

Fair Transit Stop Placement: A Clustering Perspective and Beyond

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Abstract

We study the transit stop placement (TrSP) problem in general metric spaces, where agents travel between source–destination pairs and may either walk directly or utilize a shuttle service via selected transit stops. We investigate fairness in TrSP through the lens of justified representation (JR) and the core, and uncover a structural correspondence with fair clustering. Specifically, we show that a constant-factor approximation to proportional fairness in clustering can be used to guarantee a constant-factor biparameterized approximation to core. We establish a lower bound of 1.366 on the approximability of JR, and moreover show that no clustering algorithm can approximate JR within a factor better than 3. Going beyond clustering, we propose the Expanding Cost Algorithm, which achieves a tight 2.414-approximation for JR, but does not give any bounded core guarantee. In light of this, we introduce a parameterized algorithm that interpolates between these approaches, and enables a tunable trade-off between JR and core. Finally, we complement our results with an experimental analysis using small-market public carpooling data.

1 Introduction

A municipality has decided to offer a publicly-operated shuttle to offer its residents a safe, convenient, and accessible alternative to private vehicles. Towards this end, the public infrastructure planner has been allocated a budget to construct a desired number of shuttle stops. Given data describing the common trips made by each resident that would use the shuttle service, how can the planner decide where to place the shuttle stops? This problem, which we call the *Transit Stop Placement Problem* (TrSP), is the focus of this work.

While the focus of prior public transportation research in civil engineering and operations research is typically on efficiency and cost minimization [9, 18, 30, 38], there has been a growing appreciation of equity concerns when designing and planning infrastructure in the last decade [7, 25, 26]. In terms of our problem, while each resident (referred to as agents, henceforth) would prefer to have a shuttle stop at the exact addresses of their starting point (e.g., home) and destination (e.g., work), the aim of the planner is to select a set of stops that is as fair as possible to the agents, subject to scarce resources. Bullinger et al. [7] recently studied this question in the case in which all agents and potential stops are located on a line. Inspired by the literature on committee voting [see, e.g., 3, 24], they defined fairness properties known as *core* and *justified representation* (a relaxation of core). Rather than take an egalitarian formulation of fairness, which may overcorrect for an isolated resident and hence result in stop placements which are not convenient for *any* agent, core and justified representation are based upon the ideal of proportional representation, i.e., one in which groups of agents with similar preferences are entitled to resources in proportion to their size.

The primary limitation of the work by Bullinger et al. [7] lies in its focus on the line, which while a natural introductory setting for investigating fairness in transit stop placement, fails to adequately capture the complexity of many real-world transportation networks. In this work, we address this limitation by studying transit stop placement in general metric spaces.

To make things more concrete, we offer an example. Suppose there are six agents, and each agent i follows a route from a_i to b_i . The planner has a budget to construct three shuttle stops. Each agent derives disutility from a transit stop placement equal to the sum of the agent’s walking distance to the stops they use and their transit cost between stops. There are four candidate stop locations, c_1 through c_4 . Consider the stop placements depicted in Figure 1a and Figure 1b, where the selected stops are indicated by stars. We observe that the placement in Figure 1a fails to adequately represent the agent group $\{1, 2, 3, 4\}$. Every agent in that group derives greater disutility from the transit stop placement in Figure 1a than from the placement in Figure 1b. This group constitutes two thirds of the population, and thus intuitively should have two thirds of the decision power, i.e. be able to effectively decide two of the three stops.

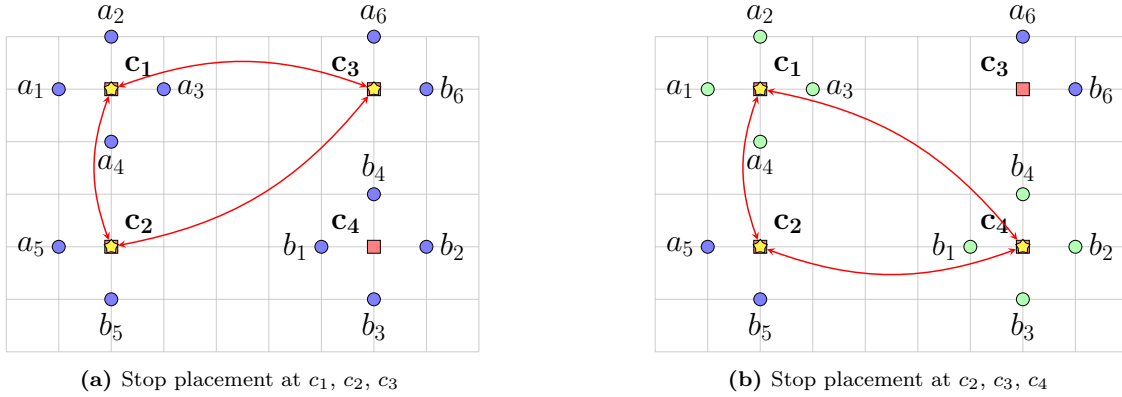


Figure 1: Transit stop placement example. Each travel route connects an agent pair (a_i, b_i) , with six agents in total (blue circles). There are four candidate stop locations, c_1 to c_4 (red squares). Panels (a) and (b) illustrate two different placement choices, with the selected stops marked by yellow stars. Red arrows indicate the shuttle transit routes.

In this work, we aim to identify algorithms which can provably guarantee fair transit stop placements in general metric spaces. Since the provably fair algorithms introduced by Bullinger et al. [7] exploit the particular structure of the line, we must take a different approach. We note that our problem shares substantial overlap with fair centroid clustering [6, 8, 14, 17, 23, 29], which also involves selecting a prescribed number of centers (or stops) for a given set of points while pursuing fairness guarantees. Indeed, the application of a fair clustering algorithm to our previous example would certainly require the selection of both c_1 and c_4 . This observation naturally raises the question of whether existing fair clustering algorithms, backed by a rich body of research on fairness, can be adapted to achieve our fairness objectives in the context of transit stop placement.

To what extent can fair clustering algorithms guarantee proportionally representative and fair transit stop placements in general metric spaces?

The two settings differ, however, in that agents are associated with a pair of points in our model whereas they are modeled as a single point in clustering, and thus clustering algorithms necessarily ignore potentially useful information on agents’ preferences. Furthermore, clustering algorithms do not capture that agents’ preferences may also depend on their transit cost between stops. We would like to define transit cost flexibly on instances defined on general metric spaces since walking and other transport infrastructure may not necessarily align.¹

Is it possible to devise algorithms which outperform clustering algorithms if we take a more fine-grained view of agents’ preferences and flexibly account for agents’ transit times?

¹For example, a footbridge may allow an agent to move between two points on foot faster than other infrastructure allows.

1.1 Our Contributions

Our first contribution is a flexible model of transit stop placement in general metric spaces. In the model of Bullinger et al. [7], transit times between stops are assumed to be proportional to the walking distance between the two stops. Our model, at its most general, allows for significantly more general transit times, requiring only that they abide by a distance function which satisfies the triangle inequality. As was done in Bullinger et al. [7], we define an approximate version of justified representation, β -JR, which requires that all agents in a deviating group prefer the deviation by a factor $\beta \geq 1$. We also define a bi-criteria approximation of core, (α, β) -core, in which the α factor strengthens the deviating group size requirement.

We first establish formal connections between centroid clustering and transit stop placement (TrSP) under the assumption of negligible transit times. By reducing TrSP to clustering, we show that any clustering algorithm satisfying ρ -Proportional Fairness (PF) can be used to guarantee a $(2, \rho)$ -core outcome for TrSP. Since the Greedy Capture algorithm gives a $(1 + \sqrt{2})$ -PF solution in clustering, a direct corollary of our theorem is that this algorithm in the context of TrSP, which we refer to as GC-TrSP, guarantees $(2, 1 + \sqrt{2})$ -core. We also prove that GC-TrSP satisfies $(2 + \sqrt{5}) \approx 4.24$ -JR, and that both of these bounds are tight. In the reverse direction, we give a mapping from clustering instances to TrSP instances which demonstrates that any β -Justified Representation (JR) algorithm for TrSP can be used to obtain a 2β -PF solution for the original clustering instance. This insight leads to an impossibility result: in general metric spaces, a JR (and therefore, core) outcome is not guaranteed to exist. We improve on this impossibility by showing that no TrSP algorithm can guarantee better than $\frac{\sqrt{3}+1}{2} \approx 1.37$ -JR in general. To understand the limitations of our reduction to clustering with respect to JR, we construct a TrSP instance for which no clustering algorithm can guarantee better than 3-JR.

To surpass this barrier, we introduce a novel algorithm, the *Expanding Cost Algorithm (ECA)*, which evaluates and selects stop *pairs* rather than singletons, and directly considers agents' costs rather than mere distances. We prove that ECA satisfies $(1 + \sqrt{2}) \approx 2.41$ -JR, again with a tight bound, thereby improving on the approximation guarantee of every clustering-based algorithm. Strikingly, this approximation factor holds for any travel times between stops that satisfy the triangle inequality. In contrast, no clustering algorithm guarantees a constant factor JR approximation under such generality. Although ECA substantially outperforms clustering algorithms in terms of JR approximation, it fails to satisfy (α, β) -core for any $\alpha, \beta \geq 1$.

Given that GC-TrSP and ECA exhibit complementary strengths in approximating JR and the core, we propose a new algorithm, λ -Hybrid, which interpolates between GC-TrSP and ECA via a tunable parameter $\lambda \geq 0$. This parameter λ controls the algorithm's preference for selecting singleton or paired stops. We show that λ -Hybrid satisfies

$$\frac{\lambda + 3 + \sqrt{\lambda^2 + 10\lambda + 9}}{2}\text{-JR and } (2, \frac{\sqrt{\lambda^2 + 6\lambda + 1} + \lambda + 1}{2\lambda})\text{-core.}$$

A comprehensive picture of JR approximations is depicted in Figure 2 while the comparative performance of GC-TrSP, ECA, and λ -Hybrid with respect to core approximations is summarized in Table 1.

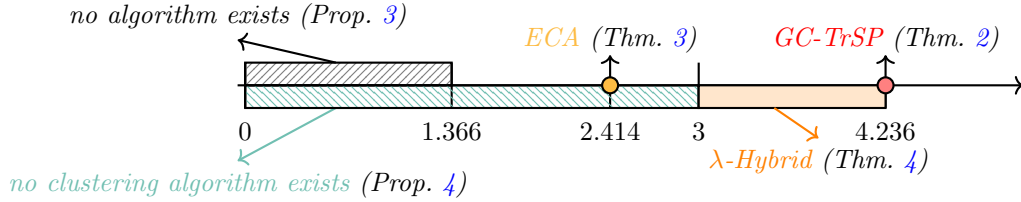


Figure 2: Overview of JR approximation ratios. The two shaded regions with diagonal lines indicate the lower bounds for general algorithms and clustering algorithms, respectively. The points at 2.414 and 4.236 correspond to the ECA and GC-TrSP algorithms. The performance of the λ -Hybrid algorithm ranges between 3 and 4.236, depending on the choice of the parameter λ .

	core	
	Lower Bound	Upper Bound
GC-TrSP	$(2, 1 + \sqrt{2})$ (Prop. 2)	$(2, 1 + \sqrt{2})$ (Prop. 2)
ECA	(γ, ρ) -core for any $\gamma, \rho \geq 1$ (Prop. 5)	–
λ-Hybrid	$(2, \frac{\sqrt{4\lambda^2+12\lambda+1}+2\lambda+1}{4\lambda})$ (Prop. 6)	$(2, \frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda})$ (Cor. 1)

Table 1: Comparison of Expanding Cost Algorithm (ECA) and Greedy Capture for TrSP (GC-TrSP) in terms of approximation ratios for core in general metric space.

We note that the algorithm by Bullinger et al. [7], which provides JR in the line instance, is in fact also a clustering algorithm on the line, in which setting it guarantees a PF solution. However, it can perform arbitrarily poorly when the candidate centers do not align with data points. To address this, we propose a new algorithm called ℓ -dictator partition algorithm, which generalizes the algorithm by Bullinger et al. [7] and satisfies PF in clustering on the line, and thus matches their $(1, 2)$ -core bound for TrSP on the line. Lastly, we prove that a solution minimizing the total cost for TrSP in general metric space is NP-hard to compute. We defer these results to appendices.

1.2 Scope and Assumptions

While our model is quite general, it does not fully encapsulate the problem of designing a public transportation network. A significant component of that task is route selection, i.e., the decision of which sequences of stops will traversed by buses, trains, etc. That decision in turn affects transit time between stops. In contrast, we focus on the placement of the stops themselves and treat transit times between stops as exogenous. As a result, our model is not well-suited to designing large-scale public transportation systems wherein transit times make up a significant portion of agents’ costs. Instead, our model is more amenable to scenarios in which either (1) a route (or set of routes) is set a priori and all candidate stops lie along these routes, or (2) total travel times are dominated by walking rather than transit time, and thus do not depend greatly on routing.

1.3 Related Work

The problem of transit stop placement has been extensively studied in the public transportation literature [9, 10, 16]. For instance, Hossein Rashidi et al. [22] investigated the optimization of stop locations in transit networks under elastic travel demand and budget constraints. They formulated the problem as a mixed integer program and developed heuristic algorithms capable of solving large-scale instances. Ceder et al. [11] addressed the challenge of bus stop placement in routes with uneven topography. They incorporated topographical variation into three distinct mathematical models that account for walking speed, the attractiveness of access paths to transit services, and vehicle acceleration at stops. To tackle it, they proposed a heuristic evolutionary algorithm to approximate optimal solutions.

Fair transit stop placement has received growing attention in recent years due to its critical role in promoting equitable access to public transportation [25, 26]. For instance, Tedjopurnomo et al. [37] studied the equitable public bus network optimization problem through a case study of Singapore’s bus system, formulating efficiency and equity metrics and conducting exploratory experiments to evaluate their real-world impact. Their work underscores the challenges of balancing fairness and efficiency in transit network design. In a complementary direction, Najmi et al. [31] explored fairness and equity from psychological and cognitive perspectives, highlighting how users’ perceptions and experiences shape their sense of fairness in public transit. Matl et al. [27] wrote a comprehensive survey on equitable vehicle routing problems (VRPs), reviewing various strategies to embed fairness into routing decisions. More recently, He et al. [21] examined

fairness in transportation network design from a welfarist perspective, focusing on selecting a subset of edges in an undirected graph to optimize network performance while maintaining fairness considerations.

In this paper, we consider fairness notions inspired by the concept of *core stability*, a foundational idea that has been extensively applied across various domains, including transferable utility cooperative games [33], coalition formation [2], exchange markets [36], and two-sided matching [1]. Although it is typically viewed as a stability concept, it is also amongst the strongest fairness notions in many social choice contexts, including committee voting [24], fair mixing [4], and fair allocation [20].

Bullinger et al. [7] were the first to explore fairness in the transit stop placement problem. Their work focused on the line metric. They presented polynomial-time algorithms for cost minimization and designed algorithms with provable fairness guarantees, including justified representation (JR) and a factor-two approximation for core stability. Importantly, they note that “an important research challenge is to develop a richer framework that can be used to reason about fairness in more realistic models of public transport.” Motivated by this, we consider a more general setting in which agents and candidate bus stops lie in an arbitrary metric space. In contrast to the positive results Bullinger et al. [7] obtained for the line, we will show that in general, the cost minimization problem becomes computationally intractable, and a solution satisfying JR need not exist.

The TrSP problem we study in this paper is closely related to the study of fairness in centroid selection for clustering [6, 14, 15, 23, 29]. Chen et al. [14] initiated the study of fairness in clustering, introducing the proportional fairness (PF) axiom, requiring that no “large enough” group of datapoints has an incentive to collectively deviate to an unselected candidate center, an idea inspired by core stability. They showed that outcomes satisfying reasonable approximations of PF are guaranteed to exist and can be computed via the “Greedy Capture” algorithm. Following this, a substantial body of work has emerged on fairness and proportionality in clustering. For example, Micha and Shah [29] extended the analysis to unconstrained centroid candidate sets; Aziz et al. [6] proposed a fairness axiom targeting proportional representation; and Kellerhals and Peters [23] established connections between fair clustering and committee voting, and analyzed the connections between several of the axioms introduced in this literature. The TrSP problem studied in this paper can be viewed as a variant of centroid selection, where centroids correspond to transit stops. However, unlike standard clustering where each datapoint represents an agent, the TrSP model associates two datapoints with each agent. In this work, we will highlight both the inherent connections and the fundamental differences between these two problem domains.

2 Preliminaries

2.1 Transit Stop Placement Model

Let \mathcal{X} be a set and d and d' be two distinct distance functions satisfying the triangle inequality. The metric space (\mathcal{X}, d) represents *walking cost*² while the metric space (\mathcal{X}, d') represents *transit cost*. An instance of the *Transit Stop Placement* (TrSP) model is defined by the tuple $\mathcal{I} = \langle N, \mathcal{C}, \{\theta_i\}_{i \in N}, k \rangle$, where $N := \{1, 2, \dots, n\}$ is a finite set of n agents, \mathcal{C} is a set of m candidate transit stops, k is a positive integer, and $\theta_i = (a_i, b_i)$ denotes the *endpoints* between which agent i travels, where $a_i, b_i \in \mathcal{X}$. We denote by $\Theta(S)$ the multiset of all endpoints associated with agents in a subset $S \subseteq N$. For simplicity, let $\Theta := \Theta(N)$. A solution to a TrSP instance is a subset of candidate transit stops $Y \subseteq \mathcal{C}$. The solution Y is said to be *feasible* if $|Y| \leq k$. For ease of exposition, we define $d(i, X) = \min_{j \in X} d(i, j)$.

Given any feasible solution Y , the cost of agent i with type (a_i, b_i) is given by

$$c_i(Y) = \min \left\{ d(a_i, b_i), \min_{y_1, y_2 \in Y} [d(a_i, y_1) + d'(y_1, y_2) + d(y_2, b_i)] \right\}$$

Intuitively, each agent i minimizes her travel cost by either walking directly from a_i to b_i , or by walking from a_i to a transit stop $y_1 \in Y$, taking the transit system from y_1 to another stop $y_2 \in Y$, and then walking

²A standard interpretation of the cost is travel time.

from y_2 to her destination b_i . Unless otherwise specified, we adopt the assumption that $d'(y_1, y_2) = 0$ for all $y_1, y_2 \in Y$. This assumption, which we refer to as *null transit times*, captures scenarios where the cost of using the transit system is negligible compared to walking. Critically, our central algorithmic result will hold for arbitrary transit times.

The total travel cost of a solution $Y \subseteq \mathcal{C}$ is defined as $c(Y) = \sum_{i \in N} c_i(Y)$. We first remark that a solution which minimizes total travel cost among all feasible solutions is NP-hard to compute, even in the special case of null transit times. Since we are mainly focused on fairness in this work, we defer the proof to Appendix A. This result marks a contrast between general metric spaces and the line, since the latter setting admits a polynomial time algorithm for any transit times which are directly proportional to the walking times [7].

Proposition 1. *Unless $P = NP$, there is no polynomial time algorithm which computes a minimum cost solution to the TrSP problem, even under null transit times, i.e., even when $d'(i, j) = 0$ for all $i, j \in \mathcal{X}$.*

We focus on the fairness axioms introduced by Bullinger et al. [7], both of which draw inspiration from the study of *core stability* in various domains including multi-winner voting, participatory budgeting, and fair clustering (see, e.g., [5, 12, 13, 19, 34, 35]). In the context of transit stop placement, core is grounded in the principle that, given an instance with n agents and a budget of k , each agent is entitled to a $\frac{k}{n}$ -proportion of the budget. Toward this end, the core comprises the set of feasible imputations in which no coalition of agents can all strictly improve their cost by using their budget on an alternative set of transit stops. That is, a solution Y is in the core if for any subset of agents $S \subseteq N$ and any transit stop set $T \subseteq \mathcal{C}$ with $|S| \geq |T| \cdot \frac{n}{k}$, there exists at least one agent $i \in S$ such that $c_i(T) \geq c_i(Y)$.

Our other fairness axiom of focus, *justified representation* (JR), weakens core by restricting its attention to deviations consisting of pairs of stops. That is, JR requires that any group of agents with size at least $\lceil \frac{2n}{k} \rceil$ should not be able to reduce their travel costs by deviating from solution Y to an alternative pair of stops. A pair of stops is considered the minimal meaningful unit since no agent can derive benefit from a single stop. Besides being significantly easier to compute, JR has another major advantage over core. Whereas the complexity of checking core in our setting is unsettled, JR can be verified in polynomial time.³

As we will see, JR (and thus core) is not guaranteed to exist in our setting. As a result, we will study multiplicative relaxations of both properties, which we will now introduce. For approximate JR, we require that at least one agent in the group improve by no more than a factor β .

Definition 1 (β -Justified Representation (β -JR)). *A solution $Y \subseteq \mathcal{C}$ is said to provide β -JR if for every subset of agents $S \subseteq N$ with $|S| \geq \frac{2n}{k}$ and every pair of transit stops $T \subseteq \mathcal{C}$, $|T| = 2$, there exists an agent $i \in S$ such that $\beta \cdot c_i(T) \geq c_i(Y)$.*

For core, we additionally parameterize the size of a deviating group of agents using multiplicative factor α .

Definition 2 ((α, β) -core). *Let $\alpha, \beta \geq 1$. A solution $Y \subseteq \mathcal{C}$ is in the (α, β) -core if for every subset of agents $S \subseteq N$ and every transit stop set $T \subseteq \mathcal{C}$ with $|S| \geq \alpha \cdot |T| \cdot \frac{n}{k}$, there exists an agent $i \in S$ such that $\beta \cdot c_i(T) \geq c_i(Y)$.*

When $\beta = 1$, the $(\alpha, 1)$ -core aligns with the β -core approximation defined by Bullinger et al. [7]. The $(\alpha, 1)$ -core represents a multiplicative *size approximation* of the core, relaxing the requirement such that a group of agents can deviate and establish $|T|$ transit stops only if its size is at least α times the number of agents who “deserve” $|T|$ stops. Conversely, when $\alpha = 1$, the $(1, \beta)$ -core introduces a relaxation on individual cost, aligning with the notion of approximate Proportional Fairness (PF) in fair clustering [14].

2.2 Fair Clustering Model

We will now review the problem description and relevant definitions of fair centroid clustering. To avoid notation confusion with the TrSP instance, we use a slightly modified notation. Similar to TrSP, clustering

³For further details, see Appendix C or Bullinger et al. [7, Appendix C.1].

instances consist of points in a metric space (\mathcal{X}, d) . A fair clustering instance is given by a tuple $\mathcal{I}' = \langle N', \mathcal{C}', k' \rangle$ where N' is a finite set of n' datapoints, \mathcal{C}' is a set of m' centers, and k' is a positive integer. A clustering solution is a subset $P \subseteq \mathcal{C}'$ of at most k' centers. We will draw and exploit connections between our fairness concepts and the clustering fairness concept known as *proportional fairness*.

Definition 3 (ρ -Proportional Fairness (ρ -PF)). *A clustering solution $P \subseteq \mathcal{C}'$ with $|P| \leq k'$ satisfies ρ -Proportional Fairness if, for all $S' \subseteq N'$ with $|S'| \geq \frac{n'}{k'}$ and for all $c \in \mathcal{C}'$, there exists a datapoint $i \in S'$ with $\rho \cdot d(i, c) \geq d(i, P)$.*

Put differently, there should be no group of agents large enough to deserve one center that would all prefer that center to their closest center under P , even when scaling their alternative distance by ρ .

Lastly, we introduce a mapping from instances of TrSP to instances of fair clustering. Given any TrSP instance $\mathcal{I} = \langle N, \mathcal{C}, k, \{\theta_i\}_{i \in N} \rangle$, we define a clustering instance $\mathcal{I}^C = \langle \Theta, \mathcal{C}, k \rangle$ within the same metric space (\mathcal{X}, d) . We call this the *clustering instance induced by \mathcal{I}* , or simply induced clustering instance, when context is clear. In words, the clustering instance induced by \mathcal{I} reinterprets the endpoints of agents in the TrSP model as datapoints in a clustering instance and maintains the same candidate set and target selection number. Notably, the set of feasible outcomes in the TrSP instance \mathcal{I} is identical to that in its induced clustering instance \mathcal{I}^C . As a result, every clustering algorithm immediately yields an algorithm for TrSP instances by the following simple procedure: given a TrSP instance \mathcal{I} , run the clustering algorithm on \mathcal{I}^C and return the output. In the next section, we will reason about the application of clustering algorithms to transit stop placement.

3 Transit Stop Placement Meets Fair Clustering

In this section, we establish connections between the TrSP problem and fair clustering, highlighting their underlying structural similarities and differences. We first show that clustering algorithms can be used to approximate fairness in our setting.

3.1 Approximate Fairness by Reduction to Clustering

To establish that fair clustering algorithms can indeed be used to guarantee fairness in TrSP, we prove a metatheorem which uses our reduction from TrSP to clustering. In particular, we show that, to guarantee a $(2, \rho)$ -core solution in TrSP, it is sufficient to apply a centroid selection algorithm that satisfies ρ -PF to the induced clustering instance.

Theorem 1. *Given a TrSP instance \mathcal{I} , if there exists a feasible solution Y satisfying ρ -PF in the induced clustering instance \mathcal{I}^C for some $\rho \geq 1$, then Y is $(2, \rho)$ -core solution for \mathcal{I} .*

Proof. Given any TrSP instance $\mathcal{I} = \langle N, \mathcal{C}, k, \{\theta_i\}_{i \in N} \rangle$, consider the induced clustering instance $\mathcal{I}^C = \langle \Theta, \mathcal{C}, k \rangle$ and let Y be a feasible centroid selection satisfying ρ -PF. For any subset of agents $S \subseteq N$ and any subset of transit stops $T \subseteq \mathcal{C}$ with $|S| \geq |T| \cdot \frac{2n}{k}$ in \mathcal{I} , let $\Theta(S)$ denote the multiset of datapoints corresponding to agents in S . Then $|\Theta(S)| = 2 \cdot |S|$, and hence we have $\frac{1}{2} \cdot |\Theta(S)| \geq |T| \cdot \frac{2n}{k}$. Define the set $Q := \{j \in \Theta(S) : \rho \cdot d(j, T) \geq d(j, Y)\}$. Since solution Y satisfies ρ -PF in \mathcal{I}^C and $|\Theta| = 2n$, it follows that for any candidate center $c \in T$, the number of datapoints $j \in \Theta(S)$ such that $\rho \cdot d(j, c) < d(j, Y)$ is strictly less than $\frac{2n}{k}$. Therefore,

$$|Q| > |\Theta(S)| - |T| \cdot \frac{2n}{k} \geq |T| \cdot \frac{4n}{k} - |T| \cdot \frac{2n}{k} = |T| \cdot \frac{2n}{k}.$$

Note that each datapoint in Q is an agent's endpoint in \mathcal{I} and $|Q| > |T| \cdot \frac{2n}{k}$. By the pigeonhole principle, there exists an agent $i^* \in S$ such that both a_{i^*} and b_{i^*} belong to Q . Hence, it holds that $\rho \cdot d(a_{i^*}, T) \geq d(a_{i^*}, Y)$ and $\rho \cdot d(b_{i^*}, T) \geq d(b_{i^*}, Y)$. This tells us the following about the cost of agent i^* :

$$\rho \cdot c_{i^*}(T) = \rho \cdot \min \left\{ d(a_{i^*}, b_{i^*}), \min_{\tau_1, \tau_2 \in T} [d(a_{i^*}, \tau_1) + d(b_{i^*}, \tau_2)] \right\}$$

$$\begin{aligned}
&= \min \{ \rho \cdot d(a_{i^*}, b_{i^*}), \rho \cdot d(a_{i^*}, T) + \rho \cdot d(b_{i^*}, T) \} \\
&\geq \min \{ \rho \cdot d(a_{i^*}, b_{i^*}), d(a_{i^*}, Y) + d(b_{i^*}, Y) \} \quad (\because \rho \cdot d(a_{i^*}, T) \geq d(a_{i^*}, Y); \rho \cdot d(b_{i^*}, T) \geq d(b_{i^*}, Y)) \\
&\geq \min \{ d(a_{i^*}, b_{i^*}), d(a_{i^*}, Y) + d(b_{i^*}, Y) \} \quad (\because \rho \geq 1) \\
&= c_{i^*}(Y).
\end{aligned}$$

Therefore, for any subset of agents $S \subseteq N$ and any subset of transit stops $T \subseteq \mathcal{C}$ with $|S| \geq |T| \cdot \frac{2n}{k}$ in \mathcal{I} , there always exists an agent i^* such that $\rho \cdot c_{i^*}(T) \geq c_{i^*}(Y)$. This implies that the solution Y satisfies $(2, \rho)$ -core in \mathcal{I} . \square

From Theorem 1, it immediately follows that $(2, 1 + \sqrt{2})$ -core solutions for the TrSP problem can be computed using the clustering algorithm known as Greedy Capture [14]. When applied to TrSP instances (through the reduction described in Section 2.2), we refer to this algorithm as Greedy Capture for TrSP (GC-TrSP).

The algorithm works by uniformly growing balls around candidate transit stops and iteratively adding stops whose balls capture a sufficient number of uncaptured endpoints. In more detail, each endpoint is first marked “active”, and GC-TrSP smoothly increases radius r and iteratively “opens” stops which are at distance at most r from at least $\lceil \frac{2n}{k} \rceil$ active endpoints. Opened stops are added to the solution and endpoints are deactivated as soon as they are contained in an opened ball. The algorithm terminates when all agents are deactivated.⁴ See Algorithm 1 for formal pseudocode.

Input: TrSP instance $\mathcal{I} = \langle N, \mathcal{C}, k, \{\theta_i\}_{i \in N} \rangle$.

Output: Solution Y .

- 1: Create $\mathcal{I}^C = \langle \Theta, \mathcal{C}, k \rangle$, the clustering instance induced by \mathcal{I} .
- 2: Denote the distance ball of candidate $c \in \mathcal{C}$ with radius r by $B(c, r) \leftarrow \{j \in \Theta : d(j, c) \leq r\}$.
- 3: Initialize $r \leftarrow 0$, $Y \leftarrow \emptyset$; $\mathcal{N} \leftarrow \Theta$.
- 4: **while** $\mathcal{N} \neq \emptyset$ **do**
- 5: Smoothly increase r .
- 6: **while** $\exists y \in Y$ s.t. $|B(y, r) \cap \mathcal{N}| \geq 1$ **do**
- 7: $\mathcal{N} \leftarrow \mathcal{N} \setminus B(y, r)$
- 8: **while** $\exists c \in \mathcal{C} \setminus Y$ s.t. $|B(c, r) \cap \mathcal{N}| \geq \lceil \frac{2n}{k} \rceil$ **do**
- 9: $Y \leftarrow Y \cup \{c\}$
- 10: $\mathcal{N} \leftarrow \mathcal{N} \setminus B(c, r)$
- 11: Return Y .

Algorithm 1: Greedy Capture for TrSP (GC-TrSP)

In fact, as we will now show, it turns out that $(2, 1 + \sqrt{2})$ -core is the best achievable approximation factor for GC-TrSP and thus the analysis provided by Theorem 1 is tight. We defer instances proving tightness of approximation to Appendix B.1.

Proposition 2. *GC-TrSP algorithm (Algorithm 1) satisfies $(2, 1 + \sqrt{2})$ -core. However, for any $\delta, \varepsilon > 0$, there exists an instance for which GC-TrSP violates $(2 - \delta, 1 + \sqrt{2} - \varepsilon)$ -core.*

One of the apparent drawbacks of the core approximation obtained by Greedy Capture is that it strengthens the coalition size requirement by a factor 2. This effectively halves each group’s representative decision power when considering deviations. It turns out that the coalition size requirement *must* be strengthened to some extent in order for GC-TrSP to give any bounded guarantee with respect to core. In particular,

⁴We note that Greedy Capture may terminate before selecting k centers, an artifact that appears in some of our lower bound arguments. This behavior can be avoided by deactivating exactly $\frac{2n}{k}$ (fractional) endpoints for each selected center. It is not clear whether the choice of which endpoints to deactivate can be used to improve the bounds. Nonetheless, we also give lower bound results like Proposition 4 which apply to all clustering algorithms.

it holds that GC-TrSP does not satisfy $(1, \rho)$ -core for any $\rho \geq 1$.⁵ As a natural next step, we investigate GC-TrSP with respect to JR, a property which maintains the proportional coalition size requirement but restricts considered deviations to those consisting of pairs of stops. The following result shows that GC-TrSP achieves a $(2 + \sqrt{5})$ -approximation to JR, and this bound is tight.

Theorem 2. *GC-TrSP satisfies $(2 + \sqrt{5})$ -JR. However, for any $\varepsilon > 0$, there exists an instance for which GC-TrSP violates $(2 + \sqrt{5} - \varepsilon)$ -JR.*

Proof. Given any TrSP instance \mathcal{I} , let $Y \subseteq \mathcal{C}$ be the solution of GC-TrSP under \mathcal{I} . Assume, for the sake of a contradiction, that Y fails to satisfy $(2 + \sqrt{5})$ -JR. That is, there exists a group of agents $S \subseteq N$ with $|S| \geq \lceil \frac{2n}{k} \rceil$ and a pair of transit stops $T = \{\tau_1, \tau_2\} \subseteq \mathcal{C}$ such that $(2 + \sqrt{5}) \cdot c_j(T) < c_j(Y)$ for every agent $j \in S$.

Without loss of generality, we assume that every agent $i \in S$ travels from transit stop τ_1 to τ_2 , walking from their starting point a_i to τ_1 and from τ_2 to their destination b_i . Define

$$r_T = \max_{j \in S} \{ \max \{ d(a_j, \tau_1), d(b_j, \tau_2) \} \}$$

as the maximum distance between any endpoint of an agent in S and its closest of the two transit stops in $\{\tau_1, \tau_2\}$. Let $i^* \in S$ be the agent that attains this maximum distance. Without loss of generality, assume that this maximum distance is realized at the starting point a_{i^*} , i.e. $r_T = d(a_{i^*}, \tau_1)$.

We begin by considering the case in which the distance radius explored by GC-TrSP never reaches r_T . That is, GC-TrSP deactivates all agent endpoints, and in particular both of the endpoints of agent i^* , with a distance radius at most r_T . In this scenario, we derive that $c_{i^*}(Y) \leq 2 \cdot r_T \leq 2 \cdot c_{i^*}(T)$, which contradicts the assumption that $(2 + \sqrt{5}) \cdot c_{i^*}(T) < c_{i^*}(Y)$.

Henceforth, we focus on the case in which GC-TrSP does consider a distance radius of r_T during its execution. We begin by examining the subcase where Y contains one of the stops in T . Without loss of generality, we assume that $\tau_1 \in Y$ and $\tau_2 \notin Y$. Since τ_1 is included in Y , all starting points a_i of agents in S are deactivated when the distance radius reaches at most r_T . Note that for stop τ_2 , if selected, could deactivate $\lceil \frac{2n}{k} \rceil$ endpoints with radius r_T . However, since $\tau_2 \notin Y$, there must exist some agent i' whose endpoint $b_{i'}$ is already deactivated by another transit stop, denoted $y \in Y$, with a radius at most r_T . Consequently, we have $c_{i'}(Y) \leq d(a_{i'}, \tau_1) + d(b_{i'}, y) \leq d(a_{i'}, \tau_1) + r_T$. With this inequality in hand, we now proceed to establish an upper bound incurred by agent i^* under the solution Y .

$$\begin{aligned} c_{i^*}(Y) &= \min_{y_1, y_2 \in Y} \{ d(a_{i^*}, y_1) + d(b_{i^*}, y_2) \} \\ &\leq d(a_{i^*}, \tau_1) + d(b_{i^*}, y) && (\because \tau_1 \in Y, y \in Y) \\ &\leq r_T + d(b_{i^*}, \tau_2) + d(\tau_2, b_{i'}) + d(b_{i'}, y) && (\because \text{triangle inequality}) \\ &\leq 3 \cdot r_T + d(b_{i'}, \tau_2). && (\because d(b_{i^*}, \tau_2) \leq r_T, d(b_{i'}, y) \leq r_T) \end{aligned}$$

We next consider the minimum multiplicative cost improvement of agent i' and i^* under T :

$$\begin{aligned} \min \left(\frac{c_{i'}(Y)}{c_{i'}(T)}, \frac{c_{i^*}(Y)}{c_{i^*}(T)} \right) &\leq \min \left(\frac{d(a_{i'}, \tau_1) + r_T}{d(a_{i'}, \tau_1) + d(b_{i'}, \tau_2)}, \frac{3 \cdot r_T + d(b_{i'}, \tau_2)}{r_T} \right) \\ &\leq \min \left(\frac{r_T}{d(b_{i'}, \tau_2)}, \frac{3 \cdot r_T + d(b_{i'}, \tau_2)}{r_T} \right) \\ &\leq \max_{z \geq 0} (\min(z, 3 + 1/z)) = \frac{3 + \sqrt{13}}{2}. \end{aligned}$$

The second inequality holds because $d(b_{i'}, \tau_2) \leq r_T$ and subtracting $d(a_{i'}, \tau_1)$ from numerator and denominator weakly increases the resulting fraction. Therefore, we have $\frac{3 + \sqrt{13}}{2} \cdot c_{i^*}(T) \geq c_{i^*}(Y)$, which contradicts that $(2 + \sqrt{5}) \cdot c_j(T) < c_j(Y)$ for every agent $j \in S$.

⁵Consider an example with two agents and $k = 3$ on the unit interval. Suppose $\mathcal{C} = \{0, 1/4, 1/2, 3/4, 1\}$ and the voters have endpoints $(0, 1/2)$ and $(0, 1)$. GC-TrSP selects $\{0, 3/4\}$, causing each agent to incur a cost of $1/4$. Note that the solution $\{0, 1/2, 1\}$ is feasible and gives each agent a cost of 0.

The remaining subcase is when neither τ_1 nor τ_2 are included in Y . In this case, since τ_1 and τ_2 are excluded from Y , we observe that there exists at least one agent $i' \in S$ such that $a_{i'}$ is deactivated by some $y' \in Y$ with a radius at most r_T and one agent $i'' \in S$ such that $b_{i''}$ is deactivated by some $y'' \in Y$ with a radius at most r_T . We next upper-bound the cost incurred by agents i^*, i' , and i'' under solution Y .

For agent i^* , we have

$$\begin{aligned} c_{i^*}(Y) &\leq d(a_{i^*}, y') + d(b_{i^*}, y'') \\ &\leq d(a_{i^*}, \tau_1) + d(\tau_1, a_{i'}) + d(a_{i'}, y') + d(b_{i^*}, \tau_2) + d(\tau_2, b_{i''}) + d(b_{i''}, y'') \\ &\leq 3 \cdot r_T + d(b_{i^*}, \tau_2) + d(\tau_1, a_{i'}) + d(\tau_2, b_{i''}). \end{aligned}$$

For agent i' , we have

$$\begin{aligned} c_{i'}(Y) &\leq d(a_{i'}, y') + d(b_{i'}, y'') \\ &\leq d(a_{i'}, y') + d(b_{i'}, \tau_2) + d(\tau_2, b_{i''}) + d(b_{i''}, y'') \\ &\leq 2 \cdot r_T + d(b_{i'}, \tau_2) + d(\tau_2, b_{i''}). \end{aligned}$$

We obtain the upper bound for agent i'' in an analogous fashion to the previous bound:

$$c_{i''}(Y) \leq 2 \cdot r_T + d(a_{i''}, \tau_1) + d(\tau_1, a_{i'}).$$

For ease of expression, we denote $x = d(a_{i'}, \tau_1)$ and $y = d(\tau_2, b_{i''})$. With the upper bounds in hand, we consider the minimum multiplicative cost improvement of agents $\{i^*, i', i''\}$ as follows.

$$\begin{aligned} &\min \left(\frac{c_{i'}(Y)}{c_{i'}(T)}, \frac{c_{i''}(Y)}{c_{i''}(T)}, \frac{c_{i^*}(Y)}{c_{i^*}(T)} \right) \\ &\leq \min \left(\frac{2 \cdot r_T + d(b_{i'}, \tau_2) + y}{x + d(b_{i'}, \tau_2)}, \frac{2 \cdot r_T + d(a_{i''}, \tau_1) + x}{d(a_{i''}, \tau_1) + y}, \frac{3 \cdot r_T + d(b_{i^*}, \tau_2) + x + y}{r_T + d(b_{i^*}, \tau_2)} \right) \\ &\leq \min \left(\frac{2 \cdot r_T + y}{x}, \frac{2 \cdot r_T + x}{y}, \frac{3 \cdot r_T + x + y}{r_T} \right). \end{aligned}$$

To optimize the above expression. We consider

$$\begin{aligned} \min_{r_T, x, y} \left(\frac{2r_T + y}{x}, \frac{2r_T + x}{y}, 3 + \frac{x + y}{r_T} \right) &\leq \min_{\substack{r_T, x, y \\ x=y}} \left(\frac{2r_T + y}{x}, \frac{2r_T + x}{y}, 3 + \frac{x + y}{r_T} \right) \\ &= \min_{r_T, x} \left(\frac{2r_T + x}{x}, 3 + \frac{2x}{r_T} \right) \\ &\leq \max_{q>0} \left(\min \left(2q + 1, 3 + \frac{2}{q} \right) \right) \\ &= 2 + \sqrt{5}. \end{aligned}$$

This implies that $\min \left(\frac{c_{i'}(Y)}{c_{i'}(T)}, \frac{c_{i''}(Y)}{c_{i''}(T)}, \frac{c_{i^*}(Y)}{c_{i^*}(T)} \right) \leq 2 + \sqrt{5}$, which contradicts that $(2 + \sqrt{5}) \cdot c_j(T) < c_j(Y)$ for every agent $j \in S$. We conclude that Greedy Capture for TrSP (Algorithm 1) satisfies $(2 + \sqrt{5})$ -JR. For the complementing lower bound, see Section B. \square

3.2 Fairness Lower Bounds

In this section, we contextualize the JR approximation obtained by GC-TrSP (Theorem 2) by establishing lower bounds on JR, both in terms of clustering algorithms and general existence. In contrast with the line metric, where JR is guaranteed to exist [7], we show that a solution satisfying 1.36-JR is not guaranteed to exist. We will begin by giving a reduction from clustering to TrSP which easily shows that JR is not guaranteed to exist.

Given any fair clustering instance $\mathcal{I}' = \langle \mathcal{N}', \mathcal{C}', k' \rangle$, where $|\mathcal{N}'| = n$ and $|\mathcal{C}'| = m$, we construct a corresponding TrSP instance as follows. We create two identical copies of \mathcal{I}' , denoted by $\mathcal{I}^a = \langle N^a, \mathcal{C}^a, k' \rangle$ and $\mathcal{I}^b = \langle N^b, \mathcal{C}^b, k' \rangle$, and place them at a sufficiently large distance from each other in a new metric space. Denote $N^a = \{a_1, a_2, \dots, a_n\}$ and $N^b = \{b_1, b_2, \dots, b_n\}$. We define the corresponding TrSP instance as $\mathcal{I}^T = \langle [n], \mathcal{C}^a \cup \mathcal{C}^b, (a_i, b_i)_{i \in [n]}, k = 2k' \rangle$.

Lemma 1. *Given a clustering instance $\mathcal{I}' = \langle N', \mathcal{C}', k' \rangle$, if there exists a solution Y satisfying β -JR in the corresponding TrSP instance \mathcal{I}^T for some $\beta \geq 1$, then there exists a solution $Y^C \subseteq Y, |Y^C| \leq k'$ such that Y^C satisfies 2β -PF in \mathcal{I}' . When $N' \cap \mathcal{C}' = \emptyset$, there exists $\varepsilon > 0$ such that Y^C satisfies $(2\beta - \varepsilon)$ -PF in \mathcal{I}' .*

Proof. Given a clustering instance $\mathcal{I}' = \langle \mathcal{N}', \mathcal{C}', k' \rangle$, consider the corresponding TrSP instance $\mathcal{I}^T = \langle [n], \mathcal{C}^a \cup \mathcal{C}^b, (a_i, b_i)_{i \in [n]}, k = 2k' \rangle$ that results from the reduction described above.

Let Y be a β -JR solution of \mathcal{I}^T and let $Y_a = Y \cap \mathcal{C}^a$ and $Y_b = Y \cap \mathcal{C}^b$. We first observe that, since points in \mathcal{C}^a and N^a are an infinite distance from points in \mathcal{C}^b and N^b , it holds for each agent i , that $c_i(Y) = d(a_i, Y_a) + d(b_i, Y_b)$. Moreover, as we know that $|Y| \leq 2k'$, it follows from the pigeonhole principle that either Y_a or Y_b has a size of at most k' . Without loss of generality, we assume $|Y_a| \leq k'$.

Consider an arbitrary set of datapoints $S' \subseteq \mathcal{N}'$ with size $|S'| \geq \frac{n}{k'}$ and an arbitrary candidate center $\tau \in \mathcal{C}'$. Let τ_a and τ_b denote the copies of τ in \mathcal{C}^a and \mathcal{C}^b , respectively. Let S denote the agents in the TrSP instance \mathcal{I}^T corresponding to the endpoints S' and note that $|S| = |S'| \geq \frac{n}{k'} = \frac{2n}{k}$. Since Y satisfies β -JR, there exists at least one agent $i \in S$ such that

$$\begin{aligned} \beta \cdot c_i(\{\tau_a, \tau_b\}) &\geq c_i(Y) \\ \Rightarrow \beta \cdot (d(a_i, \tau_a) + d(b_i, \tau_b)) &\geq d(a_i, Y_a) + d(b_i, Y_b). \end{aligned}$$

Notice that $d(a_i, \tau_a) = d(b_i, \tau_b)$ due to the construction of our reduction. This implies that

$$2 \cdot \beta d(a_i, \tau_a) \geq d(a_i, Y_a) + d(b_i, Y_b) \geq d(a_i, Y_a). \quad (1)$$

It follows that for any arbitrary group of datapoints $S' \subseteq \mathcal{N}'$ with size $|S'| \geq \frac{n}{k'}$ and candidate center $\tau \in \mathcal{C}'$, there exists a datapoint $j \in S'$ such that $2\beta \cdot d(j, \tau) \geq d(j, Y_a)$. Thus, Y_a is a 2β -PF solution to the original clustering instance \mathcal{I} .

Suppose that $\mathcal{N} \cap \mathcal{C} = \emptyset$. Let $\varepsilon = \frac{\min_{i' \in \mathcal{N}', \tau' \in \mathcal{C}'} d(i', \tau')}{\max_{i^* \in \mathcal{N}', \tau^* \in \mathcal{C}'} d(i^*, \tau^*)}$ be the minimum ratio between any pair of distances between agents and candidate centers in the clustering instance. Note that these ratios are well-defined and strictly positive for all agent-candidate pairs since all distances are strictly positive by our assumption. Then Equation (1) tells us that $(2\beta - \varepsilon) \cdot d(i, \tau) \geq 2\beta \cdot d(i, \tau) - d(i, Y_b) \geq d(i, Y_a)$, showing that Y_a satisfies $(2\beta - \varepsilon)$ -PF for $\varepsilon > 0$. \square

It follows easily from Lemma 1 that a solution exactly satisfying JR is not guaranteed to exist⁶. In fact, we are able to improve on this bound by constructing an instance for which a solution satisfying $(\frac{1+\sqrt{3}}{2} - \varepsilon)$ -JR is not guaranteed to exist for any $\varepsilon > 0$.

Proposition 3. *For any $\varepsilon > 0$, there exists a TrSP instance for which no solution satisfies $(\frac{1+\sqrt{3}}{2} - \varepsilon)$ -JR.*

Proof. Consider a TrSP instance with 3 agents, 6 transit candidate stops, and $k = 3$. Distances are specified in the following table. We first observe that the endpoints and stops respect the triangle inequality and are partitioned into two distinct regions, separated by an infinite distance. Within each region, the internal distance structure remains identical in the metric space. We note that the TrSP solutions $\{\tau_1, \tau_2, \tau_3\}$ and $\{\tau_4, \tau_5, \tau_6\}$ fail to provide any approximation of JR, as all three agents have strong incentives to deviate to any alternative solution that selects at least one transit stop from each region. Such a deviation reduces their cost from infinity to a finite constant.

⁶To see this, assume that there exists an algorithm that always outputs a JR solution. By Lemma 1, this would imply the existence of an algorithm that satisfies $(2 - \varepsilon)$ -PF in clustering for any instance in which $\mathcal{N} \cap \mathcal{C} = \emptyset$. However, Chen et al. [14] give a clustering instance in which $\mathcal{N} \cap \mathcal{C} = \emptyset$ and no $(2 - \varepsilon)$ -PF solution exists for any $\varepsilon > 0$, thereby yielding a contradiction.

$d(\cdot)$	a_1	a_2	a_3	b_1	b_2	b_3
τ_1	$2 + \sqrt{3}$	$\sqrt{3}$	1	∞	∞	∞
τ_2	$\sqrt{3}$	1	$2 + \sqrt{3}$	∞	∞	∞
τ_3	1	$2 + \sqrt{3}$	$\sqrt{3}$	∞	∞	∞
τ_4	∞	∞	∞	$2 + \sqrt{3}$	$\sqrt{3}$	1
τ_5	∞	∞	∞	$\sqrt{3}$	1	$2 + \sqrt{3}$
τ_6	∞	∞	∞	1	$2 + \sqrt{3}$	$\sqrt{3}$

Table 2: An Instance in which all solutions fail $(\frac{1+\sqrt{3}}{2} - \varepsilon)$ -JR

Due to the identical distance structure in the two regions, without loss of generality, it suffices to consider two TrSP solutions: $Y_1 = \{\tau_1, \tau_2, \tau_6\}$ and $Y_2 = \{\tau_1, \tau_2, \tau_4\}$. To see this, note that τ_4, τ_5 , and τ_6 mirror τ_1, τ_2 , and τ_3 , respectively, and that one of these groups will have exactly one selected stop. Thus, we only need to consider two types of solutions: one in which that selected stop is the counterpart of a selected stop on the other side (Y_2), and one in which it is not (Y_1).

Recall that $\lceil \frac{2n}{k} \rceil = \frac{2 \cdot 3}{3} = 2$. For the solution Y_1 , consider the deviating coalition $S_1 = \{2, 3\}$ and the alternative set of transit stops $T_1 = \{\tau_1, \tau_4\}$. For agent 2, we have $c_2(Y_1) = d(a_2, \tau_2) + d(b_2, \tau_6) = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$ and $c_2(T_1) = d(a_2, \tau_1) + d(b_2, \tau_4) = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$. Hence, we obtain $(\frac{1+\sqrt{3}}{2} - \varepsilon) \cdot c_2(T_1) < c_2(Y_1)$. Similarly, for agent 3, we compute $c_3(Y_1) = 1 + \sqrt{3}$ and $c_3(T_1) = 2$, yielding $(\frac{1+\sqrt{3}}{2} - \varepsilon) \cdot c_3(T_1) < c_3(Y_1)$. Thus, Y_1 violates $(\frac{1+\sqrt{3}}{2} - \varepsilon)$ -JR.

Now consider the solution Y_2 , with deviating coalition $S_2 = \{1, 2\}$ and alternative transit stops $T_2 = \{\tau_2, \tau_5\}$. For agent 1, we have $c_1(Y_2) = d(a_1, \tau_2) + d(b_1, \tau_4) = \sqrt{3} + 2 + \sqrt{3} = 2 + 2\sqrt{3}$ and $c_1(T_2) = d(a_1, \tau_2) + d(b_1, \tau_5) = 2\sqrt{3}$; For agent 2, we compute $c_2(Y_2) = 1 + \sqrt{3}$ and $c_2(T_2) = 2$. It follows that

$$\min \left\{ \frac{c_1(Y_2)}{c_1(T_2)}, \frac{c_2(Y_2)}{c_2(T_2)} \right\} = \min \left\{ \frac{2 + 2\sqrt{3}}{2\sqrt{3}}, \frac{1 + \sqrt{3}}{2} \right\} = \frac{1 + \sqrt{3}}{2}.$$

Therefore, Y_2 also fails to satisfy $(\frac{1+\sqrt{3}}{2} - \varepsilon)$ -JR. \square

The lower bound stated in Proposition 3 leaves open the possibility that other algorithms can significantly outperform GC-TrSP with respect to approximate JR. Recall that GC-TrSP proceeds by reducing TrSP instances to clustering instances, by reinterpreting all agents' endpoints in TrSP as datapoints in clustering, thus forfeiting information tying points to agents. Since our fairness properties ultimately consider *agent* costs, it seems likely that any clustering approach to our problem will leave significant room for improvement. The next result formalizes this intuition by showing that no clustering algorithm can achieve an approximation ratio better than 3 with respect to JR.

Proposition 4. *For any $\varepsilon > 0$, there is no clustering algorithm which satisfies $(3 - \varepsilon)$ -JR for the TrSP problem.*

Proof. Fix $\varepsilon > 0$. To prove the statement, we first define a clustering instance $\mathcal{I}^C = \langle \Theta, \mathcal{C}, k \rangle$ and then show that, no matter which solution Y the clustering algorithm returns, there exists a TrSP instance \mathcal{I} for which (1) Y violates $(3 - \varepsilon)$ -JR and (2) \mathcal{I}^C is the clustering instance induced by \mathcal{I} . We begin by defining the clustering instance \mathcal{I}^C with 12 datapoints, 9 candidate centers, and $k = 6$. Specifically, $\Theta = \{x_1, x_2, \dots, x_{12}\}$, $\mathcal{C} = \{x_1, x_2, x_3, x_5, x_6, x_7, x_9, x_{10}, x_{11}\}$, where all the datapoints and centers are partitioned into three groups, each of which are separated from each other by a sufficiently large distance. We represent the instance graphically by Figure 3.

Note that $k = 6$ and there are three candidate centers in each separated group. Since the internal structure in each separated group is exactly the same, we can limit our attention to two cases: (1) there is

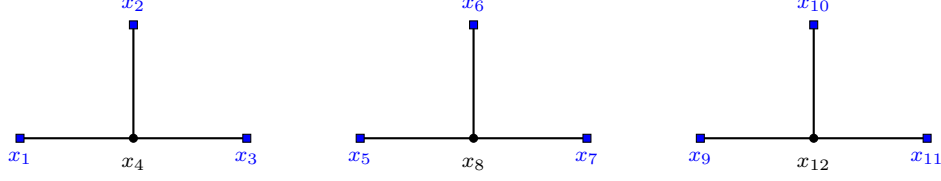


Figure 3: Graphical representation of clustering instance \mathcal{I}^C . Each edge in the graph has unit length 1 and distances between pairs of points are given by the shortest path between them (infinite distance if the pair is not connected). Datapoints which are candidate centers are labeled by blue rectangles.

a group with zero candidate centers selected; or (2) each group has at least one selected candidate center, and there are at least two groups with at most two centers selected from each.

Case (1). Without loss of generality, suppose that no center is selected from the first group, i.e., suppose $Y \subseteq \{x_5, x_6, x_7, x_9, x_{10}, x_{11}\}$. Let \mathcal{I} be a TrSP instance in which $\theta_1 = (x_1, x_8)$ and $\theta_2 = (x_4, x_5)$ and the remaining datapoints are arbitrarily assigned as the remaining four agents' endpoints. It is clear that \mathcal{I}^C is the clustering instance induced by \mathcal{I} . Let $S = \{1, 2\}$. We observe that the agent group S is large enough to deserve two transit stops, i.e., $|S| = 2 = \frac{2n}{k}$. Consider an alternative set of stops $T = \{x_1, x_5\}$. For each agent $i \in S$, it holds that $c_i(Y) = \infty$ and $c_i(T) = 1$. Therefore Y gives an arbitrarily bad approximation of JR under this instance.

Case (2). Without loss of generality, we assume that the first two groups have at most two centers selected from each and that these centers are $Y = \{x_1, x_2, x_5, x_6\}$. Note that selecting less centers from either group could only help us in finding a deviating coalition so we are analyzing the worst case. Also, while Y could also contain centers from the third group, this is irrelevant to the present case since we will not consider any endpoints in the third group when constructing our deviating coalition. Let \mathcal{I} be a TrSP instance in which $\theta_1 = (x_4, x_7)$ and $\theta_2 = (x_3, x_8)$ and the remaining datapoints are arbitrarily assigned as the remaining four agents' endpoints. Again, note that \mathcal{I}^C is the clustering instance induced by \mathcal{I} . Let $S = \{1, 2\}$ and $T = \{x_3, x_7\}$. For each agent $i \in S$, i prefers stops in T than Y as $c_i(Y) = 2 + 1 = 3$ and $c_i(T) = 1 + 0 = 1$, which gives us $\frac{c_i(Y)}{c_i(T)} = 3 > 3 - \varepsilon$. This concludes the proof. \square

We close this section with a remark showing that clustering algorithms, besides exhibiting a JR lower bound of 3, also fail to provide any guarantee with respect to JR under instances with arbitrary transit cost functions. In the next section, we will propose an algorithm which attains a JR approximation below the lower bound stated in Proposition 4, and show that this holds for arbitrary transit cost functions.

Remark 1. When allowing for arbitrary transit cost functions $d'(\cdot)$, no clustering algorithm can achieve a constant-factor approximation with respect to JR. To see this, recall the example in the proof of Proposition 4 and additionally define $d'(y_1, y_2) = H > 0$ and $d'(\tau_1, \tau_2) = 0$. For the deviation coalition $\{1, 2, 3, 4\}$, as $d'(y_1, y_2) = H$ tends to infinity, it follows immediately that for every agent $i \in S$, we have $\frac{c_i(Y)}{c_i(T)} \rightarrow \infty$, which implies arbitrarily bad approximation.

4 Expanding Cost Algorithm

In the previous section, we leveraged connections with centroid clustering to show that GC-TrSP approximates JR within a $2 + \sqrt{5}$ factor. Our lower bound on approximate JR existence of $\frac{1+\sqrt{3}}{2}$ then leaves an intriguing gap. From Proposition 4 and the ensuing remark, we know that clustering algorithms cannot hope to improve this factor beyond 3, and furthermore, are not robust to general transit cost functions. Given the gap and the shortfalls of the clustering approach, a natural question arises: can we design algorithms that achieve better approximations to JR and are robust to non-zero transit cost functions? We answer this question affirmatively by proposing the novel *Expanding Cost Algorithm* (ECA), which fully utilizes

the agent (as opposed to endpoint) information to guarantee a $1 + \sqrt{2} \approx 2.4$ approximation to JR under arbitrary transit cost functions.

ECA draws inspiration from the Greedy Capture approach of uniformly growing balls. However, instead of growing distance-based radii around individual candidate stops, it grows what we refer to as the *cost radius*, centered on *pairs* of stops. Specifically, ECA begins by enumerating all possible pairs of candidate transit stops. For each pair $T = \{\tau_1, \tau_2\}$, it uniformly expands a “cost ball”, that is a set that includes all agents whose total cost when using T is at most r .⁷ The algorithm iteratively “opens” these cost balls, and adds the associated stop locations into the solution. Agents are considered active if they are not yet covered by any previously opened ball. In each iteration, the algorithm selects and opens any ball that covers at least $\lceil \frac{2n}{k} \rceil$ active agents. We formally describe ECA in Algorithm 2.

Input: TrSP instance $\mathcal{I} = \langle N, \mathcal{C}, k, \{\theta_i\}_{i \in N} \rangle$.

Output: Solution Y .

- 1: Initialize $r \leftarrow 0$, $Y \leftarrow \emptyset$, $\mathcal{N} \leftarrow \Theta$.
- 2: **while** $\mathcal{N} \neq \emptyset$ **do**
- 3: Smoothly increase r .
- 4: **while** $\exists i \in N$ such that $c_i(Y) \leq r$ **do**
- 5: $\mathcal{N} \leftarrow \mathcal{N} \setminus \{i\}$
- 6: **while** $\exists \{\tau_1, \tau_2\} \subseteq \mathcal{C}$, $\{\tau_1, \tau_2\} \not\subseteq Y$ and $\exists S \subseteq \mathcal{N}$, $|S| \geq \lceil \frac{2n}{k} \rceil$, such that $\forall j \in S$, $c_j(Y \cup \{\tau_1, \tau_2\}) \leq r$ **do**
- 7: $Y \leftarrow Y \cup \{\tau_1, \tau_2\}$
- 8: $\mathcal{N} \leftarrow \mathcal{N} \setminus S$
- 9: Return Y .

Algorithm 2: Expanding Cost Algorithm

We next prove that ECA achieves a tight $(1 + \sqrt{2}) \approx 2.414$ -JR guarantee, and moreover, that this guarantee holds for any arbitrary transit cost function $d'(\cdot) \geq 0$.

Theorem 3. *For any arbitrary transit cost function $d'(\cdot) \geq 0$, ECA satisfies $(1 + \sqrt{2})$ -JR. However, for any $\varepsilon > 0$, there exists an instance with null transit costs for which ECA violates $(1 + \sqrt{2} - \varepsilon)$ -JR.*

Proof. Given any TrSP instance \mathcal{I} , let $Y \subseteq \mathcal{C}$ be the solution computed by ECA. Suppose, for a contradiction, that Y violates $(1 + \sqrt{2})$ -JR. It follows that there exists a group of agents $S \subseteq N$ with $|S| \geq \lceil \frac{2n}{k} \rceil$ and a pair of transit stops $T = \{\tau_1, \tau_2\} \subseteq \mathcal{C}$ such that $(1 + \sqrt{2}) \cdot c_j(T) < c_j(Y)$ for all $j \in S$. We first observe that for each agent $j \in S$, it must hold that $c_j(T) < d(a_j, b_j)$, since otherwise $c_j(Y) \leq c_j(T)$. In other words, every agent in S uses the transit stops in T for their route. Without loss of generality, for each agent $j \in S$, denote a_j as the endpoint that uses stop τ_1 and b_j as the endpoint that uses stop τ_2 .

Let $r_T = \max_{j \in S} c_j(T)$ be the maximum cost to any agent in S incurred by using transit stops T and let i^* be the agent in S that realizes this maximum. That is, $r_T = d(a_{i^*}, \tau_1) + d'(\tau_1, \tau_2) + d(b_{i^*}, \tau_2)$. If the cost radius considered by ECA never reaches r_T , then ECA returns stop placement Y covering all the agents in N with a cost radius smaller than r_T , which means that $c_{i^*}(Y) \leq c_{i^*}(T)$, yielding a contradiction.

The other case is that ECA does consider a cost radius of r_T at some point during its execution. In this situation, ECA must add some transit stop pair which gives some agent in S a cost upper bounded by r_T before it continues smoothly increasing the cost radius. To see this, consider that otherwise ECA will add the pair $T = (\tau_1, \tau_2)$ into the solution as $|S| \geq \lceil \frac{2n}{k} \rceil$ and $r_T = \max_{j \in S} c_j(T)$. Therefore, there exists an agent $i \in S$ with $c_i(Y) \leq r_T$. We next prove an upper bound on the cost of i^* under solution Y . In particular, we show $c_{i^*}(Y) \leq 2 \cdot r_T + c_{i^*}(T)$ by case analysis on agent i 's cost under Y .

Case (a). $c_i(Y) = d(a_i, b_i)$, i.e., agent i walks under Y . We prove the statement as follows:

$$c_{i^*}(Y) \leq d(a_{i^*}, b_{i^*})$$

⁷We note that what we refer to as cost balls are not in fact balls in the geometric sense. To see this, note that an agent can be located infinite distance from each stop in T but still have low cost for T if it is low cost for them to walk. We use the ball terminology nevertheless as it lends a natural interpretation to ECA, and especially the algorithm we introduce in Section 5.

$$\begin{aligned}
&\leq d(a_{i^*}, \tau_1) + d(\tau_1, a_i) + d(a_i, b_i) + d(b_i, \tau_2) + d(\tau_2, b_{i^*}) && (\because \text{triangle inequality}) \\
&\leq (d(a_{i^*}, \tau_1) + d'(\tau_1, \tau_2) + d(\tau_2, b_{i^*})) + (d(\tau_1, a_i) + d'(\tau_1, \tau_2) + d(b_i, \tau_2)) + d(a_i, b_i) \\
&\hspace{15em} (\because d'(\tau_1, \tau_2) \geq 0) \\
&= c_{i^*}(T) + c_i(T) + d(a_i, b_i) && (\because \forall i \in S, c_i(T) = d(a_i, \tau_1) + d'(\tau_1, \tau_2) + d(\tau_2, b_i)) \\
&= c_{i^*}(T) + c_i(T) + c_i(Y) && (\because c_i(Y) = d(a_i, b_i)) \\
&\leq 2 \cdot r_T + c_i(T). && (\because c_{i^*}(T) = r_T, c_i(Y) \leq r_T)
\end{aligned}$$

Case (b). $c_i(Y) < d(a_i, b_i)$, i.e., agent i uses the transit system under Y . Let (y_1, y_2) denote the pair of transit stops that agent i uses for minimizing her traveling cost, i.e., $(y_1, y_2) = \arg \min_{y, y' \in Y} d(a_i, y) + d'(y, y') + d(b_i, y')$. The upper bound follows from a similar argument to the previous case.

$$\begin{aligned}
c_{i^*}(Y) &\leq d(a_{i^*}, y_1) + d'(y_1, y_2) + d(b_{i^*}, y_2) \\
&\leq (d(a_{i^*}, \tau_1) + d(\tau_1, a_i) + d(a_i, y_1)) + (d(b_{i^*}, \tau_2) + d(\tau_2, b_i) + d(b_i, y_2)) + d'(y_1, y_2) \\
&\hspace{15em} (\because \text{triangle inequality}) \\
&\leq c_{i^*}(T) + c_i(T) + (d(a_i, y_1) + d(b_i, y_2)) + d'(y_1, y_2) && (\because \forall i \in S, c_i(T) \leq d(a_i, \tau_1) + d(\tau_2, b_i)) \\
&= c_{i^*}(T) + c_i(T) + c_i(Y) && (\because c_i(Y) = d(a_i, y_1) + d'(a_i, b_i) + d(b_i, y_2)) \\
&\leq 2 \cdot r_T + c_i(T). && (\because c_{i^*}(T) = r_T, c_i(Y) \leq r_T)
\end{aligned}$$

Lastly, with the upper bound of $c_{i^*}(Y)$ in hand, we consider the minimum multiplicative cost improvement of agents i and i^* under T :

$$\min \left(\frac{c_i(Y)}{c_i(T)}, \frac{c_{i^*}(Y)}{c_{i^*}(T)} \right) \leq \min \left(\frac{r_T}{c_i(T)}, \frac{2 \cdot r_T + c_i(T)}{r_T} \right) \leq \max_{z \geq 0} (\min(z, 2 + 1/z)) = 1 + \sqrt{2},$$

which contradicts that $(1 + \sqrt{2}) \cdot c_j(T) < c_j(Y)$ for all $j \in S$. \square

In light of the theoretical limitations of clustering algorithms (see Proposition 4 and Remark 1), Theorem 3 establishes two clear advantages of ECA over all clustering algorithms: a superior approximation to JR, and robustness of this approximation factor to arbitrary transit cost functions.

To do so, ECA explicitly considers pairs of stops at a time, and in this way, assigns agents to routes as it goes, rather than simply assigning endpoints to stops. While this approach outperforms clustering algorithms in the sense of satisfying coalitions who all desire the same pair of stops, those guarantees do not extend to coalitions who prefer to deviate to larger sets of stops. Indeed, as we will now show, the approach of ECA is too myopic to guarantee any bounded approximation to the core.

Proposition 5. *For any $\gamma, \rho \geq 1$, there exists a TrSP instance in which ECA fails (γ, ρ) -core.*

Proof. Fix arbitrary $\gamma, \rho \geq 1$ and fix an integer $r \geq 2$. We define $z = \lceil \frac{r}{r-1} \cdot \gamma + 1 \rceil$ and construct a TrSP instance $\mathcal{I} = \langle N, \mathcal{C}, \{\theta_i\}_{i \in N}, k \rangle$ where

$$|N| = \frac{(z^2 - z) \cdot r}{2}, |\mathcal{C}| = z^2, \text{ and } k = z^2 - z.$$

The instance is based on a complete graph K_z with z vertices, where each vertex $i \in [z]$ represents a candidate transit stop τ_i . For every pair of distinct vertices $i, j \in [z]$, the distance between τ_i and τ_j is assumed to be infinite. Along each of the edges (i, j) of K_z , there are r agents whose endpoints lie on that edge. Additionally, there are two extra candidate stops, denoted by $\tau_{i,j}$ and $\tau_{j,i}$ located on the edge. To illustrate, consider the edge between τ_1 and τ_2 , pictured in Figure 4. For the edge between τ_1 and τ_2 , there are two extra candidate stops τ_{12} and τ_{21} . All of these 4 candidate stops are marked as blue circles. On this edge, there are r agents with travel endpoints located along the line. Specifically, for agents $1, 2, \dots, r-1$,

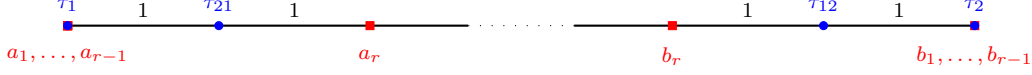


Figure 4: One edge in the complete graph K_z with endpoint vertices 1 and 2

the starting point is at τ_1 , and terminal point is at τ_2 . Agent r has a starting point at a_r and a terminal point at b_r . All endpoints are marked as red squares.

The total number of agents is $n = r \cdot \frac{z(z-1)}{2}$, since there are $\frac{z(z-1)}{2}$ edges in K_z , with r agents placed on each edge. The total number of candidate transit stops is $z + z(z-1) = z^2$, which includes the z vertices of the complete graph and two additional stops per edge. Given $k = z^2 - z$, we have $\lceil \frac{2n}{k} \rceil = r$. According to the execution procedures of ECA, when the cost radius reaches 2, each cost ball centered around a stop pair $\{\tau_{ij}, \tau_{ji}\}$ will capture all r agents on the corresponding edge (i, j) . Therefore, ECA outputs the solution $Y = \bigcup_{i,j \in [z]: i \neq j} \{\tau_{ij}, \tau_{ji}\}$, which contains $2 \cdot \frac{z(z-1)}{2} = z^2 - z = k$ stops and is thus a feasible solution. Now, consider the subset of agents S whose endpoints are on the vertices of K_z , that is, excluding agent r on each edge. We have $|S| = \frac{z(z-1)}{2} \cdot (r-1)$. Let $T = \{\tau_1, \tau_2, \dots, \tau_z\}$ be a deviation with $|T| = z$. For each agent $i \in S$, since both endpoints coincide with stops in T , it holds that $c_i(T) = 0$. On the other hand, under ECA solution Y , all agents in S incur cost $c_i(Y) = 2$. Recalling that $z = \lceil \frac{r}{r-1} \cdot \gamma + 1 \rceil$, we see that

$$\frac{|S|}{|T| \cdot \lceil \frac{n}{k} \rceil} = \frac{\frac{z(z-1) \cdot (r-1)}{2}}{z \cdot \frac{r}{2}} = \frac{r-1}{r} \cdot (z-1) \geq \gamma.$$

Therefore, in this instance, under the ECA solution Y , there exists a group of agents S and a solution T such that $|S| \geq \gamma \cdot |T| \cdot \frac{n}{k}$, and for every agent $i \in S$, $\rho \cdot c_i(T) = 0 < c_i(Y)$, which implies that ECA fails to satisfy (γ, ρ) -core for any $\gamma, \rho \geq 1$. \square

5 λ -Hybrid: Balancing Core and JR Approximations

As we saw in Section 2.2 and Section 4, ECA guarantees the best-known approximation to JR despite performing arbitrarily poorly with respect to core, while GC-TrSP obtains a worse approximation to JR but guarantees a constant-factor approximation to core. In this section, we present an algorithm, parameterized by $\lambda \in [0, 1]$, which effectively navigates the tradeoff between JR and core delineated by ECA and GC-TrSP.⁸

Intuitively, this algorithm integrates the decision-making principles of both GC-TrSP and ECA by concurrently simulating both algorithms and considering both individual transit stop candidates and pairs of stops. We call it the λ -Hybrid algorithm and give a formal description in Algorithm 3. In essence, the parameter λ allows tuning between ECA and GC-TrSP by controlling the rates of growth of the respective “radii” of each algorithm relative to each other. Specifically, λ encodes the ratio between the rate of growth of the distance radius (GC-TrSP radius) and the cost radius (ECA radius). If the rates of growth are close to equal (λ close to 1), the algorithm is closer to GC-TrSP since as the distance radius will likely dominate stop selection. On the other hand, as λ approaches 0, the distance radius grows much slower than the cost radius, and the algorithm more so mimics the behavior of ECA. We remark that the 0-hybrid algorithm is not equivalent to ECA. To see this, when $\lambda = 0$, note that when given instances where $\frac{2n}{k}$ endpoints are located at the same position, the 0-hybrid algorithm will deactivate these endpoints immediately, as they are already within distance radius zero. In contrast, this location is not guaranteed to be selected by ECA when no pair of stops with 0 cost exists.

In the remainder of this section, we will show that the λ -Hybrid algorithm offers a way of smoothly navigating the tradeoff between JR and core created by ECA and GC-TrSP. Specifically, we show that

⁸The tradeoff we remark on here is purely between these two algorithms. Theoretical evidence of such a tradeoff, for example showing the impossibility of algorithms which guarantee α -JR and (β, γ) -core for some α, β, γ , is an interesting direction for future research.

Input: $N, \{\theta_i\}_{i \in N}, \mathcal{M}, k, \lambda$.

Output: Y .

```

1: Initialize  $r \leftarrow 0, Y \leftarrow \emptyset$ .
2: Let  $\Theta$  be a multiset including all the endpoints of agents in  $N$ .
3: while  $\Theta \neq \emptyset$  do
4:   Smoothly increase  $r$ .
5:   while  $\exists \{a_i, b_i\} \subseteq \Theta$  such that  $c_i(Y) \leq r$  or  $\exists e_j \in \Theta$  such that  $d(e_j, Y) \leq \lambda \cdot r$  do
6:      $\Theta \leftarrow \Theta \setminus \{a_i, b_i\}$  or  $\Theta \leftarrow \Theta \setminus \{e_j\}$ 
7:     while  $\exists (\tau_1, \tau_2) \in \mathcal{M}^2 \setminus Y^2$  and  $\exists S \subseteq \mathcal{N}, |S| \geq \lceil \frac{2 \cdot n}{k} \rceil$ , such that  $\forall j \in S, (1) \{a_i, b_i\} \subseteq \Theta$ ; and (2)  $c_j(Y \cup \{\tau_1, \tau_2\}) \leq r$  do
8:        $Y \leftarrow Y \cup \{\tau_1, \tau_2\}$ 
9:        $\Theta \leftarrow \Theta \setminus \{a_i, b_i\}_{i \in S}$ 
10:    while  $\exists \tau_3 \in \mathcal{M} \setminus Y$  and  $\exists E \subseteq \Theta, |E| \geq \lceil \frac{2n}{k} \rceil$ , such that  $\forall e \in E, d(e, Y \cup \{\tau_3\}) \leq \lambda \cdot r$  do
11:       $Y \leftarrow Y \cup \{\tau_3\}$ 
12:       $\Theta \leftarrow \Theta \setminus E$ 
13: Return  $Y$ .
```

Algorithm 3: λ -Hybrid

λ -Hybrid satisfies

$$\frac{\lambda + 3 + \sqrt{\lambda^2 + 10\lambda + 9}}{2}\text{-JR and } (2, \frac{\sqrt{\lambda^2 + 6\lambda + 1} + \lambda + 1}{2\lambda})\text{-core}$$

where the JR approximation upper bound holds for $\lambda \in [0, 1]$ and the core approximation ratio holds for all $\lambda \in (0, 1]$. Figure 5 plots the fairness approximations obtained as a function of λ , showing that, as λ increases, the approximation ratio of JR worsens while that of core improves, as expected.

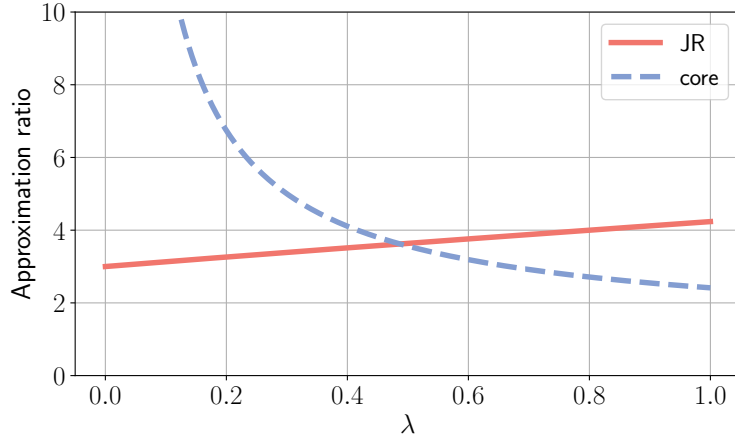


Figure 5: Parameter $\lambda \in [0, 1]$. Red solid line represents the JR approximation ratio of $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2}$ and blue dashed line represents the parameterized function $\frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda}$ of core approximation.

5.1 JR analysis of λ -Hybrid

Parameterized by $\lambda \in [0, 1]$, we begin by analyzing the extent to which the λ -Hybrid algorithm approximates JR. Building on the ideas underlying the JR analysis of the GC-TrSP algorithm and ECA, we establish that

the λ -Hybrid algorithm achieves a $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2}$ -approximation of JR. As λ approaches 0, the λ -Hybrid algorithm aligns more closely with ECA, thereby attaining a stronger approximation of JR. Conversely, as λ approaches 1, the algorithm shifts towards the behavior of GC-TrSP, which yields a weaker JR approximation but, in turn, provides a stronger guarantee with respect to core approximation (see the next subsection). For ease of exposition, for the remainder of the section, we will refer to the selection process in lines 7-9 as the “ECA loop” and the selection process in lines 10-12 as the “GC-TrSP loop”.

Theorem 4. λ -Hybrid satisfies $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2}$ -JR, where $\lambda \in [0, 1]$, and this bound is tight.

Proof. Let $Y \subseteq M$ be the transit stop solution returned by λ -Hybrid. Note that $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} \geq 3$ on the interval $[0, 1]$. Suppose for a contradiction that Y violates $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2}$ -JR. That is, there exists a group of agents $S \subseteq N$, $|S| \geq \lceil \frac{2n}{k} \rceil$, and pair of transit stops $T \subseteq \mathcal{M}$ such that for every agent $i \in S$, $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} \cdot c_i(T) < c_i(Y)$. Denote the two stops in T by τ_1 and τ_2 . For each agent $i \in S$, there is a matching between their endpoints and stops τ_1, τ_2 . Without loss of generality, for each agent $i \in S$, denote a_i as the endpoint that uses stop τ_1 and b_i as the endpoint that uses stop τ_2 . Let $r_T = \max_{i \in S} c_i(T)$ be the maximum cost of any agent in S when using transit stop pair $T = \{\tau_1, \tau_2\}$. Let i^* be an agent in S that realizes this maximum distance, i.e., $r_T = c_{i^*}(T)$.

We first consider the case when the parameter r never reaches r_T . That is, all endpoints are deactivated either by the GC-TrSP loop with a distance radius at most $\lambda \cdot r_T$ or by the ECA loop with a cost radius at most r_T . Consequently, we have $c_{i^*}(Y) \leq \max(2\lambda \cdot r_T, r_T) \leq 2 \cdot c_{i^*}(T)$, contradicting that $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} \cdot c_{i^*}(T) < c_{i^*}(Y)$.

We next consider the case in which the parameter r reaches r_T (i.e., the GC-TrSP loop reaches distance radius $\lambda \cdot r_T$ and the ECA loop reaches cost radius r_T). Notice that the algorithm reaches radius parameter r_T but does not select (τ_1, τ_2) . This implies that either (a) an agent in S , or (b) an endpoint of an agent in S , has already been deactivated during the execution of the algorithm.

Case (a). There exists an agent $j \in S$ such that j is deactivated by some pair of transit stops $\{y_1, y_2\} \subseteq Y$ in the ECA loop with a cost radius of at most r_T . Therefore, it holds that $c_j(Y) \leq r_T$. For agent i^* , we have

$$\begin{aligned} c_{i^*}(Y) &\leq d(a_{i^*}, y_1) + d(b_{i^*}, y_2) \\ &\leq d(a_{i^*}, \tau_1) + d(\tau_1, a_j) + d(a_j, y_1) + d(b_{i^*}, \tau_2) + d(\tau_2, b_j) + d(b_j, y_2) \\ &\leq 2 \cdot r_T + c_j(T). \end{aligned}$$

The minimum multiplicative cost improvement for agents i^* and j is:

$$\min \left(\frac{c_j(Y)}{c_j(T)}, \frac{c_{i^*}(Y)}{c_{i^*}(T)} \right) \leq \min \left(\frac{r_T}{c_j(T)}, \frac{2 \cdot r_T + c_j(T)}{r_T} \right) \leq \max_{z \geq 0} \left(\min \left(z, 2 + \frac{1}{z} \right) \right) \leq 1 + \sqrt{2},$$

again contradicting that $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} \cdot c_i(T) < c_i(Y)$ for all $i \in S$.

Case (b). There exists an endpoint of an agent $j \in S$ which is deactivated by some singleton transit stop candidate y_1 in the GC-TrSP loop with a distance radius at most $\lambda \cdot r_T$. The tougher subcase is that in which $\tau_2 \notin Y$. Notice that τ_2 can deactivate $|S| \geq \frac{2n}{k}$ terminal endpoints for agents in S with radius at most r_T , but is not selected by the algorithm. Hence, there must exist some selected transit stop $y_2 \in Y$ which deactivates at least one terminal endpoint of some agent in S with radius no greater than r_T . Denote the corresponding agent and endpoint by j' and $b_{j'}$. We first consider the cost of agent j under Y ,

$$\begin{aligned} c_j(Y) &\leq d(a_j, y_1) + d(b_j, y_2) \\ &\leq \lambda \cdot r_T + d(b_j, \tau_2) + d(\tau_2, b_{j'}) + d(b_{j'}, y_2) \\ &\leq (\lambda + 1) \cdot r_T + d(b_j, \tau_2) + d(\tau_2, b_{j'}). \end{aligned}$$

Similarly, we consider the cost of agent j' under Y ,

$$c_{j'}(Y) \leq d(a_{j'}, y_1) + d(b_{j'}, y_2)$$

$$\begin{aligned}
&\leq d(a_{j'}, \tau_1) + d(\tau_1, a_j) + d(a_j, y_1) + d(b_{j'}, y_2) \\
&\leq (\lambda + 1) \cdot r_T + d(\tau_1, a_j) + d(a_{j'}, \tau_1).
\end{aligned}$$

We next focus on agent i^* and show the following upper bound:

$$\begin{aligned}
c_{i^*}(Y) &\leq d(a_{i^*}, y_1) + d(b_{i^*}, y_2) \\
&\leq d(a_{i^*}, \tau_1) + d(\tau_1, a_j) + d(a_j, y_1) + d(b_{i^*}, \tau_2) + d(\tau_2, b_{j'}) + d(b_{j'}, y_2) \\
&\leq r_T + d(\tau_1, a_j) + \lambda \cdot r_T + d(\tau_2, b_{j'}) + r_T \\
&\leq (\lambda + 2) \cdot r_T + d(\tau_1, a_j) + d(\tau_2, b_{j'}).
\end{aligned}$$

To clarify the expression, let $d(\tau_1, a_j)$ be x and $d(b_{j'}, \tau_2)$ be y . With these upper bounds in hand, we derive the minimum multiplicative cost improvement of agents $\{i^*, j, j'\}$ under T .

$$\begin{aligned}
&\min \left(\frac{c_j(Y)}{c_j(T)}, \frac{c_{j'}(Y)}{c_{j'}(T)}, \frac{c_{i^*}(Y)}{c_{i^*}(T)} \right) \\
&\leq \min \left(\frac{(\lambda + 1) \cdot r_T + d(b_j, \tau_2) + y}{x + d(b_j, \tau_2)}, \frac{(\lambda + 1) \cdot r_T + x + d(a_{j'}, \tau_1)}{d(a_{j'}, \tau_1) + y}, \frac{(\lambda + 2) \cdot r_T + x + y}{r_T} \right) \\
&\leq \min \left(\frac{(\lambda + 1) \cdot r_T + y}{x}, \frac{(\lambda + 1) \cdot r_T + x}{y}, \frac{(\lambda + 2) \cdot r_T + x + y}{r_T} \right).
\end{aligned}$$

To optimize this expression, we consider

$$\begin{aligned}
&\min_{r_T, y, z} \left(\frac{(\lambda + 1) \cdot r_T + y}{x}, \frac{(\lambda + 1) \cdot r_T + x}{y}, (\lambda + 2) + \frac{x + y}{r_T} \right) \\
&\leq \min_{\substack{r_T, x, y \\ x=y}} \left(1 + \frac{(\lambda + 1) \cdot r_T}{x}, (\lambda + 2) + \frac{2x}{r_T} \right) \\
&\leq \max \left(\min_{q \geq 0} \left(1 + (\lambda + 1)q, (\lambda + 2) + \frac{2}{q} \right) \right) \\
&= \frac{\lambda + 3 + \sqrt{\lambda^2 + 10\lambda + 9}}{2},
\end{aligned}$$

where it holds with equality when $q = \frac{\lambda + 1 + \sqrt{\lambda^2 + 10\lambda + 9}}{2(\lambda + 1)}$. Therefore, we conclude that for some agent in $i' \in \{j, j', i^*\} \subseteq S$, we have $c_{i'}(Y) \leq \frac{\lambda + 3 + \sqrt{\lambda^2 + 10\lambda + 9}}{2} \cdot c_{i'}(T)$, contradicting to the assumption that for every agent $i \in S$, $\frac{\lambda + 3 + \sqrt{\lambda^2 + 10\lambda + 9}}{2} \cdot c_i(T) < c_i(Y)$.

Lastly, we just need to handle the easier subcase in which τ_2 already belongs to the stop placement, i.e. $\tau_2 \in Y$. Here we need only consider the multiplicative improvement of agents j and i^* :

$$\begin{aligned}
\min \left(\frac{c_j(Y)}{c_j(T)}, \frac{c_{i^*}(Y)}{c_{i^*}(T)} \right) &\leq \min \left(\frac{\lambda \cdot r_T + d(b_j, Y)}{d(a_j, \tau_1) + d(b_j, \tau_2)}, \frac{d(a_{i^*}, \tau_1) + d(a_j, \tau_1) + \lambda \cdot r_T + d(b_{i^*}, Y)}{c_{i^*}(T)} \right) \\
&\leq \min \left(\frac{\lambda \cdot r_T + d(b_j, \tau_2)}{d(a_j, \tau_1) + d(b_j, \tau_2)}, \frac{c_{i^*}(T) + d(a_j, \tau_1) + \lambda \cdot r_T}{r_T} \right) \\
&\leq \min \left(\frac{\lambda \cdot r_T}{d(a_j, \tau_1)}, \frac{(\lambda + 1)r_T + d(a_j, \tau_1)}{r_T} \right) \\
&\leq \max \left(\min_{q \geq 0} \left(\lambda \cdot q, 1 + \lambda + \frac{1}{q} \right) \right) \\
&= \frac{\lambda + 1 + \sqrt{\lambda^2 + 6\lambda + 1}}{2}.
\end{aligned}$$

Since this value never exceeds $1 + \sqrt{2}$ on the unit interval, this also provides a contradiction and concludes our proof. \square

5.2 Analysis of Core Fairness of the λ -Hybrid Algorithm

We next analyze the core approximation ratio of the λ -Hybrid algorithm. Our analysis leverages Theorem 1, which provides the crucial link between proportional fairness in clustering and core in TrSP. We will demonstrate that the λ -Hybrid algorithm satisfies $(\frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda})$ -PF in the induced clustering instance, which immediately implies that it satisfies $(2, \frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda})$ -core.

Lemma 2. *For any $\lambda \in (0, 1]$ and any TrSP instance \mathcal{I} , the λ -Hybrid solution under \mathcal{I} guarantees $\frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda}$ -PF in the induced clustering instance \mathcal{I}^C .*

Proof. Given any TrSP instance $\mathcal{I} = \langle N, \mathcal{C}, k, \{\theta_i\}_{i \in N} \rangle$, let Y be the TrSP solution returned by λ -Hybrid under \mathcal{I} . We show that Y satisfies $\frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda}$ -PF for the induced clustering instance $\mathcal{I}^C = \langle \Theta, \mathcal{C}, k \rangle$. Suppose, for the sake of contradiction, that there exists a set of datapoints $\theta' \subseteq \Theta$ and candidate center $c \in \mathcal{C}$ such that $|\theta'| \geq 2 \cdot \lceil \frac{n}{k} \rceil$ and $\frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda} \cdot d(e, c) < d(e, Y)$ for each $e \in \theta'$.

Let $r_T = \max_{e \in \theta'} d(e, c)$, let e^* denote the point in θ' that attains this maximum, and let i^* denote the agent which e^* is an endpoint of in the original TrSP instance, i.e., $e^* \in \theta_{i^*}$. We first consider the case in which the parameter r never reaches r_T/λ during the execution of λ -Hybrid. If e^* was deactivated during the GC-TrSP loop, then $d(e^*, Y) \leq \lambda(r_T/\lambda) = r_T = d(e^*, c)$, providing a contradiction. The other sub-case is that e^* was instead deactivated during the ECA loop, in which case it follows that $d(e^*, Y) \leq c_{i^*}(Y) \leq r_T/\lambda = (1/\lambda) \cdot d(e^*, c)$. It can be verified that $\frac{1}{\lambda} \leq \frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda}$ for all $\lambda \in (0, 1]$, meaning this also contradicts our assumption.

We now restrict our attention to the case in which the parameter r reaches r_T/λ during the execution of λ -Hybrid. Note that when $r = r_T/\lambda$, if all endpoints in θ' remain active, the ball of radius $\lambda \cdot (r_T/\lambda) = r_T$ centered at c captures at least $2 \cdot \lceil n/k \rceil$ endpoints. Since c is not selected, there must be at least one datapoint which was deactivated when the parameter r was at most r_T/λ . We term this endpoint e' , denote the agent it belongs to by i' , and denote the candidate center selected in the round e' is deactivated by y .

We claim that $d(e', y) \leq r_T/\lambda$. To see this, note that if y was selected in a GC-TrSP round, it holds that $d(e', y) \leq \lambda \cdot r_T/\lambda \leq r_T/\lambda$. Otherwise, y was selected in an ECA round, and it follows that $d(e', y) \leq c_{i'}(Y) \leq r_T/\lambda$. Using this, we now obtain a contradiction by considering the minimum multiplicative improvement attained by endpoints e^* and e' from c :

$$\begin{aligned} \min \left(\frac{d(e^*, Y)}{d(e^*, c)}, \frac{d(e', Y)}{d(e', c)} \right) &\leq \min \left(\frac{d(e^*, c) + d(c, e') + d(e', y)}{r_T}, \frac{r_T/\lambda}{d(e', c)} \right) \\ &\leq \min_{r_T, d(e', c)} \left(\frac{(1 + 1/\lambda)r_T + d(e', c)}{r_T}, \frac{r_T/\lambda}{d(e', c)} \right) \\ &\leq \max_{q \geq 0} [\min(q + 1 + 1/\lambda, 1/(\lambda \cdot q))] \\ &= \frac{\sqrt{\lambda^2 + 6\lambda + 1} + \lambda + 1}{2\lambda}. \end{aligned}$$

where the final equality holds because the maximum in the penultimate expression is obtained when $q = (\sqrt{\lambda^2 + 6\lambda + 1} - \lambda - 1)/(2\lambda)$. \square

Combining Theorem 1 and Lemma 2 yields the following corollary, which gives an upper bound on the core approximation guaranteed by the λ -Hybrid rules.

Corollary 1. *For every $\lambda \in (0, 1]$, the λ -hybrid algorithm satisfies $(2, \frac{\sqrt{\lambda^2+6\lambda+1}+\lambda+1}{2\lambda})$ -core.*

We give an almost tight lower bound which, when taken together with Remark 2, gives a bound between that of ECA and that of GC-TrSP, as one would expect.

Proposition 6. *Given any $\lambda \in (0, 1]$ and $\delta, \varepsilon > 0$, there is an instance for which λ -hybrid does not satisfy $(2 - \delta, \frac{\sqrt{4\lambda^2+12\lambda+1}+2\lambda+1}{4\lambda} - \varepsilon)$ -core.*

The lower bound implied by Proposition 6 for $\lambda \geq \sqrt{2}/2$ is weaker than the lower bound proved for GC-TrSP in Proposition 2. Given this, we complement Proposition 6 by strengthening the core lower bound of λ -hybrid for the case of $\lambda \geq 1/2$ in the following remark.

Remark 2. *Given any $\lambda \geq 1/2$, there is an instance for which λ -hybrid does not satisfy $(2 - \delta, 1 + \sqrt{2} - \varepsilon)$ -core. This follows from the exact same example and argument used to prove Proposition 2. In particular, due to the symmetry of the instance given by Table 4, as long as $\lambda \geq 1/2$, the ECA loop will not select any candidates in the execution of λ -hybrid. This means the algorithm will execute identically to GC-TrSP on this instance.*

Given Remark 2, it is likely that the lower bound in Proposition 6 is not tight. If it is the case that our upper bound is indeed tight, this suggests that the core approximation of λ -Hybrid is thanks to the algorithm’s approximation of PF (rather than any consideration involving cost). Again, we observe that while taking a clustering approach naively ignores agent-specific cost information, it serves as a very useful tool in the pursuit of core approximation.

6 Experiments

We complement our theoretical contributions by evaluating the empirical performance of the GC-TrSP algorithm, the Expanding Cost algorithm, and the λ -Hybrid algorithm on a real world dataset.

Experimental Setup We use resident travel route data from the City of Helena Capital Transit service, comprising 10,282 distinct travel routes between 3,075 unique spatial points. The initial dataset specifies only pick-up and drop-off locations. Using OpenStreetMap data [32] and the open-source *Valhalla* routing engine [39], we compute route-level travel times for both walking and public transit, which serve as the corresponding cost metrics. For the JR and core experiments, we randomly sample 400 and 40 agents⁹, respectively, together with their associated routes from the full dataset. We define the candidate stop set as the union of all observed locations among the sampled agents. For the target number of stops, we evaluate JR over k in $[20, 60, 80, 100]$ and core over k in $[5, 10, 15, 20]$. Finally, because the Helena dataset exhibits substantial disparities between walking and transit costs, we additionally rescale transit costs over a broad range to assess algorithmic performance under varying degrees of separation between transit and walking scales. For each parameter combination, we sample 50 rounds for the experiments.

JR Evaluation We evaluate the approximation performance with respect to JR by simulating GC-TrSP, ECA and $\frac{1}{2}$ -Hybrid across a range of stop selection sizes and transit cost scales.

In Figure 6, we first observe that for all of the three evaluated algorithms, the approximation ratio of JR under random sampling is close to 1, suggesting that our theoretical lower bounds may not be borne out in practice. Moreover, despite our result that ECA admits a stronger worst case bound for JR, GC-TrSP delivers approximation ratios that remain close to one across all stop selection sizes. ECA exhibits slightly larger ratios and greater variability when the transit costs remain low, with the most pronounced separation around $k = 20$, followed by improvement as k increases. The $\frac{1}{2}$ -Hybrid algorithm closely tracks GC-TrSP: both maintain averages very near 1 when k is small and exhibit only a mild upward trend as k increases. The trend shifts when the transit cost is scaled by a larger factor, representing scenarios where walking (or other modes) may be a feasible alternative for a significant number of routes. In this regime, the behavior of ECA changes substantially. Its average approximation ratio decreases as k grows, and its outcomes become more concentrated, indicating improved robustness with respect to JR approximation. In contrast, GC-TrSP and $\frac{1}{2}$ -Hybrid display a gradual increase in their averages as k increases, whereas ECA is stable and becomes competitive, achieving better approximation ratios for larger sizes.

⁹The core test uses a small sample size because verifying core membership is computationally intensive and requires solving large scale integer programs.

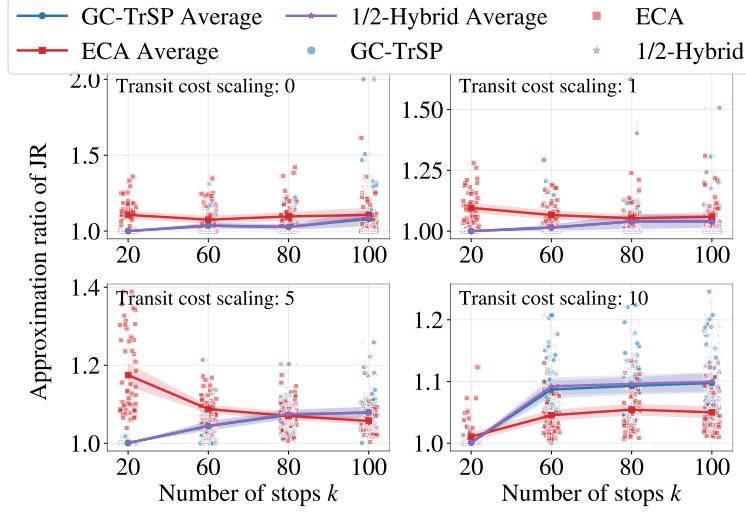


Figure 6: JR approximation evaluation for comparing GC-TrSP, ECA and $\frac{1}{2}$ -Hybrid with 400 agents. Stop selection size ranges from 20 to 100, and the transit cost scale ranges from 0 to 10. Distribution of instance approximation ratios and the mean approximation ratio with 95% confidence intervals

Core Evaluation We fix the size relaxation parameter (α) to 2 and evaluate the cost approximation ratios (β) of both algorithms and exhibit the performance of GC-TrSP, ECA and $\frac{1}{2}$ -Hybrid under the zero transit cost (transit cost scaling = 0) and regular transit cost settings (transit cost scaling = 1) in Table 3. Experimental results under various transit cost scales, GC-TrSP attains better core approximation ratios than ECA and $\frac{1}{2}$ -Hybrid. While ECA displays higher means and greater variability in several regimes, its approximation ratios remain close to 1 for most instances, which indicates that ECA is still practically effective in our experiments, despite admitting an arbitrarily poor worst-case guarantee. The $\frac{1}{2}$ -Hybrid algorithm interpolates between GC-TrSP and ECA, tracking GC-TrSP closely while showing a modest increase for larger k , but with much less variance than ECA.

Zero Transit Cost (Scaling = 0)												
Results	$k = 5$			$k = 10$			$k = 15$			$k = 20$		
	GC	ECA	$\frac{1}{2}$ -Hybrid	GC	ECA	$\frac{1}{2}$ -Hybrid	GC	ECA	$\frac{1}{2}$ -Hybrid	GC	ECA	$\frac{1}{2}$ -Hybrid
Average	1.016	1.114	1.105	1.095	1.430	1.148	1.155	1.365	1.217	1.074	1.269	1.175
Min	1	1	1	1	1.074	1	1	1.081	1.011	1	1.005	1.025
Max	1.263	1.301	1.266	1.556	2.095	1.575	1.533	1.743	1.433	1.241	1.675	1.337
Regular Transit Cost (Scaling = 1)												
Results	$k = 5$			$k = 10$			$k = 15$			$k = 20$		
	GC	ECA	$\frac{1}{2}$ -Hybrid	GC	ECA	$\frac{1}{2}$ -Hybrid	GC	ECA	$\frac{1}{2}$ -Hybrid	GC	ECA	$\frac{1}{2}$ -Hybrid
Average	1.007	1.072	1.063	1.053	1.454	1.120	1.067	1.386	1.121	1.058	1.190	1.117
Min	1	1	1	1	1.138	1	1	1.041	1	1	1	1
Max	1.082	1.256	1.256	1.240	1.889	1.369	1.241	1.787	1.253	1.257	1.499	1.338

Table 3: Core approximation evaluation with 40 agents under zero and regular transit costs. The stop selection size ranges from 5 to 20. We consider both zero and regular transit costs. GC denotes GC-TrSP. For each setting, we report the average, minimum, and maximum values over 50 sampled rounds.

7 Conclusion

In this paper, we introduce a model of transit stop placement in general metric spaces. Pursuing fair stop placements, we first explored the extent to which fair clustering algorithms guarantee fair solutions for instances of our problem. We then introduce the Expanding Cost Algorithm (ECA), which performs better than all clustering-based algorithms with respect to JR, and is robust to the incorporation of transit times between stops, but provides no guarantee with respect to core. We then combined our algorithm with the Greedy Capture algorithm from clustering to introduce a hybrid class of algorithms which navigate a tradeoff between JR and core.

Our central algorithmic result, showing that ECA satisfies $(1 + \sqrt{2})$ -JR, extends to arbitrary transit cost metrics. In contrast, our core approximation upper bounds relied on a connection to proportional fairness in clustering, and thus necessarily hold only under null transit costs. It remains to be seen whether it is possible to give constant-factor approximations to core that also apply under broader classes of transit cost metrics. This is a promising direction.

In one sense, the transit stop problem can be viewed as a generalization of the centroid selection in which each agent has two points rather than one. One possible extension of the transit stop problem is to allow each agent to have more than two location points or multiple pairs of points that need to be traversed. The first challenge of such an extension would be defining a sensible cost function for agents. It would then be interesting to see how well the results of this paper extend to such a setting.

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References

- [1] A. Alkan and D. Gale. The core of the matching game. *Games and Economic Behavior*, 2(3):203–212, 1990.
- [2] H. Aziz and R. Savani. Hedonic games. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 15. Cambridge University Press, 2016.
- [3] H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified representation in approval-based committee voting. *Social Choice and Welfare*, pages 461–485, 2017.
- [4] H. Aziz, A. Bogomolnaia, and H. Moulin. Fair mixing: the case of dichotomous preferences. *ACM Transactions on Economics and Computation (TEAC)*, 8(4):18:1–18:27, 2020.
- [5] H. Aziz, E. Micha, and N. Shah. Group fairness in peer review. In *Proceedings of the 36th Annual Conference on Neural Information Processing (NeurIPS)*, pages 64885–64895, 2023.
- [6] H. Aziz, B. E. Lee, S. M. Chu, and J. Vollen. Proportionally representative clustering. In *Proceedings of the 20th International Conference on Web and Internet Economics (WINE)*, pages 155–171, 2024.
- [7] M. Bullinger, E. Elkind, and M. Latifan. Towards fair and efficient public transportation: A bus stop model. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 427–435, 2025.
- [8] I. Caragiannis, E. Micha, and N. Shah. Proportional fairness in non-centroid clustering. In *Proceedings of the 37th Annual Conference on Neural Information Processing (NeurIPS)*, volume 37, pages 19139–19166, 2024.

- [9] A. Ceder. *Public transit planning and operation: Modeling, practice and behavior*. CRC press, 2016.
- [10] A. Ceder and N. H. Wilson. Bus network design. *Transportation Research Part B: Methodological*, 20(4):331–344, 1986.
- [11] A. A. Ceder, M. Butcher, and L. Wang. Optimization of bus stop placement for routes on uneven topography. *Transportation Research Part B: Methodological*, 74:40–61, 2015.
- [12] B. R. Chaudhury, L. Li, M. Kang, B. Li, and R. Mehta. Fairness in federated learning via core-stability. In *Proceedings of the 35th Annual Conference on Neural Information Processing (NeurIPS)*, pages 5738–5750, 2022.
- [13] B. R. Chaudhury, A. Murhekar, Z. Yuan, B. Li, R. Mehta, and A. D. Procaccia. Fair federated learning via the proportional veto core. In *Proceedings of the 41st International Conference on Machine Learning (ICML)*, pages 42245–42257, 2024.
- [14] X. Chen, B. Fain, L. Lyu, and K. Munagala. Proportionally fair clustering. In *Proceedings of the 36th International Conference on Machine Learning (ICML)*, pages 1032–1041, 2019.
- [15] A. Chhabra, K. Masalkovaitė, and P. Mohapatra. An overview of fairness in clustering. *IEEE Access*, 9:130698–130720, 2021.
- [16] S. I. Chien* and Z. Qin. Optimization of bus stop locations for improving transit accessibility. *Transportation planning and Technology*, 27(3):211–227, 2004.
- [17] B. Cookson, N. Shah, and Z. Yu. Unifying proportional fairness in centroid and non-centroid clustering. In *Proceedings of the 38th Annual Conference on Neural Information Processing (NeurIPS)*, page forthcoming, 2025.
- [18] G. Desaulniers and M. D. Hickman. Chapter 2 public transit. In C. Barnhart and G. Laporte, editors, *Transportation*, volume 14 of *Handbooks in Operations Research and Management Science*, pages 69–127. Elsevier, 2007.
- [19] B. Fain, A. Goel, and K. Munagala. The core of the participatory budgeting problem. In *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*, pages 384–399, 2016.
- [20] B. Fain, K. Munagala, and N. Shah. Fair allocation of indivisible public goods. In *19th*, 2018.
- [21] Z. He, S. Botan, J. Lang, A. Saffidine, F. Sikora, and S. Workman. Fair railway network design. Technical report, arXiv abs/2409.02152, 2024.
- [22] T. Hossein Rashidi, D. Rey, S. Jian, and T. Waller. A clustering algorithm for bi-criteria stop location design with elastic demand. *Computer-Aided Civil and Infrastructure Engineering*, 31(2):117–131, 2016.
- [23] L. Kellerhals and J. Peters. Proportional fairness in clustering: A social choice perspective. In *Proceedings of the 37th Annual Conference on Neural Information Processing (NeurIPS)*, pages 111299–111317, 2024.
- [24] M. Lackner and P. Skowron. *Multi-Winner Voting with Approval Preferences*. Springer Briefs in Intelligent Systems. Springer, 2023.
- [25] K. Martens. *Transport Justice: Designing fair transportation systems*. Routledge, 2016.
- [26] K. Martens and K. Lucas. Chapter 26: Perspectives on transport and social justice. In *Handbook on Global Social Justice*. Edward Elgar Publishing, 2018.
- [27] P. Matl, R. F. Hartl, and T. Vidal. Workload equity in vehicle routing problems: A survey and analysis. *Transportation Science*, 52(2):239–260, 2018.

- [28] N. Megiddo and K. J. Supowit. On the complexity of some common geometric location problems. *SIAM Journal on Computing*, 13(1):182–196, 1984.
- [29] E. Micha and N. Shah. Proportionally fair clustering revisited. In *Proceedings of the 47th International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 85:1–85:16, 2020.
- [30] P. Miller, A. G. de Barros, L. Kattan, and S. Wirasinghe. Public transportation and sustainability: A review. *KSCE Journal of Civil Engineering*, 20(3):1076–1083, 2016.
- [31] A. Najmi, T. Waller, M. Memarpour, D. Nair, and T. H. Rashidi. A human behaviour model and its implications in the transport context. *Transportation Research Interdisciplinary Perspectives*, 18: 100800, 2023.
- [32] OpenStreetMap contributors. Openstreetmap. [Data set], 2026. Available under ODbL. Retrieved January 24, 2026, from <https://www.openstreetmap.org>.
- [33] B. Peleg and P. Sudhölter. *Introduction to the Theory of Cooperative Games*. 2nd edition, 2007.
- [34] D. Peters. The core of approval-based committee elections with few seats. In *Proceedings of the 34th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 4014–4022, 2025.
- [35] G. Pierczyński and P. Skowron. Core-stable committees under restricted domains. In *Proceedings of the 18th International Conference on Web and Internet Economics (WINE)*, pages 311–329, 2022.
- [36] L. S. Shapley and H. Scarf. On cores and indivisibility. *Journal of Mathematical Economics*, 1(1):23–37, 1974.
- [37] D. Tedjopurnomo, Z. Bao, F. Choudhury, H. Luo, and A. K. Qin. Equitable public bus network optimization for social good: A case study of singapore. In *Proceedings of the 2022 ACM Conference on Fairness, Accountability, and Transparency*, pages 278–288, 2022.
- [38] D. Teodorovic and M. Janić. *Transportation engineering: Theory, practice and modeling*. Butterworth-Heinemann, 2016.
- [39] Valhalla contributors. Valhalla: Open source routing engine for openstreetmap. GitHub repository, 2026. URL <https://github.com/valhalla/valhalla>. Version 3.6.2, released January 15, 2026. Accessed January 24, 2026.

A Minimizing Total Travel Cost

Proposition 1. *Unless $P = NP$, there is no polynomial time algorithm which computes a minimum cost solution to the TrSP problem, even under null transit times, i.e., even when $d'(i, j) = 0$ for all $i, j \in \mathcal{X}$.*

Proof. We prove the statement by a reduction from the canonical k -median problem in general metric space, which is known to be NP-hard [28].

k -median decision problem. Given a metric space (\mathcal{X}, d) , a set N of datapoints $\{x_i\}_{i \in N}$, a set \mathcal{C}' of candidate centers, an integer k' , and a bound B , the decision problem asks whether there exists a set $Y \subseteq \mathcal{C}'$ with $|Y| \leq k'$ such that

$$\sum_{i \in N} d(x_i, Y) \leq B.$$

TrSP decision problem (with null transit times). Let $\mathcal{I} = \langle N, \mathcal{C}, k, \theta_i i \in N \rangle$ be a TrSP instance with null transit times in metric space (\mathcal{X}, d) . The τ -TrSP decision problem asks whether there exists a set $Y \subseteq \mathcal{C}$ with $|Y| \leq k$ such that

$$\sum_{i \in N} c_i(Y) \leq \tau.$$

Given an arbitrary k -median instance in metric space (\mathcal{X}, d) , we construct a corresponding TrSP instance as follows. We augment the metric space by adding additional points, x' and c' , which are both located at a distance B from every other point in the metric space and distance 0 from each other, i.e., $\forall x \in \mathcal{X} \setminus \{x', c'\}, d(x', x) = d(c', x) = B$ and $d(x', c') = 0$. We then let $\theta_i = (x_i, x')$ be the endpoints of each agent $i \in N$, and let $\mathcal{C} = \mathcal{C}' \cup \{c'\}$. Lastly, we set the number of desired transit stops to $k = k' + 1$ and let $\tau = B$.

Suppose the k -median instance is a YES instance, i.e., there exists a solution $Y \subseteq \mathcal{C}$ with $|Y| \leq k' = k - 1$ such that $\sum_{i \in N} d(x_i, Y) \leq B$. We show that $Y^* = Y \cup \{c'\}$ is a YES solution for the constructed TrSP instance:

$$\begin{aligned} \sum_{i \in N} c_i(Y^*) &= \sum_{i \in N} \min\{d(x_i, x'), \min_{y_1, y_2 \in Y^*} d(x_i, y_1) + d(x', y_2)\} \\ &= \sum_{i \in N} \min\{B, \min_{y_1 \in Y} d(x_i, y_1)\} & (\because c' \in Y^*, d(x', c') = 0) \\ &\leq \sum_{i \in N} d(x_i, Y) \\ &\leq B. \end{aligned}$$

Conversely, suppose the TrSP instance is a YES instance, i.e., there exists Y with $|Y| \leq k$ such that $\sum_{i \in N} c_i(Y) \leq B$. First observe that $c^* \in Y$, since otherwise every agent's cost will be at least B . Also note that $\min_{y \in Y} d(x_i, y) \leq B$ for every $i \in N$. Now consider the set $Y^* = Y \setminus \{c^*\}$. Since $|Y^*| = |Y| - 1 \leq k - 1 = k'$, we know that Y^* is a feasible solution to the k -median instance. We see that Y^* is a certificate that the k -median instance is a YES instance by noting that

$$\begin{aligned} \sum_{i \in N} d(x_i, Y^*) &= \sum_{i \in N} \min_{y \in Y^*} d(x_i, y) \\ &= \sum_{i \in N} \min\{d(x_i, x'), \min_{y \in Y^*} d(x_i, y)\} & (\because \min_{y \in Y} d(x_i, y) \leq B) \\ &= \sum_{i \in N} \min\{d(x_i, x'), \min_{y_1, y_2 \in Y} d(x_i, y_1) + d(x', y_2)\} \\ &= \sum_{i \in N} c_i(Y) \\ &\leq B. \end{aligned}$$

The third transition follows because $c' \in Y$ and thus choosing y_2 to be c' incurs no additional cost. Since the above reduction is polynomial time, the TrSP decision problem is NP-hard. \square

B Omitted Proofs

Some of the proofs contained herein correspond only to the lower bound portion of the statements. In these cases, the proof of the upper bound portion can be found in the main body.

B.1 Omitted Proof of Proposition 2

Proposition 2. *GC-TrSP algorithm (Algorithm 1) satisfies $(2, 1 + \sqrt{2})$ -core. However, for any $\delta, \varepsilon > 0$, there exists an instance for which GC-TrSP violates $(2 - \delta, 1 + \sqrt{2} - \varepsilon)$ -core.*

Proof. Since Chen et al. [14] has shown that the Greedy Capture algorithm achieves a $(1 + \sqrt{2})$ -PF guarantee in general fair clustering, it follows that for any TrSP instance \mathcal{I} , the GC-TrSP algorithm satisfies $(1 + \sqrt{2})$ -PF in the induced clustering instance \mathcal{I}^C . By Theorem 1, this implies that GC-TrSP satisfies the $(2, 1 + \sqrt{2})$ -core.

To prove the bound is tight, let $H = \lceil 1/(3 \cdot \delta) \rceil$. We construct an instance with $n = 15 \cdot H$ agents N , $m = 7$ candidate transit stops $\mathcal{C} = \{c_1, c_2, \dots, c_7\}$, and $k = 5$. The agent set N is partitioned into three subsets, each consisting of agents located at distinct but internally identical locations. Specifically, we define $N = N_1 \cup N_2 \cup N_3$, where $|N_1| = |N_2| = 6 \cdot H - 1$, and $|N_3| = 3 \cdot H + 2$. For each group N_i , all agents share the same travel locations, denoted (a_i, b_i) . The distance from transit stops $\{\tau_1, \dots, \tau_4\}$ to locations $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ are specified in Table 4. For the remaining stops $\{\tau_5, \tau_6, \tau_7\}$, we assign a large constant distance to each endpoint in $\{a_1, b_1, a_2, b_2, a_3, b_3\}$.

$d(\cdot)$	a_1	a_2	a_3	b_1	b_2	b_3
τ_1	1	$\sqrt{2} - 1$	1	∞	∞	∞
τ_2	$1 + \sqrt{2} - \varepsilon$	$1 - (\sqrt{2} - 1)\varepsilon$	$1 - (\sqrt{2} - 1)\varepsilon$	∞	∞	∞
τ_3	∞	∞	∞	1	$\sqrt{2} - 1$	1
τ_4	∞	∞	∞	$1 + \sqrt{2} - \varepsilon$	$1 - (\sqrt{2} - 1)\varepsilon$	$1 - (\sqrt{2} - 1)\varepsilon$

Table 4: An instance in which GC-TrSP fails $(2 - \delta, 1 + \sqrt{2} - \varepsilon)$ -core.

Keeping in mind that $\lceil \frac{2n}{k} \rceil = 6 \cdot H$, we describe the execution of GC-TrSP on this instance. The minimum radius ball that captures $6 \cdot H$ endpoints is centered at τ_2 (and τ_4) with radius $1 - (\sqrt{2} - 1)\varepsilon$. Thus, GC-TrSP selects $\{\tau_2, \tau_4\}$, and deactivates all of the endpoints of the agents in $N_2 \cup N_3$. After that, there is no candidate which can capture $6 \cdot H$ endpoints with a distance radius less than ∞ , so endpoints located at a_1 (b_1) are deactivated by τ_2 (τ_4). As a result, GC-TrSP returns solution $Y = \{\tau_2, \tau_4\}$.

Now consider the set of agents $S = N_1 \cup N_2$ and candidate stop set $T = \{\tau_1, \tau_3\}$. For each agent $i \in N_1$, it holds that $\frac{c_i(Y)}{c_i(T)} = \frac{2(1+\sqrt{2}-\varepsilon)}{2} = 1 + \sqrt{2} - \varepsilon$. For each agent $i \in N_2$, it holds that $\frac{c_i(Y)}{c_i(T)} = \frac{2(1-(\sqrt{2}-1)\varepsilon)}{2(\sqrt{2}-1)} = 1 + \sqrt{2} - \varepsilon$. Furthermore, we have

$$|S| = 12 \cdot H - 2 \geq (2 - \frac{1}{3 \cdot H}) \cdot \frac{2n}{k} \geq (2 - \delta) \cdot \frac{|T| \cdot n}{k}.$$

It follows that there exists such a blocking coalition S and a candidate stop subset T with size $|S| \geq (2 - \delta) \cdot \frac{|T| \cdot n}{k}$ such that for every agent $i \in S$, we have $(1 + \sqrt{2} - \varepsilon) \cdot c_i(T) < c_i(Y)$. This implies that the GC-TrSP algorithm violates $(2 - \delta, 1 + \sqrt{2} - \varepsilon)$ -core. \square

Theorem 2. *GC-TrSP satisfies $(2 + \sqrt{5})$ -JR. However, for any $\varepsilon > 0$, there exists an instance for which GC-TrSP violates $(2 + \sqrt{5} - \varepsilon)$ -JR.*

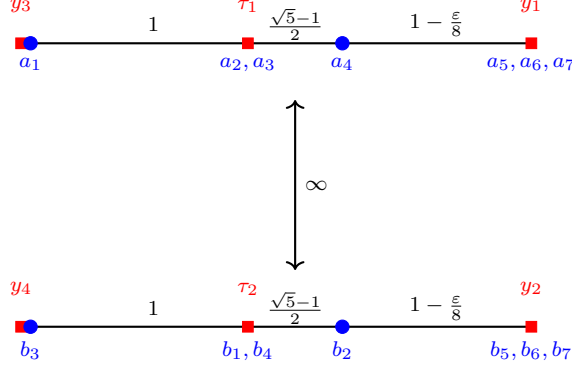


Figure 7: An instance where GC-TrSP algorithm fails to satisfy $(2 + \sqrt{5} - \varepsilon)$ -JR.

Proof. To show the tightness of the analysis, we provide the following instance in which the Greedy Capture for TrSP (Algorithm 1) fails to achieve $(2 + \sqrt{5} - \varepsilon)$ -JR. Consider the TrSP instance pictured in Figure 7.

We consider $n = 7$ agents, whose endpoints are represented by blue circles. There are 6 candidate transit stops, marked as red squares, and $k = 4$. The distances are specified in the figure, and can be considered as two lines separate by an infinite distance, i.e., the distance between any pair of points lying on the same line is the sum of the distances of the intervals between them.. We note that $\lceil \frac{2n}{k} \rceil = \lceil \frac{14}{4} \rceil = 4$. It follows that, once a candidate stop can deactivate 4 active endpoints, it is selected by GC-TrSP and the corresponding endpoints are deactivated. We observe that y_1 and y_2 are the first two candidate stops selected by GC-TrSP. Afterward, no further candidate stop is selected, as no candidate stop can deactivate at least 4 active endpoints within a distance radius of at most $2 + \frac{\sqrt{5}-1}{2} - \frac{\varepsilon}{8}$. Once the radius reaches $2 + \frac{\sqrt{5}-1}{2} - \frac{\varepsilon}{8}$, all the endpoints are deactivated by the selected stops $Y = \{y_1, y_2\}$, which forms the final solution produced by the GC-TrSP algorithm. Under this solution $Y = \{y_1, y_2\}$, the costs incurred by agents $\{1, 2, 3, 4\}$ are computed as follows:

$$c_1(Y) = c_3(Y) = 2 + \sqrt{5} - \frac{\varepsilon}{4}; \quad c_2(Y) = c_4(Y) = \frac{3 + \sqrt{5}}{2} - \frac{\varepsilon}{4}.$$

Notice that $S = \{1, 2, 3, 4\}$ forms a deviation coalition that prefers the alternative stop pair $T = \{\tau_1, \tau_2\}$. Under solution T , the costs are

$$c_1(T) = c_3(T) = 1, \quad c_2(T) = c_4(T) = \frac{\sqrt{5} - 1}{2}.$$

Thus, for each agent $i \in \{1, 3\}$, we have $\frac{c_i(Y)}{c_i(T)} = 2 + \sqrt{5} - \frac{\varepsilon}{4} > 2 + \sqrt{5} - \varepsilon$ and for agent $i \in \{2, 4\}$, we have

$$\frac{c_i(Y)}{c_i(T)} = \frac{\frac{3+\sqrt{5}}{2} - \frac{\varepsilon}{4}}{\frac{\sqrt{5}-1}{2}} = 2 + \sqrt{5} - \frac{(\sqrt{5}+1)\varepsilon}{8} > 2 + \sqrt{5} - \varepsilon.$$

Therefore, we derive that for any agent $i \in S$, $(2 + \sqrt{5} - \varepsilon) \cdot c_i(T) < c_i(Y)$, implying that solution Y by GC-TrSP violates $(2 + \sqrt{5} - \varepsilon)$ -JR. \square

Theorem 3. For any arbitrary transit cost function $d'(\cdot) \geq 0$, ECA satisfies $(1 + \sqrt{2})$ -JR. However, for any $\varepsilon > 0$, there exists an instance with null transit costs for which ECA violates $(1 + \sqrt{2} - \varepsilon)$ -JR.

Proof. To show the tightness, we provide an instance in which ECA fails to achieve $(1 + \sqrt{2} - \varepsilon)$ -JR. Fix $\varepsilon > 0$ and consider a TrSP instance with 4 agents, 4 candidate stops, and $k = 3$. The transit cost function satisfies $d'(i, j) = 0$ for any $i, j \in \mathcal{C}$. For each $i \in \{1, 2, 3, 4\}$, agent i travels from a_i to b_i . The distances between candidate stops and endpoints are specified in Table 5.

$d(\cdot)$	a_1	a_2, a_3	a_4	b_1	b_2, b_3	b_4
τ_1	1	$\sqrt{2} - 1$	$1 + \sqrt{2}$	∞	∞	∞
τ_2	$1 + \sqrt{2}$	$1 - \frac{\varepsilon}{4}$	$1 - \frac{\varepsilon}{4}$	∞	∞	∞
τ_3	∞	∞	∞	1	$\sqrt{2} - 1$	$1 + \sqrt{2}$
τ_4	∞	∞	∞	$1 + \sqrt{2}$	$1 - \frac{\varepsilon}{4}$	$1 - \frac{\varepsilon}{4}$

Table 5: A TrSP instance in which ECA fails $(1 + \sqrt{2} - \varepsilon)$ -JR

We first observe that $\{\tau_2, \tau_4\}$ is the first pair selected by ECA as when the cost radius reaches $2(1 - \frac{\varepsilon}{4})$, agents $\{2, 3, 4\}$ are deactivated by $\{\tau_2, \tau_4\}$. Afterward, no other candidate stop can be selected by ECA. Therefore, ECA returns $\{\tau_2, \tau_4\}$ as the output. However, consider a deviation coalition $S = \{1, 2, 3\}$ and an alternative pair $T = \{\tau_1, \tau_3\}$. For agent 1, we have $c_1(Y) = 2(1 + \sqrt{2})$ and $c_1(T) = 2$. Thus, we have $\frac{c_1(Y)}{c_1(T)} = 1 + \sqrt{2} > 1 + \sqrt{2} - \varepsilon$. For agents 2 and 3, we have $c_2(Y) = c_3(Y) = 2 - \frac{\varepsilon}{2}$ and $c_2(T) = c_3(T) = 2(\sqrt{2} - 1)$. Thus, we have

$$\frac{c_2(Y)}{c_2(T)} = \frac{c_3(Y)}{c_3(T)} = \frac{2 - \frac{\varepsilon}{2}}{2(\sqrt{2} - 1)} = (1 + \sqrt{2}) - \frac{(\sqrt{2} + 1)\varepsilon}{4} > (1 + \sqrt{2}) - \varepsilon.$$

Thus, ECA fails to satisfy $(1 + \sqrt{2} - \varepsilon)$ -JR for any $\varepsilon > 0$. \square

Theorem 4. λ -Hybrid satisfies $\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2}$ -JR, where $\lambda \in [0, 1]$, and this bound is tight.

Proof. To show the tightness of this approximation ratio, we slightly modify the instance originally used to prove the tightness of the GC-TrSP algorithm. In the example, we have $n = 7$ and $k = 4$ with candidate stops $\mathcal{M} = \{\tau_1, \tau_2, y_1, y_2\} \cup D$ where $D = \{y_3, y_4, y_5, y_6\}$. For candidate stops in D , we assign a large constant distance to each endpoint. The locations of endpoints and their distances (except candidate stops in D) are illustrated in Figure 8.

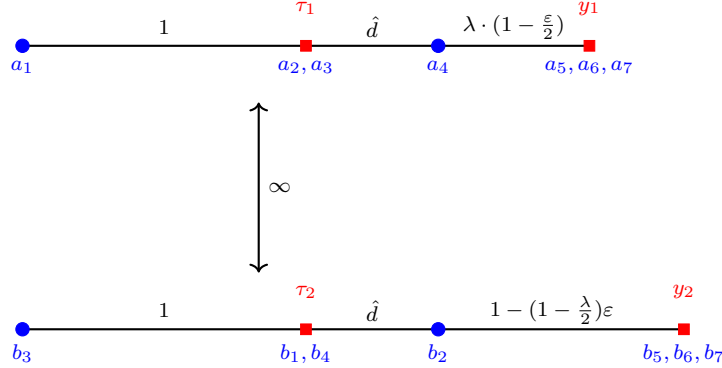


Figure 8: TrSP instance where λ -Hybrid violates $(\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} - \varepsilon)$ -JR where the distance $\hat{d} = \frac{\sqrt{\lambda^2+10\lambda+9}-\lambda-1}{4}$, which is in the range of $[\frac{1}{2}, \frac{\sqrt{5}-1}{2}]$ for $\lambda \in [0, 1]$.

Consider the execution of the λ -Hybrid algorithm, keeping in mind that $\lceil 2n/k \rceil = 4$. Since $\lambda \in [0, 1]$, we first observe that y_1 is the first selected transit stop as when the parameter r reaches $1 - \frac{\varepsilon}{2}$, y_1 will deactivate endpoints $\{a_4, a_5, a_6, a_7\}$ with a distance radius of $\lambda \cdot (1 - \frac{\varepsilon}{2})$ via the GC-TrSP loop. Notice that the pair $\{\tau_1, \tau_2\}$ are not selected because they can deactivate 4 agents only when the parameter r reaches 1. After the selection of y_1 , it is not until the parameter r reaches $(1 - (1 - \frac{\lambda}{2}) \cdot \varepsilon) / \lambda < \frac{1}{\lambda}$ that candidate stop y_2 deactivates 4 endpoints $\{b_2, b_5, b_6, b_7\}$ in a “GC-TrSP” loop. Notice that after the selection of y_1 and y_2 , no

other candidate stop or pair is selected by λ -Hybrid algorithm as all the endpoints will be deactivated when the parameter r reaches $\lambda \cdot (\hat{d} + 2 - (1 - \frac{\lambda}{2}) \cdot \varepsilon)$. Hence, λ -Hybrid algorithm finally outputs $Y = \{y_1, y_2\}$.

Let T denote the pair $\{\tau_1, \tau_2\}$ and consider agent subset $S = \{1, 2, 3, 4\}$. For agent 1, the cost of using Y is $c_1(Y) = \frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} - \varepsilon$ while the cost of using T is $c_1(T) = 1$. Then we have $\frac{c_1(Y)}{c_1(T)} = \frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} - \varepsilon$. Similarly, we have $\frac{c_3(Y)}{c_3(T)} = \frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} - \varepsilon$ as agent 3 shares the same cost of using Y and T as agent 1.

For agent 2 (the same for agent 4), we compute that $c_2(Y) = \hat{d} + \lambda \cdot (1 - \frac{\varepsilon}{2}) + 1 - (1 - \frac{\lambda}{2}) \cdot \varepsilon = 1 + \hat{d} + \lambda - \varepsilon$ and $c_2(T) = \hat{d}$. Consequently, we have

$$\begin{aligned} \frac{c_2(Y)}{c_2(T)} &= \frac{1 + \hat{d} + \lambda - \varepsilon}{\hat{d}} = 1 + \frac{1 + \lambda}{\hat{d}} - \frac{\varepsilon}{\hat{d}} \\ &= 1 + \frac{4(1 + \lambda)}{\sqrt{\lambda^2 + 10\lambda + 9} - \lambda - 1} - \frac{\varepsilon}{\hat{d}} \\ &= 1 + \frac{4(1 + \lambda) \cdot (\sqrt{\lambda^2 + 10\lambda + 9} + (\lambda + 1))}{\lambda^2 + 10\lambda + 9 - (\lambda + 1)^2} - \frac{\varepsilon}{\hat{d}} \\ &= 1 + \frac{4(1 + \lambda) \cdot (\sqrt{\lambda^2 + 10\lambda + 9} + (\lambda + 1))}{8(\lambda + 1)} - \frac{\varepsilon}{\hat{d}} \\ &\leq 1 + \frac{\sqrt{\lambda^2 + 10\lambda + 9} + (\lambda + 1)}{2} - \varepsilon \\ &= \frac{\sqrt{\lambda^2 + 10\lambda + 9} + \lambda + 3}{2} - \varepsilon. \end{aligned}$$

Hence, we now have a group of agents S with size $\lceil \frac{2n}{k} \rceil$ such that for each agent $i \in S$, it holds that $\frac{c_i(Y)}{c_i(T)} \leq \frac{\sqrt{\lambda^2+10\lambda+9}+\lambda+3}{2} - \varepsilon$. This implies that λ -Hybrid algorithm fails to satisfy $(\frac{\lambda+3+\sqrt{\lambda^2+10\lambda+9}}{2} - \varepsilon)$ -JR. \square

Proposition 6. *Given any $\lambda \in (0, 1]$ and $\delta, \varepsilon > 0$, there is an instance for which λ -hybrid does not satisfy $(2 - \delta, \frac{\sqrt{4\lambda^2+12\lambda+1}+2\lambda+1}{4\lambda} - \varepsilon)$ -core.*

Proof. Let $H = \lceil 2/\delta \rceil$. We construct an instance with $n = 4 \cdot H$ agents, $m = 12$ candidate transit stops $\mathcal{C} = \{\tau_1, \tau_2, \tau_3, \tau_4, c_1, \dots, c_8\}$, and $k = 8$. The agent set N is partitioned into six subsets, denoted N_i for $i \in [6]$. The number of agents in each group and their start and end locations are given in Table 6.

i	$ N_i $	(a_i, b_i)
1	$H - 1$	(x_1, x_3)
2	$H - 1$	(x_2, x_4)
3	$H - 1$	(y_1, y_2)
4	$H - 1$	(y_3, y_4)
5	2	(z_1, z_2)
6	2	(z_3, z_4)

Table 6: The number of agents and endpoint locations of the partition of agents, N , in the instance used to prove Proposition 6.

The distances from the locations mentioned in Table 6 to the candidate stops $\{\tau_1, \tau_2, \tau_3, \tau_4, c_1, c_2, c_3, c_4\}$ are specified in Table 7. For simplicity, we use $q := (\sqrt{4\lambda^2 + 12\lambda + 1} - 2\lambda - 1)/4\lambda$. Values in parentheses correspond to each other and any location-candidate pairs not specified are assigned an infinite distance. For the remaining stops $\{c_5, c_6, c_7, c_8\}$, we assign a large constant distance to each endpoint. Noting that $q \leq 1$ for all $\lambda \in (0, 1]$, one can check that the resulting metric space satisfies the triangle inequality.

$d(\cdot)$	$\tau_1(\tau_3)$	$c_1(c_3)$	$\tau_2(\tau_4)$	$c_2(c_4)$
$x_1(x_3)$	1	$1 + q + 1/(2 \cdot \lambda) - \varepsilon$	∞	∞
$y_1(y_3)$	q	$1/(2\lambda) - q \cdot \varepsilon$	∞	∞
$z_1(z_3)$	$1 + q$	$1/(2\lambda) - q \cdot \varepsilon$	∞	∞
$x_2(x_4)$	∞	∞	1	$1 + q + 1/(2 \cdot \lambda) - \varepsilon$
$y_2(y_4)$	∞	∞	q	$1/(2\lambda) - q \cdot \varepsilon$
$z_2(z_4)$	∞	∞	$1 + q$	$1/(2\lambda) - q \cdot \varepsilon$

Table 7: Distances for an instance in which λ -hybrid fails $(2 - \delta, \frac{\sqrt{4\lambda^2 + 12\lambda + 1} + 2\lambda + 1}{4\lambda} - \varepsilon)$ -core.

In words, there are four “zones” where endpoints and candidate stops lie proximal to each other and each of these zones has an identical structure of endpoint distances. Each agent has an endpoint in exactly two zones, and notably, N_1 and N_2 do not share their respective zone pairs with any other agent group, whereas N_3 and N_4 share their zone pairs with N_5 and N_6 , respectively. We will now explain the execution of λ -hybrid and show that it selects $Y = \{c_1, c_2, c_3, c_4\}$. Note that $\lceil 2n/k \rceil = H$. Thus, the endpoints located at any single point in the table above are not enough to trigger the GC-TrSP loop. However, any ball capturing at least two of the points in Table 7 is sufficient to trigger the GC-TrSP loop.

We begin with the case in which $\lambda > \frac{1}{2(1+\varepsilon)}$. Observe that, for each $j \in [4]$, there are $H + 1$ endpoints within a radius of $1/(2\lambda) - \varepsilon < 1$ of c_j (specifically endpoints located at y_j and z_j , whereas 1 is the minimum radius required to capture at least H endpoints with candidate τ_j for each j). Also note that the pair of stops with the smallest cost radius in this case is c_1 and c_2 (c_3 and c_4), which capture agents in $N_3 \cup N_5$ ($N_4 \cup N_6$) with a cost radius of $1/\lambda - 2 \cdot q \cdot \varepsilon < 2$. Thus, regardless of whether it is the GC-TrSP loop or ECA loop which acts first in λ -hybrid, we can assume that either c_1 or c_1 and c_2 are selected first. In the former case, c_2 , c_3 , and c_4 will remain the stops with the minimum radius which can capture H active endpoints and will thus be selected next by the GC-TrSP loop. In the latter case, c_3 and c_4 would be selected afterward by the ECA loop as they can capture $H + 1$ agents with an identical cost radius to c_1 and c_2 . This means that λ -hybrid will certainly select $Y = \{c_1, c_2, c_3, c_4\}$ first. At this point, only endpoints belonging to agents in N_1 and N_2 remain active. These agents would be deactivated by the stops selected already, since every remaining candidate requires a very large radius to capture both N_1 and N_2 , and λ -hybrid would return Y .

Next, we handle the case in which $\lambda \leq \frac{1}{2(1+\varepsilon)}$. Here, the c_j stops are not favored by the GC-TrSP loop since $1/(2\lambda) - \varepsilon \geq 1$. Instead, the GC-TrSP loop would first select τ_j for some j and it would do so when the r parameter increases so that $r \cdot \lambda = 1 \implies r = 1/\lambda$. It can be verified that the pair of stops which capture at least H agents with the minimum cost radius is c_1 and c_2 (or c_3 and c_4). This holds precisely because agent groups N_1 and N_2 are located in distinct zone pairs from N_3 and N_4 and thus it is impossible for multiple of these groups to benefit from the selection of two stops. For example, the selection of τ_1 and τ_2 can capture agents in N_3 with a cost radius of $2q$, but incurs infinite cost for agents in N_1 , N_2 , and N_4 . Note that the agents in $N_3 \cup N_5$ all incur a cost of $2(1/(2\lambda) - \varepsilon)$ by using c_1, c_2 . Thus, the radius parameter r required to select c_1 and c_2 is strictly less than $1/\lambda$ and hence the ECA loop is triggered first and c_1 and c_2 are selected. By the same argument, c_3 and c_4 are selected by the ECA loop as well. Thus, λ -hybrid selects $Y = \{c_1, c_2, c_3, c_4\}$ first and the same argument as was used in the first case applies to show that this is the set returned by λ -hybrid.

Now consider the set of agents $S = N_1 \cup N_2 \cup N_3 \cup N_4$ and transit stop set $T = \{\tau_1, \tau_2, \tau_3, \tau_4\}$. For each agent $i \in N_1 \cup N_2$, it holds that

$$c_i(Y)/c_i(T) = 1 + q + 1/(2 \cdot \lambda) - \varepsilon = \frac{\sqrt{4\lambda^2 + 12\lambda + 1} + 2\lambda + 1}{4\lambda} - \varepsilon,$$

and for each agent $i \in N_3 \cup N_4$, it holds that

$$c_i(Y)/c_i(T) = \frac{1/(2\lambda) - q \cdot \varepsilon}{q}$$

$$\begin{aligned}
&= \frac{2}{\sqrt{4\lambda^2 + 12\lambda + 1} - 2\lambda - 1} - \varepsilon \\
&= \frac{\sqrt{4\lambda^2 + 12\lambda + 1} + 2\lambda + 1}{4\lambda} - \varepsilon
\end{aligned}$$

where the final equality follows from multiplying the fraction's numerator and denominator by the conjugate of the denominator. Lastly, we have

$$|S| = 4 \cdot H - 4 = 2 \cdot H \left(2 - \frac{2}{H}\right) \geq (2 - \delta) \cdot \frac{|T| \cdot n}{k}.$$

In summary, there exists such a blocking coalition S and a candidate stop subset T with size $|S| \geq (2 - \delta) \cdot \frac{|T| \cdot n}{k}$ such that for every agent $i \in S$, we have $(\frac{\sqrt{4\lambda^2 + 12\lambda + 1} + 2\lambda + 1}{4\lambda} - \varepsilon) \cdot c_i(T) < c_i(Y)$. \square

C Missing Details for Experimental Evaluation

In this section, we present additional details of the experimental setup and further experimental results.

C.1 Detailed Experimental Setup

Solution Approximation Verification. The key step regarding the experimental analysis is to compute the approximation ratio of JR and core with respect to the solutions outputted by GC-TrSP and ECA. We generally follow the verification procedures from Bullinger et al. [7] (Appendix C).

To test whether a given solution satisfies JR and core, we rely on the following idea. Fix a solution and examine whether there exist deviations to pairs of stops that would strictly reduce agents' costs. Such deviations, if sufficiently widespread, witness a violation of JR or of core stability.

Formally, for any solution $Y \subseteq \mathcal{C}$ and agent $i \in N$, define

$$\mathcal{P}_i(Y) := \{T \subseteq \mathcal{C} : |T| = 2, c_i(T) < c_i(Y)\},$$

that is, the set of stop pairs to which agent i can deviate and obtain a strictly lower cost. Using this notation, we recall the Proposition C.1 of Bullinger et al. [7].

Proposition 7. *Consider a solution $Y \subseteq \mathcal{C}$. We have,*

- (i) Y satisfies JR if and only if there is no set $T \subseteq \mathcal{C}$ with $|T| = 2$ such that $|\{i \in N : T \in \mathcal{P}_i(Y)\}| \geq \frac{2n}{k}$.
- (ii) Y is in the core if and only if there is no set $T \subseteq \mathcal{C}$ with $T \neq \emptyset$ such that $|\{i \in N : \exists T' \in \mathcal{P}_i(Y), T' \subseteq T\}| \geq \frac{|T| \cdot n}{k}$.

Given a solution Y , checking whether it satisfies JR can be done in polynomial time. Specifically, one can compute $\mathcal{P}_i(Y)$ for each agent $i \in N$, and then verify the condition in Proposition 7 (1) by scanning over stop pairs T and counting how many agents include T in their deviation pairs. Moreover, to compute the approximation ratio, one can perform a binary search over a cost relaxation parameter and, for each candidate value, test whether there exists a pair T that is strictly improving for at least $\frac{2n}{k}$ agents. This procedure runs in polynomial time.

Regarding core testing, applying the same brute force protocol would require considering an exponential number of coalitions $T \subseteq \mathcal{C}$. Instead, we test core stability via the following integer program, denoted CORETESTING.

$$\begin{aligned}
&\max \sum_{i \in N} x_i \\
\text{s.t. } &x_i \leq \sum_{T' \in \mathcal{P}_i(Y)} y_{T'} \quad \forall i \in N
\end{aligned}$$

$$\begin{aligned}
y_{T'} &\leq y_s & \forall T' \subseteq \mathcal{C}, |T'| = 2, s \in T' \\
\sum_{i \in N} x_i &\geq \frac{n}{k} \sum_{s \in \mathcal{C}} y_s \\
x_i &\in \{0, 1\} & \forall i \in N \\
y_s &\in \{0, 1\} & \forall s \in \mathcal{C} \\
y_{T'} &\in \{0, 1\} & \forall T' \subseteq \mathcal{C}, |T'| = 2
\end{aligned}$$

We then recall the Proposition C.3 by Bullinger et al. [7] which shows how the integer program checks the core satisfaction for any given solution.

Proposition 8 ([7]). *A solution Y is in the core if and only if its corresponding integer program (CORETESTING) has an optimal value of 0.*

By Proposition 8, we can again use binary search to identify the smallest cost relaxation parameter for which, under that parameter, the instance induced by Y yields an integer program with optimal value 0.

Data and Sample. We use trip records from the City of Helena Capital Transit Service dataset, which contains 10,282 unique routes among citizens over 3,075 distinct spatial points. Each record specifies a pick up point and a drop off point. To obtain the underlying travel costs, we combine these trip endpoints with OpenStreetMap road network data and use the open source Valhalla routing engine to compute shortest path travel costs. This allows us to derive both walking costs and shuttle bus transit costs between any pair of points in the spatial set.

Our experiments are designed to compare algorithmic performance across different stop numbers k and cost scaling settings. For each parameter combination, we sample either 400 agents or 30 agents together with their associated routes, and take the union of their origin and destination points as the candidate facility location set. We repeat the sampling procedure 50 times and present, for each algorithm, the average as well as the minimum and maximum approximation ratios across the sampled instances.

C.2 Extra Core Approximation Evaluation under Different Scaling

Figure 9 shows the empirical approximation ratio to the $(2, \beta)$ -core achieved by GC-TrSP, ECA, and the $\frac{1}{2}$ -Hybrid algorithm, as a function of the stop number k and the cost relaxation parameter β under different transit cost scaling regimes. Across all panels, the three methods behave almost identically for small k (in particular $k = 2, 4$), with mean approximation ratios essentially equal to 1, indicating that the returned solutions typically lie in the exact core.

As k increases, the methods begin to separate. ECA is consistently the most sensitive to larger k , exhibiting a clear upward drift in its average approximation ratio and substantially larger dispersion across instances, especially when transit costs are small. In contrast, GC-TrSP remains highly stable across all k , with averages staying very close to 1. The $\frac{1}{2}$ -Hybrid algorithm interpolates between the two, tracking GC-TrSP closely while showing a modest increase for larger k , but with much less variance than ECA. While at scaling 10, all three algorithms are essentially indistinguishable, producing near exact core outcomes across all k . Notably, ECA admits an unbounded worst case core approximation ratio, yet in our experiments it remains well behaved, never exceeding an approximation ratio of 2.

D Transit Stop Placement on a Line

Although the algorithm by Bullinger et al. [7] was originally designed for TrSP, it naturally extends to the clustering problem when the metric space is a line. Specifically, the algorithm processes the data sequentially from the leftmost point, selecting the nearest point on the right to form clusters of size $\lceil \frac{n}{k} \rceil$. We demonstrate that this algorithm satisfies PF when $\mathcal{M} = \mathcal{X}$.

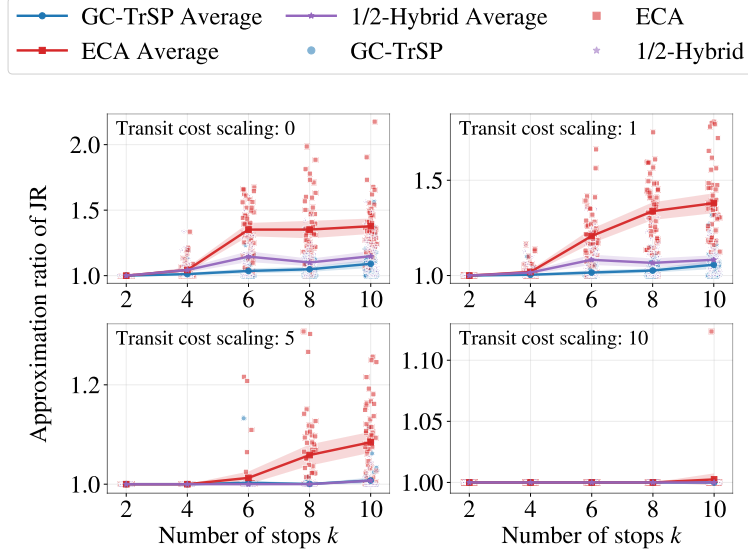


Figure 9: Core approximation evaluation for comparing GC-TrSP, ECA and $\frac{1}{2}$ -Hybrid with 30 agents. Fixing the size relaxation parameter α as 2. Stop selection size ranges from 2 to 10, and the transit cost scale ranges from 0 to 10. Distribution of instance approximation ratios and the mean approximation ratio with 95% confidence intervals

Proposition 9. *Algorithm 1 by Bullinger et al. [7] satisfies PF in clustering on a line when $\mathcal{M} = \mathcal{X}$.*

Proof. Assume for contradiction that Algorithm 1 by Bullinger et al. [7] does not satisfy PF, that is, for the solution P by the algorithm, there exists a subset of agents $S \subseteq N$ of size $|S| \geq \lceil \frac{n}{k} \rceil$ and some center $y \in \mathcal{M}$ such that $d(i, y) < D_i(P)$ for every $i \in S$. Obviously, we have $y \notin P$. Now consider y lies between any two centers c and c' in P . In this case, there will be at least $\lceil \frac{n}{k} \rceil$ data points lie strictly between c and c' as $d(i, y) < D_i(P)$ holds for every $i \in S$. However, according to the center selection by Algorithm 1 by Bullinger et al. [7], there are at most $\lceil \frac{n}{k} \rceil - 1$ data points between c and c' (otherwise c' cannot be selected as a center), which implies contradiction. For the case that y lies on the left (right) side of the leftmost (rightmost) center in P , since there are at least $\lceil \frac{n}{k} \rceil$ agents on the left (right) side of leftmost (rightmost) center in P , a contradiction shows up as one more center should be selected by the design of the algorithm. \square

Proposition 9 establishes that Algorithm 1 by Bullinger et al. [7] guarantees PF solutions for clustering on a line when the candidate set coincides with the set of agents, i.e., $\mathcal{M} = \mathcal{X}$. Combined with Theorem 1, Proposition 9 immediately yields the upper bound of Theorem 4.4 of Bullinger et al. [7], i.e., these statements show that Algorithm 1 by Bullinger et al. [7] satisfies 2-core when $\mathcal{M} = \mathcal{X}$. However, when the candidate set is arbitrary (allowing agents to be distinct from candidates), the performance of their algorithm can deteriorate significantly. To illustrate this, we present the following example in Figure 10.

Example 1. *Consider an instance with $N = \{1, 2, 3, 4\}$, $k = 2$, and candidates $\{c_1, c_2, c_3, c_4\}$. All the data points and candidates are shown in Figure 10. According to the algorithm by Bullinger et al. [7], candidates*

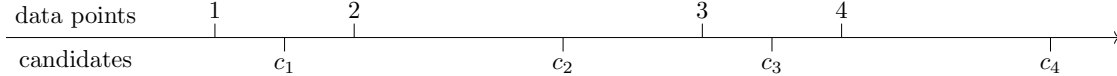


Figure 10: Example of Algorithm by Bullinger et al. [7] fails PF

c_2 and c_4 will be selected as the centers. However, agents $\{1, 2\}$ (resp. agents $\{3, 4\}$) form a deviation coalition that prefers center c_1 (resp. center c_3). The approximation ratio of PF can be arbitrarily poor as $d(1, c_1)$ and $d(2, c_1)$ can be arbitrarily small while $d(1, c_2)$ and $d(2, c_2)$ can be arbitrarily large.

As a fitting follow-up to this observation, we propose a novel algorithm named ℓ -dictator partition algorithm, which ensures PF for general cases on a line.

Input: $N, (\mathcal{X}, d), \mathcal{M}, k$.

Output: P .

- 1: Initialize $P \leftarrow \emptyset$, ℓ be a constant such that $\ell \leq \lfloor n/k \rfloor$.
- 2: **for** $i = 0, \dots, k-1$ **do**
- 3: Let j be the ℓ -th agent among $\{i\lceil n/k \rceil + 1, i\lceil n/k \rceil + 2, \dots, (i+1)\lceil n/k \rceil\}$.
- 4: $c_i \leftarrow \min_{c \in \mathcal{M} \setminus P} d(j, c)$.
- 5: Update $P \leftarrow P \cup \{c_i\}$.

Algorithm 4: ℓ -dictator partition algorithm

Theorem 5. *The ℓ -dictator partition algorithm satisfies 1-PF in clustering on a line.*

Proof. Consider any arbitrary subset S , of agents with size at least $\lceil \frac{n}{k} \rceil$. For any subset of candidates T , we prove that there always exists an agent i in S such that i prefers the clustering P returned by the ℓ -dictator partition algorithm to T , i.e., $\exists i \in S, D_i(P) < D_i(T)$. Without loss of generality, we denote the location of the leftmost and rightmost agents in S by $\text{lm}(S)$ and $\text{rm}(S)$.

Case 1: There exists $c \in P$ such that $c \in (\text{lm}(S), \text{rm}(S))$.

Consider any unselected candidate $y \in \mathcal{M} \setminus P$. Assuming that y is on the left (right) side of c , then there always exists at least one agent i located on the right (left) side of c , meaning that $D_i(y) > D_i(c) \geq D_i(P)$.

Case 2. There is no candidate $c \in P$ satisfying $c \in (\text{lm}(S), \text{rm}(S))$.

First note that, due to the size of S , there must be some ℓ -dictator agent in the interval $[\text{lm}(S), \text{rm}(S)]$. If S contains some ℓ -dictator agent, then P contains this agent's closest center and we are done. We assume henceforth that there is instead some agent $q \in (\text{lm}(S), \text{rm}(S)) \setminus S$ who is an ℓ -dictator.

The first sub-case is that all of the agents in S are placed between two selected centers, denoted as c and c' . It is apparent that S would not deviate to a candidate y outside of the interval between c and c' , i.e. some candidate left of c or right of c' . Hence, we focus on an arbitrary candidate y in the interval between c and c' . Notice that y is not selected by agent q , implying that the distance from agent q to c (c') is strictly smaller than that to y . Hence, there must exist at least one agent i on the left (right) side of q in S who prefers c (c');

The second sub-case is that all of the agents in S are placed on the left of the leftmost selected center or on the right of the rightmost selected center. Supposing all of the agents in S are placed on the left of the leftmost selected center c , then q always selects c . No agent in S will deviate to any candidate center y on the right side of c . Furthermore, there must not be a candidate center between the location of q and c , since otherwise, q would not select c . Lastly, note that the agent $\text{rm}(S)$ would prefer c to any candidate center y left of q (since q also prefers c to y). If the agents are on the right of the rightmost center, the statement holds by an analogous argument. This completes the proof. \square