

Best of Both Worlds: Fair Lotteries over Core Committees

Haris Aziz

School of Computer Science and Engineering, UNSW Sydney, Australia

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Abstract

We study lotteries over fixed-size committees under approval preferences. We prove that ex ante property of group resource proportionality (GRP) is compatible with core stability ex post, and that such a lottery can be computed in polynomial time with support of size at most one more than the number of candidates. This strengthens several earlier best-of-both-worlds guarantees for committee voting. Although strict Droop-core committees can be computed in polynomial time, we show that any fixed reduction below the Hare quota makes even ex-post justified representation incompatible with ex-ante positive share.

Keywords: Approval-based committee voting, Core stability, Group resource proportionality, Randomized social choice

1. Introduction

Approval-based committee voting selects a fixed number of candidates from voters' approval sets. Each voter values a committee by the number of its members that she approves. The central fairness objective is proportional representation: groups of voters should receive representation commensurate with their size [8]. Randomization allows this objective to be imposed at two levels. Ex-ante guarantees concern the marginal selection probabilities, whereas ex-post guarantees must hold for every realized committee. A fair lottery should provide meaningful representation in expectation while ensuring that every possible outcome is itself fair.

This best-of-both-worlds perspective was developed in fair allocation [2] and introduced to committee voting by Aziz et al. [3]. Subsequent work has strengthened both sides of the guarantee. On the ex-ante side, Suzuki and Vollen [10] introduced *group resource proportionality* (GRP), which protects every coalition's share of the candidates approved by its members. GRP

implies group fair share (GFS) and strong unanimous fair share (Strong UFS) [10, Proposition 3.6], the strongest axioms in the two fair-share hierarchies of Aziz et al. [3], and hence also their individual fair share (IFS) and unanimous fair share (UFS). On the ex-post side, the *core* rules out a coalition that can make all its members better off by selecting an alternative set using its proportional share of the seats. Unlike representation axioms restricted to cohesive groups, the core permits arbitrary coalitions and proposals. Becker et al. [5] recently proved that a core committee always exists and can be computed in polynomial time.

We show that these two guarantees can be achieved simultaneously. Every approval election admits a polynomial-time computable lottery whose marginals satisfy GRP and whose support consists entirely of core committees. The support has size at most $m + 1$, where m is the number of candidates. Our proof combines the harmonic-entropy method of Becker et al. [5] with the affordable-set completion theorem of Suzuki and Vollen [10]. The additional step is to find an affordable partial committee whose every completion lies in the core. We obtain it by proving a deletion bound for every subset that violates affordability and analyzing a finite harmonic-entropy objective directly. A local search then establishes existence and polynomial-time computation simultaneously. Conceptually, a partial committee fixed with certainty carries ex-post stability, while randomization over the remaining seats carries ex-ante proportionality. We call such a set a *core anchor*: an affordable set of candidates such that every committee containing it is in the core. As a structural result of independent interest, every election admits a core anchor, computable in polynomial time (Theorem 2). This strengthens the existence of core committees [5], since the seats outside an anchor can be filled arbitrarily.

We also show that the Hare quota cannot be reduced while retaining this compatibility. The method of Becker et al. [5] computes core committees even under the strict Droop quota, but after any fixed reduction below Hare, even ex-post justified representation is incompatible with the weakest ex-ante guarantee, positive share (Theorem 3).

The choice of the ex-ante guarantee matters. Jin et al. [7] show that even ex-ante weak Pareto optimality is incompatible with ex-post justified representation, a requirement weaker than the core. Consequently, ex-ante core stability cannot be combined with ex-post core stability. GRP provides a strong ex-ante proportionality guarantee that remains attainable under this ex-post requirement.

2. Related work

Aziz et al. [3] initiated the study of fair-share axioms for randomized approval-based committee voting. They gave a polynomial-time lottery combining ex-post extended justified representation with ex-ante group fair share and strong unanimous fair share. Their work also established compatibility of ex-post full justified representation with strong unanimous fair share. The same authors extended the approach to participatory budgeting, where unequal project costs create additional rounding difficulties [4].

The ex-post representation hierarchy began with justified and extended justified representation (EJR) [1]. Full justified representation (FJR), introduced by Peters et al. [9], further relaxes the agreement required within a protected coalition. All these requirements are weaker than the core. The harmonic-entropy construction of Becker et al. [5] establishes core existence and polynomial-time computation, as well as a stronger property called core+. Its payment certificate and addition bound are the starting point of our proof. We depart from their argument in three places. We price seats instead of fixing the committee size, so that the search ranges over sets of all sizes; we prove a deletion bound for every subset that violates affordability, not only for the committee as a whole; and we analyze a finite objective directly, so that one local search yields both existence and polynomial-time computation.

The closest ex-ante result is due to Suzuki and Vollen [10]. They characterize GRP by maximum flows and prove that every affordable partial committee can be completed to a lottery satisfying GRP. They use this to obtain GRP together with ex-post full justified representation, as well as a polynomial-time construction with ex-post EJR+, a strengthening of extended justified representation. We use their completion theorem directly. Our contribution is the core anchor, an affordable partial committee that already ensures core stability for every completion, together with a polynomial-time procedure for finding it. The impossibilities of Jin et al. [7] provide the contrasting boundary for ex-ante core stability and efficiency.

3. Preliminaries

Let $N = \{1, \dots, n\}$ be a set of $n \geq 1$ voters, let C be a set of m candidates, and let $1 \leq k \leq m$ be the committee size. Voter i approves $A_i \subseteq C$ and has utility $u_i(W) = |A_i \cap W|$ for a set $W \subseteq C$. We reserve $q = n/k$ for the Hare quota and write q' for any other quota. Write $W + c = W \cup \{c\}$,

$W - c = W \setminus \{c\}$, and

$$A(S) = \bigcup_{i \in S} A_i, \quad N(D) = \{i \in N : A_i \cap D \neq \emptyset\}.$$

A *committee* has exactly k members, whereas a *partial committee* has at most k members.

A *lottery* is a distribution $(\lambda_h, W_h)_{h=1}^s$ over committees, with $\lambda_h > 0$ and $\sum_h \lambda_h = 1$. Its marginal vector is $p = \sum_h \lambda_h \mathbf{1}_{W_h}$. A *fractional committee* is a vector $p \in [0, 1]^C$ with $p(C) = k$, where $p(D) = \sum_{c \in D} p_c$. Every such vector admits an implementing lottery. For any fractional vector z , write $u_i(z) = z(A_i)$. In particular, $u_i(p)$ is voter i 's expected utility.

Definition 1 (Core [1]). A committee W is in the *core* if there are no nonempty $S \subseteq N$ and $T \subseteq C$ such that $q|T| \leq |S|$ and $u_i(T) > u_i(W)$ for every $i \in S$. A lottery satisfies the core ex post if every committee in its support is in the core.

For a quota $q' > 0$, core stability at q' replaces the affordability condition in Definition 1 by $q'|T| \leq |S|$. The *strict Droop core* instead uses $q' = n/(k+1)$ and the strict condition $q'|T| < |S|$. Unless stated otherwise, core stability and GRP use the Hare quota q .

Definition 2 (Ex-ante core [6]). A fractional committee p is in the *fractional core* if there are no nonempty $S \subseteq N$ and $z \in [0, 1]^C$ such that $qz(C) \leq |S|$ and $u_i(z) > u_i(p)$ for every $i \in S$. A lottery satisfies the core ex ante if its marginals are in the fractional core.

Definition 3 (Group resource proportionality [10]). A fractional committee p satisfies *group resource proportionality (GRP)* if, for every $S \subseteq N$,

$$p(A(S)) \geq \frac{|S|}{q} - \max_{T \subseteq S} \left(\frac{|T|}{q} - |A(T)| \right). \quad (1)$$

A lottery satisfies GRP ex ante if its marginals satisfy (1).

The left side counts selected candidates approved by at least one member of S , in expectation. The subtraction accounts for subgroups whose approved candidates cannot absorb their entire proportional share. The empty subgroup is allowed in the maximum, so this correction is nonnegative. Equivalently, measure money in seats: each voter holds $1/q = k/n$ of a seat, may spend it only on candidates she approves, and each candidate can absorb at most one seat. The right side of (1) is then the largest amount

that the voters in S could spend if they were alone, which is the value of a maximum flow in the network of Section 4.1 restricted to S [10]. GRP thus requires that, in expectation, the candidates approved by S fill at least as many seats as S could buy on its own. GRP and fractional core stability are incomparable [10].

Affordability ensures that a deterministically fixed part of the committee does not consume representation owed to other voters. Regard q as the price of a seat and give each voter a budget of one. A set is affordable if approving voters can simultaneously pay the full price of all its members, with no voter spending more than one in total. Since budgets cannot be counted twice, every subset must be fundable by the voters who approve at least one of its members.

Definition 4 (Affordability [10]). A set $W_0 \subseteq C$ is *affordable* if approving voters, each with budget one, can pay exactly q for every member of W_0 . Equivalently, the capacitated Hall condition holds:

$$|N(D)| \geq q|D| \quad \text{for every } D \subseteq W_0. \quad (2)$$

Affordability implies $|W_0| \leq k$. Dividing all payments by q gives the equivalent normalization with voter budgets k/n and candidate costs one, used in the completion network below. Next, we propose the concept of a core anchor.

Definition 5 (Core anchor). An affordable set $W_0 \subseteq C$ is a *core anchor* if every committee $W \supseteq W_0$ is in the core.

A core anchor fixes part of the outcome in a way that its own supporters can fund, while leaving every completion core-stable.

4. Main result

Theorem 1. *Every approval election admits a lottery satisfying GRP ex ante and the core ex post. Such a lottery, including its probabilities, can be computed in polynomial time and has support of size at most $m + 1$.*

Remark 1. The construction satisfies the stronger core+ property of Becker et al. [5] ex post. Core+ relaxes the variables in the integral blocking test, including coalition participation, while retaining the integer utility-improvement threshold. It is distinct from the fractional core in Definition 2. We use its payment certificate without needing its full definition.

Theorem 2. *Every approval election admits a core anchor W_0 such that every committee containing W_0 satisfies core+. Such a set can be computed in polynomial time.*

Theorem 2 is the main step towards Theorem 1. Fixing a core anchor with certainty secures the core for every realized committee, and completing the anchor by a maximum flow of voters’ budgets, as in Suzuki and Vollen [10], secures GRP. We first outline the resulting algorithm and derive Theorem 1 from these two components; the rest of the paper then proves them.

Overview of the algorithm. Algorithm 1 has three phases. Phase 1 supplies the anchor W_0 , Phase 2 constructs the marginals p , and Phase 3 implements them.

Phase 1: find an anchor. Find a set W_0 that is affordable and admits the payment certificate (4). The search starts from the empty set and adds or removes one candidate whenever this improves the finite potential Ψ_L in (16) by at least a fixed threshold. The potential is a finite harmonic-entropy value minus a price slightly below q per seat. A failure of the certificate yields a profitable addition, while a failure of affordability yields a profitable deletion. Thus the terminal set W_0 is a core anchor (Theorem 2). Unlike the local search of Becker et al. [5], which exchanges candidates within committees of size exactly k , Phase 1 adds and removes single candidates in sets of any size: the price per seat replaces the size constraint, and affordability, a size constraint imposed on every group of voters, emerges at termination.

Phase 2: complete. Compute a maximum flow of voters’ budgets to approved candidates that pays for W_0 in full. Starting from the amounts received by candidates, increase coordinates within $[0, 1]$ until their sum is k . The resulting marginals satisfy GRP and equal one on W_0 (Lemma 1).

Phase 3: decompose. Write the marginals as a lottery over at most $m + 1$ committees, each containing W_0 .

Proof of Theorem 1. Assuming Theorem 2 and Lemma 1, let W_0 be the core anchor of Theorem 2, and apply Lemma 1 to it. The resulting marginals satisfy GRP. Every committee in the support contains W_0 , and hence satisfies core+ and in particular the core. Both steps take polynomial time, and Lemma 1 gives the support bound. $\square$

The analysis proceeds in the reverse order of the algorithm. Section 4.1 shows that the two requirements on W_0 suffice: affordability permits the

completion of Phases 2 and 3, and the certificate makes every completion core-stable. The certificate is sufficient but not necessary for a set to be a core anchor. Section 4.2 proves the addition and deletion bounds for the finite objective. Section 5 uses these bounds to prove correctness and polynomial termination of Phase 1 in one argument. Appendix A runs all three phases on a small election.

4.1. What the anchor must satisfy

Lemma 1 (Suzuki and Vollen, Theorem 5.3). *Given an affordable set W_0 , one can compute in polynomial time GRP marginals and an implementing lottery over committees containing W_0 , with support of size at most $m + 1$.*

Proof. Consider the network in which money flows from a source s through voters and candidates to a sink t . The arc $s \rightarrow i$ has capacity $1/q$, voter i 's budget measured in seats; the arc $i \rightarrow c$ exists for $c \in A_i$ and has unlimited capacity, so voters fund only approved candidates; and the arc $c \rightarrow t$ has capacity one, the price of a seat. A flow is thus a way for voters to spend their budgets on approved candidates, and $f(c, t)$ is the amount that candidate c receives. A maximum flow maximizes total spending. By the maximum-flow characterization of GRP [10, Theorem 3.3], a fractional committee p satisfies GRP if and only if $p_c \geq f(c, t)$ for every c , for some maximum flow f .

Affordability supplies a feasible flow saturating (c, t) for every $c \in W_0$, that is, paying every member of W_0 in full. Augment it along shortest residual s - t paths to obtain a maximum flow f^* . Each augmenting path is a reallocation chain. A voter i_1 with unspent budget pays more to an approved candidate c_1 ; if c_1 is full, a voter i_2 who pays c_1 shifts the same amount from c_1 to another approved candidate c_2 ; and so on, until the chain reaches a candidate that is not full. Each intermediate candidate gains from one voter what it loses to another, so only the last candidate receives more. Formally, an s - t path never uses an arc leaving t . Augmentation therefore never reduces what a candidate receives, and every member of W_0 remains fully paid in f^* . A flow that admits no augmenting path is maximum, and choosing shortest paths (the Edmonds–Karp rule) ensures polynomially many augmentations.

Starting from $p_c = f^*(c, t)$, increase coordinates within $[0, 1]$ until $p(C) = k$. This is possible because the flow value is at most $k \leq m$. Then $p_c \geq f^*(c, t)$ for every c , so p satisfies GRP.

For completeness, a direct decomposition gives the support bound. An integral vector already specifies a committee. For nonintegral p , choose a size- k set W containing all coordinates equal to one and no coordinate equal

to zero, and put

$$\alpha = \min(\{p_c : c \in W\} \cup \{1 - p_c : c \notin W\}).$$

Then $0 < \alpha < 1$ and $p = \alpha \mathbf{1}_W + (1 - \alpha)p'$ for another fractional committee p' . At least one additional coordinate of p' is integral, and already integral coordinates remain unchanged. Repeating on p' terminates after at most m steps, giving at most $m + 1$ terms. All operations have polynomial bit complexity: with a common denominator for the initial p , the unnormalized remaining mass and coordinates remain multiples of its reciprocal. Since $p_c = 1$ for $c \in W_0$, every positive-weight term contains W_0 . $\square$

Conversely, a set whose candidate arcs are saturated by some maximum flow is affordable. Affordable sets are thus exactly those that a maximum flow can pay for in full, so committing to them costs nothing in terms of GRP.

For any $W \subseteq C$, let $\mathcal{P}_q(W)$ be the set of nonnegative payments π_{ic} on approved members of W and reserves r_i satisfying

$$r_i + \sum_{c \in A_i \cap W} \pi_{ic} = 1 \quad (i \in N), \quad \sum_{i: c \in A_i} \pi_{ic} \leq q \quad (c \in W). \quad (3)$$

Unspecified payments are zero. The payment characterization of Becker et al. [5, Theorem 3.3] implies that every size- k completion of W satisfies core+ if some $(\pi, r) \in \mathcal{P}_q(W)$ additionally satisfies

$$\pi_{ic} \leq r_i \quad (c \in A_i \cap W), \quad \sum_{i: c \in A_i} r_i < q \quad (c \notin W). \quad (4)$$

Indeed, retain the reserves and give zero payments to added winners. The characterization then applies to the completed committee. This upward closure is what lets a partial committee, rather than a full committee as in Becker et al. [5], carry the core. These payments need not witness affordability: they may leave a selected candidate with less than q . A core anchor therefore needs two payment systems: the certificate, which keeps reserves large, and affordability payments, which pay every member in full. Core stability alone, as in Becker et al. [5], requires only the first. Informally, voters' spare money is too thin to buy any excluded candidate, and no voter is committed to a winner beyond her spare money. A coalition improving on a completion would have to fund its proposal, and the certificate prices every improving proposal above what the coalition can mobilize.

4.2. Finding the anchor

We use the payment and water-filling arguments of Becker et al. [5], working throughout with a finite harmonic-entropy objective. Becker et al. [5] work with the infinite series. Because our objective is finite, the potential we analyze is exactly the one computed by the linear program of Section 5, so no approximation argument is needed. The price is a remainder term in the water-filling identity (10). For a probability vector x of dimension d , let $S_t(x)$ be the sum of its t largest coordinates. For an integer $L \geq 1$, define

$$f_\ell(x) = \max_{1 \leq t \leq d} \frac{S_t(x)}{\ell + t}, \quad F_L(x) = \sum_{\ell=0}^{L-1} \left(\frac{1}{\ell + 1} - f_\ell(x) \right). \quad (5)$$

Each f_ℓ is continuous, convex, and symmetric. Hence F_L is continuous, concave, and symmetric, and adding a zero coordinate does not change its value. Writing $H_0 = 0$ and $H_j = \sum_{h=1}^j 1/h$, the bounds $1/(\ell + d) \leq f_\ell(x) \leq 1/(\ell + 1)$ give

$$0 \leq F_L(x) \leq H_{d-1} + H_L - H_{L+d-1} \leq H_{d-1}. \quad (6)$$

For a payment system on $W \subseteq C$, let x_i list voter i 's payments to approved members of W , followed by her reserve r_i . Define

$$\mathcal{V}_L(W) = \max_{(\pi, r) \in \mathcal{P}_q(W)} \sum_{i \in N} F_L(x_i). \quad (7)$$

The maximum exists because $\mathcal{P}_q(W)$ is nonempty and compact. Harmonic entropy rewards spreading a voter's money evenly. By symmetry and concavity, a uniform vector maximizes F_L among vectors of the same dimension, attaining the first upper bound in (6). Thus, whenever equal splitting among approved winners and the reserve respects the candidate capacities,

$$\mathcal{V}_L(W) = \sum_{i \in N} \left(H_{u_i(W)} - \sum_{a=1}^{u_i(W)} \frac{1}{L + a} \right). \quad (8)$$

This is a finite version of the Proportional Approval Voting (PAV) score $\sum_i H_{u_i(W)}$. In general, the candidate capacities require some voters to split their money unevenly.

Becker et al. [5] use harmonic entropy to establish core+ existence and compute a core+ committee in polynomial time by local search. A full core committee cannot simply be fixed here, since doing so leaves no room for the randomization that GRP generally requires. Nor can members be

removed arbitrarily: the remaining set may no longer ensure core stability for every completion. We instead find an affordable set satisfying the payment certificate. The following canonical-payment argument is the same averaging argument as in Becker et al. [5, Lemma 5.2]; it applies directly to F_L .

Lemma 2 (Canonical payments). *For every $W \subseteq C$, an optimum in (7) can be chosen so that*

$$\begin{aligned} \pi_{ic} &\leq r_i && (c \in A_i \cap W), \\ \sum_{i:c \in A_i} \pi_{ic} &< q &\implies \pi_{ic} = r_i && \text{for every } i \text{ with } c \in A_i. \end{aligned} \quad (9)$$

Proof. Since F_L is continuous, the set of optimal payment systems is compact. Among them, choose one minimizing $\sum_i (r_i^2 + \sum_{c \in A_i \cap W} \pi_{ic}^2)$. By symmetry and concavity of F_L , moving two unequal coordinates of a voter vector towards their average cannot decrease F_L , and strictly decreases this secondary objective. If $\pi_{ic} > r_i$, averaging these two coordinates is feasible: the candidate's payment only decreases. Hence $\pi_{ic} \leq r_i$. If candidate c receives less than q and $\pi_{ic} < r_i$, moving a sufficiently small amount from r_i to π_{ic} is likewise feasible, again a contradiction. This proves both conditions. $\square$

We call such an optimum canonical. Its reserves satisfy $r_i \geq 1/(u_i(W) + 1) > 0$ by the unit budget and $\pi_{ic} \leq r_i$. The water-filling operation of Becker et al. [5] is

$$\Phi(x) = (\min\{x_1, f_1(x)\}, \dots, \min\{x_d, f_1(x)\}, f_1(x)).$$

Here $f_1(x)$ is the unique positive level τ satisfying $\sum_j (x_j - \tau)_+ = \tau$, where $(a)_+ = \max\{a, 0\}$. The operation weakly decreases existing coordinates and creates one new coordinate, preserving total mass. By Becker et al. [5, Lemma 4.2], $f_\ell(\Phi(x)) = f_{\ell+1}(x)$. Telescoping the finite sum in (5) therefore gives the exact identity

$$F_L(\Phi(x)) - F_L(x) = \max_j x_j - f_L(x), \quad 0 \leq f_L(x) \leq \frac{1}{L+1}. \quad (10)$$

Opening one more share is thus worth the voter's largest coordinate, less a term of at most $1/(L+1)$. We use this identity both forwards, to add a candidate, and backwards, to delete one. For the infinite series the remainder vanishes. Here it works against additions and in favor of deletions, so the level L matters only for additions.

Lemma 3 (Addition). *Let (π, r) be a canonical optimum for W . If $c \notin W$ has reserve load $R_c = \sum_{i:c \in A_i} r_i \geq q$, then*

$$\mathcal{V}_L(W + c) - \mathcal{V}_L(W) \geq q - \frac{n}{L+1}. \quad (11)$$

Proof. As in Becker et al. [5, Lemma 5.3], apply Φ to the vector of each voter approving c , interpreting the new coordinate as her payment to c . Other voters keep their original vectors. Since r_i is a largest coordinate of x_i , (10) gives a total gain of at least $R_c - n/(L+1)$. Existing candidate payments weakly decrease, and the payment to c is $\sum_{i:c \in A_i} f_1(x_i) \leq R_c$, since $f_1(x_i) \leq \max_j x_{ij} = r_i$. Mix the new vectors with the original vectors, extended by zero payments to c , with weight $\alpha = q/R_c$ on the new vectors. This restores feasibility. Concavity and invariance under adding a zero coordinate yield a gain of at least

$$\alpha \left(R_c - \frac{n}{L+1} \right) = q - \frac{\alpha n}{L+1} \geq q - \frac{n}{L+1}. \quad \square$$

For the untruncated objective, the corresponding bound of Becker et al. [5] is q . The shortfall $n/(L+1)$ comes from the remainder in (10). It means that at the undiscounted price q , an addition with reserve load exactly q may lower the potential, which is why Section 5 discounts the price.

The deletion bound localizes Becker et al. [5, Lemma 5.4], which bounds the loss by $n/|W|$ when $q|W| > n$; taking $D = W$ recovers it, since $|N(W)| \leq n$. It applies inside any subset violating affordability, accounting only for that subset's approving voters. The global bound does not suffice, since a set may satisfy $q|W| \leq n$ while a subset is unaffordable: in the election of Appendix A, $W = \{a, b, c\}$ has $q|W| = n$ but $|N(W)| = 4 < q|W|$. The proof below redistributes reserves among underfunded candidates. Inside D , however, a voter can redistribute only to approved members of the underfunded part, whose capacities need not accommodate every voter's full reserve at once. We need full redistribution only for voters approving the candidate eventually deleted; a balancing argument ensures exactly this.

Lemma 4 (Deletion from a deficient subset). *Let $W \subseteq C$ and let $D \subseteq W$ satisfy $|N(D)| < q|D|$. Some $c \in D$ satisfies*

$$\mathcal{V}_L(W) - \mathcal{V}_L(W - c) \leq \frac{|N(D)|}{|D|} < q. \quad (12)$$

Proof. Fix a canonical optimum (π, r) for W , and let

$$U = \left\{ c \in D : \sum_{i:c \in A_i} \pi_{ic} < q \right\}, \quad \Delta = q|D| - |N(D)| > 0.$$

Unit voter budgets imply $U \neq \emptyset$, and canonicity gives $\pi_{ic} = r_i$ on approved members of U . We let each voter in $N(U)$ redistribute part of her reserve among approved members of U . Consider payments y to approved members of U satisfying

$$\begin{aligned} y_{ic} &\geq r_i & (c \in A_i \cap U), \\ \sum_{c \in A_i \cap U} y_{ic} &\leq (|A_i \cap U| + 1)r_i & (i \in N(U)), \\ \sum_{i: c \in A_i} y_{ic} &\leq q & (c \in U). \end{aligned} \tag{13}$$

The original payments are feasible, and every row's upper bound is at most one. Hence the feasible set is nonempty and compact, with all coordinates in $[0, 1]$. Choose a feasible y minimizing

$$\sum_{i \in N(U)} \sum_{c \in A_i \cap U} (1 - y_{ic})^2.$$

All redistributed money comes from voters in $N(D)$, while candidates in $D \setminus U$ retain their original payments of q . Thus

$$\begin{aligned} \sum_{c \in U} \sum_{i: c \in A_i} y_{ic} &\leq \sum_{i \in N(U)} \left(r_i + \sum_{c \in A_i \cap U} \pi_{ic} \right) \\ &\leq |N(D)| - q|D \setminus U| = q|U| - \Delta. \end{aligned}$$

Choose $c^* \in U$ receiving the smallest total payment. Then

$$\sum_{i: c^* \in A_i} y_{ic^*} \leq q - \frac{\Delta}{|U|} < q. \tag{14}$$

For every voter i approving c^* , two consequences of optimality now suffice. First, her row in (13) is tight: otherwise $y_{ic^*} < 1$, and increasing it slightly would decrease the objective while preserving feasibility. Second, $y_{ic^*} \geq y_{ic}$ for every $c \in A_i \cap U$. Otherwise, moving a small amount from a larger payment to y_{ic^*} preserves the row sum and all bounds, while strictly decreasing the objective. Both moves are possible because c^* has spare capacity.

Delete c^* . For each voter approving it, use y_{ic^*} as her new reserve and y_{ic} as her payments to surviving members of U , keeping her payments outside U unchanged. Other voters keep their original vectors. Tightness preserves

each affected row sum. Every surviving column in U is bounded by its total in y , since unchanged payments equal $r_i \leq y_{ic}$, so the new system is feasible.

Let x_i^- and x_i be an affected voter's new and original vectors. All unchanged coordinates are at most r_i , and tightness gives

$$\sum_j (x_{ij}^- - r_i)_+ = \sum_{c \in A_i \cap U} (y_{ic} - r_i) = r_i.$$

Since $r_i > 0$, it follows that $f_1(x_i^-) = r_i$ and $\Phi(x_i^-) = x_i$ up to permutation. The largest coordinate of x_i^- is y_{ic^*} . By (10),

$$F_L(x_i) - F_L(x_i^-) = y_{ic^*} - f_L(x_i^-) \leq y_{ic^*}.$$

Summing over affected voters and using (14),

$$\begin{aligned} \mathcal{V}_L(W) - \mathcal{V}_L(W - c^*) &\leq \sum_{i: c^* \in A_i} y_{ic^*} \\ &\leq q - \frac{\Delta}{|U|} \leq q - \frac{\Delta}{|D|} = \frac{|N(D)|}{|D|}. \end{aligned} \quad \square$$

The redistribution is only a proof device: the algorithm need not find a deficient subset or solve the auxiliary minimization. It evaluates single-candidate additions and deletions. The two lemmas yield the following result, whose existence and computational claims are proved together in Section 5.

Proposition 1. *Every approval election admits an affordable partial committee W_0 satisfying the payment certificate (4). Thus every size- k completion of W_0 satisfies core+.* Such a set W_0 can be computed in polynomial time.

5. Polynomial-time computation

We use one finite potential throughout. Set

$$\varepsilon = \frac{1}{2km}, \quad L = 4nkm, \tag{15}$$

and define

$$\Psi_L(W) = \mathcal{V}_L(W) - (q - \varepsilon)|W| \quad (W \subseteq C). \tag{16}$$

The small discount in the price of a seat makes additions profitable even when an excluded candidate's reserve load is exactly q . It is small enough that deletions remain profitable whenever affordability fails. Choosing L as above makes the final term in (10) small enough for additions to be detected; the

deletion bound (12) does not depend on L . The discount has no counterpart in Becker et al. [5], where the committee size is fixed. It is possible because integrality bounds the deletion margin $q - |N(D)|/|D|$ below by $1/(km)$, as the proof of Proposition 1 shows.

Algorithm 1 states the full procedure. Phase 1 adds or removes one candidate at a time. Intermediate sets may have any cardinality; affordability, and hence size at most k , is guaranteed at termination. Phases 2 and 3 carry out the construction in the proof of Lemma 1. Appendix A traces all three phases on a small election.

Algorithm 1 (GRP ex ante and core ex post).

Input: Approval sets $(A_i)_{i \in N}$ and committee size k .

Output: GRP marginals p and a lottery over at most $m+1$ core committees.

Phase 1: find a chore anchor.

1. Set ε and L as in (15), $W \leftarrow \emptyset$, and $\Psi_L(W) \leftarrow 0$.
2. For each $c \in C$, let W' be $W - c$ if $c \in W$ and $W + c$ otherwise. Obtain $\mathcal{V}_L(W')$ from the linear program (3), (17), and (18), and set $\Psi_L(W') = \mathcal{V}_L(W') - (q - \varepsilon)|W'|$.
3. If some W' satisfies $\Psi_L(W') \geq \Psi_L(W) + \varepsilon/2$, set $W \leftarrow W'$, retain its evaluated potential, and return to Step 2. Otherwise set $W_0 \leftarrow W$.

Phase 2: complete.

4. In the network of Lemma 1 restricted to the candidates in W_0 , compute a maximum flow. By affordability, it saturates (c, t) for every $c \in W_0$, paying every member of W_0 in full.
5. In the full network, augment this flow along shortest residual s - t paths to a maximum flow f^* ; members of W_0 remain fully paid. Set $p_c = f^*(c, t)$ and raise coordinates within $[0, 1]$ until $p(C) = k$.

Phase 3: decompose.

6. Decompose p into a lottery over at most $m+1$ committees, each containing W_0 , as in the proof of Lemma 1. Return p and the lottery.

5.1. Evaluating the objective

The value $\mathcal{V}_L(W)$ is computed exactly by a linear program, as in Becker et al. [5, Section 7]. Let $d_i = |A_i \cap W| + 1$ be the dimension of x_i . For each voter vector x_i , level $0 \leq \ell < L$, and coordinate j , introduce nonnegative variables $t_{i\ell}$, $a_{i\ell j}$ and impose

$$a_{i\ell j} \geq x_{ij} - t_{i\ell}, \quad \sum_j a_{i\ell j} \leq \ell t_{i\ell}. \quad (17)$$

These constraints are equivalent to $t_{i\ell} \geq f_\ell(x_i)$. Indeed, eliminating a gives $\sum_j (x_{ij} - t_{i\ell})_+ \leq \ell t_{i\ell}$. For any $t \geq 0$, the left side equals $\max_{0 \leq h \leq d_i} (S_h(x_i) -$

ht), where $S_0(x_i) = 0$. The inequality therefore holds exactly when $S_h(x_i) \leq (\ell + h)t$ for every $1 \leq h \leq d_i$, which is the defining condition $t \geq f_\ell(x_i)$. This also covers $\ell = 0$.

Maximize

$$nH_L - \sum_{i \in N} \sum_{\ell=0}^{L-1} t_{i\ell} \quad (18)$$

subject to (3) and (17), where $x_i = ((\pi_{ic})_{c \in A_i \cap W}, r_i)$. For fixed payments, maximization sets every $t_{i\ell}$ to its smallest feasible value $f_\ell(x_i)$. Thus the optimum is exactly $\mathcal{V}_L(W)$. The linear program has $O(nmL)$ variables and constraints. Its rational coefficients have polynomial encoding length, as does H_L (whose denominator divides $L!$). Hence it can be solved exactly in polynomial time.

5.2. Correctness and running time

Proof of Proposition 1. Let W be any set reached in Phase 1. Suppose first that W is unaffordable. Some $D \subseteq W$ satisfies $|N(D)| < q|D|$, and the positive integer $n|D| - k|N(D)|$ gives

$$q - \frac{|N(D)|}{|D|} = \frac{n|D| - k|N(D)|}{k|D|} \geq \frac{1}{k|D|} \geq \frac{1}{km} = 2\varepsilon.$$

By Lemma 4, some deletion loses at most $q - 2\varepsilon$ in $\mathcal{V}_L$, and hence increases Ψ_L by at least ε .

Next, suppose that a canonical optimum for W has an excluded candidate with reserve load at least q . Lemma 3 gives an addition increasing Ψ_L by at least

$$\varepsilon - \frac{n}{L+1} > \varepsilon - \frac{n}{L} = \frac{\varepsilon}{2}.$$

Thus either failure produces a move accepted in Phase 1. At termination, W_0 is affordable and a canonical optimum satisfies (4). Every size- k completion therefore satisfies core+.

Finally, $\Psi_L(\emptyset) = 0$ and $q - \varepsilon > 0$. By (6), $\Psi_L(W) \leq \mathcal{V}_L(W) \leq nH_m$. Every accepted move increases Ψ_L by at least $\varepsilon/2$, so the search terminates after at most

$$\frac{2nH_m}{\varepsilon} = 4nkmH_m$$

moves. Each iteration evaluates at most m neighbors, using linear programs of polynomial size and encoding length because $L = 4nkm$. This proves existence and polynomial-time computation of W_0 . $\square$

In Becker et al. [5], existence follows from a maximizer of the untruncated objective over committees of size k , and computation from a separate local search; here, termination of Phase 1 gives both.

Proof of Theorem 2. By Proposition 1, the set W_0 found in Phase 1 of Algorithm 1 is affordable and satisfies (4). Hence every committee containing W_0 satisfies core+, and in particular the core, so W_0 is a core anchor. Together with the proof of Theorem 1 in Section 4, this completes the analysis of Algorithm 1. $\square$

6. Implications and limits

We have shown that GRP ex ante and the core ex post are compatible: a lottery achieving both can be computed in polynomial time and has support of size at most $m + 1$ (Theorem 1). The key ingredient is a core anchor (Theorem 2), an affordable set whose every completion is core-stable. Fixing the anchor supplies ex-post stability, and maximum-flow completion of the remaining seats supplies ex-ante proportionality. We close by relating Theorem 1 to earlier results and by showing that neither of its guarantees can be strengthened: the ex-ante guarantee cannot be raised to the fractional core, and the ex-post quota cannot be lowered.

6.1. Earlier best-of-both-worlds results

Theorem 1 subsumes earlier best-of-both-worlds results for approval-based committee voting. Numbering follows the EC version of Suzuki and Vollen [10] and, since the WINE version of Aziz et al. [3] is a one-page abstract, the full version of the latter.

Corollary 1. *The lottery of Theorem 1 satisfies GFS and Strong UFS ex ante and FJR ex post. It therefore strengthens each of the following results to GRP ex ante and the core ex post, computed in polynomial time:*

- (a) Aziz et al. [3, Theorem 4.1]: *GFS and Strong UFS ex ante with EJR ex post, in polynomial time;*
- (b) Aziz et al. [3, Theorem 5.3]: *Strong UFS ex ante with FJR ex post, by an algorithm that does not run in polynomial time and whose output need not satisfy GFS;*
- (c) Suzuki and Vollen [10, Corollary 5.4]: *existence of a lottery with GRP ex ante and FJR ex post, which answered the question of Aziz et al. [3] whether GFS ex ante is compatible with FJR ex post.*

Proof. GRP implies GFS and Strong UFS [10, Proposition 3.6]. The core implies FJR [9]: a violation of FJR gives $S \subseteq N$, $T \subseteq C$ and a positive integer β with $|S| \geq q|T|$ and $u_i(T) \geq \beta > u_i(W)$ for every $i \in S$, so (S, T) blocks W . EJR is the special case of FJR in which every member of S approves all of T and $\beta = |T|$. $\square$

Since GFS implies UFS and IFS [3], the lottery also satisfies *positive share*: every voter with a nonempty approval set approves a selected candidate with positive probability.

6.2. The ex-ante core is unattainable

GRP cannot be replaced by the fractional core. By Jin et al. [7, Theorem 1], there are elections in which no lottery satisfying justified representation ex post is weakly Pareto optimal ex ante. The fractional core implies weak Pareto optimality, since otherwise the grand coalition could improve every voter's utility within its full budget, and the core implies justified representation. Hence the core ex ante and the core ex post are incompatible in general. The same argument excludes ex-ante Pareto optimality, and Jin et al. [7, Theorem 2] excludes ex-ante average justified representation.

6.3. The Hare quota is tight

Becker et al. [5, Theorem 6.3] prove that the strict Droop core is nonempty. Their local-search method also yields polynomial-time computation of a strict Droop-core committee.¹ Nevertheless, no fixed reduction below the Hare quota is compatible even with positive share ex ante. A committee W satisfies *justified representation (JR) at q'* if every group of more than q' voters with a commonly approved candidate contains a voter who approves a member of W [1].

Theorem 3. *For every $k \geq 1$ and $\delta > 0$, positive share ex ante is incompatible with JR ex post at $q' = n/(k + \delta)$.*

Proof. Choose an integer $g > 1/\delta$. There are k groups of g voters, the j th approving only a_j , and one further voter approving only b . Then $n = kg + 1$ and $g(k + \delta) = n - 1 + g\delta > n$, so each group has more than q' voters. JR at q' therefore requires every committee in the support to be $\{a_1, \dots, a_k\}$, and

¹In the local search of Becker et al. [5, Lemma 6.2 and Section 7], take $q' = n/(k + 1) + 1/(2k(k + 1))$. Every integral strict Droop blocker has $|T| \leq k$ and $|S| - n|T|/(k + 1) \geq 1/(k + 1)$, so it also blocks at this q' . The improvement gap $q' - n/(k + 1) = 1/(2k(k + 1))$ is inverse polynomial, so their truncation and local search still run in polynomial time.

the last voter never approves a selected candidate. For $\delta \leq 1$, this committee does satisfy JR at q' , since $q' \geq 1$. $\square$

The core at q' implies JR at q' , and so does the strict Droop core at $q' = n/(k+1)$: take T to be a single commonly approved candidate. Hence Theorem 1 fails for every fixed reduction of the Hare quota, and in particular for the strict Droop core.

Declaration of generative AI assistance

The main result was obtained with the assistance of OpenAI’s GPT Astra 6.1. The author worked interactively with OpenAI’s GPT Astra 6.1 and Anthropic’s Opus 5.5 to simplify and streamline the proof. The author takes full responsibility for the content of the paper.

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Appendix A. Worked example

We run Algorithm 1 on an election with $n = 6$ voters, $m = 5$ candidates a, b, c, d, e , and $k = 3$, so that $q = 2$. Voters 1–4 approve $\{a, b, c\}$, voter 5 approves $\{d\}$, and voter 6 approves $\{e\}$. Let $G = \{1, 2, 3, 4\}$. Proportionally, G deserves two seats, whereas voters 5 and 6 deserve half a seat each and cannot pool their halves, since they approve different candidates. The algorithm fixes two seats for G and lets voters 5 and 6 share the third by lottery:

$$\underbrace{\emptyset \rightarrow \{a\} \rightarrow \{a, b\}}_{\text{Phase 1: anchor } W_0} \implies \underbrace{p = (1, 1, 0, \frac{1}{2}, \frac{1}{2})}_{\text{Phase 2: marginals}} \implies \underbrace{\frac{1}{2}\{a, b, d\} + \frac{1}{2}\{a, b, e\}}_{\text{Phase 3: lottery}}.$$

Figure A.1 shows Phases 1 and 2; the paragraphs below explain each step.

The two requirements. A committee is in the core if and only if it contains at least two of a, b, c . If it contains at most one, then G blocks it with $T = \{a, b\}$, since $q|T| = 4 = |G|$. If it contains two, a blocking coalition that includes a member of G needs $T \supseteq \{a, b, c\}$, hence $|S| \geq 3q = 6$, so $S = N$ and $T = \{a, b, c\}$, which does not improve voters 5 and 6. A coalition within $\{5, 6\}$ can afford at most one candidate, and no candidate improves both of them.

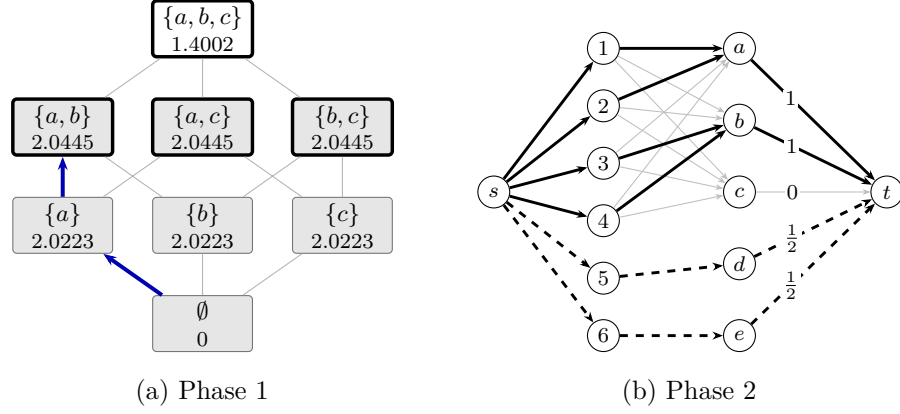


Figure A.1: The run of Algorithm 1 on the example. (a) Phase 1 in the lattice of subsets of $\{a, b, c\}$, with $\Psi_L(W)$ below each set ($L = 360$). Shaded sets are affordable, and sets with a thick border admit the certificate (4). The search (arrows) moves $\emptyset \rightarrow \{a\} \rightarrow \{a, b\}$ and stops at the first set that is both. Sets containing d or e are never affordable and are omitted. (b) Phase 2 in seat units, where each voter holds $\frac{1}{2}$. Solid arcs carry the Step 4 flow that pays for $W_0 = \{a, b\}$; dashed arcs are the Step 5 augmenting paths; grey arcs are approvals that carry no flow. Labels on the sink arcs are the marginals.

For $S = G$, $\{5\}$ and $\{6\}$, the right side of (1) is $|S|/q$, so GRP requires

$$p_a + p_b + p_c \geq 2, \quad p_d \geq \frac{1}{2}, \quad p_e \geq \frac{1}{2}.$$

Since $p(C) = 3$, all three hold with equality. Hence no single committee satisfies GRP. On the other hand, a lottery with GRP marginals may contain a blocked committee: the lottery $\frac{1}{2}\{a, b, c\} + \frac{1}{2}\{a, d, e\}$ has marginals $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which satisfy GRP, but $\{a, d, e\}$ is blocked.

The table tests some candidate anchors against Definition 5 and the two conditions of Section 4.1.

W_0	Every $W \supseteq W_0$ in the core	Certificate (4)	Affordable (2)
$\emptyset$	no: $\{c, d, e\}$ is blocked	no: $R_a = 4 \geq q$	yes
$\{a\}$	no: $\{a, d, e\}$ is blocked	no: $R_b \geq q$	yes
$\{a, b\}$	yes	yes	yes
$\{a, b, c\}$	yes	yes	no: $ N(W_0) = 4 < 6$

The sets $\emptyset$ and $\{a\}$ are too small to secure the core, and $\{a, b, c\}$ is too large to be affordable. Among subsets of $\{a, b, c\}$, exactly the three pairs are both certified and affordable; sets containing d or e are never affordable, since a single voter cannot pay q . For $\{a\}$, every payment system with $\pi_{ia} \leq r_i$ gives each voter in G a reserve of at least $\frac{1}{2}$, so $R_b \geq q$.

Phase 1. Here $\varepsilon = 1/30$ and $L = 360$. For every W , splitting each voter's unit equally among her approved members of W and her reserve gives each candidate at most q ; the maximum, 2, is attained by a when $W = \{a\}$. Hence (8) gives the exact value of $\mathcal{V}_L(W)$. In particular, adding a candidate approved by a voter who currently approves $j-1$ selected candidates increases that voter's contribution by $1/j - 1/(L+j)$. The table lists changes in Ψ_L when each candidate is added or, if marked *, removed; accepted moves are in bold, and Figure A.1(a) shows the same run as a walk. Values are rounded to four decimal places; the acceptance threshold is $\varepsilon/2 = 1/60$.

W	a	b	c	d	e	Move
$\emptyset$	2.0223	2.0223	2.0223	-0.9694	-0.9694	add a
$\{a\}$	-2.0223*	0.0223	0.0223	-0.9694	-0.9694	add b
$\{a, b\}$	-0.0223*	-0.0223*	-0.6444	-0.9694	-0.9694	stop

Ties are broken alphabetically; other choices end at $\{a, c\}$ or $\{b, c\}$, which work equally well. The second move shows why the discount is needed. At $W = \{a\}$, candidate b has reserve load exactly $4 \cdot \frac{1}{2} = q$. Adding it increases $\mathcal{V}_L$ by $2 - 4/(L+2)$, which is less than the undiscounted price $q = 2$. With the discount, its increase in Ψ_L is

$$\frac{1}{30} - \frac{4}{362} = \frac{121}{5430} > \frac{1}{60},$$

so the move is accepted. Without the discount, the search would stop at $\{a\}$, which is not a core anchor: its completion $\{a, d, e\}$ is blocked.

At $W_0 = \{a, b\}$ both requirements hold, witnessed by two different payment systems:

Voters	Certificate payments			Affordability payments		
	to a	to b	reserve	to a	to b	reserve
1, 2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	0	0
3, 4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	1	0
5, 6	—	—	1	—	—	1
Received	$\frac{4}{3}$	$\frac{4}{3}$		2	2	

The certificate payments, an equal split, satisfy (4): each payment equals the payer's reserve, and the reserve loads of the excluded candidates are $R_c = \frac{4}{3}$ and $R_d = R_e = 1$, all below $q = 2$. They pay a and b only $\frac{4}{3}$ each. The affordability payments pay a and b in full, but violate $\pi_{ic} \leq r_i$.

Phase 2. In seat units, each voter holds $\frac{1}{2}$ (Figure A.1(b)). In Step 4, the maximum flow restricted to $\{a, b\}$ sends $\frac{1}{2}$ along each of $1 \rightarrow a$, $2 \rightarrow a$, $3 \rightarrow b$ and $4 \rightarrow b$, paying a and b in full. In Step 5, the augmenting paths $s \rightarrow 5 \rightarrow d \rightarrow t$ and $s \rightarrow 6 \rightarrow e \rightarrow t$ each carry $\frac{1}{2}$. Every source arc is then saturated, so the flow is maximum, with value $3 = k$. The marginals are $(p_a, \dots, p_e) = (1, 1, 0, \frac{1}{2}, \frac{1}{2})$, and no top-up is needed. They satisfy GRP, with equality in the three constraints above.

Phase 3. Take $W = \{a, b, d\}$. Then $\alpha = \min\{1, 1, \frac{1}{2}, 1 - 0, 1 - \frac{1}{2}\} = \frac{1}{2}$, and the remainder $p' = (p - \frac{1}{2}\mathbf{1}_W)/\frac{1}{2}$ is the committee $\{a, b, e\}$. The output is the following lottery, whose average is the marginal vector:

	a	b	c	d	e	Probability
$\{a, b, d\}$	1	1	0	1	0	$\frac{1}{2}$
$\{a, b, e\}$	1	1	0	0	1	$\frac{1}{2}$
p	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	

Both committees contain $W_0 = \{a, b\}$ and are therefore in the core, and the marginals satisfy GRP.