Symbolic Computation of Sequential Equilibria

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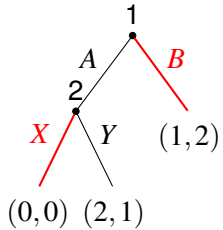
AAMAS 2024

Motivation

- Extensive games with imperfect information.
- Standard solution concept: Sequential Equilibrium.
- No general solver available.

Nash equilibrium

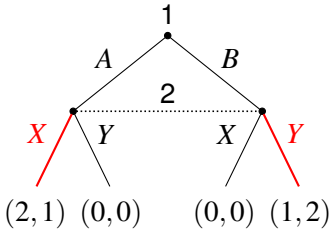
No player can improve their expected utility by deviating.



Problem: Actions in unreached parts of the tree do not have to be optimal.

Subgame perfect equilibrium

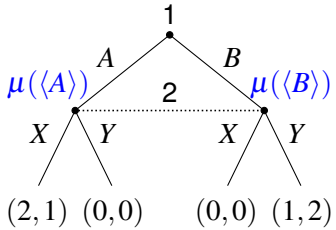
No player can improve their expected utility **in any subgame** by deviating.



Problem: Not suitable for games with imperfect information.

Assessments

Assessments consisting of a strategy profile β and system of beliefs μ.



Believed utility at information set I:

$$U_i^B(\beta, \mu \mid I) = \sum_{h \in I} \mu(I)(h) U_i(\beta \mid h)$$

Sequential Equilibrium

Sequential Equilibria [Kreps & Wilson 1982]

- **Sequential rationality**
  No player can improve their believed payoff at any information set by deviating from the equilibrium strategy given their belief.
- **Consistency**
  The players beliefs should be sensible given the played strategies.

Sequential rationality

$$U_i^B(\beta', \mu \mid I) \leq U_i^B(\beta, \mu \mid I)$$

- Similar to subgame perfectness.
- Can be reduced to polynomial equations and inequalities.

$$U_i^B(\beta, \mu | I, a) - U_i^B(\beta, \mu | I) \quad \leq 0$$

$$\beta(I)(a) \cdot \left(U_i^B(\beta, \mu | I, a) - U_i^B(\beta, \mu | I)\right) = 0$$

Consistent beliefs

- At information sets that are reached by playing β: Conditional probabilities.

$$\mu(I)(h) = P_\beta(h \mid I) = \frac{P_\beta(h)}{P_\beta(I)}$$

- But what if the information set is never reached?

Consistency definition

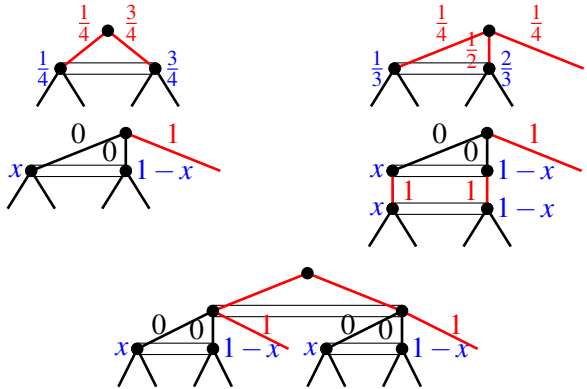
There exists a series of assessments (β^n, μ^n) such that:

$$\beta^n(I)(a) > 0$$

$$\mu^n(I)(h) = \frac{P_{\beta^n}(h)}{P_{\beta^n}(I)}$$

$$\lim_{n \to \infty} (\beta^n, \mu^n) = (\beta, \mu)$$

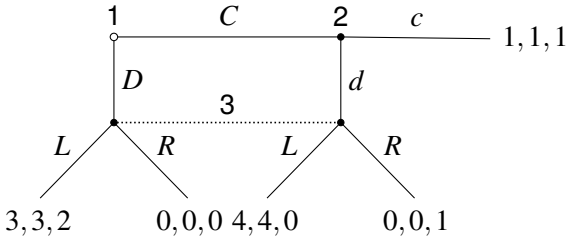
Examples of consistency



Dealing with Consistency

- Working with consistency is difficult.
- Different characterization of consistency is required.
- Process to transform consistency to polynomial equations [Kohlberg & Reny, 1997].
- Involves the calculation of extreme directions of polyhedral cones.

Selten's horse



$$\mu(h_3)\beta(C)\beta(d) = \mu(h_4)\beta(D)$$

System of Equations

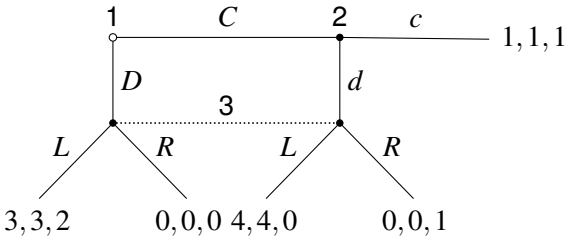
- Sequential rationality as polynomial equations and inequalities.
- Consistency as polynomial equations.

$\Rightarrow$ Single system of equations that describes all sequential equilibria of the game.

Finding Solutions

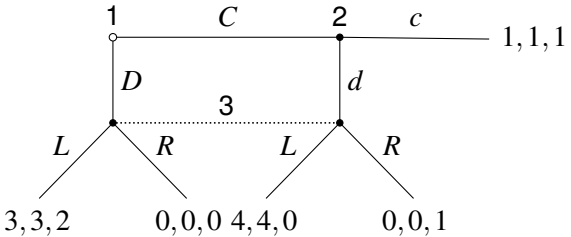
- Algorithm: cylindrical algebraic decomposition (*Mathematica*)
- Compact representation of all solutions.

Selten's horse solutions



$$\beta(D) = 0 \quad \beta(d) = 0 \quad \beta(L) = 0 \qquad 0 \leq \mu(\langle D \rangle) \leq \tfrac{1}{3}$$
$$\beta(C) = 1 \quad \beta(c) = 1 \quad \beta(R) = 1 \quad \mu(\langle C, d \rangle) = 1 - \mu(\langle C \rangle)$$

Selten's horse solutions



$$\beta(D) = 0 \quad \beta(d) = 0 \quad 0 < \beta(L) \leq \tfrac{1}{4} \quad \mu(\langle D \rangle) = \tfrac{1}{3}$$
$$\beta(C) = 1 \quad \beta(c) = 1 \quad \beta(R) = 1 - \beta(L) \quad \mu(\langle C, d \rangle) = \tfrac{2}{3}$$

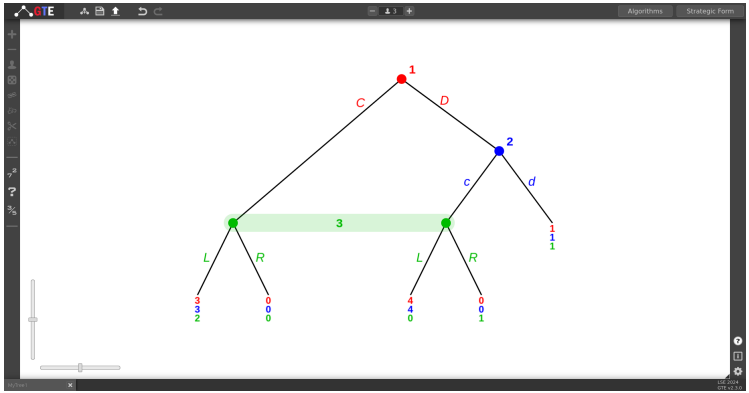
Manipulating the system of equations

- Solve parameterized games by adding additional variables.
- Compute interesting subsets of equilibria.
- Reduce computation time by looking only for pure strategy equilibria.

Results

- Single system of polynomial equations and inequalities that describes all sequential equilibria of the game.
- Implemented a general solver for sequential equilibria.
- Integrated into *Game Theory Explorer*.
- Computational complexity double exponential in number of variables.
- Small examples are feasable.

Game Theory Explorer



Try out yourself at `gte.engesser.xyz`
`github.com/tengesser/GTE-sequential`