A Polynomial-time Algorithm to Achieve Extended Justified Representation

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Abstract

We consider a committee voting setting in which each voter approves of a subset of candidates and based on the approvals, a target number of candidates are to be selected. In particular we focus on the axiomatic property called extended justified representation (EJR). Although a committee satisfying EJR is guaranteed to exist, the computational complexity of finding such a committee has been an open problem and explicitly mentioned in multiple recent papers. We settle the complexity of finding a committee satisfying EJR by presenting a polynomial-time algorithm for the problem. Our algorithmic approach may be useful for constructing other multi-winner voting rules.

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JEL: C63, C70, C71, and C78

1. Introduction

The topic of multi-winner/committee voting has witnessed a renaissance with a number of new and interesting developments in the last few years (see [1, 7] for recent surveys). We consider a committee voting setting in which each voter approves of a subset of candidates and based on the approvals, a target k number of candidates are selected. The setting has been referred to as approval-based multi-winner voting or committee voting with approvals. The setting has inspired a number of natural voting rules [9, 5, 10, 4, 15]. Many of the voting rules attempt to satisfy some notion of representation. However it has been far from clear what axiom captures the representation requirements.

Aziz et al. [2, 3] proposed two compelling representation axioms called *justified representation (JR)* and *extended justified representation (EJR)*. Interestingly, Sánchez-Fernández et al. [13] presented an intermediate property called *proportional justified representation (PJR)*. The idea behind all the three properties is that a cohesive and large enough group deserves sufficient number of approved candidates in the winning set of candidates. Interestingly, it is known that there always exists a committee satisfying the strongest property *EJR* [3]. However to date, it has been unknown whether a committee satisfying *EJR* can be computed in polynomial time. For the two weaker representation notions, polynomial-time algorithms have been presented for finding a committee satisfying *JR* [2, 3][1] and *PJR* [6, 14][2]. On the other hand, the computational complexity of finding a committee satisfying *EJR* has been open. Aziz et al. [2, 3] mentioned the problem in their original paper. The problem has been reiterated in subsequent work. Brill et al. [6] state that

> "*it remains an open problem whether committees providing EJR can be computed efficiently.*"

Sánchez-Fernández et al. [14] mention the same problem:

> "*Whether a voting rule exists that satisfies the extended justified representation and can be computed in polynomial time remains an open issue.*"

In a different paper, Sánchez-Fernández et al. [12] state the following.

> "*In contrast, it is conjectured that finding committees that provide EJR is computationally hard.*"

Incidentally, there exists an interesting rule called *PAV* (*Proportional Approval Voting*) that satisfies *EJR* [3]. In *PAV*, each voter is viewed as getting an additional score of $1/j$ for getting the j-th approved candidate in the committee. The *PAV* rule returns a committee with the highest total *PAV* score for the voters. The *PAV* rule has a fascinating history as it was proposed by the Danish polymath Thorvald N. Thiele in the 19th century and then rediscovered by Forrest Simmons [8]. Finding a *PAV* outcome is NP-hard [4, 15] and W[1]-hard even if each voter approves of 2 candidates [4]. Thiele also presented a greedy sequential version of PAV. The rule that is referred to as SeqPAV (Sequential PAV) or RAV (reweighted approval voting) does not even satisfy *JR* [3].

One natural approach to find a committee satisfying *EJR* is to enumerate possible committees and then test them for *EJR*. However the number

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[1] For *JR*, a simple linear-time algorithm call GreedyAV finds a committee satisfying *JR*.

[2] It has recently been shown that a committee satisfying *PJR* can be computed in polynomial time. Brill et al. [6] proved that SeqPhragmén (an algorithm proposed by Swedish mathematician Phragmén in the 19th century) is polynomial-time and returns a committee satisfying *PJR*. Independently and around the same time as the result by Brill et al. [6], Sánchez-Fernández et al. [14] presented a different algorithm that finds a *PJR* committee and also satisfies other desirable monotonicity axioms.

of committees is exponential and even testing whether a committee satisfies *EJR* is coNP-complete [3]. Aziz et al. [3] presented a result that implies that if k is a constant, then a committee satisfying *EJR* can be computed in time $poly(n \cdot |C|^k)$. The result does not show that finding a committee satisfying *EJR* is polynomial-time solvable in general or whether it is fixed parametrized tractable.

Contributions. We present the first polynomial-time algorithm to find a committee that satisfies *EJR*. The result implies that there exists a polynomial-time algorithm to find a committee that satisfies the weaker property of *PJR*. As mentioned earlier, it has only recently been proven in two independent papers that a committee satisfying *PJR* can be computed in polynomial time [6, 14]. Both of the algorithms in [6] and [14] sequentially build a committee while optimizing some flow or load balancing objective. In contrast, our algorithm uses an approach based on swapping candidates from inside a committee with candidates from outside the committee. The correctness of our algorithm relies on a careful insight on the connection between *EJR* and a property we refer to as *PAV*-swap-freeness. We feel that this simple idea of allowing swaps may lead to other interesting algorithms for *EJR* as well as other compelling properties in multi-winner voting problems.

2. Approval-based Committee Voting and Representation Properties

We consider a social choice setting with a set $N = \{1, \ldots, n\}$ of voters and a set C of m candidates. Each voter $i \in N$ submits an approval ballot $A_i \subseteq C$, which represents the subset of candidates that she approves of. We refer to the list $\vec{A} = (A_1, \ldots, A_n)$ of approval ballots as the *ballot profile*. We will consider *approval-based multi-winner voting rules* that take as input a quadruple $(N, C, \vec{A}, k)$, where k is a positive integer that satisfies $k \leq m$, and return a subset $W \subseteq C$ of size k, which we call the *winning set*, or *committee*.

Definition 1 (Justified representation (JR)). *Given a ballot profile $\vec{A} = (A_1, \ldots, A_n)$ over a candidate set C and a target committee size k, we say that a set of candidates W of size $|W| = k$ satisfies justified representation for $(\vec{A}, k)$ if*

$$\forall X \subseteq N : |X| \geq \frac{n}{k} \ and \ |\cap_{i \in X} A_i| \geq 1 \implies (|W \cap (\cup_{i \in X} A_i)| \geq 1)$$

The rationale behind this definition is that if k candidates are to be selected, then, intuitively, each group of $\frac{n}{k}$ voters "deserves" a representative. Therefore, a set of $\frac{n}{k}$ voters that have at least one candidate in common should not be completely unrepresented.

Definition 2 (Proportional Justified Representation (PJR)). *Given a ballot profile $(A_1, \ldots, A_n)$ over a candidate set C, a target committee size k, $k \leq m$,*

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and integer ℓ we say that a set of candidates W, $|W| = k$, satisfies ℓ-proportional justified representation for $(\vec{A}, k)$ if

$$\forall X \subseteq N : |X| \geq \ell \frac{n}{k} \ and \ |\cap_{i \in X} A_i| \geq \ell \implies (|W \cap (\cup_{i \in X} A_i)| \geq \ell)$$

We say that W satisfies proportional justified representation for $(\vec{A}, k)$ if it satisfies ℓ-proportional justified representation for $(\vec{A}, k)$ and all integers $\ell \leq k$.

Definition 3 (Extended justified representation (*EJR*))**.** *Given a ballot profile $(A_1, \ldots, A_n)$ over a candidate set C, a target committee size k, $k \leq m$, we say that a set of candidates W, $|W| = k$, satisfies ℓ-extended justified representation for $(\vec{A}, k)$ and integer ℓ if*

$$\forall X \subseteq N : |X| \geq \ell \frac{n}{k} \ and \ |\cap_{i \in X} A_i| \geq \ell \implies (\exists i \in X : |W \cap A_i| \geq \ell).$$

We say that W satisfies extended justified representation for $(\vec{A}, k)$ if it satisfies ℓ-extended justified representation for $(\vec{A}, k)$ and all integers $\ell \leq k$.

It is easy to observe that *EJR* implies *PJR* which implies *JR*. So any committee that satisfies *EJR* also satisfies the other two properties.

3. PAV-score and Swaps

The *PAV*-score of a voter i for a committee W is

$$H(|W \cap A_i|)$$

where

$$H(p) = \begin{cases} 0, \ \text{for } p = 0 \\ \sum_{j=1}^{p} \frac{1}{j}, \ \text{for } p > 0. \end{cases}$$

The *PAV*-score of a committee $W \subseteq C$ is defined as

$$\sum_{i \in N} H(|W \cap A_i|).$$

The *PAV* rule that we discussed in the introduction outputs a set $W \subseteq C$ of size k with the highest *PAV*-score.

We say that a committee W such that $|W| = k$ satisfies *PAV-swap-freeness* if there exists no $c' \in W, c \in C \setminus W$ s.t. *PAV-score*$((W \setminus \{c'\}) \cup \{c\}) > PAV\text{-}score((W))$. Note that if a committee W has the highest possible *PAV*-score, it satisfies *PAV*-swap-freeness.

We now define marginal contribution as used in [3]. For each candidate $w \in W$, we define $MC(w, W)$ its *marginal contribution* as the difference between the *PAV*-score of W and that of $W \setminus \{w\}$:

$$MC(w, W) = PAV\text{-}score((W) - PAV\text{-}score((W \setminus \{w\}).$$

Let $MC(W)$ denote the sum of marginal contributions of all candidates in W:

$$MC(W) = \sum_{w \in W} MC(w, W).$$

We now formally state as a lemma an observation that was already made in [3].

Lemma 1. *For any committee W such that $|W| = k$, $\sum_{c \in W} MC(c, W) \leq |\{i \in N : A_i \cap W \neq \emptyset\}|$. Moreover there exists at least one $c \in W$ such that $MC(c, W) \leq |\{i \in N : A_i \cap W \neq \emptyset\}|/k \leq n/k$.*

Proof. Pick a voter $i \in N$, and let $j = |A_i \cap W|$. If $j > 0$, this voter contributes exactly $\frac{1}{j}$ to the marginal contribution of each candidate in $A_i \cap W$, and hence her contribution to $MC(W)$ is exactly 1. If $j = 0$, this voter does not contribute to $MC(W)$ at all. Therefore, we have $MC(W) = \sum_{c \in W} MC(c, W) \leq |\{i \in N : A_i \cap W \neq \emptyset\}| \leq n$. Since there are exactly k candidates, there exists some $c \in W$ such that $MC(c, W) \leq |\{i \in N : A_i \cap W \neq \emptyset\}|/k \leq n/k$. □

We now prove that if a committee satisfies *PAV*-swap-freeness, then it satisfies *EJR*. The argument is almost identical to the argument that the outcome of *PAV* satisfies *EJR* [3]. However, we reproduce it just for the sake of completeness because we will further refine this argument.

Lemma 2. *If a committee satisfies PAV-swap-freeness, then it satisfies EJR.*

Proof. Suppose that there is a committee W such that $|W| = k$ that satisfies *PAV*-swap-freeness but violates *EJR*. Since W violates *EJR*, there is a value of $\ell \geq 1$ and a set of voters N^*, $|N^*| = s \geq \ell \cdot \frac{n}{k}$. We know that at least one of the ℓ candidates approved by all voters in N^* is not elected; let c be some such candidate. Each voter in N^* has at most $\ell - 1$ representatives in W, so the marginal contribution of c (if it were to be added to W) would be at least $s \cdot \frac{1}{\ell} \geq \frac{n}{k}$. On the other hand, by Lemma 1, we have $\sum_{c \in W} MC(c, W) \leq n$.

Now, consider some candidate $w \in W$ with the smallest marginal contribution; clearly, his marginal contribution is at most $\frac{n}{k}$. If it is strictly less than $\frac{n}{k}$, we are done, as we can improve the total *PAV*-score by swapping w and c, a contradiction.

Therefore suppose it is exactly $\frac{n}{k}$, and therefore the marginal contribution of each candidate in W is exactly $\frac{n}{k}$. We know that $A_i \cap W \neq \emptyset$ for each $i \in N^*$, because otherwise $\sum_{w' \in W} MC(w', W) \leq n - 1$ (by Lemma 1) which implies that the marginal contribution of w is less than $\frac{n}{k}$. Hence we know that $A_i \cap W \neq \emptyset$ for all $i \in N^*$. Pick some candidate $w' \in W \cap A_i$ for some $i \in N^*$, and set $W' = (W \setminus \{w'\}) \cup \{c\}$. Observe that after w' is removed, adding c increases the total *PAV*-score by at least

$$(s - 1) \cdot \frac{1}{\ell} + \frac{1}{\ell - 1} > \frac{s}{\ell} \geq n/k.$$

Thus, the *PAV*-score of W' is higher than that of W, a contradiction again. □

Although *PAV*-swap-freeness is a much weaker property than maximizing total *PAV*-score, it is surprising that it already implies *EJR*. In the next section, this insight helps us to come up with useful algorithms.

4. MaxSwapPAV

Based on *PAV*-score improving swaps, one can formulate the following algorithm called SwapPAV.

> SwapPAV: Start from a random committee of size k. Keep implementing swaps that increase the total *PAV*-score of the committee while such a swap is possible. Return the committee if no more improving swaps are possible.

Our first observation is that Swap-PAV always terminates. The reason is that each time we implement the swap, the PAV-score of the committee increases. This can only happen finitely often as $PAV\text{-}score((W) \leq nH(k) \leq n(\ln k + 1)$ for any committee W of size k. In fact, one can easily prove that with each improving swap, the *PAV*-score increases by at least $1/k!$ so that that the total number of swaps cannot exceed $n(\ln k + 1)k!$. This observation already gives us the first FPT algorithm for finding a committee satisfying *EJR*.

We now show how we can modify SwapPAV to find a committee satisfying *EJR* in polynomial time. We modify SwapPAV as follows. If W is not *PAV*-swap-free, then we look at all possible swaps and only implement the swap which makes biggest difference to the *PAV*-score. We impose an extra condition that we only swap if the improvement in the total *PAV*-score is at least $\frac{1}{2k^3}$. The algorithm is specified as Algorithm 1 (MaxSwapPAV).

Algorithm 1 MaxSwapPAV

Require: $(N, \vec{A}, k)$.
Ensure: W
1: $W \longleftarrow$ any committee of size k.
2: **while** $\exists c' \in W, c \in C \backslash W$ s.t. $PAV\text{-}score((W \backslash \{c'\}) \cup \{c\}) - PAV\text{-}score(W) \geq 1/2k^3$ **do**
3:     **for** each $c' \in W, c \in C \setminus W$ **do**
4:         $diff(c, c') = PAV\text{-}score((W \setminus \{c'\}) \cup \{c\}) - PAV\text{-}score(W)$
5:     **end for**
6:     Find $c' \in W, c \in C \setminus W$ with the maximum $diff(c, c')$.
7:     $W \longleftarrow (W \setminus \{c'\}) \cup \{c\}$
8: **end while**
9: **return** W.

We now argue why MaxSwapPAV returns a committee satisfying *EJR* and it terminates in polynomial time. The most crucial lemma for both statements is Lemma 3. The lemma is stronger than Lemma 2 and requires a more careful analysis.

Lemma 3. *Suppose that W does not satisfy EJR, then there exist $c' \in W, c \in C \setminus W$ s.t. PAV-score$((W \setminus \{c'\}) \cup \{c\}) -$ PAV-score$((W)) \geq 1/2k^3$.*

Proof. Since W violates *EJR*, there is a value of $\ell \geq 1$ and a set of voters N^*, $|N^*| = s \geq \ell \cdot \frac{n}{k}$. We know that at least one of the ℓ candidates approved by all voters in N^* is not elected; let c be some such candidate. Each voter in N^* has at most $\ell - 1$ representatives in W, so the marginal contribution of c (if it were to be added to W) would be at least $s \cdot \frac{1}{\ell} \geq \frac{n}{k}$.

Let $w \in W$ be the candidate with smallest marginal contribution. By Lemma 1, $MC(w, W) \leq \frac{n}{k}$. If $MC(w, W) \leq \frac{n}{k} - \frac{1}{2k^3}$, then replacing w with c results in a committee W' with PAV-score increasing by at least $\frac{1}{2k^3}$, and so we are done.

So, we assume that $MC(w, W) > \frac{n}{k} - \frac{1}{2k^3}$. This implies that $MC(w', W) > \frac{n}{k} - \frac{1}{2k^3}$ for every $w' \in W$. Since $\sum_{w' \in W} MC(w', W) \leq n$, we use this fact to find the maximum possible marginal contribution among all candidates in W. Let the maximum marginal contribution be $MC(b, W)$ of candidate b. In that case we know that

$$\sum_{w' \in W} MC(w', W) \leq n$$

$$\iff \sum_{w' \in W \setminus \{b\}} MC(w', W) + MC(b, W) \leq n$$

$$\iff MC(b, W) \leq n - \sum_{w' \in W \setminus \{b\}} MC(w', W)$$

$$\implies MC(b, W) < n - (k-1)(\frac{n}{k} - \frac{1}{2k^3})$$

$$\iff MC(b, W) < \frac{(2nk^3 - 2nk^3 + 2nk^2 + k - 1)}{2k^3}$$

$$\iff MC(b, W) < \frac{n}{k} + \frac{1}{2k^2} - \frac{1}{2k^3} < \frac{n}{k} + \frac{1}{2k^2}.$$

Hence, it follows that $MC(w', W) < \frac{n}{k} + \frac{1}{2k^2}$ for every $w' \in W$.

We now claim that there is a candidate $w' \in W$ that is also in $\bigcup_{i \in N^*} A_i$. Suppose not. This means that no one in N^* approves of anybody in W and so by Lemma 1, $\sum_{w' \in W} MC(w', W) \leq |N \setminus N^*| \leq n - \frac{n}{k}$. Thus, $MC(w, W) \leq \frac{n}{k} - \frac{n}{k^2} < \frac{n}{k} - \frac{1}{2k^3}$, contradicting our assumption that $MC(w, W) > \frac{n}{k} - \frac{1}{2k^3}$. Now pick any $w' \in W \cap \bigcup_{i \in N^*} A_i$. As $w \in \bigcup_{i \in N^*} A_i$, this implies that $|A_i \cap (W \setminus \{w'\})| \leq \ell - 2$ for some $i \in N^*$. Hence, $MC(c, (W \setminus \{w'\}) \cup \{c\})$ would be at least $\frac{n}{k} + \frac{1}{\ell - 1} - \frac{1}{\ell} \geq \frac{n}{k} + \frac{1}{k^2}$. Therefore, replacing w' with c results in a committee W' with PAV-score increasing by at least $\frac{1}{k^2} - \frac{1}{2k^2} = \frac{1}{2k^2} \geq \frac{1}{2k^3}$. This proves the lemma. $\qquad\square$

Lemma 3 is the foundation for proving the main properties of the MaxSwap-PAV algorithm.

Proposition 1. *MaxSwapPAV returns a committee that satisfies EJR.*

Proof. MaxSwapPAV returns a committee W such that there exist no $c' \in W$ and $c \in C \setminus W$ s.t. *PAV-score*$((W \setminus \{c'\}) \cup \{c\}) - PAV\text{-}score((W)) \geq 1/2k^3$. By Lemma 3, such a committee satisfies *EJR*. □

Proposition 2. *MaxSwapPAV runs in polynomial time* $O(n^2 mk^4 \ln k)$.

Proof. We first show that the total number of swaps in MaxSwapPAV cannot exceed $2n(\ln k + 1)k^3$. In Lemma 3, we proved that each swap in the algorithm improves the *PAV*-score by at least $\frac{1}{2k^3}$. Since the *PAV* score of any committee cannot exceed $n(\ln k + 1)$, there can be at most $2n(\ln k + 1)k^3$ swaps. Each swap requires examining $O(km)$ pairs of candidates. For each pair, we need to make $O(n)$ operations. □

It follows from the two propositions above that MaxSwapPAV returns a committee satisfying *EJR* in polynomial time.

5. Discussion

To conclude, we presented the first polynomial-time algorithm for finding a committee that satisfies *EJR*. In Table 1, we summarize the justified representation related properties satisfied by different polynomial-time algorithms in the literature.

Rules	JR	PJR	EJR
MaxSwapPAV (this paper)	✓	✓	✓
SeqPhragmén [6, 11]	✓	✓	✗
Open DHondt (ODH) [14]	✓	✓	✗
GreedyAV [3, 16]	✓	✗	✗
SeqPAV [16]	✗	✗	✗

Table 1: Related and known polynomial-time algorithms for approval-based committee voting.

Our result shows that *EJR* is as amenable to efficient computation as *PJR*. Depending on particular specifications, our algorithm to find a committee satisfying *EJR* can also used to formulate particular voting rules.

Skowron et al. [15] mentioned that SeqPAV can be seen as a desirable approximation algorithm for PAV. Our alternative approach of allowing exchanges rather than sequentially building a committee seems to be closer to one of the defining features of *PAV* that it satisfies *EJR*.

The approach of implementing swaps of candidates also makes it possible to move towards fairer representation from a default committee without having to disband the whole committee. The swapping procedure can also be used as post-processing step after running any other committee rule. If the initial committee is the outcome of SeqPAV, then we know that the committee already

guarantees at least $(1 - \frac{1}{e})$ of the maximum possible *PAV*-score [15]. Hence it follows that subsequent *PAV*-score improving swaps can only further increase the score.

Our algorithmic result also adds a new talking point to the debate between the harmonic scoring approach of Thiele versus the load balancing approach of Phragmén that started over a hundred years ago [8]. Note that Brill et al. [6] showed that SeqPhragmén—one of the efficient algorithms within Phragmén's framework of multi-winner rules—satisfies *PJR*. On the other hand, SeqPAV the well-known polynomial-time algorithm using Thiele's approach of harmonic weights does not even satisfy *JR*. However, we have shown that by allowing swaps of candidates, one can satisfy *EJR* which is stronger than *PJR*.

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