NICTA Advanced Course

Theorem Proving

Principles, Techniques, Applications

$$\{\}$$

CONTENT

- ➜ Intro & motivation, getting started with Isabelle
- ➜ Foundations & Principles
  - Lambda Calculus
  - Higher Order Logic, natural deduction
  - Term rewriting
- ➜ **Proof & Specification Techniques**
  - **Inductively defined sets, rule induction**
  - Datatypes, recursion, induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

LAST TIME

➜ Conditional term rewriting

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- ➜ Congruence and AC rules

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- ➜ Isar: fix, obtain, abbreviations, moreover, ultimately

SETS IN ISABELLE

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- $f`A \equiv \{y.\ \exists x \in A.\ y = f\ x\}$
- $\ldots$

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Natural deduction proofs:

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- . . .   (see Tutorial)

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- ballI: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \Longrightarrow \forall x \in A.\ P\ x$
- bspec: $[\![\forall x \in A.\ P\ x; x \in A]\!] \Longrightarrow P\ x$

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- ➜ bexI: $[\![P\ x; x \in A]\!] \Longrightarrow \exists x \in A.\ P\ x$
- ➜ bexE: $[\![\exists x \in A.\ P\ x; \bigwedge x.\ [\![x \in A; P\ x]\!] \Longrightarrow Q]\!] \Longrightarrow Q$

Demo: Sets

INDUCTIVE DEFINITIONS

EXAMPLE

$$\frac{}{\langle \mathsf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{[\![e]\!]\sigma = v}{\langle \mathsf{x} := \mathsf{e}, \sigma\rangle \longrightarrow \sigma[x \mapsto v]}$$

$$\frac{\langle c_1, \sigma\rangle \longrightarrow \sigma' \quad \langle c_2, \sigma'\rangle \longrightarrow \sigma''}{\langle c_1; c_2, \sigma\rangle \longrightarrow \sigma''}$$

$$\frac{[\![b]\!]\sigma = \mathsf{False}}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma\rangle \longrightarrow \sigma}$$

$$\frac{[\![b]\!]\sigma = \mathsf{True} \quad \langle c, \sigma\rangle \longrightarrow \sigma' \quad \langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma'\rangle \longrightarrow \sigma''}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma\rangle \longrightarrow \sigma''}$$

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But which set?

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

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Why the smallest set?

- ➜ Objective: **no junk**. Only what must be in X shall be in X.
- ➜ Gives rise to a nice proof principle (rule induction)
- ➜ Alternative (greatest set) occasionally also useful: coinduction

Formally

Rules $\dfrac{a_1 \in X \quad \ldots \quad a_n \in X}{a \in X}$ with $a_1, \ldots, a_n, a \in A$

define set $X \subseteq A$

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Example:

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\begin{aligned}
R \quad &\equiv \quad \{(\{\}, 0)\} \cup \{(\{n\}, n+1).\ n \in \mathbb{R}\} \\
\hat{R}\, \{3, 6, 10\} \quad &=
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$$\begin{array}{lcl} R & \equiv & \{(\{\}, 0)\} \cup \{(\{n\}, n+1).\ n \in \mathbb{R}\} \\ \hat{R}\,\{3, 6, 10\} & = & \{0, 4, 7, 11\} \end{array}$$

THE SET

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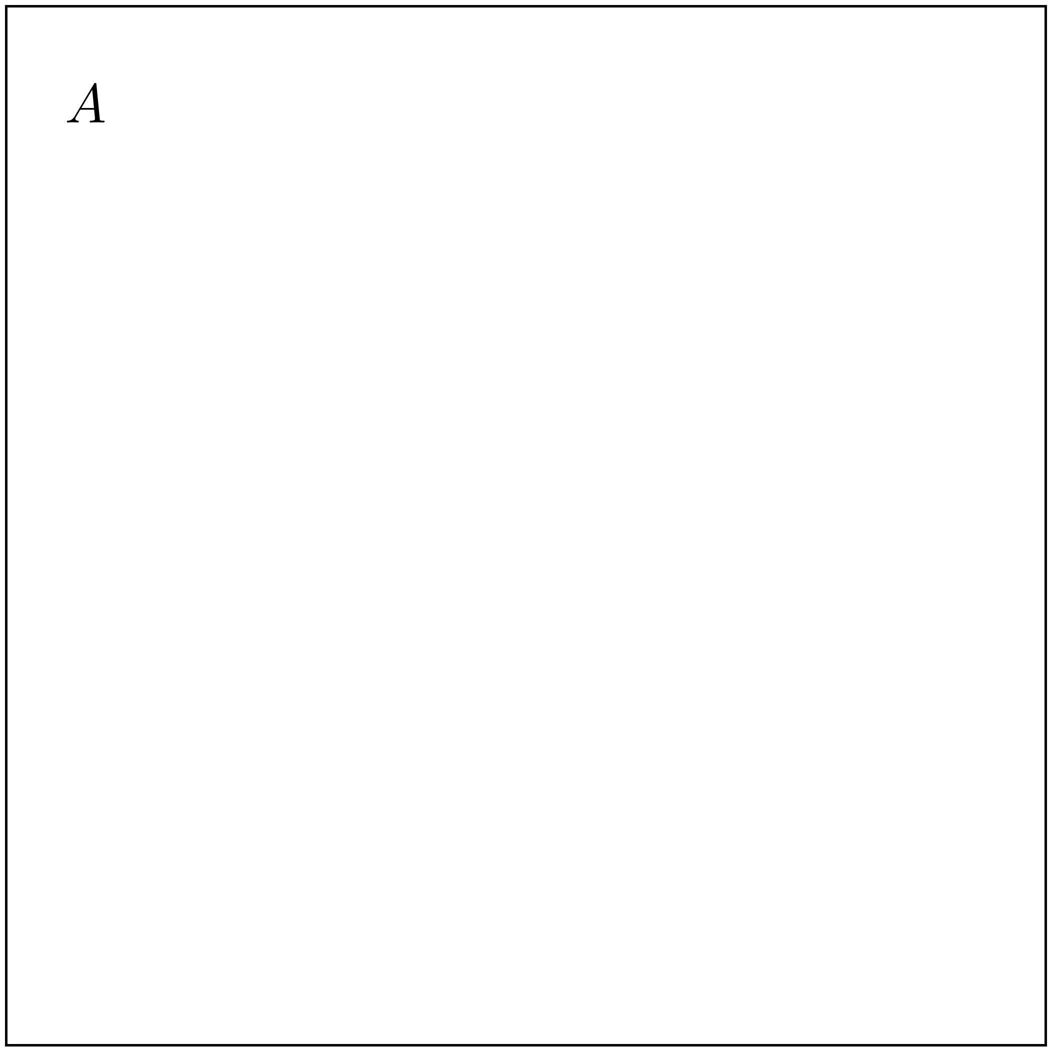
This does always exist:

Fact: B_1 R-closed $\wedge$ B_2 R-closed $\Longrightarrow B_1 \cap B_2$ R-closed

Hence: $X = \bigcap\{B \subseteq A.\ B\ R{-}\text{closed}\}$

Generation from Above



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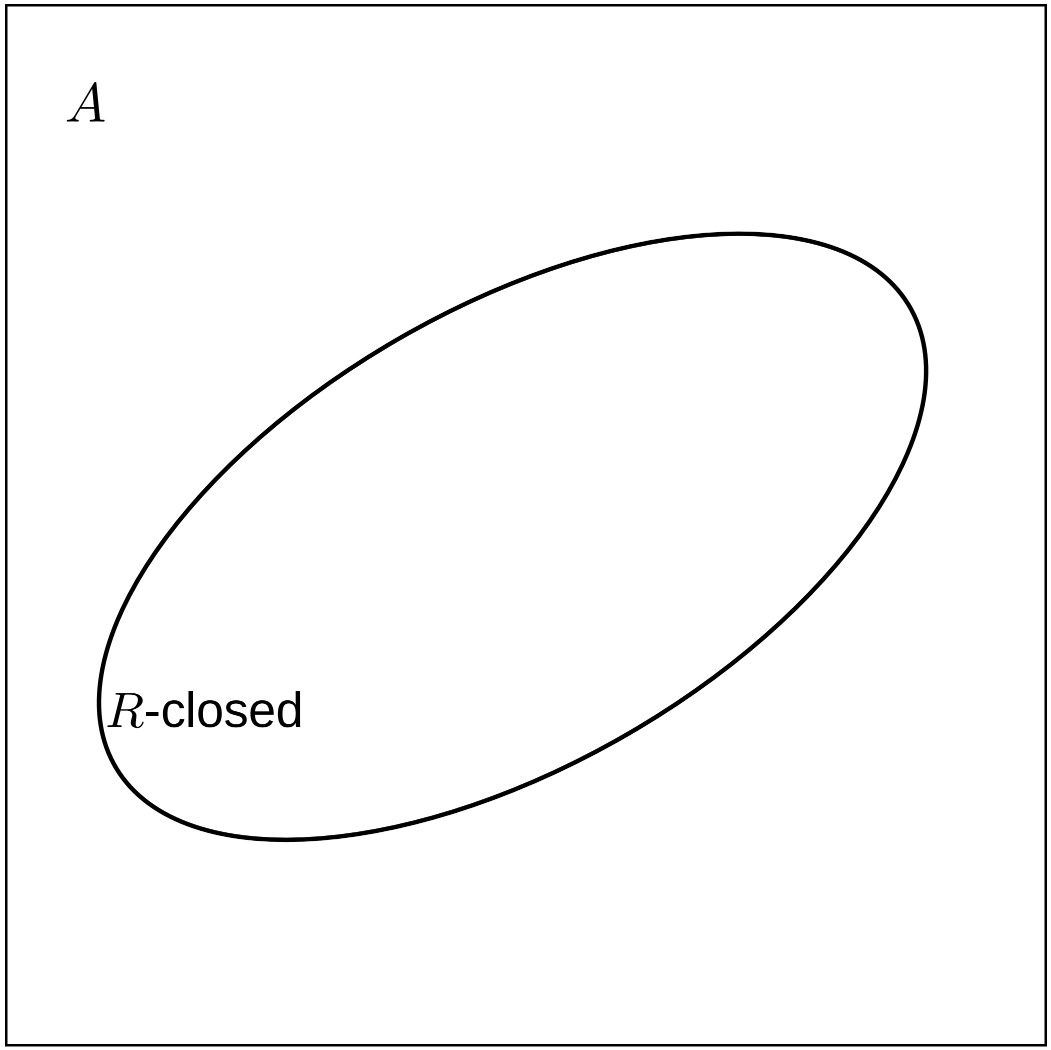

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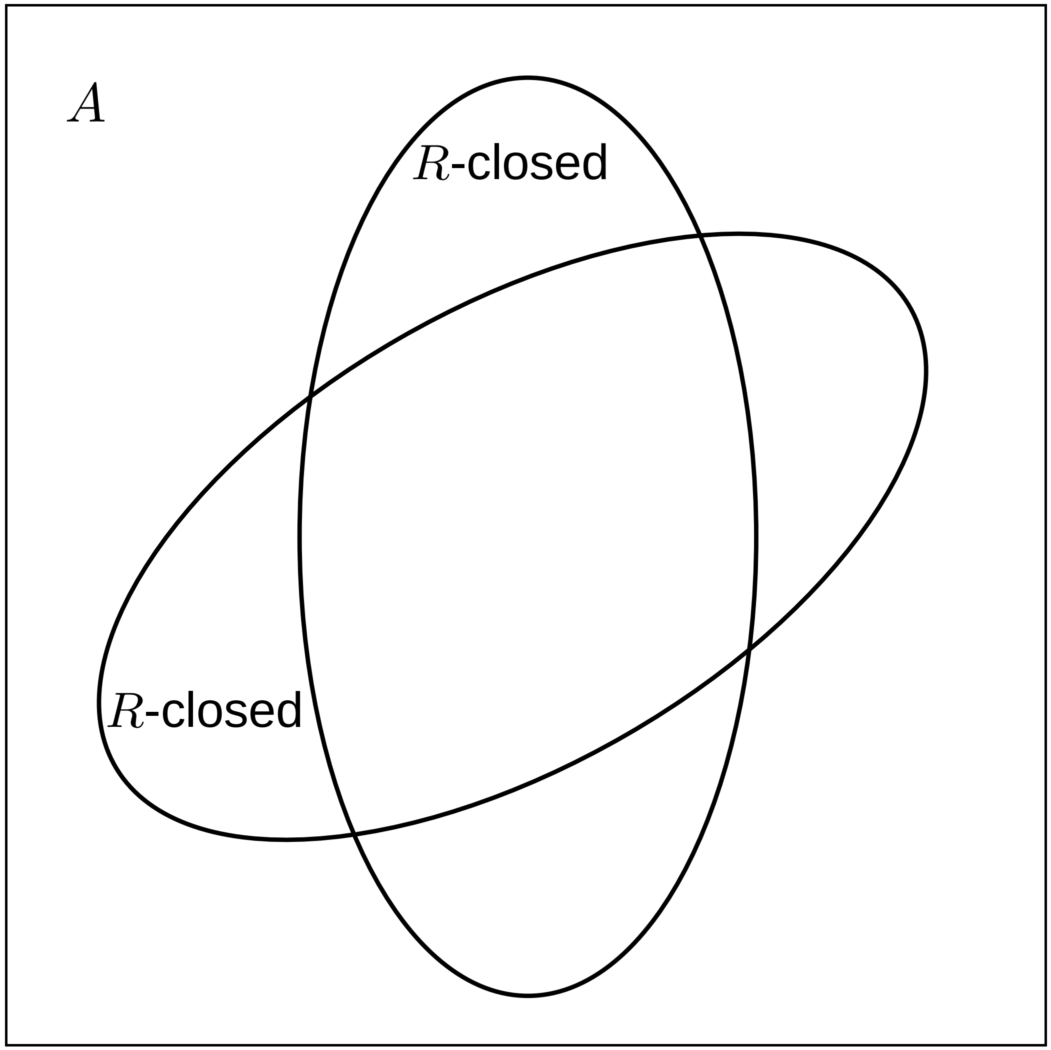

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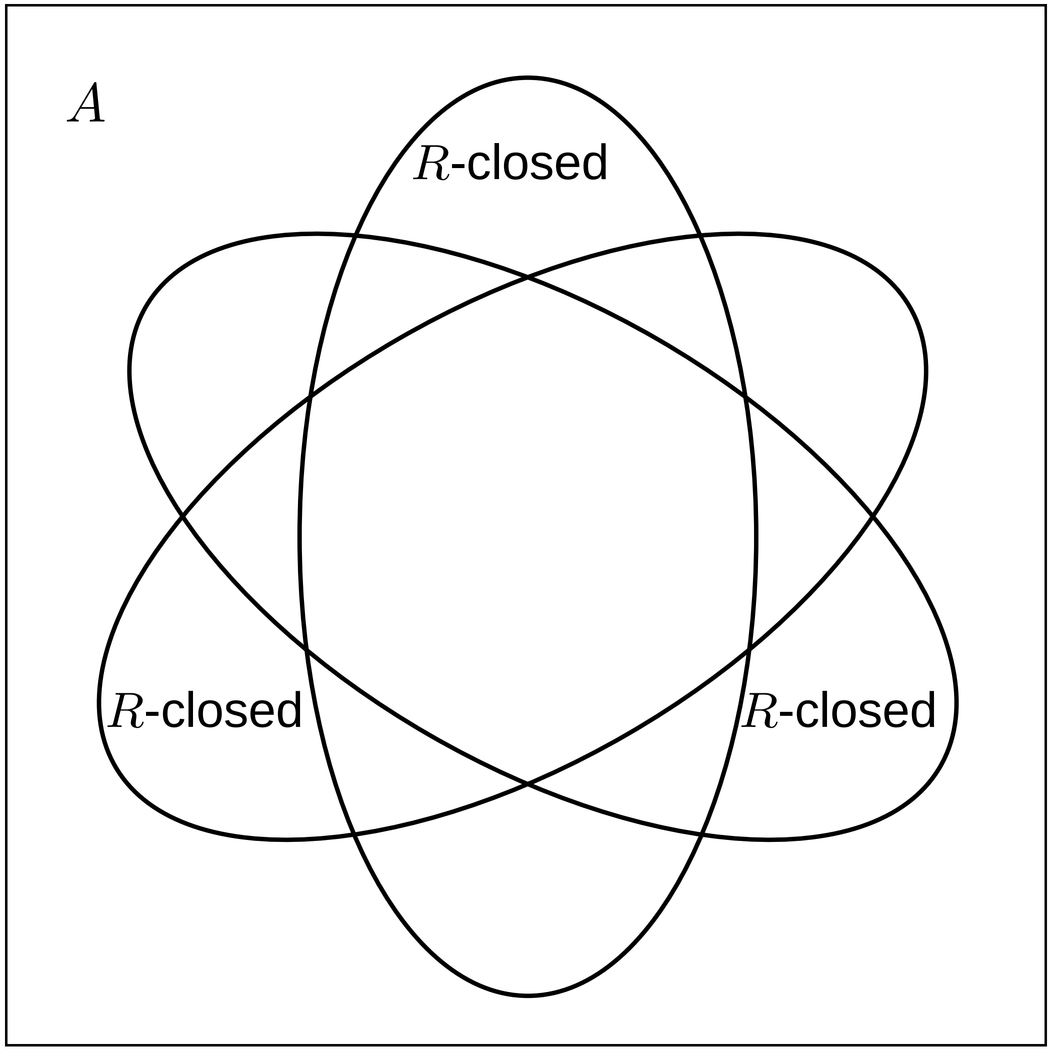

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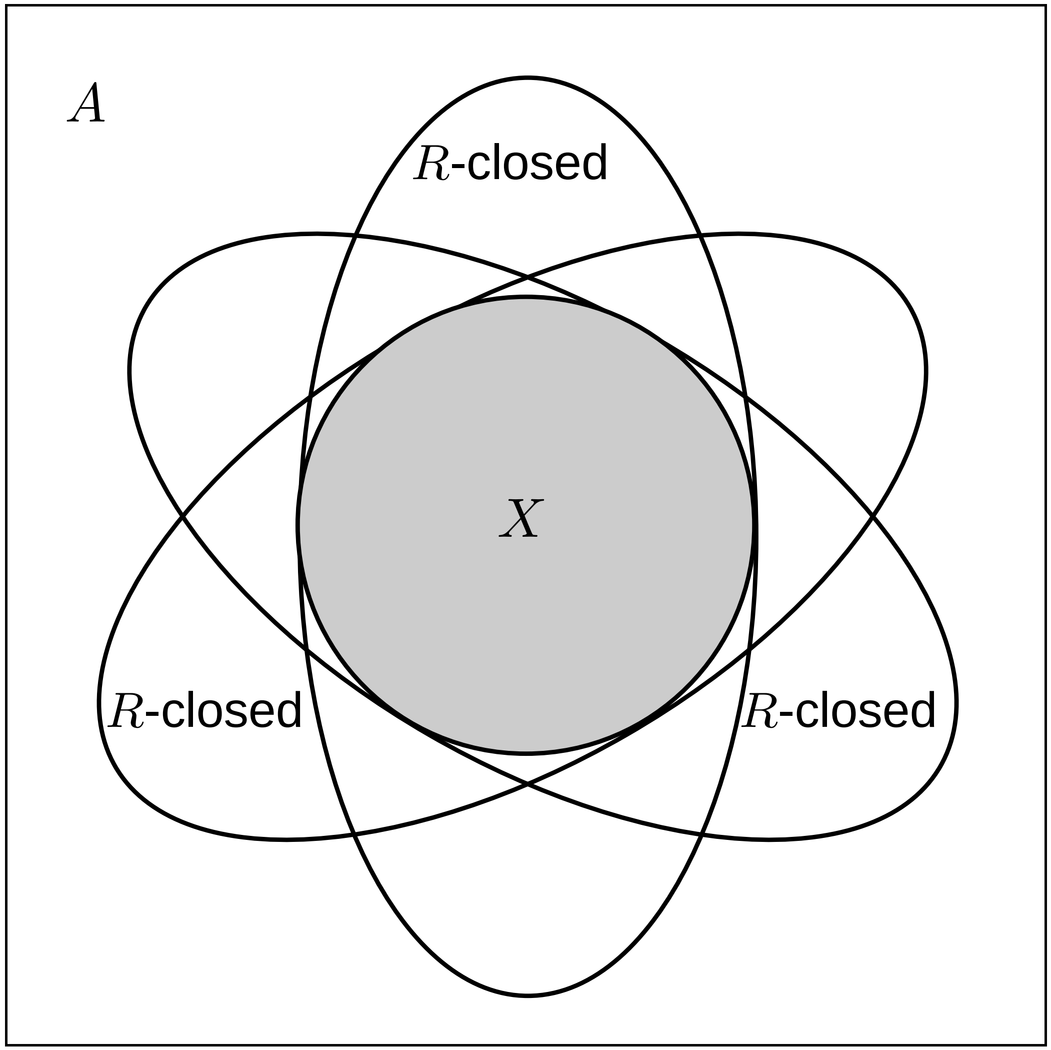

RULE INDUCTION

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

induces induction principle

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In general:

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

WHY DOES THIS WORK?

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Rules with side conditions

$$\frac{a_1 \in X \quad \ldots \quad a_n \in X \qquad C_1 \quad \ldots \quad C_m}{a \in X}$$

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induction scheme:

$$(\forall (\{a_1, \dots a_n\}, a) \in R.\ P\ a_1 \wedge \dots \wedge P\ a_n \wedge$$
$$C_1 \wedge \dots \wedge C_m \wedge$$
$$\{a_1, \dots, a_n\} \subseteq X \Longrightarrow P\ a)$$

$$\Longrightarrow$$

$$\forall x \in X.\ P\ x$$

X AS FIXPOINT

How to compute X?

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$X = \bigcap\{B \subseteq A.\ B\ R-\text{closed}\}$ hard to work with.

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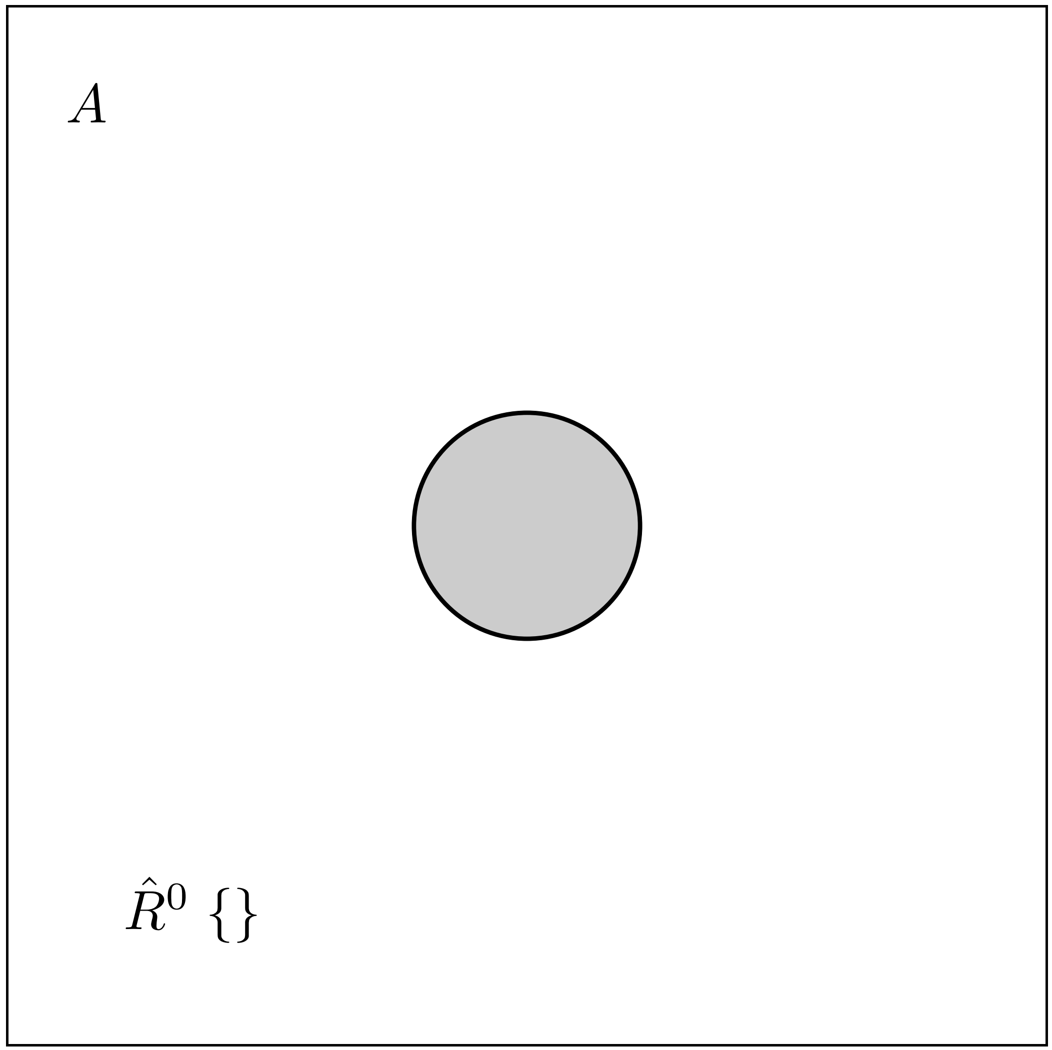
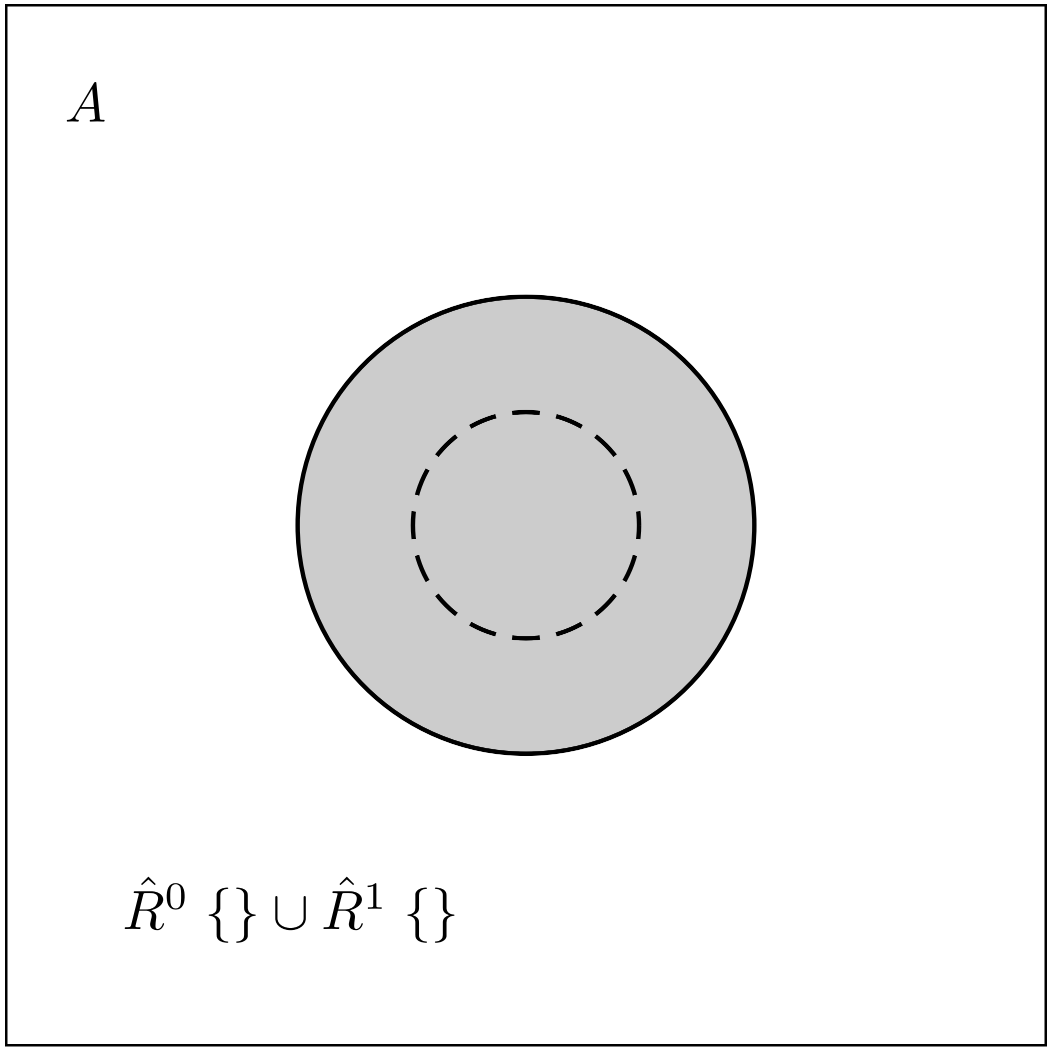
$$X_\omega = \bigcup_{n \in \mathbb{N}} (R^n\ \{\}) = X$$

Generation from Below



GENERATION FROM BELOW



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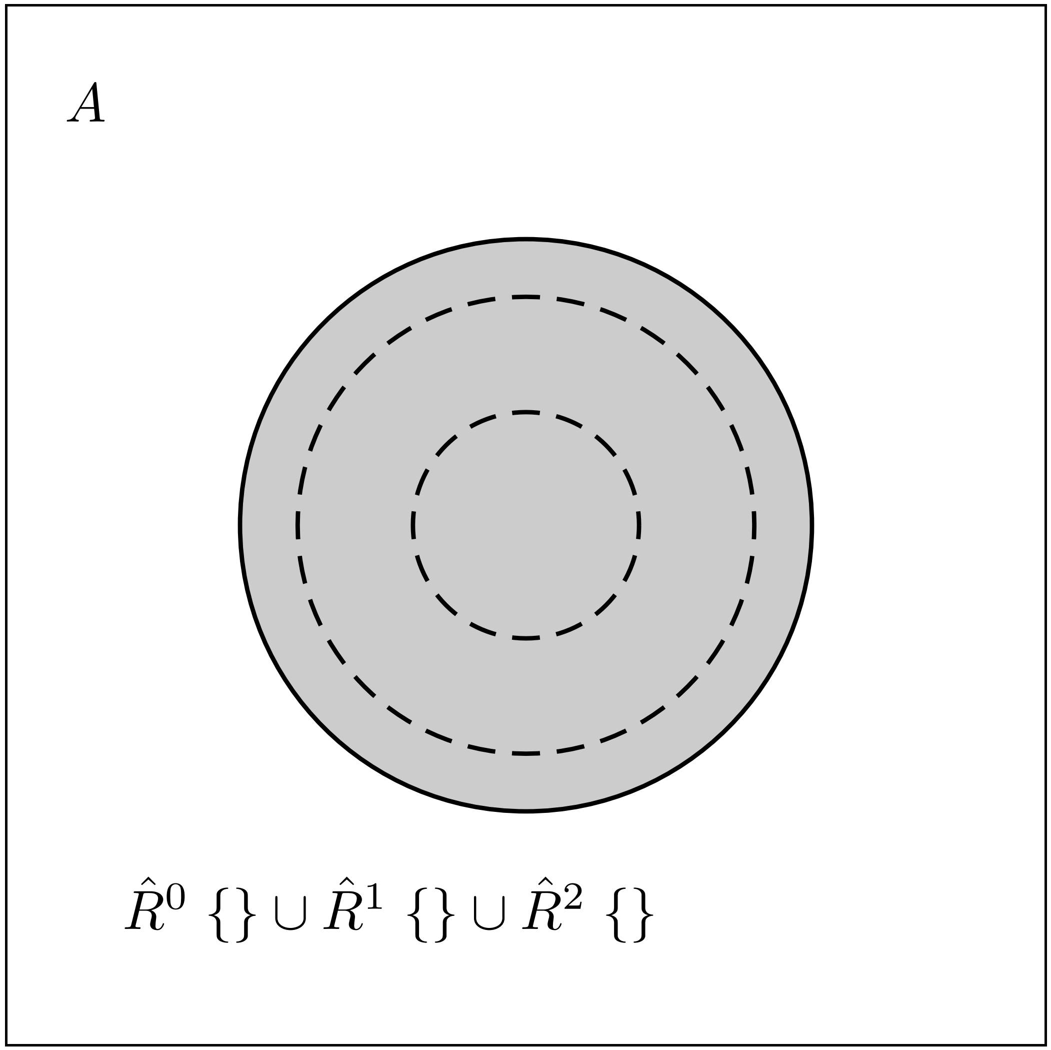

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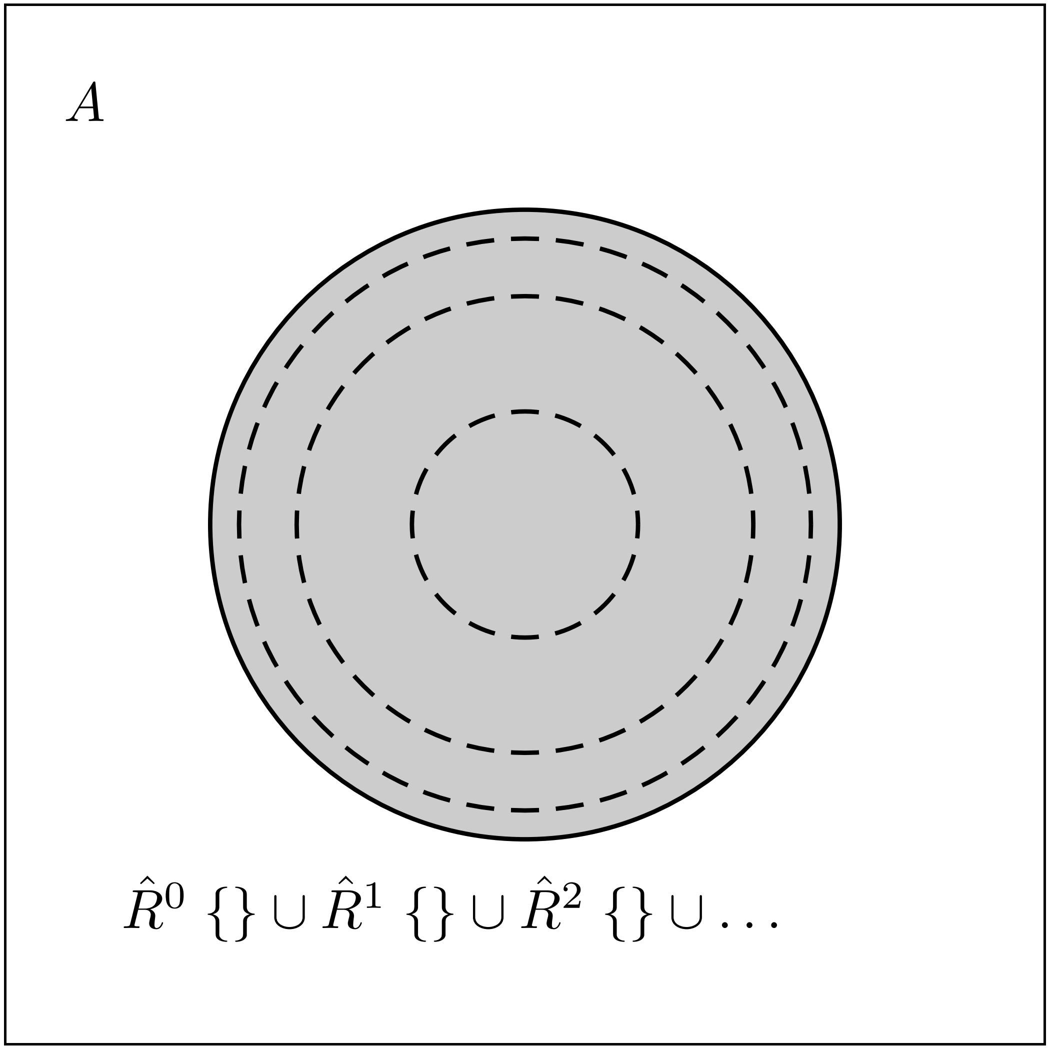

DEMO: INDUCTIVE DEFINITONS

ISAR

INDUCTIVE DEFINITION IN ISABELLE

inductive S

intros

rule$_1$: "$[\![s \in S; A]\!] \Longrightarrow s' \in S$"

$\vdots$

rule$_n$: $\ldots$

Rule induction

show "$x \in S \Longrightarrow P\ x$"

proof (induct rule: S.induct)

  fix s and s' **assume** "$s \in S$" and "A" and "$P\ s$"

  ...

  show "$P\ s'$"

next

⋮

qed

Abbreviations

show "$x \in S \Longrightarrow P\ x$"
proof (induct rule: S.induct)
  case rule$_1$
  . . .
  show ?case
next
⋮
next
  case rule$_n$
  . . .
  show ?case
qed

Implicit selection of induction rule

assume A: "$x \in S$"

⋮

show "$P\ x$"

using A **proof** induct

⋮

qed

IMPLICIT SELECTION OF INDUCTION RULE

assume A: "$x \in S$"

⋮

show "$P\ x$"

using A **proof** induct

⋮

qed

lemma assumes A: "$x \in S$" **shows** "$P\ x$"

using A **proof** induct

⋮

qed

Renaming free variables in rule

case (rule$_i$ $x_1 \ldots x_k$)

Renames first k (alphabetically!) variables in rule$_i$ to $x_1 \ldots x_k$.

A REMARK ON STYLE

➜ **case** (rule$_i$ x y) . . . **show** ?case
is easy to write and maintain

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➜ **case** (rule$_i$ x y) . . . **show** ?case
is easy to write and maintain

➜ **fix** x y **assume** $formula$ . . . **show** $formula'$
is easier to read:

- all information is shown locally
- no contextual references (e.g. ?case)

DEMO

We have seen today ...

➜ Sets in Isabelle

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- ➜ Sets in Isabelle
- ➜ Inductive Definitions

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- ➜ Fixpoints

We have seen today ...

- ➜ Sets in Isabelle
- ➜ Inductive Definitions
- ➜ Rule induction
- ➜ Fixpoints
- ➜ Isar: induct and cases

EXERCISES

Formalize this lecture in Isabelle:

➜ Define **closed** $f\ A :: (\alpha \text{ set} \Rightarrow \alpha \text{ set}) \Rightarrow \alpha \text{ set} \Rightarrow \text{bool}$

➜ Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)

➜ Define **lfpt** f as the intersection of all f-closed sets

➜ Show that lfpt f is a fixpoint of f if f is monotone

➜ Show that lfpt f is the least fixpoint of f

➜ Declare a constant $R :: (\alpha \text{ set} \times \alpha) \text{ set}$

➜ Define $\hat{R} :: \alpha \text{ set} \Rightarrow \alpha \text{ set}$ in terms of R

➜ Show soundness of rule induction using R and lfpt $\hat{R}$