NICTA Advanced Course

Theorem Proving

Principles, Techniques, Applications

$$\longrightarrow$$

CONTENT

- Intro & motivation, getting started with Isabelle
- **Foundations & Principles**
  - Lambda Calculus
  - **Higher Order Logic, natural deduction**
  - **Term rewriting**
- Proof & Specification Techniques
  - Inductively defined sets, rule induction
  - Datatypes, recursion, induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

LAST TIME ON HOL

➜ Defining HOL

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➜ Higher Order Abstract Syntax

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- ➜ More automation

THE THREE BASIC WAYS OF INTRODUCING THEOREMS

➜ **Axioms**:

Expample: **axioms** refl: "$t = t$"

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  The harder, but safe choice.

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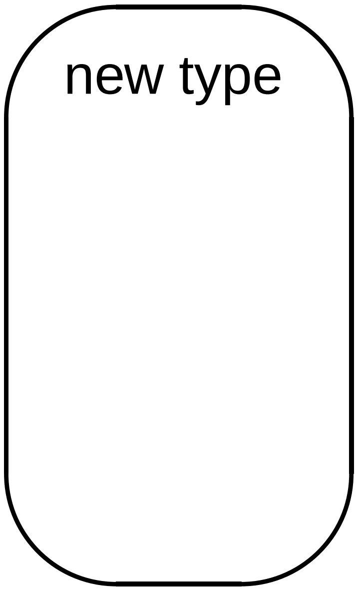
- **typedef**: by definiton as a set

  Example: **typdef** new_type = "$\{$some set$\}$" $<$proof$>$

  Introduces a new type as a subset of an existing type.

  The proof shows that the set on the rhs in non-empty.

How typedef Works



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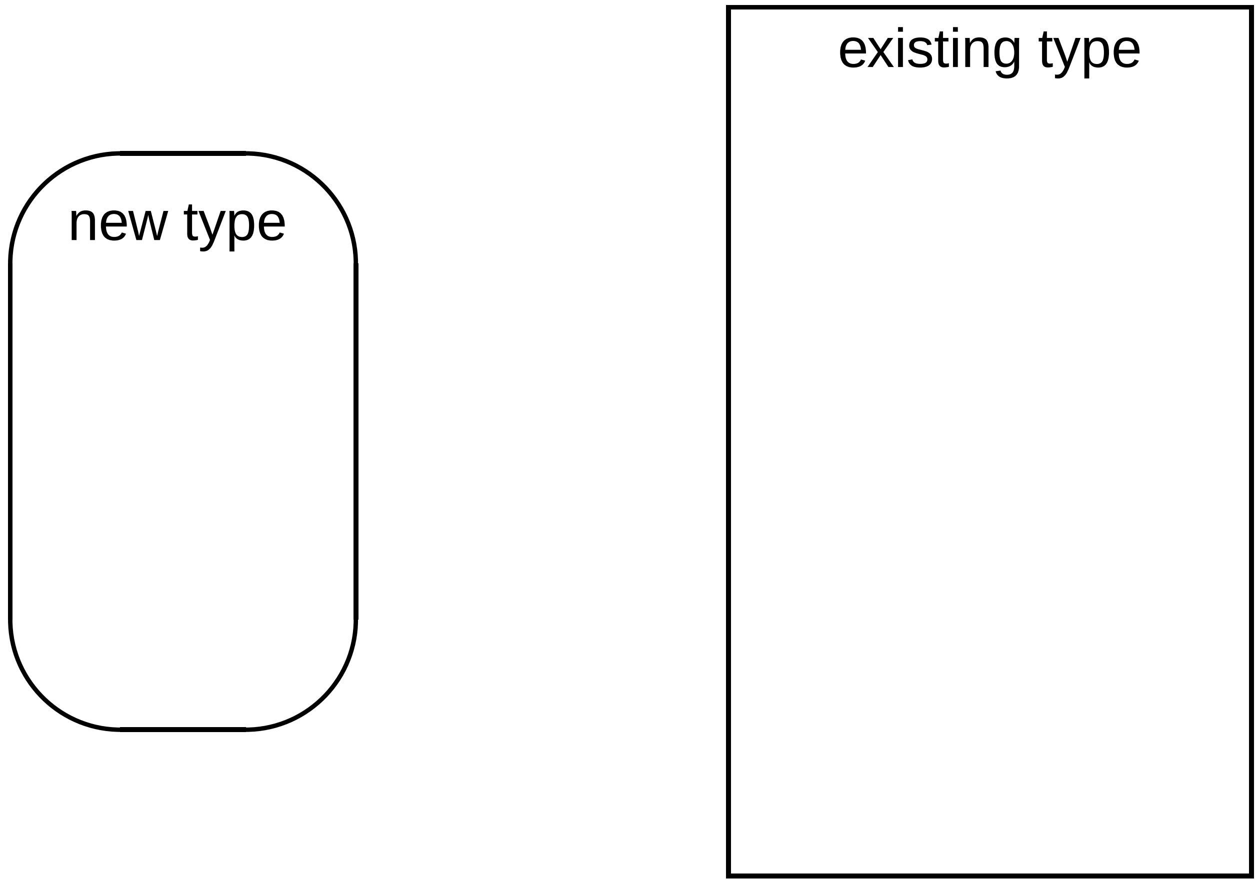

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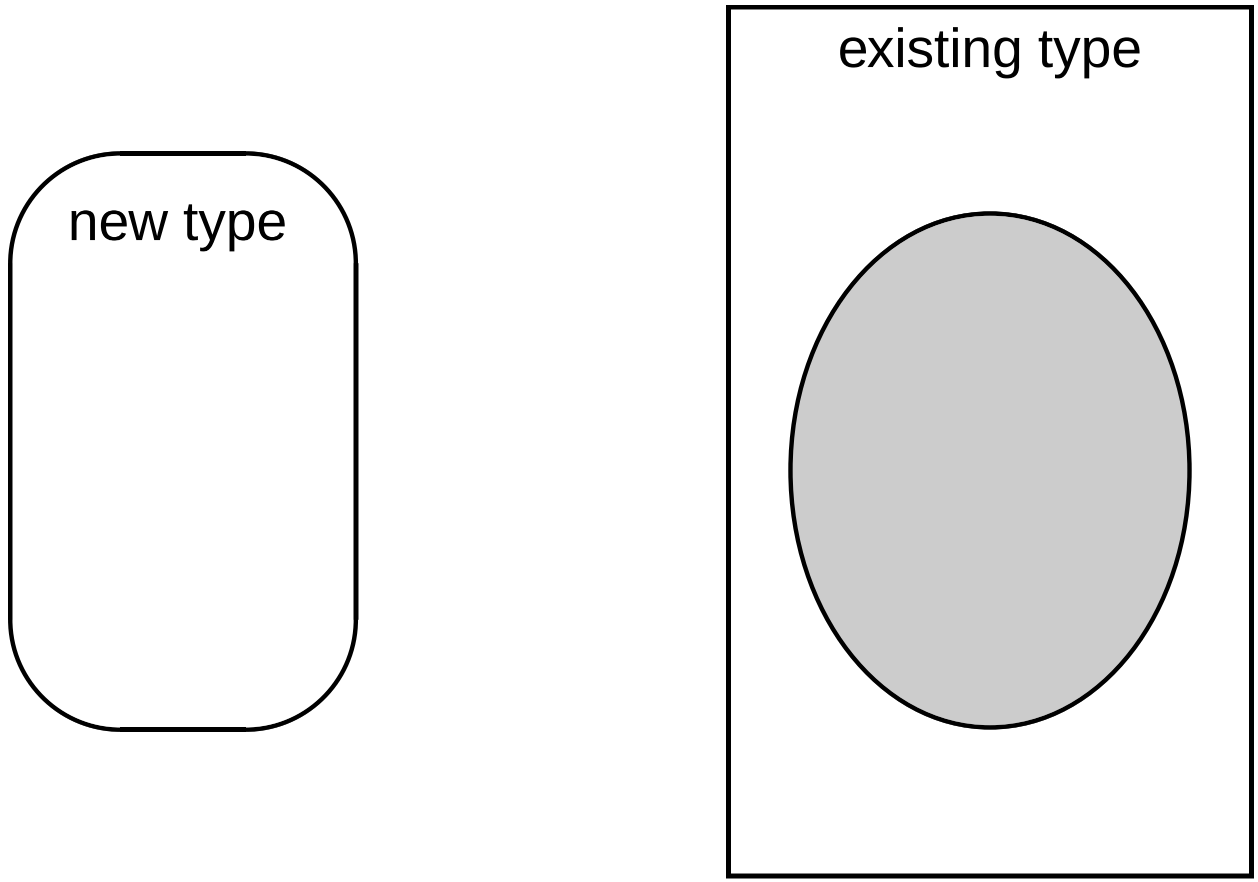

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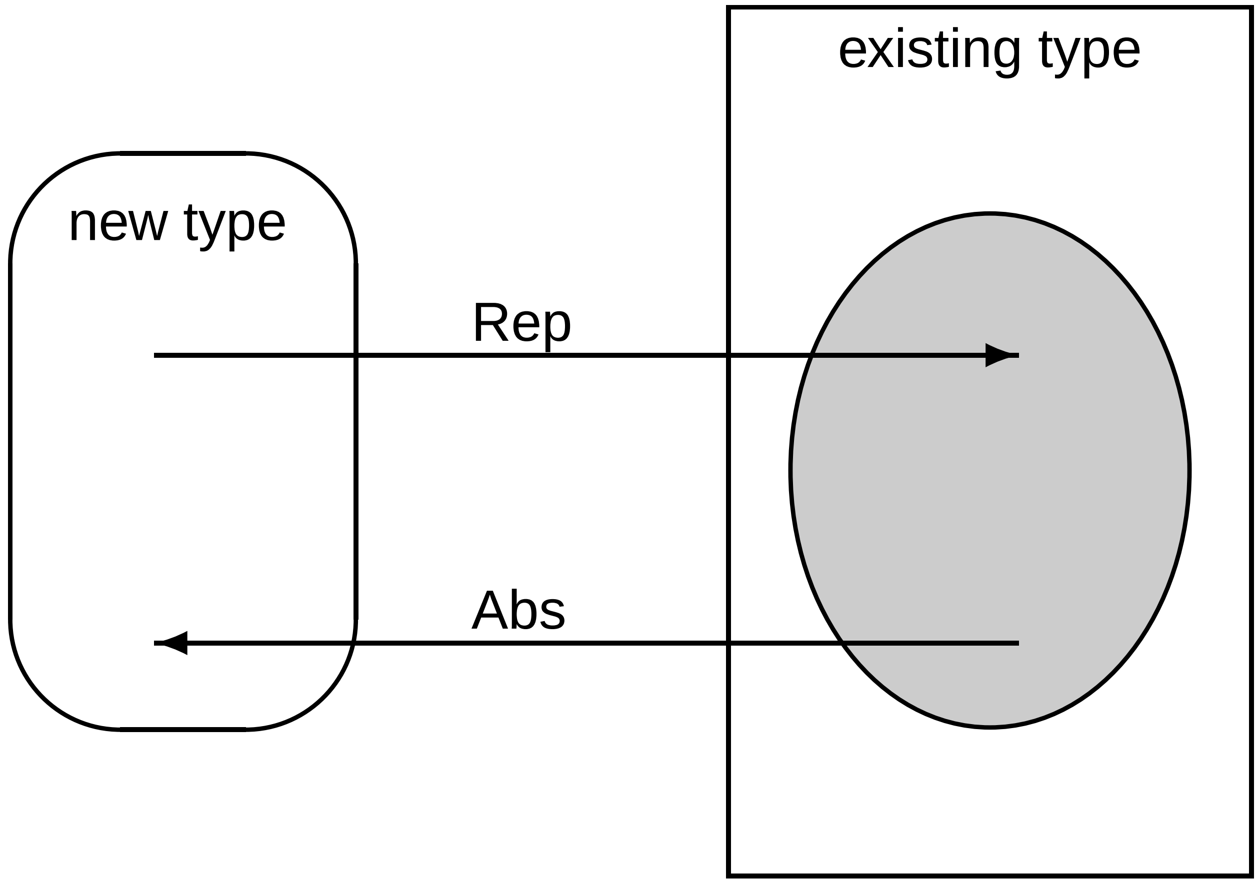

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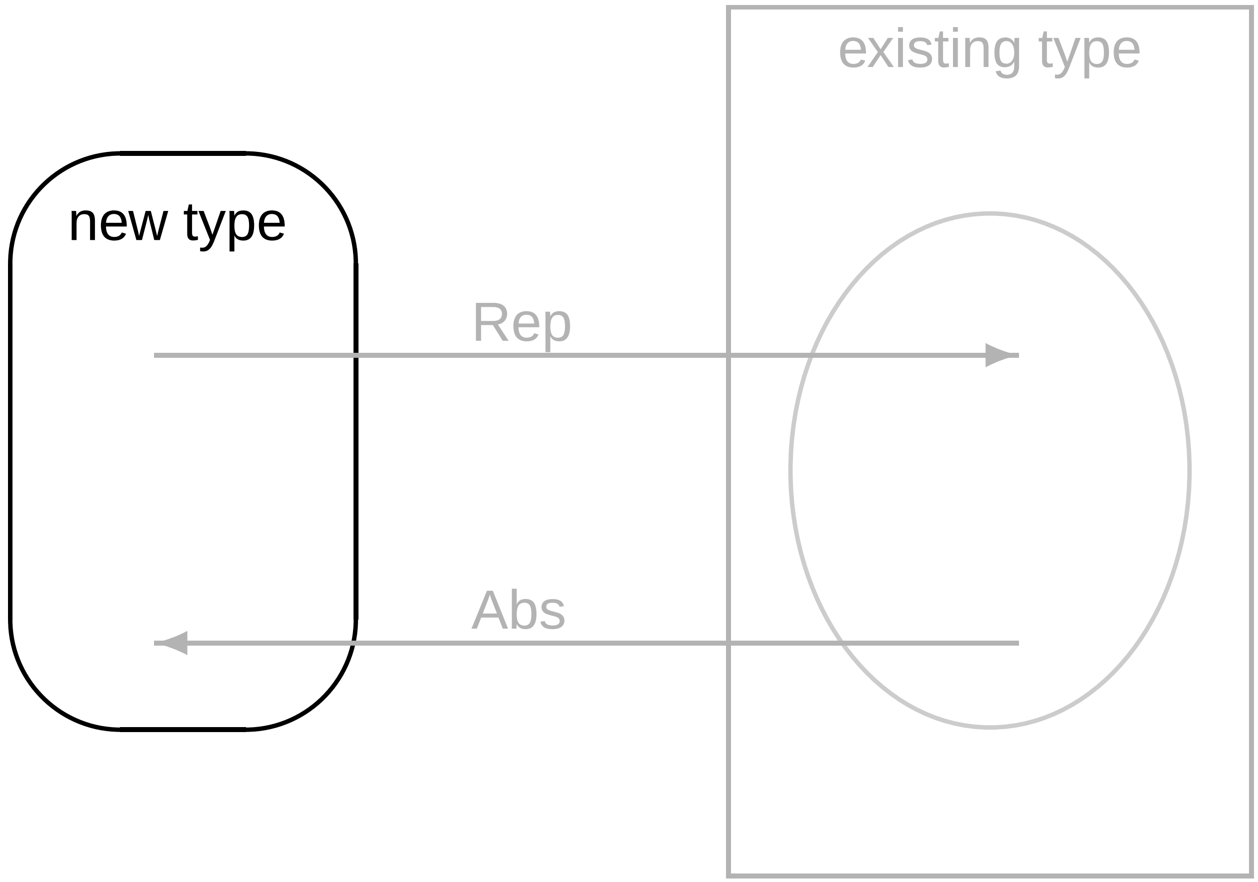

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④ We now can:

- define constants Pair, fst, snd in terms of Abs_Prod and Rep_Prod
- derive all characteristic theorems
- forget about Rep/Abs, use characteristic theorems instead

Demo: Introducting new Types

Term Rewriting

The Problem

Given a set of equations

$$l_1 = r_1$$

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Applications in:

➜ **Mathematics** (algebra, group theory, etc)

➜ **Functional Programming** (model of execution)

➜ **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$l_1 \longrightarrow r_1$$

$$l_2 \longrightarrow r_2$$

$$\vdots$$

$$l_n \longrightarrow r_n$$

decide $l = r$ **by deciding** $l \stackrel{*}{\longleftrightarrow} r$

Arrow Cheat Sheet

$$\xrightarrow{0} \quad = \quad \{(x,y) | x = y\} \qquad \text{identity}$$

ARROW CHEAT SHEET

$\xrightarrow{0}$	$=$	$\{(x,y)\|x=y\}$	identity
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How to Decide $l \stackrel{*}{\longleftrightarrow} r$

Same idea as for β:

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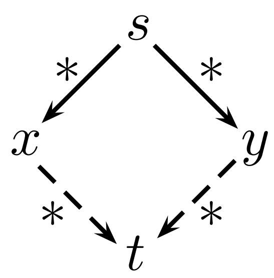
Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.



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is a given set of reduction rules confluent?

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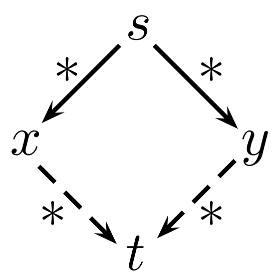

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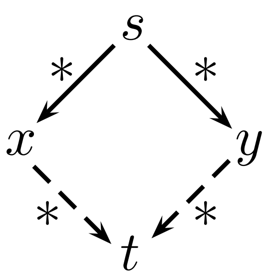

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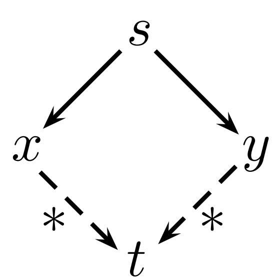

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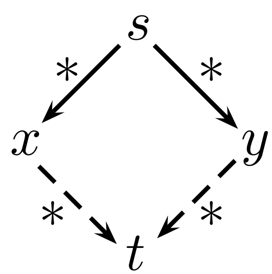

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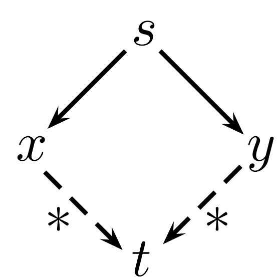

Fact: local confluence and termination $\Longrightarrow$ confluence

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$\longrightarrow$ is **terminating** if there are no infinite reduction chains

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① $g\ x <_r f\ (g\ x)$ and $f\ x <_r g\ (f\ x)$
② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

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confluence:	not guaranteed (result may depend on which rule is used first)

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  apply (simp add: <rules>) and **apply** (simp del: <rules>)

- ➜ Using only the specified set of equations:
  apply (simp only: <rules>)

Demo

Isar

A Language for Structured Proofs

apply scripts

➜ unreadable

apply scripts

➜ unreadable

➜ hard to maintain

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- ➜ Elegance?

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- ➜ Explaining deeper insights?

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apply scripts	**What about..**
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What about..

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- ➜ Explaining deeper insights?
- ➜ Large developments?

Isar!

A TYPICAL ISAR PROOF

proof

  assume $formula_0$

  have $formula_1$ **by** simp

  $\vdots$

  have $formula_n$ **by** blast

  show $formula_{n+1}$ **by** $\ldots$

qed

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proves $formula_0 \Longrightarrow formula_{n+1}$

proof

  assume $formula_0$

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  show $formula_{n+1}$ **by** $\dots$

qed

proves $formula_0 \Longrightarrow formula_{n+1}$

(analogous to **assumes/shows** in lemma statements)

ISAR CORE SYNTAX

proof = **proof** [method] statement* **qed**

| **by** method

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method = (simp . . .) | (blast . . .) | (rule . . .) | . . .

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statement = **fix** variables $\quad\quad$ ($\bigwedge$)

$\quad\quad$ | **assume** proposition $\quad\quad$ ($\Longrightarrow$)

$\quad\quad$ | [**from** name$^+$] (**have** | **show**) proposition proof

$\quad\quad$ | **next** $\quad\quad$ (separates subgoals)

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proposition = [name:] formula

PROOF AND QED

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"

PROOF AND QED

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➜ **proof** ($<$method$>$) applies method to the stated goal

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➜	**proof** ($<$method$>$)	applies method to the stated goal
➜	**proof**	applies a single rule that fits
➜	**proof -**	does nothing to the goal

How do I know what to Assume and Show?

Look at the proof state!

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- so we need 2 shows: **show** "A" and **show** "B"
- We are allowed to **assume** A,
  because A is in the assumptions of the proof state.

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goal has been stated, proof needs to follow.

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proof (rule conjI) **[state]**
  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** **. . .**

HAVE

Can be used to make intermediate steps.

Example:

HAVE

Can be used to make intermediate steps.

Example:

lemma "$(x :: \mathsf{nat}) + 1 = 1 + x$"

proof -

have A: "$x + 1 = \mathsf{Suc}\ x$" **by** simp

have B: "$1 + x = \mathsf{Suc}\ x$" **by** simp

show "$x + 1 = 1 + x$" **by** (simp only: A B)

qed

Demo: Isar Proofs

We have learned today ...

➜ Introducing new Types

WE HAVE LEARNED TODAY ...

- ➜ Introducing new Types
- ➜ Equations and Term Rewriting

We have learned today ...

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WE HAVE LEARNED TODAY ...

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- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle
- ➜ First structured proofs (Isar)

EXERCISES

- ➜ use **typedef** to define a new type v with exactly one element.
- ➜ define a constant u of type v
- ➜ show that every element of v is equal to u
- ➜ design a set of rules that turns formulae with $\wedge, \vee, \longrightarrow, \neg$
  into disjunctive normal form
  (= disjunction of conjunctions with negation only directly on variables)
- ➜ prove those rules in Isabelle
- ➜ use **simp only** with these rules on $(\neg B \longrightarrow C) \longrightarrow A \longrightarrow B$