NICTA Advanced Course

Theorem Proving
Principles, Techniques, Applications

λ

Content

- ➜ Intro & motivation, getting started with Isabelle
- ➜ **Foundations & Principles**
  - **Lambda Calculus**
  - Higher Order Logic, natural deduction
  - Term rewriting
- ➜ Proof & Specification Techniques
  - Datatypes, recursion, induction
  - Inductively defined sets, rule induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

Alonzo Church

- ➜ lived 1903–1995
- ➜ supervised people like Alan Turing, Stephen Kleene
- ➜ famous for Church-Turing thesis, lambda calculus, first undecidability results
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λ-calculus

- ➜ originally meant as foundation of mathematics
- ➜ important applications in theoretical computer science
- ➜ foundation of computability and functional programming

UNTYPED λ-CALCULUS

- ➜ turing complete model of computation
- ➜ a simple way of writing down functions

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Basic intuition:

instead of	$f(x) = x + 5$
write	$f = \lambda x.\ x + 5$

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- ➜ a nameless function

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- ➜ a nameless function
- ➜ that adds 5 to its parameter

FUNCTION APPLICATION

For applying arguments to functions

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(computation!)

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For applying arguments to functions

instead of	$f(x)$
write	$f\ x$

Example: $(\lambda x.\ x + 5)\ a$

Evaluating: in $(\lambda x.\ t)\ a$ replace x by a in t

(computation!)

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

THAT'S IT!

NOW FORMAL

Syntax

Terms: $$t \quad ::= \quad v \;\mid\; c \;\mid\; (t\ t) \;\mid\; (\lambda x.\ t)$$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

SYNTAX

Terms: $t \quad ::= \quad v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- ➜ v, x variables
- ➜ c constants
- ➜ $(t\ t)$ application
- ➜ $(\lambda x.\ t)$ abstraction

CONVENTIONS

- ➜ leave out parentheses where possible
- ➜ list variables instead of multiple λ

Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

CONVENTIONS

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Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

Rules:

➜ list variables: $\lambda x.\ (\lambda y.\ t) = \lambda x\ y.\ t$

➜ application binds to the left: $x\ y\ z = (x\ y)\ z \neq x\ (y\ z)$

➜ abstraction binds to the right: $\lambda x.\ x\ y = \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$

➜ leave out outermost parentheses

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

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$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

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GETTING USED TO THE SYNTAX

Example:

$$\lambda x\ y\ z.\ x\ z\ (y\ z) =$$

$$\lambda x\ y\ z.\ (x\ z)\ (y\ z) =$$

$$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$$

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

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$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

Intuition: replace parameter by argument

this is called β-reduction

Example

$$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$$

Intuition: replace parameter by argument

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$$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$$
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Intuition: replace parameter by argument

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Example

$$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$$

$$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$$

$$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$$

Intuition: replace parameter by argument

this is called β-reduction

Example

$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$

$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$

$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$

$f\ 5$

Defining Computation

β reduction:

$$
\begin{array}{lllllll}
 & & & & (\lambda x.\ s)\ t & \longrightarrow_\beta & s[x \leftarrow t] \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\
t & \longrightarrow_\beta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\
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\end{array}
$$

Still to do: define $s[x \leftarrow t]$

Easy concept. Small problem: variable capture.

Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

DEFINING SUBSTITUTION

Easy concept. Small problem: variable capture.

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We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

Defining Substitution

Easy concept. Small problem: variable capture.

Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

FREE VARIABLES

Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

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Free variables FV of a term:

$$\begin{aligned} FV\ (x) &= \{x\} \\ FV\ (c) &= \{\} \\ FV\ (s\ t) &= FV(s) \cup FV(t) \\ FV\ (\lambda x.\ t) &= FV(t) \setminus \{x\} \end{aligned}$$

Example: $FV(\quad \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x \quad)$

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Example: $FV(\quad \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x \quad) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

Substitution

$$x\ [x \leftarrow t] = t$$

$$y\ [x \leftarrow t] = y \qquad \text{if } x \neq y$$

$$c\ [x \leftarrow t] = c$$

$$(s_1\ s_2)\ [x \leftarrow t] =$$

Substitution

$$
\begin{array}{lll}
x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
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$$

$$(s_1\ s_2)\ [x \leftarrow t] = (s_1[x \leftarrow t]\ s_2[x \leftarrow t])$$

$$(\lambda x.\ s)\ [x \leftarrow t] =$$

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$$(\lambda y.\ s)\,[x \leftarrow t] = (\lambda z.\ s[y \leftarrow z][x \leftarrow t]) \qquad \text{if } x \neq y \text{ and } z \notin FV(t) \cup FV(t)$$

SUBSTITUTION EXAMPLE

$$(x\ \ (\lambda x.\ x)\ \ (\lambda y.\ z\ x))[x \leftarrow y]$$

Substitution Example

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= \quad & y\ (\lambda x.\ x)\ (\lambda y'.\ z\ y)
\end{aligned}
$$

α Conversion

Bound names are irrelevant:

$\lambda x.\ x$ and $\lambda y.\ y$ denote the same function.

α **conversion:**

$s =_\alpha t$ means $s = t$ up to renaming of bound variables.

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$$s =_\alpha t \quad \text{iff} \quad s \longrightarrow_\alpha^* t$$

($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

Equality in Isabelle is equality modulo α conversion:

if $s =_\alpha t$ then s and t are syntactically equal.

Examples:

$x\ (\lambda x\ y.\ x\ y)$

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We have defined β reduction: $\longrightarrow_\beta$

Some notation and concepts:

➜ β **conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$

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➜ t is **reducible** if there is an s such that $t \longrightarrow_\beta s$

BACK TO β

We have defined β reduction: $\longrightarrow_\beta$

Some notation and concepts:

- ➜ β **conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- ➜ t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- ➜ $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)

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➜ if it is not reducible, t is in **normal form**

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- ➜ $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- ➜ t is reducible iff it contains a redex
- ➜ if it is not reducible, t is in **normal form**
- ➜ t **has a normal form** if there is an irreducible s such that $t \longrightarrow^*_\beta s$

Example:

$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)\ \longrightarrow_\beta$$

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$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)\ \longrightarrow_\beta$$
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)\ \longrightarrow_\beta$$

Does every λ term have a normal form?

No!

Example:

$$(\lambda x.\, x\, x)\, (\lambda x.\, x\, x) \longrightarrow_\beta$$
$$(\lambda x.\, x\, x)\, (\lambda x.\, x\, x) \longrightarrow_\beta$$
$$(\lambda x.\, x\, x)\, (\lambda x.\, x\, x) \longrightarrow_\beta \ldots$$

No!

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(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x))\ \longrightarrow_\beta\ \lambda y.\ y$)

No!

Example:

$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)\ \longrightarrow_\beta$$
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)\ \longrightarrow_\beta$$
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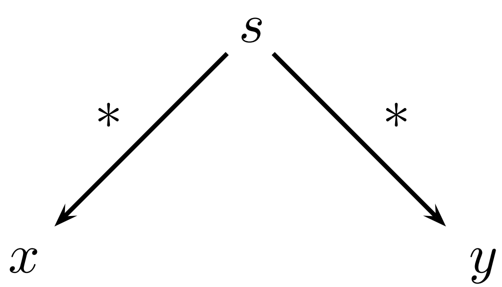
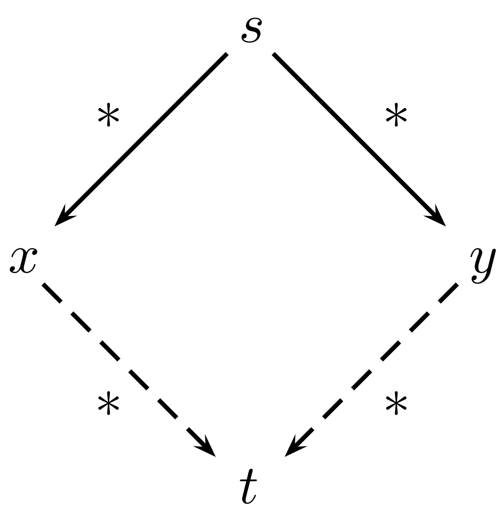
λ calculus is not terminating

β REDUCTION IS CONFLUENT

Confluence: $s \longrightarrow_\beta^* x \wedge s \longrightarrow_\beta^* y \Longrightarrow \exists t.\ x \longrightarrow_\beta^* t \wedge y \longrightarrow_\beta^* t$



Confluence: $s \longrightarrow_{\beta}^{*} x \land s \longrightarrow_{\beta}^{*} y \Longrightarrow \exists t.\ x \longrightarrow_{\beta}^{*} t \land y \longrightarrow_{\beta}^{*} t$



Order of reduction does not matter for result

Normal forms in λ calculus are unique

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a)$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a)$

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a)$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta \lambda y.\ y$

Example:

$$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$$

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η CONVERSION

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

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Defi nition:

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$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta$

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$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta (\lambda x.\ f\ x)\ g \longrightarrow_\eta f\ g$

- ➜ η reduction is confluent and terminating.
- ➜ $\longrightarrow_{\beta\eta}$ is confluent.
  $\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.
- ➜ **Equality in Isabelle is also modulo η conversion.**

Equality in Isabelle is modulo α, β, and η conversion.

We will see next lecture why that is possible.

So, what can you do with λ calculus?

λ calculus is very expressive, you can encode:

- ➜ logic, set theory
- ➜ turing machines, functional programs, etc.

Examples:

SO, WHAT CAN YOU DO WITH λ CALCULUS?

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- ➜ logic, set theory
- ➜ turing machines, functional programs, etc.

Examples:

$$\texttt{true} \equiv \lambda x\ y.\ x$$

$$\texttt{false} \equiv \lambda x\ y.\ y$$

$$\texttt{if} \equiv \lambda z\ x\ y.\ z\ x\ y$$

SO, WHAT CAN YOU DO WITH λ CALCULUS?

λ calculus is very expressive, you can encode:

➜ logic, set theory

➜ turing machines, functional programs, etc.

Examples:

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Encoding natural numbers (Church Numerals)

$$0 \equiv \lambda f\ x.\ x$$

$$1 \equiv \lambda f\ x.\ f\ x$$

$$2 \equiv \lambda f\ x.\ f\ (f\ x)$$

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$$\ldots$$

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\begin{aligned}
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\texttt{add} &\equiv \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)
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Fix Points

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$(\lambda xf.\ f\ (x\ x\ f))\ (\lambda xf.\ f\ (x\ x\ f))$ is Turing's fi x point operator

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  allows arbitrary quantification over predicates
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  Fix the problem
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Problem:

with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x \qquad x \in M \equiv M\ x$
you can write	$R \equiv \lambda x.$ `not` $(x\ x)$
and get	$(R\ R) =_\beta$ `not` $(R\ R)$

We have learned so far...

- ➜ λ calculus syntax
- ➜ free variables, substitution
- ➜ β reduction
- ➜ α and η conversion
- ➜ β reduction is confluent
- ➜ λ calculus is very expressive (turing complete)
- ➜ λ calculus is inconsistent

ISABELLE DEMO

EXERCISES

- ➜ Play around with the syntax. Enter a number of λ terms into Isabelle.
- ➜ Not all λ terms are accepted by Isabelle. Which are not? Why?
- ➜ Evaluate the substitution $(y\ (\lambda v.\ x\ v))[x \leftarrow (\lambda y.\ v\ y)]$ on paper.
- ➜ Reduce $(\lambda n.\ \lambda f\ x.\ f\ (n\ f\ x))\ ((\lambda n.\ \lambda f\ x.\ f\ (n\ f\ x))\ (\lambda f\ x.\ x))$ to its β normal form on paper and in Isabelle.
- ➜ Pairs in λ calculus: define functions fs, sn, and $pair$ such that $fs\ (pair\ a\ b)\ \longrightarrow_\beta^*\ a$ and $sn\ (pair\ a\ b)\ \longrightarrow_\beta^*\ b$
- ➜ What can be done to fix the inconsistency in λ calculus?