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CHROMATIC DERIVATIVES, EXPANSIONS AND ASSOCIATED HILBERT SPACES

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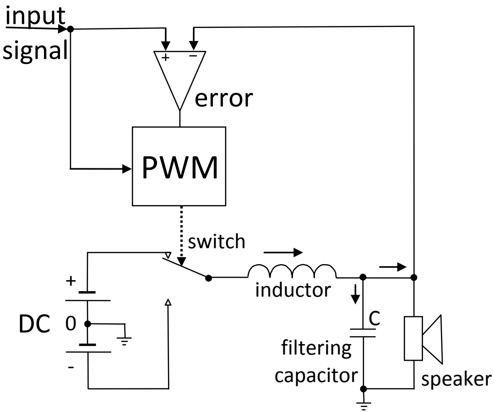
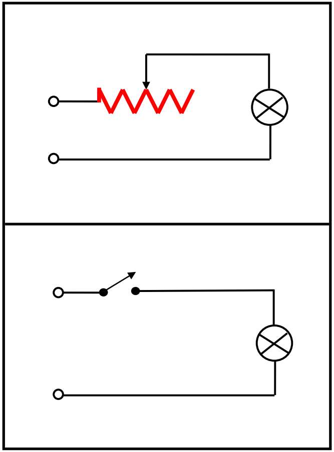
Details can be found in:

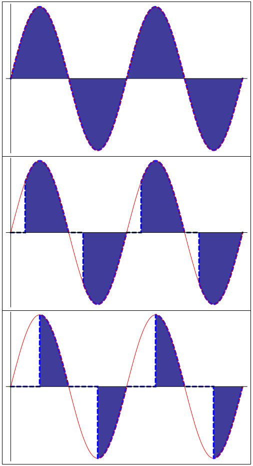
- US Patents 6313778 (1998) and 6115726. (1999)
- A. Ignjatovic: Local Approximations Based on Orthogonal Differential Operators, **Journal of Fourier Analysis and Applications**, Vol. 13, Issue 3 (2007).
- — "— : Chromatic derivatives and local approximations, **IEEE Transactions on Signal Processing**, Volume 57, Issue 8 (2009).
- — "— : Chromatic derivatives, chromatic expansions and associated spaces, **East Journal on Approximations**, Volume 15, Number 3 (2009).
- A. Ignjatovic and A. Zayed: Multidimensional chromatic derivatives and series expansions, to appear in the **Proceedings of the American Mathematical Society**.

The sad story of my life: how the whole thing started

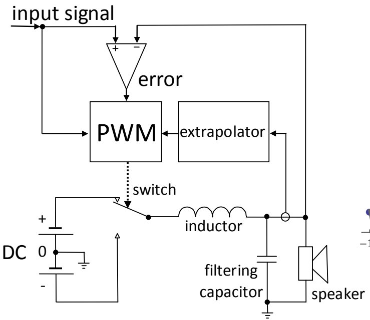


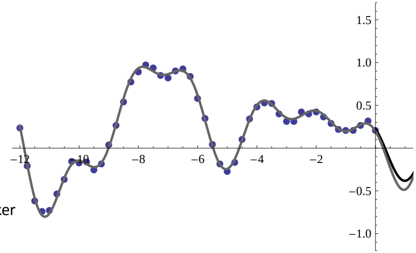
How a dimmer works





Solution: predicting the very near future





- What kind of extrapolation functions should we use?

- This is a very unusual signal processing problem: we are interested in extremely local, “microscopic” signal behavior.

Towards local signal representation

- Let $f \in \mathbf{BL}(\pi)$, i.e., $f \in L^2$ with $\widehat{f(\omega)}$ supported on $[-\pi, \pi]$

Shannon's Expansion:
(Whittaker–Kotelnikov–Nyquist–Shannon)

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \, \frac{\sin \pi(t-n)}{\pi(t-n)}$$

- **global in nature** – requires samples $f(n)$ for all n;
- **fundamental** to signal processing;
- poorly represents local signal behavior

Taylor's Expansion:

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \, \frac{t^n}{n!}$$

- **local in nature** – requires $f^{(n)}(t)$ at a single instant $t = 0$.
- **very little use** in signal processing – **why ?**

Problems with Taylor's expansion of $\mathbf{BL}(\pi)$ signals

1. Numerical evaluation of derivatives of high orders of a noisy sampled signal is **unfeasible**.

2. Truncations of Shannon's expansion of an $f \in \mathbf{BL}(\pi)$

– belong to $\mathbf{BL}(\pi)$

– converge to f both uniformly and in L^2

– if A is a filter, then

$$A[f](t) = \sum_{n=-\infty}^{\infty} f(n)\, A\,[\mathrm{sinc}\,]\,(t-n), \qquad (1)$$

– In comparison, truncations of Taylor's expansion of an $f \in \mathbf{BL}(\pi)$ **have none of these important properties**

Can we fix all of these problems???

Numerical differentiation of band limited signals

Let $f \in \mathbf{BL}(\pi)$; then $\dfrac{f^{(n)}(t)}{\pi^n} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{i}^n \left(\frac{\omega}{\pi}\right)^n \widehat{f(\omega)} e^{\mathrm{i}\,\omega t} d\omega.$

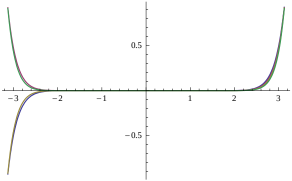


Figure: $(\omega/\pi)^n$ for $n = 15 - 18$

- derivatives of high order **obliterate the spectrum**.
- transfer functions of the (normalized) derivatives **cluster together and are nearly indistinguishable.**
- can we find a better base for the space of linear differential operators? An **orthogonal base**??

Orthogonal base for the space of linear diff. operators

- Start with normalized and re-scaled Legendre polynomials:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P^L_n(\omega)P^L_m(\omega)\mathrm{d}\omega = \delta(m-n).$$

- Obtain operator polynomials by replacing ω^k with $\mathrm{i}^k\, d^k/dt^k$:

$$\mathcal{K}^n_t = (-\mathrm{i})^n P^L_n\left(\mathrm{i}\ \frac{d}{dt}\right)$$

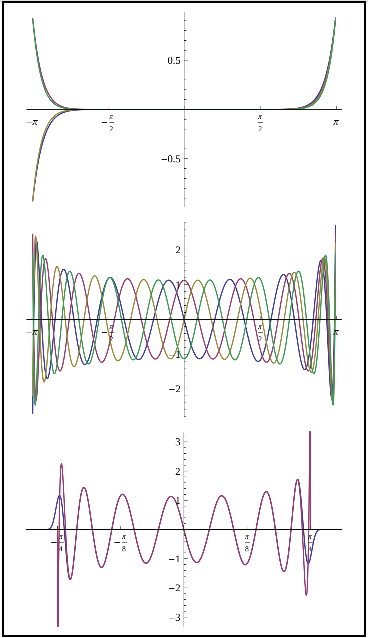
- Definition of $\mathcal{K}^n$ chosen so that

$$\mathcal{K}^n_t[e^{\mathrm{i}\,\omega t}] = \mathrm{i}^n P^L_n(\omega)\, e^{\mathrm{i}\,\omega t}.$$

- Thus, for $f \in \mathbf{BL}(\pi)$,

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}^n P^L_n(\omega)\widehat{f(\omega)} e^{\mathrm{i}\,\omega t} d\omega.$$

Why are chromatic derivatives a better base?



- Compare the graphs of the transfer functions of $1/\pi^n d^n/dt^n$, i.e., $(\omega/\pi)^n$ (first graph) and of $\mathcal{K}^n$, i.e., $P_n^L(\omega)$ (second graph).

- Transfer functions of $\mathcal{K}^n$ form a sequence of **well separated comb filters** which **preserve spectral features** of the signal, thus we call them the **chromatic derivatives**.

- Third graph: transfer function of the ideal filter $\mathcal{K}^{15}$ (red) vs. transfer function of a transversal filter (blue), (128 taps, 2× oversampling.)

Local representation of the scalar product in $\mathbf{BL}(\pi)$

Proposition: Assume that $f, g \in \mathbf{BL}(\pi)$; then the sums on the left hand side of the following equations do not depend on the choice of the instant t, and

$$\sum_{n=0}^{\infty} K^n[f](t)^2 = \int_{-\infty}^{\infty} f(x)^2 dx = \|f\|^2$$

$$\sum_{n=0}^{\infty} K^n[f](t)\overline{K^n[g](t)} = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx = \langle f, g \rangle$$

$$\sum_{n=0}^{\infty} K^n[f](t) K_t^n[g(u-t)] = \int_{-\infty}^{\infty} f(x) g(u-x) dx = (f * g)(u)$$

- These are the **local equivalents** of the usual, "globally defined" norm, scalar product and convolution!
- **Aim: "maximally localized"** signal processing, suitable for transient analysis.

Fixing Taylor's Expansion: Chromatic Expansion

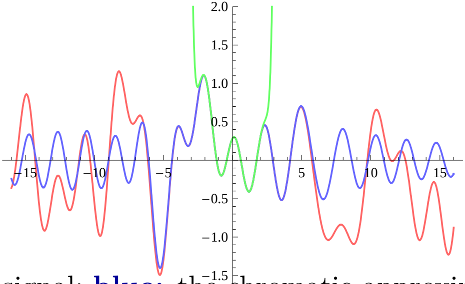
Proposition: Let $\text{sinc}\,(t) = \dfrac{\sin(\pi t)}{\pi t}$ and let $f(t)$ be **any analytic function**. Then,

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[\text{sinc}\,(t)] \\ &= \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t) \end{aligned}$$

$j_n(t)$ — the spherical Bessel functions of the first kind

- **The truncations of the series belong to BL**(π).
- **If** $f \in$ **BL**(π) **the series converges to** $f(t)$ **both uniformly and in** L^2.
- **Both a good and a bad news??**

Chromatic approximation versus Taylor's approximation

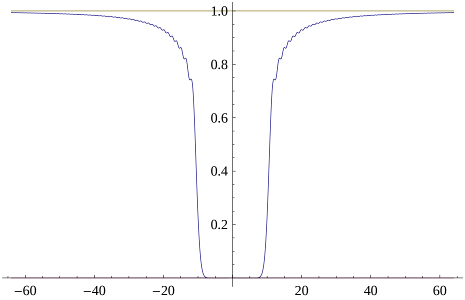


- **red:** the signal; **blue:** the chromatic approximation of order 15; **green:** the Taylor approximation of order 15.

- $f^{(k)}(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\sum_{m=0}^{n}(-1)^m\,\mathcal{K}^m[f](0)\,\mathcal{K}^m[\mathrm{sinc}\,](t)\right]_{t=0}$

- Chromatic approximations are **local approximations**
- k^{th} derivative approximated with order $n-k$ expansion
- We can locally alias any waveform

Approximation error behavior



$$\left| f(t) - \sum_{m=0}^{n} (-1)^m \, \mathcal{K}^m[f](0) \, \mathcal{K}^m[\mathrm{sinc}\,](t) \right| \leq \|f\|_2 \; E(t) \qquad (2)$$

$$E(t) = \sqrt{1 - \sum_{m=0}^{n} \mathcal{K}^m[\boldsymbol{m}](t)^2}$$

Chromatic expansion vs. Shannon's expansion

How is Shannon expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, \frac{\sin \pi(t-n)}{\pi(t-n)}$$

related to the chromatic expansion

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t)$$

- Transformation $\{f(n)\}_{n\in\mathbb{N}} \Leftrightarrow \{\mathcal{K}^n[f](0)\}_{n\in\mathbb{N}}$ by an unitary operator defined by the infinite matrix

$$\left[\sqrt{2k+1}\, j_k(n\pi)\ :\ k \in \mathbb{N}, n \in \mathbb{Z}\right]:$$

$$\begin{aligned} f(n) &= \sum_{k=0}^{\infty} \mathcal{K}^k[f](0)\, \sqrt{2k+1}\, j_k(n\pi); \\ \mathcal{K}^k[f](0) &= \sum_{n=-\infty}^{\infty} f(n)\, \sqrt{2k+1}\, j_k(n\pi). \end{aligned}$$

- In practice one CANNOT evaluate $\mathcal{K}^k[f](0)$ using Shannon rate samples via

$$\mathcal{K}^k[f](0) \approx \sum_{n=-N}^{N} f(n)\,\sqrt{2k+1}\,j_k(n\pi)$$

because $\sqrt{2n+1}\,j_n(\pi t)$ decay very slowly and we would need huge N.

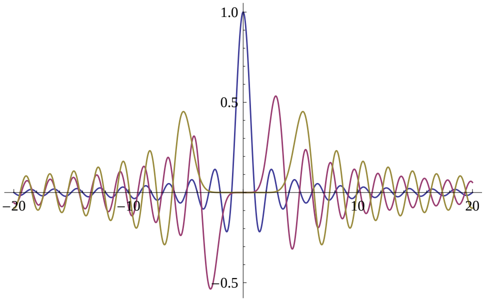


Figure: $n = 0$, $n = 7$, $n = 14$

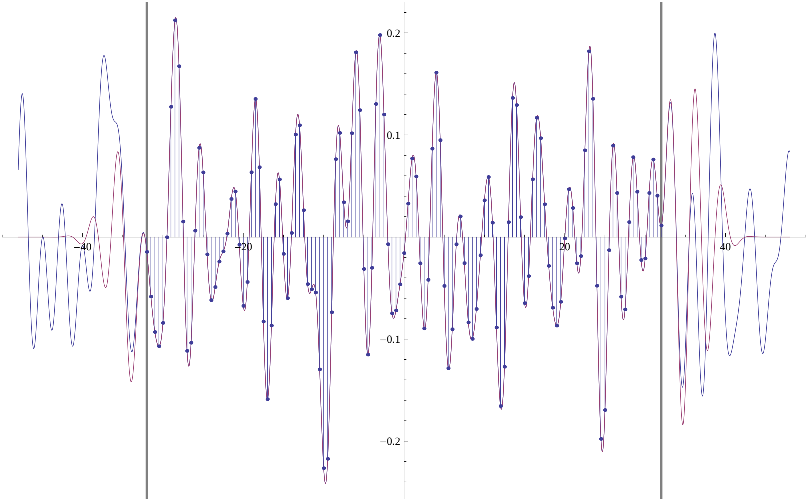
Chromatic derivatives are non-redundant!

- This is a good news because this means that chromatic derivatives are non redundant to the Nyquist rate samples;

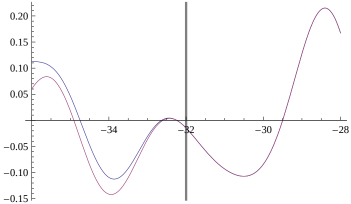
- They provide additional "information which can be used **in addition** to the standard Nyquist rate methods making them **more powerful**.

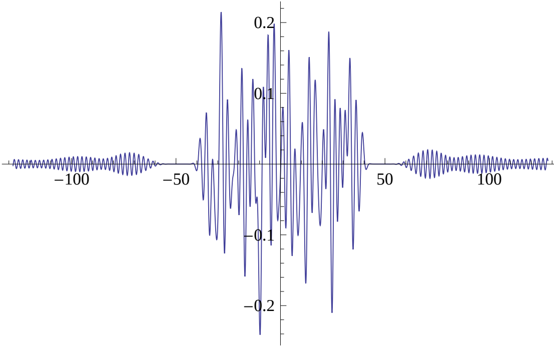
- Extremely convenient for **regularized least squares**:
  - Minimize $\sum_j (\sum_{n=0}^{N} X_n \sqrt{2n+1}\, j_n(\pi t_j) - s_j)^2 + \mu \sum_{n=0}^{N} X_n^2$
  - signal in $\mathbf{BL}(\pi)$ iff $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2 < \infty$
  - interpolation functions are bounded;
  - extremely robust for choices of N and μ.

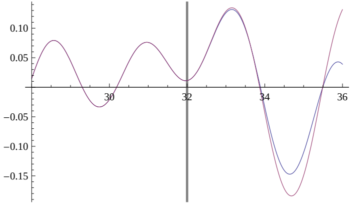
So did the amplifier work: chromatic extrapolation



View of the left and right ends of extrapolation $a_f(t)$







Truncation of $a(t)$ is time limited and **nearly band limited**:

$$|\widehat{a}(\omega)| \leq \frac{|\mathcal{K}^n[a]^\frown(\omega)|}{|P_n(\omega)|} \leq \frac{M}{|P_n(\omega)|}$$

General families of chromatic derivatives

- Given a family of orthonormal polynomials $P_n(\omega)$ we can always define differential operators

$$\mathcal{K}_t^n = (-\mathrm{i})^n \; P_n^L\left(\mathrm{i} \; \frac{d}{dt}\right)$$

Question:

What are the families of orthogonal polynomials such that for the corresponding differential operators K^n and some associated function $\boldsymbol{m}(t)$ we have

$$f(t) = \sum_{n=0}^{\infty} (-1)^n K^n[f](u) K^n[\boldsymbol{m}](t-u)$$

for important classes of functions, and when is the convergence uniform?

Examples:
Legendre Polynomials/Spherical Bessel functions

- For the (normalized) **Legendre polynomials**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) \mathrm{d}\omega = \delta(m-n)$$

and for $\boldsymbol{m}(t) = \frac{\sin(\pi t)}{\pi t}$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \sqrt{2n+1}\, j_n(\pi t)$
and

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t)$$

holds for all analytic functions;

- The convergence is uniform for functions in $\mathbf{BL}(\pi)$

Examples:
Chebyshev polynomials /Bessel functions

- For the (normalized) **Chebyshev polynomials** of the first kind:

$$\int_{-\pi}^{\pi} \frac{P_n^T(\omega)P_m^T(\omega)}{\pi^2\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}}\mathrm{d}\omega = \delta(n-m).$$

for $\boldsymbol{m}(t) = J_0(\pi t)$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{2}\,J_n(\pi t)$ and

$$f(t) = f(u)J_0(\pi t) + \sqrt{2}\,\sum_{n=1}^{\infty}\mathcal{K}^n[f](0)J_n(\pi t)$$

- The Neumann series - converges for all analytic functions;

- Convergence uniform for band limited functions which satisfy

$$\int_{-\pi}^{\pi} |\widehat{f}(\omega)|^2\,\sqrt{1-(\omega/\pi)^2}\,\mathrm{d}\omega < \infty$$

Examples:
Hermite polynomials/Gaussian monomials

- For the (normalized) **Hermite polynomials**

$$\int_{-\infty}^{\infty} P_n^H(\omega) P_m^H(\omega) \, \frac{\mathrm{e}^{-\omega^2}}{\sqrt{\pi}} \, \mathrm{d}\omega = \delta(n-m)$$

and $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \, \frac{t^n}{\sqrt{2^n \, n!}} \, \mathrm{e}^{-t^2/4}$

- chromatic expansion converges for analytic functions s.t.

$$\limsup_{n\to\infty} \frac{|f^{(n)}(z)|^{1/n}}{\sqrt{n}} < \infty$$

- converges uniformly for all analytic functions s.t.

$$\int_{-\infty}^{\infty} |\widehat{f(\omega)}|^2 \, \mathrm{e}^{\omega^2} \mathrm{d}\omega < \infty$$

Examples: the hyperbolic family

If $L_n(\omega)$ satisfy

$$\frac{1}{2}\int_{-\infty}^{\infty} L_n(\omega)\; L_m(\omega)\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)\mathrm{d}\omega = \delta(m-n)$$

and $\boldsymbol{m}(z) = \mathrm{sech}(z)$ then $\mathcal{K}^n[\boldsymbol{m}](z) = (-1)^n\,\mathrm{sech}(z)\tanh^n(z)$
and

$$f(z) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\,\mathrm{sech}(z)\tanh^n(z)$$

converges uniformly inside **the disc** $|z| < \pi/2$ for functions analytic inside this disc, and whose Fourier transform satisfies

$$\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\cosh(\omega)\mathrm{d}\omega < \infty$$

General families of chromatic derivatives

Definition: A family of polynomials $P_n(\omega)$ which is orthonormal with respect to a non-decreasing bounded **moment distribution function** $a(\omega)$:

$$\int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) \mathrm{d}a(\omega)$$

is **chromatic** if the moments μ_n of $a(\omega)$,

$$\mu_n = \int_{-\infty}^{\infty} \omega^n \mathrm{d}a(\omega)$$

satisfy

$$\rho = \limsup_{n\to\infty} \frac{\mu_n^{1/n}}{n} < \infty$$

Lemma: $P_n(\omega)$ are chromatic if and only if for every $0 \leq \alpha < \rho$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|} \mathrm{d}a(\omega) < \infty$$

General families of chromatic derivatives

Theorem: Let $P_n(\omega)$ be a chromatic family of polynomials orthonormal with respect to $a(\omega)$, and let

$$\boldsymbol{m}(z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}a(\omega)$$

Then $\boldsymbol{m}(z)$ is analytic on the strip $S_{\rho/2} = \{z : |\mathrm{Im(z)}| < \rho/2\}$.

Definition: $L^2_{a(\omega)}$ is the space of functions $\phi(\omega)$ satisfying

$$\int_{-\infty}^{\infty} |\phi(\omega)|^2 \mathrm{d}a(\omega) < \infty.$$

General families of chromatic derivatives

Theorem: If $P_n(\omega)$ are a chromatic family of polynomials orthonormal with respect to $a(\omega)$, then they are a complete base of the space $L^2_{a(\omega)}$.

Definition: Λ^2 is the space of functions $f(t)$ analytic on $S_{\rho/2}$ such that for the chromatic derivatives $\mathcal{K}^n$ which correspond to $P_n(\omega)$ we have

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 < \infty.$$

Theorem: A function $f(z)$ is in Λ^2 if and only if there exists a function $\phi_f(\omega)$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi_f(\omega) \mathrm{e}^{\mathrm{i}\,\omega z} \mathrm{d}a(\omega)$$

in which case

$$\phi_f(\omega) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) P_n(\omega)$$

General families of chromatic derivatives

Theorem: If $f(z) \in \Lambda^2$, then

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\mathcal{K}^n[\boldsymbol{m}](t)$$

with the series converging uniformly on strips $S_{\rho/2-\epsilon}$.

A geometric interpretation

- Let $f \in L_2^{\mathcal{M}}$ and $t \in \mathbb{R}$; then $f \mapsto \langle\!\langle \mathcal{K}^m[f](t) \rangle\!\rangle_{m\in\mathbb{N}}$ is an isomorphism between $L_2^{\mathcal{M}}$ and l_2.
- The orthonormal base $\{\mathcal{K}^n[\boldsymbol{m}](t)\}_{n\in\mathbb{N}}$ of $L_2^{\mathcal{M}}$ is then mapped into an orthonormal base $\{\langle\!\langle (\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t) \rangle\!\rangle_{m\in\mathbb{N}}\}_{n\in\mathbb{N}}$.
- For every t,

$$\langle\!\langle \mathcal{K}^m[f](t) \rangle\!\rangle_{m\in\mathbb{N}} = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) \langle\!\langle (\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t) \rangle\!\rangle_{m\in\mathbb{N}}$$

- Let $\vec{e}_{n+1}(t) = \langle\!\langle (\mathcal{K}^n \circ \mathcal{K}^m)[\boldsymbol{m}](t) \rangle\!\rangle_{m\in\mathbb{N}}$ and let $H(t)$ be an antiderivative of $\vec{e}_1$. Then $\vec{e}_1(t) = \vec{H}'(t)$ and

$$\vec{e}_1{}'(t) = \gamma_0 \, \vec{e}_2(t) \qquad (\gamma_n\text{- three term recursion coeff's})$$
$$\vec{e}_k{}'(t) = -\gamma_{k-2} \, \vec{e}_{k-1}(t) + \gamma_{k-1} \, \vec{e}_{k+1}(t), \quad \text{for } k \geq 2.$$

- These are the Frenet–Serret formulas; $\{\vec{e}_{n+1}(t)\}_{n\in\mathbb{N}}$ are the moving frame of the helix $H(t)$ in l_2, with curvatures = the recursion coefficients of the three term recurrence formula!

General families of chromatic derivatives

Theorem: If $f(z) \in \Lambda^2$, then

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\mathcal{K}^n[\boldsymbol{m}](t)$$

with the series converging uniformly on strips $S_{\rho/2-\epsilon}$.

How about the **local (non-uniform) convergence** of the chromatic series??

For example, in the case of the Chebyshev polynomials $T_n(\omega)$ and the Bessel functions of the first kind $J_n(\omega)$, we know that the chromatic series is just the Newmann series, and that the above equality holds **for every analytic function** $f(z)$!

Weakly bounded families

Theorem: A family of polynomials is orthonormal with respect to a moment distribution function $a(\omega)$ with all odd moments $\mu_{2n+1} = 0$ if and only if there exist $\gamma_n > 0$ such that

$$P_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\,P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P_{n-1}(\omega).$$

Definition: Such family of polynomials $P_n(\omega)$ is:

1. **bounded** if for some M and all n we have $\frac{1}{M} \leq \gamma_n \leq M$.
2. **weakly bounded** if for some $0 \leq p < 1$ we have

$$\frac{1}{M} < \gamma_n < M\,n^p \quad \text{and} \quad \frac{\gamma_n}{\gamma_{n+1}} < M$$

- Bounded families are also weakly bounded with $p = 0$.

Examples:

- **Bounded families** $(p = 0)$:
  - **Legendre** polynomials: $\gamma_n = \frac{\pi(n+1)}{\sqrt{4(n+1)^2-1}} \to \frac{\pi}{2}$
  - **Chebyshev** polynomials: $\gamma_0 = \frac{\pi}{\sqrt{2}}$ and $\gamma_{n+1} = \frac{\pi}{2}$
- **Weakly bounded family** $(p = 1/2)$:
  - **Hermite** polynomials: $\gamma_n = \sqrt{(n+1)/2}$;
- **Non - weakly bounded family** $(p = 1)$:
  - **Hyperbolic** family: $\gamma_n = n + 1$;
- This shows that if we want $\boldsymbol{m}(z)$ to be entire, then the bound $p < 1$ is sharp.

Lemma: Every weakly bounded family of orthonormal polynomials is also chromatic.

Theorem: Let $\{P_n(\omega)\}_{n\in\mathbb{N}}$ be a weakly bounded family and let $f(z)$ be an entire function. If

$$\lim_{n\to\infty}\left|\frac{f^{(n)}(0)}{n!^{1-p}}\right|^{1/n} = 0$$

then for every $z\in\mathbb{C}$

$$f(z) = \sum_{j=0}^{\infty}(-1)^j\mathcal{K}^j[f](0)\,\mathcal{K}^j[\boldsymbol{m}](z).$$

The convergence is uniform on every disc of finite radius.

Corollary: If $\mathcal{M}$ is bounded then the chromatic expansion of every entire function $f(z)$ point-wise converges to $f(z)$ for all z.

- It turns out that many of the classical formulas such as

$$\mathrm{e}^{\mathrm{i}\,\omega t} = \sum_{n=0}^{\infty} \mathrm{i}^n\, T_n(\omega) J_n(t)$$
$$J_0(t+u) = J_0(u)J_0(t) + 2\sum_{n=1}^{\infty}(-1)^n J_n(u)J_n(t)$$
$$J_0(t)^2 + 2\sum_{k=1}^{\infty} J_n(t)^2 = 1$$
$$J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1$$

are special cases of chromatic expansions valid **for all weakly bounded families** of polynomials and their associated $\boldsymbol{m}(z)$:

$$\mathrm{e}^{\mathrm{i}\,\omega t} = \sum_{n=0}^{\infty} \mathrm{i}^n P_n(\omega)\mathcal{K}^n[\boldsymbol{m}](t)$$
$$\boldsymbol{m}(t+u) = \sum_{n=0}^{\infty}(-1)^n \mathcal{K}^n[\boldsymbol{m}](u)\mathcal{K}^n[\boldsymbol{m}](t)$$
$$\sum_{k=1}^{\infty} \mathcal{K}^n[\boldsymbol{m}](t)^2 = 1$$
$$\boldsymbol{m}(z) + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}}\right) \mathcal{K}^{2n}[\boldsymbol{m}](z) = 1$$

Theorem: Assume $\mathcal{M}$ is weakly bounded and let k be such that $k \geq 1/(1-p)$; then:

(a) there exists $K > 0$ such that

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < \frac{|Kz|^n}{n!^{1-p}}\, \mathrm{e}^{|Kz|^k};$$

(b) for every $f(t) \in \Lambda_2$ there exists $C, L > 0$ such that

$$|f(z)| \leq C\mathrm{e}^{L|z|^k}.$$

A (mild) generalization of the Paley-Wiennner Theorem??

Assume that $f(z)$ is an entire function for which there exist a symmetric moment distribution function $a(\omega)$ and a function $\varphi(\omega) \in L^2_{a(\omega)}$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi(\omega) \mathrm{e}^{\mathrm{i}\, z\omega} \mathrm{d}a(\omega).$$

Are the following are equivalent:

(a) f is of exponential type, i.e., there exist $C, L > 0$ such that

$$|f(z)| < C\mathrm{e}^{L|z|}, \quad (z \in \mathbb{C});$$

(b) $a(\omega)$ can be chosen such that $\mathrm{d}a(\omega)$ is finitely supported.

A real generalization of the Paley - Wienner Theorem??

Assume that $f(z)$ is an entire function for which there exist a symmetric moment distribution function $a(\omega)$ and a function $\varphi(\omega) \in L^2_{a(\omega)}$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi(\omega) \mathrm{e}^{\mathrm{i}\, z\omega} \mathrm{d}a(\omega),$$

and let $k \geq 1$ be an integer. Are the following equivalent:

(c) there exist $C, L > 0$ such that

$$|f(z)| < C\mathrm{e}^{L|z|^m}, \quad (z \in \mathbb{C});$$

(d) $a(\omega)$ can be chosen such that the corresponding γ_n satisfy $\gamma_n < Mn^p$ for some $0 \leq p \leq 1 - 1/k$.

Some more open questions:

Qestion: Is it possible to characterize weakly bounded families purely in terms of the properties of the corresponding $a(\omega)$?

Qestion: If not, is it possible to characterize functionals $\mathcal{M}$ for which

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|} \mathrm{d}\,a(\omega) < \infty$$

purely in terms of the asymptotic behavior of the recursion coefficients γ_n of the corresponding family of orthonormal polynomials?

Periodic functions

- Trigonometric functions do not belong to the spaces Λ_2:

$$\|e^{i\omega t}\|_{\Lambda}^2 = \sum_{n=0}^{\infty} |\mathcal{K}^n[e^{i\omega t}]|^2 = \sum_{n=0}^{\infty} P_n(\omega)^2 \to \infty$$

Definition: Assume $\mathcal{M}$ is weakly bounded. We denote by $\mathcal{C}$ the vector space of analytic functions such that the sequence

$$\nu_n^f(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2$$

converges uniformly on every finite interval.

Definition: Let $\mathcal{C}_0 \subset \mathcal{C}$ consists of $f(t)$ such that

$$\lim_{n\to\infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2 = 0.$$

We define $\mathcal{C}_2 = \mathcal{C}/\mathcal{C}_0$.

Theorem: Let $f, g \in \mathcal{C}$ and

$$\sigma_n^{fg}(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t);$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function.

Definition: For $f, g \in \mathcal{C}$ we define

$$\langle f, g\rangle = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t)$$

- Do the trigonometric functions belong to $\mathcal{C}_2$?

$$\frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} |\mathcal{K}^k[\mathrm{e}^{\mathrm{i}\omega t}]|^2 = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} P_n(\omega)^2$$

- Chebyshev polynomials: $(p = 0)$ if $0 < \omega < \pi$ then

$$\|e^{i\omega t}\| = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P_n^T(\omega)^2 = 1$$

- for all $0 < \sigma, \omega < \pi$, $\sigma \neq \omega$

$$\langle e^{i\sigma t}, e^{i\omega t}\rangle = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P_k^T(\sigma) P_k^T(\omega) = 0$$

- Hermite polynomials: $(p = 1/2)$ for all $\omega, \sigma > 0$, $\omega \neq \sigma$,

$$\|e^{i\omega t}\| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} P_k^H(\omega)^2 = \sqrt{\frac{2}{\pi}}\, e^{\omega^2},$$

$$\langle e^{i\sigma t}, e^{i\omega t}\rangle = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} P_k^H(\sigma) P_k^H(\omega) = 0$$

Thus, in this space every two pure harmonic oscillations with distinct positive frequencies are mutually orthogonal!

Conjecture: Assume that for some $0 \le p < 1$ the recursion coefficients γ_n satisfy

$$0 < \lim_{n\to\infty} \frac{\gamma_n}{n^p} < \infty.$$

Then for the corresponding family of orthogonal polynomials we have

$$0 < \lim_{n\to\infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} P_k(\omega)^2 < \infty$$

for all ω in the support $sp(a)$ of $a(\omega)$.

Numerical experiments indicate that this is true...

It turns out that the special case with $p = 0$ is a previously well known, still open problem (P. Nevai).

New developments:

Definition: A family of polynomials $P_n(\omega)$ is **weakly bounded** if for some $0 \leq p \leq 1$ we have

$$\frac{1}{M} < \gamma_n, \qquad \frac{\gamma_n}{\gamma_{n+1}} < M \qquad \text{and} \qquad \gamma_n < M\, n^p.$$

It turns out that we can replace this with the more general

$$\frac{1}{M} < \gamma_n, \qquad \frac{\gamma_n}{\gamma_{n+1}} < M \qquad \text{and} \qquad \sum_{k=1}^{n} \frac{1}{\gamma_k} \quad \text{diverges}$$

Then, if in all theorems we replace sums of the form $\dfrac{\sum_{k=0}^{n} \cdots}{(n+1)^{1-p}}$ with $\dfrac{\sum_{k=0}^{n} \cdots}{\sum_{k=0}^{n} \frac{1}{\gamma_k}}$ all proofs go through!

The more general conjecture:

Conjecture: Assume that the recursion coefficients γ_n are such that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{\gamma_k}$$

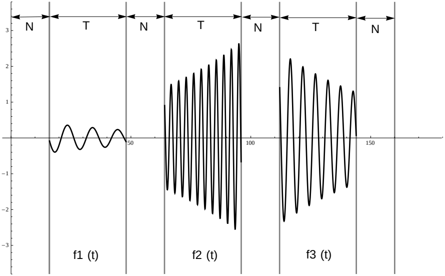
diverges, plus some mild other condition; (for example the assumption that γ_n are non-decreasing suffices but is way an overkill.) Then for the corresponding family of orthogonal polynomials we have

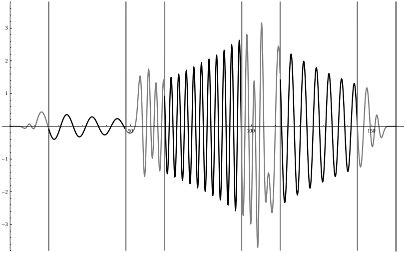
$$0 < \lim_{n\to\infty}\frac{\sum_{k=0}^{n} P_k(\omega)^2}{\sum_{k=0}^{n}\frac{1}{\gamma_k}} < \infty$$

for all ω in the support $sp(a)$ of $a(\omega)$.

Numerical experiments indicate that this is also true...

Application: signal interpolation





Given pieces of band limited signals join them so that the out of band energy is minimal.

We use chromatic expansions to ensure that the resulting signal is N times continuously differentiable Then

$$|\widehat{f}(\omega)| \leq \frac{|\mathcal{K}^n[f]\widehat{\ }(\omega)|}{|P_n(\omega)|} \leq \frac{M}{|P_n(\omega)|}$$

Application: frequency estimation

Idea: *A signal is a sum of at most N shifted and damped sine waves iff it is a solution to a homogeneous linear differential equation with constant coefficients of order at most $2N$.*

A rough sketch of the frequency estimation algorithm:

- Choose the chromatic derivatives which are orthogonal with respect to the power spectrum density of the noise:
  - take polynomials $P_n(\omega)$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(\omega)P_m(\omega)S(\omega)\,d\omega = \delta(m-n)$$

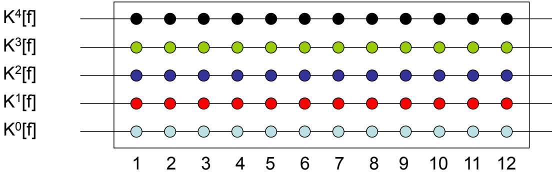
  - Let $\mathcal{K}^n$ be the chromatic derivatives corresponding to the polynomials $P_n(\omega)$, i.e., let

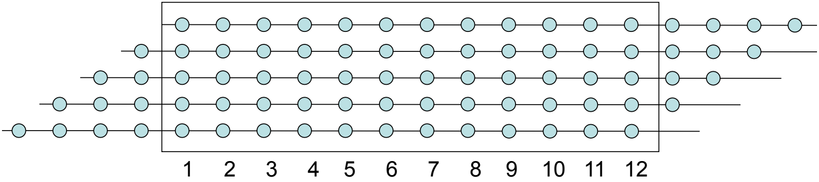
$$\mathcal{K}^n = (-\mathrm{i})^n P_n(-\mathrm{i}\, d/dt).$$

Then, assuming $E[\nu(n)^2] = \rho^2$, we have

$$E\{\mathcal{K}^n[\nu](n)\mathcal{K}^m[\nu](n)\} = \delta(m-n)\rho^2$$

so we can apply the standard SVD or ED methods.





error Cazdow's method: 0.0025;
error Cazdow's method+CD method: 0.0018
(SNR= $-10db$; 10 000 runs)

What if we allow time varying coefficients? We can easily detect chirps, ets. In fact, transients can be classified according to what type of differential eqation they satisfy!

CONJECTURE:

Classification via the minimal degree linear differential equation (with time varying coefficients) satisfied by a transient can play the role which the spectrum plays for the "steady state" signals!!

My website

`http://www.cse.unsw.edu.au/~ignjat/diff/`

contains papers on chromatic derivatives as well as some programs. The most complete presentation is in " Chromatic Derivatives, Chromatic Expansions and Associated Spaces", available as

`http://www.cse.unsw.edu.au/~ignjat/diff/ChromaticDerivatives.pdf`

The programs are mostly an uncommented mess, except perhaps for the tutorial available at the above web page, but I will clean them up and and comment them before the end of the year, hopefully.

If you have a slightest interest in this stuff please do get in touch, I'd love to collaborate!