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NICTA Advanced Course

Theorem Proving
Principles, Techniques, Applications

λ

CONTENT

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- ➜ Intro & motivation, getting started with Isabelle
- ➜ **Foundations & Principles**
  - **Lambda Calculus**
  - Higher Order Logic, natural deduction
  - Term rewriting
- ➜ Proof & Specification Techniques
  - Datatypes, recursion, induction
  - Inductively defined sets, rule induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

λ-CALCULUS

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Alonzo Church

- ➜ lived 1903–1995
- ➜ supervised people like Alan Turing, Stephen Kleene
- ➜ famous for Church-Turing thesis, lambda calculus, first undecidability results
- ➜ invented λ calculus in 1930's

λ-calculus

- ➜ originally meant as foundation of mathematics
- ➜ important applications in theoretical computer science
- ➜ foundation of computability and functional programming

UNTYPED λ-CALCULUS

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- ➜ turing complete model of computation
- ➜ a simple way of writing down functions

Basic intuition:

instead of	$f(x) = x + 5$
write	$f = \lambda x.\ x + 5$

$\lambda x.\ x + 5$

- ➜ a term
- ➜ a nameless function
- ➜ that adds 5 to its parameter

FUNCTION APPLICATION

For applying arguments to functions

instead of $f(x)$
write $f\ x$

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Example: $(\lambda x.\ x + 5)\ a$

Evaluating: in $(\lambda x.\ t)\ a$ replace x by a in t
(computation!)

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

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THAT'S IT!

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NOW FORMAL

SYNTAX

Terms: $t \ ::= \ v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

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- ➜ v, x variables
- ➜ c constants
- ➜ $(t\ t)$ application
- ➜ $(\lambda x.\ t)$ abstraction

CONVENTIONS

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- ➜ leave out parentheses where possible
- ➜ list variables instead of multiple λ

Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

Rules:

- ➜ list variables: $\lambda x.\ (\lambda y.\ t)\ =\ \lambda x\ y.\ t$
- ➜ application binds to the left: $x\ y\ z\ =\ (x\ y)\ z \neq x\ (y\ z)$
- ➜ abstraction binds to the right: $\lambda x.\ x\ y\ =\ \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$
- ➜ leave out outermost parentheses

GETTING USED TO THE SYNTAX

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Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

$\lambda x\ y\ z.\ (x\ z)\ (y\ z) =$

$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$

$\lambda x.\ \lambda y.\ \lambda z.\ ((x\ z)\ (y\ z)) =$

$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

COMPUTATION

Intuition: replace parameter by argument
this is called β-reduction

Example

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$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$

$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$

$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$

$f\ 5$

DEFINING COMPUTATION

β reduction:

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$$
\begin{array}{lllll}
 & & (\lambda x.\ s)\ t & \longrightarrow_\beta & s[x \leftarrow t] \\
s \longrightarrow_\beta s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\
t \longrightarrow_\beta t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\
s \longrightarrow_\beta s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\beta & (\lambda x.\ s')
\end{array}
$$

Still to do: defi ne $s[x \leftarrow t]$

Defining Substitution

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Easy concept. Small problem: variable capture.

Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

Free Variables

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Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

Free variables FV of a term:

$$
\begin{aligned}
FV\ (x) &= \{x\} \\
FV\ (c) &= \{\} \\
FV\ (s\ t) &= FV(s) \cup FV(t) \\
FV\ (\lambda x.\ t) &= FV(t) \setminus \{x\}
\end{aligned}
$$

Example: $FV(\quad \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x \quad) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

Substitution

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$$
\begin{aligned}
x\ [x \leftarrow t] &= t \\
y\ [x \leftarrow t] &= y && \text{if } x \neq y \\
c\ [x \leftarrow t] &= c
\end{aligned}
$$

$$(s_1\ s_2)\ [x \leftarrow t] = (s_1[x \leftarrow t]\ s_2[x \leftarrow t])$$

$$
\begin{aligned}
(\lambda x.\ s)\ [x \leftarrow t] &= (\lambda x.\ s) \\
(\lambda y.\ s)\ [x \leftarrow t] &= (\lambda y.\ s[x \leftarrow t]) && \text{if } x \neq y \text{ and } y \notin FV(t) \\
(\lambda y.\ s)\ [x \leftarrow t] &= (\lambda z.\ s[y \leftarrow z][x \leftarrow t]) && \text{if } x \neq y \text{ and } z \notin FV(t) \cup FV(t)
\end{aligned}
$$

Substitution Example

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$$
\begin{aligned}
& (x\ \ (\lambda x.\ x)\ \ (\lambda y.\ z\ x))[x \leftarrow y] \\
= \ & (x[x \leftarrow y])\ \ ((\lambda x.\ x)[x \leftarrow y])\ \ ((\lambda y.\ z\ x)[x \leftarrow y]) \\
= \ & y\ (\lambda x.\ x)\ (\lambda y'.\ z\ y)
\end{aligned}
$$

α CONVERSION

Bound names are irrelevant:
$\lambda x.\ x$ and $\lambda y.\ y$ denote the same function.

α conversion:
$s =_\alpha t$ means $s = t$ up to renaming of bound variables.

Formally:

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$$(\lambda x.\ t) \longrightarrow_\alpha (\lambda y.\ t[x \leftarrow y]) \text{ if } y \notin FV(t)$$
$$s \longrightarrow_\alpha s' \Longrightarrow (s\ t) \longrightarrow_\alpha (s'\ t)$$
$$t \longrightarrow_\alpha t' \Longrightarrow (s\ t) \longrightarrow_\alpha (s\ t')$$
$$s \longrightarrow_\alpha s' \Longrightarrow (\lambda x.\ s) \longrightarrow_\alpha (\lambda x.\ s')$$

$s =_\alpha t$ iff $s \longrightarrow_\alpha^* t$
($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

α CONVERSION

Equality in Isabelle is equality modulo α conversion:

if $s =_\alpha t$ then s and t are syntactically equal.

Examples:

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$$\begin{array}{ll} & x\ (\lambda x\ y.\ x\ y) \\ =_\alpha & x\ (\lambda y\ x.\ y\ x) \\ =_\alpha & x\ (\lambda z\ y.\ z\ y) \\ \neq_\alpha & z\ (\lambda z\ y.\ z\ y) \\ \neq_\alpha & x\ (\lambda x\ x.\ x\ x) \end{array}$$

BACK TO β

We have defined β reduction: $\longrightarrow_\beta$

Some notation and concepts:

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- β **conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- t is reducible iff it contains a redex
- if it is not reducible, t is in **normal form**
- t **has a normal form** if there is an irreducible s such that $t \longrightarrow_\beta^* s$

DOES EVERY λ TERM HAVE A NORMAL FORM?

No!

Example:

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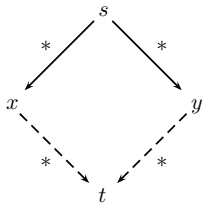
$$\begin{array}{l} (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \ldots \end{array}$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \longrightarrow_\beta \lambda y.\ y$)

λ calculus is not terminating

β REDUCTION IS CONFLUENT

Confluence: $s \longrightarrow_\beta^* x \wedge s \longrightarrow_\beta^* y \Longrightarrow \exists t.\ x \longrightarrow_\beta^* t \wedge y \longrightarrow_\beta^* t$



Order of reduction does not matter for result
Normal forms in λ calculus are unique

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β REDUCTION IS CONFLUENT

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta \lambda y.\ y$

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η CONVERSION

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

Defi nition:

$$\begin{array}{llllll} & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t & \text{if } x \notin FV(t) \\ s \longrightarrow_\eta s' \Longrightarrow & & (s\ t) & \longrightarrow_\eta & (s'\ t) & \\ t \longrightarrow_\eta t' \Longrightarrow & & (s\ t) & \longrightarrow_\eta & (s\ t') & \\ s \longrightarrow_\eta s' \Longrightarrow & & (\lambda x.\ s) & \longrightarrow_\eta & (\lambda x.\ s') & \end{array}$$

$s =_\eta t$ iff $\exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta (\lambda x.\ f\ x)\ g \longrightarrow_\eta f\ g$

- ➜ η reduction is confluent and terminating.
- ➜ $\longrightarrow_{\beta\eta}$ is confluent.
  $\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.
- ➜ **Equality in Isabelle is also modulo η conversion.**

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IN FACT ...

Equality in Isabelle is modulo α, β, and η conversion.

We will see next lecture why that is possible.

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So, what can you do with λ calculus?

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λ calculus is very expressive, you can encode:

- ➜ logic, set theory
- ➜ turing machines, functional programs, etc.

Examples:

$\texttt{true} \equiv \lambda x\ y.\ x$ $\qquad$ $\texttt{if true}\ x\ y \longrightarrow^*_\beta x$

$\texttt{false} \equiv \lambda x\ y.\ y$ $\qquad$ $\texttt{if false}\ x\ y \longrightarrow^*_\beta y$

$\texttt{if} \equiv \lambda z\ x\ y.\ z\ x\ y$

Now, `not`, `and`, `or`, etc is easy:

$\texttt{not} \equiv \lambda x.\ \texttt{if}\ x\ \texttt{false}\ \texttt{true}$
$\texttt{and} \equiv \lambda x\ y.\ \texttt{if}\ x\ y\ \texttt{false}$
$\texttt{or} \equiv \lambda x\ y.\ \texttt{if}\ x\ \texttt{true}\ y$

More Examples

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Encoding natural numbers (Church Numerals)

$0 \equiv \lambda f\ x.\ x$
$1 \equiv \lambda f\ x.\ f\ x$
$2 \equiv \lambda f\ x.\ f\ (f\ x)$
$3 \equiv \lambda f\ x.\ f\ (f\ (f\ x))$
$\ldots$

Numeral n is takes arguments f and x, applies f n-times to x.

$\texttt{iszero} \equiv \lambda n.\ n\ (\lambda x.\ \texttt{false})\ \texttt{true}$
$\texttt{succ} \equiv \lambda n\ f\ x.\ f\ (n\ f\ x)$
$\texttt{add} \equiv \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)$

Fix Points

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$(\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ \ t \longrightarrow_\beta$
$(\lambda f.\ f\ ((\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ f))\ \ t \longrightarrow_\beta$
$t\ \ ((\lambda x\ f.\ f\ (x\ x\ f))\ (\lambda x\ f.\ f\ (x\ x\ f))\ t)$

$\mu = (\lambda x f.\ f\ (x\ x\ f))\ (\lambda x f.\ f\ (x\ x\ f))$
$\mu\ t \longrightarrow_\beta t\ (\mu\ t) \longrightarrow_\beta t\ (t\ (\mu\ t)) \longrightarrow_\beta t\ (t\ (t\ (\mu\ t))) \longrightarrow_\beta \ldots$

$(\lambda x f.\ f\ (x\ x\ f))\ (\lambda x f.\ f\ (x\ x\ f))$ is Turing's fi x point operator

Nice, but ...

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As a mathematical foundation, λ does not work. **It is inconsistent.**

- ➜ **Frege** (Predicate Logic, $\sim$ 1879): allows arbitrary quantification over predicates
- ➜ **Russel** (1901): Paradox $R \equiv \{X | X \notin X\}$
- ➜ **Whitehead & Russel** (Principia Mathematica, 1910-1913): Fix the problem
- ➜ **Church** (1930): λ calculus as logic, `true`, `false`, $\wedge$, ... as λ terms

Problem:

with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x$	$x \in M \equiv M\ x$
you can write	$R \equiv \lambda x.\ \texttt{not}\ (x\ x)$	
and get	$(R\ R) =_\beta \texttt{not}\ (R\ R)$	

WE HAVE LEARNED SO FAR...

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- ➜ λ calculus syntax
- ➜ free variables, substitution
- ➜ β reduction
- ➜ α and η conversion
- ➜ β reduction is confluent
- ➜ λ calculus is very expressive (turing complete)
- ➜ λ calculus is inconsistent

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ISABELLE DEMO

EXERCISES

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- ➜ Play around with the syntax. Enter a number of λ terms into Isabelle.
- ➜ Not all λ terms are accepted by Isabelle. Which are not? Why?
- ➜ Evaluate the substitution $(y\ (\lambda v.\ x\ v))[x \leftarrow (\lambda y.\ v\ y)]$ on paper.
- ➜ Reduce $(\lambda n.\ \lambda f\ x.\ f\ (n\ f\ x))\ ((\lambda n.\ \lambda f\ x.\ f\ (n\ f\ x))\ (\lambda f\ x.\ x))$ to its β normal form on paper and in Isabelle.
- ➜ Pairs in λ calculus: define functions fs, sn, and $pair$ such that $fs\ (pair\ a\ b) \longrightarrow^*_\beta a$ and $sn\ (pair\ a\ b) \longrightarrow^*_\beta b$
- ➜ What can be done to fix the inconsistency in λ calculus?