Ron van der Meyden
Ka-shu Wong

Complete Axiomatizations for Reasoning about Knowledge and Branching Time

Abstract

Sound and complete axiomatizations are provided for a number of different logics involving modalities for the knowledge of multiple agents and operators for branching time, extending previous work of Halpern, van der Meyden and Vardi [to appear, SIAM Journal on Computing] for logics of knowledge and linear time. The paper considers the system constraints of synchrony, perfect recall and unique initial states, which give rise to interaction axioms. The language is based on the temporal logic CTL*, interpreted with respect to a version of the bundle semantics.

Introduction

Systems in which agents operate with incomplete information about their environment, which might include other agents, are of significance in computer science, philosophy and economics. A common approach to the analysis of such systems has been to use modal logics of *knowledge* [Hin62, FHMV95]. A key concern is how states of knowledge change over time, as agents acquire information, and as the environment changes. This has lead to the study of modal logics combining operators for knowledge and time. A variety of logics for the combination have been proposed [Sat77, FHV91, Leh84, LR86, PR85, Spa90] each defined in subtly different ways, but with large divergences in the complexity of the resulting logics.

An effort to classify this variety of logics, show how they are related and explain the differences in complexity, was initiated by Halpern and Vardi [HV89, HV88a]. They identify two parameters of variance in the logics: the language used and the assumptions made on the underlying distributed system. The linguistic differences include whether there is a single agent or multiple agents, whether an operator for common knowledge is included (which makes sense only when there are multiple agents), and whether the temporal logic is a *linear* time or a *branching* time logic. They consider four independent assumptions that can be made of the underlying distributed

Studia Logica **0**: 1–**??**, 2003.

system: unique initial state, synchrony, perfect recall, and no-learning. The different settings for these parameters lead to a total of 96 logics: 48 involving linear time and 48 involving branching time.

In [HV89, HV88a], the complexity of the logics resulting from different choices for these parameters is completely characterized. The complexity results show that some of these logics cannot be given a recursive axiomatization, since the set of valid formulas for these logics is not recursively enumerable. Complete axiomatizations for all the axiomatizable logics with linear time temporal operators were developed in [HvdMVar], incorporating earlier results from [HV86, HV88b, Mey94].

In this paper, we develop complete axiomatizations for the axiomatizable cases amongst the 24 logics with branching time operators, and the assumptions of unique initial state, synchrony, perfect recall. (That is, we do not treat the logics making the semantic assumption of no learning.) The language is based on the temporal logic CTL*, interpreted with respect to a version of the bundle semantics.

A number of the classes of systems studied in this paper (the cases in which the synchrony assumption is made, and in which perfect recall is not assumed) have been previously axiomatized [HV89, HV88a], but for a weaker temporal language: we defer a more detailed comparison to the end of the paper. The completeness proofs we present use a somewhat different technique from that used in this previous work. We present our proof in a uniform framework that is an extension to logics of knowledge and branching time of the framework for completeness proofs for the logics of knowledge and linear time developed in [HvdMVar]. Our arguments show that this framework extends very smoothly to the branching time logics.* The classes of systems we treat with perfect recall but excluding the assumption of synchrony do not appear to have been previously considered in the branching time case. We show that an interaction axiom identified by van der Meyden [Mey94] for the logic of knowledge and linear time in such systems can be used to give a complete axiomatization in the branching time case also.

The rest of this paper is organized as follows. In the following section, we define the syntax and semantics of all the logics of knowledge and branching time that we consider. We also define the semantic assumptions of unique initial state, synchrony and perfect recall. This is followed by a section in

*While using the same framework, the completeness proofs of [HvdMVar] for the linear time cases with no learning are somewhat more complex than the cases in which this assumption is not made. We believe the general framework will still be applicable, but the branching time logics with no learning seem to require somewhat greater modification of the constructions in [HvdMVar] than those we consider in this paper.

which we state the axioms for all the axiomatizable logics corresponding to the classes of systems in which we make some subset of these assumptions. The section following this recalls from [HvdMVar] the notion of *enriched systems*, which form the basis for all our completeness proofs. Having defined this notion, we are in a position to prove soundness and completeness for the axiom systems. The final section concludes with a discussion of related work and open problems.

The Formal Model: Language and Systems

Significant portions of the material in this section are taken from [HV89, HvdMVar], and is repeated here to make this paper self-contained. The reader is encouraged to consult [HV89] for further details and motivation.

The logics we are considering are all propositional. Thus, we start out with primitive propositions p, q, ... and we close the logics under negation and conjunction, so that if φ and ψ are formulas, so are $\neg\varphi$ and $\varphi \wedge \psi$. In addition, we close off under modalities for knowledge and time, as discussed below. As usual, we view *true* as an abbreviation for $\neg(p \wedge \neg p)$, $\varphi \vee \psi$ as an abbreviation for $\neg(\neg\varphi \wedge \neg\psi)$, and $\varphi \Rightarrow \psi$ as an abbreviation for $\neg\varphi \vee \psi$.

If we have m agents we add the modalities $K_1, \ldots, K_m$. Thus, if φ is a formula, so is $K_i\varphi$ (read "agent i knows φ"). We take $L_i\varphi$ to be an abbreviation for $\neg K_i \neg\varphi$. In some cases we also want to talk about common knowledge, so we add the modalities E and C into the language; $E\varphi$ says that everyone knows φ, while $C\varphi$ says φ is common knowledge.

The temporal fragment of the language is a variant of the branching time logic CTL* [EH85]. There are two types of temporal construct: *path quantifiers* and linear time temporal modalities.

The linear time modalities are a unary operator $\bigcirc$ and a binary operator U. Thus, if φ and ψ are formulas, then so are $\bigcirc\varphi$ (read "next time φ") and $\varphi\, U\, \psi$ (read "φ until ψ"). We take $\Diamond\varphi$ to be an abbreviation for $true\, U\, \varphi$, and $\Box\varphi$ to be an abbreviation for $\neg\Diamond\neg\varphi$. Intuitively, $\bigcirc\varphi$ says that φ is true at the next point (one time unit later), $\varphi\, U\, \psi$ says that φ holds until ψ does, $\Diamond\varphi$ says that φ is eventually true (either in the present or at some point in the future), and $\Box\varphi$ says that φ is always true (in the present and at all points in the future).

The path quantifiers $\forall$ and $\exists$ act like unary operators: if φ is a formula, then so is $\forall\varphi$ (read "on all paths φ") and $\exists\varphi$ (read "on some path φ"). Intuitively, $\forall\varphi$ says that for all computation paths, or branches of the computation, that extend the computation from the current point, satisfy φ. We treat $\exists\varphi$ as an abbreviation for $\neg\forall\neg\varphi$. Intuitively, $\exists\varphi$ says that there exists a computation path, or branch of the computation, that extends the computation from the current point, and satisfies φ.

We take CKB^*_m to be the language for m agents with all the modal operators for knowledge and branching time discussed above; KB^*_m is the restricted version without the common knowledge operator. This notation indicates the constructs making up the language: C indicates the common knowledge operator, K indicates operators for knowledge, and B^* indicates that we include the branching time operators from CTL*. The subscript m indicates the number of agents in the system. For purposes of discussion, we also use the notation KL_m for the language containing the knowledge operators and the linear time temporal operators (but not the branching operator '$\forall$'); CKL_m adds the common knowledge operator to KL_m.

For purposes of discussion, we also define some fragments of these languages, in which we restrict the branching operators to occur in the patterns $\forall\bigcirc\varphi$, $\exists\bigcirc\varphi$, $\forall\varphi\, U\, \psi$ and $\exists\varphi\, U\, \psi$, where φ and ψ are formulas constructed subject to a similar restriction. The resulting temporal logic is called CTL. We write CKB_m to be the language for m agents with all the modal operators for knowledge and these restricted branching time temporal operators; KB_m is the restricted version without the common knowledge operator.

We now turn to semantics for CKB^*_m (and hence, KB^*_m, CKB_m and KB_m). Branching time temporal logics are generally given semantics over transition systems, or over infinite trees. In order to deal with the knowledge operators, we use a slightly different approach, identifying branching structure in the sets of runs typically used in the semantics for the logic of knowledge. We give our definitions first, and relate them to the usual approaches below.

A *system* for m agents consists of a set $\mathcal{R}$ of runs, where each run $r \in \mathcal{R}$ is a function from $I\!N$ to L^{m+1}, where L is some set of *local states*. There is a local state for each agent, together with a local state for the *environment*; intuitively, the environment keeps track of all the relevant features of the system not described by the agents' local states, such as messages in transit but not yet delivered. Thus, $r(n)$ has the form $\langle l_e, l_1, \ldots, l_m\rangle$, where l_e is the state of the environment, and l_i is the local state of agent i, for $i = 1, \ldots, m$; such a tuple is called a *global state*. If $i = 1 \ldots m$ is an agent, we write $r_i(n)$ for agent i's local state l_i in the global state $r(n)$, and similarly write $r_e(n)$ for the state of the environment l_e. If r is a run, the prefix of r to time n, written $r[0 \ldots n]$ is the sequence $r(0), r(1), \ldots, r(n)$. Two runs r, r' are said to be equivalent to time n, if $r[0 \ldots n] = r'[0 \ldots n]$. A *point* in a system $\mathcal{R}$ is a pair (r, n) where $r \in \mathcal{R}$ and $n \in I\!N$. Intuitively, a point denotes a particular moment of time within a particular possible evolution of the system.

An *interpretation* for a system $\mathcal{R}$ is a function π that maps every point $(r, n) \in \mathcal{R} \times I\!\!N$ to a truth assignment $\pi(r, n)$ to the primitive propositions (so that $\pi(r, n)(p) \in \{\mathit{true}, \mathit{false}\}$ for each primitive proposition p). An interpretation π is said to *depend only on global states* if for all points $(r, n), (r', n')$ in $\mathcal{R}$, if $r(n) = r'(n')$ then $\pi(r, n)(p) = \pi(r', n')(p)$ for all propositions p. Intuitively, this says that the truth value of a proposition does not depend on historical information, or on information about what will happen in the future. (This means that we could have defined π to be a function from global states (not points) to truth values.)

We assume for the remainder of the paper that all interpretations depend only on global states. Some work in the literature on reasoning about knowledge (e.g. [FHMV95]) makes this assumption, some (e.g., [HV89]) does not. In the context of the logic of knowledge and linear time, the decision does not affect the set of valid formulas: see [HvdMVar] for an explanation. However, the situation is quite different for branching time logic: the axioms B1,B2 below would not be valid if we did not assume that interpretations depend only on global states.

An *interpreted system* $\mathcal{I}$ *for* m *agents* is a tuple $(\mathcal{R}, \pi)$ where $\mathcal{R}$ is a system for m agents, and π is an interpretation for $\mathcal{R}$. Given an interpreted system $\mathcal{I} = (\mathcal{R}, \pi)$, we write $(\mathcal{I}, r, n) \models \varphi$ if the formula φ is true at (or *satisfied by*) the point (r, n) of interpreted system $\mathcal{I}$. We define $\models$ inductively for formulas of CKB^*_m (for KB^*_m we just omit the clauses involving C and E). For the part of the definition dealing with formulas of the form $K_i\varphi$, we need to introduce one new notion. If $r(n) = \langle l_e, l_1, \ldots, l_m \rangle$, $r'(n') = \langle l'_e, l'_1, \ldots, l'_m \rangle$, and $l_i = l'_i$, then we say that $r(n)$ and $r'(n')$ are *indistinguishable to agent* i and write $(r, n) \sim_i (r', n')$. Of course, $\sim_i$ is an equivalence relation on global states (inducing an equivalence relation on points). The formula $K_i\varphi$ is defined to be true at (r, n) exactly if φ is true at all the points whose associated global state is indistinguishable to i from that of (r, n). We proceed as follows:

- $(\mathcal{I}, r, n) \models p$ for a primitive proposition p iff $\pi(r, n)(p) = \mathbf{true}$
- $(\mathcal{I}, r, n) \models \varphi \wedge \psi$ iff $(\mathcal{I}, r, n) \models \varphi$ and $(\mathcal{I}, r, n) \models \psi$
- $(\mathcal{I}, r, n) \models \neg\varphi$ iff $(\mathcal{I}, r, n) \not\models \varphi$
- $(\mathcal{I}, r, n) \models K_i\varphi$ iff $(\mathcal{I}, r', n') \models \varphi$ for all (r', n') such that $(r, n) \sim_i (r', n')$
- $(\mathcal{I}, r, n) \models E\varphi$ iff $(\mathcal{I}, r', n') \models K_i\varphi$ for $i = 1, \ldots, m$

- $(\mathcal{I}, r, n) \models C\varphi$ iff $(\mathcal{I}, r', n') \models E^k\varphi$, for $k = 1, 2, \ldots$ (where $E^1\varphi = E\varphi$ and $E^{k+1}\varphi = EE^k\varphi$)
- $(\mathcal{I}, r, n) \models \bigcirc\varphi$ iff $(\mathcal{I}, r, n+1) \models \varphi$
- $(\mathcal{I}, r, n) \models \varphi U \psi$ iff there is some $n' \geq n$ such that $(\mathcal{I}, r, n') \models \psi$, and for all n'' with $n \leq n'' < n'$, we have $(\mathcal{I}, r, n'') \models \varphi$.
- $(\mathcal{I}, r, n) \models \forall\varphi$ if for all runs r' of $\mathcal{I}$ that are equivalent to r to time n, we have $(\mathcal{I}, r', n) \models \varphi$.

The semantics we have presented for the branching time operator is a version of one of the several semantics that have been considered in the literature. There are two main dimensions of variation: the set of runs considered in the definition, and the set of points on those runs. There is also a less essential difference, concerning the structure used as the fundamental basis of the semantics. We have taken this to be a set of runs, but the usual semantics begins with an (infinite) tree, and defines runs to be maximal chains in this tree. This difference is minor, since it is possible to derive a tree structure from a set of runs: the nodes of the tree are the prefixes $r[0 \ldots n]$ for runs r in the set, and the order is the natural prefix order. Conversely, a run can be constructed from each branch (maximal chain) in this tree. However, the set of runs obtained from branches in this way is not necessarily equal to the original set of runs: it may be a much larger set. Thus, our semantics for the branching operator corresponds to quantifying over a subset of the set of branches of the tree. This is known in the literature as the *bundle* approach [Bur79, Sti92]. The alternative, quantifying over all branches of the tree, we call the *complete tree* approach.

The other possible dimension of variation in the semantics of the branching quantifier '$\forall$' is the set of points considered on the runs. We will call the approach we have adopted, which uses the points identical to time n, the *narrow* approach. There is an alternative definition, which we call the *broad* approach: $(\mathcal{I}, r, n) \models \forall\varphi$ if for all runs r' (in the appropriate set, which depends on whether we are using the bundle or complete tree semantics) and times n' with $r(n) = r'(n')$, we have $(\mathcal{I}, r', n) \models \varphi$. See [Sti92] for a discussion of this approach, which is related to interpreting branching time logic in transition systems. The two dimensions of variation give four possible semantics for the branching quantifier. The definitions we have adopted above correspond to just one of these possibilities: the narrow, bundled semantics.

There is a graphical interpretation of the semantics of C which we shall find useful in the sequel. Fix an interpreted system $\mathcal{I}$. A point (r', n') in $\mathcal{I}$

is *reachable* from a point (r, n) if there exist points $(r_0, n_0), \ldots, (r_k, n_k)$ such that $(r, n) = (r_0, n_0)$, $(r', n') = (r_k, n_k)$, and for all $j = 0, \ldots, k-1$ there exists i such that $(r_j, n_j) \sim_i (r_{j+1}, n_{j+1})$.

The following result is well known (and easy to check).

LEMMA 1. [HM92] *$(\mathcal{I}, r, n) \models C\varphi$ iff $(\mathcal{I}, r', n') \models \varphi$ for all points (r', n') reachable from (r, n).*

As usual, we define a formula φ to be *valid with respect to a class $\mathcal{C}$ of interpreted systems* iff $(\mathcal{I}, r, n) \models \varphi$ for all interpreted systems $\mathcal{I} \in \mathcal{C}$ and points (r, n) in $\mathcal{I}$. A formula φ is *satisfiable with respect to $\mathcal{C}$* iff for some $\mathcal{I} \in \mathcal{C}$ and some point (r, n) in $\mathcal{I}$, we have $(\mathcal{I}, r, n) \models \varphi$.

We now turn our attention to formally defining the classes of interpreted systems of interest.

Perfect recall means, intuitively, that an agent's local state encodes everything that has happened (for that agent's point of view) thus far in the run. To make this precise, we need to define "what has happened so far from the agent's point of view". Let *agent i's local-state sequence at the point (r, n)* be the sequence $l_0, \ldots, l_k$ of states that agent i takes on in run r up to and including time n, with consecutive repetitions omitted. For example, if from time 0 through 4 in run r agent i goes through the sequence l, l, l', l, l of states, its history at $(r, 4)$ is just l, l', l. Agent i's local-state sequence at a point (r, m) essentially describes what has happened in the run up to time m, from i's point of view. Omitting consecutive repetitions from the local-state is intended to model asynchrony; "stuttering" is ignored.

Roughly speaking, agent i has perfect recall if i's current state encodes its history, i.e., i's whole local-state sequence. More formally, we say that *agent i has perfect recall* (alternatively, *agent i does not forget*) in system $\mathcal{R}$ if at all points (r, n) and (r', n') in $\mathcal{R}$, if $(r, n) \sim_i (r', n')$, then r has the same local-state sequence at both (r, n) and (r', n').

There are a number of equivalent characterizations of perfect recall. The following characterization proves useful for the purposes of our completeness proofs. Let $S = (s_0, s_1, s_2, \ldots)$ and $T = (t_0, t_1, t_2, \ldots)$ be two (finite or infinite) sequences and let $\sim$ be a relation on the elements of S and T. Then we say that S and T are $\sim$-concordant if S consists of a sequence of nonempty consecutive intervals $S_1, \ldots$ and T consists of a sequence of nonempty consecutive intervals $T_1, \ldots$ of the same length (possibly infinite) such that for all $s \in S_j$ and $t \in T_j$, we have $s \sim t$, for all indices j.

LEMMA 2. [HV86, Mey94] *The following are equivalent.*

(a) Agent i has perfect recall in system $\mathcal{R}$.

(b) For all points $(r,n) \sim_i (r',n')$ in $\mathcal{R}$, $((r,0),\ldots,(r,n))$ is $\sim_i$-concordant with $((r',0),\ldots,(r',n'))$.

(c) For all points $(r,n) \sim_i (r',n')$ in $\mathcal{R}$, if $n > 0$, then either $(r,n-1) \sim_i (r',n')$ or there exists a number $l < n'$ such that $(r,n-1) \sim_i (r',l)$ and for all k with $l < k \leq n'$ we have $(r,n) \sim_i (r',k)$.

(d) For all points $(r,n) \sim_i (r',n')$ in $\mathcal{R}$, if $k \leq n$, then there exists $k' \leq n'$ such that $(r,k) \sim_i (r',k')$.

Proof: The implications from (a) to (b), from (b) to (c) and from (c) to (d) are straightforward. The implication from (d) to (a) can be proved by a straightforward induction on $n + n'$. ∎

This lemma shows that perfect recall requires an unbounded number of local states in general, since agent i may have an infinite number of distinct histories in a given system. We remark that the official definition of perfect recall given here is taken from [FHMV95]. In [HV86], part (d) of Lemma 2 was taken as the definition of perfect recall (which was called no forgetting in that paper).

In a *synchronous* system, we assume that every agent has access to a global clock that ticks at every instant of time, and the clock reading is part of its state. Thus, in a synchronous system, each agent always "knows" the time. More formally, we say that a system $\mathcal{R}$ is *synchronous* if for all agents i and all points (r,n) and (r',n'), if $(r,n) \sim_i (r',n')$, then $n = n'$. Observe that in a synchronous system where i has perfect recall, an easy induction on n shows that if $(r,n) \sim_i (r',n)$ and $n > 0$, then $(r,n-1) \sim_i (r',n-1)$.

Finally, we say that a system $\mathcal{R}$ has a *unique initial state* if for all runs $r, r' \in \mathcal{R}$, we have $r(0) = r'(0)$. Thus, if $\mathcal{R}$ is a system with a unique initial state, then we have $(r,0) \sim_i (r',0)$ for all runs r, r' in $\mathcal{R}$ and all agents i.

We say that $\mathcal{I} = (\mathcal{R}, \pi)$ is an interpreted system where agents have perfect recall (resp., are synchronous, there is a unique initial state) exactly if $\mathcal{R}$ is a system with that property. We use $\mathcal{C}_m$ to denote the class of all interpreted systems for m agents, and add the superscripts *pr*, *sync*, and *uis* to denote particular subclasses of $\mathcal{C}_m$. Thus, for example, we use $\mathcal{C}_m^{pr,sync}$ to denote the set of all interpreted systems with m agents that have perfect recall and are synchronous. We omit the subscript m when it is clear from context.

The results of [HV89, HV88a] (some of which are based on earlier results of Ladner and Reif [LR86]) concerning the complexity of logics of knowledge and branching time in systems, subject to some subset of the three assumptions defined above, are summarized in Table 1. Each entry states a complexity class for which the corresponding problem is complete. The languages considered in these results are the restricted languages KB_m and CKB_m. For $\varphi \in KB_m^*$ (or in its subset KB_m), we define $ad(\varphi)$ to be the greatest number of alternations of distinct K_i's along any branch in φ's parse tree. For example, $ad(K_1 \neg K_2 K_1 p) = 3$; temporal operators are not considered, so that $ad(K_1 \forall \bigcirc K_1 p) = 1$. (In Table 1, we do not consider the language CKB_1. This is because if $m = 1$, then $C\varphi$ is equivalent to $K_1\varphi$. Thus, CKB_1 is equivalent to KB_1.) Since our main concern is with axiomatizability rather than complexity, we omit the definitions of complexity classes such as Π_1^1 and nonelementary time $ex(ad(\varphi) + 1, c|\varphi|)$ here. (Note that c is a constant in the latter expression.)

The complexity of validity for the languages KB_m^* and CKB_m^* have not been investigated.[†] However, is clear that for the cases where the complexity is Π_1^1, the validity problem is too hard, and there can be no recursive axiomatization, for the language CKB_m with restricted branching time operators, and *a fortiori*, for the language CKB_m^* with the unrestricted operators. We provide complete axiomatizations here for all the remaining cases, for the corresponding languages KB_m^* and CKB_m^* with the *unrestricted* branching time operators.

	CKB_m, $m \geq 2$	KB_m, $m \geq 2$	KB_1
$\mathcal{C}_m^{pr}, \mathcal{C}_m^{pr,sync}$, $\mathcal{C}_m^{pr,uis}, \mathcal{C}_m^{pr,sync,uis}$	Π_1^1	nonelementary time $ex(ad(\varphi) + 1, c\lvert\varphi\rvert)$	double-exponential time
$\mathcal{C}_m, \mathcal{C}_m^{sync}$, $\mathcal{C}_m^{sync,uis}, \mathcal{C}_m^{uis}$	EXPTIME	EXPTIME	EXPTIME

Table 1. The complexity of the validity problem for logics of knowledge and branching time

Axiom Systems

In this section, we describe the axioms and inference rules that we need for reasoning about knowledge and time for various classes of systems, and state the completeness results. The proofs of these results are deferred to the following sections. As usual, we write $\vdash \varphi$ if the formula φ can be

[†]It is believed the complexities may increase by an exponential [HV86].

derived from the axioms by application of the rules of inference. This relation depends on the particular axioms and rules under consideration, but as these will be clear from the context, we generally leave this implicit.

For reasoning about knowledge alone, the following system, with axioms K1–K5 and rules of inference R1–R2, is well known to be sound and complete [FHMV95, Hin62]:

K1. All tautologies of propositional logic

K2. $K_i\varphi \wedge K_i(\varphi \Rightarrow \psi) \Rightarrow K_i\psi$, $i = 1, \ldots, m$

K3. $K_i\varphi \Rightarrow \varphi$, $i = 1, \ldots, n$

K4. $K_i\varphi \Rightarrow K_iK_i\varphi$, $i = 1, \ldots, m$

K5. $\neg K_i\varphi \Rightarrow K_i\neg K_i\varphi$, $i = 1, \ldots, m$

R1. From φ and $\varphi \Rightarrow \psi$ infer ψ

R2. From φ infer $K_i\varphi$, $i = 1, \ldots, m$

This axiom system is known as $\mathrm{S5}_m$. (It is well known that axiom K4 can be derived from the others.)

For reasoning about the linear time temporal operators individually, the following system (together with K1 and R1), is well known to be sound and complete [FHMV95, GPSS80]:

T1. $\bigcirc(\varphi) \wedge \bigcirc(\varphi \Rightarrow \psi) \Rightarrow \bigcirc\psi$

T2. $\bigcirc(\neg\varphi) \Leftrightarrow \neg\bigcirc\varphi$

T3. $\varphi\, U\, \psi \Leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi\, U\, \psi))$

RT1. From φ infer $\bigcirc\varphi$

RT2. From $\varphi' \Rightarrow \neg\psi \wedge \bigcirc\varphi'$ infer $\varphi' \Rightarrow \neg(\varphi\, U\, \psi)$

Following [HvdMVar], we call the system for knowledge and linear time $\mathrm{S5}_m^U$.

The following axioms B1-B6 and inference rule RB, together for the axioms for the linear time temporal operators, are known to provide a sound and complete axiomatization for the logic of branching time with the broad semantics for the branching operators, and with respect to the bundle semantics [Sti92]:

B1. $p \Rightarrow \forall p$, where p is atomic

B2. $\exists p \Rightarrow p$, where p is atomic

B3. $\forall\varphi \Rightarrow \varphi$

B4. $\forall(\varphi \Rightarrow \psi) \Rightarrow (\forall\varphi \Rightarrow \forall\psi)$

B5. $\forall\varphi \Rightarrow \forall\forall\varphi$

B6. $\exists\varphi \Rightarrow \forall\exists\varphi$

RB. From φ infer $\forall\varphi$.

We note that substitution instances of axioms B1 and B2 are not axioms; indeed, the instance $\bigcirc\varphi \Rightarrow \forall\bigcirc\varphi$ of B1 is not valid. The validity of B1 and B2 depends on the assumption that the interpretation depends only on global states. There is some redundancy in the above axioms. Note that B3-B6 and RB are just the axioms and rule of S5, so B5 can be eliminated. Moreover, B2 follows from B1. (To see this, note that we have $\vdash \forall\varphi \Leftrightarrow \forall\neg\neg\varphi$ by propositional logic, RB and B4. Thus, we obtain $\vdash \exists\neg p \Rightarrow \neg p$ from B1. Applying RB and B4, we get $\vdash \forall\exists\neg p \Rightarrow \forall\neg p$. Now by B3 and B6 we have $\vdash \neg p \Rightarrow \forall\exists\neg p$. It follows that $\vdash \neg p \Rightarrow \forall\neg p$, from which we get B2 by contraposition.)

The following axiom concerns an interaction between the branching operator and the next time operator. It is valid with respect to the narrow semantics we have adopted for the branching operators, and is also valid with respect to the broad semantics subject to the condition that the set of runs is *fusion closed*.[‡]

FC. $\forall\bigcirc\varphi \Rightarrow \bigcirc\forall\varphi$

The following axiom concerns an interaction between knowledge and branching time

KB. $K_i\varphi \Rightarrow \forall K_i\varphi$

The logic containing the above axioms and inference rules for both knowledge and time is called $S5_m^B$. The axiomatization $S5_m^B$ is easily seen to be

[‡] $\mathcal{R}$ is defined to be fusion closed if for all runs $r, r' \in \mathcal{R}$ and natural numbers n, m, if $r(n) = r'(m)$, then the function r'' defined by $r''(k) = r(k)$ for $k \leq n$ and $r''(k) = r'(k)$ for $k > n$ is a run in $\mathcal{R}$.

sound for $\mathcal{C}_m$, the class of all systems for m agents. The logic also turns out to be complete for this class of systems. Moreover, $\mathrm{S5}^B_m$ is also complete even if we impose the requirements of synchrony or uis. This indicates that our language is not expressive enough to capture these conditions.

THEOREM 1. $\mathrm{S5}^B_m$ *is a sound and complete axiomatization for the language* KB^*_m *with respect to* $\mathcal{C}_m$, $\mathcal{C}^{sync}_m$, $\mathcal{C}^{uis}_m$, *and* $\mathcal{C}^{sync,uis}_m$.

We defer the proof of all results in this section to the following sections. Theorem 1 mirrors a result for the linear time case, where $\mathrm{S5}^U_m$ is known [HvdMVar] to be sound and complete for the language KL_m with respect to $\mathcal{C}_m$, $\mathcal{C}^{sync}_m$, $\mathcal{C}^{uis}_m$, and $\mathcal{C}^{sync,uis}_m$.

It is well known that the following two axioms and inference rule characterize common knowledge [FHMV95, HM92]:

C1. $E\varphi \Leftrightarrow \bigwedge_{i=1}^m K_i\varphi$

C2. $C\varphi \Rightarrow E(\varphi \wedge C\varphi)$

RC1. From $\varphi \Rightarrow E(\psi \wedge \varphi)$ infer $\varphi \Rightarrow C\psi$

Let $\mathrm{S5C}^B_m$ be the result of adding C1, C2, and RC1 to $\mathrm{S5}^B_m$. We then have the following extension of Theorem 1.

THEOREM 2. $\mathrm{S5C}^B_m$ *is a sound and complete axiomatization for the language* CKB^*_m *with respect to* $\mathcal{C}_m$, $\mathcal{C}^{sync}_m$, $\mathcal{C}^{uis}_m$, *and* $\mathcal{C}^{sync,uis}_m$.

Thus, adding the common knowledge operator does not help to distinguish the classes $\mathcal{C}_m$, $\mathcal{C}^{sync}_m$, $\mathcal{C}^{uis}_m$, and $\mathcal{C}^{sync,uis}_m$. (This again mirrors the situation in the linear time case.) However, in the presence of perfect recall, we do get an ability to distinguish systems with synchrony from systems without. The following axioms are discussed in [HvdMVar].

KT1. $K_i\Box\varphi \Rightarrow \Box K_i\varphi$, $i = 1, \ldots, m$

KT2. $K_i\bigcirc\varphi \Rightarrow \bigcirc K_i\varphi$, $i = 1, \ldots, m$.

KT3. $K_i\varphi_1 \wedge \bigcirc(K_i\varphi_2 \wedge \neg K_i\varphi_3) \Rightarrow L_i((K_i\varphi_1)\, U\, [(K_i\varphi_2)\, U \neg\varphi_3])$, $i = 1, \ldots, m$

Axiom KT1 was first discussed by Ladner and Reif [LR86]. It was for some time believed that $\mathrm{S5}^U_m$+KT1 would be complete for the language KL_m with respect to $\mathcal{C}^{pr}_m$, but this was shown to be false in [Mey94], where KT3 was introduced and where it was shown $S5_m^U + KT3$ is sound and complete for $\mathcal{C}_m^{pr}$. For intuition concerning KT3, a proof of its soundness in systems with perfect recall, and a proof that KT1 is derivable in $S5_m^U + KT3$, we refer the reader to [Mey94, HvdMVar].

Just as in the linear time case, KT3 turns out to be strong enough to give us completeness, with or without the condition uis.

THEOREM 3. $S5_m^B + KT3$ *is a sound and complete axiomatization for the language KB_m^* with respect to $\mathcal{C}_m^{pr}$ and $\mathcal{C}_m^{pr,uis}$.*

Theorem 1 shows that requiring synchrony or uis does not have an impact when we consider the class of all systems—$\mathcal{C}_m$, $\mathcal{C}_m^{sync}$, $\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$ are all axiomatized by $S5_m^B$—and Theorem 3 shows that adding uis does not have an impact in the presence of perfect recall. However, requiring synchrony does have an impact in the presence of perfect recall. It is easy to see that KT2 is valid in $\mathcal{C}_m^{pr,sync}$, and it clearly is not valid in $\mathcal{C}_m^{pr}$. Moreover, just as in the linear time case [HvdMVar], KT2 suffices for completeness in $\mathcal{C}_m^{pr,sync}$; we do not need the complications of KT3.

THEOREM 4. $S5_m^B + KT2$ *is a sound and complete axiomatization for the language KB_m^* with respect to $\mathcal{C}_m^{pr,sync}$ and $\mathcal{C}_m^{pr,sync,uis}$.*

A Framework for Completeness Proofs

In this section we describe a general framework for completeness proofs that reduces the work required in each of the different completeness results to a single lemma. The framework is a generalization of one developed in [HvdMVar] for completeness proofs for logics of knowledge and linear time, and significant portions of this section are drawn from this work. The principal changes concern the branching operator '$\forall$'. Where the proofs of certain lemmas concerning this framework are unchanged, we omit the details, and refer the reader to [HvdMVar].

With respect to the temporal dimension, our constructions resemble those previously used for completeness of dynamic logic [KP81] and temporal logics, in that we construct a model for a consistent formula ψ out of consistent subsets of a finite set of formulas, called the *closure* of ψ. However, in order to deal with the knowledge modalities, we need a number of distinct levels of closure, having a tree-like structure. At the leaves of this tree-like structure, the closure is like the usual closure for temporal logic. As we move towards the root, we add formulas to the closure that increase the level of nesting of the knowledge modalities.

A formula ψ is said to be *consistent* in a logic L if it is not the case that $\vdash_L \neg\psi$. For each of the pairs of logic L and class $\mathcal{C}$ of systems we consider, the proof that L is complete with respect to $\mathcal{C}$ proceeds by constructing, for every formula ψ consistent with respect to L, a system in $\mathcal{C}$ containing a point at which ψ is true. All the results in this section hold for every logic containing $\mathrm{S5}^U_m$, except for Lemma 9, which mentions common knowledge. This lemma holds for every logic containing $\mathrm{S5C}^U_m$. Rather than mentioning the logic L explicitly in each case, we just write $\vdash$ rather than $\vdash_L$; the intended logic(s) will be clear from context. For the remainder of this section and the next, we also fix the formula ψ, which is assumed to be consistent with respect to L.

A finite sequence $\sigma = i_1 i_2 \ldots i_k$ of agents, possibly equal to the null sequence ϵ, is called an *index* if $i_l \neq i_{l+1}$ for all $l < k$. We write $|\sigma|$ for the length k of such a sequence; the null sequence has length equal to 0.

If S is a set, and S^* is the set of all finite sequences over S, we define the absorptive concatenation function $\#$ from $S^* \times S$ to S^* as follows. Given a sequence σ in S^* and an element x of S, we take $\sigma\#x = \sigma$ if the final element of σ is x. If the final element of σ is not equal to x then we take $\sigma\#x$ to be σx, i.e. the result of concatenating x to σ. We write simply $x_1\#x_2\#x_3 \ldots \#x_n$ for $(\ldots((x_1\#x_2)\#x_3)\ldots)\#x_n$. We shall have two distinct uses for this function, applying it primarily to sequences of agents, and sometimes to sequences of "instantaneous states" of agents in the context of asynchronous systems.

If $\psi \in CKB^*_m$, for each $k \geq 0$, we define the k-closure $cl_k(\psi)$, and for each agent i, we define the k,i-closure $cl_{k,i}(\psi)$. The definitions are similar to those used for the linear time case, but we add some formulas for the branching operator to the lowest level closure $cl_0(\psi)$. First, we let the *basic closure* $cl(\psi)$ be the smallest set containing ψ that is closed under subformulas, contains $\neg\varphi$ if it contains φ and φ is not of the form $\neg\varphi'$, contains $EC\varphi$ if it contains $C\varphi$, and contains $K_1\varphi, \ldots, K_m\varphi$ if it contains $E\varphi$. (The last two clauses do not apply if ψ is in KB^*_m, and thus does not mention common knowledge.) Now define the 0-closure $cl_0(\psi)$ to be the union of $cl(\psi)$ with the set of formulas $\forall(\varphi_1 \vee \ldots \vee \varphi_n)$ or $\neg\forall(\varphi_1 \vee \ldots \vee \varphi_n)$, where the φ_l are distinct formulas in $cl(\psi)$. If i is an agent, we take $cl_{k,i}(\psi)$ to be the union of $cl_k(\psi)$ with the set of formulas of the form $K_i(\varphi_1 \vee \ldots \vee \varphi_n)$ or $\neg K_i(\varphi_1 \vee \ldots \vee \varphi_n)$, where the φ_l are distinct formulas in $cl_k(\psi)$. Finally, $cl_{k+1}(\psi)$ is defined to be $\cup_{i=1}^m cl_{k,i}(\psi)$.

If X is a finite set of formulas we write φ_X for the conjunction of the formulas in X. A finite set X of formulas is said to be consistent if φ_X is consistent. If X is a finite set of formulas and φ is a formula we write $X \Vdash \varphi$

when $\vdash \varphi_X \Rightarrow \varphi$.[§] Clearly if $X \Vdash \varphi_1$ and $\vdash \varphi_1 \Rightarrow \varphi_2$ then $X \Vdash \varphi_2$.

Suppose Cl is a finite set of formulas with the property that for all $\varphi \in Cl$, either $\neg\varphi \in Cl$ or φ is of the form $\neg\varphi'$ and $\varphi' \in Cl$. (Note that the sets $cl_k(\psi)$ and $cl_{k,i}(\psi)$ have this property.) We define an *atom* of Cl to be a maximal consistent subset of Cl. Evidently, if X is an atom of Cl and $\varphi \in Cl$, then either $X \Vdash \varphi$ or $X \Vdash \neg\varphi$. Using only propositional reasoning (K1 and R1), $\bigvee_{X \text{ an atom of } Cl} \varphi_X$ is easily provably equivalent to *true*. Thus, we have that

Lemma 3. $\vdash \bigvee_{X \text{ an atom of } Cl} \varphi_X$.

We begin the construction of the model of ψ by first constructing a *pre-model,* which is a structure $\langle S, \rightarrow, \approx_1, \ldots, \approx_n \rangle$ consisting of a set S of states, a binary relation $\rightarrow$ on S, and for each agent i an equivalence relation $\approx_i$ on S. Recall that for a formula $\varphi \in KB^*_m$, the alternation depth $ad(\varphi)$ is the number of alternations of distinct operators K_i in φ. Let $d = ad(\psi)$ if $\psi \in KB^*_m$; otherwise (that is, if ψ mentions the modal operator C), let $d = 0$.

The set S consists of all the pairs (σ, X) such that σ is an index, $|\sigma| \leq d$, and

1. if $\sigma = \epsilon$ then X is an atom of $cl_d(\psi)$, and

2. if $\sigma = \tau i$ then X is an atom of $cl_{k,i}(\psi)$, where $k = d - |\sigma|$.

We may view the set S of states as the union of a collection of sets $S_\sigma = \{(\sigma, X) \mid (\sigma, X) \in S\}$, with the indices σ providing a tree-like structure on this collection. Note that as σ approaches the root, the size of the closure from which the atoms X are drawn increases.

The relation $\rightarrow$ is defined so that $(\sigma, X) \rightarrow (\tau, Y)$ iff $\tau = \sigma$ and the formula $\varphi_X \wedge \bigcirc\varphi_Y$ is consistent. If X is an atom we write X/K_i for the set of formulas φ such that $K_i\varphi \in X$. We say that states (σ, X) and (τ, Y) are *i-adjacent* if $\sigma\#i = \tau\#i$. The relation $\approx_i$ is defined so that $(\sigma, X) \approx_i (\tau, Y)$ iff σ and τ are i-adjacent and $X/K_i = Y/K_i$. Clearly, i-adjacency is an equivalence relation, as is the relation $\approx_i$.

A *σ-state (for ψ)* is a pair (σ, X) as above. A state (for ψ) is a σ-state for some index σ with $|\sigma| \leq d$. Thus (σ, X) is the unique σ-state with atom X. If $s = (\sigma, X)$ is a state, we define φ_s to be the formula φ_X, and write

[§]Note that $X \Vdash \varphi$ is not equivalent to $X \vdash \varphi$ (under perhaps the most natural definition of $\vdash$ with sets of formulas on the left-hand side) because of generalization rules like R2 and RT1. For example, although $\varphi \vdash K_i\varphi$, it is not the case that $\vdash \varphi \Rightarrow K_i\varphi$.

$s \Vdash \varphi$ for $\vdash \varphi_s \Rightarrow \varphi$. We say that the state s *directly decides* a formula φ if either (a) $\varphi \in X$ or (b) $\neg\varphi \in X$ or (c) $\varphi = \neg\varphi'$ and $\varphi' \in X$. We say that s *decides* φ if either $s \Vdash \varphi$ or $s \Vdash \neg\varphi$. Evidently, if s directly decides φ then s decides φ. Note that if $\sigma = \tau i$ then each σ-state directly decides every formula in $cl_{d-|\sigma|,i}(\psi)$. Also, every ϵ-state directly decides every formula in $cl_d(\sigma)$.

LEMMA 4. *[HvdMVar] If s and t are i-adjacent states, then the same formulas of the form $K_i\varphi$ are directly decided by s and t.*

If s is a σ-state, we take $\Phi_{s,i}$ to be the disjunction of the formulas φ_t, where t ranges over the σ-states satisfying $s \approx_i t$, and we take $\Phi^+_{s,i}$ to be the disjunction of the formulas φ_t, where t ranges over the $(\sigma\#i)$-states satisfying $s \approx_i t$.¶ Observe that because $\approx_i$ is an equivalence relation we have that if $s \approx_i t$ then $\Phi_{s,i} = \Phi_{t,i}$ and $\Phi^+_{s,i} = \Phi^+_{t,i}$. The following result lists a number of knowledge formulas decided by states.

LEMMA 5. *[HvdMVar]*

(a) If s is a σ-state and t is a σ-state or $(\sigma\#i)$-state such that $s \not\approx_i t$, then $s \Vdash K_i\neg\varphi_t$.

(b) For all σ-states s, we have $s \Vdash K_i\Phi_{s,i}$; in addition, if $|\sigma\#i| \leq d$, then $s \Vdash K_i\Phi^+_{s,i}$.

(c) For all σ-states s and $(\sigma\#i)$-states t with $s \approx_i t$, we have $s \Vdash L_i\varphi_t$.

(d) If s is a σ-state and t is a $(\sigma\#i)$-state such that $s \not\approx_i t$, then $t \Vdash \neg K_i\Phi^+_{s,i}$.

If T is a set of states, then we write φ_T for the disjunction of the formulas φ_t for t in T. Using RT1, T1, and T2, the following result is immediate from the fact that $s \not\to t$ implies $\vdash \varphi_s \Rightarrow \neg\bigcirc\varphi_t$, together with the fact that $\vdash \bigvee_{s \text{ a } \sigma-\text{state}} \varphi_s$, which follows from Lemma 3.

LEMMA 6. *Let s be a state and let T be the set of states t such that $s \to t$. Then $s \Vdash \bigcirc\varphi_T$*

The next result provides a useful way to derive formulas containing the until operator.

LEMMA 7. *For all formulas α, β and γ, if $\vdash \alpha \Rightarrow \neg\gamma$ and $\vdash \alpha \Rightarrow \bigcirc(\alpha \vee (\neg\beta \wedge \neg\gamma))$ then $\vdash \alpha \Rightarrow \neg(\beta\, U\, \gamma)$.*

¶It can be shown that if $|\sigma\#i| \leq d$, then $\Phi_{s,i}$ is logically equivalent to $\Phi^+_{s,i}$, but we do not need this fact here.

The following shows that the pre-model has properties resembling those for the truth definitions for formulas in the basic closure. Note that every state directly decides all formulas in the basic closure. Define a $\rightarrow$-*sequence* of states to be a (finite or infinite) sequence $s_1, s_2, \ldots$ such that $s_1 \rightarrow s_2 \rightarrow \ldots$.

LEMMA 8. *[HvdMVar] For all σ-states s, we have*

(a) if $\bigcirc\varphi \in cl_0(\psi)$, then for all states t such that $s \rightarrow t$, we have $s \Vdash \bigcirc\varphi$ iff $t \Vdash \varphi$,

(b) If $K_i\varphi \in cl_0(\psi)$, then $s \Vdash \neg K_i\varphi$ iff there is some σ-state t such that $s \approx_i t$ and $t \Vdash \neg\varphi$. Moreover, if $|\sigma\#i| \leq d$, then $s \Vdash \neg K_i\varphi$ iff there is some $(\sigma\#i)$-state t such that $s \approx_i t$ and $t \Vdash \neg\varphi$.

(c) if $\varphi_1 U \varphi_2 \in cl_0(\psi)$ then $s \Vdash \varphi_1 U \varphi_2$ iff there exists a $\rightarrow$-sequence $s = s_0 \rightarrow s_1 \rightarrow \ldots \rightarrow s_n$, where $n \geq 0$, such that $s_n \Vdash \varphi_2$, and $s_k \Vdash \varphi_1$ for all $k < n$.

For the next result, recall that when the formula ψ contains the common knowledge operator we take $d = 0$, so that all states are ϵ-states.

LEMMA 9. *[HvdMVar] If $C\varphi \in cl_0(\psi)$, then $s \Vdash \neg C\varphi$ iff there is a state t reachable from s through the relations $\approx_i$ such that $t \Vdash \neg\varphi$.*

We say that an infinite $\rightarrow$-sequence of states $(s_0, s_1, \ldots)$, where $s_n = (\sigma, X_n)$ for all n, is *acceptable* if for all $n \geq 0$, if $\varphi_1 U \varphi_2 \in X_n$ then there exists an $m \geq n$ such that $s_m \Vdash \varphi_2$ and $s_k \Vdash \varphi_1$ for all k with $n \leq k < m$.

We can prove a similar set of results for the branching operator $\forall$. First, define a *state formula* to be a formula φ such that $\vdash \varphi \Rightarrow \forall\varphi$.[‖] We write X_{ST} for the set of state formulas in a set X. Define the relation $\approx_\forall$ on states by $(\sigma, X) \approx_\forall (\sigma', X')$ if $\sigma = \sigma'$ and $X_{ST} = X'_{ST}$.

LEMMA 10. *If $\forall\varphi \in cl_0(\psi)$ and $s \Vdash \forall\varphi$, then for all states t such that $s \approx_\forall t$, we have $t \Vdash \varphi$*

Proof: If $s = (\sigma, X) \approx_\forall (\sigma', X') = t$ and $\forall\varphi \in cl_0(\psi) \cap X$, then $\forall\varphi \in X'$, hence $\varphi \in X'$ by B3. It follows that $t \Vdash \varphi$. ∎

[‖]Stirling [Sti92] defines φ to be a state formula if $\vdash \varphi \Rightarrow \forall\varphi$ or $\vdash \exists\varphi \Rightarrow \varphi$. This is equivalent to our definition, since if $\vdash \exists\varphi \Rightarrow \varphi$ then $\vdash \varphi \Rightarrow \forall\varphi$. To see this, assume $\vdash \exists\varphi \Rightarrow \varphi$. Then $\vdash \forall\exists\varphi \Rightarrow \forall\varphi$ by RB and B4. Thus, using B6, we get $\vdash \exists\varphi \Rightarrow \forall\varphi$. But, by B3 we have $\vdash \varphi \Rightarrow \exists\varphi$, so we conclude $\vdash \varphi \Rightarrow \forall\varphi$.

LEMMA 11. *If* $\forall\varphi \in cl_0(\psi)$ *and* $s \Vdash \neg\forall\varphi$*, then there exists a state* t *such that* $s \approx_\forall t$ *and* $t \Vdash \neg\varphi$.

Proof: Let $s = (\sigma, X)$ and suppose $\forall\varphi \in cl_0(\psi)$ and $s \Vdash \neg\forall\varphi$. We show that $X_{ST} \cup \{\neg\varphi\}$ is consistent. For, if not, then $\vdash \varphi_{X_{ST}} \Rightarrow \varphi$. Noting that by B4 and RB we have $\vdash \forall\varphi_1 \wedge \forall\varphi_2 \Leftrightarrow \forall(\varphi_1 \wedge \varphi_2)$ for all formulas φ_1, φ_2, we see that $\varphi_{X_{ST}}$ is a state formula. Because $\vdash \varphi_X \Rightarrow \varphi_{X_{ST}}$, we get that $\vdash \varphi_X \Rightarrow \forall\varphi_{X_{ST}}$, and hence that $\vdash \varphi_X \Rightarrow \forall\varphi$ by B4. But this contradicts the assumption that $s \Vdash \neg\forall\varphi$. Thus $X_{ST} \cup \{\neg\varphi\}$ is consistent, and there exists a maximal consistent subset X' of $cl_d(\psi)$ containing this set. The state $t = (\sigma, X')$ clearly satisfies $t \Vdash \neg\varphi$. Moreover, $s \approx_\forall t$. This is because X' contains X_{ST}, which in turn contains either φ or $\neg\varphi$ for every state formula φ in $cl_d(\psi)$. It follows from this that $X'_{ST} = X_{ST}$. ∎

Next, we generalize the definition of *enriched system* of [HvdMVar] by adding two clauses (clauses 5 and 6) that deal with the branching time operator.

DEFINITION 1. An *enriched system* for ψ is a pair $(\mathcal{R}, \Sigma)$, where $\mathcal{R}$ is a set of runs and Σ is a *partial* function mapping points in $\mathcal{R} \times I\!N$ to states for ψ such that the following hold, for all runs $r \in \mathcal{R}$:

1. If $\Sigma(r, n)$ is defined then $\Sigma(r, n')$ is defined for all $n' > n$, and the sequence $\Sigma(r, n), \Sigma(r, n+1), \ldots$ is an acceptable $\rightarrow$-sequence.
2. For all points $(r, n) \sim_i (r', n')$, if $\Sigma(r, n)$ is defined then $\Sigma(r', n')$ is defined and $\Sigma(r, n) \approx_i \Sigma(r', n')$.
3. If $\Sigma(r, n)$ and s are σ-states such that $\Sigma(r, n) \approx_i s$, then there exists a point (r', n') such that $(r, n) \sim_i (r', n')$ and $\Sigma(r', n') = s$.
4. if $C\varphi \in cl_0(\psi)$ and $\Sigma(r, n) \Vdash \neg C\varphi$, then there exists a point (r', n') reachable from (r, n) such that $\Sigma(r', n') \Vdash \neg\varphi$.
5. if $\Sigma(r, n)$ is defined then for all runs r' such that $r'[0 \ldots n] = r[0 \ldots n]$, we have that $\Sigma(r', n)$ is defined and $\Sigma(r, n) \approx_\forall \Sigma(r', n)$.
6. if $\Sigma(r, n)$ is defined and s is a state such that $\Sigma(r, n) \approx_\forall s$ then there exists a run r' such that $r'[0 \ldots n] = r[0 \ldots n]$, and $\Sigma(r', n) = s$.

An *enriched*$^+$ *system* for ψ is a pair $(\mathcal{R}, \Sigma)$ satisfying conditions 1, 2, 5, 6 and the following modification of 3:

3′. If $\Sigma(r,n)$ is a σ-state and s is a $(\sigma\#i)$-state such that $\Sigma(r,n) \approx_i s$, then there exists a point (r',n') such that $(r,n) \sim_i (r',n')$ and $\Sigma(r',n') = s$.

∎

Intuitively, in an enriched (enriched$^+$) system, the points where Σ is defined are the points that are "relevant" to the truth of certain formulas at certain points.

Given an enriched (resp., enriched$^+$) system $(\mathcal{R},\Sigma)$, we define an interpreted system $\mathcal{I} = (\mathcal{R}',\pi)$.[**] Given a run r, define the run r^* by $r^*_e(n) = r[0\ldots n]$ and, for each agent i, $r^*_i(n) = r_i(n)$. The set of runs $\mathcal{R}'$ is defined to be the set of runs r^* for $r \in \mathcal{R}$. We define the interpretation π for $\mathcal{R}'$ on basic propositions p by $\pi(r^*,n)(p) = \mathbf{true}$ just when $\Sigma(r,n)$ is defined and $\Sigma(r,n) \Vdash p$.[††] Note that if $r^* = (r')^*$ then $r = r'$, so the runs of $\mathcal{R}'$ and $\mathcal{R}$ are in one-to-one correspondence, and the interpretation π is well-defined. Moreover, the interpretation depends only on global states. For, if $r^*(n) = (r')^*(n')$, then $n = n'$ and $r[0\ldots n] = r'[0\ldots n]$. Thus, by condition 5 of Definition 1, $\Sigma(r,n)$ is defined iff $\Sigma(r',n)$ is defined, and if defined, we have $\Sigma(r,n) \approx_\forall \Sigma(r',n)$. Since basic propositions are state formulas, it follows that $\pi(r,n)(p) = \pi(r',n')(p)$.

The following theorem gives a sufficient condition for a formula in the basic closure to hold at a point in the interpreted system $(\mathcal{R}',\pi)$. If σ is the index $i_1\ldots i_k$, let $K_\sigma\varphi$ be an abbreviation for $K_{i_1}\ldots K_{i_k}\varphi$. (If $\sigma = \epsilon$, then we take $K_\sigma\varphi$ to be φ.)

THEOREM 5. *(a) If $(\mathcal{R},\Sigma)$ is an enriched system for $\psi \in CKB^*_m$, $\mathcal{I}$ is the associated interpreted system, φ is in the basic closure $cl_0(\psi)$, and $\Sigma(r,n)$ is defined, then $(\mathcal{I},r^*,n) \models \varphi$ if and only if $\Sigma(r,n) \Vdash \varphi$.*

(b) If $(\mathcal{R},\Sigma)$ is an enriched$^+$ system for $\psi \in KB^*_m$, $\mathcal{I}$ is the associated interpreted system, φ is in the basic closure $cl_0(\psi)$, $\Sigma(r,n)$ is a σ-state, and $ad(K_\sigma\varphi) \le d$, then $(\mathcal{I},r^*,n) \models \varphi$ if and only if $\Sigma(r,n) \Vdash \varphi$.

Proof: The proof of both parts is by induction on the construction of φ. We consider just the base case and the case for the branching operator '$\forall$'.

[**] In the linear time case treated in [HvdMVar], the interpreted system obtained from an enriched system was obtained by adding to the set of runs $\mathcal{R}$ the interpretation defined by $\pi(r,n)(p) = \mathbf{true}$ just when $\Sigma(r,n)$ is defined and $\Sigma(r,n) \Vdash p$. We also make an adjustment to the set of runs here, in order to ensure that the proposition depends only on the global state.

[††] This definition makes p false at points where Σ is undefined. We could just as well have made p true at such points, without changing our results.

(In these cases the proof is the same for parts (a) and (b).) All other cases are exactly as in [HvdMVar] (modulo changing r to r^* in assertions of the form $(\mathcal{I}, r, n) \models \varphi$).

Suppose p is an atomic proposition in $cl_0(\psi)$ and $\Sigma(r,n)$ is defined. Then $(\mathcal{I}, r^*, n) \models p$ iff $\pi(r^*, n)(p) = \mathit{true}$ iff $p \in \Sigma(r,n)$ iff $\Sigma(r,n) \Vdash p$.

When φ is of the form $\forall\varphi'$, we first show that if $\Sigma(r,n) \Vdash \forall\varphi'$, then $(\mathcal{I}, r^*, n) \models \forall\varphi'$. By Lemma 10, we have $t \Vdash \varphi'$ for all states t such that $\Sigma(r,n) \approx_\forall t$. Let r' be a run such that $(r')^*[0 \dots n] = r^*[0 \dots n]$. Then we have $r[0 \dots n] = r'[0 \dots n]$. By Condition 5 of Definition 1, we have $\Sigma(r',n) \approx_\forall \Sigma(r,n)$, so $\Sigma(r',n) \Vdash \varphi'$, hence, by the induction hypothesis, $(\mathcal{I}, (r')^*, n) \models \varphi'$. This shows that $(\mathcal{I}, r^*, n) \models \forall\varphi'$.

Conversely, we show that if $\Sigma(r,n) \Vdash \neg\forall\varphi'$ then $(\mathcal{I}, r^*, n) \models \neg\forall\varphi'$. By Lemma 11, there exists a state t such that $t \Vdash \neg\varphi'$ and $\Sigma(r,n) \approx_\forall t$. By Condition 6 of Definition 1, there exists a run r' such that $r[0 \dots n] = r'[0 \dots n]$ and $\Sigma(r',n) = t$. By the induction hypothesis, $(\mathcal{I}, (r')^*, n) \models \neg\varphi'$. Moreover, it follows from $r[0 \dots n] = r'[0 \dots n]$ that $r^*[0 \dots n] = (r')^*[0 \dots n]$. Thus, we have shown $(\mathcal{I}, r^*, n) \models \neg\forall\varphi'$.

∎

Corollary 1. *If $(\mathcal{R}, \Sigma)$ is an enriched (resp., enriched$^+$) system for ψ, $\mathcal{I}$ is the associated interpreted system, and (r,n) is a point of $\mathcal{R}$ such that $\Sigma(r,n)$ is an ϵ-state and $\Sigma(r,n) \Vdash \psi$, then $(\mathcal{I}, r^*, n) \models \psi$.*

We apply this corollary in all our completeness proofs, constructing an appropriate enriched or enriched$^+$ system in all cases.

Some further observations and lemmas concerning the branching operator will be useful for our constructions. Observe that a formula of the form $K_i\varphi$ or $\neg K_i\varphi$ is a state formula. For formulas $K_i\varphi$ we see this directly from KB. For formulas $\neg K_i\varphi$, note that $\vdash \neg K_i\varphi \Rightarrow K_i\neg K_i\varphi$, hence by KB and B4 we have $\vdash \neg K_i\varphi \Rightarrow \forall K_i\neg K_i\varphi$, from which we obtain $\vdash \neg K_i\varphi \Rightarrow \forall\neg K_i\varphi$ by K3 and B4.

Lemma 12. *Suppose that s and t are σ-states such that $s \approx_\forall t$. Then $s \Vdash \exists\varphi_t$.*

Proof: Let $s = (\sigma, X)$ and $t = (\sigma, Y)$, and suppose $X_{ST} = Y_{ST}$. Write $Y = \{\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_l\}$, where $Y \cap cl(\psi) = \{\varphi_1, \dots, \varphi_k\}$. Note that all the remaining formulas ψ_j are of the form $\forall\varphi$ or $K_i\varphi$, or the negations of these forms, hence state formulas. It follows that $s \Vdash \forall(\bigwedge_{j=1}^{l} \psi_j)$.

Let $\alpha = \forall(\bigvee_{j=1}^{k} \neg\varphi_j)$. By construction, $\alpha \in cl_0(\psi)$. Hence $t \Vdash \alpha$ or $t \Vdash \neg\alpha$. However, we cannot have $t \Vdash \alpha$, for then Y would be inconsistent, by B3. It follows that $\neg\alpha \in Y$, hence $\neg\alpha \in X$, since $\neg\alpha$ is a state formula by B6. Thus $s \Vdash \neg\alpha$, hence, $s \Vdash \exists(\bigwedge_{j=1}^{l} \varphi_j)$. By the fact proved above that $s \Vdash \forall(\bigwedge_{j=1}^{l} \psi_j)$, RB and B4, it follows that $s \Vdash \exists(\bigwedge_{j=1}^{l} \varphi_j \wedge \bigwedge_{j=1}^{l} \psi_j)$, i.e., $s \Vdash \exists\varphi_t$. ∎

LEMMA 13. *Suppose that s, s' and t are states such that $s \to s'$ and $s' \approx_\forall t'$. Then there exists a state t such that $t \to t'$ and $s \approx_\forall t$.*

Proof: We proceed by contradiction. Let $s = (\sigma, X)$, $s' = (\sigma, X')$, and $t' = (\sigma, Y')$. Suppose that $s \to s'$ and $s' \approx_\forall t'$ and there does not exist a state t such that $t \to t'$ and $s \approx_\forall t$. Then for all σ-states $t = (\sigma, Y)$, if $s \approx_\forall t$ then $\vdash \varphi_t \Rightarrow \neg\bigcirc\varphi_{t'}$. Now by Lemma 3 and the fact that if $s \not\approx_\forall t$ then $s \Vdash \neg\varphi_t$, we have

$$\vdash \varphi_{X_{ST}} \Rightarrow \bigvee_{t:\ s\approx_\forall t} \varphi_t,$$

where the disjunction is over σ-states t. It follows (using T2) that $\vdash \varphi_{X_{ST}} \Rightarrow \bigcirc\neg\varphi_{t'}$. By RB and B4, we get $\vdash \forall\varphi_{X_{ST}} \Rightarrow \forall\bigcirc\neg\varphi_{t'}$. Using FC, it follows that $\vdash \forall\varphi_{X_{ST}} \Rightarrow \bigcirc\forall\neg\varphi_{t'}$. Since all formulas in X_{ST} are state formulas, and the conjunction of state formulas is a state formula (by RB and B4), we have $\vdash \varphi_s \Rightarrow \bigcirc\forall\neg\varphi_{t'}$.

Since $s \to s'$, the formula $\varphi_s \wedge \bigcirc\varphi_{s'}$ is consistent. It follows using T1,T2 and RT1 that $\varphi_{s'} \wedge \forall\neg\varphi_{t'}$ is consistent, i.e., $\varphi_{s'} \wedge \neg\exists\varphi_{t'}$ is consistent. But by Lemma 12 and the assumption that $s' \approx_\forall t'$, we have that $\vdash \varphi_{s'} \Rightarrow \exists\varphi_{t'}$. This is a contradiction. ∎

The following lemma will help us to construct a well-defined mapping Σ in building an enriched system.

LEMMA 14. *Let $s_0 \to s_1 \to \ldots$ and $s'_0 \to s'_1 \to \ldots$ be acceptable sequences of σ-states with $s_k \approx_\forall s'_k$ for all k. Then $s_k = s'_k$ for all k.*

Proof: Let $s_k = (\sigma, X_k)$ and $s'_k = (\sigma, X'_k)$, and assume that for all $k \geq 0$ and all state formulas φ, we have $\varphi \in X_k$ iff $\varphi \in X'_k$. We show by induction on the complexity of $\varphi \in cl_d(\psi)$ that $\varphi \in X_k$ iff $\varphi \in X'_k$. For primitive propositions p, this holds because primitive propositions are state formulas by axiom B1. Similarly, the claim holds for formulas of the form $\forall\varphi$ since these are state formulas by axiom B5. For formulas of the form $\neg\varphi$ and $\varphi_1 \wedge \varphi_2$, the conclusion follows from the induction hypothesis and the fact that X_k and X'_k are consistent. Formulas of the form $K_i\varphi$ are state formulas by KB, so the result is immediate in this case. This leaves formulas of the form $\bigcirc\varphi$ and $\varphi_1 \, U \, \varphi_2$. Here we use the induction hypothesis (for times in the future of k), and the fact that the sequences $s_0 \to s_1 \to \ldots$ and $s'_0 \to s'_1 \to \ldots$ are acceptable. ∎

Proofs of Soundness and Completeness

We are now in a position to prove the completeness results claimed above. The proofs require some lemmas concerning the axioms KT2 and KT3. The proofs of these lemmas are identical to the proofs given in [HvdMVar], to which we refer the reader for details. Throughout this section (as in the previous section) ψ is the fixed consistent formula which we are attempting to show to be satisfiable.

Dealing with $\mathcal{C}_m$, $\mathcal{C}_m^{sync}$, $\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$ (Theorems 1 and 2)

It is straightforward to show that S5C$_m^B$ is sound for $\mathcal{C}_m$, the class of all systems; we leave details to the reader. We prove completeness of S5$_m^B$ for the language KB_m^* and of S5C$_m^B$ for the language CKB_m^* with respect to $\mathcal{C}_m$, $\mathcal{C}_m^{sync}$, $\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$, by constructing an enriched system, and using Corollary 1. We assume here that the language includes common knowledge and that we are dealing with the axiom system S5C$_m^B$ when constructing the states in the enriched structure. Recall that in this case we work with ϵ-states only. The proof proceeds in the same way when common knowledge is not in the language.

The runs in the enriched system we construct will be generated from a set of acceptable sequences of states. The following lemma establishes the existence of acceptable sequences. We omit the proof as it is a well-known result in the context of linear time temporal logic; details can also be found in [HvdMVar].

Lemma 15. *Every finite $\to$-sequence of states can be extended to an infinite acceptable sequence.*

For each agent i, define the function O_i to map the state (σ, X) to the pair $(\sigma\#i, X/K_i)$. (In this section $\sigma = \epsilon$, but O_i is also used later in our other constructions.) Given a state s, we call $O_i(s)$ agent i's *current information* at s. Let x be a new object not equal to any state. We say that a sequence $S = (x, x, \ldots, x, s_N, s_{N+1}, \ldots)$ is an *acceptable sequence from N* if it starts with N copies of x and the suffix $(s_N, s_{N+1}, \ldots)$ is an acceptable $\rightarrow$-sequence of states for ψ. Given a sequence S acceptable from N, we define a run r as follows. For each agent i, take $r_i(n) = (n, x)$ when $n < N$ and $r_i(n) = (n, O_i(s_n))$ otherwise. For the environment component e, take $r_e(n) = x$ if $n < N$ and $r_e(n) = X_{ST}$ where $s_n = (\epsilon, X)$ for $n \geq N$. (Recall that X_{ST} is the set of state formulas in X.)

Let $\mathcal{R}^{sync}$ be the set of all runs so obtained, and define the partial function Σ on points in $\mathcal{R}^{sync} \times I\!N$ so that $\Sigma(r, n) = s_n$ when r is derived from a sequence $(x, x, \ldots, x, s_N, s_{N+1}, \ldots)$ acceptable from N and $n \geq N$, and $\Sigma(r, n)$ is undefined otherwise. Note that Σ is well defined, because if r is derived from both from a sequence $(x, x, \ldots, x, s_N, s_{N+1}, \ldots)$ and from a sequence $(x, x, \ldots, x, s'_{N'}, s'_{N'+1}, \ldots)$, then we have $N = N'$ and $s_k \approx_\forall s'_k$ for $k > N$, so by Lemma 14, it follows that $s_k = s'_k$ for all $k \geq N$.

LEMMA 16. *The pair $(\mathcal{R}^{sync}, \Sigma)$ is an enriched system.*

Proof: It is immediate from the construction that $(\mathcal{R}^{sync}, \Sigma)$ satisfies conditions 1 and 2 of Definition 1. To see that it satisfies Condition 3, suppose that (r, n) is a point such that $\Sigma(r, n)$ is defined and $\Sigma(r, n) \approx_i s$. By Lemma 15 there exists an acceptable sequence $(s_n, s_{n+1}, \ldots)$ with $s = s_n$. Let r' be the run obtained from the sequence $(x, \ldots, x, s_n, s_{n+1}, \ldots)$. Then it is immediate that $(r, n) \sim_i (r', n')$ and $\Sigma(r', n') = s$. To see that it satisfies Condition 4, suppose that $C\varphi \in cl_0(\psi)$ and $\Sigma(r, n) \Vdash \neg C\varphi$. By Lemma 9, there is a state t reachable from $\Sigma(r, n)$ through the relations $\approx_i$ such that $t \Vdash \neg\varphi$. An easy inductive argument on the length of the path from $\Sigma(r, n)$ to t, using Condition 3, shows that there is a point (r', n') reachable from (r, n) through the relations $\sim_i$ such that $\Sigma(r', n') = t$. Thus, the enriched system satisfies condition 4.

It is clear from the definition of r_e and $\approx_\forall$ that if $r[0 \ldots n] = r'[0 \ldots n]$ and $\Sigma(r, n)$ is defined then $\Sigma(r, n) \approx_\forall \Sigma(r', n)$, and thus condition 5 is satisfied.

Finally, to see that condition 6 is satisfied, let (r, n) be a point such that $\Sigma(r, n) = s$ and $s \approx_\forall s'$. Then r is the run obtained from a sequence $(x, \ldots, x, s_N, s_{N+1}, \ldots)$ where $N \leq n$ and $s_n = s$. By repeated application of Lemma 13, there exist states $s'_N, s'_{N+1}, \ldots, s'_n$ such that $s'_n = s'$, $s'_i \approx_\forall s_i$ for $N \leq i < n$, and $s'_N \rightarrow s'_{N+1} \rightarrow \ldots \rightarrow s'_n$. By Lemma 15, we may extend this sequence to an acceptable sequence $(s'_N, \ldots, s'_n, s'_{n+1}, \ldots)$. Let r' be the run obtained from the sequence $(x, \ldots, x, s'_N, \ldots, s'_n, s'_{n+1}, \ldots)$, where there are N copies of x. By definition of Σ, we have $\Sigma(r', n) = s'$. We show that $r[0 \cdots n] = r'[0 \cdots n]$. For $k < N$, we have $r(k) = \langle x, (k, x), \ldots, (k, x) \rangle = r'(k)$. For $N \leq k \leq n$, let $s_k = (\epsilon, X_n)$ and $s'_k = (\epsilon', X'_n)$. Since $s_k \approx_\forall s'_k$, we have $(X_k)_{ST} = (X'_k)_{ST}$. By the observation that formulas of the form $K_i\varphi$ are state formulas, it follows that $X_k/K_i = X'_k/K_i$ also. Thus, for $N \leq k \leq n$, we have $r(k) = \langle (X_k)_{ST}, (k, X_k/K_1), \dots, (k, X_k/K_m)\rangle = r'(k)$, completing the proof that $r[0\cdots n] = r'[0\cdots n]$. ∎

Clearly the system $\mathcal{R}^{sync}$ is synchronous, so the interpreted system $\mathcal{I}$ derived from $(\mathcal{R}^{sync}, \Sigma)$ is also synchronous. Let s be an ϵ-state such that $s \Vdash \psi$. Such a state must exist because ψ was assumed consistent. By Lemma 15 there exists an acceptable sequence $(s_0, s_1, \dots)$ with $s = s_0$. Let r be the corresponding run in $\mathcal{R}^{sync}$. Corollary 1 implies that $(\mathcal{I}, r^*, 0) \models \psi$. This establishes the completeness of the axiomatization $\mathrm{S5C}^B_m$ for the language CKB^*_m (resp., of $\mathrm{S5}^B_m$ for the language KB^*_m) with respect to the classes of systems $\mathcal{C}_m$ and $\mathcal{C}^{sync}_m$. To establish completeness of these axiomatizations for the corresponding languages with respect to the classes of systems $\mathcal{C}^{uis}_m$ and $\mathcal{C}^{sync,uis}_m$, we make use of the following result, which shows that sound and complete axiomatizations for the class of systems satisfying some subset of the properties of perfect recall and synchrony are also sound and complete axiomatizations for the class of systems with the same subset of these properties, but with unique initial states in addition. This completes the proofs of Theorem 1 and 2.

LEMMA 17. *Suppose x is a subset of $\{pr, sync\}$. If $\varphi \in CKB^*_m$ is satisfiable with respect to $\mathcal{C}^x_m$, then it is also satisfiable with respect to $\mathcal{C}^{x,uis}_m$.*

Proof: Suppose $\mathcal{I} = (\mathcal{R}, \pi) \in \mathcal{C}^x_m$. We define a system $\mathcal{I}'$ by adding a new initial state to each run in $\mathcal{R}$. Formally, we define the system $\mathcal{I}' = (\mathcal{R}', \pi')$ as follows. Let l be some local state that does not occur in $\mathcal{I}$ and let s_e be any state of the environment. For each run $r \in \mathcal{R}$, let r^+ be the run such that $r^+(0) = (s_e, l, \dots, l)$ and $r^+(n+1) = r(n)$. Let $\mathcal{R}' = \{r^+ : r \in \mathcal{R}\}$. The interpretation π' is given by $\pi'(r, 0)(p) = \mathit{false}$ and $\pi'(r, n+1)(p) = \pi(r, n)(p)$, for $n \geq 0$ and propositions p. It is clear that $\mathcal{I}'$ is a system with unique initial states. Plainly, π' depends only on the global state if π depends only on the global state. Moreover, if $\mathcal{I}$ is synchronous, then so is $\mathcal{I}'$, and if $\mathcal{I}$ is a system with perfect recall then so is $\mathcal{I}'$. A straightforward induction on the construction of the formula $\varphi \in CKB^*_m$ now shows that, for all points (r, n) in $\mathcal{I}$, we have $(\mathcal{I}, r, n) \models \varphi$ iff $(\mathcal{I}', r^+, n+1) \models \varphi$. (For the case of the branching time operator, note that we have $r^+_1[0\dots n+1] = r^+_2[0\dots n+1]$ iff $r_1[0\dots n] = r_2[0\dots n]$.) ∎

Dealing with $\mathcal{C}_m^{pr}$ and $\mathcal{C}_m^{pr,uis}$ (Theorem 3)

In this section we show that $\mathrm{S5}_m^B + \mathrm{KT3}$ is sound and complete with respect to $\mathcal{C}_m^{pr}$ and $\mathcal{C}_m^{pr,uis}$. All axioms and rules of inference other than KT3 have already been noted to be sound in *all* systems, so their soundness in systems $\mathcal{C}_m^{pr}$ is immediate. Soundness of KT3 in $\mathcal{C}_m^{pr}$ (hence in $\mathcal{C}_m^{pr,uis}$) is straightforward; see [HvdMVar] for details.

For completeness, we construct an enriched$^+$ system and invoke Corollary 1. To satisfy the perfect recall requirement in constructing an interpreted system, we use a lemma characterizing the interaction of knowledge and time in the pre-model. The proof is exactly as in [HvdMVar], so is omitted. The reader is encouraged to compare the following result with condition (c) of Lemma 2.

LEMMA 18. *[Mey94, HvdMVar] Suppose that the axiomatization includes KT3. Then for all σ-states s, t and for all $(\sigma\#i)$-states t', if $s \rightarrow t$ and $t \approx_i t'$, then either (a) $s \approx_i t'$ or (b) there exists a $(\sigma\#i)$-state s' such that $s \approx_i s'$ and there exists a sequence of $(\sigma\#i)$-states $u_0 \rightarrow u_1 \rightarrow \ldots \rightarrow u_n = t'$, where $n \geq 0$, such that $s' \rightarrow u_0$ and $u_l \approx_i u_{l+1}$ for all $l = 0 \ldots n-1$.*

We are now ready to define, for each consistent ψ, an enriched$^+$ system for ψ that establishes completeness of $\mathrm{S5}_m^U + KT3$ with respect to $\mathcal{C}_m^{pr}$. The runs of this system are those derived from the acceptable sequences $(s_0, s_1, \ldots)$ (of states for ψ) as follows. The local states in the run r corresponding to the sequence $(s_0, s_1, \ldots)$ are defined by $r_i(n) = O_i(s_0)\# \ldots \#O_i(s_n)$, for each agent i and $n \geq 0$. Thus, $r_i(n)$ is the sequence of current information that agent i has had up to time n. For the environment component e, we define $r_e(n) = (\sigma, X_{ST})$ where $s_n = (\sigma, X)$. Let $\mathcal{R}^{pr}$ be the set of runs defined in this way. The function Σ is given by $\Sigma(r, n) = s_n$ for each $n \geq 0$, where r is constructed from the sequence of states $(s_0, s_1, \ldots)$. The function Σ is well-defined, because if r is derived both from an acceptable sequence of σ-states $(s_0, s_1, \ldots)$ and an acceptable sequence of σ'-states $(s'_0, s'_1, \ldots)$, then $\sigma = \sigma'$ is the index included in the global states of r and $s_k \approx_\forall s'_k$ for all $k \geq 0$, so it follows using Lemma 14 that $s_k = s'_k$ for all $k \geq 0$.

LEMMA 19. *Suppose that the axiomatization includes KT3. Then $(\mathcal{R}^{pr}, \Sigma)$ is an enriched$^+$ system.*

Proof: It is clear that $(\mathcal{R}^{pr}, \Sigma)$ satisfies Conditions 1 and 2 of Definition 1.

We show that Condition $3'$ holds. Suppose that $\Sigma(r,n)$ is a σ-state and $\Sigma(r,n) \approx_i s$ for some $(\sigma\#i)$-state s. We must find a point (r',n') such that $\Sigma(r',n') = s$. The proof proceeds by induction on n. The result for $n=0$ is immediate, since we can take r' to be an acceptable sequence starting from s (such a sequence exists by Lemma 15), so $\Sigma(r',0) = s$ and clearly $(r,0) \sim_i (r',0)$.

Now suppose $n > 0$ and the result holds for $n-1$. Because $\Sigma(r,n-1) \rightarrow \Sigma(r,n)$ and $\Sigma(r,n) \approx_i s$, it follows by Lemma 18 that either (a) $\Sigma(r,n-1) \approx_i s$ or (b) there exists a $(\sigma\#i)$-state s' such that $\Sigma(r,n-1) \approx_i s'$ and there exists a sequence of $(\sigma\#i)$-states $u_0 \rightarrow u_1 \rightarrow \ldots \rightarrow u_k$ such that $s' \rightarrow u_0$, $u_l \approx_i u_{l+1}$ for $l = 0 \ldots k-1$, and $u_k = s$. By the induction hypothesis, there exists for any $(\sigma\#i)$-state t with $\Sigma(r,n-1) \approx_i t$ a point (r',n') such that $(r,n-1) \sim_i (r',n')$ and $\Sigma(r',n') = t$. In case (a), we take $t = s$, and we then have that $\Sigma(r,n-1) \approx_i \Sigma(r,n)$ and $\Sigma(r',n') = s$. It follows that $(r,n) \sim_i (r,n-1)$, and by the transitivity of $\sim_i$, we also have $(r,n) \sim_i (r',n')$. Hence we are done. In case (b), we take $t = s'$. Suppose that r' is derived from the sequence $(v_0, v_1, \ldots)$. Let r'' be any run derived from an acceptable sequence with initial segment $(v_0, \ldots, v_{n'}, u_0, \ldots, u_k)$. Again, such a run exists by Lemma 15. By construction, $\Sigma(r'', n'+k+1) = u_k = s$. Moreover, since $r''_i(n') = r'_i(n') = r_i(n-1)$ and $O_i(u_l) = O_i(s)$ for all $l = 0, \ldots, k$, we have $r''_i(n'+k+1) = r''_i(n')\#O_i(u_0)\# \ldots \#O_i(u_k) = r_i(n-1)\#O_i(s) = r_i(n)$, and hence $(r,n) \sim_i (r'', n'+k+1)$.

It is clear from the definition of r_e and $\approx_\forall$ that if $r[0\ldots n] = r'[0\ldots n]$ then $\Sigma(r,n) \approx_\forall \Sigma(r',n)$, and thus condition 5 is satisfied.

Finally, to see that condition 6 is satisfied, let (r,n) be a point such that $\Sigma(r,n) = s$ and $s \approx_\forall s'$. Let the run r be obtained from the sequence $(s_0, s_1, \ldots)$. By repeated application of Lemma 13, there exist states $s'_0, \ldots, s'_n$ such that $s'_n = s'$, $s'_k \approx_\forall s_k$ for $0 \le k < n$, and $s'_0 \rightarrow s'_1 \rightarrow \ldots \rightarrow s'_n$. By Lemma 15, we may extend this sequence to an acceptable sequence $(s'_0, \ldots, s'_n, s'_{n+1}, \ldots)$. Let r' be the run obtained from this acceptable sequence.

By definition of Σ, we have $\Sigma(r',n) = s'$. We show that $r[0\cdots n] = r'[0\cdots n]$. For $0 \le k \le n$, let $s_k = (\sigma, X_n)$ and $s'_k = (\sigma', X'_n)$. Since $s_k \approx_\forall s'_k$, we have $\sigma = \sigma'$ and $(X_k)_{ST} = (X'_k)_{ST}$. Thus $r_e(k) = r'_e(k)$. By the observation that formulas of the form $K_i\varphi$ are state formulas, it follows that $X_k/K_i = X'_k/K_i$ also, hence $O_i(s_k) = O_i(s'_k)$. It follows that $r_i(k) = O_i(s_0)\# \ldots \#O_i(s_k) = O_i(s'_0)\# \ldots \#O_i(s'_k) = r'_i(k)$. Thus, $r(k) = r'(k)$ for $k \le n$, completing the proof that $r[0\cdots n] = r'[0\cdots n]$. ∎

Now take any ϵ-state s such that $s \Vdash\!\!- \ \psi$, and let r be a run derived from an acceptable sequence starting with s. By construction, the system $\mathcal{I}$ obtained from the enriched$^+$ system is in $\mathcal{C}_m^{pr}$, and by Corollary 1, we have $(\mathcal{I}, r, 0) \models \psi$. Thus, ψ is satisfiable in $\mathcal{C}_m^{pr}$. By Lemma 17, ψ is also satisfiable in systems in $\mathcal{C}_m^{pr,uis}$. Since this argument applies to any formula ψ consistent with respect to $\mathrm{S5}_m^B + KT3$, this completes the proof of Theorem 3.

Dealing with $\mathcal{C}_m^{pr,sync}$ and $\mathcal{C}_m^{pr,sync,uis}$ (Theorem 4)

We now show that $\mathrm{S5}_m^B + KT2$ is sound and complete with respect to KB_m^* for the classes of systems $\mathcal{C}_m^{pr,sync}$ and $\mathcal{C}_m^{pr,sync,uis}$. The arguments for soundness of $KT2$ are exactly as in the linear time case, and we refer the reader to [HvdMVar] for details.

Before constructing an enriched$^+$ system for the completeness part, we first note a property of the pre-model, analogous to Lemma 18.

LEMMA 20. *[HvdMVar] Suppose the axiomatization includes KT2. Then for all σ-states s, t with $s \to t$, and for all $(\sigma\#i)$-states t' with $t \approx_i t'$ there exists a $(\sigma\#i)$-state s' such that $s \approx_i s'$ and $s' \to t'$.*

To construct the enriched$^+$ system, we now take $\mathcal{R}^{pr,sync}$ to be the set of runs r derived from acceptable sequences $(s_0, s_1, \ldots)$ of states for the formula ψ by putting $r_e(n) = (\sigma, X_{ST})$ when $s_n = (\sigma, X)$, and putting $r_i(n) = O_i(s_0) \ldots O_i(s_n)$, for each agent i and $n \geq 0$. The notation $O_i(s_0) \ldots O_i(s_n)$ is meant to denote the sequence formed by concatenating $O_i(s_0), \ldots, O_i(s_n)$. Thus, the length of the sequence is $n + 1$, which enforces synchrony. Again, the function Σ is given by $\Sigma(r, n) = s_n$ for each $n \geq 0$. (As above, it follows using Lemma 14 that Σ is well defined.)

LEMMA 21. *Suppose that the axiomatization includes KT2. Then the pair $(\mathcal{R}^{pr,sync}, \Sigma)$ is an enriched$^+$ system.*

Proof: Conditions 1 and 2 of the definition of an enriched$^+$ system are immediate. To show that Condition $3'$ holds, suppose that $\Sigma(r, n)$ is a σ-state, and t is a $(\sigma\#i)$-state such that $\Sigma(r, n) \approx_i t$. Suppose that r is derived from the acceptable sequence $(s_0, s_1, \ldots)$, so $\Sigma(r, n) = s_n$. It follows from Lemma 20 that there exists a $\to$-sequence $t_0 \to \ldots \to t_n$ such that $t_n = t$ and $s_j \approx_i t_j$ for $j = 1 \ldots n$. By Lemma 15, this sequence may be extended to an infinite acceptable sequence. Taking r' to be the run derived from this sequence, we see that $(r, n) \sim_i (r', n)$ and $\Sigma(r', n) = t$. The proof of conditions 5 and 6 are similar to the arguments in the proof of Lemma 19, but using the concatenation operator '$\cdot$' instead of the absorptive concatenation operator '#'. ∎

Take $\mathcal{I}^{pr,sync}$ to be the system obtained from $(\mathcal{R}^{pr,sync}, \Sigma)$. By construction, this system is in $\mathcal{C}^{pr,sync}$. Now take any ϵ-state s such that $s \Vdash \psi$, and let r be a run derived from an acceptable sequence starting with s. By construction, the system $\mathcal{I}$ obtained from the enriched$^+$ system is in $\mathcal{C}_m^{pr,sync}$, and by Corollary 1, we have $(\mathcal{I}, r, 0) \models \psi$. Thus, ψ is satisfiable in $\mathcal{C}_m^{pr,sync}$. By Lemma 17, ψ is also satisfiable in systems in $\mathcal{C}_m^{pr,sync,uis}$. Since this argument applies to any formula ψ consistent with respect to $\mathrm{S5}_m^B + KT2$, this completes the proof of Theorem 4.

Discussion

We conclude by discussing related work and open questions. We limit our discussion to the specific issue of logics of knowledge and branching time, omitting some closely related topics such as logics of knowledge and linear time, logics combining knowledge operators and operators from dynamic logic, logics for topological spaces using knowledge and temporal-like operators, logics containing operators for specific types of knowledge update, and temporal logics of belief.

Several authors have studied logics of knowledge with branching time operators from the point of view of complexity and axiomatization. Parikh and Ramanujam [PR85] consider essentially the class of systems $\mathcal{C}^{pr}$ with respect to the language KB_m. Ladner and Reif [LR86] define two logics. One is the *tree logic of protocols*, which corresponds roughly to the class of systems C^{pr} with respect to a language that contains operators K_i for knowledge and the restricted temporal constructs $\forall\bigcirc\varphi$ and $\forall\Box\varphi$ (i.e. $\forall\neg(true\, U\, \neg\varphi)$. They also consider the same language with another semantics, yielding what they call the *linear logic of protocols*. This amounts to the class $C^{pr,nl,uis}$ of structures with perfect recall, no learning and unique initial states. Their focus is on complexity: they do not attempt a complete axiomatization, though they discuss the axioms

KT1$'$. $K_i\forall\bigcirc\varphi \Rightarrow \forall\bigcirc K_i\varphi$.

KT2$'$. $K_i\forall\Box\varphi \Rightarrow \forall\Box K_i\varphi$.

which can be seen as restricted versions of the axioms KT1 and KT2.

As discussed above, Halpern and Vardi [HV89, HV88a] have characterized the complexity of a large space of logics of knowledge and time. They also provided axiomatizations for the class of systems $\mathcal{C}$ and $\mathcal{C}^{sync,pr}$, using, in the branching time case, the language KB_m and the axiom KT1′ [HV86, HV88b]. Thus, our results in this paper are slightly more general, since we deal with the broader language KB^*_m. Our proof technique is also somewhat different. We note that the axiom KB does not appear in these earlier axiomatizations (it is not expressible in the language KB_m.)

Spaan [Spa90] optimizes the complexity lower-bound constructions of some of the results of [HV89, HV88a, LR86], showing that use of the next-time and until operators can be eliminated. In the branching time cases, this leaves the operators $\forall\Diamond$ and $\exists\Diamond$. The complexity of logics of knowledge and branching time with respect to the full languages KB^*_m and CKB^*_m does not appear to have been studied in detail, but Halpern and Vardi [HV86] suggest that the extension adds at least an exponential to the complexities indicated in Table 1.

In addition to the Hilbert style axiomatizations we have studied here, there has been work on resolution [DFW98] and tableaux-based [WDF98] proof systems for logics of knowledge and *linear* time: the branching time does not appear to have been studied for these approaches.

We have noted that the semantics we have adopted for the temporal fragment of our logics corresponds to the *narrow, bundle* semantics for CTL*. The problem of axiomatizing combinations of CTL* with logics of knowledge with respect to the other versions of the semantics of the branching operator remains open. We expect that the techniques of this paper can be used for the *broad, bundle* semantics. The complete tree approach to the semantics is likely to be more complex: the problem of axiomatizing CTL* with respect to this semantics has only recently been solved [Rey01]. Similarly, it has only recently been shown [BHWZ02] that CTL* is decidable with respect to a semantics that does not assume that propositions depend only on global states. Zanardo [Zan85, Zan96] has axiomatized branching time logic with respect to a semantics that does not assume that propositions depend only on global states, but with respect to a weaker language than we have considered here, that takes $\Box$ (and its past time version) as the linear time basis.

It is interesting to note the strong resemblance between the branching time axiom FC and the axiom KT2 for synchronous perfect recall. Indeed, this is not surprising: if we consider an agent i that has full information about the global state, i.e., for which $r_i(n) = r'_i(n')$ implies $r(n) = r'(n)$, then we see that the semantic definitions of $K_i\varphi$ and $\forall\varphi$ coincide when the agent has synchronous perfect recall.[‡‡] From this perspective, branching time logic is

[‡‡]It is tempting to try to define "the agent has full information about the global state"

a special case of the logic of knowledge! It is also worth noting that the same observation applies for the broad semantics of the branching operator, which can be understood as an operator for the knowledge of an agent that has full information about the global state (but is neither synchronous, nor has perfect recall.)

Finally, we note that we have not considered systems with no learning in this paper. As in the linear time case, not all the branching time logics of knowledge that combine this assumption with the assumptions of unique initial state, synchrony and perfect recall are axiomatizable. All the axiomatizable linear time logics are axiomatized in [HvdMVar], but the proofs use some techniques that we have not touched upon in this paper, so we leave the branching time cases with no learning to future work.

Acknowledgements: Research supported by a Discovery Grant from the Australian Research Council. We also acknowledge our obvious debt to the first author's earlier collaborative work with Joseph Halpern and Moshe Vardi [HvdMVar].

References

[Bar88] J. Barwise. Three views of common knowledge. In M. Y. Vardi, editor, *Proc. Second Conference on Theoretical Aspects of Reasoning about Knowledge*, pages 365–379. Morgan Kaufmann, San Francisco, Calif., 1988.

[BHWZ02] Sebastian Bauer, Ian Hodkinson, Frank Wolter, and Michael Zakharyaschev. On non-local and weak monodic quantified CTL*. submitted for publication, 2002.

[Bur79] J. Burgess. Logic and time. *Journal of Symbolic Logic*, 44:556–582, 1979.

[DFW98] C. Dixon, M. Fisher, and M. Wooldridge. Resolution for temporal logics of knowledge. *Journal of Logic and Computation*, 8(3):345–372, 1998.

[EH85] E. A. Emerson and J. Y. Halpern. Decision procedures and expressiveness in the temporal logic of branching time. *Journal of Computer and System Sciences*, 30(1):1–24, 1985.

as $r_i(n) = r(n)$, but this is hard to model in a standard set theory. However, it could be done in set theories lacking the axiom of foundation. Barwise [Bar88] has considered the modelling of knowledge situations using such set theories.

[FHMV95] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning about Knowledge*. MIT Press, Cambridge, Mass., 1995.

[FHV91] R. Fagin, J. Y. Halpern, and M. Y. Vardi. A model-theoretic analysis of knowledge. *Journal of the ACM*, 91(2):382–428, 1991. A preliminary version appeared in *Proc. 25th IEEE Symposium on Foundations of Computer Science*, 1984.

[GPSS80] D. Gabbay, A. Pnueli, S. Shelah, and J. Stavi. On the temporal analysis of fairness. In *Proc. 7th ACM Symp. on Principles of Programming Languages*, pages 163–173, 1980.

[Hin62] J. Hintikka. *Knowledge and Belief.* Cornell University Press, Ithaca, N.Y., 1962.

[HM92] J. Y. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.

[HV86] J. Y. Halpern and M. Y. Vardi. The complexity of reasoning about knowledge and time. In *Proc. 18th ACM Symp. on Theory of Computing*, pages 304–315, 1986.

[HV88a] J. Y. Halpern and M. Y. Vardi. The complexity of reasoning about knowledge and time in asynchronous systems. In *Proc. 20th ACM Symp. on Theory of Computing*, pages 53–65, 1988.

[HV88b] J. Y. Halpern and M. Y. Vardi. The complexity of reasoning about knowledge and time: synchronous systems. Research Report RJ 6097, IBM, 1988.

[HV89] J. Y. Halpern and M. Y. Vardi. The complexity of reasoning about knowledge and time, I: lower bounds. *Journal of Computer and System Sciences*, 38(1):195–237, 1989.

[HvdMVar] J.Y. Halpern, R. van der Meyden, and M.Y. Vardi. Complete axiomatizations for reasoning about knowledge and time. *SIAM Journal on Computing*, to appear.

[KP81] D. Kozen and R. Parikh. An elementary proof of the completeness of PDL. *Theoretical Computer Science*, 14(1):113–118, 1981.

[Leh84] D. Lehmann. Knowledge, common knowledge, and related puzzles. In *Proc. 3rd ACM Symp. on Principles of Distributed Computing*, pages 62–67, 1984.

[LR86] R. E. Ladner and J. H. Reif. The logic of distributed protocols (preliminary report). In J. Y. Halpern, editor, *Theoretical Aspects of Reasoning about Knowledge: Proc. 1986 Conference*, pages 207–222. Morgan Kaufmann, San Francisco, Calif., 1986.

[Mey94] R. van der Meyden. Axioms for knowledge and time in distributed systems with perfect recall. In *Proc. 9th IEEE Symp. on Logic in Computer Science*, pages 448–457. 1994.

[PR85] R. Parikh and R. Ramanujam. Distributed processing and the logic of knowledge. In R. Parikh, editor, *Proc. Workshop on Logics of Programs*, pages 256–268, 1985.

[Rey01] M. Reynolds. An axiomatization of full computation tree logic. *Journal of Symbolic Logic*, 66(3):1011–1057, Sept 2001.

[Sat77] M. Sato. A study of Kripke-style methods for some modal logics by Gentzen's sequential method. *Publications Research Institute for Mathematical Sciences, Kyoto University*, 13(2), 1977.

[Spa90] E. Spaan. Nexttime is not necessary. In R. J. Parikh, editor, *Theoretical Aspects of Reasoning about Knowledge: Proc. Third Conference*, pages 241–256. Morgan Kaufmann, San Francisco, Calif., 1990.

[Sti92] C. Stirling. Modal and temporal logics. In S. Abramsky, Dov M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science Volume 2: Background: Computational Structures*, pages 477–563. Clarendon Press, Oxford, 1992.

[WDF98] M. Wooldridge, C. Dixon, and M. Fisher. A tableau-based proof method for temporal logics of knowledge and belief. *Journal of Applied Non-Classical Logics*, 8(3), 1998.

[Zan85] A. Zanardo. A finite axiomatization of the set of strongly valid Ockhamist formulas. *Journal of Philosophical Logic*, 14(447-468), 1985.

[Zan96] A. Zanardo. Branching-time logic with quantification over branches: the point of view of modal logic. *Journal of Symbolic Logic*, 61(1):1–39, March 1996.

Ron van der Meyden, Ka-shu Wong
School of Computer Science and Engineering
University of New South Wales
Sydney 2052, Australia
meyden@cse.unsw.edu.au,
kswong@cse.unsw.edu.au