Maximal Recursive Rule: A New Social Decision Scheme

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Abstract

In social choice settings with strict preferences, *random dictatorship* rules were characterized by Gibbard [1977] as the only randomized social choice functions that satisfy strategyproofness and ex post efficiency. In the more general domain with indifferences, *RSD (random serial dictatorship)* rules are the well-known and perhaps only known generalization of random dictatorship. We present a new generalization of random dictatorship for indifferences called *Maximal Recursive (MR)* rule as an alternative to RSD. We show that MR is polynomial-time computable, weakly strategyproof with respect to stochastic dominance, and, in some respects, outperforms RSD on efficiency.

1 Introduction

Social choice theory is one of the main pillars of microeconomic theory which helps understand and devise methods to aggregate preferences of multiple agents. Although the field is sometimes viewed through a restricted lens of voting theory, social choice theory is broad enough to include various well-studied multi-agent settings including resource allocation, matchings, and coalition formation. The main setting in social choice concerns a set of alternatives *A*, a set of agents *N*, and each agent in *N* expresses preferences over *A*. Based on the preferences of the agents, an alternative is chosen according to some social choice function.

Two of the most fundamental criteria of social choice functions are *Pareto efficiency* and *strategyproofness*. An outcome is Pareto efficient if there is no other outcome that all agents weakly prefer the latter outcome and at least one agent strictly prefers. Strategyproofness captures the idea that no agent should have an incentive to misreport his preferences to get a more preferred outcome. Pareto optimality has been termed "the most important tool of normative economic analysis" [Page 8, Moulin, 2003]. Similarly, seeking strategyproof mechanisms in different domains is a long-standing project within microeconomics. In this regard, a central result is that no non-trivial deterministic resolute social choice function satisfying a weaker property than Pareto optimality is strategyproof [Gibbard, 1973; Satterthwaite, 1975]. Here, we use the standard notion of strategyproofness due to Gibbard, i.e., there are no vNM utilities consistent with an agent's preference profile such that when an agent misreports his preferences, he can increase his expected utility.

In this paper, we focus on randomized social choice functions — also called *social decision schemes* [see e.g., Gibbard, 1977; Barberà, 1979b]. A *social decision scheme (SDS)* is a function which takes a preference profile and returns a lottery over *A* based on the preference profile. A probabilistic approach to social choice has various virtues. Randomization is one of the most common ways to regain ex ante fairness. Consider two agents having opposite preferences over two alternatives. Selecting an alternative with equal probability seems to be the fair course of action. SDSs are especially important when agents want to decide on sharing a divisible resource like time or space. In this case, the probability which an alternative gets from the lottery can be interpreted as the proportion of time or space the agents decide to allot to some alternative [Example 3.6, Moulin, 2003]. Social decision schemes have another motivation that the probability of each alternative can also be used as a suggestion for the proportional representation of the alternative in representative democracy or seat allocation of a parliament [Tullock, 1967]. Non-deterministic voting rules have been used in history, for example, in Ancient Greece and Renaissance Italy [see, e.g., Stone, 2011; Walsh and Xia, 2012]. SDSs are not only a topic of recent research in political science [see, e.g., Stone, 2011] but also in computer science [e.g., Conitzer and Sandholm, 2006; Procaccia, 2010]. In view of the negative strategyproofness results regarding deterministic social choice functions, it is also natural to consider social decision schemes.

In this paper, we propose a new social decision scheme called *MR (Maximal Recursive)* which satisfies a number of desirable properties pertaining to economic efficiency, strategyproofness, and efficient computability. The rule is based on using (generalized) plurality scores of the alternatives and the idea of *inclusion minimal subsets* to recursively adjust the probability weights of the alternatives. We will show that MR is an interesting generalization of the well-studied *random dictatorship* rule and has some favourable properties in comparison to *RSD (random serial dictatorship)* rule which is a well-known generalization of random dictatorship.

2 Seeking efficiency and strategyproofness

For social decision schemes, *ex post efficiency* is a minimal requirement of efficiency in the sense that no Pareto dominated alternative gets non-zero probability. Gibbard [1977] proved that for social choice settings where agents express strict preferences over alternatives, there exist no ex post efficient and strategyproof decision schemes other than RD (random dictatorship).* In RD, an agent is chosen randomly as the dictator who then has the privilege to select his unique maximally preferred alternative. If one requires anonymity so that the rule does not depend on the names of the agents, then we obtain the unique RD which gives equal probability to each agent being the dictator. The resulting rule simply chooses an alternative in proportion to its plurality score. This rule has a number of merits. In political science, it is generally referred to as *lottery voting* [see e.g., Saunders, 2010; Stone, 2011]. The probability which each alternative achieves is proportional to the voting weight it gets in Tullock's method for representative democracy where alternatives are considered representatives of the voters [Potthof and Brams, 1998].

Although much of work in the voting literature assumes that agents have strict preferences over alternatives, it is natural to consider domains where agents may express indifferences among alternatives. The indifference could be because an agent does not have enough information to decide which alternative is better or it could simply be the case that the agent values both alternatives equally. In fact, in many important domains of social choice such as resource allocation and matching, ties are unavoidable because agents are indifferent among all outcomes in which their allocation or match is the same [e.g., Sönmez and Ünver, 2011; Bouveret and Lang, 2008]. If agents express indifferences over (pure) alternatives, then RD needs to be reformulated so that ex post efficiency is satisfied. In the literature there is a well-known generalization of RD called *RSD (random serial dictatorship)* for the domain with indifferences. RSD uniformly randomizes over $n!$ lotteries — one for each permutation π over N. A lottery for a particular π is computed as follows. Agent $\pi(1)$ chooses the set of maximally preferred alternatives from A, $\pi(2)$ chooses the maximally preferred alternatives from the refined set and so on so until a final subset of A is left. Then, the uniform lottery over the alternatives the final refined set is chosen as the lottery corresponding to a given permutation π. RSD is well-established in resource allocation and especially in the restricted domain of social choice called the assignment problem [see e.g., Svensson, 1994; Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2001].† For social choice settings with indifferences, RSD is strategyproof in the Gibbard sense and it is also ex post efficient.

Despite the central position of RSD as the natural extension of RD to the general domain, RSD is not without some criticism. Critiques of RSD include efficiency losses [Aziz *et al.*, 2013b; Bogomolnaia and Moulin, 2001] and computational overhead. Although RSD is a well-established mechanism, Bogomolnaia and Moulin [2001] demonstrated that RSD suffers from unambiguous efficiency losses even in the restricted social choice domain of assignment problems. In particular, it does not satisfy SD-efficiency (stochastic dominance efficiency) — an efficiency notion that is stronger than ex post efficiency. Although RSD can be used efficiently to randomly return a single alternative, it is computationally demanding to analyze since computing exactly the probability of an alternative being chosen requires enumerating $|N|!$ permutations: no polynomial-time algorithm is known. In fact, recently, computing the lottery returned by RSD has been shown to be #P-complete to compute [Aziz *et al.*, 2013a] which implies that there exists no polynomial-time algorithm unless a statement even stronger than P=NP holds. In light of these issues, we are motivated to find some other generalization of RD that is computationally efficient to handle and fares better than RSD in terms of efficiency. We present a new generalization of random dictatorship for indifferences, namely the Maximal Recursive (MR) rule.

In order to analyze MR, we undertake a more nuanced investigation of social decision schemes by considering strategyproofness (abbreviated as SP) and efficiency with respect to different *lottery extensions* just like in [Aziz *et al.*, 2013b; Cho, 2012]. A lottery extension specifies how preferences over alternatives are extended to preferences over lotteries. We consider three lottery extensions: SD (stochastic dominance), DL (downward lexicographic) and DL^1 (downward lexicographic one). All of these lottery extensions define relations on lotteries. One lottery *stochastically dominates* another if for all utility representations consistent with the ordinal preferences, the former yields as much utility as the latter [see, e.g., Cho, 2012]. As the name suggests DL is based on lexicographic comparisons. DL is a refinement of the SD lottery extension. DL is also a refinement of DL^1—a lottery extension in which lotteries are compared only with respect to the probability assigned to the maximally preferred alternatives. DL^1 is a natural preference relation and models situations in which an agents wants to maximize the probability of his most preferred alternatives. Maximizing the probability of the most preferred alternative has been examined before [see e.g., Conitzer *et al.*, 2007].

Our main contribution is proposing the *Maximal Recursive (MR)* Rule and showing that MR satisfies various desirable axiomatic properties. MR is computationally-efficient and ex post efficient. In fact for many instances, it returns an SD-efficient lottery even though RSD returns a lottery that is not SD-efficient. On the other hand, MR satisfies DL-strategyproofness—a weaker notion of strategyproofness than the one satisfied by RSD. MR applies to any setting in which agents have preferences over outcomes and these preferences need to be aggregated into a probability distribution over the outcomes. Our results are summarized in the following central theorem.

*SDSs that assign probabilities to alternatives in proportion to their Borda or Copeland scores [see e.g., Barberà, 1979a; Conitzer and Sandholm, 2006; Procaccia, 2010] are strategyproof but they are not ex post efficient.

†In his original paper, Gibbard [1977] already gave a partial description of RSD in which a dictator chooses his maximally preferred alternatives and then his 'henchman' refined the set further. In the resource allocation literature, RSD is also referred to as *RP (random priority)* [see e.g., Bogomolnaia and Moulin, 2001].

Theorem 1 *MR is*

(*i*) *single-valued, anonymous, neutral and monotonic;*

(*ii*) *computable in polynomial time;*

(*iii*) *equivalent to RD for strict preferences and hence both SD-SP and SD-efficient for strict preferences;*

(*iv*) *DL-SP (and hence weak SD-SP) but not SD-SP;*

(*v*) *ex post efficient;*

(*vi*) *SD-SP for dichotomous preferences;*

(*vii*) *SD-efficient for some instances for which RSD is not SD-efficient.*

The statement is proved in a series of propositions. Before we proceed, the main concepts are defined and then MR is presented.

3 Preliminaries

Social choice and lotteries. Consider the social choice setting in which there is set of agents $N = \{1, \dots, n\}$, a set of alternatives $A = \{a_1, \dots, a_m\}$ and a preference profile $R = (R_1, \dots, R_n)$ such that each R_i is a complete and transitive relation over A. We have $a\ R_i\ b$ denote that agent i values alternative a at least as much as alternative b and write P_i for the strict part of R_i, that is, $a\ P_i\ b$ if $a\ R_i\ b$ but not $b\ R_i\ a$. I_i denotes i's indifference relation, that is, $a\ I_i\ b$ if both $a\ R_i\ b$ and $b\ R_i\ a$. Whenever all R_i are antisymmetric, we say that agents have *strict* preferences. For any $S \subseteq A$, we will denote by $\max_{R_i}(S)$ the alternatives in S that are maximally preferred by i. The relation R_i results in equivalence classes $E_i^1, E_i^2, \dots, E_i^{k_i}$ for some k_i such that $a\ P_i\ a'$ for $a \in E_i^l$ and $a' \in E_i^{l'}$ for $l < l'$. Often, we will use equivalent classes to represent the preference relation of an agent as a preference list

$$i : E_i^1, E_i^2, \dots, E_i^{k_i}.$$

For example, we will denote the preference $a\ I_i\ b\ P_i\ c$ by following list

$$i : \quad \{a, b\}, \{c\}.$$

An agent's preferences are *dichotomous*, if he partitions the alternatives into two equivalence classes E_i^1 and E_i^2, i.e., $k_i = 2$. An agent's preferences are *strict* if he is not indifferent between any alternatives, i.e., $k_i = m$.

Let $\Delta(A)$ denote the set of all *lotteries* (or *probability distributions*) over A. The support of a lottery $p \in \Delta(A)$, denoted by $\text{supp}(p)$, is the set of all alternatives to which p assigns a positive probability, i.e., $\text{supp}(p) = \{x \in A \mid p(x) > 0\}$. We will represent a lottery in the following way

$$[a_1 : p_1, \dots, a_m : p_m],$$

where p_j is the probability of alternative a_j for $j \in \{1, \dots, m\}$.

Lottery extensions. In order to reason about the outcomes of social decision schemes, we need to reason about how how agents compare lotteries. A *lottery extension* maps preferences over alternatives to (possibly incomplete) preferences over lotteries. Given R_i over A, one can extend R_i to R_i^e over $\Delta(A)$ where R_i^e is a relation between lotteries based on lottery extension e. We now define particular lottery extensions including the standard extensions *SD (stochastic dominance)* and *DL (downward lexicographic)* [see e.g., Bogomolnaia and Moulin, 2001; Schulman and Vazirani, 2012; Abdulkadiroğlu and Sönmez, 2003; Cho, 2012]. We also consider an extension DL[1] which is coarser than DL. For a lottery p and $A' \subseteq A$, we will denote $\sum_{a \in A'} p(a)$ by $p(A')$. Consider two lotteries $p, q \in \Delta(A)$. Then, the SD lottery extension is defined as follows.

$$p\ R_i^{SD}\ q \text{ iff } \sum_{j=1}^{l} p(E_i^j) \geq \sum_{j=1}^{l} q(E_i^j) \text{ for all } l \in \{1, \dots, k_i\}.$$

Next we define the DL lottery extension. Let $l \in \{1, \dots k_i\}$ be the smallest integer such that $p(E_i^l) \neq q(E_i^l)$. Then,

$$p\ P_i^{DL}\ q \text{ iff } p(E_i^j) > q(E_i^j).$$

If there exists no integer $l \in \{1, \dots k\}$ such that $p(E_i^l) \neq q(E_i^l)$ then $p\ I_i^{DL}\ q$.

Finally, we define the lottery extension DL[1].

$$p\ R_i^{DL^1}\ q \text{ iff } p(E_i^1) \geq q(E_i^1).$$

It is clear that $p\ R_i^{SD}\ q \implies p\ R_i^{DL}\ q$ i.e., DL is a refinement of SD. Also DL is a refinement of DL[1].

Fact 1 $R^{SD} \subset R^{DL}$. *Also,* $R^{DL^1} \subset R^{DL}$.

To clarify the differences between the three lottery extensions, we provide the following example.

Example 2 *For $A = \{a, b, c, d\}$, consider lotteries p and q over A: $p = [a : 0.4, b : 0.1, c : 0.5, d : 0]$ and $q = [a : 0.4, b : 0.3, c : 0, d : 0.3]$. Then for an agent i with preferences $a\ P_i\ b\ P_i\ c\ P_i\ d$, his preferences between the lotteries p and q are as follows: $q\ P_i^{DL}\ p$; $p\ I_i^{DL^1}\ q$; and $\neg(p\ P_i^{SD}\ q)$ and $\neg(q\ P_i^{SD}\ p)$.*

Efficiency and strategyproofness based on lottery extensions. Based on a lottery extension, one can define an appropriate notion of efficiency and strategyproofness. For a lottery extension e, lottery p is *e-efficient* if there exists no lottery q such that $q\ R_i^e\ p$ for all $i \in N$ and $q\ P_i^e\ p$ for some $i \in N$. An SDS which always returns an e-efficient lottery is termed e-efficient. A standard efficiency notion that cannot be phrased in terms of lottery extensions is *ex post efficiency* also known as *Pareto-optimality*. A lottery is ex post efficient if it is a lottery over Pareto optimal alternatives.

Fact 2 *DL-efficiency $\implies$ SD-efficiency $\implies$ ex post efficiency. Also SD-efficiency $\implies$ weak SD-efficiency, and DL-efficiency $\implies$ DL[1]-efficiency.*

An SDS φ is *weak e-SP* if for every preference profile R, there exists no R_i' for some agent $i \in N$ such that $\varphi(R_i', R_{-i})\ P_i^e\ \varphi(R_i, R_{-i})$. An SDS φ is *e-SP* if for every preference profile R, and every R_i' for all $i \in N$, $\varphi(R_i, R_{-i})\ R_i^e\ \varphi(R_i', R_{-i})$. For a complete lottery extension such as DL or DL[1], SP and the weak notion of SP coincide. This is not the case for SD for which SD-SP is strictly stronger than weak SD-SP.

Fact 3 *SD-SP $\implies$ DL-SP $\implies$ weak SD-SP. Also DL-SP $\implies$ DL^1-SP.*

Note that SD-SP is equivalent to strategyproofness in the Gibbard sense.

We also present a formal definition of RSD. Let Π^N be the set of permutations over N and $\pi(i)$ be the i-th agent in permutation $\pi \in \Pi^N$. Then,

$$RSD(N, A, R) = \sum_{\pi \in \Pi^N} \frac{1}{n!} \delta_{\text{uniform}}(Prio(N, A, R, \pi))$$

where

$$Prio(N, A, R, \pi) = \max_{R_{\pi(n)}}(\max_{R_{\pi(n-1)}}(\cdots(\max_{R_{\pi(1)}}(A))\cdots)),$$

$\delta_{\text{uniform}}(B)$ is the uniform lottery over the given set B, and $\max_{R_i}(A')$ is the set of alternatives in A' maximally preferred by agent i.

4 Maximal Recursive Rule

We now present the Maximal Recursive (MR) rule. MR relies on the concept of IMS (inclusion minimal subsets). For $S \subseteq A$, let $A_1, \ldots, A_{m'}$ be subsets of S. Let $I(A_1, \ldots, A_{m'})$ be the set of non-empty intersection of elements of some subset of $\{A_1, \ldots, A_{m'}\}$.

$$\begin{aligned}&I(A_1, \ldots, A_{m'})\\ =&\{X \in 2^A \setminus \emptyset : X = \bigcap_{A_j \in A'} A_j \text{ for some } A' \subseteq \{A_1, \ldots, A_{m'}\}\}.\end{aligned}$$

Then, the inclusion minimal subsets of $S \subseteq A$ with respect to $(A_1, \ldots, A_{m'})$ are defined as follows.

$$\begin{aligned}&IMS(A_1, \ldots, A_{m'})\\ =&\{X \in I(A_1, \ldots, A_{m'}) : \nexists X' \in I(A_1, \ldots, A_{m'})\\ &\text{ s.t. } X' \subset X\}.\end{aligned}$$

Example 3 *Consider the sets $A_1 = \{a, b, c, d\}$, $A_2 = \{a, b\}$, $A_3 = \{d, e, f\}$, $A_4 = \{f\}$. Then, $IMS(A_1, \ldots, A_4) = \{\{a, b\}, \{d\}, \{f\}\}$.*

Inclusion minimal subsets can be computed efficiently.

Lemma 1 *The set $IMS(A_1, \ldots, A_{m'})$ can be computed in polynomial time.*

Proof sketch: We first identify alternatives that are dominated. We say that $a \in A$ is dominated by $b \in A$ if for each A_j such that $a \in A_j$, it holds that $b \in A_j$ and there exists some A_j such that $a \notin A_j$ but $b \in A_j$. We put all the alternatives not dominated in set S. We now partition S as follows. If there exist no A_j such that only one of a or b is in A_j then we connect a and b. By doing this we can partition S into connected components. Each component corresponds to a member of $IMS(A_1, \ldots, A_{m'})$. □

Input: (A, N, R)
Call MR-subroutine$(A, 1, (A, N, R))$ to compute $p(a)$ for each $a \in A$.
return $[a_1 : p(a_1), \ldots, a_m : p(a_m)]$

Algorithm 1: MR

Input: $(S, v, (A, N, R))$
if $\max_{R_i}(S) = S$ for all $i \in N$ **then**
  $p(a) = v/|S|$ for all $a \in S$
else
  $T(i, S, R) \longleftarrow \{a : a \in \arg\max_{a \in \max_{R_i}(S)} s^1(a, S, R)\}$ for all $i \in N$
  $t(i, a, R) \longleftarrow 1/|T(i, S, R)|$ if $a \in T(i, S, R)$ & zero otherwise for all $i \in N$ & $a \in S$
  $\gamma(a) \longleftarrow \sum_{i \in N} t(i, a, R)$ for all $a \in S$
  $p(a, R) \longleftarrow v(\gamma(a, R))/|N|$ for all $a \in S$
  $\{S_1, \ldots, S_k\} \longleftarrow IMS(\max_{R_1}(S), \ldots, \max_{R_n}(S))$
  for each $S_j \in \{S_1, \ldots, S_k\}$ **do**
    return MR-subroutine$(S_j, p(S_j), (A, N, R))$
  end for
end if

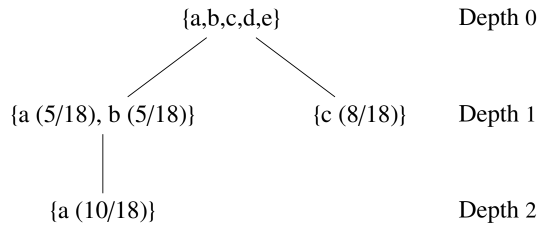
Algorithm 2: MR-subroutine

Now that we have defined IMS, we are ready to present MR. MR is summarized as Algorithm 1 which calls a subroutine (Algorithm 2) and computes a lottery p such that $p(a)$ is the probability of $a \in A$. We will denote by $s^1(a, S, R)$ the *generalized plurality score* of a according to R when the alternative set and the preference profile is restricted to S.

$$s^1(a, S, R) = |\{i \in N : a \in \max_{R_i}(S)\}|.$$

In MR, the probability weight of a set of alternatives is recursively redistributed over disjoint subsets of the alternatives. MR starts with the set of all alternatives for which the probability weight of one is to be redistributed. In MR-subroutine, S is a subset of A, and v is the total probability weight of S that is to be redistributed among the elements of S. In the first call of MR-subroutine, $S = A$ and $v = 1$. Each agent i selects $T(i, S, R)$ the set of his most preferred alternatives from within the set S with the maximum generalized plurality score with respect to R. Each $i \in N$ uniformly divides a total score of one among some of the most favoured alternatives in S: each $i \in N$ gives $1/|T(i, S, R)|$ to each element of $T(i, S, R)$. Based on these contributions of scores, each alternative a gets a total score $\gamma(a, R)$ which is then normalized by the sum of the total scores of all alternatives. The resultant fraction $\gamma(a, R)/\sum_{a_j \in A} \gamma(a_j, R) = \gamma(a, R)/|N|$ is the fraction of probability weight v which alternative a gets. The intersections of $T(i, S, R)$ for each $i \in N$ result in minimal intersecting sets each with a total weight equaling the sum of the weights of alternatives in the intersecting sets. The process is recursively repeated until the maximal elements of each set for each agent equal the set itself.

The algorithm gives rise to a recursion tree in which the root is labelled by A and is at depth zero. Each node is labeled by a subset of A. At any depth of the tree, the sets of the nodes of the tree are disjoint. Consider the leaves $L_1, \ldots, L_k$. Then



$1: \{a,b,c,d\},\{e\}$ $\quad$ $2: \{a,b\},\{c,d\},\{e\}$

$3: \{c,e\},\{a\},\{d\},\{b\}$

Figure 1: Running MR on the preference profile above. The lottery returned is $[a : 10/18, b : 0, c : 8/18]$. RSD would have returned $[a : 1/2, b : 0, c : 1/2]$.

$\bigcup_{i\in\{1,\ldots,k\}} L_i = \mathrm{supp}(p)$. Also note that $a \in S$ for some node labeled S, then a is in each ancestor of S. Consider any node labeled S, and consider a child S' of S such that $a, a' \in S'$. Then, for each agent i, $a \in \max_{R_i}(S) \iff a' \in \max_{R_i}(S)$.

Note that we start the whole set A at depth 0. At depth k, the amount of weight put on the first k equivalence classes is fully set for each agent i.e., it will not change.

Proposition 1 *MR is single-valued and runs in polynomial time.*

In each depth of the recursion tree, a lottery is maintained. In each depth there are at most $|A|$ function calls to the MR-subroutine, each taking time $O(|A|^3 n)$. The depth of the recursion tree can be at most $|A|$. Hence the total time taken is polynomial in the input. A unique lottery is returned in the end.

It can easily be seen that MR does not depend on the identities of the agents and alternatives but only on the preference profile. MR is also monotonic i.e., if an alternative is reinforced, then its probability cannot decrease.

Proposition 2 *MR is anonymous, neutral, and monotonic.*

Just like RSD, MR is a proper extension of the anonymous random dictator rule.

Proposition 3 *MR is an RD rule for strict preferences and is therefore SD-SP and SD-efficient for strict preferences.*

5 Efficiency

Ex post efficiency is a minimal efficiency requirement for social decision schemes. We show that MR is ex post efficient.

Proposition 4 *MR is ex post efficient.*

Proof Sketch: Let p be the lottery returned by MR. Consider an alternative a such that a is Pareto dominated by b. Without loss of generality, assume that b is Pareto optimal. We will show that a gets zero probability. If a is a maximally preferred alternative for some agent, then so is b. If $a \in S^1$ for some $S^1 \in \{S_1, \ldots, S_l\}$ for the sets in depth one, then $b \in S^1$. By the same argument, if $a \in S^k \subset S^{k-1} \subset \cdots \subset S^1$, then $b \in S^k \subset S^{k-1} \subset \cdots \subset S^1$ where S^k is the set containing a and b at recursion depth k. Finally, since b Pareto dominates a, there exists some $i \in N$ such that $b \ P_i \ a$. Hence, there exists some $k' < |A|$ such that $a \notin S^{k'} \subset S^{k'-1} \subset \cdots \subset S^1$ and $b \in S^{k'} \subset S^{k'-1} \subset \cdots \subset S^1$. Therefore a Pareto dominated alternative gets probability zero. Hence MR is ex post efficient. □

On a number of instances, MR fares much better than RSD from the point of view of economic efficiency.

Proposition 5 *There exist instances for which the MR lottery is SD-efficient but the RSD lottery is neither SD-efficient nor DL^1-efficient.*

Proof: Consider the following preference profile presented first in [Aziz *et al.*, 2013b].

$1: \{a,c\},\{b\},\{d\}$ $\quad$ $2: \{b,d\},\{a\},\{c\}$

$3: \{a\},\{d\},\{b,c\}$ $\quad$ $4: \{b\},\{c\},\{a,d\}$

The outcome of RSD is $[a : 5/12, b : 5/12, c : 1/12, d : 1/12]$ which is SD dominated by the MR outcome $[a : 6/12, b : 6/12, c : 0, d : 0]$. □

Proposition 6 *There exist instances with indifferences for which the MR lottery is the same as the RSD lottery and the lottery is not SD-efficient.*

Proof: Consider the following preference profile.

$1: \{a_1\},\{a\},\{d\},\{b,c\}$ $\quad$ $2: \{a_1\},\{b\},\{c\},\{a,d\}$

$3: \{a,c\},\{a_1,b,d\}$ $\quad$ $4: \{b,d\},\{a_1,a,c\}$

Then, the MR outcome as well as the RSD outcome is $[a_1 : 1/2, a : 1/8, b : 1/8, c : 1/8, d : 1/8]$. The outcome is not SD-efficient since it is SD dominated by $[a_1 : 1/2, a : 1/4, b : 1/4$. □

6 Strategyproofness

In this section we examine the strategyproofness aspects of MR. We show that MR is DL-SP and hence weak SD-SP.

Proposition 7 *MR is DL-SP.*

Proof Sketch: We show that at each depth in the recursion tree starting from depth 0, and for each set $S \subseteq A$ corresponding to the node in the recursion tree, each agent $i \in N$ has no incentive other than to express $\max_{R_i}(S)$ as the maximally preferred alternatives in S. We will denote by $MR(A', (R_i, R_{-i}))$ the sum of probability weight of alternatives in A' with respect to the lottery returned by MR on preference profile (R_i, R_{-i}). Note that in MR, for any set $S \subseteq A$ and any $i \in N$, once the probability weight for $\max_{R_i}(S)$ has been fixed, it cannot decrease.

Let us assume that agent i expresses preferences $R_i' \neq R_i$. We show that $MR(\max_{R_i}(S), (R_i'', R_{-i})) \geq MR(\max_{R_i}(S), (R_i', R_{-i}))$ if R_i'' is such that $\max_{R_i''}(S) = \max_{R_i'}(S) \cup \max_{R_i}(S)$ for all $S \subseteq A$ or if R_i'' is such that $\max_{R_i''}(S) = \max_{R_i'}(S) \cap \max_{R_i}(S)$ for all $S \subseteq A$. Note that the only sets S that matter are the IMS sets encountered during the MR algorithm.

Consider the preference profile R_i'' such that $\max_{R_i''}(S) = \max_{R_i'}(S) \cup \max_{R_i}(S)$ for all $S \subseteq A$. For all $a \in \max_{R_i}(S)$, if $a \in \max_{R_i'}(S)$, then $a \in \max_{R_i''}(S)$. For any agent $j \neq i$, if $a \in T(j, S, (R_i', R_{-i}))$, then it must be the case that $a \in T(j, S, (R_i'', R_{-i}))$. Thus, $\sum_{a\in\max_{R_i}(S)} t(j, a, (R_i', R_{-i}) \leq \sum_{a\in\max_{R_i}(S)} t(j, a, (R_i'', R_{-i})$ for all $j \in N$. Hence, $MR(\max_{R_i}(S), (R_i'', R_{-i})) \geq MR(\max_{R_i}(S), (R_i', R_{-i}))$.

Consider the preference profile R_i'' such that $\max_{R_i''}(S) = \max_{R_i'}(S) \cap \max_{R_i}(S)$ for all $S \subseteq A$. Thus, $\sum_{a\in\max_{R_i}(S)} t(j, a, (R_i', R_{-i}) \leq \sum_{a\in\max_{R_i}(S)} t(j, a, (R_i'', R_{-i})$ for all $j \in N$. Hence, $MR(\max_{R_i}(S), (R_i'', R_{-i})) \geq MR(\max_{R_i}(S), (R_i', R_{-i}))$.

Since DL preferences are strict and complete, any preference R_i' that yields a different lottery will be strictly less preferred by agent i. □

Corollary 1 *MR is weak SD-SP and weak DL[1]-SP.*

Corollary 2 *MR is SD-SP for dichotomous preferences.*

In other words, an agent cannot misreport his preferences to get an unambiguous improvement in expected utility or to maximize the probability of his most preferred alternatives.

Whereas MR is DL-SP, it is not SD-SP like RSD.

Proposition 8 *MR is not SD-SP.*

Proof: Consider the following preference profile.

$1 : \{a_1\}, \{a_2\}, \{a_3\}$	$2 : \{a_1\}, \{a_2\}, \{a_3\}$
$3 : \{a_2, a_3\}, \{a_1\}$	$4 : \{a_2, a_3\}, \{a_1\}$
$5 : \{a_3\}, \{a_2\}, \{a_1\}$	

Then MR returns the lottery $[a_1 : 2/5, a_2 : 0/5, a_3 : 3/5]$. If agent 1 submits the preferences $R_1' : \{a_1, a_2\}, \{a_3\}$, then, the MR lottery is $[a_1 : 1/5, a_2 : 2/5, a_3 : 2/5]$. Therefore it is not the case that $MR(R_1, R_{-1})\ R_1^{SD}\ MR(R_1', R_{-1})$. □

7 Discussion

Properties	RSD	MR
SD-SP & SD-efficient for strict preferences	+	+
SD-efficient	-	-
Ex post efficient	+	+
SD-SP	+	-
DL-SP	+	+
Weak SD-SP	+	+
Polynomial-time algorithm to compute the lottery	–[a]	+
Monotonic, anonymous, and neutral	+	+

[a] #P-complete

Table 1: Properties satisfied by RSD and MR.

We presented a new social decision scheme called MR as an alternative to RSD. We showed that MR has both computational and efficiency advantages over RSD.

MR also fares well against other social decision schemes. Recently *strict maximal lotteries* [Kreweras, 1965; Fishburn, 1984] have been proposed as an alternative to RSD [Aziz *et al.*, 2013b]. However, strict maximal lotteries are not even weak SD-SP for strict preferences and they are P-complete to compute (hence at least as computationally hard as linear programming). On the other hand, maximal lotteries are attractive from an efficiency point of view since they are SD-efficient. It will be interesting to see whether there is a way to modify MR so that it maintains its strategic properties but additionally satisfies SD-efficiency. Just like MR, another weak SD-SP rule is *Condorcet* which gives probability 1 to a Condorcet winner or else gives uniform probability to all the alternatives [Theorem 1, Postlewaite and Schmeidler, 1986]. However the Condorcet social decision scheme is not ex post efficient.

Some interesting questions still remain to be answered. For example, does there exist an anonymous non-RD rule that is both weak SD-SP and ex post efficient? Does there exist an anonymous rule that is both DL-efficient and DL-SP? Finally, most of the research on strategyproofness and efficiency concepts based on lottery extensions has been conducted in the restricted domain of assignments problems. The same framework, when applied to the general domain of social choice, will open up new research frontiers.

Acknowledgements

This material is based upon work supported by the Australian Government's Department of Broadband, Communications and the Digital Economy, the Australian Research Council, the Asian Office of Aerospace Research and Development through grant AOARD-124056. The author would like acknowledge the useful feedback of Markus Brill, Nick Mattei, and the anonymous referees. He also thanks Markus Brill and Felix Brandt for a number of insightful discussions on the topic.

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