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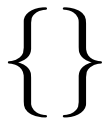
NICTA Advanced Course

Theorem Proving
Principles, Techniques, Applications



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Content

- ➜ Intro & motivation, getting started with Isabelle
- ➜ Foundations & Principles
  - Lambda Calculus
  - Higher Order Logic, natural deduction
  - Term rewriting
- ➜ **Proof & Specification Techniques**
  - **Inductively defined sets, rule induction**
  - Datatypes, recursion, induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

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Last Time

- ➜ Conditional term rewriting
- ➜ Congruence and AC rules
- ➜ More on confluence
- ➜ Completion
- ➜ Isar: fix, obtain, abbreviations, moreover, ultimately

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Sets in Isabelle

Type **'a set**: sets over type 'a

- ➜ $\{\}, \quad \{e_1, \ldots, e_n\}, \quad \{x.\ P\ x\}$
- ➜ $e \in A, \quad A \subseteq B$
- ➜ $A \cup B, \quad A \cap B, \quad A - B, \quad -A$
- ➜ $\bigcup x \in A.\ B\ x, \quad \bigcap x \in A.\ B\ x, \quad \bigcap A, \quad \bigcup A$
- ➜ $\{i..j\}$
- ➜ insert :: $\alpha \Rightarrow \alpha$ set $\Rightarrow \alpha$ set
- ➜ $f`A \equiv \{y.\ \exists x \in A.\ y = f\ x\}$
- ➜ ...

PROOFS ABOUT SETS

Natural deduction proofs:

- ➜ equalityI: $[\![A \subseteq B;\ B \subseteq A]\!] \Longrightarrow A = B$
- ➜ subsetI: $(\bigwedge x.\ x \in A \Longrightarrow x \in B) \Longrightarrow A \subseteq B$
- ➜ ... (see Tutorial)

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BOUNDED QUANTIFIERS

- ➜ $\forall x \in A.\ P\ x \equiv \forall x.\ x \in A \longrightarrow P\ x$
- ➜ $\exists x \in A.\ P\ x \equiv \exists x.\ x \in A \wedge P\ x$

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- ➜ ballI: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \Longrightarrow \forall x \in A.\ P\ x$
- ➜ bspec: $[\![\forall x \in A.\ P\ x; x \in A]\!] \Longrightarrow P\ x$
- ➜ bexI: $[\![P\ x; x \in A]\!] \Longrightarrow \exists x \in A.\ P\ x$
- ➜ bexE: $[\![\exists x \in A.\ P\ x; \bigwedge x.\ [\![x \in A; P\ x]\!] \Longrightarrow Q]\!] \Longrightarrow Q$

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DEMO: SETS

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INDUCTIVE DEFINITIONS

EXAMPLE

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$$\frac{}{\langle \mathsf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{[\![e]\!]\sigma = v}{\langle \mathsf{x} := \mathsf{e}, \sigma\rangle \longrightarrow \sigma[x \mapsto v]}$$

$$\frac{\langle c_1, \sigma\rangle \longrightarrow \sigma' \quad \langle c_2, \sigma'\rangle \longrightarrow \sigma''}{\langle c_1; c_2, \sigma\rangle \longrightarrow \sigma''}$$

$$\frac{[\![b]\!]\sigma = \mathsf{False}}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma\rangle \longrightarrow \sigma}$$

$$\frac{[\![b]\!]\sigma = \mathsf{True} \quad \langle c, \sigma\rangle \longrightarrow \sigma' \quad \langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma'\rangle \longrightarrow \sigma''}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma\rangle \longrightarrow \sigma''}$$

WHAT DOES THIS MEAN?

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- ➜ $\langle c, \sigma\rangle \longrightarrow \sigma'$ fancy syntax for a relation $(c, \sigma, \sigma') \in E$
- ➜ relations are sets: $E :: (\mathsf{com} \times \mathsf{state} \times \mathsf{state})\ \mathsf{set}$
- ➜ the rules define a set inductively

But which set?

SIMPLER EXAMPLE

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

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- ➜ N is the set of natural numbers $\mathbb{N}$
- ➜ But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n+1 \in \mathbb{R}$
- ➜ $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

- ➜ Objective: **no junk**. Only what must be in X shall be in X.
- ➜ Gives rise to a nice proof principle (rule induction)
- ➜ Alternative (greatest set) occasionally also useful: coinduction

FORMALLY

Rules $\dfrac{a_1 \in X \quad \ldots \quad a_n \in X}{a \in X}$ with $a_1, \ldots, a_n, a \in A$

define set $X \subseteq A$

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Formally: set of rules $R \subseteq A\ \mathsf{set} \times A$ (R, X possibly infinite)

Applying rules R to a set B: $\hat{R}\, B \equiv \{x.\ \exists H.\ (H, x) \in R \wedge H \subseteq B\}$

Example:

$$\begin{array}{lcl} R & \equiv & \{(\{\}, 0)\} \cup \{(\{n\}, n+1).\ n \in \mathbb{R}\} \\ \hat{R}\,\{3, 6, 10\} & = & \{0, 4, 7, 11\} \end{array}$$

THE SET

Definition: B is R-closed iff $\hat{R}\ B \subseteq B$

Definition: X is the least R-closed subset of A

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This does always exist:

Fact: B_1 R-closed $\wedge$ B_2 R-closed $\Longrightarrow B_1 \cap B_2$ R-closed

Hence: $X = \bigcap\{B \subseteq A.\ B\ R\text{-closed}\}$

GENERATION FROM ABOVE

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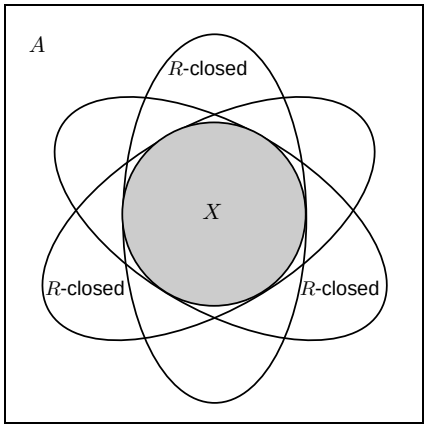

RULE INDUCTION

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

induces induction principle

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$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n+1)]\!] \Longrightarrow \forall x \in X.\ P\ x$$

In general:

$$\frac{\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

WHY DOES THIS WORK?

$$\frac{\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

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$\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a$
says
$\{x.\ Px\}$ is R-closed

but: X is the least R-closed set
hence: $X \subseteq \{x.\ P\ x\}$
which means: $\forall x \in X.\ P\ x$

qed

Rules with side conditions

$$\frac{a_1 \in X \quad \ldots \quad a_n \in X \qquad C_1 \quad \ldots \quad C_m}{a \in X}$$

induction scheme:

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$$(\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \wedge C_1 \wedge \ldots \wedge C_m \wedge \{a_1, \ldots, a_n\} \subseteq X \Longrightarrow P\ a) \Longrightarrow \forall x \in X.\ P\ x$$

X as Fixpoint

How to compute X?
$X = \bigcap\{B \subseteq A.\ B\ R - \text{closed}\}$ hard to work with.
Instead: view X as least fixpoint, X least set with $\hat{R}\ X = X$.

Fixpoints can be approximated by iteration:

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$$X_0 = \hat{R}^0\ \{\} = \{\}$$
$$X_1 = \hat{R}^1\ \{\} = \text{rules without hypotheses}$$
$$\vdots$$
$$X_n = \hat{R}^n\ \{\}$$

$$X_\omega = \bigcup_{n \in \mathbb{N}}(R^n\ \{\}) = X$$

Generation from Below

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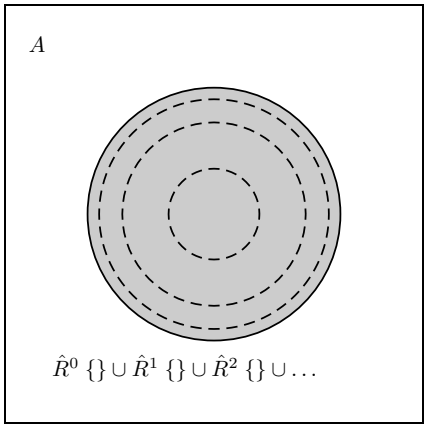

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Demo: inductive definitons

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ISAR

INDUCTIVE DEFINITION IN ISABELLE

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inductive S
intros
rule$_1$: "$[\![s \in S; A]\!] \Longrightarrow s' \in S$"
⋮
rule$_n$: ...

RULE INDUCTION

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show "$x \in S \Longrightarrow P\ x$"
proof (induct rule: S.induct)
  fix s and s' **assume** "$s \in S$" and "A" and "$P\ s$"
  ...
  show "$P\ s'$"
next
⋮
qed

ABBREVIATIONS

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show "$x \in S \Longrightarrow P\ x$"
proof (induct rule: S.induct)
  case rule$_1$
  ...
  show ?case
next
⋮
next
  case rule$_n$
  ...
  show ?case
qed

IMPLICIT SELECTION OF INDUCTION RULE

assume A: "$x \in S$"
⋮
show "$P\ x$"
using A **proof** induct
⋮
qed

lemma assumes A: "$x \in S$" **shows** "$P\ x$"
using A **proof** induct
⋮
qed

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RENAMING FREE VARIABLES IN RULE

case (rule$_i$ $x_1 \ldots x_k$)

Renames first k (alphabetically!) variables in rule$_i$ to $x_1 \ldots x_k$.

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A REMARK ON STYLE

➜ **case** (rule$_i$ x y) . . . **show** ?case
is easy to write and maintain

➜ **fix** x y **assume** $formula$ . . . **show** $formula'$
is easier to read:

- all information is shown locally
- no contextual references (e.g. ?case)

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DEMO

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WE HAVE SEEN TODAY ...

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- ➜ Sets in Isabelle
- ➜ Inductive Definitions
- ➜ Rule induction
- ➜ Fixpoints
- ➜ Isar: induct and cases

EXERCISES

Formalize this lecture in Isabelle:

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- ➜ Define **closed** $f\ A :: (\alpha \text{ set} \Rightarrow \alpha \text{ set}) \Rightarrow \alpha \text{ set} \Rightarrow \text{bool}$
- ➜ Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)
- ➜ Define **lfpt** f as the intersection of all f-closed sets
- ➜ Show that lfpt f is a fixpoint of f if f is monotone
- ➜ Show that lfpt f is the least fixpoint of f
- ➜ Declare a constant $R :: (\alpha \text{ set} \times \alpha) \text{ set}$
- ➜ Define $\hat{R} :: \alpha \text{ set} \Rightarrow \alpha \text{ set}$ in terms of R
- ➜ Show soundness of rule induction using R and lfpt $\hat{R}$