Pairwise Maximin Share Allocations Need Not Exist

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Abstract

We study the fair allocation of indivisible goods among agents with strictly positive additive valuations. Pairwise maximin share fairness (PMMS) asks that, for every ordered pair of agents, the first agent value her own bundle at least as highly as the best worst-case share she could secure by repartitioning the two agents' combined bundles into two parts. Whether a PMMS allocation always exists for positive additive valuations has remained open since the notion was introduced. We resolve this question in the negative. We construct an instance with four agents and strictly positive additive valuations that admits no complete PMMS allocation.

1 Introduction

Cut-and-choose is the canonical protocol for dividing a resource between two agents. One agent partitions the resource into two parts and the other agent chooses first. The cutter can therefore guarantee the value of her best balanced two-way partition. For divisible resources this idea underlies envy-free cake cutting. With indivisible goods, exact envy-freeness can fail for the elementary reason that a single valuable good cannot be split.

Pairwise maximin share (PMMS) fairness imports the robust part of cut-and-choose into a multi-agent allocation. Fix two agents i and j and pool the bundles they receive. Agent i asks what value she could guarantee if she repartitioned that pool into two bundles and received the less valuable one. An allocation is PMMS if i's actual bundle meets this benchmark for every other agent j (Caragiannis et al., 2019).

This definition is appealing for several reasons. First, it has the direct procedural interpretation of pairwise cut-and-choose. Second, it treats every pair symmetrically while evaluating fairness from each agent's own cardinal perspective. Third, for strictly positive additive valuations, PMMS implies EFX and hence also EF1. PMMS is thus a particularly demanding local fairness benchmark, yet one that is meaningful even when exact envy-freeness is impossible. Considering that the existence of an EFX allocation remains an open problem for positive additive valuations, it also motivates to settle the guaranteed existence of allocations satisfying the stronger PMMS concept.

The strength of PMMS has made its existence difficult to understand. Exact PMMS is immediate for two agents and for identical valuations, but the general additive problem remained unresolved. Work therefore concentrated on multiplicative approximations, culminating in a 0.781 existential guarantee for additive valuations (Kurokawa, 2017). Exact existence was subsequently established in several structured domains, while a recent counterexample showed failure for monotone valuations when two of the three valuations are nonadditive (Byrka et al., 2026). None of these results settled the additive case.

Our contribution. We prove that a complete PMMS allocation need not exist even when every agent has a strictly positive additive valuation.

Theorem 1 (Main result)**.** *There exists an instance with four agents and strictly positive additive valuations for which no complete PMMS allocation exists.*

Theorem 1 resolves the exact-existence question for additive valuations in the negative. Strict positivity is worth emphasizing: the obstruction is not caused by dummy goods or zero-value degeneracies. Since a PMMS allocation always exists for two additive agents and the three-agent additive case remains open, four agents are the smallest number for which additive nonexistence is currently known.

The result also clarifies the relationship with EFX. Under strictly positive additive valuations, PMMS implies EFX, but the converse fails. Our theorem therefore rules out PMMS as a route to proving universal additive EFX existence; it does not provide an EFX counterexample.

2 Related Work

Origins and neighboring notions. Caragiannis et al. (2019) introduced PMMS together with EFX in their study of the fairness properties of maximum Nash welfare allocations. For a broader survey of fairness notions for indivisible goods, see Amanatidis et al. (2023).

For positive additive valuations, PMMS implies EFX. Quantitatively, Amanatidis et al. (2018) showed that every EFX allocation is $2/3$-PMMS and every EF1 allocation is $1/2$-PMMS; both implications are tight in the appropriate sense. As EF1 allocations can be computed efficiently (Lipton et al., 2004), the latter implication already gives a simple polynomial-time baseline for approximate PMMS. In the other direction, approximate PMMS alone need not give any constant EFX guarantee (Amanatidis et al., 2018). Very recent work determines that exact PMMS guarantees a tight $10/17$ fraction of the global MMS benchmark (Qin, 2026).

In the chore (additive negative valuations) setting, He and Tao (2026) prove that EFX and hence PMMS allocations need not exist for any $(n \geq 4)$.

Approximate PMMS for additive valuations. The best general existential guarantee is 0.781 (Kurokawa, 2017), improving the $(\varphi - 1) \approx 0.618$ guarantee of Nash-welfare maximization (Caragiannis et al., 2019). Simultaneous guarantees include an efficiently computable 0.618-EFX and 0.717-PMMS allocation (Amanatidis et al., 2020), and an existential allocation that is 0.618-EFX, 0.618-PMMS, and achieves 0.618 of the optimal Nash welfare (Feldman et al., 2024). Instances with range parameter γ admit efficiently computable $(5\gamma/6)$-PMMS allocations; in particular, restricted additive instances admit efficiently computable $5/6$-PMMS allocations (Barman et al., 2024). Common-ranking instances admit efficiently computable $4/5$-PMMS allocations, originally claimed by Dai et al. (2024) and subsequently proved using a nondegeneracy reduction by Feldman et al. (2026).

Exact existence and relaxations. Before the present work, no additive counterexample to exact PMMS was known. Byrka et al. (2026) constructed a three-agent instance with no PMMS allocation, but only one valuation in that instance is additive; the other two are monotone and nonadditive. The same paper proves polynomial-time existence for factored personalized bivalued valuations, for binary-valued MMS-feasible valuations, and for pair-demand valuations. In another sparse domain, exact PMMS exists when each good is valued by at most two agents (Christodoulou and Mastrakoulis, 2026).

3 Model and Definitions

Let $N = \{1, \dots, n\}$ be a finite set of agents and let $M = \{g_1, \dots, g_m\}$ be a finite set of indivisible goods. Each agent $i \in N$ has a valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$, normalized so that $v_i(\varnothing) = 0$.

The valuation v_i is *additive* if $v_i(S) = \sum_{g \in S} v_i(g)$ for every $S \subseteq M$, where $v_i(g)$ abbreviates $v_i(\{g\})$. It is *strictly positive* if $v_i(g) > 0$ for every $g \in M$. Unless stated otherwise, all valuations in our counterexample are additive and strictly positive.

An *allocation* is a tuple $A = (A_1, \dots, A_n)$ of pairwise disjoint bundles. It is *complete* if $\bigcup_{i \in N} A_i = M$. All allocations in the main result are complete.

Definition 1 (Pairwise maximin share fairness)**.** For $\alpha \in [0, 1]$, an allocation $A = (A_1, \dots, A_n)$ is *α-pairwise maximin share fair* (α-PMMS) if, for every pair of distinct agents $i, j \in N$,

$$v_i(A_i) \geq \alpha \max_{B \subseteq A_i \cup A_j} \min\bigl\{v_i(B), v_i((A_i \cup A_j) \setminus B)\bigr\}.$$

It is *pairwise maximin share fair* (PMMS) when the condition holds with $\alpha = 1$. Otherwise, if the inequality fails for (i, j), agent i *PMMS-envies* agent j.

For comparison, an allocation is EFX if $v_i(A_i) \geq v_i(A_j \setminus \{g\})$ for every $i, j \in N$ and every $g \in A_j$. Because our valuations are strictly positive, there is no distinction here between EFX and the convention that quantifies only over goods of positive value to the potentially envious agent.

4 The Counterexample

We now give the instance establishing Theorem 1. The proof is self-contained and uses only the values in Table 1, additivity, and the definition of PMMS.

There are four agents and 52 goods. Partition the goods into four classes M_A, M_B, M_C, M_D of respective sizes $3, 5, 5, 39$. Goods in the same class have the same value. Table 1 gives the value of one good from each class.

Table 1: The strictly positive additive counterexample.

	M_A (3 goods)	M_B (5 goods)	M_C (5 goods)	M_D (39 goods)
Agents $1, 2$	81	100	94	101
Agents $3, 4$	81	94	100	101

It is useful to express values relative to the common baseline 101. For each agent i and good g, define its *deficit* by

$$c_i(g) := 101 - v_i(g),$$

and extend c_i additively to bundles. Thus the deficits are

$$\begin{array}{c|cccc} & M_A & M_B & M_C & M_D \\ \hline i \in \{1,2\} & 20 & 1 & 7 & 0 \\ i \in \{3,4\} & 20 & 7 & 1 & 0 \end{array} \tag{1}$$

The total deficit of all goods is the same for every agent:

$$c_i(M) = 3 \cdot 20 + 5 \cdot 1 + 5 \cdot 7 + 39 \cdot 0 = 100. \tag{2}$$

Since each singleton value satisfies $v_i(g) = 101 - c_i(g)$ and both v_i and c_i are additive, summing over the goods in a bundle gives, for every $S \subseteq M$,

$$v_i(S) = \sum_{g \in S} \bigl(101 - c_i(g)\bigr) = 101|S| - c_i(S).$$

The deficit of a bundle is bounded on both sides. Every singleton deficit is nonnegative, so $c_i(S) \geq 0$; nonnegativity also gives monotonicity, $c_i(S) \leq c_i(M)$. Hence $0 \leq c_i(S) \leq c_i(M) = 100$ for every bundle S, and substituting into the identity above yields

$$v_i(S) = 101|S| - c_i(S), \qquad 101|S| - 100 \leq v_i(S) \leq 101|S|. \tag{3}$$

The upper bound is attained exactly when S consists of padding goods only, and the lower bound exactly when S contains all thirteen goods of $A \cup B \cup C$. The essential feature of (3) is that the total deficit, 100, is *strictly smaller* than the base value 101 of a single good: a bundle is worth 101 per good up to an error smaller than one good's worth. Consequently a bundle with more goods is always strictly more valuable—a set of $p+1$ goods is worth at least $101p+1$, whereas any set of p goods is worth at most $101p$—and this is what lets cardinality dominate composition throughout the proof.

In particular, among bundles of the same size, agent i prefers precisely the bundle with the smaller i-deficit.

Proof of Theorem 1. Suppose, for contradiction, that $X = (X_1, X_2, X_3, X_4)$ is a complete PMMS allocation.

Step 1: every bundle has size 13. If $|X_i| = p$ and $|X_j| \geq p+2$, then $X_i \cup X_j$ contains at least $2p+2$ goods and admits a bipartition into parts of at least $p+1$ goods each. By (3), each part has i-value at least $101(p+1) - 100 = 101p + 1 > 101p \geq v_i(X_i)$. Hence

$$\max_{B \subseteq X_i \cup X_j} \min\{v_i(B), v_i((X_i \cup X_j) \setminus B)\} > v_i(X_i),$$

contradicting PMMS. Bundle sizes therefore differ by at most one, and since they sum to 52, every bundle has exactly 13 goods. Put $R := M_A \cup M_B \cup M_C$ and $Y_i := X_i \cap R$.

Step 2: a necessary deficit inequality. For every ordered pair (i, j) and every bipartition (U, V) of $Y_i \cup Y_j$,

$$\max\{c_i(U), c_i(V)\} \geq c_i(Y_i). \tag{4}$$

Otherwise both deficits are strictly below $c_i(Y_i)$. Since $Y_i \cup Y_j \subseteq R$ and $|R| = 13$, both $|U|$ and $|V|$ are at most 13. The union $X_i \cup X_j$ contains exactly

$$26 - |Y_i \cup Y_j| = (13 - |U|) + (13 - |V|)$$

padding goods, so (U, V) can be changed to a bipartition (P, Q) of $X_i \cup X_j$ with $|P| = |Q| = 13$. Padding goods have deficit zero, so

$$v_i(P) = 13 \cdot 101 - c_i(U) > 13 \cdot 101 - c_i(Y_i) = v_i(X_i),$$
$$v_i(Q) = 13 \cdot 101 - c_i(V) > 13 \cdot 101 - c_i(Y_i) = v_i(X_i),$$

contradicting PMMS.

Call the goods in $M_B \cup M_C$ *ordinary*; their total deficit is $5 \cdot 1 + 5 \cdot 7 = 40$ for every agent, while each M_A-good has deficit 20.

Step 3: no agent receives two M_A-goods. Suppose agent i receives $a \geq 2$ of the M_A-goods, and let s be the i-deficit of the ordinary goods in Y_i, so $c_i(Y_i) = 20a + s \geq 40$. At least two agents receive no M_A-good. Since the ordinary goods have total deficit 40, one of them, say j, satisfies $t := c_i(Y_j) \leq 20$.

Choose $h \in Y_i \cap M_A$ and partition $Y_i \cup Y_j$ into

$$U = Y_i \setminus \{h\}, \qquad V = Y_j \cup \{h\}.$$

Then

$$c_i(U) = c_i(Y_i) - 20 < c_i(Y_i)$$

and

$$c_i(V) = 20 + t \leq 40 \leq c_i(Y_i).$$

Thus both deficits are strictly below $c_i(Y_i)$ unless all three equalities $a = 2$, $s = 0$, and $t = 20$ hold. Outside this exceptional case, (4) is contradicted.

In the exceptional case, Y_i consists of two M_A-goods, say h_1, h_2, and Y_j is a nonempty set of ordinary goods of total i-deficit 20. Pick $g \in Y_j$ and write $z := c_i(g) \in \{1, 7\}$. Partition $Y_i \cup Y_j$ instead into

$$\{h_1, g\} \qquad \text{and} \qquad \{h_2\} \cup (Y_j \setminus \{g\}).$$

Their deficits are

$$20 + z \leq 27 < 40 = c_i(Y_i)$$

and

$$20 + (20 - z) = 40 - z < 40 = c_i(Y_i),$$

again contradicting (4). Hence every agent receives at most one M_A-good.

The three M_A-goods therefore lie with three distinct agents, and exactly one agent receives none. By exchanging agents within the two identical types, and if necessary exchanging the two types together with the names M_B, M_C, we may assume agent 1 receives no M_A-good. Set

$$S := Y_1, \qquad x := |S \cap M_B|, \qquad y := |S \cap M_C|.$$

Thus

$$c_1(S) = c_2(S) = x + 7y, \qquad c_3(S) = c_4(S) = 7x + y.$$

Step 4: $x \geq 3$ and $y \geq 3$. If $x + y \geq 8$, then $x, y \leq 5$ immediately gives $x, y \geq 3$. Suppose therefore that $x + y \leq 7$. At least three ordinary goods then lie outside S, all held by agents $2, 3, 4$.

First assume some $j \in \{2, 3, 4\}$ holds nothing beyond her M_A-good. The other two M_A-holders then share the ordinary goods outside S, so one of them, say k, holds at least two. Split the ordinary goods of Y_k into two nonempty parts of k-deficits $t_1, t_2 \geq 1$, and pair one of the two M_A-goods of $Y_k \cup Y_j$ with each part. The resulting bipartition has k-deficits

$$20 + t_1 < 20 + t_1 + t_2 = c_k(Y_k)$$

and

$$20 + t_2 < 20 + t_1 + t_2 = c_k(Y_k),$$

contradicting (4).

Hence every $j \in \{2, 3, 4\}$ holds an ordinary good, so $\sigma_j := c_j(Y_j) - 20 \geq 1$. Partition $Y_j \cup S$ into the M_A-good of j and everything else. The two j-deficits are

$$20 < 20 + \sigma_j = c_j(Y_j)$$

and

$$\sigma_j + c_j(S).$$

By (4), the second part must have deficit at least $c_j(Y_j) = 20 + \sigma_j$. Therefore

$$\sigma_j + c_j(S) \geq 20 + \sigma_j, \qquad \text{so} \qquad c_j(S) \geq 20.$$

For $j = 2$ and $j = 3$, respectively, this gives

$$x + 7y \geq 20, \qquad 7x + y \geq 20.$$

Since $x, y \leq 5$,

$$x \leq 2 \implies 7x + y \leq 14 + 5 = 19,$$

so $x \geq 3$; symmetrically,

$$y \leq 2 \implies x + 7y \leq 5 + 14 = 19,$$

so $y \geq 3$.

Step 5: the final violating partition. Because $y \geq 3$, at most two M_C-goods lie outside S, so some $j \in \{2, 3, 4\}$ holds none. Because $x \geq 3$, at most two M_B-goods lie outside S, and this j holds at most two of them. Hence

$$c_1(Y_j) \leq 20 + 2 \cdot 1 = 22.$$

Also

$$c_1(S) = x + 7y \geq 3 + 7 \cdot 3 = 24.$$

Since $x, y \geq 3$, choose $T \subseteq S$ consisting of three M_C-goods and two M_B-goods. Then

$$c_1(T) = 3 \cdot 7 + 2 \cdot 1 = 23 < 24 \leq c_1(S).$$

Partition $S \cup Y_j$ into T and its complement. The complementary part has deficit

$$\begin{aligned} c_1((S \cup Y_j) \setminus T) &= c_1(S) + c_1(Y_j) - c_1(T) \\ &\leq c_1(S) + 22 - 23 \\ &= c_1(S) - 1 \\ &< c_1(S). \end{aligned}$$

Thus both parts have deficit strictly below $c_1(Y_1) = c_1(S)$, violating (4) for the ordered pair $(1, j)$ and completing the contradiction. □

Remark 2 (Origin of the construction). The 13-item deficit pattern in (1) is a modification of the additive-chore instance used by He and Tao (2026). Their instance has three chores of cost 20, five chores with costs 1 and 7 for the two agent types, and five chores with the two costs reversed. We turn an item of cost $c_i(g)$ into a good of value $101 - c_i(g)$ and add 39 common goods of value 101. The added goods force every PMMS allocation to have equal bundle sizes, after which value comparisons become the reverse of the corresponding deficit comparisons.

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