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COMP3152/9152
Lecture 11
Variants of Common Knowledge
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An apparent paradox:

Common Knowledge seems to be esential for coordinated action,

BUT, in many contexts (e.g. asynchronous message passing, unreliable message passing, unbounded message delivery), common knowledge is hard to attain.

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Say that $\mathcal{I}$ is a *finite state interpreted system* if for all agents i, the set

$$L_i = \{r_i(n) \mid (r,n) \text{ a point of } \mathcal{I}\}$$

is finite.

(Note that the number of points is still infinite, as may be the number of environment states $r_e(n)$.

Example: Let E be a finite state environment, as defined in Lecture 5. Then $\mathcal{I}^{\mathsf{obs}}(E)$ is a finite state interpreted system.

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Theorem: Let $\mathcal{I}$ be an interpreted finite state system, let l be the minimum of $|L_i|$, and define $k = 2l - 1$. Then

$$\mathcal{I} \models C_G\phi \iff E_G^k\phi.$$

So common knowledge collapses to a finite level of knowledge in such systems.

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Recall from Lecture 5:

In a system where agents have perfect recall, and a "hello" message is either delivered immediately, or with a delay of one tick of the clock, the fact "the message has been delivered" is never common knowledge.

More generally,

Scenario 1: if a message sent at time m is delivered at a random time between $m+1$ and $m+\epsilon$, then for a "hello" message, the fact "the message has been delivered" is never common knowledge.

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A variant:

Scenario 2: Suppose a message sent at time m is delivered by time $m+\epsilon$. Suppose Alice and Bob use the same global clock and have synchronous perfect recall.

At time m, Alice sends to Bob: "this message is being sent at time m: hello".

Then at time $m+\epsilon$, it is common knowledge to Alice and Bob that the message has been delivered.

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What is the difference in the two scenarios?

In the first, Bob has uncertainty about when the message was sent.

In the second, Bob is certain about the sending time, and there is a *simultaneous event* common to Alice and Bob: $time = m+\epsilon$.

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Events

An *event* in a system is a subset e of the set of points

Event e is *local* to agent i if there is a set $X \subseteq L_i$ of agent i's local states such that $(r,m) \in e$ iff $r_i(m) \in X$.

A *state event* is an event e such that there exists a set X of global states such that $(r,m) \in e$ iff $r(m) \in X$.

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Given a state event e, we can define an atomic proposition ϕ_e such that $(\mathcal{I}, r, m) \models \phi_e$ iff $(r, m) \in e$.

Conversely, for every formula ϕ, there is an event

$$ev_{\mathcal{I}}(\phi) = \{(r, m) \mid (\mathcal{I}, r, m) \models \phi\}$$

Lemma: An event e is local to process i in a system $\mathcal{I}$ iff $\mathcal{I} \models \psi_e \Rightarrow K_i\psi_e$.

Lemma: $\mathcal{I} \models \phi \Rightarrow K_i\phi$ iff $ev_{\mathcal{I}}(\phi)$ is local to process i in $\mathcal{I}$.

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Simultaneous Events

An *event ensemble* for a group G is a mapping $\mathbf{e}$ from $i \in G$ to events $\mathbf{e}(i)$ local to i.

An event ensemble is *perfectly coordinated* if $\mathbf{e}(i) = \mathbf{e}(j)$ for all $i, j \in G$.

(Thus for all points (r, m), we have $(r, m) \in \mathbf{e}(i)$ iff $(r, m) \in \mathbf{e}(j)$.

For an even ensemble $\mathbf{e}$, define the proposition $\psi_{\mathbf{e}}$ by

$$\psi_{\mathbf{e}} = \bigvee_{i \in G} \psi_{\mathbf{e}(i)}$$

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Theorem:

(a) For every formula ϕ the ensemble for G defined by $\mathbf{e}(i) = ev_{\mathcal{I}}(K_iC_G\phi)$ is perfectly coordinated.

(b) If $\mathbf{e}$ is a perfectly coordinated ensemble for G, then for every $i \in G$ we have $\mathcal{I} \models \psi_{\mathbf{e}(i)} \Rightarrow K_iC_G\psi_{\mathbf{e}(i)}$.

(b) If $\mathbf{e}$ is a perfectly coordinated ensemble for G, then we have $\mathcal{I} \models \psi_{\mathbf{e}} \Rightarrow K_iC_G\psi_{\mathbf{e}}$.

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Temporal Imprecision

In real systems, clock drift is an issue.

A system $\mathcal{R}$ has *temporal imprecision* if for all points (r, m) and processes $i \neq j$, there exists a point (r', m') such that $r'_i(m) = r_i(m)$ but $r'_j(m') = r_j(m+1)$.

(i is unsure about j's rate of progress.)

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Define ensemble $\mathbf{e}$ to be *nontrivial* if there exists a run r and times m, m' such that $(r, m) \in \cup_{i \in G} \mathbf{e}(i)$ but $(r, m') \notin \cup_{i \in G} \mathbf{e}(i)$

Proposition: In a system with temporal imprecision, there are no nontrivial perfectly coordinated ensembles for G, if $|G| \geq 2$.

Corollary: If $\mathcal{I}$ is a system with temporal imprecision, ϕ is any formula and $|G| \geq 2$ then for all runs r and times m, if $(\mathcal{I}, r, m) \models C_G\phi$ then $(\mathcal{I}, r, 0) \models C_G\phi$.

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Fixpoints

Consider the equivalence

$$C_G\phi \iff E_G(\phi \wedge C_G\phi)$$

We can understand this as saying that $C_G\phi$ is a solution of the "equation"

$$x \iff E_G(\phi \wedge x)$$

This can be made precise using functions

$$f : \mathcal{P}(Points(\mathcal{I})) \to \mathcal{P}(Points(\mathcal{I}))$$

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Extend the language by adding a propostion x

For each formula ϕ in the extended language, define

$$f_\phi : \mathcal{P}(Points(\mathcal{I})) \to \mathcal{P}(Points(\mathcal{I}))$$

such that intuitively, $f_\phi(A)$ is the set of points satisfying ϕ if we interpret x as being true at a point (r, m) iff $(r, m) \in A$.

Inductively,

$$\begin{aligned}
f_x(A) &= A \\
f_p(X) &= \{(r, m) \mid \pi(r(m), p) = 1\} \\
f_{\neg\phi}(A) &= Points(\mathcal{I}) \setminus f_\phi(A) \\
f_{\phi\wedge\psi}(A) &= f_\phi(A) \cap f_\psi(A) \\
f_{K_i\phi}(A) &= \{(r, m) \mid (r, m) \sim_i (r', m') \Rightarrow (r'm') \in f_\phi(A)\}
\end{aligned}$$

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$$f_{E_G\phi}(A) = \bigcap_{i \in G} f_{K_i\phi}(A)$$

and

$$f_{C_G\phi}(A) = \bigcap_{l > 0} f^l_{E_G\phi}(A)$$

where $f^0(A) = A$ and $f^{l+1}(A) = f(f^l(A))$.

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The following justifies the intuitive interpretation of f_ϕ given above.

Write $\phi^{\mathcal{I}}$ for $\{(r,m) \mid (\mathcal{I},r,m) \models \phi\}$. Then for every formula ϕ that does not contain x, we have

$$f_\phi(A) = \phi^{\mathcal{I}}$$

More generally, for formulas ϕ that contain x, write $\phi[x/\psi]$ for the result of replacing all occurrences of x in ϕ by ψ. Then

$$f_\phi(\psi^{\mathcal{I}}) = (\phi[x/\psi])^{\mathcal{I}}$$

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Now note that

$$\mathcal{I} \models C_G\phi \iff E_G(\phi \wedge C_G\phi)$$

is equivalent to

$$(C_G\phi)^{\mathcal{I}} = (E_G(\phi \wedge C_G\phi))^{\mathcal{I}}$$

which is equivalent to

$$f_{E_G(\phi\wedge x)}(A) = A$$

where $A = (C_G\phi)^{\mathcal{I}}$.

Say: $(C_G\phi)^{\mathcal{I}}$ is a *fixpoint* of $f_{E_G(\phi\wedge x)}$

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Suppose $f : \mathcal{P}(S) \to \mathcal{P}(S)$ and $A \subseteq S$

A is a fixpoint of f if $f(A) = A$

B is a *greatest fixpoint* of f if B is a fixpoint of f and for all fixpoints A of f, we have $A \subseteq B$.

B is a *least fixpoint* of f if B is a fixpoint of f and for all fixpoints A of f, we have $B \subseteq A$.

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f is *monotone* if for all sets $A \subseteq B$, we have $f(A) \subseteq f(B)$.

Lemma: If every occurrence of x in ϕ occurs in the scope of an even number of negation symbols, then the function f_ϕ is monotone.

(Example: x occurs in the scope of an even number (viz. 0) of negation symbols in $E_G(\phi \wedge x)$.)

Lemma: Every monotone function has a greatest and a least fipoint.

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Mu-calculus

Extend the language by adding operators for least and greatest fixpoints.

For every *monotone* formula ϕ (such that x occurs in the scope of an even number of negation symbols), the following are formulas:

$\mu x[\phi]$ - the least fixpoint of $\phi(x)$

$\nu x[\phi]$ - the greatest fixpoint of $\phi(x)$

Semantically:
$f_{\mu x[\phi]}(A)$ is the least fixpoint of f_ϕ
$f_{\nu x[\phi]}(A)$ is the greatest fixpoint of f_ϕ

$(\mathcal{I}, r, m) \models \mu x[\phi]$ iff $(r, m) \in f_{\mu x[\phi]}(\emptyset)$
$(\mathcal{I}, r, m) \models \nu x[\phi]$ iff $(r, m) \in f_{\nu x[\phi]}(\emptyset)$

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Common Knowledge expressed in mu-calculus

$$\models C_G(\phi) \iff \nu x(E_G(\phi \wedge x))$$

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Properties of mu-calculus operators

Lemma: let ψ be a monotone formula and let $\mathcal{I}$ be an interpreted system. Then

(a) $\models \psi[x/\nu x[\psi]] \iff \nu x[\psi]$, and

(b) if $\mathcal{I} \models \phi \Rightarrow \psi[x/\phi]$ then $\mathcal{I} \models \phi \Rightarrow \nu x[\psi]$.

Note that if we take ψ to be $E_G(\phi \wedge x)$, where $C_G\phi = \nu x\psi$, (a) says:

$$\models E_G(\phi \wedge C_G\phi) \iff C_G\phi$$

If we take ψ to be $E_G(\psi \wedge x)$, then (b) says: if $\mathcal{I} \models \phi \Rightarrow E_G(\psi \wedge \phi)$ then $\mathcal{I} \models \phi \Rightarrow C_G\psi$.

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Approximations to Common Knowledge

We have seen
— simultaneous agreement requires common knowledge of agreement
— common knowledge is sometimes impossible to obtain.

Questions:
— Can we make do with weaker notions of agreement?
— What types of knowledge do such weaker forms of agreement require?

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Epsilon-coordination

Say that an ensemble $\mathbf{e}$ for G is *ϵ-coordinated* if the local events in $\mathbf{e}$ never happen more than ϵ time units apart, i.e., if $(r, m) \in \mathbf{e}(i)$ for some $i \in G$, then there exists an interval $I = [m', m' + \epsilon]$ such that $m \in I$ and for all $j \in G$ there exists $m_j \in I$ such that $(r, m_j) \in \mathbf{e}(j)$.

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Example: suppose that messages sent by A at time m are guaranteed to be delivered to B by time $m + \epsilon$.

Then the event $\mathbf{e}$ for $\{A, B\}$ defined by $\mathbf{e}(i) = (K_i \mathtt{sent}_A(\mu))^{\mathcal{I}}$ for $i \in \{A, B\}$ is ϵ-coordinated.

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Epsilon-group knowledge

Define $E^{\epsilon}_G \phi$ by

$(\mathcal{I}, r, m) \models E^{\epsilon}_G \phi$ if there exists an interval $I = [m', m' + \epsilon]$ such that $m \in I$ and for all $j \in G$ there exists $m_j \in I$ such that $(\mathcal{I}, r, m_j) \models K_j \phi$.

In the example: $\mathcal{I} \models \mathtt{sent}_A(\mu) \Rightarrow E^{\epsilon}_{\{A,B\}}(\mathtt{sent}_A(\mu))$

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Epsilon-Common Knowledge

$$C^{\epsilon}_G \phi = \nu x [E^{\epsilon}_G (\phi \wedge x)]$$

This has the following properties:

$\models E^{\epsilon}_G(\phi \wedge C^{\epsilon}_G \phi) \iff C^{\epsilon}_G \phi$

If $\mathcal{I} \models \phi \Rightarrow E^{\epsilon}_G(\psi \wedge \phi)$ then $\mathcal{I} \models \phi \Rightarrow C^{\epsilon}_G \psi$.

$\models C^{\epsilon}_G(\phi) \Rightarrow C^{\epsilon}_G(C^{\epsilon}_G \phi)$ (positive introspection)

(but not the other S5 properties!)

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Relating Epsilon-coordination and Epsilon-Common Knowledge

Proposition: Let $\mathcal{I}$ be an interpreted system and G a group of agents.

(a) For every formula ϕ, the ensemble $\mathbf{e}$ for G defined by $\mathbf{e}(i) = ev_{\mathcal{I}}(K_i C^{\epsilon}_G \phi)$ is ϵ-coordinated.

(b) If $\mathbf{e}$ is an ϵ-coordinated ensemble for G, then for every $i \in G$ we have $\mathcal{I} \models \psi_{\mathbf{e}(i)} \Rightarrow K_i C^{\epsilon}_G \psi_{\mathbf{e}(i)}$.

(c) If $\mathbf{e}$ is an ϵ-coordinated ensemble for G, then $\mathcal{I} \models \psi_{\mathbf{e}} \Rightarrow K_i C^{\epsilon}_G \psi_{\mathbf{e}}$.

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Eventual Coordination

An ensemble $\mathbf{e}$ for G is *eventually-coordinated* if for every run r, if $(r, m) \in \mathbf{e}(i)$ for some $i \in G$, then for all $j \in G$ there exists m_j such that $(r, m_j) \in \mathbf{e}(j)$.

This similarly corresponds to *eventual common knowledge*, defined by

$$C^{\Diamond}_G \phi = \nu x[E^{\Diamond}_G(\phi \wedge x)]$$

where

$(\mathcal{I}, r, m) \models E^{\Diamond}_G \phi$ if for each $i \in G$ there exists m_i such that $(\mathcal{I}, r, m_i) \models K_i \phi$.

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More coordination concepts

Time-stamped coordination: each agent has a local clock. There exists a time t such that the event $\mathbf{e}(i)$ happens for each $i \in G$ at time t on agent i's local clock.

Probabilistic Coordination: there exists an ϵ such that the events $\mathbf{e}(i)$ happen simultaneously with probability $1 - \epsilon$.

A notion of common knowledge can be defined for each such that a similar correspondence holds.