An Impossibility Result for Housing Markets with Fractional Endowments

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Abstract

The housing market setting constitutes a fundamental model of exchange economies of goods. House allocation with fractional endowments of houses was considered by Athanassoglou and Sethuraman (2011) who posed the open problem whether individual rationality, efficiency, and weak strategyproofness are compatible for the setting. We show that the three axioms are incompatible.

Keywords: House allocation, Housing markets, Strategyproofness, Pareto optimality, Individual rationality
JEL: C62, C63, and C78

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1. Introduction

The housing market is a fundamental model of exchange economies of goods. It has been used to model online barter markets and nation-wide kidney markets [11, 14]. The housing market (also called the *Shapley-Scarf* market) consists of a set of agents each of whom owns a house and has preferences over the set of houses. The goal is to redistribute the houses among the agents in an efficient and stable manner. The desirable properties include the following ones: *Pareto optimality* (there exists no other assignment which each agent weakly prefers and at least one agent strictly prefers); *individual rationality* (the resultant allocation is at least as preferred by each agent as his endowment); and *core stability* (there exists no subset of agents who could have redistributed their endowments among themselves so as to get a more preferred outcome than the resultant assignment).

Shapley and Scarf [13] showed that for housing markets with strict preferences, an elegant mechanism called *Gale's Top Trading Cycles (TTC)* (that is based on multi-way exchanges of houses between agents) is strategyproof and finds an allocation that is in the core [13, 8]. Along with the Deferred Acceptance Algorithm, TTC has provided the foundations for many of the developments in matching market design [9, 14]. The Shapley-Scarf market has been used to

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model important real-world problems for allocation of human organs and seats at public schools. Since the formalization of TTC, considerable work has been done to extend and generalize TTC for more general domains that allow indifference in preferences [1, 5, 3, 10, 12] or multiple units in endowment [4, 7, 15, 16].

Despite recent progress on house allocation and housing market mechanisms, most of the literature assumes that fractions of houses cannot be allocated. The assumption is restrictive for the following reasons. Agents have the right to use different facilities for different fractions of the time and fractional allocation of resources is helpful in obtaining more equitable outcomes. Allowing for exchanges of fractions of services can also be helpful in modeling time-bank models in which agents perform services in order to receive other services for the same time duration. Fractional allocation of houses can also be interpreted as the relative right of an agent over a house [2]. Finally, fractional allocations can be used to model randomized allocation of indivisible resources where agents exchange probabilities of getting particular houses.

Athanassoglou and Sethuraman [2] considered housing markets with fractional endowments and presented a desirable mechanism for the setting. They posed the following open question: *"a natural question to ask is whether there exists a mechanism that is individually rational, ordinally efficient, and weakly strategyproof."* In this paper, we answer the open question in the negative. The result implies a number of results in the literature including theorems in [2, 17].

2. Preliminaries

2.1. Model

Consider a market with a set of agents $N = \{1, \dots, n\}$ and a set of houses $H = \{h_1, \dots, h_m\}$. Each agent has complete and transitive preferences $\succsim_i$ over the houses and $\succsim = (\succsim_1, \dots, \succsim_n)$ is the preference profile of the agents. Agents may be indifferent among houses. Although the model we consider allows for weak preferences, the impossibility result we prove will be for the restricted domain of strict preferences. When we represent a strict preference order, we will represent it by a comma separated list in decreasing order of preference.

Each agent i is endowed with an allocation $e(i)$ where there are $e(i)(h_j)$ units of house h_j given to agent i. The quadruple $(N, H, \succsim, e)$ is an instance of a *housing market with fractional endowments*. Note that in the basic housing market, each agent is endowed with and is allocated one house and the endowments are discrete: $n = m$, $e(i)(h_j) \in \{0, 1\}$ and $\sum_{h \in H} e(i)(h) = 1$ for all $i \in N$ and $\sum_{i \in N} e(i)(h) = 1$ for all $h \in H$. When allocations are discrete we will also abuse notation and denote $e(i)$ as a set.

A *fractional assignment* is an $n \times m$ matrix $[x(i)(h_j)]_{1 \leq i \leq n, 1 \leq j \leq m}$ such that for all $h \in H$, $\sum_{i \in N} x(i)(h) = \sum_{i \in N} e(i)(h)$. The value $x(i)(h_j)$ is the fraction or units of house h_j that agent i gets. We will use fraction or unit interchangeably since we do not assume that exactly one unit of each house is in the market. Each row $x(i) = (x(i)(h_1), \dots, x(i)(h_m))$ represents the *allocation* of agent i.

Given two allocations $x(i)$ and $y(i)$, $x(i) + y(i)$ is the point-wise sum of the allocations $x(i)$ and $y(i)$. If $\sum_{i \in N} x(i)(h) = 1$ for each $h \in H$, a fractional assignment can also be interpreted as a random assignment where $x(i)(h_j)$ is the probability of agent i getting house h_j. Note that endowment e itself can be considered as the initial assignment of houses to the agents with $e(i)$ being the initial allocation of agent $i \in N$. An assignment x is called *single unit* if for each agent $i \in N$, $\sum_{h\in H} x(i)(h) = 1$. A *fractional housing market* mechanism is a function that takes as input $(N, H, \succsim, e)$ and returns an assignment or vector of allocations $(x(1), \dots, x(n))$ such that $\sum_{i\in N} x(i) = \sum_{i\in N} e(i)$. We do not require in general that $\sum_{i\in N} e(i)(h)$ or $\sum_{h\in H} e(i)(h)$ are integers.

2.2. Properties of assignments and mechanisms

Before defining various properties, we need consider agents' preferences over allocations. A standard method to compare random allocations is to use the *SD (stochastic dominance)* relation. Given two random assignments x and y, $x(i) \succsim_i^{SD} y(i)$ i.e., an agent i *SD prefers* allocation $x(i)$ to $y(i)$ if

$$\forall h \in H : \textstyle\sum_{h_j\in\{h_k : h_k \succsim_i h\}} x(i)(h_j) \geq \sum_{h_j\in\{h_k : h_k \succsim_i h\}} y(i)(h_j).$$

The SD relation is not complete in general. We define normative properties of assignments as well as mechanisms.

- *SD-efficiency*: an assignment x is *SD-efficient* if there exists no other assignment y such that $y(i) \succsim_i^{SD} x(i)$ for all $i \in N$ and $y(i) \succ_i^{SD} x(i)$ for some $i \in N$. A mechanism is SD-efficient if it always returns an SD-efficient outcome.

- *SD individually rational (SD-IR)*: an assignment x is *SD-individually rational* if $x(i) \succsim_i^{SD} e(i)$ for all $i \in N$. A mechanism is SD-efficient if it always returns an SD-IR outcome.

- A mechanism f is *SD-manipulable* if there exists an agent $i \in N$ and preference profiles $\succsim$ and $\succsim'$ with $\succsim_j = \succsim'_j$ for all $j \neq i$ such that $f(\succsim') \succ_i^{SD} f(\succsim)$. A mechanism is *weakly SD-strategyproof* if it is not *SD-manipulable*.

3. The result

We show that there does not exist an SD-efficient, SD-individually rational and weak SD-strategyproof mechanism. The theorem below answers a question raised by Athanassoglou and Sethuraman [2].

Theorem 1. *There does not exist a weak SD-strategyproof, SD-efficient and SD-individually rational fractional housing market mechanism even for single unit allocations and endowments, and for strict preferences.*

Proof. Consider the housing market $(N, H, \succsim, e)$ where $N = \{1, 2, 3, 4, 5\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$, the preference profile $\succsim$ is

$$\begin{aligned}
\succsim_1&: \quad h_3, h_1, h_2, h_4, h_5\\
\succsim_2&: \quad h_5, h_1, h_2, h_3, h_4\\
\succsim_3&: \quad h_1, h_4, h_2, h_3, h_5\\
\succsim_4&: \quad h_2, h_4, h_1, h_3, h_5\\
\succsim_5&: \quad h_5, h_3, h_1, h_2, h_4
\end{aligned}$$

and

$$e = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

First we can establish the claim that agents $3, 4, 5$ get exactly their endowment as long as the following conditions hold:

A the resultant assignment is SD-IR.

B agents $3, 4$, and 5 report truthfully.

C agent 2 reports h_4 as his least preferred house.

D agent 2 reports h_5 as his most preferred house.

E agent 1 expresses h_4 as a house less preferred than h_1, h_2, h_3.

We argue for the claim as follows. Suppose the assignment is

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}.$$

Since h_5 is the most preferred house of agent 2 (Assumption D) and agent 5 (truthful preference of agent 5 according to Assumption B) who also happen to hold 0.5 each of h_5, SD-IR of assignment a implies that $a_{25} = a_{55} = 0.5$ and $a_{15} = a_{35} = a_{45} = 0$. House h_4 is the second most preferred house of agents 3 and 4 and they get 0.5 of their second most preferred house and 0.5 of their most preferred house. If agent 3 or agent 4 get an SD-improvement over their endowment, then they must get less of h_4 and more of their most preferred house. However, due to B, C, E, and the assumption that agent 5 reports truthfully, no agent among 1, 2, and 5 can take any fraction of h_4 or else SD-IR is violated. Hence, $a_{34} = a_{44} = 0.5$ and $a_{14} = a_{24} = a_{54} = 0$. Since h_1 is the most preferred house of agent 3 and her allocation of h_4 is fixed, it follows from SD-IR that $a_{31} = 0.5$. Hence $a(3) = e(3)$. Since h_2 is the most preferred house of agent 4 and her allocation of h_4 is fixed, it follows from SD-IR that $a_{42} = 0.5$. Hence $a(4) = e(4)$. It remains to be established that $a_{53} = 0.5$. Since $a_{55} = 0.5$ is already fixed and since h_3 is the next most preferred house of agent 5 after house h_5, SD-IR implies that $a_{53} = 0.5$. Thus $a(5) = e(5)$ and we have proved the claim.

From now on, we will consider a preference profile in which the conditions above are met so that by SD-IR, we get that agents 3, 4, 5 get exactly their endowments.

Assuming that agents $3, 4, 5$ get the same allocation as their endowment, agent 1 must get $1/2$ of h_3, and agent 2 must get $1/2$ of h_5 in any SD-efficient and SD-IR assignment under profile $\succsim$. Thus the only SD-efficient and SD-IR assignments under $\succsim$ are:

$$x = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix},$$

and

$$z_\lambda = \begin{pmatrix} \lambda & 1/2-\lambda & 1/2 & 0 & 0 \\ 1/2-\lambda & \lambda & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

for $0 \leq \lambda < 1/2$.

If the outcome of $\succsim = (\succsim_1, \succsim_2, \succsim_3, \succsim_4, \succsim_5)$ is assignment x, then agent 2 can report $\succsim_2'$:

$$\begin{aligned} \succsim_1&: \quad h_3, h_1, h_2, \ldots \\ \succsim_2'&: \quad h_5, h_1, h_3, h_2, h_4 \\ \succsim_3&: \quad h_1, h_4, h_2, h_3, h_5 \\ \succsim_4&: \quad h_2, h_4, h_1, h_3, h_5 \\ \succsim_5&: \quad h_5, h_3, h_1, h_2, h_4 \end{aligned}$$

For the profile $(\succsim_1, \succsim_2', \succsim_3, \succsim_4, \succsim_5)$, SD-efficiency and SD-individual rationality imply that the outcome is z_0 (z_λ for the particular value of $\lambda = 0$).

$$z_0 = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

We note that assignment z_0 provides an SD-improvement over assignment x for agent 2 with respect to his real preference relation $\succsim_2$. Thus weak SD-strategyproofness is violated.

If the outcome of $\succsim = (\succsim_1, \succsim_2, \succsim_3, \succsim_4, \succsim_5)$ is assignment z_λ, then agent 1 can report $\succsim_1'$:

$$
\begin{aligned}
\succsim_1'&: \quad h_1, h_3, h_2, \ldots \\
\succsim_2&: \quad h_5, h_1, h_2, h_3, h_4 \\
\succsim_3&: \quad h_1, h_4, h_2, h_3, h_5 \\
\succsim_4&: \quad h_2, h_4, h_1, h_3, h_5 \\
\succsim_5&: \quad h_5, h_3, h_1, h_2, h_4
\end{aligned}
$$

Note that z_λ is not SD-IR for agent 1 with respect to preference $\succsim_1'$. The only SD-IR and SD-efficient outcome of $(\succsim_1', \succsim_2, \succsim_3, \succsim_4, \succsim_5)$ is assignment x which is an SD-improvement for agent 1 over the truthful outcome z_λ. Thus weak SD-strategyproofness is violated. □

We also get as corollaries previous impossibility results in the literature:

Corollary 1 (Theorem 4, Yilmaz [17])**.** *There does not exist an SD-IR, SD-efficient, weak SD-strategyproof, and weak SD-envy-free fractional housing market mechanism.*

Corollary 2 (Theorem 2, Athanassoglou and Sethuraman [2])**.** *There does not exist an SD-IR, SD-efficient, weak justified envy-free and weak SD-strategyproof fractional housing market mechanism.*

Corollary 3 (Theorem 3, Athanassoglou and Sethuraman [2])**.** *There does not exist an SD-IR, SD-efficient, and SD-strategyproof fractional housing market mechanism.*

We remark that the three properties used in Theorem 1 are independent from each other. SD-efficiency and weak SD-strategypoofness can be simultaneously satisfied by the *multi-unit eating probabilistic serial mechanism* [6] if preferences are strict. SD-individual rationality and weak SD-strategyproofness (even SD-strategyproofness) are satisfied by the mechanism that returns the endowment. SD-individual rationality and SD-efficiency can be satisfied by imposing linear constraints for SD-IR and then maximizing sum of utilities that are consistent with the ordinal preferences.

Acknowledgements

The author is supported by a UNSW Scientia Fellowship.

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