Sampford Apportionment Satisfies Threshold Monotonicity

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Abstract

Apportionment distributes a fixed number of legislative seats among political parties in proportion to their vote shares. Randomization can satisfy quota in every realization and exact proportionality in expectation, but these requirements determine only the parties' marginal seat distributions. We study threshold monotonicity, which requires that when every fractional quota inside a coalition weakly increases and every fractional quota outside it weakly decreases, the coalition's new seat total stochastically dominates its old one. We prove that Sampford apportionment satisfies threshold monotonicity, resolving the threshold-monotonicity conjecture of Correa et al. (2024). Complementing this result, we prove that no quota-respecting, ex-ante proportional method satisfies pairwise threshold monotonicity even for disjoint coalitions when there are at least seven parties. This impossibility applies whether or not the method has full support.

Keywords: Apportionment, Randomized rounding, Sampford sampling, Threshold monotonicity, Stochastic dominance

1 Introduction

> "*The proper basis of apportioning or dividing representation in political assemblies among the various units entitled to membership in such bodies has become one of the most difficult questions in the working of representative government*" — Edmund J. James (1897) [22].

An apportionment problem asks how a house of size h should be divided among n political parties in proportion to their vote totals. If party i has fractional quota q_i, integrality generally prevents it from receiving exactly q_i seats. A classical requirement is *quota*: party i must receive either $\lfloor q_i \rfloor$ or $\lceil q_i \rceil$ seats. Deterministic methods face a classical conflict between quota and the most natural population-monotonicity requirement [5]. Randomization avoids this particular impasse. As observed by Grimmett [18], a method may satisfy quota in every realization while awarding each party exactly its fractional quota in expectation.

Exact expectations describe only individual marginals, while political questions are often joint. A coalition may care about a majority, a blocking minority, or another important threshold. Two randomized methods with the same marginal inclusion probabilities can assign very different probabilities to these events. More troublingly, a coalition whose parties all become proportionally stronger can become *less* likely to reach a given threshold if the rounding decisions are correlated in the wrong way. Such a reversal can also create strategic incentives for voters who evaluate outcomes through the total number of seats won by an approved set of parties. To exclude this behaviour, Correa et al. [12, 13] introduced *threshold monotonicity*. In its quota-based form, the axiom requires the total number of seats received by a reinforced coalition to increase in stochastic dominance. This is the relevant comparison for every nondecreasing utility of the coalition's seat total, and hence simultaneously controls all threshold events.

Example 1.1. The following example explains why monotonicity should concern *seat thresholds*, not merely which parties are rounded up. Consider a house of four seats and parties A, B, C, where A and B form a coalition. Fractional quotas $(0.9, 0.9, 2.2)$ leave two seats to be awarded at random. In this instance, quota and ex ante proportionality already determine the relevant probabilities, independently of the method: exactly two parties are rounded up, with marginal probabilities $0.9, 0.9, 0.2$. The coalition therefore receives two seats exactly when C is not rounded up, an event of probability 0.8.

Suppose the fractional quotas change to $(1.03, 0.92, 2.05)$: both coalition members gain and their opponent loses. Party C now receives a third seat with probability 0.05, so the coalition receives at least two seats with probability 0.95. Its seat distribution has improved. Nevertheless, the probability that A and B are both *rounded up* falls from 0.8 to zero, because A's formerly random seat has become part of its lower quota. Comparing seat thresholds correctly treats guaranteed and randomly awarded seats alike. Figure 1 displays both comparisons.

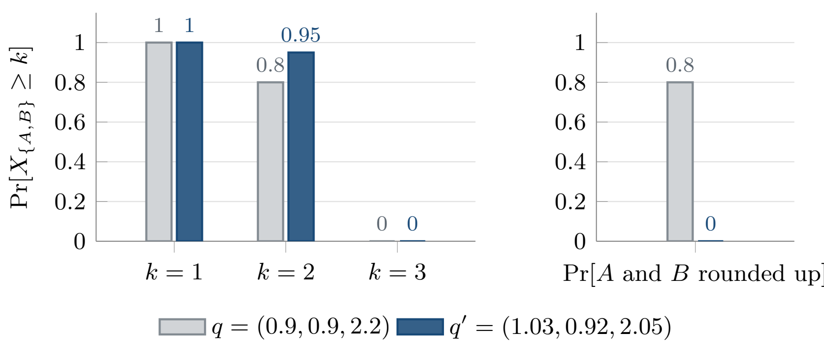


Figure 1: The coalition $\{A, B\}$ of Example 1.1 with house size four. Left: upper-tail probabilities of the coalition's seat total before and after the change from q to q'; every tail weakly increases, so the seat distribution improves in stochastic dominance. Right: the probability that both coalition members are rounded up falls from 0.8 to 0, because A's formerly random seat has become part of its lower quota. Monotonicity must therefore be stated in terms of seat thresholds rather than rounding events.

Correa et al. [13] describe threshold monotonicity as a general notion of strategy-proofness for randomized apportionment: a voter cannot profit by supporting a disapproved party, and the same protection extends to coalitions of voters. Correa et al. [12, 13] conjectured that Sampford's quota-compliant, ex-ante proportional method satisfies threshold monotonicity for arbitrary coalitions. The conjecture also appears in a 2025 survey [2].

Sampford sampling [26] is a classical unequal-probability sampling scheme: here it selects the parties to be rounded up with inclusion probabilities equal to their quota residues. Threshold monotonicity is stronger than the selection monotonicity studied by Correa et al. [13], who proved that Sampford's method satisfies selection monotonicity. Threshold monotonicity implies this property. The empty-sample case is immediate. Otherwise, use $q = p$, $q' = p'$, and $h = \sum_i p_i = \sum_i p'_i$. A feasible set is selected exactly when its h members receive h seats in total. Selection monotonicity therefore compares the probability of a specified feasible set when its members' residues increase and all other residues decrease. It controls one specified set of rounded-up parties. However, a threshold event is a union of many such outcomes. After a coalition is reinforced, some outcomes in this union can become less likely because they also round up parties outside the coalition. The selection-monotonicity inequalities therefore cannot simply be added: a proof for threshold monotonicity must show that losses in some outcomes are compensated by gains in others.

1.1 Contributions

We prove the main conjecture of Correa et al. [12, 13], giving the first established compatibility of quota, ex-ante proportionality, and threshold monotonicity (Theorem 1). The proof moves along the straight segment from the original vector of fractional quotas to the reinforced one. Between integer crossings, we condition on whether the varied party is selected. The only possibly negative contribution to the derivative of a threshold probability is then a single covariance. Newton's inequalities confine that contribution to outcomes falling exactly one coalition member short of the threshold. A coefficient bound and double count show that the positive term in the derivative is at least as large. Along the segment, coalition residues rise and outside residues fall, so every threshold probability is nondecreasing between crossings. At a crossing, a seat merely changes its description, from residual to guaranteed or conversely, without changing the complete allocation law. Continuity there joins the pieces.

We also prove a complementary impossibility (Theorem 2) for the pairwise threshold monotonicity axiom introduced by Correa et al. [13]. This axiom compares a coalition whose members gain with a second coalition whose members lose, leaves all other parties unrestricted, and requires at least one of the two coalition seat distributions to move in the expected direction. Even when the tested coalitions are required to be disjoint, quota and ex-ante proportionality are incompatible with this axiom for every $n \geq 7$. Unlike the impossibility theorem of Correa et al. [13], our argument does not assume full support. Our impossibility result can be viewed as a randomized and coalitional analogue of the well-known Balinski–Young impossibility theorem, which says that no quota-compliant deterministic method satisfies population monotonicity [5]. The proof is graph-theoretic: proportionality turns the two-seat lottery into a fractional matching with prescribed vertex degrees, and, at the constructed profile, pairwise threshold monotonicity forces the positive edges between the small parties to be pairwise intersecting.

2 Related work

Apportionment. Classical apportionment theory balances quota, population monotonicity, and house monotonicity [5, 24, 25, 28]. House monotonicity requires that an increase in the total number of seats should not hurt any party. Population monotonicity, as Young [31] succinctly puts it, requires that "a growing state should not give up seats to a shrinking state"; its violation is the population paradox [5, pp. 42–43]. The Balinski–Young impossibility theorem shows that the deterministic quota requirement is incompatible with population monotonicity [5, 16]. More basically, seat indivisibility prevents any deterministic method from giving every party its fractional quota. Randomization changes the comparison.

Randomized apportionment. Although there is relatively less work on randomized apportionment, even countries committed to deterministic apportionment rely on chance at the margins. Almost every divisor or quota method needs a tie-break rule, and a common one is the random draw which is used in some countries including Germany [24, Chapter 4]. Grimmett [18] combines ex-post quota with exact proportionality in expectation; Gölz et al. [16] add ex-post house monotonicity through cumulative rounding; and Cembrano et al. [9] characterize the randomized methods satisfying quota, ex-ante proportionality, and house monotonicity via network-flow formulations of quota-compliant, house-monotone paths and lotteries over them. Proportional lotteries also arise in peer selection and random allocation [2, 3, 8]. These guarantees either vary the house size or concern a single vote profile. Correa et al. [13] instead keep the house fixed and ask how the *joint* distribution of rounded-up parties responds when a coalition is reinforced. Their threshold monotonicity can be viewed as a coalition-level generalization of population monotonicity for randomized methods. They proved selection monotonicity for Sampford rounding and conjectured the arbitrary-coalition threshold result proved here.

Unequal-probability sampling. After lower quotas have been awarded, the residual allocation is exactly a fixed-size unequal-probability (πps) sampling problem: construct a distribution over k-element samples whose first-order inclusion probabilities equal prescribed values $\pi_i \in [0, 1]$ with $\sum_i \pi_i = k$, here with $\pi_i = p_i$ [7]. Sampford's design [26] is one of these methods. The selection and threshold axioms of Correa et al. [13] draw on systematic and unequal-probability sampling [23, 21, 20], maximum-entropy designs [10], and the pivotal method [14, 29, 30]. Whereas the sampling problem starts from one list of prescribed individual probabilities, the present question compares every coalition threshold across two such profiles.

Dependent rounding. In computer science, related procedures are studied under the name dependent rounding and use negative dependence for concentration and approximation guarantees [1, 6, 11, 15]. These describe one lottery at one profile, whereas threshold monotonicity compares two.

Fair Division. Apportionment can also be viewed as a special case of fair allocation with entitlements where the items are identical and homogeneous. For more on weighted fair division, see the survey of Suksompong [27]. In that respect, randomized apportionment can be viewed as best-of-both-worlds fair division [4] where we aim for fairness guarantees ex-ante and ex-post.

3 Preliminaries

Let $N = [n] = \{1, \ldots, n\}$ be the set of parties. For a finite set G and an integer r, write $\binom{G}{r} = \{D \subseteq G : |D| = r\}$.

3.1 From quota residues to a fixed-size lottery

Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n_{\geq 0}$ be a vote vector with $\sum_{i \in N} v_i > 0$, and let $h \in \mathbb{N}$ be the house size. An *apportionment* is a vector $x \in \mathbb{N}_0^n$ satisfying $\sum_{i \in N} x_i = h$. An *apportionment method* $\mathcal{A}$ maps every pair (v, h) to a random apportionment $\mathcal{A}(v, h)$.

Party i's *fractional quota*, *lower quota*, and *residue* are, respectively,

$$q_i = \frac{hv_i}{\sum_{j \in N} v_j}, \qquad b_i = \lfloor q_i \rfloor, \qquad p_i = q_i - b_i. \tag{1}$$

The number of seats remaining after all lower quotas have been awarded is

$$\alpha = h - \sum_{i \in N} b_i = \sum_{i \in N} p_i. \tag{2}$$

Thus $p \in [0,1)^n$ and $\alpha \in \{0, \ldots, n-1\}$.

An apportionment method satisfies *quota* if every party receives either its lower or upper quota in every realization. It is *ex ante proportional* if $\mathbb{E}[\mathcal{A}(v,h)_i] = q_i$ for every party i.

Following Correa et al. [13], a *selection lottery* r takes as input a vector $p \in [0,1)^n$ whose coordinates sum to an integer α and returns a random set

$$r(p) \in \binom{N}{\alpha} \quad \text{such that} \quad \Pr[i \in r(p)] = p_i \quad (i \in N).$$

Every selection lottery induces an apportionment method by awarding lower quotas and then one additional seat to each selected party:

$$\mathcal{A}^r(v,h)_i = b_i + \mathbf{1}\{i \in r(p)\}. \tag{3}$$

The induced method satisfies quota and ex ante proportionality. Selection lotteries are also studied as unequal-probability sampling without replacement and as dependent randomized rounding [7, 14, 15].

3.2 Sampford rounding and its prescribed marginals

For a residue vector $p \in [0,1)^n$, let

$$N' = \{i \in N : 0 < p_i < 1\}.$$

Thus N' is the set of parties with nonintegral fractional quotas. The marginal condition in the definition of a selection lottery already implies that parties outside N' are never selected.

Remark 3.1 (Why not simply condition independent rounding?). Selecting each party in N' independently with probability p_i gives the desired individual probabilities, but the number selected is random. Conditioning on selecting exactly α parties fixes the size. For $S \in \binom{N'}{\alpha}$,

$$\prod_{i \in S} p_i \prod_{j \in N' \setminus S} (1 - p_j) = \left(\prod_{j \in N'} (1 - p_j) \right) \prod_{i \in S} \frac{p_i}{1 - p_i}.$$

The first factor is common to every S, so the conditioned sampling weights S by the product of the odds $w_i = p_i/(1-p_i)$. This explains the odds in Sampford's rule, but it does not recover the desired marginals: conditioning generally changes each party's selection probability. The odds merely rewrite the original independent-rounding after conditioning; they are not recalibrated probabilities. Conditional Poisson sampling can instead search for different starting probabilities whose conditional marginals are p_i [19, 10, 29]. Sampford instead corrects the conditioned set weights themselves.

Definition 3.2 (Sampford rounding)**.** For $\alpha = 0$, set $r^{\mathrm{Samp}}(p) = \varnothing$. For $\alpha \geq 1$, repeat the following experiment until it is accepted. Make α independent draws from N', with replacement. One *distinguished draw* selects party $i \in N'$ with probability p_i/α. Each of the other $\alpha - 1$ draws selects $i \in N'$ with probability $w_i / \sum_{j \in N'} w_j$, where $w_i = p_i/(1-p_i)$. If the α drawn parties are all distinct, output their set as $r^{\mathrm{Samp}}(p)$; otherwise discard all the draws and repeat the experiment. The resulting selection lottery is *Sampford rounding*, and the method induced through (3) is the *Sampford method*. Parties outside N' have zero residue and are never selected.

The word "distinguished" refers only to the single draw that uses a different probability distribution; the position of that draw is irrelevant. Polynomial-time implementations that avoid restarts are also available [17]. Assume for the rest of this derivation that $\alpha \geq 1$. For any $D \subseteq N'$, define its *correction factor*

$$c_p(D) = \sum_{i \in D} (1 - p_i). \tag{4}$$

If $S \in \binom{N'}{\alpha}$, then

$$c_p(S) = \sum_{j \in N' \setminus S} p_j,$$

because $|S| = \sum_{i \in N'} p_i = \alpha$. Thus $c_p(S)$ is both the total amount by which the selected parties are rounded up and the fractional mass left outside S.

The distinguished draw produces the following correction. Fix a possible output S. If $i \in S$ is the outcome of that draw, its contribution, apart from factors common to every output, is

$$p_i \prod_{j \in S \setminus \{i\}} w_j = (1 - p_i) \prod_{j \in S} w_j.$$

The remaining elements can occur in any order, but the number of orderings is the same for every S and cancels when we condition on distinct draws. Summing over the possible outcomes of the distinguished draw gives

$$\sum_{i \in S} p_i \prod_{j \in S \setminus \{i\}} w_j = c_p(S) \prod_{j \in S} w_j.$$

Thus the distinguished draw multiplies the conditioned independent-rounding weight by exactly the correction factor $c_p(S)$. The restart construction therefore gives, for $S \in \binom{N'}{\alpha}$,

$$\Pr_p[r^{\mathrm{Samp}}(p) = S] = \frac{c_p(S) \prod_{i \in S} p_i \prod_{j \in N' \setminus S} (1 - p_j)}{\sum_{R \in \binom{N'}{\alpha}} c_p(R) \prod_{i \in R} p_i \prod_{j \in N' \setminus R} (1 - p_j)}. \tag{5}$$

For $\alpha \geq 1$, this formula covers every residue vector in $[0, 1)^n$: zero-residue parties have already been omitted. The case $\alpha = 0$ was defined separately. One-sided limits in which a residue approaches one are treated in the proof of Theorem 1; see also Correa et al. [13, Equation (2)].

The correction factor restores the prescribed marginals. To see this, fix a party $k \in N'$ and use the equivalent raw weights $c_p(S) \prod_{i \in S} w_i$. Dividing the total weight of sets containing k by w_k gives

$$\begin{aligned}
&\frac{1}{w_k} \sum_{\substack{S \in \binom{N'}{\alpha} \\ k \in S}} c_p(S) \prod_{i \in S} w_i \\
&= \sum_{D \in \binom{N' \setminus \{k\}}{\alpha - 1}} \left(\sum_{\ell \in N' \setminus (D \cup \{k\})} p_\ell \right) \prod_{i \in D} w_i \\
&= \sum_{S \in \binom{N' \setminus \{k\}}{\alpha}} c_p(S) \prod_{i \in S} w_i.
\end{aligned}$$

The middle equality pairs D with a party $\ell \in N' \setminus (D \cup \{k\})$, puts $S = D \cup \{\ell\}$, and uses $p_\ell = (1 - p_\ell) w_\ell$. The final expression is the total weight of sets that omit k. Sets containing k

therefore have w_k times as much total weight as sets omitting it, and hence

$$\Pr[k \in r^{\mathrm{Samp}}(p)] = \frac{w_k}{1+w_k} = p_k.$$

Thus the correction factor created by the distinguished draw exactly repairs the marginal distortion caused by conditioning. Parties outside N' have both selection probability and prescribed marginal equal to zero, so Sampford rounding has the prescribed inclusion probabilities [26].

For example, let $p = (0.9, 0.9, 0.2)$ and $\alpha = 2$. Conditioning independent rounding gives the pairs $\{A, B\}, \{A, C\}, \{B, C\}$ odds-product weights $(81, 2.25, 2.25)$, hence probabilities $(18/19, 1/38, 1/38)$ and marginals $(37/38, 37/38, 1/19)$—not $(0.9, 0.9, 0.2)$. The three correction factors are $(0.2, 0.9, 0.9)$. Multiplying by them and normalizing gives Sampford probabilities $(0.8, 0.1, 0.1)$, whose marginals are exactly $(0.9, 0.9, 0.2)$.

3.3 Coalition reinforcement and stochastic dominance

Definition 3.3 (Stochastic dominance)**.** For nonnegative integer-valued random variables X and Y, write $X \succeq_{\mathrm{SD}} Y$ if

$$\Pr[X \geq t] \geq \Pr[Y \geq t] \quad \text{for every } t \in \mathbb{N}_0.$$

Equivalently, $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for every nondecreasing function u for which the expectations exist. This is also called first-order stochastic dominance.

For a coalition $T \subseteq N$, define its random seat total under method $\mathcal{A}$ by

$$X_T(v, h) = \sum_{i \in T} \mathcal{A}(v, h)_i. \tag{6}$$

Definition 3.4 (Threshold monotonicity [13])**.** Let $v, v' \in \mathbb{R}^n_{\geq 0}$ be two vote vectors, each with positive total, let $h \in \mathbb{N}$, and let q, q' be the corresponding vectors of fractional quotas. An apportionment method $\mathcal{A}$ satisfies *threshold monotonicity* if, for every coalition $T \subseteq N$,

$$q'_i \geq q_i \quad (i \in T), \qquad q'_j \leq q_j \quad (j \notin T) \tag{7}$$

implies

$$X_T(v', h) \succeq_{\mathrm{SD}} X_T(v, h).$$

For fixed q, q', if $G = \{i : q'_i > q_i\}$ and $Z = \{i : q'_i = q_i\}$, the qualifying coalitions are exactly those satisfying $G \subseteq T \subseteq G \cup Z$; in particular, if $Z = \varnothing$, the only one is $T = G$. The house size h is fixed in the comparison, but the residual seat counts $\alpha(q)$ and $\alpha(q')$ need not coincide, as Example 1.1 illustrates.

The Sampford method depends on (v, h) only through the vector of fractional quotas. Accordingly, we also write $X_T(q)$ for the random variable in (6).

Under Sampford rounding,

$$X_T(q) = \sum_{i \in T} \lfloor q_i \rfloor + |r^{\mathrm{Samp}}(p) \cap T|.$$

Thus, while the lower quotas are fixed, the only random part of the coalition's seat total is its number of members in the residual sample.

4 Threshold monotonicity of the Sampford method

In this section, we prove Theorem 1.

Theorem 1 (Sampford threshold monotonicity)**.** *The Sampford method satisfies threshold monotonicity.*

Between integer crossings, lower quotas are fixed. We will show that the partial derivative of an upper-tail probability with respect to a party's residue is nonnegative for a party in the coalition and nonpositive for a party outside it. Along the straight reinforcement path, coalition residues rise and outside residues fall, so these signs make every upper-tail probability nondecreasing.

4.1 Adjacent product-weight sample sizes

Although the Sampford method itself includes the correction factor $c_p(S)$, our proof uses auxiliary fixed-size samples whose probabilities are proportional only to the product of their odds weights. We need the following elementary fact about such samples.

Lemma 4.1. *Let G be a finite set with positive weights. For $0 \le s \le |G|$, let H_s be an s-subset drawn with probability proportional to the product of its members' weights; in particular, $H_0 = \varnothing$. Then, for every $U \subseteq G$ and $1 \le s \le |G|$,*

$$|H_s \cap U| \ \preceq_{\mathrm{SD}} \ |H_{s-1} \cap U| + 1.$$

Proof. If U is empty, the conclusion is immediate. Assume from now on that U is nonempty. For $A \subseteq G$, let $e_\ell(A)$ be the sum of the products of the weights over all ℓ-subsets of A, with $e_0(A) = 1$. Then

$$\Pr[|H_s \cap U| = \ell] = \frac{e_\ell(U) e_{s-\ell}(G \setminus U)}{e_s(G)}.$$

On the common support of $|H_s \cap U| - 1$ and $|H_{s-1} \cap U|$,

$$\frac{\Pr[|H_s \cap U| - 1 = \ell]}{\Pr[|H_{s-1} \cap U| = \ell]} = \frac{e_{s-1}(G)}{e_s(G)} \frac{e_{\ell+1}(U)}{e_\ell(U)}.$$

For $1 \le \ell \le |U| - 1$, Newton's inequalities give

$$e_\ell(U)^2 \ge e_{\ell-1}(U) e_{\ell+1}(U),$$

so $e_{\ell+1}(U)/e_\ell(U)$ is nonincreasing wherever the ratio is defined. The two probability masses therefore cross at most once: the first may exceed the second only at lower values. Any mass outside their common support lies below it for $|H_s \cap U| - 1$ and above it for $|H_{s-1} \cap U|$. Since both distributions have total mass one, every upper tail of the first is no larger than the corresponding upper tail of the second. This proves the lemma. □

4.2 How one residue changes a threshold probability

We first work with a fixed set of fractional parties and at least two residual seats. The cases with at most one residual seat and the integer crossings themselves are handled in the final subsection.

For the partial derivatives below, extend the Sampford probabilities off the hyperplane where the residues sum to α by assigning each $S \in \binom{N'}{\alpha}$ probability proportional to $c_p(S) \prod_{x \in S} p_x/(1 - p_x)$. This gives a smooth distribution for every $p \in (0,1)^{N'}$ and agrees with Sampford rounding whenever $\sum_{x \in N'} p_x = \alpha$.

Lemma 4.2. *Let $N' \subseteq N$, let $p \in (0,1)^{N'}$ satisfy*

$$\sum_{x \in N'} p_x = \alpha \geq 2$$

for an integer α, and let $T \subseteq N$. For every integer $k \geq 0$,

$$\frac{\partial}{\partial p_i} \Pr_p[|r^{\mathrm{Samp}}(p) \cap T| \geq k] \geq 0 \quad \textit{for } i \in T \cap N',$$

whereas

$$\frac{\partial}{\partial p_j} \Pr_p[|r^{\mathrm{Samp}}(p) \cap T| \geq k] \leq 0 \quad \textit{for } j \in N' \setminus T.$$

Proof. All derivatives are evaluated at the feasible vector. We use $\sum_{x \in N'} p_x = \alpha$ only after differentiation.

It is enough to prove the first inequality for a nonconstant threshold. Fix $i \in T \cap N'$, remove i, and draw an $(\alpha - 1)$-subset H of $G = N' \setminus \{i\}$ with probability proportional to the product of its members' odds. Write $Y = |H \cap T|$ for its coalition count.

Separating outcomes according to whether i is selected. Divide every aggregate raw weight below by the product-weight normalizer used for H. If p_i is varied to t while the other residues are held fixed, then, for a possible value D of H, the raw weight of $D \cup \{i\}$ is

$$\left(\frac{t}{1-t} c_p(D) + t\right) \prod_{x \in D} w_x.$$

Consequently the aggregate weights of all outcomes containing i and of the threshold outcomes containing i are, respectively,

$$\frac{t}{1-t}\mathbb{E}[c_p(H)] + t \quad \text{and} \quad \frac{t}{1-t}\mathbb{E}[c_p(H)\mathbf{1}\{Y \geq k-1\}] + t \Pr[Y \geq k-1]. \tag{8}$$

The weights of outcomes omitting i do not depend on t. Their values at the feasible vector follow from

$$c_p(S) \prod_{x \in S} w_x = \sum_{x \in S} p_x \prod_{y \in S \setminus \{x\}} w_y. \tag{9}$$

After writing $H = S \setminus \{x\}$, the aggregate weight of all such outcomes is

$$\mathbb{E}\left[\sum_{x \in G \setminus H} p_x\right] = \mathbb{E}[c_p(H)] + 1 - p_i,$$

where the equality uses feasibility. For the threshold outcomes, the same reindexing gives

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x \in G \setminus H} p_x \mathbf{1}\{Y + \mathbf{1}\{x \in T\} \geq k\}\right] \\
&\quad = \mathbb{E}[c_p(H)\mathbf{1}\{Y \geq k-1\}] + (1-p_i)\Pr[Y \geq k-1] - \mathbb{E}\left[\mathbf{1}\{Y = k-1\} \sum_{\substack{x \in G \setminus H \\ x \notin T}} p_x\right]. \qquad (10)
\end{aligned}$$

The final term removes precisely the outside completions of a partial set that is one coalition member short.

The weights of outcomes omitting i contain no t. Feasibility was used only to rewrite their constant values at the base vector in the forms above. Adding the containing and omitting weights and then setting $t = p_i$ shows that the total raw weight and its derivative are

$$\frac{\mathbb{E}[c_p(H)] + 1 - p_i}{1 - p_i} \quad \text{and} \quad \frac{\mathbb{E}[c_p(H)]}{(1 - p_i)^2} + 1, \tag{11}$$

whereas the threshold raw weight and its derivative are

$$\begin{aligned} &\frac{\mathbb{E}[c_p(H)\mathbf{1}\{Y \geq k-1\}]}{1 - p_i} + \Pr[Y \geq k-1] - \mathbb{E}\left[\mathbf{1}\{Y = k-1\} \sum_{\substack{x \in G \setminus H \\ x \notin T}} p_x\right], \\ \text{and} \qquad &\frac{\mathbb{E}[c_p(H)\mathbf{1}\{Y \geq k-1\}]}{(1 - p_i)^2} + \Pr[Y \geq k-1]. \end{aligned} \tag{12}$$

The quotient rule now gives the exact derivative

$$\begin{aligned} &\frac{\partial}{\partial p_i} \Pr_p[|r^{\mathrm{Samp}}(p) \cap T| \geq k] = \\ &= \frac{(\mathbb{E}[c_p(H)] + (1-p_i)^2)\mathbb{E}\left[\mathbf{1}\{Y = k-1\} \sum_{\substack{x \in G \setminus H \\ x \notin T}} p_x\right] + p_i \operatorname{Cov}(c_p(H), \mathbf{1}\{Y \geq k-1\})}{\left(\mathbb{E}[c_p(H)] + 1 - p_i\right)^2}. \end{aligned} \tag{13}$$

Only the covariance in this formula can be negative.

Accounting for the correction factor. Expanding $c_p(H) = \sum_{x \in H}(1 - p_x)$ gives

$$\operatorname{Cov}(c_p(H), \mathbf{1}\{Y \geq k-1\}) = \sum_{x \in G} (1 - p_x) \Pr[x \in H] \left(\Pr[Y \geq k-1 \mid x \in H] - \Pr[Y \geq k-1]\right). \tag{14}$$

Conditional on $x \notin H$, the set H is a product-weighted $(\alpha - 1)$-set on $G \setminus \{x\}$. Conditional on $x \in H$, the set $H \setminus \{x\}$ is a product-weighted $(\alpha - 2)$-set on the same ground set. These are the two adjacent sizes in Lemma 4.1. When $x \in T$, the latter sample also comes with the fixed coalition member x, so its summand in (14) is nonnegative. When $x \notin T$, the same lemma says

$$\Pr[Y \geq k-1 \mid x \in H] \geq \Pr[Y \geq k \mid x \notin H].$$

Therefore

$$\begin{aligned} &\Pr[Y \geq k-1] - \Pr[Y \geq k-1 \mid x \in H] \\ &\quad = \Pr[x \notin H] \left(\Pr[Y \geq k-1 \mid x \notin H] - \Pr[Y \geq k-1 \mid x \in H]\right) \\ &\quad \leq \Pr[Y = k-1,\ x \notin H]. \end{aligned} \tag{15}$$

Thus an outside party can contribute negatively only when the partial sample has exactly $k - 1$ coalition members.

We also need the following coefficient bound, which follows from feasibility. For every $x \in G$,

$$\mathbb{E}[c_p(H)] = \frac{\sum_{J \in \binom{G}{\alpha-2}} \left(\prod_{z \in J} w_z\right) \sum_{r \in G \setminus J} p_r}{\sum_{D \in \binom{G}{\alpha-1}} \prod_{z \in D} w_z} \geq \frac{\sum_{J \in \binom{G \setminus \{x\}}{\alpha-2}} \prod_{z \in J} w_z}{\sum_{D \in \binom{G}{\alpha-1}} \prod_{z \in D} w_z} = \frac{1-p_x}{p_x} \Pr[x \in H]. \tag{16}$$

The first equality expands $c_p(H)$ party by party, removes that party from H, and uses $(1-p_r)w_r = p_r$. For every $J \in \binom{G}{\alpha-2}$, we have

$$\sum_{r \in G \setminus J} p_r = c_p(J) + 2 - p_i \geq 1,$$

which proves the inequality. Equivalently,

$$(1-p_x)\Pr[x \in H] \leq p_x \mathbb{E}[c_p(H)].$$

Thus, for each outside party, the magnitude of its negative summand in (14) is at most

$$\mathbb{E}[c_p(H)] p_x \Pr[Y = k-1,\ x \notin H].$$

Summing this bound over the outside parties gives

$$\operatorname{Cov}(c_p(H), \mathbf{1}\{Y \geq k-1\}) \geq -\mathbb{E}[c_p(H)]\mathbb{E}\left[\mathbf{1}\{Y = k-1\} \sum_{\substack{x \in G \setminus H \\ x \notin T}} p_x\right]. \tag{17}$$

The expectation in (17) is the same boundary term that appears positively in (13). Substitution leaves

$$\frac{\partial}{\partial p_i} \Pr_p[|r^{\mathrm{Samp}}(p) \cap T| \geq k] \geq \frac{1-p_i}{\mathbb{E}[c_p(H)] + 1 - p_i} \mathbb{E}\left[\mathbf{1}\{Y = k-1\} \sum_{\substack{x \in G \setminus H \\ x \notin T}} p_x\right] \geq 0. \tag{18}$$

Finally, let $j \in N' \setminus T$. If $k > \alpha$, the event in the lemma is impossible, so its derivative is zero. If $k \leq \alpha$, apply the inequality just proved to the complementary coalition. Since every residual sample has size α,

$$\Pr_p[|r^{\mathrm{Samp}}(p) \cap T| \geq k] = 1 - \Pr_p[|r^{\mathrm{Samp}}(p) \cap (N' \setminus T)| \geq \alpha - k + 1].$$

Increasing p_j makes the probability after the minus sign nondecreasing, so the derivative of the original upper tail is nonpositive. □

4.3 Proof of Theorem 1

Figure 2 illustrates the interval decomposition used below and the corresponding coalition upper-tail probabilities along one representative path.

We divide the straight path of fractional quotas at its integer crossings. On each resulting open interval, the set of parties with nonintegral fractional quotas is fixed, so the derivative calculation applies there. We then prove continuity of the complete allocation at the crossings and join the interval comparisons. No derivative at a crossing is needed.

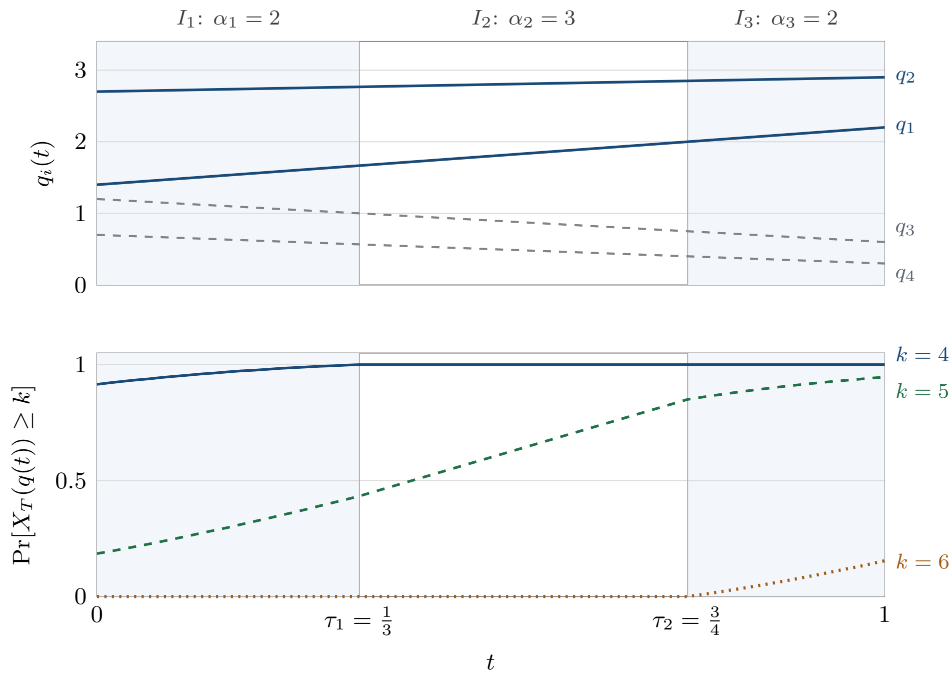


Figure 2: The straight path of fractional quotas $q(t) = (1-t)q + tq'$ for four parties, $h = 6$, coalition $T = \{1, 2\}$, $q = (1.4, 2.7, 1.2, 0.7)$ and $q' = (2.2, 2.9, 0.6, 0.3)$; the fractional quotas of coalition members are solid, and the fractional quotas of outsiders are dashed. Top: the two interior breakpoints τ_1 and τ_2 are the times at which a fractional quota crosses an integer; between them the lower quotas, the set of fractional parties and the number α_ℓ of residual seats are fixed. Bottom: the Sampford upper-tail probabilities $F_k(t) = \Pr[X_T(q(t)) \geq k]$, computed directly from (5), for three thresholds. Each F_k is nondecreasing on every open interval by Lemma 4.2 and continuous at the breakpoints, which together give $F_k(1) \geq F_k(0)$.

Proof of Theorem 1. The statement is immediate when $T = \varnothing$ or $T = N$, so assume that both T and its complement are nonempty. Let q, q' satisfy (7), set

$$q(t) = (1-t)q + tq' \qquad (0 \leq t \leq 1),$$

and fix a complete-seat threshold $k \in \mathbb{N}_0$. Write

$$F_k(t) = \Pr[X_T(q(t)) \geq k].$$

We will prove that $F_k(1) \geq F_k(0)$.

Let

$$0 = \tau_0 < \tau_1 < \cdots < \tau_M = 1$$

consist of the endpoints together with all distinct interior times at which a nonconstant coordinate of $q(t)$ is an integer. On

$$I_\ell = (\tau_{\ell-1}, \tau_\ell),$$

the lower quotas are fixed. The set

$$N_\ell = \{i \in N : q_i(t) \notin \mathbb{Z}\}, \qquad t \in I_\ell,$$

is also fixed, and

$$p(t) = \big(q_i(t) - \lfloor q_i(t) \rfloor\big)_{i \in N_\ell} \in (0,1)^{N_\ell}.$$

Parties outside N_ℓ have constant integral values of $q_i(t)$ on this interval and receive only their deterministic lower-quota seats. Moreover,

$$\alpha_\ell = \sum_{i \in N_\ell} p_i(t)$$

is a fixed integer on I_ℓ.

We first show that F_k is nondecreasing on I_ℓ. The residual threshold

$$\kappa_\ell = k - \sum_{i \in T} \lfloor q_i(t) \rfloor$$

is fixed there. If $\kappa_\ell \le 0$ or $\kappa_\ell > \alpha_\ell$, the corresponding upper tail is constant. In particular, this covers every case when $\alpha_\ell = 0$. If $\alpha_\ell = 1$, the only nonconstant case is $\kappa_\ell = 1$, when

$$F_k(t) = \sum_{i \in T \cap N_\ell} p_i(t).$$

This is nondecreasing because each fractional quota in T is nondecreasing.

Suppose now that $\alpha_\ell \ge 2$. The normalized raw-weight formula is smooth throughout $(0,1)^{N_\ell}$. The chain rule and Lemma 4.2, applied on the ground set N_ℓ, give

$$F_k'(t) = \sum_{i \in T \cap N_\ell} \dot{p}_i(t) \frac{\partial}{\partial p_i} \Pr[|r^{\mathrm{Samp}}(p(t)) \cap T| \ge \kappa_\ell] + \sum_{j \in N_\ell \setminus T} \dot{p}_j(t) \frac{\partial}{\partial p_j} \Pr[|r^{\mathrm{Samp}}(p(t)) \cap T| \ge \kappa_\ell] \ge 0.$$

Indeed, residues inside T are nondecreasing, residues outside T are nonincreasing, and the two partial derivatives have the corresponding signs. Thus F_k is nondecreasing on every open interval I_ℓ.

It remains to join these comparisons at the breakpoints. Approach one such breakpoint from an adjacent interval I_ℓ, whose fixed set of fractional parties is N_ℓ. If the interior residue vector converges to $\bar{p} \in [0,1]^{N_\ell}$, then

$$r^{\mathrm{Samp}}(p(t)) \xrightarrow{d} \{x \in N_\ell : \bar{p}_x = 1\} \cup r^{\mathrm{Samp}}\big((\bar{p}_x)_{x \in N_\ell :\, 0 < \bar{p}_x < 1}\big). \tag{19}$$

Here $\bar{p}$ is only a one-sided limiting vector; an actual residue vector still lies in $[0,1)^n$. If $\bar{p}_i = 0$, then party i's inclusion probability tends to zero, so party i is absent from the limiting sample.

When fractional coordinates remain in the limit, (19) follows from (5): every set of positive limiting weight contains all unit coordinates and no zero coordinate, and its remaining factor is exactly the Sampford weight on the fractional coordinates. Their residues sum to

$$\alpha_\ell - \big|\{x \in N_\ell : \bar{p}_x = 1\}\big|,$$

an integer strictly between zero and their number, so the reduced Sampford sampling has a positive normalizer. If $\bar{p}$ is a zero–one vector, the prescribed marginals give, for every interior feasible sequence $p^{(m)} \to \bar{p}$,

$$\Pr\Big[r^{\mathrm{Samp}}(p^{(m)}) \ne \{x \in N_\ell : \bar{p}_x = 1\}\Big] \le \sum_{\substack{x \in N_\ell \\ \bar{p}_x = 1}} (1 - p_x^{(m)}) + \sum_{\substack{x \in N_\ell \\ \bar{p}_x = 0}} p_x^{(m)} \longrightarrow 0.$$

A fractional quota whose residue tends to one approaches the breakpoint integer from below. Its lower quota on the interval is one less than at the breakpoint, while its residual seat becomes certain.

A fractional quota whose residue tends to zero approaches the breakpoint integer from above, has the same lower quota as at the breakpoint, and is omitted in the limit. For the coordinates that remain fractional, the limiting distribution of the selected subset is precisely the distribution obtained by applying the Sampford method to their residues at the breakpoint. These limits hold jointly for all coordinates crossing an integer. At every interior breakpoint, the complete allocation law therefore converges from both sides to the law at the breakpoint; at each endpoint it converges from the one adjacent interval. Hence $F_k(t)$ is continuous on $[0, 1]$.

Continuity extends each open-interval comparison to its endpoints. Concatenating them yields $F_k(1) \geq F_k(0)$. Since this holds for every $k \in \mathbb{N}_0$, we obtain

$$X_T(q') \succeq_{\mathrm{SD}} X_T(q),$$

as required. □

The two extreme thresholds have useful interpretations for the residual lottery.

Corollary 4.3 (Extreme threshold events)**.** *Let $p, p' \in [0, 1)^n$ have the same integer coordinate sum, and let $T \subseteq N$. If*

$$p'_i \geq p_i \quad (i \in T), \qquad p'_j \leq p_j \quad (j \notin T),$$

then

$$\Pr[T \subseteq r^{\mathrm{Samp}}(p')] \geq \Pr[T \subseteq r^{\mathrm{Samp}}(p)], \qquad \Pr[r^{\mathrm{Samp}}(p') \subseteq T] \geq \Pr[r^{\mathrm{Samp}}(p) \subseteq T].$$

Proof. If the common coordinate sum is zero, both selected sets are empty and the claim is immediate. Otherwise, let the common sum be α and apply Theorem 1 to the vectors of fractional quotas $q = p$ and $q' = p'$. They are realized by house size $h = \alpha$ and vote vectors $v = p$ and $v' = p'$. The theorem therefore orders the two coalition counts in stochastic dominance. If $|T| \leq \alpha$, selecting every member of T is their upper-tail event at threshold $|T|$; if $|T| > \alpha$, that event is impossible. If $|T| \geq \alpha$, selecting no outsider is their upper-tail event at threshold α; if $|T| < \alpha$, that event is impossible. □

The first inequality says that the probability of selecting every coalition member cannot fall; the second says the same about selecting no outsider. These are *joint-inclusion monotonicity* and *joint-exclusion monotonicity*, respectively. When $|T|$ equals the residual sample size, both events say that the selected set is exactly T. Thus either inequality recovers Sampford's known selection-monotonicity guarantee, while Theorem 1 also controls every intermediate coalition count.

5 Limits of pairwise threshold monotonicity

Theorem 1 concerns one reinforced coalition and requires every party outside it to weaken. Pairwise threshold monotonicity instead tests one gaining and one losing coalition while allowing all remaining quotas to move in either direction; it requires at least one of the two coalition distributions to move as prescribed. Even when the tested coalitions must be disjoint, this requirement is incompatible with quota and ex-ante proportionality from seven parties onward. The impossibility applies to every such apportionment method, not only Sampford, and follows from support constraints in a two-seat problem.

Definition 5.1 (Disjoint pairwise threshold monotonicity)**.** Let $v, v' \in \mathbb{R}^n_{\geq 0}$ be two vote vectors, each with positive total, let $h \in \mathbb{N}$, and let q, q' be their vectors of fractional quotas. An apportionment method satisfies *pairwise threshold monotonicity for disjoint coalitions* if, for every pair of disjoint coalitions $A, B \subseteq N$ satisfying

$$q'_i \geq q_i \quad (i \in A), \qquad q'_j \leq q_j \quad (j \in B), \tag{20}$$

at least one of

$$X_A(v', h) \succeq_{\mathrm{SD}} X_A(v, h) \qquad \text{or} \qquad X_B(v, h) \succeq_{\mathrm{SD}} X_B(v', h) \tag{21}$$

holds. The same comparison in (21) must hold at every threshold; the selected branch cannot depend on the threshold.

The word "pairwise" refers to a pair of coalitions, not to pairwise correlation between parties. Definition 5.1 is weaker than the unrestricted pairwise axiom of Correa et al. [13], because it tests only disjoint coalitions. It nevertheless contains ordinary threshold monotonicity: take $B = N \setminus A$. Since $X_B = h - X_A$, the two alternatives in (21) are then equivalent.

An apportionment method $\mathcal{A}$ has *full support* if, for every instance (v, h) with $\sum_i v_i > 0$ and every apportionment x that gives each party either its lower or upper quota, $\Pr[\mathcal{A}(v, h) = x] > 0$. Theorem 2 does not require full support.

Theorem 2 (Pairwise impossibility with or without full support)**.** *For every $n \geq 7$, no randomized apportionment method on all strictly positive vote profiles simultaneously satisfies quota, ex-ante proportionality, and pairwise threshold monotonicity for disjoint coalitions. The contradiction already uses house size two, vectors of fractional quotas strictly between zero and one, and a comparison in which every member of the first coalition strictly gains while every member of the second strictly loses.*

Set $h = 2$ and let $q \in (0, 1)^n$ satisfy $\sum_i q_i = 2$. Quota then makes every outcome a two-party set $S_q \in \binom{N}{2}$, and ex-ante proportionality becomes

$$\Pr_q[i \in S_q] = q_i \qquad (i \in N). \tag{22}$$

Here and below, the subscript q means that the method is evaluated at the specific instance $(v, h) = (q, 2)$; no homogeneity or quota-only assumption is being made. For a pair $C = \{i, j\}$, since $X_C = |S_q \cap C| \in \{0, 1, 2\}$,

$$\Pr_q[X_C \geq 2] = \Pr_q[S_q = C], \qquad \Pr_q[X_C \geq 1] = q(C) - \Pr_q[S_q = C], \tag{23}$$

where $q(C) = \sum_{i \in C} q_i$. Thus, when a pair's mean is fixed, an increase in its joint-selection probability improves its threshold-two probability but worsens its threshold-one probability.

This two-seat problem has a natural weighted-graph representation. The parties are vertices, and each possible allocation $C \in \binom{N}{2}$ is an edge with weight $\Pr_q[S_q = C]$. Equation (22) says that the total weight of the edges incident to vertex i is exactly q_i. After zero-weight edges are removed, the remaining edges form the *support graph*. We call an edge family *intersecting* when every two of its edges share a vertex.

The contradiction is encoded by the support graph on the small parties. Pairwise threshold monotonicity forces this graph to be intersecting, but every intersecting edge family lies in a star or a triangle, and neither form can carry the prescribed weighted degrees. To force intersection, make party 1 almost certain. Jointly selecting either tested pair then becomes almost impossible:

the growing pair fails at threshold two, while the shrinking pair fails in the opposite direction at threshold one.

Lemma 5.2 (Two disjoint small-party pairs cannot both occur)**.** *Suppose a quota and ex-ante proportional method satisfies Definition 5.1. Fix $n \geq 7$ and consider the two-seat profile of fractional quotas*

$$p = \left(\frac{3}{5}, \frac{7}{5(n-1)}, \dots, \frac{7}{5(n-1)}\right). \tag{24}$$

Then the support graph induced by $\{2, \dots, n\}$ is intersecting: there cannot be two disjoint pairs $A, B \subseteq \{2, \dots, n\}$ with $\Pr_p[S_p = A] > 0$ and $\Pr_p[S_p = B] > 0$.

Proof. Put $m = n - 1$ and $b = 7/(5m)$, so that the profile in (24) is $(3/5, b, \dots, b)$. Suppose otherwise. Choose

$$0 < \varepsilon < \min\left\{\Pr_p[S_p = A], \frac{\Pr_p[S_p = B]}{2}, \frac{1}{2}\right\},$$

and then choose

$$0 < \delta < \min\left\{b, 1-b, \frac{\Pr_p[S_p = B] - \varepsilon}{2}\right\}.$$

Let $R = \{2, \dots, n\} \setminus (A \cup B)$ and put

$$\rho = \frac{1 + \varepsilon - 4b}{m - 4}.$$

Because $m \geq 6$, we have $1 - 4b > 0$ and $m - 4 \geq 2$; the choice $\varepsilon < 1/2$ therefore gives $0 < \rho < 1$. The following table displays the comparison between p and the perturbed vector of fractional quotas p':

party or block	fractional quota under p	fractional quota under p'
1	$3/5$	$1-\varepsilon$
each $i \in A$	b	$b+\delta$
each $i \in B$	b	$b-\delta$
each $i \in R$	b	ρ

All entries of p' are strictly between zero and one and sum to two. Every member of A strictly gains and every member of B strictly loses.

At the new profile, selecting either pair A or pair B requires omitting party 1. Exact proportionality therefore gives

$$\Pr_{p'}[S_{p'} = A],\ \Pr_{p'}[S_{p'} = B] \leq \Pr_{p'}[1 \notin S_{p'}] = \varepsilon. \tag{25}$$

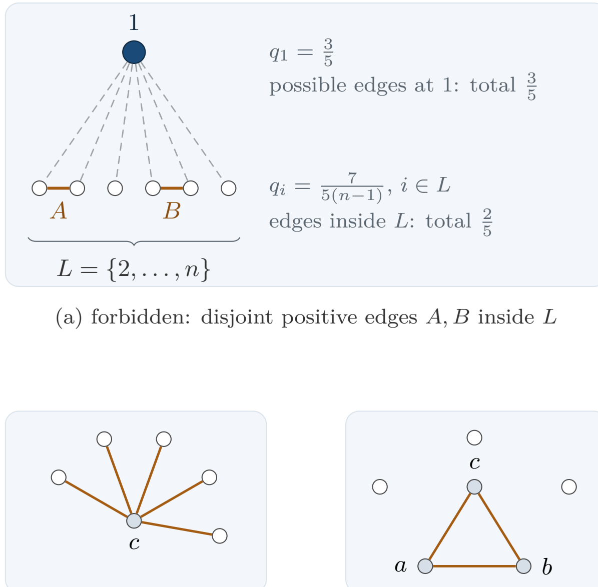
The first branch of (21) fails at threshold two because

$$\Pr_{p'}[X_A \geq 2] = \Pr_{p'}[S_{p'} = A] \leq \varepsilon < \Pr_p[S_p = A] = \Pr_p[X_A \geq 2].$$

The second branch would require the old seat total of B to dominate the new one. But (23) and (25) give

$$\begin{aligned}\Pr_{p'}[X_B \geq 1] &= 2b - 2\delta - \Pr_{p'}[S_{p'} = B] \\ &\geq 2b - 2\delta - \varepsilon \\ &> 2b - \Pr_p[S_p = B] \\ &= \Pr_p[X_B \geq 1].\end{aligned}$$

Thus the second branch fails at threshold one, a contradiction. □



(a) forbidden: disjoint positive edges A, B inside L

(b) star: $q_c \geq \frac{2}{5} > \frac{7}{5(n-1)}$

(c) triangle: $q_a + q_b + q_c \geq \frac{4}{5} > \frac{21}{5(n-1)}$

Figure 3: The two-seat support graph for $n = 7$, where edge weights are allocation probabilities and weighted degrees are fractional quotas. By Lemma 5.2, the positive edges inside L are intersecting and hence, by Lemma 5.3, form a star or lie in a triangle. Their total weight is $2/5$, so a star forces one marginal to be at least $2/5$, while a triangle forces three marginals to sum to at least $4/5$. Both contradict the prescribed small-party fractional quotas. Dashed lines denote possible edges incident to party 1.

Lemma 5.2 makes the small-party support graph intersecting. Figure 3 summarizes the forbidden configuration and the two possible shapes of an intersecting edge family. The next elementary lemma gives the classification.

Lemma 5.3 (Intersecting edge families)**.** *If every two edges in a simple graph intersect, then all of its edges are incident to one common vertex or all of its edges lie in one triangle.*

Proof. The claim is immediate if the graph has at most one edge. Otherwise, take two distinct intersecting edges and label them ab and ac. Every edge not incident to a must intersect both of them and hence must be bc. If bc is absent, every edge uses a. If bc is present, any further edge must meet all three of ab, ac, and bc, so it lies in the triangle on $\{a, b, c\}$. $\square$

Proof of Theorem 2. Fix $n \geq 7$ and let p be the profile in (24). Since $\sum_i p_i = 2$, the instance $(v, h) = (p, 2)$ has vector of fractional quotas p. Consider the support graph introduced above, restricted to the small-party block $L = \{2, \ldots, n\}$. Lemma 5.2 says that this graph is intersecting.

The total weight of its edges is

$$\Pr_p[S_p \subseteq L] = \Pr_p[1 \notin S_p] = 1 - p_1 = \frac{2}{5}.$$

By Lemma 5.3, the positive edges lie in a star or a triangle. In the star case, the center is selected in every event whose edge lies entirely in L, so its marginal is at least $2/5$. This contradicts its prescribed marginal $7/[5(n-1)] < 2/5$. In the triangle case, every event counted among the small-party edges selects two of the triangle's three vertices. Their marginal probabilities therefore sum to at least 4/5, whereas exact proportionality makes their sum $21/[5(n-1)] < 4/5$. This is again a contradiction. □

Consequently, the impossibility assumes neither continuity, neutrality, nor full support. It applies to whatever positive support the method chooses: no such support can realize the prescribed inclusion probabilities.

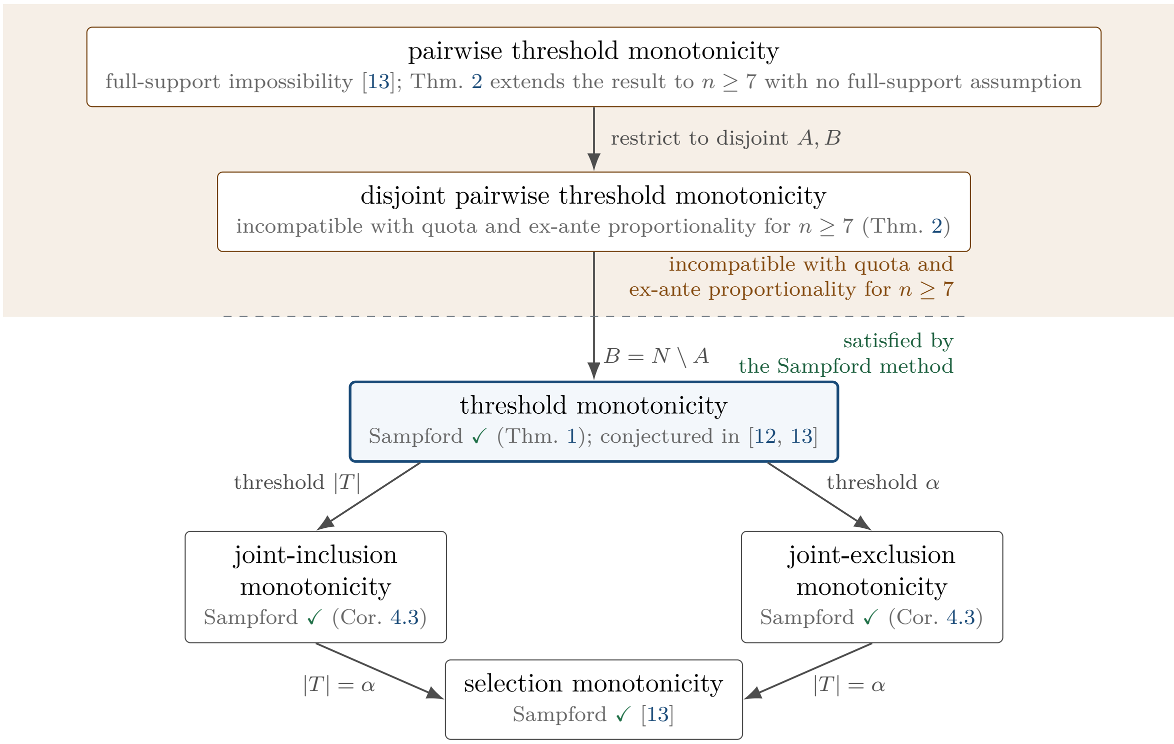


Figure 4: Implications among the monotonicity axioms; arrows point from stronger to weaker requirements and are labelled by the relevant specialization. The shaded axioms are incompatible with quota and ex-ante proportionality (Theorem 2), whereas the Sampford method satisfies the unshaded requirements (Theorem 1, Corollary 4.3, and 13). Joint-inclusion, joint-exclusion, and selection monotonicity concern the residual selection lottery. The implications from threshold monotonicity to the residual axioms use the standard embedding $q = p$ and $h = \alpha$.

6 Conclusions

Our two results locate threshold monotonicity within randomized apportionment (Figure 4). The first is positive. Quota, ex-ante proportionality, and threshold monotonicity for arbitrary coalitions are compatible. Sampford sampling, a classical method, satisfies all three. This settles the conjecture of Correa et al. [12, 13]. On the negative side, the pairwise strengthening fails from seven parties onward, even when restricted to disjoint coalitions. This impossibility applies whether or not the method has full support.

Our study on randomized apportionment sits at an exciting intersection of statistics and axiomatic social choice. We have barely scratched the surface of the axiomatic landscape. Pertinent to our results, several questions remain. For example, are there other natural rounding rules that satisfy threshold monotonicity? Finally, can threshold monotonicity be simultaneously achieved alongside house monotonicity?

Declaration on the use of generative AI

Both results in the paper were derived via extensive and interactive use of ChatGPT (Sol 5.6, OpenAI). The authors verified all arguments; simplified, framed and wrote the results; and take full responsibility for the manuscript.

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