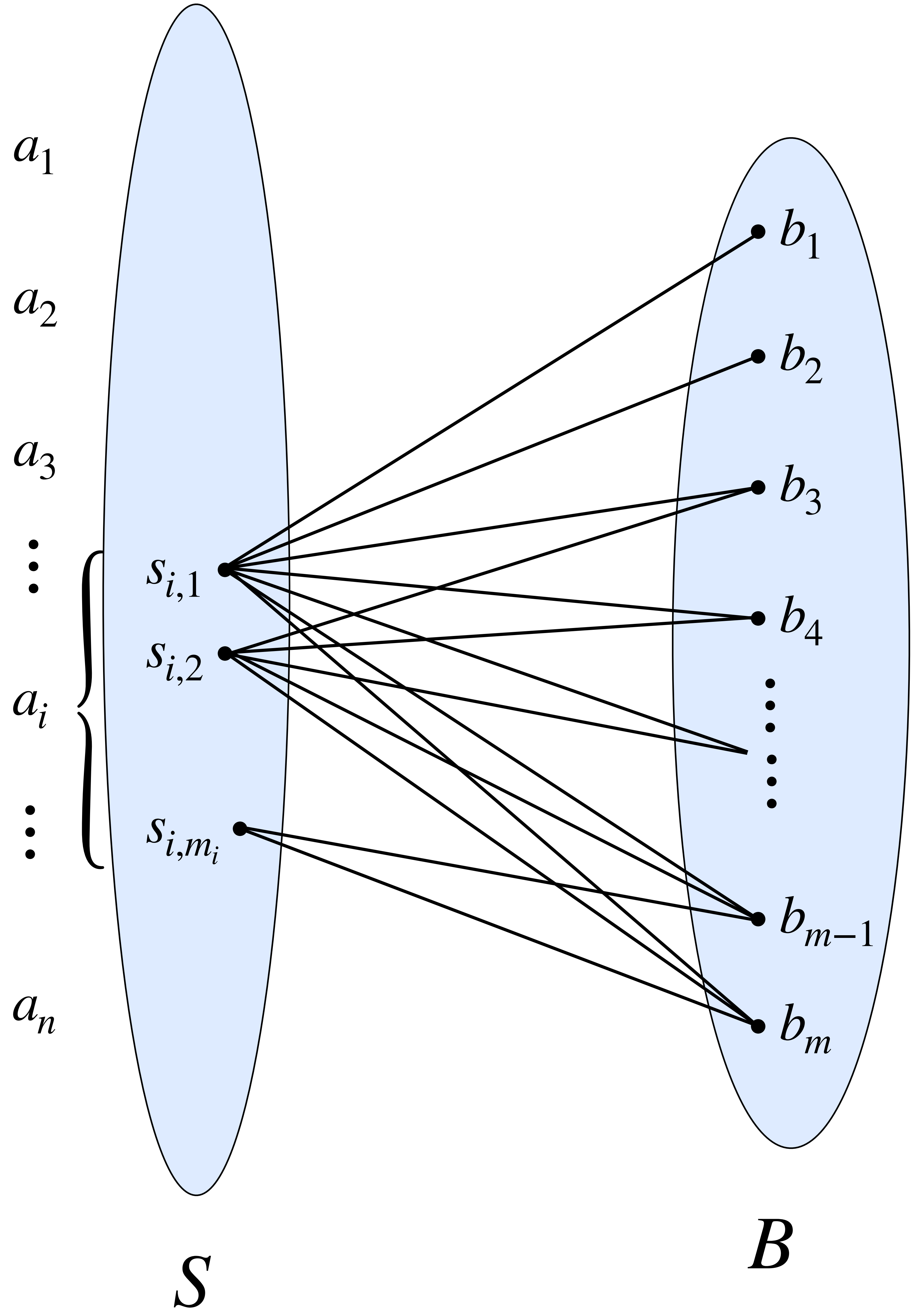


Finding WSD-PROP1 Allocations

Lemma 1:

A matching that matches all the vertices in B (B-perfect) corresponds to a WSD-PROP1 allocation and vice-versa

Matching Polytope ≡ WSD-PROP1 Polytope



Existence of WSD-PROP1 Allocations

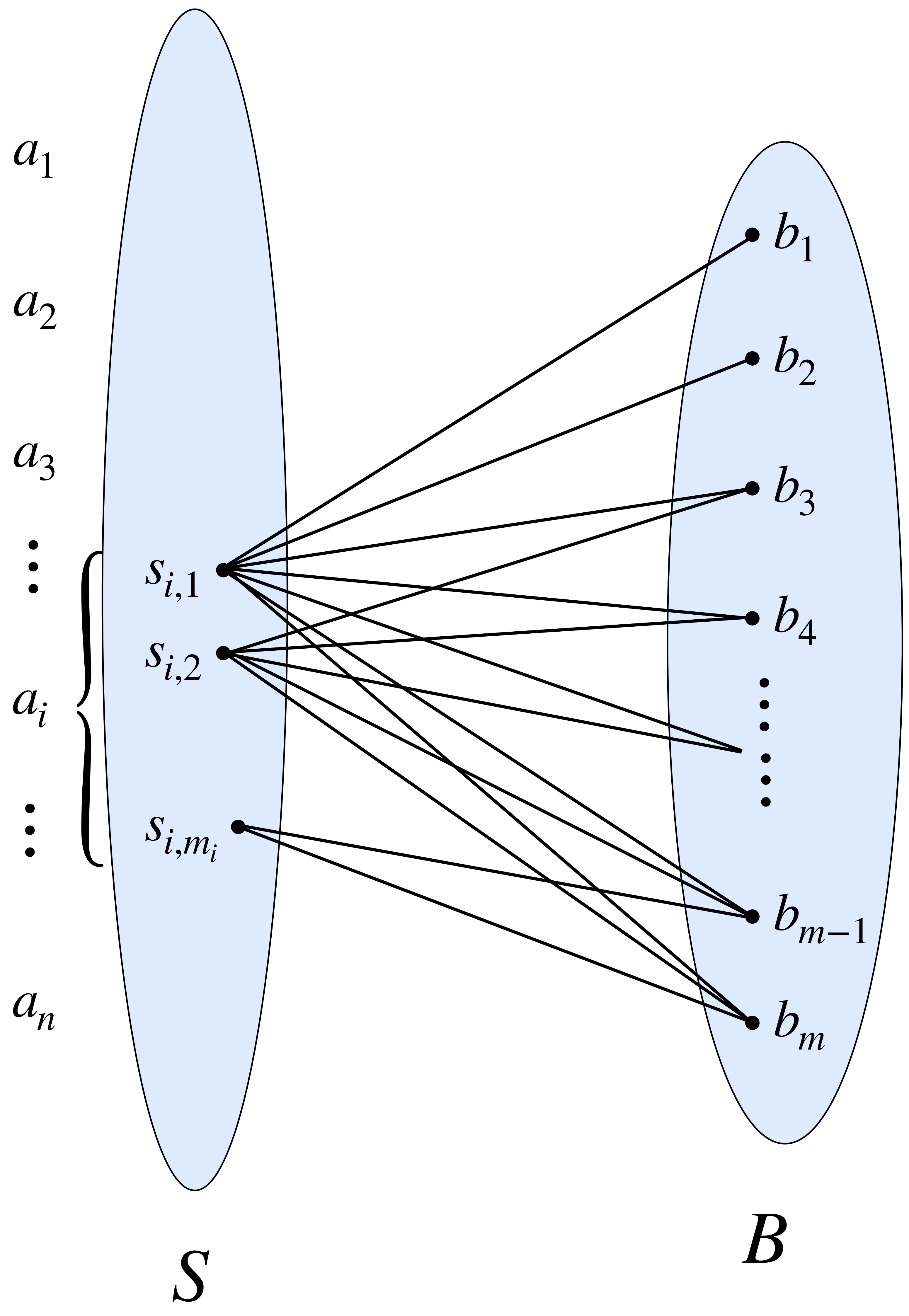
Lemma 2:

The allocation graph always admits a B-perfect matching.

Proof:

Application of Hall's Theorem.

Given a bipartite graph $G = (X \cup Y, E)$, there exists a Y-perfect matching in G iff $\forall S \subseteq Y$, $|N(S)| \geq |S|$



Finding WSD-PROP1 Allocations
Using Matchings

Algorithm to find a WSD-PROP1 allocation:

Input: $I = \langle A, B, \Pi, \overrightarrow{\alpha} \rangle$

Output: A WSD-PROP1 allocation

1. Construct the allocation graph $G_c = (S \cup B, E)$
2. Find a B-perfect matching M in G_c
3. Return the allocation corresponding to M

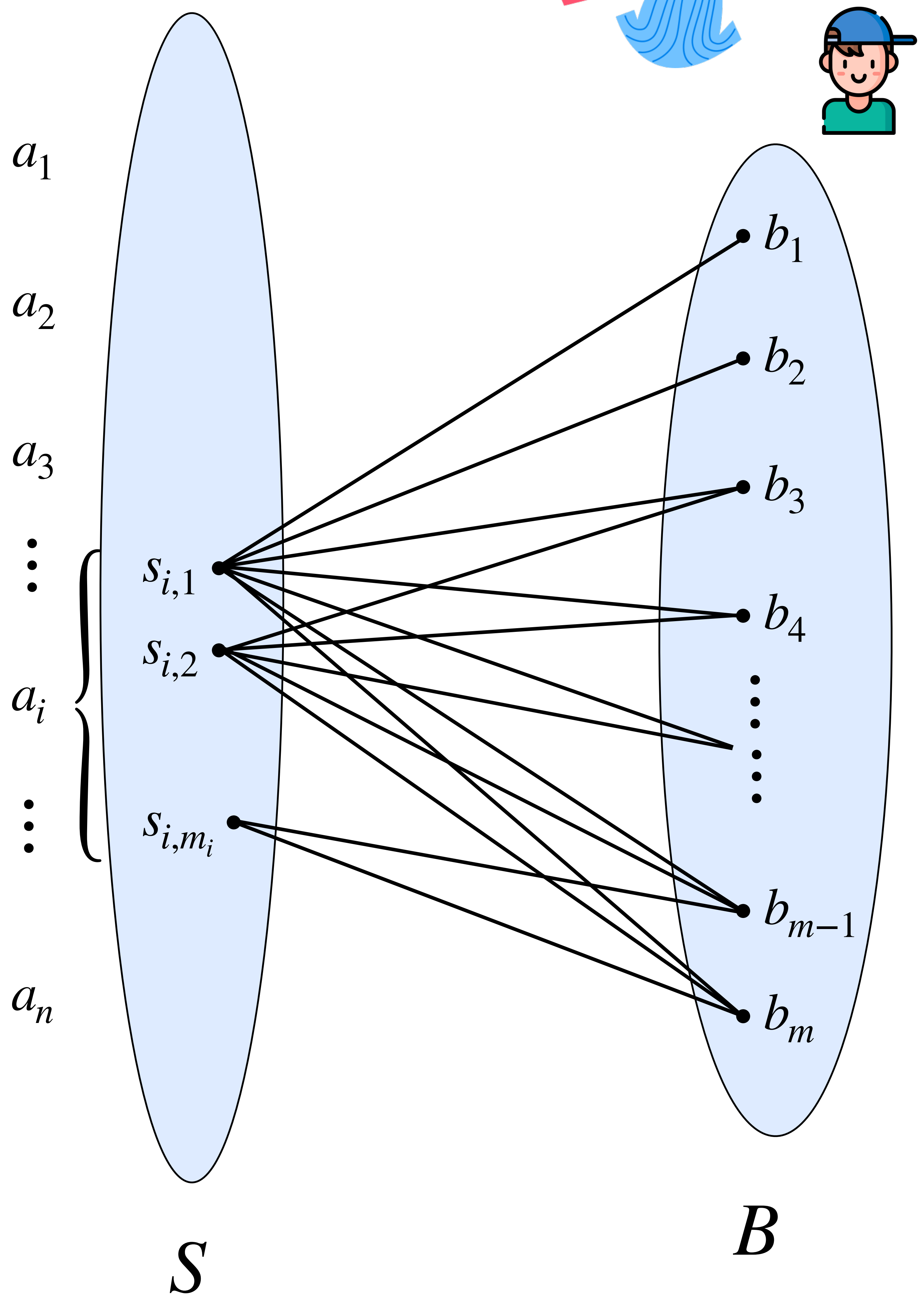
WSD-PROP1 $\in$ P, RNC, Quasi-NC

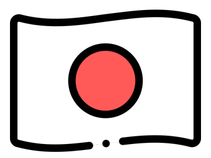
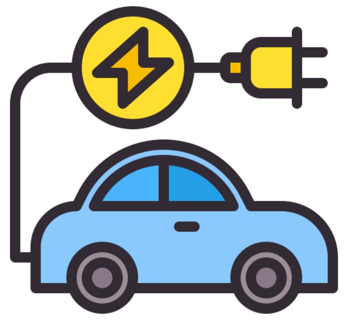
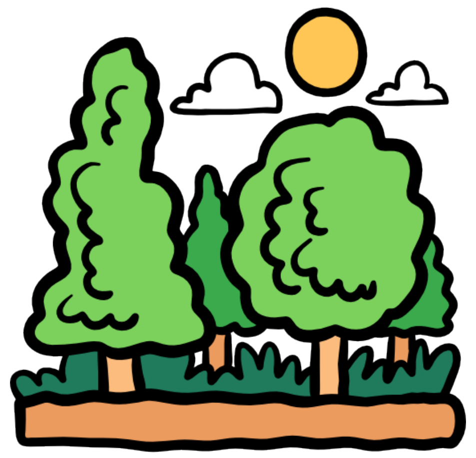
$\blacksquare$

Optimizing over WSD-PROP1 Polytope

- Let $u_i : B \to [0,1]$ denote how efficiently agent a_i can do chores.
  - u_i can be treated as edge weights.
  - Maximum weight B-perfect matching in G_c



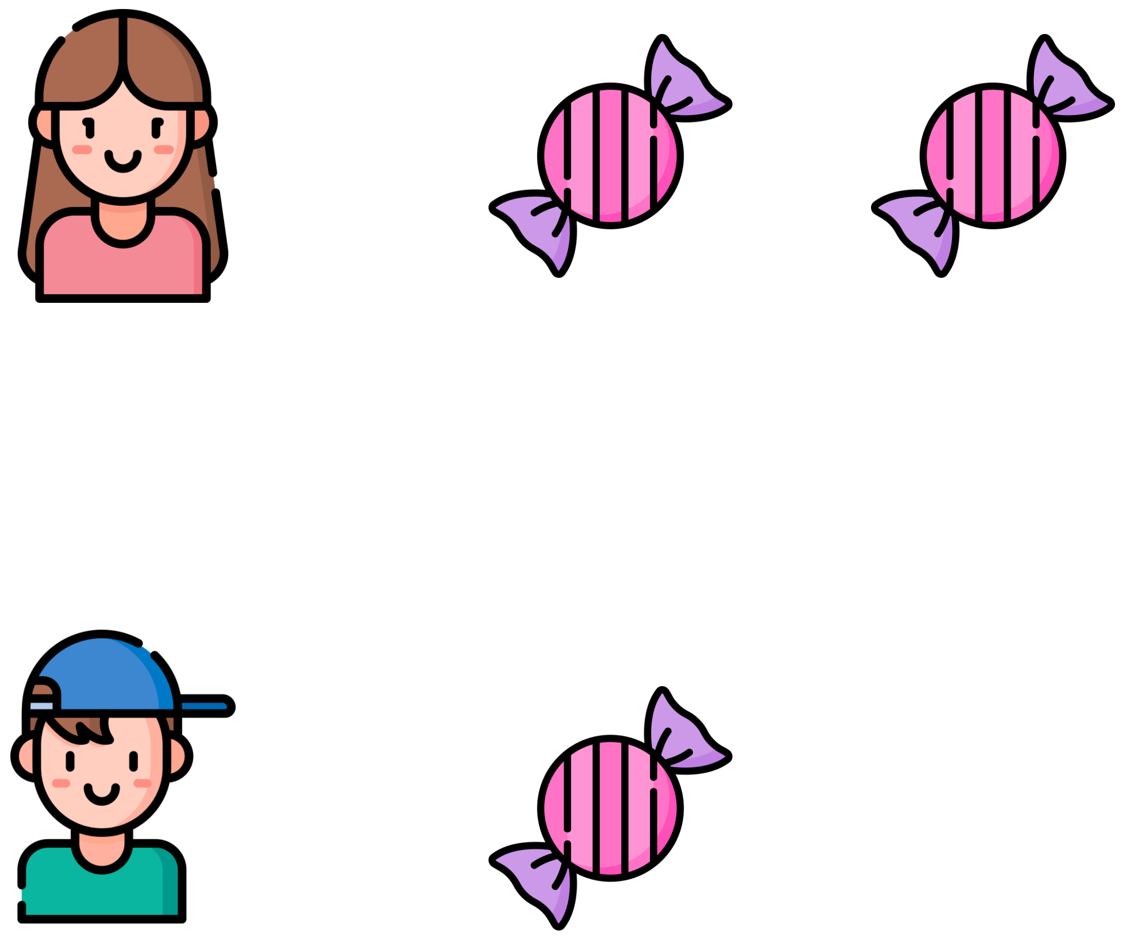


Best of Both Worlds

Using Randomization

Using Randomization



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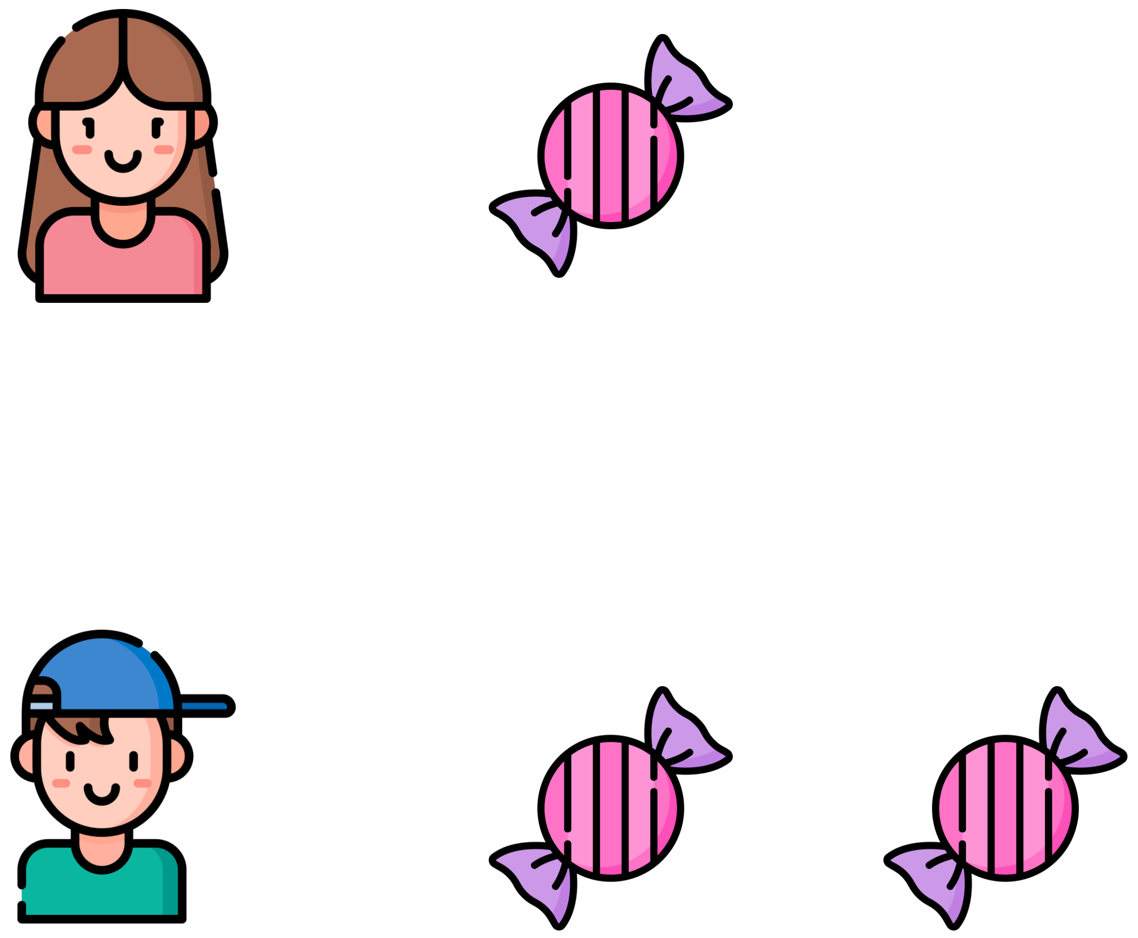

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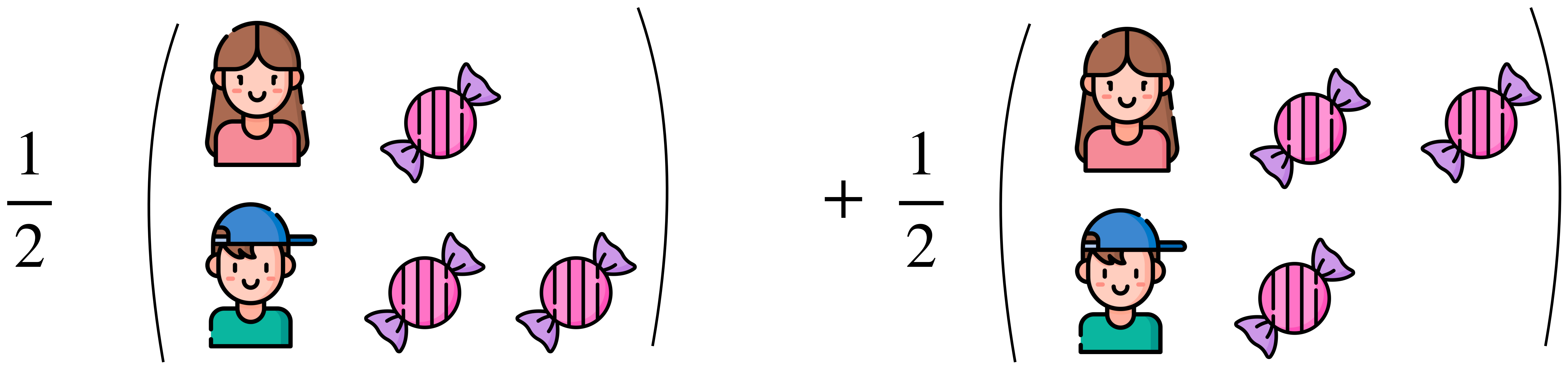


Best of Both Worlds Fairness

A tuple $((p_1, Y^1), (p_2, Y^2), \cdots, (p_q, Y^q))$ where $\sum_{i \in q} p_i = 1$ and $p_i \in [0,1]$

Ex-Post **WSD-PROP1 :** If every Y^i is WSD-PROP1

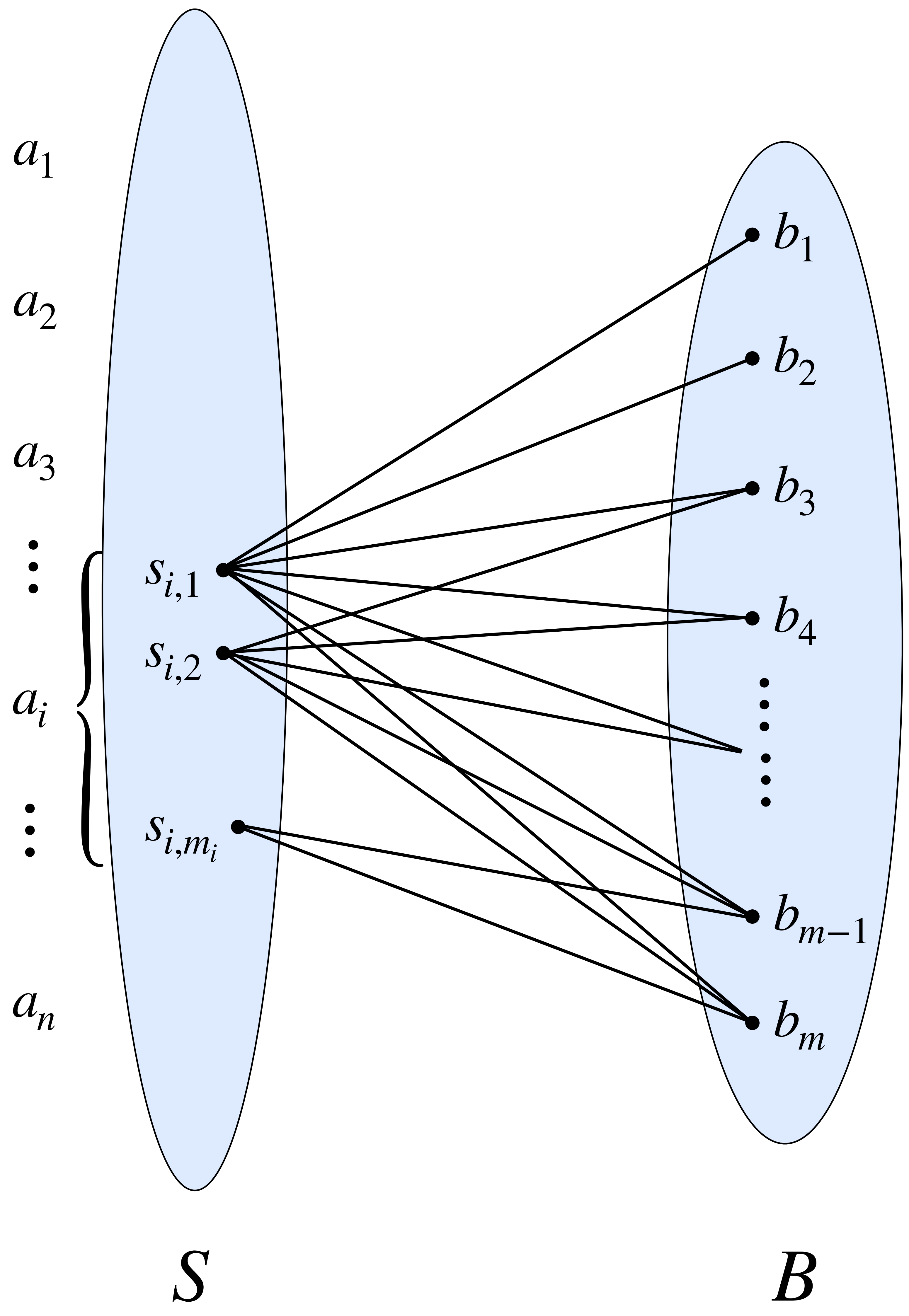
Ex-Ante **WSD-PROP :** If Expected bundle value is **WSD-PROP** for all agents.

Does there exist such a tuple?

Balancing the Allocation Graph

1. $m_i \leq \lfloor m\alpha_i \rfloor + 1$

2. $r_\ell \geq \left\lceil \dfrac{\ell - 1}{\alpha_i} \right\rceil$



Balancing the Allocation Graph

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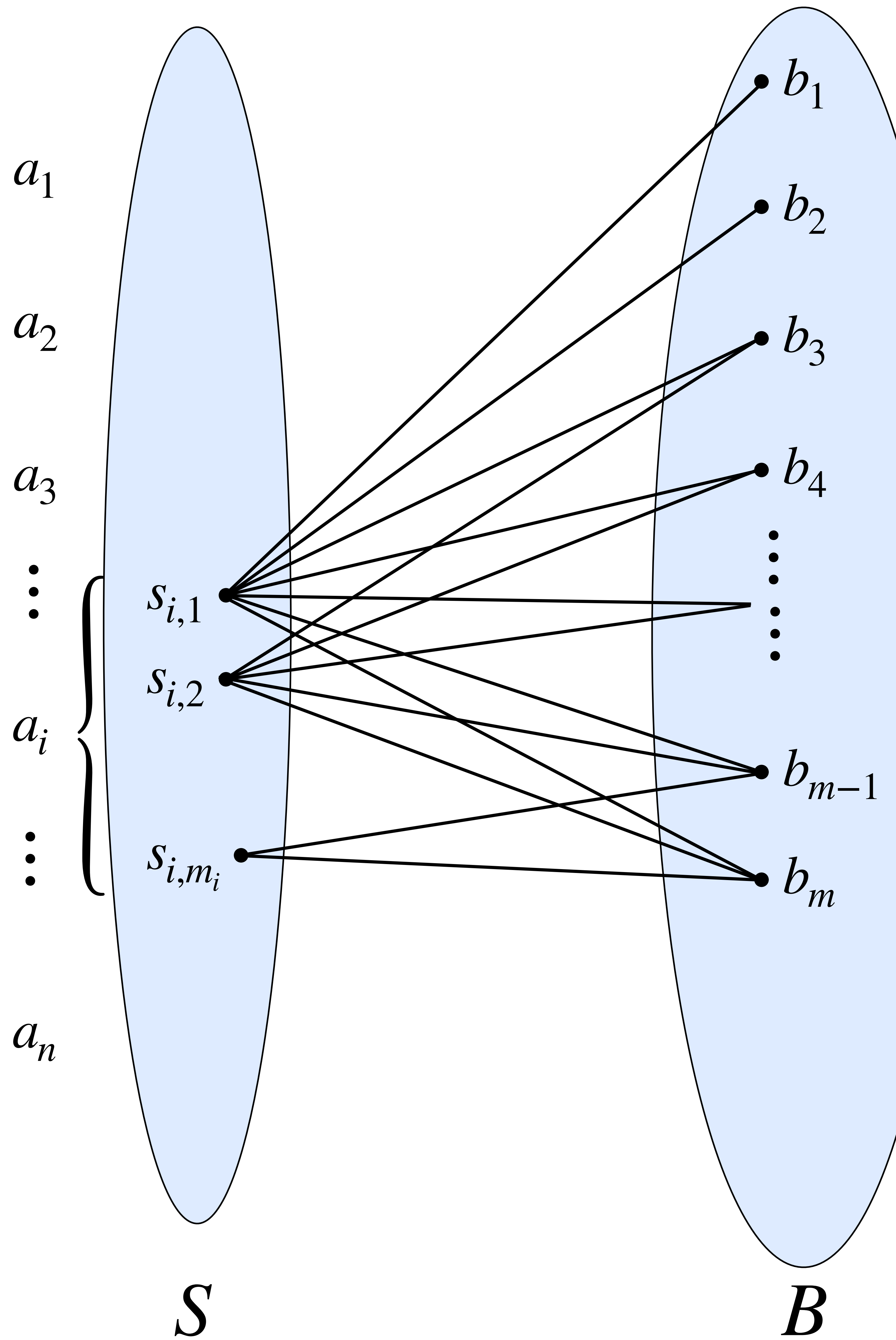


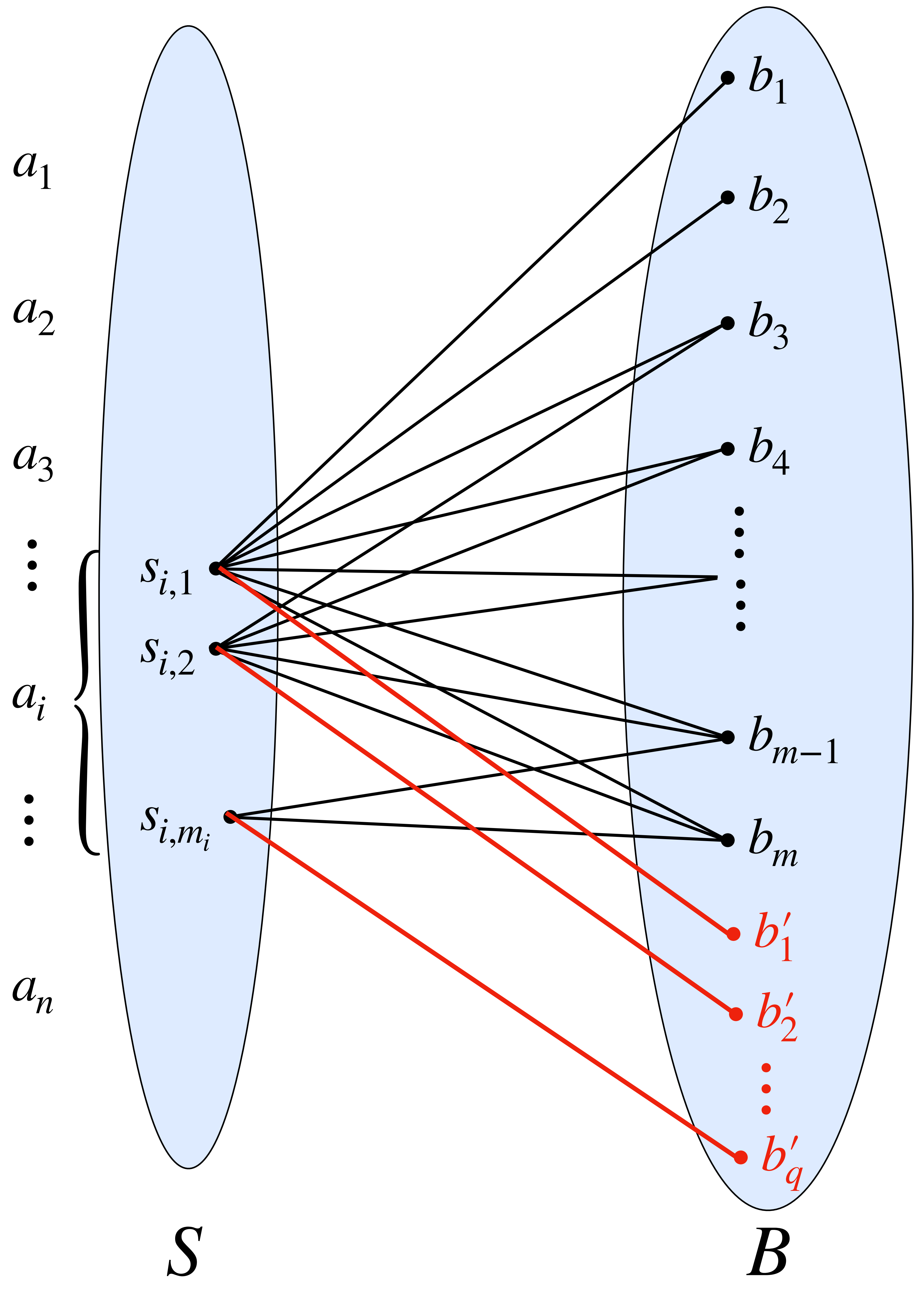
Balancing the Allocation Graph

A perfect matching in G_c^+ corresponds to a WSD-PROP1 allocation and vice-versa.

For a bipartite graph $G = (X \cup Y, E)$:

Matching Polytope

$$\begin{cases} \sum_{x \in N(y)} e_{xy} = 1 & \forall y \in Y \\ \sum_{y \in N(x)} e_{xy} = 1 & \forall x \in X \\ e_{xy} \geq 0 & \end{cases}$$

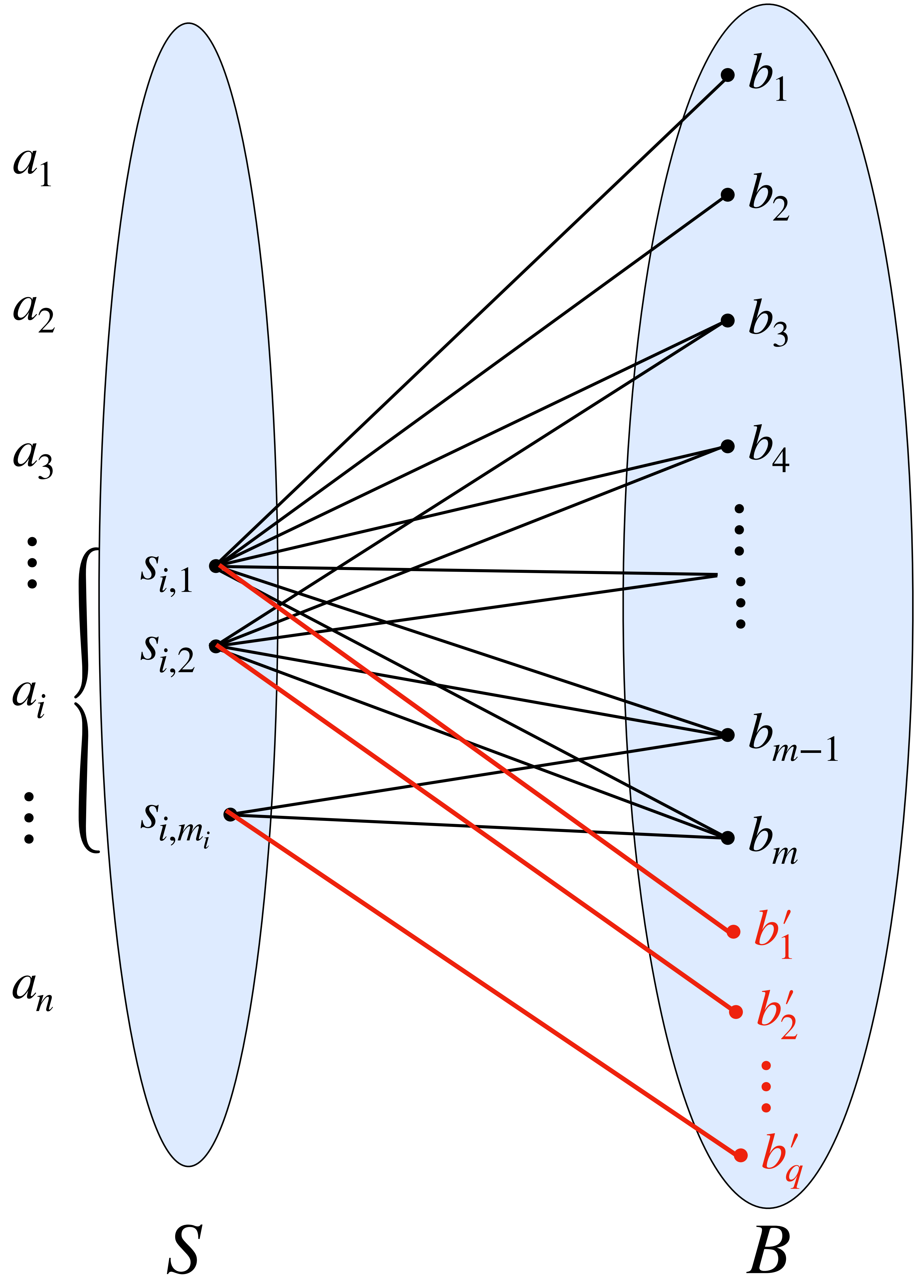


Best of Both Worlds Fairness

Fractional allocation Y: Agent a_i gets α_i fraction of every real chore.

Y is WSD-PROP (and WSD-EF)

There exists a **fractional perfect matching** M_y in G_c^+ corresponding to the above allocation

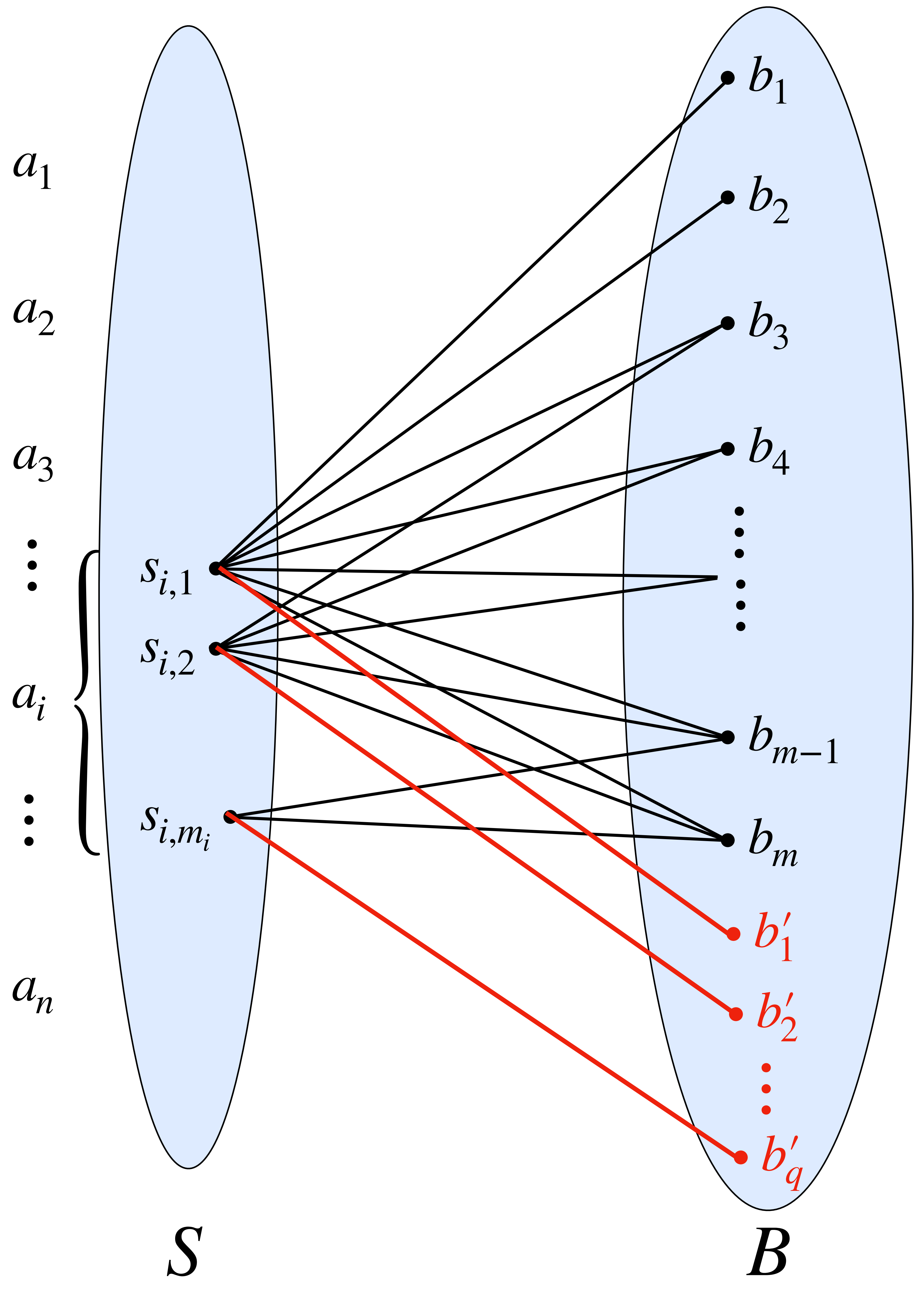


Best of Both Worlds Fairness

Theorem [Birkhoff-von Neumann]:

A fractional perfect matching M can be expressed as a convex combination of polynomially many integral perfect matchings

$$M = p_1 M_1 + p_2 M_2 + \cdots + p_q M_q$$



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Economic Guarantees

Ordinal Pareto Optimal: An allocation **X** is ordinary Pareto optimal if there does not exist any other allocation **Y** such that under all order-respecting valuations <u>no agent gets a worse bundle and at least one agent gets a better bundle in **Y**.</u>

Rank Maximal Matching $\implies$ Ordinal Pareto optimal

Rank Maximal Matchings can be found in time $\mathcal{O}(m+n)^{3.5}$ [Irving 2003, Irving, Kavita, Mehlhorn, Michail 2006].

Economic Guarantees

Cardinal Pareto Optimal: An allocation **X** is Cardinally Pareto optimal if there does not exist any other allocation **Y** such that under some order-respecting valuations no agent gets a worse bundle and at least one agent gets strictly better bundle in **Y**.

Result:

Cardinally PO allocations do not always exist

Popularity

An allocation **X** is said to be Popular if **X** does not lose a head-to-head election with any other allocation **Y**.

Popular $\Longrightarrow$ Pareto optimal

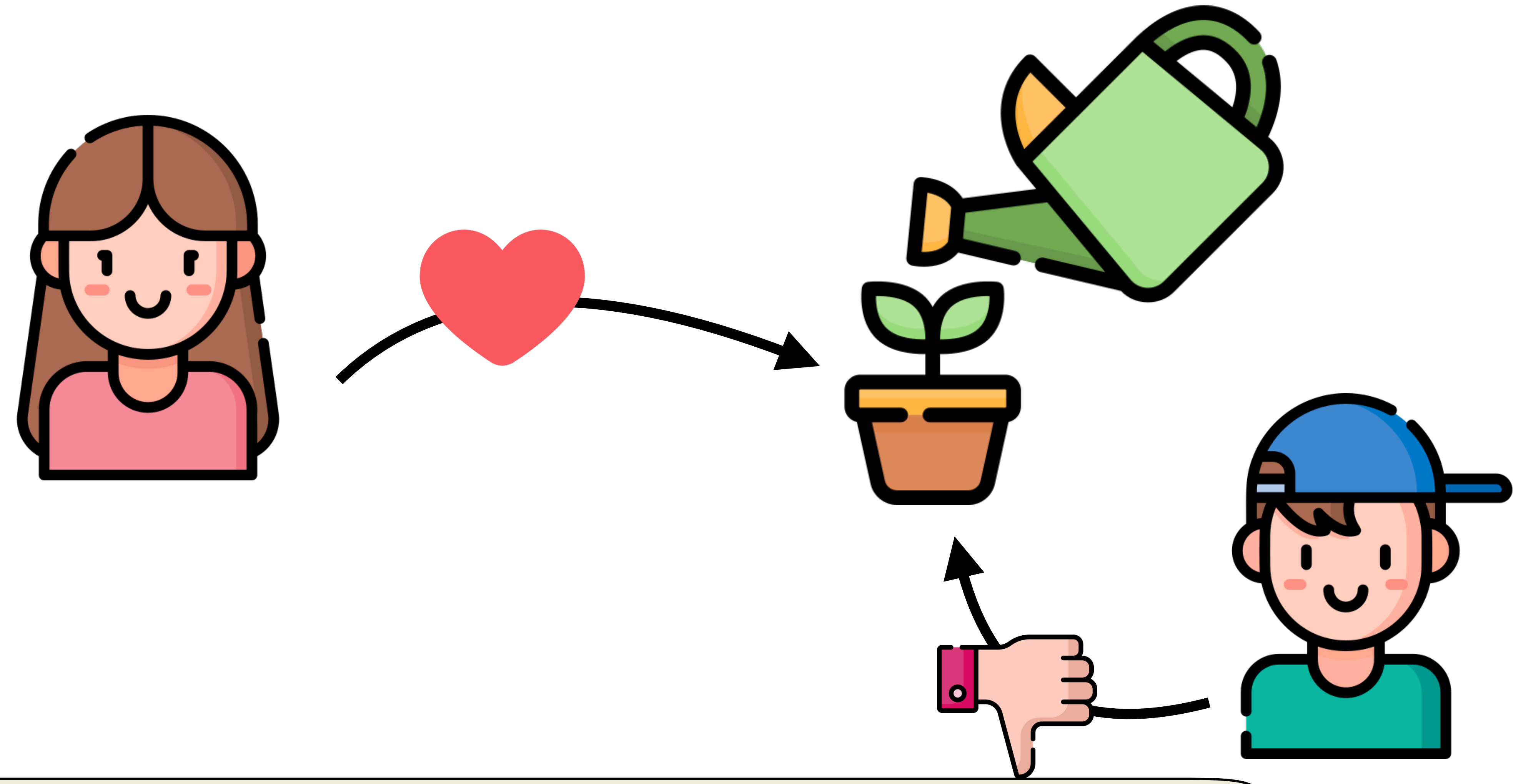
Popularity

Maximum cardinality Popular matchings in One-sided preference $\in \mathbb{P}$
[Abraham, Irving, Kavitha, Mehlhorn; SODA 2005]

Therefore, Finding a Popular WSD-PROP1 allocation $\in \mathbb{P}$

Open Questions

- Mixed Setting: An item can be a chore for one agent and good for another.
- Matching based approaches for other fairness notions?



Conclusion:

Matching approach:

- Works for both Goods and Chores. (Alternate proof of existence using Hall's Theorem)
- Gives an integral polytope of all WSD-PROP1 allocations.
- Also gives economic efficiency guarantees.
- Best of Both World fairness notions.
- Is Parallelizable. That is, WSD-PROP1 is in **RNC, Quasi-NC**
- Brings along notions from Matching Theory Literature - Popularity, Matchings with quotas...

Thank You!

Questions?