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NICTA Advanced Course

Theorem Proving
Principles, Techniques, Applications

Gerwin Klein
Formal Methods

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ORGANISATORIALS

When	Mon	14:00 – 15:30
	Wed	10:30 – 12:00
	7 weeks	ends Mon, 20.9.2004
Exceptions	Mon 6.9., 13.9., 20.9. at 15:00 – 16:30	

Web page:
`http://www.cse.unsw.edu.au/~kleing/teaching/thprv-04/`

free – no credits – no assigments

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WHAT YOU WILL LEARN

- ➜ how to use a theorem prover
- ➜ background, how it works
- ➜ how to prove and specify

Health Warning

Theorem Proving is addictive

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WHAT YOU WILL NOT LEARN

- ➜ semantics / model theory
- ➜ soundness / completeness proofs
- ➜ decision procedures

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CONTENT

- ➜ Intro & motivation, getting started with Isabelle (today)
- ➜ Foundations & Principles
  - Lambda Calculus
  - Higher Order Logic, natural deduction
  - Term rewriting
- ➜ Proof & Specification Techniques
  - Datatypes, recursion, induction
  - Inductively defined sets, rule induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

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CREDITS

material (in part) shamelessly stolen from

Tobias Nipkow, Larry Paulson, Markus Wenzel

David Basin, Burkhardt Wolff

Don't blame them, errors are mine

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WHAT IS A PROOF?

to prove (Marriam-Webster)

- ➜ from Latin probare (test, approve, prove)
- ➜ to learn or find out by experience (archaic)
- ➜ to establish the existence, truth, or validity of (by evidence or logic)
  prove a theorem, the charges were never proved in court

pops up everywhere

- ➜ politics (weapons of mass destruction)
- ➜ courts (beyond reasonable doubt)
- ➜ religion (god exists)
- ➜ science (cold fusion works)

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WHAT IS A MATHEMATICAL PROOF?

In mathematics, a proof is a demonstration that, given certain axioms, some statement of interest is necessarily true. (Wikipedia)

Example: $\sqrt{2}$ is not rational.

Proof: assume there is $r \in \mathbb{Q}$ such that $r^2 = 2$.

Hence there are mutually prime p and q with $r = \frac{p}{q}$.

Thus $2q^2 = p^2$, i.e. p^2 is divisible by 2.

2 is prime, hence it also divides p, i.e. $p = 2s$.

Substituting this into $2q^2 = p^2$ and dividing by 2 gives $q^2 = 2s^2$. Hence, q is also divisible by 2. Contradiction. Qed.

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NICE, BUT..

- ➜ still not rigorous enough for some
  - what are the rules?
  - what are the axioms?
  - how big can the steps be?
  - what is obvious or trivial?
- ➜ informal language, easy to get wrong
- ➜ easy to miss something, easy to cheat

Theorem. A cat has nine tails.
Proof. No cat has eight tails. Since one cat has one more tail than no cat, it must have nine tails.

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WHAT IS A FORMAL PROOF?

A derivation in a formal calculus

Example: $A \wedge B \longrightarrow B \wedge A$ derivable in the following system

Rules: $\frac{X \in S}{S \vdash X}$ (assumption) $\frac{S \cup \{X\} \vdash Y}{S \vdash X \longrightarrow Y}$ (impI)

$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y}$ (conjI) $\frac{S \cup \{X, Y\} \vdash Z}{S \cup \{X \wedge Y\} \vdash Z}$ (conjE)

Proof:

1.	$\{A, B\} \vdash B$	(by assumption)
2.	$\{A, B\} \vdash A$	(by assumption)
3.	$\{A, B\} \vdash B \wedge A$	(by conjI with 1 and 2)
4.	$\{A \wedge B\} \vdash B \wedge A$	(by conjE with 3)
5.	$\{\} \vdash A \wedge B \longrightarrow B \wedge A$	(by impI with 4)

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WHAT IS A THEOREM PROVER?

Implementation of a formal logic on a computer.

- ➜ fully automated (propositional logic)
- ➜ automated, but not necessarily terminating (first order logic)
- ➜ with automation, but mainly interactive (higher order logic)
- ➜ based on rules and axioms
- ➜ can deliver proofs

There are other (algorithmic) verifi cation tools:

- ➜ model checking, static analysis, ...
- ➜ usually do not deliver proofs

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WHY THEOREM PROVING?

- ➜ Analysing systems/programs thoroughly
- ➜ Finding design and specification errors early
- ➜ High assurance (mathematical, machine checked proof)
- ➜ it's not always easy
- ➜ it's fun

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Main theorem proving system for this course:

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What is Isabelle?

A generic interactive proof assistant

- ➜ **generic:**
  not specialised to one particular logic
  (two large developments: HOL and ZF, will mainly use HOL)
- ➜ **interactive:**
  more than just yes/no, you can interactively guide the system
- ➜ **proof assistant:**
  helps to explore, find, and maintain proofs

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Why Isabelle?

- ➜ free
- ➜ widely used system
- ➜ active development
- ➜ high expressiveness and automation
- ➜ reasonably easy to use
- ➜ (and because I know it best ;-))

We will see other systems, too: HOL4, Coq, Waldmeister

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If I prove it on the computer, it is correct, right?

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No, because:

① hardware could be faulty

② operating system could be faulty

③ implementation runtime system could be faulty

④ compiler could be faulty

⑤ implementation could be faulty

⑥ logic could be inconsistent

⑦ theorem could mean something else

If I prove it on the computer, it is correct, right?

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No, but:

probability for

➜ 1 and 2 reduced by using different systems

➜ 3 and 4 reduced by using different compilers

➜ faulty implementation reduced by right architecture

➜ inconsistent logic reduced by implementing and analysing it

➜ wrong theorem reduced by expressive/intuitive logics

No guarantees, but assurance way higher than manual proof

If I prove it on the computer, it is correct, right?

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Soundness architectures

careful implementation	PVS
LCF approach, small proof kernel	HOL4 Isabelle
explicit proofs + proof checker	Coq Twelf Isabelle

Meta Logic

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Meta language:
The language used to talk about another language.

Examples:
English in a Spanish class, English in an English class

Meta logic:
The logic used to formalize another logic

Example:
Mathematics used to formalize derivations in formal logic

Meta Logic – Example

Syntax:

Formulae: $F ::= V \mid F \longrightarrow F \mid F \wedge F \mid False$

$V ::= [A - Z]$

Derivable: $S \vdash X$ X a formula, S a set of formulae

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logic / meta logic

$$\frac{X \in S}{S \vdash X} \qquad \frac{S \cup \{X\} \vdash Y}{S \vdash X \longrightarrow Y}$$

$$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y} \qquad \frac{S \cup \{X, Y\} \vdash Z}{S \cup \{X \wedge Y\} \vdash Z}$$

Isabelle's Meta Logic

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$\bigwedge$ $\Longrightarrow$ λ

$\bigwedge$

Syntax: $\bigwedge x.\ F$ (F another meta level formula)

in ASCII: `!!x. F`

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- ➜ universial quantifier on the meta level
- ➜ used to denote parameters
- ➜ example and more later

$\Longrightarrow$

Syntax: $A \Longrightarrow B$ (A, B other meta level formulae)

in ASCII: `A ==> B`

Binds to the right:

$$A \Longrightarrow B \Longrightarrow C \quad = \quad A \Longrightarrow (B \Longrightarrow C)$$

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Abbreviation:

$$[\![A; B]\!] \Longrightarrow C \quad = \quad A \Longrightarrow B \Longrightarrow C$$

- ➜ read: A and B implies C
- ➜ used to write down rules, theorems, and proof states

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EXAMPLE: A THEOREM

mathematics: if $x < 0$ and $y < 0$, then $x + y < 0$

formal logic: $\vdash \; x < 0 \wedge y < 0 \longrightarrow x + y < 0$

variation: $x < 0; y < 0 \;\vdash\; x + y < 0$

Isabelle: **lemma** "$x < 0 \wedge y < 0 \longrightarrow x + y < 0$"

variation: **lemma** "$[\![x < 0; y < 0]\!] \Longrightarrow x + y < 0$"

variation: **lemma**

assumes "$x < 0$" and "$y < 0$" shows "$x + y < 0$"

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EXAMPLE: A RULE

logic: $$\frac{X \quad Y}{X \wedge Y}$$

variation: $$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y}$$

Isabelle: $[\![X; Y]\!] \Longrightarrow X \wedge Y$

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EXAMPLE: A RULE WITH NESTED IMPLICATION

logic: $$\frac{X \vee Y \quad \begin{array}{c} X \\ \vdots \\ Z \end{array} \quad \begin{array}{c} Y \\ \vdots \\ Z \end{array}}{Z}$$

variation: $$\frac{S \cup \{X\} \vdash Z \quad S \cup \{Y\} \vdash Z}{S \cup \{X \vee Y\} \vdash Z}$$

Isabelle: $[\![X \vee Y; X \Longrightarrow Z; Y \Longrightarrow Z]\!] \Longrightarrow Z$

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λ

Syntax: $\lambda x.\ F$ (F another meta level formula)

in ASCII: `%x. F`

- ➜ lambda abstraction
- ➜ used to for functions in object logics
- ➜ used to encode bound variables in object logics
- ➜ more about this in the next lecture

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ENOUGH THEORY!
GETTING STARTED WITH ISABELLE

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SYSTEM ARCHITECTURE

Proof General – user interface

HOL, ZF – object-logics

Isabelle – generic, interactive theorem prover

Standard ML – logic implemented as ADT

User can access all layers!

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SYSTEM REQUIREMENTS

- ➜ **Linux**, **MacOS X** or **Solaris**
- ➜ **Standard ML**
  (PolyML fastest, SML/NJ supports more platforms)
- ➜ **XEmacs** or **Emacs**
  (for ProofGeneral)

If you do not have Linux, MacOS X or Solaris, try **IsaMorph**:
`http://www.brucker.ch/projects/isamorph/`

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DOCUMENTATION

Available from `http://isabelle.in.tum.de`

- ➜ Learning Isabelle
  - Tutorial on Isabelle/HOL (LNCS 2283)
  - Tutorial on Isar
  - Tutorial on Locales
- ➜ Reference Manuals
  - Isabelle/Isar Reference Manual
  - Isabelle Reference Manual
  - Isabelle System Manual
- ➜ Reference Manuals for Object-Logics

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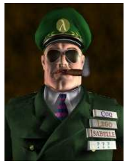
ProofGeneral



- ➜ User interface for Isabelle
- ➜ Runs under XEmacs or Emacs
- ➜ Isabelle process in background

Interaction via

- ➜ Basic editing in XEmacs (with highlighting etc)
- ➜ Buttons (tool bar)
- ➜ Key bindings
- ➜ ProofGeneral Menu (lots of options, try them)

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X-Symbol Cheat Sheet

Input of funny symbols in ProofGeneral

- ➜ via menu ("X-Symbol")
- ➜ via ASCII encoding (similar to LaTeX): `\<and>`, `\<or>`, ...
- ➜ via abbreviation: `/\`, `\/`, `-->`, ...
- ➜ via *rotate*: `l C-.` = λ (cycles through variations of letter)

	$\forall$	$\exists$	λ	$\neg$	$\wedge$	$\vee$	$\longrightarrow$	$\Rightarrow$
①	`\<forall>`	`\<exists>`	`\<lambda>`	`\<not>`	`/\`	`\/`	`-->`	`=>`
②	ALL	EX	%	~	&	|		

① converted to X-Symbol

② stays ASCII

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Demo

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Exercises

- ➜ Download and install Isabelle from `http://isabelle.in.tum.de` or `http://mirror.cse.unsw.edu.au/pub/isabelle/`
- ➜ Switch on X-Symbol in ProofGeneral
- ➜ Step through the demo file from the lecture web page
- ➜ Write an own theory file, look at some theorems, try 'find theorem'