Approximate Strategyproofness in Budget Division

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Abstract

In budget division, the task is to distribute a divisible resource (such as money or time) to a set of candidates based on the voters' preferences over such distributions. Under various utility models, far-reaching impossibility results show that no distribution rule can be simultaneously strategyproof, efficient, and fair. Examples include the settings with approval utilities [15], overlap utilities [10], and linear utilities [40]. In this paper, we aim to circumvent these impossibility theorems by focusing on approximate strategyproofness. To this end, we analyze the incentive ratio of distribution rules, which quantifies the maximum multiplicative utility gain of a voter by manipulating. We prove that the Nash product rule (NASH) has an incentive ratio of 2 under arbitrary concave utility reports, thereby bypassing each of the aforementioned impossibility theorems. We also show that NASH has an improved incentive ratio of $e^{1/e}$ in the special case of Cobb–Douglas utilities. We then investigate whether other fair and efficient rules can achieve a better incentive ratio than NASH in the special case of approval-based budget division. To that end, we show that an incentive ratio of 2 is in fact optimal subject to some of the fairness and efficiency properties of NASH. Furthermore, it turns out that other classical distribution rules prone to manipulation have an incentive ratio that scales with the number of voters or candidates. Finally, we complement our results with an experimental analysis.

1 Introduction

Participatory budgeting is a democratic process through which residents help decide how public funds are spent, with the aim of increasing civic engagement and legitimizing decision-making through direct and participatory democracy. First developed in Porto Alegre, Brazil, in 1989, participatory budgeting has since been adopted in cities and public institutions around the world, including New York, Chicago, San Francisco, Toronto, Paris, and Madrid [19, 55]. In many contemporary implementations, participatory budgeting is modeled as a project-selection problem: citizens vote over projects with fixed costs, and the outcome is a subset of projects whose total cost does not exceed the available budget [4, 50]. This suitably captures standalone projects like a playground or public facility. However, some budgetary decisions are more fluid and do not fit this all-or-nothing framework, for example the allocation of a budget across various funds, such as affordable housing trusts, small business grants, and subsidized youth programs. In such applications, the budget can be distributed essentially arbitrarily across the available alternatives. In fact, such budget allocation exercises have been present since the beginnings of participatory budgeting. For example, in Porto Alegre, residents identified and prioritized spending needs across districts and policy areas, and these priorities informed the municipal budget and annual expenditure plan [1, 8, 28].

In social choice theory, such budget division problems have been formally studied as preference aggregation tasks. In particular, budget division is commonly formalized as an allocation problem in this field: a continuous resource, such as money or time, needs to be distributed across some candidates based on the voters' preferences over these candidates. Existing works investigate this problem under different assumptions on the voters' utilities. The first strand assumes approval preferences, i.e., voters only distinguish between liked and disliked candidates. This model, which we refer to as *approval-based* budget division, was introduced by Bogomolnaia *et al.* [14], and has garnered significant attention [e.g., 5, 6, 15, 41, 46], in part because approval ballots offer a favorable tradeoff between the cognitive burden

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for voters and the expressive power of the ballot format [32, 57]. In a second line of work, authors have studied the more general model of *linear utilities*, in which voters may prefer candidates with varying intensities, and these intensity values weight the utility they attain from each candidate's allocation [17, 31, 41]. While more expressive, this approach encounters significant challenges related to preference elicitation. A third model, often referred to as budget aggregation, counters these challenges by letting voters submit ballots containing their ideal budget distribution [10, 18, 20, 21, 34, 35, 36]. Voters then derive utility only up to their desired allocation level for each candidate. Hence, we refer to this setting as budget division with *overlap utilities*.[1] In each of these strands, researchers study *distribution rules*, which map each profile of the voters' reported preferences to a distribution of the resource over the candidates.

The literature on such distribution rules [e.g., 5, 15, 31, 35] has identified three fundamental properties known as *strategyproofness*, *efficiency*, and *fairness*. Roughly, strategyproofness requires that voters cannot benefit by lying about their true preferences. Or, put differently, this property ensures that it is in the best interest of the voters to reveal their preferences truthfully. Efficiency, on the other hand, states that we need to choose a distribution that is on the Pareto frontier, i.e., there is no other distribution that makes all voters weakly better off and some voters strictly better off. Lastly, fairness demands that the chosen distribution should be fair to all groups of voters: an appropriate amount of the resource should be spent on behalf of each group of voters. To see why fairness is important when distributing a divisible resource, we may consider an example with two candidates in which 51% of the voters prefer candidate a and 49% prefer candidate b. Allocating the entire budget to a single candidate seems patently unfair in this case.[2] Instead, it seems reasonable to allocate 51% of the budget to a and 49% to b. This intuition has been formalized and generalized through hierarchies of fairness properties, in particular in approval-based budget division [see, e.g., 5] and, more recently, in budget division with overlap utilities [10, 21].

Unfortunately, sweeping impossibility theorems demonstrate that strategyproofness, efficiency, and fairness cannot be jointly satisfied in any of the settings mentioned above. In the approval-based setting, Brandl *et al.* [15] showed that no strategyproof and efficient distribution rule can even guarantee every voter strictly positive utility, a property known as *positive share*. In the setting with overlap utilities, Bei *et al.* [10] recently showed that every strategyproof and efficient distribution rule violates a property even weaker than positive share, strengthening a previous impossibility of Brandt *et al.* [18]. When voters have linear utilities, a characterization of Hylland [40] shows that the only strategyproof and efficient rule is a dictatorship. Moreover, even if we relax efficiency or fairness, the landscape does not greatly improve. For example, in the approval-based special case, the known strategyproof distribution rules, such as the utilitarian rule and the conditional utilitarian rule, all severely violate efficiency or fairness conditions [5, 6].

On the other hand, little is known about how severely fair and efficient distribution rules fail strategyproofness. This question is made more intriguing by the fact that, unlike Pareto efficiency and positive share, which are minimal notions of efficiency and fairness, strategyproofness is a very strong incentive property. As prior works have observed [e.g., 3, 45], a rule need only give a small reward to a deviating voter on one occasion to violate strategyproofness. The natural question, which we tackle in this paper, is summarized succinctly as follows: if we instead require that a rule not allow a voter to benefit *too much* from misreporting, can we identify rules which are both fair and efficient?

1.1 Our Contribution

In this paper, we investigate whether there are distribution rules that are efficient, fair, and approximately strategyproof. To this end, we introduce the *incentive ratio* of a distribution rule as the worst-case ratio between the utility of a voter when manipulating and when voting truthfully. Put differently, if a rule has an incentive ratio of γ, a voter can get at most γ times her original utility by manipulating. Hence, the lower the incentive ratio, the less manipulable a rule is, and an incentive ratio of 1 means that the rule is strategyproof. The incentive ratio has previously been studied in private good settings, such as Fisher markets [e.g., 23, 24, 26] and assignment problems [e.g., 9, 38], where it led to several positive results: desirable but manipulable mechanisms often have a low incentive ratio. In practice, this may suffice to prohibit manipulations as the cost of a manipulation (e.g., to learn the other voters' preferences or to compute a beneficial manipulation) may exceed its benefit.

[1] This is formally equivalent to the model in which voters derive ℓ_1 distance disutilities. That is, each voter's disutility is equal to the ℓ_1 distance between their ideal budget distribution and the selected distribution. Since we study non-negative utilities in this paper, we use the term overlap utilities for this model.

[2] This marks a contrast with the setting with a deterministic single winner, where majoritarian considerations assert that selecting a is the only reasonable outcome.

Our main result is to show that the Nash product rule (NASH), which maximizes the product of the voters' utilities, has an incentive ratio of 2 even when the voters are allowed to report arbitrary concave utilities over distributions (Theorem 2). Since this rule satisfies efficiency and demanding fairness axioms, this result answers our question about the existence of fair, efficient, and approximately strategyproof distribution rules in the affirmative. The concave utility assumption is general enough to capture several well-studied utility models. In particular, it allows us to circumvent the aforementioned impossibility theorems in the settings with approval utilities [15], overlap utilities [10], and linear utilities [40]. Furthermore, this result extends to group manipulations: no group of voters can manipulate such that each voter in the group more than doubles her utility. Hence, NASH may also disincentivize groups from strategically misreporting their preferences, in particular because such manipulations come with additional efforts to coordinate the voters in the group. Additionally, we prove that NASH has an improved incentive ratio of $e^{1/e} \approx 1.44$ under a special case of concave utilities known as Cobb–Douglas utilities (Section B).

Motivated by this positive result, we further explore whether there are distribution rules that have an incentive ratio smaller than 2, and satisfy some of the desirable properties of NASH. To this end, we focus on the case of approval-based budget division as this setting is often the most permissive one and offers a rich body of axioms and rules to compare to. Perhaps surprisingly, it turns out that, under various side conditions, the incentive ratio of NASH is optimal. In more detail, we show that every rule in the following three classes has an incentive ratio of at least 2, even under approval utilities (Theorem 4):

(i) distribution rules that satisfy average fair share, one of the fairness conditions satisfied by NASH;

(ii) distribution rules that maximize an additively separable and strictly concave welfare function;

(iii) distribution rules that satisfy efficiency, a mild fairness axiom called group fair share, and three basic side conditions called anonymity, neutrality, and independence of losers.

These results demonstrate that, subject to its desirable properties, no distribution rule has a lower incentive ratio than NASH, even in the special case of approval-based budget division.

In fact, the incentive ratio of NASH is even more striking when compared to the incentive ratio of other known approval-based distribution rules. Specifically, we analyze the incentive ratios of three established rules in the literature: the fair utilitarian rule (FUT) by Bogomolnaia *et al.* [13], the maximum payment rule (MP) by Aziz *et al.* [6], and the egalitarian rule (EGAL). These rules are known to be fair and/or efficient and additionally satisfy axiomatic weakenings of strategyproofness, so they are natural alternatives to NASH. However, we prove that the incentive ratios of FUT and MP grow linearly in the number of voters and that of EGAL grows linearly in the number of candidates (Theorem 3). Thus, in contrast to NASH, these rules are in the worst case severely manipulable. For a concise comparison of the incentive ratio of NASH to other approval-based rules and, in addition, information on the fairness and efficiency guarantees of these rules, we refer to Table 1 in Section 3.2.

Lastly, we experimentally analyze the manipulability of distribution rules under approval preferences (cf. Section 4). Specifically, we measure the average incentive ratio and the percentage of manipulable profiles of FUT, MP, EGAL, and NASH for approval profiles sampled from several distributions. These experiments show that all four rules are frequently manipulable, with NASH and EGAL allowing for manipulations in almost all considered profiles. This emphasizes the importance of understanding when voters truly have an incentive to lie about their preferences. Further, with the exception of FUT, our theoretical results on the incentive ratio match the trends observed in the experiments. That is, while NASH is effectively always manipulable, voters never gain substantially by lying about their preferences, whereas EGAL and MP allow for much more beneficial manipulations. The outlier to this trend is FUT, whose average incentive ratio is much lower than our theoretical worst-case result indicates. Hence, except for FUT, this experimental analysis confirms our theoretical results and underlines the robustness of NASH to strategic behavior of the voters.

1.2 Related Work

Our work fits into a larger literature on voting to decide the distribution of a divisible resource, be it money, time, or probability. We refer to the survey by Suksompong and Teh [52] for extensive coverage of these topics as well as related models. In what follows, we focus on approval-based budget division and budget division with overlap utilities in more detail, in particular because strong impossibilities have been proved in each of these special cases, and subsequent work

has attempted to circumvent these impossibilities via approximation. Moreover, we briefly review some important insights on more general utility models and discuss other applications of the incentive ratio.

Approval-based utilities. The work on approval-based budget division was initiated by Bogomolnaia *et al.* [13, 14], who introduced several distribution rules and analyzed them with respect to strategyproofness, efficiency, and fairness. They define a number of fairness properties, such as *individual fair share* (IFS), which guarantees that every voter receives at least utility $1/n$, and *group fair share* (GFS), which guarantees that for every group of voters S there is at least $|S|/n$ of the resource distributed to candidates approved by at least one voter in the group. Moreover, these authors observe a trade-off between strategyproofness, efficiency, and fairness as none of their studied rules simultaneously satisfy all three. This trade-off was formalized by an impossibility result using IFS [14], which was subsequently strengthened by Brandl *et al.* [15] to demonstrate the following: no strategyproof and efficient approval-based distribution rule satisfies positive share (i.e., every voter always receives strictly positive utility).

Furthermore, Aziz *et al.* [5] analyze several rules in terms of efficiency, fairness, and strategyproofness. They prove that two strategyproof rules — the random priority rule (RP) and the conditional utilitarian rule (CUT) — are severely inefficient. In a similar vein, Michorzewski *et al.* [46] show that all fair distribution rules under approval utilities may have a low utilitarian social welfare compared with the optimum. However, they also note that, among fair rules, NASH gives almost optimal guarantees regarding the utilitarian social welfare. Aziz *et al.* [5] also introduce more restrictive fairness properties known as *average fair share* (AFS) and *core*, and show that NASH satisfies these conditions. By contrast, Aziz *et al.* [6] have shown that most other rules severely fail AFS, even in an approximate sense. In particular, to date, NASH is the only approval-based distribution rule that is known to satisfy either AFS or core. Due to these positive results, NASH is seen favorably in approval-based budget division, which is further supported by a characterization of this rule based on a mild fairness notion, efficiency, and several auxiliary conditions [37].

Closer in spirit to our work, other authors have circumvented the impossibility theorem of Brandl *et al.* [15] by studying axiomatic relaxations of strategyproofness. In particular, Aziz *et al.* [5] introduce excludable strategyproofness, a weakening of strategyproofness where voters are assumed to only obtain utility from candidates they approve both before and after the manipulation. They show that the egalitarian rule (EGAL), which maximizes the minimum utility of the voters, is efficient, excludable strategyproof, and satisfies IFS. Similarly, Brandl *et al.* [15] study *monotonicity*, which requires that if a voter approves an additional candidate, the selected distribution should not decrease the share of that candidate. They prove that monotonicity is weaker than strategyproofness. Moreover, these authors rediscover the fair utilitarian rule (FUT) of Bogomolnaia *et al.* [13], and show that it satisfies monotonicity, efficiency, and GFS. In this work, we will show that all of these rules (excluding the strategyproof ones) have large incentive ratios, meaning that the axiomatic weakenings of strategyproofness do not give any meaningful guarantee in terms of the incentive ratio.

Overlap utilities. The setting in which voters derive overlap utilities — that is, wherein each voter has an ideal distribution and derives utility up to their ideal level for each candidate — is commonly referred to as *budget aggregation*. Lindner *et al.* [43], Lindner [44] and Goel *et al.* [36] each study variants of the utilitarian rule, which maximizes the utilitarian social welfare, and show this rule to be strategyproof. Freeman *et al.* [35] note that the utilitarian rule is majoritarian and thus neglects fairness considerations. Motivated by this observation, they introduce a class of strategyproof rules called *moving-phantom rules*, which generalize the ideas of the "phantom median rules" from facility location on the line [47, 49] to the setting with arbitrarily many dimensions. Moreover, they introduce a basic proportionality criterion called *single-minded proportionality* — which requires that, when voters' ideal distributions are standard basis vectors, each candidate receives a share of the resource in proportion to the number of voters who favor that candidate — and identify a moving-phantom rule satisfying this criterion. However, they show that the utilitarian rule is the only efficient moving-phantom rule, ruling out the possibility of a fair and efficient rule within this class. More generally, Brandt *et al.* [18] show that no distribution rule satisfies all of strategyproofness, efficiency, and single-minded proportionality. Bei *et al.* [10] strengthened this impossibility by proving that no rule satisfies strategyproofness, efficiency, and a property requiring positive share only on single-minded profiles.

In light of these negative results, researchers have also examined how well distribution rules approximate efficient and fair outcomes. In budget division with overlap utilities, Caragiannis *et al.* [20] and Freeman and Schmidt-Kraepelin [34] use the mean distribution — that is, the distribution which selects the mean of the voters' ideal allocations for each candidate — as a fairness benchmark. Caragiannis *et al.* [20] measure the worst-case ℓ_1 distance of the distribution chosen by a rule from the mean distribution (with maximum distance 2). Among other results, these authors introduce a

novel moving-phantom rule for which this distance is at most $2/3$ for the case of two candidates, and they show that no strategyproof rule achieves a worst-case ℓ_1 distance from the mean distribution of less than $1/2$. Freeman and Schmidt-Kraepelin [34] instead measure the worst-case ℓ_∞ distance from the mean distribution (with maximum distance 1). They introduce a moving-phantom rule whose ℓ_∞ distance from the mean distribution is at most $1/2 - 1/m$. Taking a different approach, Cembrano *et al.* [21] study the tradeoff between efficiency, fairness, and strategyproofness via the price of proportionality. This quantity measures the worst-case welfare ratio between a single-minded proportional rule and the utilitarian rule, and Cembrano *et al.* identify a moving-phantom rule that achieves the optimal worst-case ratio. Finally, Bei *et al.* [10] take an axiomatic approach to fairness more akin to the approval-based budget division literature. They define fairness axioms stronger than single-minded proportionality and give strategyproof and efficient rules satisfying their fairness properties in special cases with a low number of candidates and/or voters.

More general utility models. A few works study budget division under more general utility models, but fairness and strategyproofness are largely treated separately in such models. In particular, Hylland [40] and follow-up works [29, 48] show that, under linear utilities, only dictatorships satisfy strategyproofness and efficiency, leaving no room for fairness considerations. On the other hand, Fain *et al.* [31] showed that NASH computes a Lindahl equilibrium, and thus satisfies a fairness notion called core [33], as long as voters' utility functions are homogeneous of degree one. This includes linear utilities as a special case. More recently, Chaudhury *et al.* [22] have shown that NASH even satisfies core under concave utilities. These works, however, ignore strategic aspects. Furthermore, when voters have linear utilities, Brandl *et al.* [16] proved that NASH gives strong participation incentives. Finally, for the case of Leontief utilities, Brandt *et al.* [17] have shown that NASH is strategyproof, efficient, and fair, thus identifying one of the rare cases in which all our desiderata are jointly compatible.

Incentive Ratio. The incentive ratio has been studied for different settings before. To our knowledge, it was first introduced for Fisher markets, where goods are sold to agents [23, 24, 25, 26]. For example, for linear utilities, the incentive ratio of the corresponding market mechanism is 2. Such Fisher markets can be seen as the private good analog of Lindahl equilibria and thus of NASH [41, 53]. Hence, in some sense, our results mirror the insights from private good settings in a public good model. We note, however, that the underlying techniques for deriving our results differ significantly from those used for private goods settings and that none of the private good results considers the full class of concave utility functions. Moreover, the incentive ratio has also been analyzed for rank aggregation [30] and assignment mechanisms, such as the probabilistic serial rule [38, 42], picking sequences [56], and cake cutting [9]. For example, Li *et al.* [42] showed that the problabilistic serial rule for random assignment has also an incentive ratio of 2, which allows to escape an impossibility theorem by Bogomolnaia and Moulin [12] for this setting. The extension of the incentive ratio to group manipulations has also been studied in private allocation [39].

2 Preliminaries

For each integer $t \in \mathbb{N}$, we define $[t] = \{1, \dots, t\}$. We assume there is a set of n voters $N = [n]$ and a set of m candidates $C = \{x_1, \dots, x_m\}$. Our goal is to distribute a continuous resource among the candidates. We will formalize this with the help of *distributions*, which are functions $p : C \to [0, 1]$ such that $\sum_{x \in C} p(x) = 1$. Further, the set of all distributions over the set of candidates C is denoted by $\Delta(C)$. Less formally, a distribution p specifies for every candidate $x \in C$ the share $p(x)$ of the resource that is allocated to x.

In the most general case, each voter i is allowed to report a *utility function* $u_i : \Delta(C) \to \mathbb{R}_{\geq 0}$ over the distributions. We impose two standard conditions on utility functions, namely that they are continuous (i.e., for all distributions $p \in \Delta(C)$ and sequences of distributions$(p^k)_{k \in \mathbb{N}}$ in $\Delta(C)$ with $\lim_{k \to \infty} p^k = p$, it holds that $\lim_{k \to \infty} u_i(p^k) = u_i(p)$) and non-zero (i.e., there is a distribution $p \in \Delta(C)$ with $u_i(p) > 0$). Moreover, we emphasize that utility functions are non-negative by definition. A *utility profile* $U = (u_i)_{i \in N}$ is a vector specifying the utility function of every voter $i \in N$. Given a utility profile U, we denote by $U_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$ the profile derived from U by removing voter i and by $(U_{-i}, u_i') = (u_1, \dots, u_{i-1}, u_i', u_{i+1}, \dots, u_n)$ the profile where we change the utility function of voter i to u_i'. To select a distribution for an input profile, we will use *(distribution) rules*, which map each utility profile U (over any finite sets of voters N and candidates C) to a distribution $p \in \Delta(C)$. We denote by $f(U, x)$ the share of the resource that a rule f assigns to a candidate x given a profile U.

2.1 Classes of Utility Functions

We will next discuss several classes of utility functions that we will consider throughout the paper. Specifically, we first introduce the class of concave utility functions and then consider several special cases of this class.

Concave utilities. As our most general class, we will consider *concave* utility functions, which satisfy that $u_i(\lambda p + (1-\lambda)q) \geq \lambda u_i(p) + (1-\lambda)u_i(q)$ for all distributions $p, q \in \Delta(C)$ and mixing coefficients $\lambda \in [0,1]$. Intuitively, this means that the voters have (weakly) diminishing marginal returns: the closer a distribution gets to a voter's most preferred one, the smaller the additional utility she derives from further improvements. Furthermore, we note that the class of concave utility functions is very large and contains many commonly considered models.

Approval utilities. Many of our complementary results will hold even in the special case of *approval-based budget division*, which we now formalize. In this model, to indicate her preferences, each voter $i \in N$ reports an *approval ballot* $A_i \subseteq C$, which is the non-empty set of her approved candidates. The utility of a voter i is then assumed to be the total share assigned to her approved candidates by the selected distribution p, i.e., $u_i(p) = \sum_{x \in A_i} p(x)$. Further, an *approval profile* $A = (A_i)_{i \in N}$ contains the approval ballot of each voter $i \in N$, and we adopt notation analogous to utility profiles. In particular, (A_{-i}, A_i') is the approval profile derived from A by changing the approval ballot of voter i to A_i', and $f(A, x)$ is the share of the resource that a rule f assigns to a candidate x in a profile A.

Overlap utilities. Overlap utilities are another influential model, where each voter $i \in N$ indicates her preferences by reporting her *ideal distribution* $a_i \in \Delta(C)$. The utility of each voter for a distribution is then equal to the total amount of the resource allocated up to the voter's ideal level for each candidate. That is, the utility of voter $i \in N$ for distribution p is $u_i(p) = \sum_{x \in C} \min(p(x), a_i(x))$. In other words, the utility function sums the overlap between the ideal distribution and the selected distribution for each candidate. Overlap utilities induce the same ordinal rankings over distributions as if the voters derive disutility equal to the ℓ_1 distance between the selected and their ideal distribution.

Further utility models. Lastly, concave utility functions also include a range of further models. For example, approval utilities are, in fact, a special case of *linear utilities*, wherein each voter $i \in N$ places a non-negative value v_{ix} on each candidate $x \in C$ and receives utility from distribution p equal to $u_i(p) = \sum_{x \in C} v_{ix} \cdot p(x)$. Another recently considered class is *Leontief utilities*, for which each voter $i \in N$ specifies coefficients $a_{ix} \geq 0$ for each candidate $x \in C$, not all zero, and receives utility from distribution p equal to $u_i(p) = \min_{x \in C : a_{ix} > 0} \frac{p(x)}{a_{ix}}$. Under Leontief utilities, candidates are perfect complements and the voter evaluates an allocation by the least well-funded required component. Finally, under *Cobb–Douglas* utility functions, there are exponents $\alpha_{ix} \geq 0$ with $\sum_{x \in C} \alpha_{ix} = 1$ such that $u_i(p) = \prod_{x \in C} p(x)^{\alpha_{ix}}$. Linear utilities capture perfect substitutes, Leontief utilities capture perfect complements, and Cobb–Douglas utilities model an intermediate case with diminishing marginal returns across candidates.

2.2 Strategyproofness and Incentive Ratio

We now turn to the central properties of this paper, namely strategyproofness and the incentive ratio of distribution rules. First, strategyproofness requires that voters cannot increase their utility by misreporting their true utility functions. More formally, a distribution rule f is *strategyproof* if $u_i(f(U)) \geq u_i(f(U_{-i}, u_i'))$ for all voters $i \in N$, utility profiles U, and utility functions u_i'. Conversely, a rule is said to be *manipulable* if it is not strategyproof. Strategyproofness is desirable because, for manipulable rules, we cannot expect voters to report their true preferences, which may lead to suboptimal outcomes.

Unfortunately, strategyproofness is a rather prohibitive axiom, even for approval-based budget division and budget division with overlap utilities. Specifically, in their respective settings, Brandl *et al.* [15] and Bei *et al.* [10] have shown that strategyproofness is incompatible with efficiency and a minimal fairness notion called positive share. To formally state these results, we say a distribution rule f is *(Pareto) efficient* if, for every profile U, there is no distribution q such that $u_i(q) \geq u_i(f(U))$ for all voters $i \in N$ and $u_i(q) > u_i(f(U))$ for some voter $i \in N$. Put differently, this axiom states that it should never be possible to make a voter strictly better off without making another voter worse off. Moreover, *positive share* requires that all voters always receive strictly positive utility, i.e., $u_i(f(U)) > 0$ for all profiles U and voters $i \in N$.

Theorem 1 (Brandl *et al.*, 2021; Bei *et al.*, 2026)**.** *No strategyproof distribution rule satisfies both efficiency and positive share. This holds even in approval-based budget division and budget division with overlap utilities.*[3]

To circumvent such impossibility theorems, we will study an approximate variant of strategyproofness called the incentive ratio. Intuitively, this ratio quantifies the worst-case multiplicative utility gain of a voter for a distribution rule over all possible utility profiles, voters, and deviations.

Definition 1 (Incentive Ratio)**.** *The* incentive ratio *of a distribution rule f is defined by* $\gamma(f) = \sup_{U,i,u_i'} \frac{u_i(f(U_{-i},u_i'))}{u_i(f(U))}$.

Put differently, if a distribution rule has an incentive ratio of γ, a voter can get at most γ times her original utility by manipulating. Since the utility of a voter can be 0 in the truthful profile U, we use the convention that $\frac{0}{0} = 1$ and $\frac{x}{0} = \infty$ for all $x > 0$. Further, we note that a distribution rule has an incentive ratio of 1 if and only if it is strategyproof, and a smaller incentive ratio means that a rule is less manipulable. Hence, to circumvent the impossibility theorems by Brandl *et al.* [15] and Bei *et al.* [10], we will try to find fair and efficient rules that have a small incentive ratio.

3 Theoretical Results

We are now ready to present our theoretical results. We prove in Section 3.1 that the Nash product rule has an incentive ratio of 2 even when voters are allowed to report concave utility functions. We then complement this positive result by narrowing in on the special case of approval-based budget division, and showing that other reasonably fair and efficient distribution rules have large incentive ratios (Section 3.2) and that the incentive ratio of 2 is optimal subject to the desirable properties of the Nash product rule (Section 3.3).

3.1 The Incentive Ratio of NASH

In this section, we will show that the Nash product rule has an incentive ratio of 2 when the voters report concave utility functions. We begin by formalizing the Nash product rule (NASH).

Nash product rule. The Nash product rule (NASH) returns a distribution p that maximizes the Nash welfare $\prod_{i \in N} u_i(p)$. The existence of such a distribution is guaranteed by the continuity and non-zero assumptions on the utility functions and compactness of the space of distributions $\Delta(C)$. Further, while there can be multiple distributions that maximize the Nash welfare, our main result will hold under any tie-breaking scheme, so we do not specify it.[4]

We point out that NASH satisfies efficiency and provides substantial fairness guarantees. For example, Chaudhury *et al.* [22] have shown that the distribution chosen by NASH always satisfies a fairness notion called core. Intuitively, this axiom requires that no group of voters can benefit by reallocating a part of the budget that is proportional to the size of the group. More formally, a distribution p is in the *core* if there does not exist another distribution q and group of voters $S \subseteq N$ such that $u_i(p) \leq (|S|/n)u_i(q)$ for every $i \in S$ and $u_i(p) < (|S|/n)u_i(q)$ for some $i \in S$. Moreover, if the voters' utility functions are homogeneous of degree one, the outcome of NASH is a Lindahl equilibrium [31], and this rule is known to satisfy numerous other fairness notions in, e.g., the approval-based special case [5]. Thus, the result of this section answers our original research question in the affirmative: the Nash product rule is an efficient, fair, and approximately strategyproof distribution rules so we can escape the impossibility results discussed in the introduction by relaxing strategyproofness.

To prove that the incentive ratio of NASH is 2 for concave utility functions, we first discuss three technical auxiliary claims that give insights into the distribution chosen by NASH. We believe that these claims may be of independent interest, as they seem helpful to prove further results about NASH for concave utility functions. The proof of the following lemma is deferred to Section C as it is a lengthy mathematical exercise.

[3]For overlap utilities, Bei *et al.* [10] show that their impossibility even holds under a weakening of positive share, which only applies this axiom to profiles where each voters wants to assign the full budget to a single candidate.

[4]A well-versed reader may even expect that tie-breaking is irrelevant for our result, since for every concave utility profile U, each distribution p that maximizes the Nash welfare $\prod_{i \in N} u_i(p)$ induces the same utility vector $(u_i(p))_{i \in N}$. However, this utility equivalence is not enough in the context of the incentive ratio, as a manipulator evaluates the outcome of NASH with a utility function u_i that she does not report in the considered profile (U_{-i}, u_i').

Lemma 1. *Fix an arbitrary concave utility profile U.*

(1) It holds that $u_i(\mathsf{NASH}(U)) > 0$ for all voters $i \in N$.

(2) For all distributions $q \in \Delta(C)$, it holds that $\sum_{i \in N} \frac{u_i(q) - u_i(\mathsf{NASH}(U))}{u_i(\mathsf{NASH}(U))} \leq 0$.

(3) For all voters $i \in N$, it holds that $e^{-1} \leq \frac{\prod_{j \in N \setminus \{i\}} u_j(\mathsf{NASH}(U))}{\prod_{j \in N \setminus \{i\}} u_j(\mathsf{NASH}(U_{-i}))} \leq 1$.

Less formally, Claim (1) shows that NASH guarantees each voter strictly positive utility. Claim (2) states the separation inequality witnessing that the distribution $p = \mathsf{NASH}(U)$ maximizes the Nash welfare $\prod_{i \in N} u_i(p)$. While this inequality appears also in earlier works [e.g., 2, 5], we derive it under more general conditions than these works since we do not even require that utility functions are differentiable. Lastly, Claim (3) demonstrates that the Nash welfare of the voters $N \setminus \{i\}$ under NASH cannot decrease too much when voter i joins the election. A similar observation has been proven by Cole *et al.* [27] for a private goods setting.

Based on these insights, we now prove our main theorem.

Theorem 2. *The incentive ratio of* NASH *is* 2*, even if voters are allowed to report concave utility functions.*

Proof. We will show here only that the incentive ratio of NASH is at most 2 since a matching lower bound follows from Theorem 4. For this, we assume for contradiction that there is a concave utility profile U, a voter i, and a concave utility function u_i' such that $\frac{u_i(\mathsf{NASH}(U_{-i}, u_i'))}{u_i(\mathsf{NASH}(U))} > 2$. This immediately implies that there are at least two voters in U, since NASH is easily seen to be strategyproof when $n = 1$. Further, to ease the notation, we let $p = \mathsf{NASH}(U)$ and $q = \mathsf{NASH}(U_{-i}, u_i')$. We will show that $\sum_{j \in N} \frac{u_j(q)}{u_j(p)} > n$, which contradicts Claim (2) in Lemma 1 as $\sum_{j \in N} \frac{u_j(q) - u_j(p)}{u_j(p)} = \sum_{j \in N} \frac{u_j(q)}{u_j(p)} - n$.

To prove this inequality, we first derive a lower bound on $\sum_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)}$. For this, we consider the term $\prod_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)}$, which is well-defined by Claim (1) of Lemma 1. Next, let $\hat{p} = \mathsf{NASH}(U_{-i})$ denote the distribution chosen by NASH when voter i is absent. Clearly, it holds that $\prod_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)} = \frac{\prod_{j \in N \setminus \{i\}} u_j(q)}{\prod_{j \in N \setminus \{i\}} u_j(\hat{p})} \cdot \frac{\prod_{j \in N \setminus \{i\}} u_j(\hat{p})}{\prod_{j \in N \setminus \{i\}} u_j(p)}$. Further, by Claim (3) of Lemma 1, we have that $\frac{1}{e} \leq \frac{\prod_{j \in N \setminus \{i\}} u_j(q)}{\prod_{j \in N \setminus \{i\}} u_j(\hat{p})}$ and $1 \leq \frac{\prod_{j \in N \setminus \{i\}} u_j(\hat{p})}{\prod_{j \in N \setminus \{i\}} u_j(p)}$. We hence conclude that

$$\frac{1}{e} \leq \frac{\prod_{j \in N \setminus \{i\}} u_j(q)}{\prod_{j \in N \setminus \{i\}} u_j(\hat{p})} \cdot \frac{\prod_{j \in N \setminus \{i\}} u_j(\hat{p})}{\prod_{j \in N \setminus \{i\}} u_j(p)} = \prod_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)}.$$

By applying the logarithm, it follows that $\log \frac{1}{e} \leq \sum_{j \in N \setminus \{i\}} \log \frac{u_j(q)}{u_j(p)}$. Next, Jensen's inequality implies that

$$\frac{1}{n-1} \sum_{j \in N \setminus \{i\}} \log \frac{u_j(q)}{u_j(p)} \leq \log \left(\frac{1}{n-1} \sum_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)} \right).$$

By chaining our inequalities, we infer that

$$\log \frac{1}{e} \leq (n-1) \log \left(\frac{1}{n-1} \sum_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)} \right).$$

Solving for $\sum_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)}$ then shows that

$$(n-1) e^{-\frac{1}{n-1}} \leq \sum_{j \in N \setminus \{i\}} \frac{u_j(q)}{u_j(p)}.$$

We now conclude that $\sum_{j \in N} \frac{u_j(q)}{u_j(p)} \geq (n-1) e^{-\frac{1}{n-1}} + 2$ as $\frac{u_i(q)}{u_i(p)} \geq 2$. We will next show that $(n-1) e^{-\frac{1}{n-1}} + 2 > n$ for all $n \geq 2$, or equivalently that $(n-1)(e^{-\frac{1}{n-1}} - 1) > -1$. For this, we note that $x(e^{-\frac{1}{x}} - 1) > x(1 - \frac{1}{x} - 1) = -1$ for all $x > 0$ since $e^y > 1 + y$ for all $y \neq 0$. By setting $x = n - 1$, this proves our inequality. Hence, the assumption that $\frac{u_i(q)}{u_i(p)} > 2$ contradicts Claim (2) of Lemma 1, thus proving our theorem. □

Remark 1. Theorem 2 generalizes to group manipulations: under NASH, no group of voters can misreport their utility functions such that each member of the group obtains more than twice her original utility. More formally, it holds for all concave utility profiles U and groups of voters $S \subseteq N$ that there is no group manipulation U'_S such that $\frac{u_i(\mathsf{NASH}(U_{-S}, U'_S))}{u_i(\mathsf{NASH}(U))} > 2$ for all $i \in S$. This statement can be proven similar to Theorem 2 because Claim (3) of Lemma 1 can be generalized to show that $e^{-|S|} \leq \frac{\prod_{i \in N \setminus S} u_i(\mathsf{NASH}(U))}{\prod_{i \in N \setminus S} u_i(\mathsf{NASH}(U_{-S}))} \leq 1$ for all groups of voters $S \subseteq N$. Based on this insight, one can then essentially repeat the proof of Theorem 2 to prove this variant for group manipulations.

Remark 2. The incentive ratio of Theorem 2 is tight for the special cases of approval-based and linear utilities as well as overlap utilities. The claim for approval-based utilities follows from Theorem 4, and it immediately also implies the claim for linear utilities since approval-based utilities are a special case of linear utilities. The claim for overlap utilities is proved in Section A and holds even for the case with two candidates. On the other hand, for Cobb–Douglas utility functions, we show that NASH has an improved incentive ratio of $e^{1/e} \approx 1.44$ (Theorem 5). Moreover, for Leontief utility functions, NASH is even known to be strategyproof [17], so its incentive ratio is 1 for this case. We further note that linear, Cobb-Douglas, and Leontief utility functions are all special cases of constant elasticity of substitution (CES) utility functions. In particular, voters have CES utility functions if, for every voter i, the utility for distribution p is given by $u_i(p) = (\sum_{x \in C} v_i(x) \cdot p(x)^r)^{1/r}$, where v_i are voter-specific weights and r an exogenous parameter. CES utilties reduce to linear utilties when $r = 1$, to Cobb–Douglas utilities when r approaches 0 and to Leontief utilities when r approaches $-\infty$. Since we have shown that the incentive ratio of NASH is 2 under linear utilities, $e^{1/e}$ under Cobb-Douglas utilities, and 1 under Leontief utilities, it seems plausible to conjecture that the incentive ratio of NASH gradually improves as the elasticity of substitution decreases.

3.2 Approval-Based Distribution Rules with Large Incentive Ratios

Given the encouraging result for NASH in the general setting, it is natural to wonder whether other fair and efficient rules can achieve a similar or better incentive ratio than NASH. To answer this question, we turn to the special case of approval-based budget division, for which numerous distribution rules have been suggested. We first note that, for this special case, there are reasonable strategyproof rules, such as the utilitarian rule (UTIL) and conditional utilitarian rule (CUT). However, these rules are known to severely violate fairness and/or efficiency. Instead, we will subsequently consider three established rules that are known to be fair but not strategyproof: the egalitarian rule (EGAL), the maximum payment rule (MP), and the fair utilitarian rule (FUT). In more detail, all of these rules are efficient (except for MP), EGAL satisfies positive share, and MP and FUT satisfy a stronger fairness notion called group fair share (see Section 3.3 for a definition). As we will see, none of these three rules has a constant-factor incentive ratio. Strikingly, this means that even when we restrict to approval-based instances, the well-studied rules which satisfy reasonable fairness and efficiency properties are vastly outperformed by NASH in terms of the incentive ratio. We will now formally introduce each of the three rules.

Egalitarian rule. The egalitarian rule (EGAL) lexicographically maximizes the minimal utility of a voter: we first maximize the utility of the worst-off voter, subject to that we maximize the utility of the second worst-off voter, and so on. To make this more formal, let $u^{\min}(p)$ denote a vector that contains the utilities $u_i(p)$ of all voters $i \in N$ in increasing order. Then, EGAL chooses a distribution p that lexicographically maximizes $u^{\min}(p)$, i.e., for every other distribution q, it either holds that $u^{\min}(p) = u^{\min}(q)$ or there is an index $\ell \in [n]$ such that $u^{\min}(p)_i = u^{\min}(q)_i$ for all $i \in [\ell - 1]$ and $u^{\min}(p)_\ell > u^{\min}(q)_\ell$. Just as for NASH, there can be multiple distributions that lexicographically optimize $u^{\min}$. However, the tie-breaking will not matter for our subsequent result, so we do not specify it.

Maximum payment rule. The maximum payment rule (MP) was introduced by Speroni di Fenizio and Gewurz [51] and recently studied for budget division by Aziz *et al.* [6]. It first gives to each voter control over a $\frac{1}{n}$ share of the budget. Then, it repeatedly identifies the most approved candidate x (with ties broken lexicographically), assigns the shares of the voters who approve x to this candidate, and deletes x as well as all voters that approve x from the profile. This process is repeated until each voter has allocated her $\frac{1}{n}$ share and the final distribution corresponds to the total shares accumulated by each candidate.

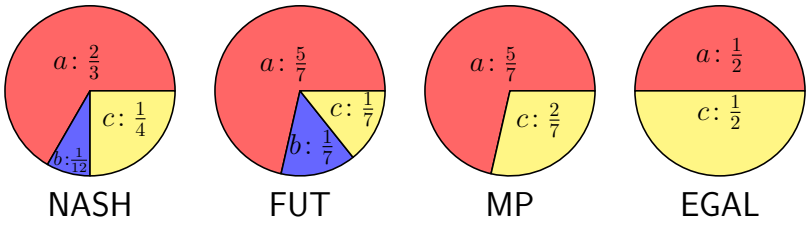


Figure 1: Example of our distribution rules. The pie charts display the distributions chosen by NASH, FUT, MP, and EGAL for the profile $A = (2\colon \{a\}, 3\colon \{a, b\}, 1\colon \{b, c\}, 1\colon \{c\})$. First, for EGAL, we choose the distribution that maximizes the utility of the worst-off voter. Since there are 2 voters who only approve a and 1 voter who only approves c, it is immediate that the worst-off voter has a utility of at most $\frac{1}{2}$, and the distribution p_1 with $p_1(a) = p_1(c) = \frac{1}{2}$ gives every voter exactly this utility. Second, MP chooses the distribution p_2 with $p_2(a) = \frac{5}{7}$ and $p_2(c) = \frac{2}{7}$. In more detail, this rule assigns to a the $\frac{1}{7}$ shares of the five voters approving it as it is approved by the most voters. After deleting the corresponding voters, all remaining voters approve c, so it gets the remaining $\frac{2}{7}$. Third, FUT also first assigns $\frac{5}{7}$ to a as it is the approval winner. After this, we set $t = 5$ and start increasing the weights λ_i of the two voters who still have their budget. When $\lambda_i = 2$, b also has a weighted utilitarian welfare of 5, so the single voter reporting $\{b, c\}$ sends her $\frac{1}{7}$ to b. Finally, the remaining $\frac{1}{7}$ share has to go to the last candidate c, so FUT picks the distribution p_3 with $p_3(a) = \frac{5}{7}$, $p_3(b) = \frac{1}{7}$, and $p_3(c) = \frac{1}{7}$. Lastly, for NASH, we need to solve the underlying maximization problem. Doing so yields that it selects the distribution p_4 with $p_4(a) = \frac{2}{3}$, $p_4(b) = \frac{1}{12}$, and $p_4(c) = \frac{1}{4}$.

Fair utilitarian rule. The fair utilitarian rule (FUT) was first suggested by Bogomolnaia *et al.* [13] and later rediscovered by Brandl *et al.* [15]. In this rule, each voter $i \in N$ controls a share of $\frac{1}{n}$ and is assigned a weight λ_i, which is initially equal to 1. Moreover, we let $t = \max_{x \in C} |\{i \in N\colon x \in A_i\}|$ denote the maximum approval score of a candidate. Then, FUT repeats the following routine until all voters have spent their share: we first increase the weights λ_i of all voters who have not spent their share yet until the set of candidates X with $\sum_{i \in N : x \in A_i} \lambda_i = t$ is non-empty. Note that in the first round, we do not need to increase the weights for this. Next, each voter i who has not spent her share and approves a candidate in X allocates her $\frac{1}{n}$ share uniformly to the candidates in $X \cap A_i$. Lastly, we delete all candidates in X from the profile and repeat this process if some voter has not spent her share yet.

We refer to Figure 1 for an illustration of our distribution rules. In approval-based budget division, the incentive ratio definition follows analogously to Definition 1, i.e., $\gamma(f) = \sup_{A,i,A'_i} \frac{u_i(f(A_{-i}, A'_i))}{u_i(f(A))}$. We will now prove that the incentive ratios of MP and FUT grow linearly in the number of voters and the incentive ratio of EGAL grows linearly in the number of candidates. This means that, in the worst-case, these rules can be severely manipulated. Moreover, we note that all three rules satisfy axiomatic weakenings of strategyproofness: MP and FUT are monotonic [6, 15], whereas EGAL satisfies a notion called excludable strategyproofness [5]. Our result thus also demonstrates that these properties do not give convincing guarantees with respect to approximate strategyproofness.

Theorem 3. *The following claims hold:*

(1) The incentive ratio of MP *is in* $\Theta(n)$.

(2) The incentive ratio of FUT *is in* $\Theta(n)$.

(3) The incentive ratio of EGAL *is in* $\Theta(m)$.

Proof. We prove all three claim independently of each other.

Claim (1): For the upper bound of the incentive ratio of MP, we note that every voter spends a share of $\frac{1}{n}$ on her approved candidates, so the distribution p chosen by MP satisfies that $u_i(p) \geq \frac{1}{n}$ for all voters $i \in N$. Since the utility of a voter is at most 1, this implies that the incentive ratio of MP is upper bounded by n.

For the lower bound, we fix an integer $k \geq 2$ and consider the following profiles with $m = 3$ candidates and $n = 2k + 4$ voters, which only differ in the ballot of the right-most voter. Further, we suppose the tie-breaking of MP favors a to c to b.

$$\begin{array}{lllll} A\colon & k\colon \{b, c\} & 3\colon \{a, b\} & k-1\colon \{a\} & 2\colon \{c\} \\ A'\colon & k\colon \{b, c\} & 3\colon \{a, b\} & k-1\colon \{a\} & 1\colon \{c\} \quad 1\colon \{a, c\} \end{array}$$

For A, MP chooses the distribution p with $p(b) = \frac{k+3}{2k+4}$, $p(a) = \frac{k-1}{2k+4}$, and $p(c) = \frac{2}{2k+4}$. Specifically, MP first assigns a share of $\frac{k+3}{2k+4}$ to b as it is approved by the most voters. After deleting b and the corresponding voters, the remaining voters report singleton ballots. It thus follows that a obtains a share of $\frac{k-1}{2k+4}$ and c of $\frac{2}{2k+4}$.

By contrast, both a and b are approved by $k+3$ voters in A'. By our tie-breaking assumption, this means that MP first assigns a share of $\frac{k+3}{2k+4}$ to a. Next, after deleting a and the corresponding voters, all remaining voters approve c, so MP sends the shares of these voter to c. Thus, MP selects the distribution q with $q(a) = \frac{k+3}{2k+4}$ and $q(c) = \frac{k+1}{2k+4}$ for A'.

Finally, let i be the voter who deviates from $\{c\}$ to $\{a, c\}$ in our profiles. Assuming that $\{c\}$ is the true ballot of i, the utility of this voter is $u_i(p) = \frac{2}{2k+4}$ in A and $u_i(q) = \frac{k+1}{2k+4}$ in A'. This means for the incentive ratio of MP that $\gamma(\mathsf{MP}) \geq \frac{u_i(q)}{u_i(p)} = \frac{k+1}{2} = \frac{n-3}{4}$. Hence, we conclude that $\gamma(\mathsf{MP}) \in \Theta(n)$.

Claim (2): For the upper bound, we note that, just as for MP, under FUT, every voter assigns a $\frac{1}{n}$ share to her approved candidates. Hence, it holds for every profile A and voter i that $u_i(\mathsf{FUT}(A)) \geq \frac{1}{n}$. Since the maximal utility of every voter is 1, it follows that the incentive ratio of FUT is at most n.

For the lower bound, we choose an integer $k \geq 6$ and consider the following two profiles $\mathcal{A}$ and $\mathcal{A}'$ with four candidates $C = \{a, b, c, d\}$ and $2k + 7$ voters. We note that these profiles only differ in the approval ballot of the left-most voter i.

A:	1: $\{a\}$	$k-2$: $\{a, b\}$	2: $\{b\}$	2: $\{b, d\}$	3: $\{a, c\}$	k: $\{c, d\}$	1: $\{d\}$
A':	1: $\{a, d\}$	$k-2$: $\{a, b\}$	2: $\{b\}$	2: $\{b, d\}$	3: $\{a, c\}$	k: $\{c, d\}$	1: $\{d\}$

We claim that FUT selects the distribution p with $p(a) = \frac{1}{2k+7}$, $p(b) = \frac{k}{2k+7}$, and $p(c) = p(d) = \frac{6+k}{4k+14}$ for A. To see this, we note that both c and d maximize the approval score with $k+3$ approvals each. Moreover, there are k voters who approve both c and d and thus split their shares between these candidates, so $p(c) = p(d) = \frac{6+k}{2\cdot(2k+7)}$. After this step, k voters approving b and $k-1$ voters approving a remain active. Since both candidates have the same approval scores in A, candidate b will be assigned its share next. In particular, all voters reporting $\{a, b\}$ send their $\frac{1}{n}$ budget to b, so $p(b) = \frac{k}{2k+7}$. Finally, a only gets the share of the single voter who reports $\{a\}$, so $p(a) = \frac{1}{2k+7}$.

On the other hand, FUT chooses the distribution q with $q(a) = \frac{k+1}{2k+7}$, $q(b) = \frac{2}{2k+7}$, $q(c) = 0$, and $q(d) = \frac{k+4}{2k+7}$ for A'. To see this, we observe that d has now the uniquely maximal approval score of $k+4$. Hence, all voters approving d send their share to this candidate and $q(d) = \frac{k+4}{2k+7}$. This leaves $k+1$ active voters approving a, k active voters approving b, and 3 active voters approving c. Consequently, a is chosen next when $\lambda = \frac{k+3}{k+1}$ because $(k+1)\lambda + 1 = k + 4$. We also note that, for this λ, it holds that $k\lambda + 2 = k\frac{k+3}{k+1} + 2 = k + 2 + \frac{2k}{k+1} < k + 4$ and $3\lambda + k = 3\frac{k+3}{k+1} + k < 3 \cdot \frac{4}{3} + k = k + 4$ as $k \geq 6$. These inequalities show that a is indeed assigned its share before b and c, so $q(a) = \frac{k+1}{2k+7}$. Finally, as no active voter approves c after this step, the remaining budget goes to b and $q(b) = \frac{2}{2k+7}$.

Lastly, we consider the utility of the left-most voter i in A and A'. Assuming that A is the truthful profile, we have that $u_i(p) = \frac{1}{2k+7}$ and $u_i(q) = \frac{k+1}{2k+7}$, so the incentive ratio of FUT is lower bounded by $\frac{u_i(q)}{u_i(p)} = k + 1 = \frac{n-5}{2}$. This completes the proof that $\gamma(\mathsf{FUT}) \in \Theta(n)$.

Claim (3): For the upper bound on the incentive ratio of EGAL, we note that, under this rule, every voter has a utility of at least $\frac{1}{m}$. In more detail, it holds for the uniform distribution over all candidates p_{uni} that $u_i(p_{uni}) \geq \frac{1}{m}$ for all $i \in N$. Since EGAL maximizes the minimal utility of a voter, this means that $u_i(p) \geq \frac{1}{m}$ for the distribution p chosen by EGAL in every profile. Thus, the incentive ratio of EGAL is at most m since the maximal utility of a voter is 1.

For the lower bound, we choose an integer $k \geq 3$ and consider the following profile A with k^2 voters and $2k + 2$ candidates $C = \{x_1, \ldots, x_k, y_1, \ldots, y_{k+2}\}$. For notational convenience, we define the set of voters as $N = [k]^2$, i.e., every voter is denoted by at tuple (i, j), where i is best interpreted as row index and j as column index.

- Voter $(1, 1)$ reports $\{x_1, y_1, \ldots, y_{k+2}\}$.
- For all $i \in [k]$, $j \in [k-1]$ with $(i, j) \neq (1, 1)$, voter (i, j) reports $\{x_i, y_j\}$.
- For all $i \in [k-2]$, the voters (i, k) report $\{x_i, y_k, y_{k+1}\}$.
- Voter $(k-1, k)$ reports $\{x_{k-1}, y_{k+1}, y_{k+2}\}$ and voter (k, k) reports $\{x_k, y_k, y_{k+2}\}$.

Further, the profile A' is derived from A by changing the ballot of voter $(1,1)$ to $\{y_1\}$. We subsequently display the profiles A and A' for $k = 4$.

A:	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$\{x_1, y_1, \dots, y_6\}$	$\{x_1, y_2\}$	$\{x_1, y_3\}$	$\{x_1, y_4, y_5\}$
$i=2$	$\{x_2, y_1\}$	$\{x_2, y_2\}$	$\{x_2, y_3\}$	$\{x_2, y_4, y_5\}$
$i=3$	$\{x_3, y_1\}$	$\{x_3, y_2\}$	$\{x_3, y_3\}$	$\{x_3, y_5, y_6\}$
$i=4$	$\{x_4, y_1\}$	$\{x_4, y_2\}$	$\{x_4, y_3\}$	$\{x_4, y_4, y_6\}$

A':	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$\{y_1\}$	$\{x_1, y_2\}$	$\{x_1, y_3\}$	$\{x_1, y_4, y_5\}$
$i=2$	$\{x_2, y_1\}$	$\{x_2, y_2\}$	$\{x_2, y_3\}$	$\{x_2, y_4, y_5\}$
$i=3$	$\{x_3, y_1\}$	$\{x_3, y_2\}$	$\{x_3, y_3\}$	$\{x_3, y_5, y_6\}$
$i=4$	$\{x_4, y_1\}$	$\{x_4, y_2\}$	$\{x_4, y_3\}$	$\{x_4, y_4, y_6\}$

We claim that EGAL chooses for A the distribution p with $p(x_i) = \frac{1}{k}$ for all $i \in [k]$. To see this, we observe that p guarantees each voter a utility of $\frac{1}{k}$. We will next show that p is indeed the only distribution that satisfies this. To this end, we let p' denote a distribution such that $u_{i,j}(p') \geq \frac{1}{k}$ for all $(i,j) \in N$ and show that $p' = p$. Consider the following three sets of voters for this: $S_1 = \{(i, k+1-i) \colon i \in [k]\}$ are the voters on the counterdiagonal, $S_2 = \{(i, k-i) \colon i \in [k-1]\} \cup \{(k,k)\}$ are the voters that are one "above" the counterdiagonal and (k,k), and $S_3 = \{(i, k-i) \colon i \in [k-2]\} \cup \{(k,1), (k-1,k)\}$ is derived from S_2 by replacing the voters $(k-1,1)$ and (k,k) with $(k,1)$ and $(k-1,k)$. Since each set contains k voters, it holds that $\sum_{(i,j)\in S_\ell} u_{i,j}(p') \geq k \cdot \frac{1}{k} = 1$ for all $\ell \in \{1,2,3\}$. Next, each set contains for each row i and each column j exactly one voter, and none of them contains voter $(1,1)$. By the construction of A, this means for each $\ell \in \{1,2,3\}$ that the approval ballots of the voters in S_ℓ are pairwise disjoint. We thus compute

$$
\begin{aligned}
1 \leq \sum_{(i,j)\in S_1} u_{i,j}(p') &= \sum_{i\in[k]} p'(x_i) + \sum_{j\in[k+2]\setminus\{k+2\}} p'(y_j), \\
1 \leq \sum_{(i,j)\in S_2} u_{i,j}(p') &= \sum_{i\in[k]} p'(x_i) + \sum_{j\in[k+2]\setminus\{k+1\}} p'(y_j), \\
1 \leq \sum_{(i,j)\in S_3} u_{i,j}(p') &= \sum_{i\in[k]} p'(x_i) + \sum_{j\in[k+2]\setminus\{k\}} p'(y_j).
\end{aligned}
$$

Since p' is a distribution, the shares of all candidates sum up to 1, so we infer form these inequalities that $p'(y_{k+2}) = p'(y_{k+1}) = p'(y_k) = 0$. This means for the voters in the k-th column that $u_{i,k}(p') = p'(x_i)$. Because $u_{i,k}(p') \geq \frac{1}{k}$ for all $i \in [k]$, we hence derive that $p'(x_i) = \frac{1}{k}$ for all $i \in [k]$ and $p' = p$.

Next, for the profile A', EGAL chooses the distribution q with $q(y_j) = \frac{2}{2k+1}$ for all $j \in [k-1]$ and $q(y_j) = \frac{1}{2k+1}$ for $j \in \{k, k+1, k+2\}$. We first observe that this distribution guarantees each voter in A' a utility of $\frac{2}{2k+1}$. We will again show that q is the only distribution that meets this condition. Thus, we fix a distribution q' such that $u_{i,j}(q') \geq \frac{2}{2k+1}$ for all (i,j) in A' and show that $q' = q$. Similar to the last case, we consider two sets of voters: $S_1 = \{(i,i) \colon i \in [k]\}$ are the voters on the main diagonal and $S_2 = \{(i,i) \in [k-2]\} \cup \{(k-1,k), (k,k-1)\}$ replaces the voters $(k-1,k-1)$ and (k,k) with $(k-1,k)$ and $(k,k-1)$. Now, we have again k voters per set, so $\sum_{(i,j)\in S_\ell} u_{i,j}(q') \geq k \cdot \frac{2}{2k+1}$ for $\ell \in \{1,2\}$. Moreover, both sets contain exactly one voter for each row and each column and both of them contain voter $(1,1)$. We hence compute that

$$
\begin{aligned}
\frac{2k}{2k+1} \leq \sum_{(i,j)\in S_1} u_{i,j}(q') &= \sum_{i=2}^{k} q'(x_i) + \sum_{j\in[k+2]\setminus\{k+1\}} q'(y_j), \\
\frac{2k}{2k+1} \leq \sum_{(i,j)\in S_2} u_{i,j}(q') &= \sum_{i=2}^{k} q'(x_i) + \sum_{j\in[k+2]\setminus\{k\}} q'(y_j).
\end{aligned}
$$

Since q' is a distribution, we infer from the first inequality that $q'(x_1) + q'(y_{k+1}) \leq \frac{1}{2k+1}$ and from the second one that $q'(x_1) + q'(y_k) \leq \frac{1}{2k+1}$. On the other hand, it holds that $u_{1,k}(q') = q'(x_1) + q'(y_k) + q'(y_{k+1}) \geq \frac{2}{2k+1}$. These three inequalities (and the condition that $q'(x_1) \geq 0$) are only satisfied if $q'(x_1) = 0$ and $q'(y_k) = q'(y_{k+1}) = \frac{1}{2k+1}$. Further, since $q'(x_1) = 0$, it follows for all $j \in [k-1]$ that $u_{1,j}(q') = q'(y_j)$. Hence, we conclude that $q'(y_j) \geq \frac{2}{2k+1}$ for all $j \in [k-1]$. Lastly, the unallocated budget so far is at most $1 - \sum_{j\in[k+1]} q'(y_j) \leq \frac{1}{2k+1}$, which needs to be

Rule	Strategyproofness	Efficiency	Fairness (AFS)
UTIL	$1^\dagger$	$1^\dagger$	$\infty^\dagger$
CUT	$1^\dagger$	$\Theta(\sqrt[3]{n})^\dagger$	$\Theta(n)^\ddagger$ (GFS)†
NASH	2	$1^\dagger$	$1^\dagger$
EGAL	$\Theta(m)$ (Excl. strategypr.)†	$1^\dagger$	$\Theta(m)^\dagger$ (IFS)†
FUT	$\Theta(n)$ (Monotonicity)*	1^*	$\Theta(n)^\ddagger$ (GFS)*
MP	$\Theta(n)$ (Monotonicity)‡	$\Theta(\log n)^\ddagger$	$2^\ddagger$ (GFS)‡

Table 1: Approximation ratios of common approval-based distribution rules to strategyproofness, efficiency, and fairness. In each row, we show for the given rule its approximation ratio to strategyproofness (i.e., its incentive ratio), efficiency, and fairness. If a rule satisfies an axiomatic weakening of strategyproofness or AFS (defined in Section 3.3), this is indicated in brackets (one exception being GFS, which is incomparable to AFS as noted in footnote 5). Approximate efficiency and fairness are measured by the minimal values α and β such that, when multiplying the utilities of all voters under the given rule by α (resp. β), efficiency (resp. AFS) is guaranteed to be satisfied. Our results are marked in blue. Results marked by a dagger ($\dagger$), asterisk ($*$), and double dagger ($\ddagger$) can be found in Aziz *et al.* [5], Brandl *et al.* [15], and Aziz *et al.* [6], respectively. These references also provide the definitions of all rules and axioms in the table, most of which are recapped in Sections 3.2 and 3.3.

used so that the voters $(k-1,k)$ and (k,k) meet the utility threshold of $\frac{2}{2k+1}$. In the current allocation, both voters obtain a utility of precisely $\frac{1}{2k+1}$ from y_k and y_{k+1}, respectively. Hence, we need to allocate the remaining budget to y_{k+2} as this is the only commonly approved candidate for both voters. Thus, we conclude that $q' = q$, as desired.

As the last point, we compute the utility gain for voter $(1,1)$ when deviating from A to A'. To this end, we note that utility of this voter in A is $u_{1,1}(p) = p(x_1) = \frac{1}{k}$, whereas her utility for q is $u_{1,1}(q) = \sum_{j\in[k+2]} q(y_j) = 1$. Hence, the incentive ratio γ of EGAL is lower bounded by $\frac{u_{1,1}(q)}{u_{1,1}(p)} = k = \frac{m-2}{2}$. This proves that $\gamma(\mathsf{EGAL}) \in \Theta(m)$. □

Remark 3. To put the incentive ratios of NASH and the other approval-based distribution rules into perspective, it is insightful to also consider how well these rules fare with respect to fairness and efficiency criteria. We present such a summary in Table 1, where we review the approximation ratios of six of the most common approval-based distribution rules to strategyproofness (measured by the incentive ratio), efficiency, and fairness (specifically, an axiom called average fair share defined in Section 3.3) as well as axiomatic properties satisfied by these rules. This table again highlights the strong performance of NASH: it is the only rule among the six listed ones that obtains a constant-factor approximation to each property and even fully satisfies efficiency and the given fairness axiom.

3.3 Optimality of NASH

A natural follow-up question to Theorems 2 and 3 is whether it is at all possible to design fair and efficient rules with a lower incentive ratio than NASH. As we will show next, such rules do not exist within three natural classes of distribution rules, even when only allowing for approval utilities. This suggests that the incentive ratio of NASH may already be optimal in approval-based budget division subject to fairness and efficiency. Specifically, we will focus on the following classes of rules.

Rules satisfying AFS. Average fair share requires that, if all voters in a group S approve a common candidate, then the average utility of the group S should be at least proportional to the size of S. More formally, a distribution p satisfies *average fair share (AFS)* for a profile A if $\frac{1}{|S|}\sum_{i\in S} u_i(p) \geq \frac{|S|}{n}$ for all groups of voters $S \subseteq N$ with $\bigcap_{i\in S} A_i \neq \emptyset$. This is a rather demanding property and, among all commonly studied rules, only the Nash product rule satisfies it. By

showing that each rule satisfying AFS has an incentive ratio of at least 2, it follows that the incentive ratio of NASH is optimal subject to its fairness guarantees.

Strictly concave welfare maximizers (SCWMs). As the second class, we study strictly concave welfare maximizers. To define these rules, we say a function $h : [0,1] \to \mathbb{R} \cup \{-\infty\}$ is a *welfare function* if *(i)* $h(x) \neq -\infty$ for all $x > 0$ and *(ii)* h is strictly increasing. Then, a distribution rule f is an (additively separable) *welfare maximizer* if there is a welfare function h such that f always returns a distribution maximizing the h-welfare, i.e., $f(A) \in \arg\max_{p \in \Delta(C)} \sum_{i \in N} h(u_i(p))$ for all profiles A. Further, f is a *strictly concave welfare maximizer (SCWM)* if h is strictly concave (i.e., $h(\lambda x + (1-\lambda)y) > \lambda h(x) + (1-\lambda)h(y)$ for all distinct $x, y \in [0,1]$ and $\lambda \in (0,1)$). For example, this class contains all rules that optimize a power welfare $\sum_{i \in N}(u_i(p))^\alpha$ for $\alpha \in (0,1)$.

We are interested in SCWMs since it contains most known rules that are efficient. Indeed, the only efficient distribution rule that is not part of this class but has been seriously considered in the literature is FUT, for which we already have shown that it has a poor incentive ratio. Moreover, SCWMs may seem a promising route to obtaining an incentive ratio of less than 2. For example, one may conjecture that the SCWM defined by $h(x) = x^{1-\epsilon}$ for a very small $\epsilon > 0$ has an incentive ratio less than 2 as it is close to maximizing the utilitarian welfare. As an alternative approach, this family allows one to tweak the objective of NASH in the hope of improving the incentive ratio. As our result for SCWMs shows, these approaches do not allow to construct a rule with an incentive ratio of less than 2.

Regular rules satisfying GFS and efficiency. Thirdly, we will focus on rules that satisfy efficiency and a fairness property known as group fair share. The idea of group fair share is that, for each group of voters S, we should spend a share proportional to the size of S on the approved candidates of the group. More formally, a distribution p satisfies *group fair share (GFS)* for a profile A if, for all groups of voters $S \subseteq N$, it holds that $\sum_{x \in \bigcup_{i \in S} A_i} p(x) \geq \frac{|S|}{n}$. GFS is a fairly mild fairness condition which is satisfied by many known rules, including MP, FUT, and NASH.[5] Furthermore, we recall that a distribution rule f is *(Pareto) efficient* if, for each profile A, there is no distribution p such that $u_i(p) \geq u_i(f(U))$ for all voters $i \in N$ and $u_i(p) > u_i(f(U))$ for some voter $i \in N$. However, since both of these conditions are rather mild, we will additionally require three basic auxiliary conditions, which we summarize by regularity. Specifically, a distribution rule f is *regular* if it is

- *anonymous*: $f(\pi(A)) = f(A)$ for all profiles A and permutations $\pi : N \to N$. Intuitively, this condition requires that the names of voters do not matter for the outcome of the rule.
- *neutral*: $f(\tau(A), \tau(x)) = f(A, x)$ for all profiles A and permutations $\tau : C \to C$. Here, $A' = \tau(A)$ denotes the profile such that, for every voter $i \in N$ and candidate $x \in C$, it holds that $\tau(x) \in A'_i$ if and only if $x \in A_i$. This axiom captures that the names of the candidates should not affect the outcome of our rule.
- *independent of losers*: $f(A) = f(A')$ for all profiles A, A' and candidates x such that $f(A, x) = 0$ and $A' = (A_i \setminus \{x\})_{i \in N}$ is obtained from A by deleting x from the approval ballot of every voter. Put differently, this axiom demands that removing candidates that have not obtained any budget from the profile does not affect the outcome of our distribution rule.

These three properties are very mild and satisfied by all rules in this paper (except for MP which fails neutrality). Our third claim in Theorem 4 thus shows that the incentive ratio of NASH is already optimal when only requiring efficiency, a mild fairness axiom, and some basic auxiliary conditions.

We show next that every rule that belongs to one of these classes has an incentive ratio of at least 2. Since NASH is a member of each class, all of our bounds are tight.

Theorem 4. *The following claims are true:*

(1) Every distribution rule that satisfies AFS has an incentive ratio of at least 2*.*

(2) Every SCWM has an incentive ratio of at least 2*.*

(3) Every regular distribution rule that satisfies GFS and efficiency has an incentive ratio of at least 2*.*

[5] Despite GFS being quite permissive compared to AFS, it is not logically implied by AFS [54].

Proof. We prove each of the three claims separately.

Proof of Claim (1): Let f denote a rule satisfying AFS, and fix two integers $\ell, k \in \mathbb{N}$ with $k \geq \ell \geq 2$. We consider the following profile A with $2k + \ell + 1$ voters and $\ell + 1$ candidates $C = \{x, y_1, \ldots, y_\ell\}$:

- For each $i \in [\ell]$, there is one voter who reports $\{x, y_i\}$.
- There are $k + 1$ voters who only approve x.
- There are k voters who report $\{y_1, \ldots, y_\ell\}$.

Next, the profile A' is derived from A by changing the approval ballot of the voter i^* who reports $\{x, y_1\}$ in A to $\{y_1\}$. For example, if $\ell = 3$ and $k = 9$, the profiles look as follows.

A:	$1\colon\{x, y_1\}$	$1\colon\{x, y_2\}$	$1\colon\{x, y_3\}$	$10\colon\{x\}$	$9\colon\{y_1, y_2, y_3\}$
A':	$1\colon\{y_1\}$	$1\colon\{x, y_2\}$	$1\colon\{x, y_3\}$	$10\colon\{x\}$	$9\colon\{y_1, y_2, y_3\}$

We first analyze the distribution $p = f(A)$. To this end, we observe that AFS implies for the k voters who report $\{y_1, \ldots, y_\ell\}$ that $\frac{k}{k} \sum_{i \in [\ell]} p(y_i) \geq \frac{k}{2k+\ell+1}$. Without loss of generality, we assume that y_1 is the candidate with the minimal probability in $\{y_1, \ldots, y_\ell\}$, which means that $p(y_1) \leq \frac{1}{\ell} \sum_{i \in [\ell]} p(y_i)$ and $\sum_{i \in [\ell] \setminus \{1\}} p(y_i) \geq \frac{\ell-1}{\ell} \sum_{i \in [\ell]} p(y_i) \geq \frac{\ell-1}{\ell} \cdot \frac{k}{2k+\ell+1}$. In turn, this implies for the utility of voter i^* (who reports $\{x, y_1\}$ in A) that $u_{i^*}(p) \leq 1 - \frac{\ell-1}{\ell} \cdot \frac{k}{2k+\ell+1}$.

Next, we turn to $q = f(A')$. By applying AFS to the sets $N_x = \{i \in N \colon x \in A'_i\}$ and $N_{y_1} = \{i \in N \colon y_1 \in A'_i\}$, respectively, we infer the following two inequalities for q.

$$\sum_{i \in N_x} u_i(q) = (k + \ell) q(x) + \sum_{i \in [\ell] \setminus \{1\}} q(y_i) \geq \frac{(k+\ell)^2}{2k + \ell + 1}$$

$$\sum_{i \in N_{y_1}} u_i(q) = (k + 1) q(y_1) + k \sum_{i \in [\ell] \setminus \{1\}} q(y_i) \geq \frac{(k+1)^2}{2k + \ell + 1}$$

By summing up these two inequalities, we derive that

$$(k + \ell) q(x) + (k + 1) \sum_{i \in [\ell]} q(y_i) \geq \frac{(k+\ell)^2 + (k+1)^2}{2k + \ell + 1}.$$

Since $q(x) + \sum_{i \in [\ell]} q(y_i) = 1$, this means that

$$\begin{aligned}(\ell - 1) q(x) &\geq \frac{(k+\ell)^2 + (k+1)^2 - (k+1)(2k+\ell+1)}{2k + \ell + 1} \\ &= \frac{k(\ell-1) + \ell(\ell-1)}{2k + \ell + 1}.\end{aligned}$$

This shows that $q(x) \geq \frac{k+\ell}{2k+\ell+1}$. In turn, we derive that $\sum_{i \in [\ell]} q(y_i) \leq \frac{k+1}{2k+\ell+1}$ and by the AFS inequality for y_1 that $q(y_1) \geq \frac{k+1}{2k+\ell+1}$. Since voter i^* reports $\{x, y_1\}$ in A, this proves that $u_{i^*}(q) = 1$. Hence, the incentive ratio of f is at least $\frac{u_{i^*}(q)}{u_{i^*}(p)} \geq \frac{1}{1 - \frac{\ell-1}{\ell} \cdot \frac{k}{2k+\ell+1}}$. When setting $k = \ell^2$ and letting ℓ go to infinity, this fraction converges to 2, so every rule satisfying AFS has an incentive ratio of at least 2.

Proof of Claim (2): Fix a strictly concave welfare function h and let f denote the induced SCWM. Further, we fix an integer $\ell \in \mathbb{N}$ with $\ell \geq 2$, and consider the following profile A with 2ℓ voters and $\ell + 2$ candidates $C = \{x_1, x_2, y_1, \ldots, y_\ell\}$.

- For each $i \in [\ell]$, there is one voter who approves x_1 and all candidates $\{y_1, \ldots, y_\ell\}$ except y_i. Put differently, these voters approve $C \setminus \{x_2, y_i\}$ for $i \in [\ell]$.

- There is one voter who approves all candidates but x_1. We denote this voter by i^*.
- There are $\ell - 1$ voters who only approve x_2.

Further, the profile A' is derived from A by changing the ballot of voter i^* to $C \setminus \{x_1, x_2\}$, i.e., this voter additionally disapproves x_2. We subsequently illustrate the profiles A and A' for $\ell = 3$.

$$\begin{array}{lccccc} A\colon & 1\colon \{x_1, y_2, y_3\} & 1\colon \{x_1, y_1, y_3\} & 1\colon \{x_1, y_1, y_2\} & 1\colon \{x_2, y_1, y_2, y_3\} & 2\colon \{x_2\} \\ A'\colon & 1\colon \{x_1, y_2, y_3\} & 1\colon \{x_1, y_1, y_3\} & 1\colon \{x_1, y_1, y_2\} & 1\colon \{y_1, y_2, y_3\} & 2\colon \{x_2\} \end{array}$$

We claim that f chooses for A the distribution p with $p(x_1) = p(x_2) = 0.5$. To prove this, we first observe that every candidate is approved by exactly half of the voters in A. Hence, for every distribution p', it holds that $\sum_{i \in N} u_i(p') = \sum_{x \in C} \sum_{i \in N\colon x \in A_i} p'(x) = \ell \sum_{x \in C} p'(x) = \ell$. Now, assume for contradiction that f chooses a distribution p' for A such that $u_i(p') \neq 0.5$ for some voter $i \in N$. By the strict concavity of h, we have that

$$\sum_{i \in N} h(0.5) = \sum_{i \in N} h\left(\frac{1}{2\ell} \sum_{j \in N} u_j(p')\right) > \sum_{i \in N} \frac{1}{2\ell} \sum_{j \in N} h(u_j(p')) = \sum_{i \in N} h(u_j(p')).$$

This proves that p generates a larger h-welfare than p', so p' cannot be chosen by h. Finally, to ensure that every voter has a utility of 0.5, we must have $p(x_2) = 0.5$ because there are $\ell - 1$ voters who only approve x_2. Further, since all voters need to have utility of 0.5, this implies that $p(y_i) = 0$ for all $i \in [\ell]$ as voter i^* approves all candidates but x_1. In turn, this proves that $p(x_1) = 0.5$.

Next, for A', we claim that f has to choose a distribution q with $q(x_1) = q(y_1) = \cdots = q(y_\ell)$. To see this, we set $y_0 = x_1$ and define $a +_{\ell+1} b = (a + b) \mod (\ell + 1)$ for all $a, b \in \mathbb{N}$. Now, fix a distribution q' that violates our equation, and denote by q^k for $k \in \{0, \dots, \ell\}$ the distribution with $q^k(x_2) = q'(x_2)$ and $q^k(y_j) = q'(y_{j+_{\ell+1}k})$ for all $j \in \{0, \dots, \ell\}$. We note that, for all candidates y_i, y_j, it holds that $\sum_{k=0}^{\ell} q^k(y_i) = \sum_{k=0}^{\ell} q'(y_k) = \sum_{k=0}^{\ell} q^k(y_j)$. Lastly, let q^* be the distribution defined by $q^* = \frac{1}{\ell+1} \sum_{k=0}^{\ell} q^k$ and observe that $q^*(y_i) = q^*(y_j)$ for all $i, j \in \{0, \dots, \ell\}$. This also means that $q^* \neq q'$ as q' fails this condition.

We will show that $\sum_{i \in N} h(u_i(q^*)) > \sum_{i \in N} h(u_i(q'))$, which proves that f cannot choose q' for A'. First, for the voters who approve only x_2, we clearly have that $u_i(q') = u_i(q^*)$ since $q'(x_2) = q^*(x_2)$. On the other hand, for the remaining $\ell + 1$ voters, we observe that, for every candidate $y_i \in \{y_0 = x_1, y_1 \dots, y_\ell\}$, there is exactly one voter who reports $C \setminus \{y_i, x_2\}$. This means that, for every voter i and integer k, there is precisely one voter j such that $u_i(q^k) = u_j(q^0)$. Hence, when letting S denote the set of these $\ell + 1$ voters, we have for each $i \in S$ that $u_i(q^*) = \frac{1}{\ell+1} \sum_{j \in S} u_j(q^0)$. Using the strict concavity of h, we compute that

$$\sum_{i \in S} h(u_i(q^*)) = \sum_{i \in S} h\left(\frac{1}{\ell+1} \sum_{j \in S} u_j(q^0)\right) > \sum_{i \in S} \frac{1}{\ell+1} \sum_{j \in S} h(u_j(q^0)) = \sum_{i \in S} h(u_i(q')).$$

In summary, we conclude that $\sum_{i \in N} h(u_i(q^*)) > \sum_{i \in N} h(u_i(q'))$, as desired. Consequently, the distribution q chosen for A' satisfies that $q(x_1) = q(y_1) = \cdots = q(y_\ell)$.

Finally, by our analysis so far, we conclude that voter i^* (who reports $C \setminus \{x_1\}$ in A) has utility $u_i(p) = \frac{1}{2}$ in A and utility $u_i(q) = q(x_2) + \frac{\ell}{\ell+1}(1 - q(x_2)) = \frac{\ell}{\ell+1} + \frac{1}{\ell+1} q(x_2)$ in A'. For the second expression, we use that the probability not assigned to x_2 by q is spread uniformly over the remaining $\ell + 1$ candidates, and that voter i^* approves ℓ of them. As a consequence, the incentive ratio of f is lower bounded by

$$\frac{u_i(q)}{u_i(p)} = \frac{\frac{\ell}{\ell+1} + \frac{1}{\ell+1} q(x_2)}{\frac{1}{2}} \geq \frac{2\ell}{\ell + 1}.$$

Finally, by letting ℓ grow to infinity, this term converges to 2, which proves that the incentive ratio of f is at least 2.

Proof of Claim (3): Let f denote a rule that satisfies efficiency, group fair share, anonymity, neutrality, and independence of losers. Further, we fix some value $k \in \mathbb{N}$ with $k \geq 3$, define the set of voters by $N = [k + 1]^2$, and the

A^1:	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$\{x_1, y_{1,1}\}$	$\{x_1, y_{1,2}\}$	$\{x_1, y_{1,3}\}$	$\{x_1, x_2, x_3, x_4\}$
$i=2$	$\{x_2, y_{2,1}\}$	$\{x_2, y_{2,2}\}$	$\{x_2, y_{2,3}\}$	$\{z, y_{1,1}, y_{2,1}, y_{3,1}, y_{1,3}, y_{2,3}, y_{3,3}\}$
$i=3$	$\{x_3, y_{3,1}\}$	$\{x_3, y_{3,2}\}$	$\{x_3, y_{3,3}\}$	$\{z, y_{1,1}, y_{2,1}, y_{3,1}, y_{1,2}, y_{2,2}, y_{3,2}\}$
$i=4$	$\{z, x_4\}$	$\{x_4\}$	$\{x_4\}$	$\{x_1, x_2, x_3, x_4\}$

A^2:	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$\{x_1, y_{1,1}\}$	$\{x_1\}$	$\{x_1\}$	$\{x_1, x_2, x_3, x_4\}$
$i=2$	$\{x_2, y_{2,1}\}$	$\{x_2\}$	$\{x_2\}$	$\{z, y_{1,1}, y_{2,1}, y_{3,1}\}$
$i=3$	$\{x_3, y_{3,1}\}$	$\{x_3\}$	$\{x_3\}$	$\{z, y_{1,1}, y_{2,1}, y_{3,1}\}$
$i=4$	$\{z, x_4\}$	$\{x_4\}$	$\{x_4\}$	$\{x_1, x_2, x_3, x_4\}$

A^3:	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$\{x_1, y_{1,1}\}$	$\{x_1, y_{1,2}\}$	$\{x_1, y_{1,3}\}$	$\{x_1, x_2, x_3, x_4\}$
$i=2$	$\{x_2, y_{2,1}\}$	$\{x_2, y_{2,2}\}$	$\{x_2, y_{2,3}\}$	$\{z, y_{1,1}, y_{2,1}, y_{3,1}, y_{1,3}, y_{2,3}, y_{3,3}\}$
$i=3$	$\{x_3, y_{3,1}\}$	$\{x_3, y_{3,2}\}$	$\{x_3, y_{3,3}\}$	$\{z, y_{1,1}, y_{2,1}, y_{3,1}, y_{1,2}, y_{2,2}, y_{3,2}\}$
$i=4$	$\{z, y_{1,2}, y_{2,2}, y_{3,2}, y_{1,3}, y_{2,3}, y_{3,3}\}$	$\{x_4\}$	$\{x_4\}$	$\{x_1, x_2, x_3, x_4\}$

Table 2: Profiles for the proof of Claim (3) of Theorem 4

set of candidates by $C = \{x_i \colon i \in [k+1]\} \cup \{y_{i,j} \colon i,j \in [k]\} \cup \{z\}$. For this proof, we again denote voters by tuples (i,j), where i and j are best interpreted as row and column indexes, respectively. Further, to simplify notation, we define $X = \{x_i \colon i \in [k+1]\}$ and $Y = \{y_{i,j} \colon i,j \in [k]\}$. Lastly, for our proof, we will focus on three profiles, which are illustrated in Table 2 for $k = 3$. First, the main profile A^1 is constructed as follows:

- Every voter $(i,j) \in [k]^2$ approves candidate x_i and $y_{i,j}$.
- For all $i \in [k]$ with $i > 1$, each voter $(i, k+1)$ approves $\{z\} \cup Y \setminus \{y_{j,i} \colon j \in [k]]\}$. Further, voters $(1, k+1)$ and $(k+1, k+1)$ approve X.
- For all $j \in [k]$ with $j > 1$, each voter $(k+1, j)$ only approves x_{k+1}. Further, voter $(1, k+1)$ reports $\{x_{k+1}, z\}$.

We start by analyzing the distribution p^1 chosen by f for A^1. To this end, we observe that for all $i \in [k]$, $j, j' \in [k] \setminus \{1\}$, the candidates $y_{i,j}$ and $y_{i,j'}$ are symmetric to each other in A^1. To make this more formal, we define the permutation $\pi^{j,j'} : C \to C$ for all $j, j' \in [k] \setminus \{1\}$ by *(i)* $\pi^{j,j'}(y_{i,j}) = y_{i,j'}$ and $\pi^{j,j'}(y_{i,j'}) = y_{i,j}$ for all $i \in [k]$ and *(ii)* $\pi^{j,j'}(c) = c$ for all other candidates c. Applying this permutation to A^1 has two effects: we exchange the j-th and j'-th columns, and the voters $(j, k+1)$ and $(j', k+1)$ swap their ballots. Hence, up to renaming voters, the resulting profile is equal to A^1. By anonymity and neutrality, it thus follows that $p^1(y_{i,j}) = p^1(y_{i,j'})$ for all $i \in [k]$ and $j, j' \in [k] \setminus \{1\}$.

Next, we will show that $p^1(y_{i,j}) = 0$ for all $i \in [k]$ and $j \in [k] \setminus \{1\}$. Assume for contradiction that this is not true, i.e., there are indices $i \in [k]$, $j \in [k] \setminus \{1\}$ such that $p^1(y_{i,j}) = \epsilon > 0$. By our previous observation, this means that $p^1(y_{i,j'}) = \epsilon$ for all $j' \in [k] \setminus \{1\}$. We will show that p^1 is inefficient. To this end, we define the distribution q by $q(x_i) = p^1(x_i) + \epsilon$, $q(y_{i,1}) = p^1(y_{i,1}) + (k-2)\epsilon$, $q(y_{i,j'}) = 0$ for all $j' \in [k] \setminus \{1\}$, and $q(c) = p^1(c)$ for all other candidates. We claim that $u_{i',j'}(q) \geq u_{i',j'}(p^1)$ for all voters and $u_{i',j'}(q) > u_{i',j'}(p^1)$ for at least one voter. To see the second claim, we note that

$$u_{1,k+1}(q) = \sum_{\ell \in [k+1]} q(x_\ell) = \epsilon + \sum_{\ell \in [k+1]} p^1(x_\ell) > u_{1,k+1}(p^1).$$

On the other hand, to see that the utility of no voter decreases, we first observe that $u_{i',j'}(q) \geq u_{i',j'}(p^1)$ for all voters who do not approve any candidate $y_{i,j}$ with $j \in [k] \setminus \{1\}$. This leaves us with the voters (i,ℓ) for $\ell \in [k] \setminus \{1\}$ (i.e., those in the i-th row) and the voters $(\ell, k+1)$ for $\ell \in [k] \setminus \{1\}$ (i.e., those in the $k+1$-th column). For the voters (i,ℓ) with $\ell \in [k] \setminus \{1\}$, it holds that

$$u_{i,\ell}(q) = q(x_i) + q(y_{i,\ell}) = p^1(x_i) + \epsilon + 0 = u_{i,\ell}(p^1).$$

On the other hand, for the voters $(\ell, k+1)$ with $\ell \in [k] \setminus \{1\}$, we note that all of these voters approve $y_{i,1}$ and exactly $k-2$ candidates of $\{y_{i,2}, \dots, y_{i,k}\}$. Thus, the utilities of these voters satisfies that

$$\begin{aligned} u_{\ell,k+1}(q) &= \sum_{c \in A^1_{\ell,k+1}} q(c) \\ &= (k-2)\epsilon - \sum_{j' \in [k] \setminus \{\ell\}} p^1(y_{i,j'}) + \sum_{c \in A^1_{\ell,k+1}} p^1(c) \\ &= u_{\ell,k+1}(p^1). \end{aligned}$$

This proves that q indeed Pareto-dominates p^1, so our assumption that $p^1(y_{i,j}) > 0$ must have been wrong. Put differently, this means that $p^1(y_i, j) = 0$ for all $i \in [k]$, $j \in [k] \setminus \{1\}$.

Now, let A^2 denote the profile derived from A^1 by deleting all candidates $y_{i,j}$ with $i \in [k]$, $j \in [k] \setminus \{1\}$ from our profile and let p^2 denote the distribution chosen by f for this profile. An example of A^2 for $k = 3$ is depicted in Table 2. It is easy to see that, in A^2, all candidates in X and the candidates in $Y' = \{y_{1,1}, \dots, y_{k,1}, z\}$ are symmetric to each other, respectively. In particular, this means that $p^2(x_1) = \dots = p^2(x_{k+1})$ and $p^2(y_{1,1}) = \dots = p^2(y_{1,k}) = p^2(z)$ by anonymity and neutrality. For voter $(k+1, 1)$, this means that $u_{k+1,1}(p^2) = p^2(z) + p^2(x_{k+1}) = \frac{1}{k+1} \sum_{c \in X} p^2(c) + \frac{1}{k+1} \sum_{c \in Y'} p^2(c) = \frac{1}{k+1}$. Further, since we derived A^2 from A^1 by only deleting candidates with probability 0 in p^1, independence of losers requires that $p^1 = p^2$ and thus $u_{k+1,1}(p^1) = \frac{1}{k+1}$, too.

Lastly, consider the profile A^3 derived from A^1 by assigning voter $(k+1, 1)$ the ballot $\{z\} \cup Y \setminus \{y_{1,1}, \dots, y_{k,1}\}$ and let $p^3 = f(A^3)$. Using an analogous argument as for A^1, it can be shown that $p^3(y_{i,j}) = p^3(y_{i,j'})$ for all $i, j, j' \in [k]$ based on anonymity and neutrality. Note that, in contrast to A^1, this inequality now also includes the candidates $y_{i,1}$ as voter $(k+1, 1)$ mimics the behavior of the voters in the $k+1$-th row for the candidates $y_{i,1}$. We claim that $p^3(y_{i,j}) = 0$ for all $i, j \in [k]$. Assume for contradiction that there are some indices $i, j \in [k]$ such that $p^3(y_{i,j}) > 0$ and let $p^3(y_{i,j}) = \epsilon$. By our previous observation, this implies that $p^3(y_{i,j'}) = \epsilon > 0$ for all $j' \in [k]$. Now, consider the distribution q defined by $q(x_i) = p^3(x_i) + \epsilon$, $q(z) = p^3(z) + (k-1)\epsilon$, $q(y_{i,j'}) = 0$ for all $j' \in [k]$, and $q(c) = p^3(c)$ for all remaining candidates. First, we note that voter $(1, k+1)$ strictly prefers q to p^3 since $\sum_{x \in X} q(x) = \epsilon + \sum_{x \in X} p^3(x)$. Next, for every voter who does not approve any candidate in $\{y_{i,1}, \dots, y_{i,k}\}$, it holds that $u_{i',j'}(q) \geq u_{i',j'}(p^3)$ because we only reduce the shares of the candidates in this set. This leaves us with the voters in the i-th row (i.e,. (i, ℓ) with $\ell \in [k]$), the voters in the $(k+1)$-th column (i.e., $(\ell, k+1)$ with $\ell \in [k] \setminus \{1\}$), and voter $(k+1, 1)$. Now, for all voters (i, ℓ) with $\ell \in [k]$, it holds that

$$u_{i,\ell}(q) = q(x_i) + q(y_{i,\ell}) = p^3(x_i) + \epsilon + 0 = u_{i,\ell}(p^3).$$

Further, for the the voters $(\ell, k+1)$ with $\ell \in [k] \setminus \{1\}$ and voter $(k+1, 1)$, it holds that they approve precisely $k-1$ candidates in $Y_i = \{y_{i,1}, \dots, y_{i,k}\}$. Hence, we get for all these voters that

$$u_{\ell,k+1}(q) = \sum_{c \in A^3_{\ell,k}} q(c) \;=\; (k-1)\epsilon \;-\; \sum_{y_{i,j'} \in Y_i \cap A^3_{\ell,k+1}} \epsilon \;+\; \sum_{c \in A^3_{\ell,k+1}} p^3(c) \;=\; u_{\ell,k+1}(p^3).$$

This proves that q indeed dominates p^3, which contradicts the efficiency of f. Hence, it holds that $p^3(y_{i,j}) = 0$ for all $i, j \in [k]$.

Lastly, GFS for the set of voters $S = \{(k+1, j) \colon j \in [k]\} \cup \{(i, k+1) \colon i \in [k] \setminus \{1\}\}$ requires that the total share allocated to approved candidates of these voters is at least $\frac{|S|}{n}$. Since $p^3(y) = 0$ for all $y \in Y$, this means that $p^3(x_{k+1}) + p^3(z) \geq \frac{|S|}{n} = \frac{2k-1}{(k+1)^2}$. Hence, it holds that $u_{k+1,1}(p^3) \geq \frac{2k-1}{(k+1)^2}$ for the true utility function of voter $(k+1, 1)$. This means that the incentive ratio of f is lower bounded by

$$\frac{u_{k+1,1}(p^3)}{u_{k+1,1}(p^1)} = \frac{(2k-1)/(k+1)^2}{1/(k+1)} = \frac{2k-1}{k+1}.$$

For $k \to \infty$, this term approaches 2, thus showing that the incentive ratio of f is at least 2. □

Remark 4. We observe that all the lower bounds given in this section only hold asymptotically when the number of voters and candidates grows to infinity. This is unavoidable, because a careful inspection of the proof of Theorem 2 shows that, for any finite number of voters n, the incentive ratio of NASH is strictly lower than 2. This means that, for finite n, it may be possible to construct desirable distribution rules that have a lower incentive ratio than NASH.

Remark 5. Under approval utilities, it is possible to design distribution rules with an incentive ratio of γ for every $\gamma \in (1, 2]$ by mixing NASH with the uniform distribution. Specifically, consider the rule $f_\lambda(A) = \lambda\mathsf{NASH}(A) + (1-\lambda)p_{uni}$ for $\lambda \in (0, 1)$, which returns the convex combination of NASH and the uniform distribution p_{uni} over all candidates. For example, when setting $\lambda = \frac{1}{m+1}$, one can show that f_λ has an incentive ratio between $\frac{6}{5}$ and $\frac{4}{3}$, where the lower bounds follow by using the example of Claim (1) of Theorem 4. However, to obtain a notable reduction in the incentive ratio, one needs to use a very small λ, so we lose the desirable properties of NASH. Moreover, we note that this mixing argument does not work with every strategyproof distribution rule. For example, mixing the utilitarian rule with NASH does not reduce the incentive ratio. To see this, we can consider any counterexample of this section and simply add a new candidate and new voters that ensure that the new candidate is approved by the most voters. Both before and after the manipulation, UTIL will assign the whole share to the new candidate, while NASH behaves identically for the old profile (but scales down the assigned resource), so a voter can still double her utility.

4 Experiments

As our last contribution, we experimentally evaluate the manipulability of the four distribution rules in Section 3.2. In particular, we are interested in how often and how severely our distribution rules are manipulable in more realistic profiles. We note that we focus here again on approval-based utility functions, as this is the domain for which most of the established distribution rules have been introduced.

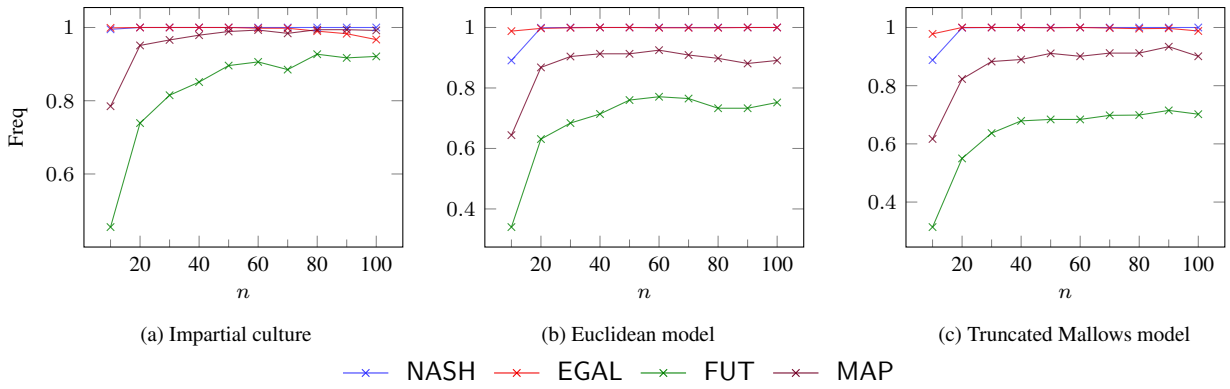
4.1 Experimental Setup

To answer this question, we sample 1000 approval profiles for each number of voters $n \in \{10, 20, \ldots, 100\}$ and $m = 10$ candidates. For each rule $f \in \{\mathsf{NASH}, \mathsf{EGAL}, \mathsf{MP}, \mathsf{FUT}\}$ and each sampled profile A, we then compute the maximal multiplicative utility gain of a voter, defined by $IR(f, A) = \max_{i, A_i'} \frac{u_i(f(A_{-i}, A_i'))}{u_i(f(A))}$. We note that a rule f is manipulable in a profile A if and only if $IR(f, A) > 1$.

We note that the number of voters and candidates in our simulations is relatively low when compared to real-world elections. This is necessary due to computational issues: to compute the value $IR(f, A)$ for a rule f and a profile A, we compute f for the original profile A and each possible deviation of each voter. For example, if $n = 100$ and $m = 10$, we need to compute f (in the worst-case) $(2^{10} - 2) \cdot 100 \approx 100,000$ times. This is computationally challenging since, e.g., EGAL requires us to repeatedly solve linear programs. Also, in particular the exponential blow-up for m prevents us from running computer experiments for more candidates. A second reason why we focus on small values of n is that, for large n, the effect of a single voter becomes more and more negligible and it seems more plausible to reason about group manipulations. Lastly, by using small numbers of voters, we also avoid numerical issues that may arise otherwise as the utilities of voters can become very small if m and n are large. That said, we find more extensive experiments and the study of the underlying computational problems desirable and leave this for future work.

We repeat these simulations with the following three distributions. We note that, for sampling preference profiles from these distributions, we rely on the PrefSampling package by Boehmer *et al.* [11]. Moreover, for each of the subsequent models, we make sure that every voter approves at least one candidate: if we sample a profile where a voter approves no candidate (and thus fails to have a non-zero utility function), we resample the entire profile until we derive a valid one.

Impartial culture model. Each voter approves each candidate with probability $p = 0.3$ independently at random. This means that, in expectation, each candidate is approved by the same number of voters (namely $0.3n$) and that there is no correlation between candidates. While this model is arguably not very realistic, it is a common baseline for experiments [see, e.g., 11].



(a) Impartial culture (b) Euclidean model (c) Truncated Mallows model

Figure 2: Frequency of manipulation. For each of our four distribution rules and each of our three culture, we plot the perecentage of profiles in our simulations for which a given rule was manipulable, as a function of the number of voters n.

Euclidean model. Candidates and voters are placed uniformly at random in the 3-dimensional unit ball $\{x \in \mathbb{R}^3 \colon |x|_2 \leq 1\}$. Further, a voter approves each candidate with an ℓ_2 distance of at most 0.4 from her position. We note that this radius is empirically chosen so that voters approve roughly three to four candidates on average. While the approval scores of the candidates are, in expectation, still roughly the same, this model introduces correlation between candidates. Specifically, candidates that are close to each other in the unit ball will be jointly approved by many voters, whereas candidates that are far apart from each other cannot be jointly approved.

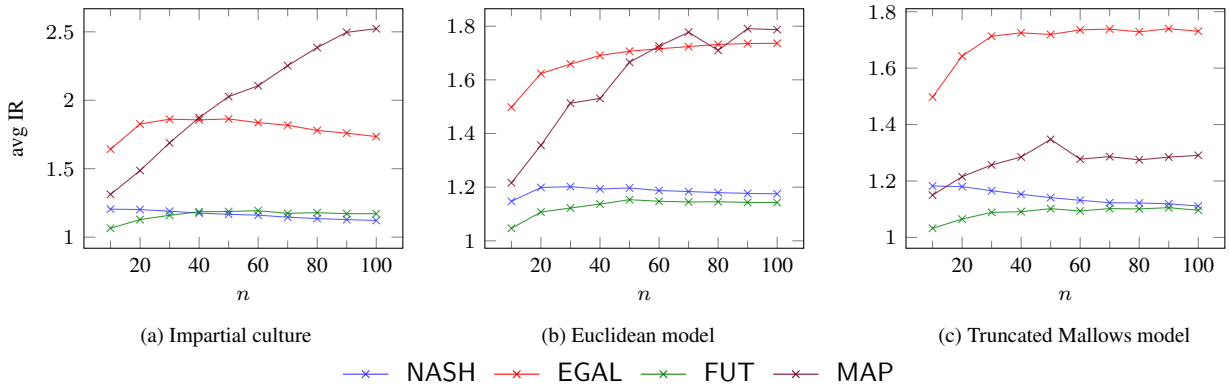
Truncated Mallows model In our last model, we sample first strict preference relations $\succ_i$ according to the Mallows model with a dispersion parameter of $\phi = 0.75$ for each voter i. Specifically, the Mallows model is defined by a fixed preference relation $\triangleright = x_1, \ldots, x_m$ over the candidates. The probability to assign a voter a preference relation $\succ_i$ is then proportional to $\phi^{|\{(x,y) \in A^2 \colon x \triangleright y \cap y \succ_i x\}|}$. As a second step, we sample a threshold value $t_i \in \{1, \ldots, m-1\}$ uniformly at random and each voter i approves the t_i most-preferred candidates in her preference relation. We empirically chose the dispersion parameter ϕ such that the approval score of the most approved candidate is roughly 2.5 times that of the least approved candidate. Further, this model introduces both a correlation between candidates and differences in the approval scores. Specifically, candidates that are close to each other in the ranking $\triangleright$ are more likely to be jointly approved and higher-ranked candidates in this ranking obtain more approvals.

4.2 Experimental Results

Our findings are summarized in Figures 2 and 3. Specifically, for each of our three distributions over profiles and our four distribution rules, we plot the percentage of manipulable profiles (cf. Figure 2) and the average incentive ratio in our simulations (cf. Figure 3) as a function of the number of voters $n \in \{10, \ldots, 100\}$. Additional statistics, such as the standard deviation, the maximum measured incentive ratio, an the 90%-percentile of the incentive ratio, can be found in Section D. We next discuss our two figures in more depth.

Frequency of manipulation. One of the most striking insights of Figure 2 is that, for all distributions, all our rules are frequently manipulable, with EGAL and NASH being almost always manipulable. A possible explanation for this is that each of our rules satisfies some fairness axioms, thereby guaranteeing that each voter can influence the outcome and thus also manipulate. Moreover, there is a clear hierarchy between our rules in terms of how often they are manipulable: FUT is consistently the least manipulable rule, followed by MP, and both EGAL and NASH can essentially always be manipulated. Lastly, for all rules, the share of manipulable profiles tends to increase as n grows, which is expected as a larger number of voters means a larger number of possible approval ballots and thus a larger number of possible manipulations.

We next take a closer look at our rules. First, for NASH and EGAL, we believe the percentage of manipulable profiles is consistently large due to free-riding manipulations. In particular, for these rules, a voter can often manipulate



(a) Impartial culture (b) Euclidean model (c) Truncated Mallows model

Figure 3: Average incentive ratio. For each of our three cultures, we plot the average incentive ratio of each rule as a function of the number of voters n.

by removing some approvals from their ballots. For example, for EGAL, if there is a candidate that is truthfully approved by some voters but gets assigned no budget, the voters approving this candidate can frequently manipulate by pretending that they only approve of this one candidate. This may also explain why EGAL is actually the only rule for which the share of manipulable profiles starts decreasing when $n \geq 70$ under the impartial culture: in this model, as n increases, EGAL becomes more and more likely to return the uniform distribution over all candidates, where such free-riding manipulations are often not possible. By contrast, for the other models, such effects would simply need more voters. We believe that similar effects also take part for NASH, but the share of manipulable profiles does not decrease as it always remains responsive to the preferences of individual voters, whereas the distribution of EGAL is determined by the most conflicting voters, especially if n is large.

Lastly, MP and FUT are significantly more frequently manipulable under the impartial culture model than under the Euclidean model or the truncated Mallows model. A possible explanation for this is that, under the impartial culture model, all candidates have roughly the same approval score. Moreover, both MP and FUT assign shares sequentially to the candidates, and this sequence is roughly guided by the approval scores. Hence, under the impartial culture model, it is particularly likely that this sequence can be manipulated by a voter, which is necessary for a successful manipulation. By contrast, under more structured distributions, such as the Euclidean model and especially the truncated Mallows model, such manipulations become less likely.

Average incentive ratio. The second main observation of our simulations is that, while NASH is almost always manipulable, its incentive ratio is very low, with an average of consistently less than 1.2 (see Figure 3). This shows that voters indeed have little incentive to manipulate under NASH. That said, surprisingly, the average incentive ratio of FUT is often even lower than that of NASH. A possible explanation for the low average incentive ratio of FUT is that this rule is notably less often manipulable than NASH, so its average incentive ratio is decreased by profiles where it cannot be manipulated. Indeed, if we condition on manipulable profiles, NASH often has a lower average incentive ratio than FUT. In line with this, we observe that the maximum incentive ratio of FUT in our simulations (typically between 2 and 3) is usually higher than that of NASH (typically around 1.5); we refer to Section D for details. Lastly, both MP and EGAL are severely manipulable in our experiments. Hence, we again get a clear hierarchy of our rules which, with the exception of FUT, aligns with our theoretical results: NASH and FUT have, on average, the lowest incentive ratio, whereas MP and EGAL perform significantly worse.

We next discuss the individual rules in more detail. First, for NASH, we observe that the incentive ratio for each of our sampling models takes a relatively similar shape: it is consistently low and slowly decreases as the number of voters increases. A possible explanation for this is that NASH is more "smooth" than the other rules, in that a single voter can only modestly influence the outcome. This intuition is, to some extent, even formalized by Claim (3) of Lemma 1, which shows that a single voter has only limited influence on the Nash welfare of the other voters. Moreover, in addition to having a low average, NASH also has a low maximum incentive ratio in our experiments. Hence, we believe it is fair to say that, under NASH, voters have in practice limited incentive to manipulate.

Next, FUT also behaves similarly for each of our three distributions over profiles. However, in contrast to NASH, its average incentive ratio seems to increase or stagnate in n, thus demonstrating that having more voters does not necessarily remove the incentive of a single voter to manipulate. Additionally, we emphasize again that FUT has a larger maximum incentive ratio in all experiments than NASH (see Section D for details).

Third, for MP, we first note that this rule has, in comparison, a large average incentive ratio in all our experiments, and that the incentive ratio of this rule strongly depends on the considered sampling model. In particular, for the impartial culture model, the average incentive ratio of this rule increases almost linearly in the number of voters up to a value of 2.5, whereas the curve for the truncated Mallows model increases only slowly in n and saturates around 1.3. We believe that the explanation for this behavior lies in the fact that MP regularly allows for very large manipulations: for example, under the impartial culture model, our simulations exhibited a profile A with $IR(\mathsf{MP}, A) = 37$. Sincer also the 90% percentile of the incentive ratio of this rule is large in our experiments, such strong manipulations happen regularly and thus drive the average. However, such strong manipulations happen much less frequently in more structured preferences. A possible explanation for this is that the sequence in which MP assigns shares to the candidates is guided by the approval scores of the active voters. Moreover, this rule can only be manipulated if this sequence is affected. Now, under unstructured models such as the impartial culture, it can happen that, e.g., a voter can influence one of the first steps of MP and thus benefit strongly. By contrast, under the Mallows model and the Euclidean model, this is less likely, thus ruling out such strong manipulations.

Finally, we also observe that EGAL has a consistently high average incentive ratio when compared to NASH and FUT. In particular, for all our models, this value is between 1.7 and 1.8 for $n = 100$. We believe that this is again due to the free-riding manipulations, which are often also rather beneficial. Indeed, we note that the maximum incentive ratio of EGAL is typically below 3 and the 90%-percentile is often even below 2. Given an average incentive ratio of roughly 1.7, this means that most profiles in our simulations admit a rather beneficial manipulation. Furthermore, in line with our observation on the frequency of manipulation of EGAL, the average incentive ratio of this rule decreases in n after a peak at $n = 30$ under the impartial culture model, whereas it increases or stagnates for the other two models.

5 Conclusion

In this paper, we analyze whether there are distribution rules that are fair, efficient, and approximately strategyproof. To this end, we study the incentive ratio of such rules, which quantifies the maximal multiplicative utility gain of a voter when manipulating. While it turns out that several known rules have a large incentive ratio, even in the special case of approval utilities, we prove that the Nash product rule (NASH) has an incentive ratio of 2 under concave utilities. Moreover, we also observe that this result extends to group manipulations and prove an improved incentive ratio of $e^{1/e}$ under Cobb–Douglas utility functions. Since this rule is efficient and satisfies the strongest fairness conditions in the literature, this answers our research question in the affirmative. We further show that the incentive ratio of NASH is optimal subject to some of its desirable properties. Lastly, we conduct an experimental analysis to understand the manipulability of our rules for realistic profiles.

Our paper points to several appealing follow-up questions. As highlighted in Remark 2, a complete analysis of the incentive ratio of NASH under CES utility functions would give a more fine-grained understanding of the relationship between the elasticity of substitution and the manipulability of NASH. Further, for approval-based budget division, it may be desirable to analyze the tradeoff between strategyproofness, efficiency, and fairness in more depth by simultaneously approximating some of these properties. More broadly, we believe the incentive ratio is worth studying in other public good settings where strategyproofness is difficult to achieve. For example, one could study the incentive ratio also in classical randomized social choice, where voters report ordinal preferences instead of utility functions and the output is a probability distribution over the candidates. In this setting, strategyproofness is also known to be in conflict with desirable properties, such as strong fairness considerations or Condorcet-consistency, so one may attempt to circumvent these impossibility results by considering the incentive ratio.

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A Tightness of Theorem 2 under Overlap Utilities

Proposition 1. *Under overlap utilities, even when $m = 2$, the incentive ratio of* NASH *is at least* $2 - 1/n$.

Proof. Fix $n \geq 3$, and let there be two candidates. Let voter 1's ideal distribution be $a_1 = (s, 1-s)$, where $s = 1 - \frac{1}{n-1}$, and let every other voter $i \in N \setminus \{1\}$ have ideal distribution $a_i = (0, 1)$. For an arbitrary distribution $(t, 1-t)$, voter 1 receives utility $\min(s, t) + \min(1-s, 1-t) = 1 - |t - s|$.

Hence, if voter 1 reports truthfully, the Nash welfare is $(1 - |t - s|)(1 - t)^{n-1}$. We will solve for the Nash distribution in this case. Clearly, it must be that $t \leq s$, since otherwise shifting mass from the first candidate to the second will be a Pareto improvement. On the interval $t \in [0, s]$, the Nash welfare is $(1 - s + t)(1 - t)^{n-1}$, so the derivative of its logarithm is

$$\frac{1}{1 - s + t} - \frac{n - 1}{1 - t}.$$

Since $1 - s = 1/(n - 1)$, this derivative is equal to 0 at $t = 0$, and it is negative for every $t > 0$. Thus the Nash welfare is maximized at $t = 0$ on $[0, s]$, which means the truthful NASH distribution is $p = (0, 1)$, and voter 1's utility is $u_1(p) = 1 - s = \frac{1}{n-1}$.

Now suppose voter 1 misreports $a'_1 = (1, 0)$. The Nash welfare becomes $t(1 - t)^{n-1}$, which is maximised at $t = 1/n$. Hence, the distribution under the manipulated profile is $q = (1/n, 1 - 1/n)$. Since $1/n < s$ for $n \geq 3$, voter 1's utility at q is

$$u_1(q) = 1 - s + \frac{1}{n} = \frac{1}{n-1} + \frac{1}{n}.$$

Therefore the utility ratio is

$$\frac{u_1(q)}{u_1(p)} = \frac{\frac{1}{n-1} + \frac{1}{n}}{\frac{1}{n-1}} = 1 + \frac{n-1}{n} = 2 - \frac{1}{n}.$$

As $n \to \infty$, this ratio converges to 2, demonstrating that the upper bound given by Theorem 2 is tight as a supremum. □

B Cobb-Douglas Utilities

Recall that under Cobb–Douglas utility functions, there are exponents $\alpha_i(x) \geq 0$ with $\sum_{x \in C} \alpha_i(x) = 1$ such that the utility of voter i for distribution p is given by $u_i(p) = \prod_{x \in C} p(x)^{\alpha_i(x)}$. When all voters have Cobb–Douglas utility functions, we represent the utility profile by the exponent vectors, i.e., $(\alpha_1, \ldots, \alpha_n)$.

Theorem 5. *The incentive ratio of* NASH *is $e^{1/e} \approx 1.44$ when the voters are allowed to report Cobb–Douglas utility functions.*

Proof. Fix a Cobb–Douglas profile $(\alpha_1, \ldots, \alpha_n)$. For a distribution p, we can rewrite the logarithm of the Nash welfare as

$$\sum_{i \in N} \log u_i(p) = \sum_{i \in N} \sum_{x \in C} \alpha_i(x) \log p(x) = \sum_{x \in C} \left(\sum_{i \in N} \alpha_i(x) \right) \log p(x).$$

The first-order conditions of the Lagrangian then yield $p(x) = (1/\lambda) \sum_{i \in N} \alpha_i(x)$. This then implies that

$$1 = \sum_{x \in C} p(x) = \frac{1}{\lambda} \sum_{x \in C} \sum_{i \in N} \alpha_i(x) = \frac{1}{\lambda} \sum_{i \in N} \sum_{x \in C} \alpha_i(x) = n/\lambda,$$

and thus $\lambda = n$. Summarizing, this means that, on the Cobb–Douglas domain, NASH coincides with averaging the exponent vectors, i.e., the distribution p computed by NASH is given by $p(x) = \frac{1}{n} \sum_{i \in N} \alpha_i(x)$.

We now bound the gain from an arbitrary manipulation on an arbitrary Cobb–Douglas profile $(\alpha_1, \ldots, \alpha_n)$. Fix a voter i with true exponent vector $\alpha_i \in \Delta(C)$. Let $\bar{\alpha}_{-i} = \sum_{j \in N \setminus \{i\}} \alpha_j$ denote the sum of the exponent vectors of

the other voters, and let $\alpha_i' \in \Delta(C)$ be an arbitrary misreport by voter i. If voter i reports truthfully, the outcome is $(\alpha_i + \bar{\alpha}_{-i})/n$, whereas if she reports α_i', the outcome is $(\alpha_i' + \bar{\alpha}_{-i})/n$. Hence her multiplicative utility gain is

$$\gamma = \frac{\prod_{x \in C} \left(\frac{\alpha_i'(x) + \bar{\alpha}_{-i}(x)}{n} \right)^{\alpha_i(x)}}{\prod_{x \in C} \left(\frac{\alpha_i(x) + \bar{\alpha}_{-i}(x)}{n} \right)^{\alpha_i(x)}} = \prod_{x \in C} \left(\frac{\alpha_i'(x) + \bar{\alpha}_{-i}(x)}{\alpha_i(x) + \bar{\alpha}_{-i}(x)} \right)^{\alpha_i(x)}. \tag{1}$$

Let $T = \{x \in C : \alpha_i'(x) > \alpha_i(x)\}$. For every $x \notin T$, the corresponding factor is at most 1. Moreover, for every $x \in T$ with $\alpha_i(x) > 0$, we have

$$\begin{aligned} \alpha_i(x) < \alpha_i'(x) &\implies \alpha_i(x) \cdot \bar{\alpha}_{-i}(x) + \alpha_i(x) \cdot \alpha_i'(x) \leq \alpha_i'(x) \cdot \bar{\alpha}_{-i}(x) + \alpha_i(x) \cdot \alpha_i'(x) \\ &\implies \alpha_i(x)(\bar{\alpha}_{-i}(x) + \alpha_i'(x)) \leq \alpha_i'(x)(\alpha_i(x) + \bar{\alpha}_{-i}(x)) \\ &\implies \frac{\bar{\alpha}_{-i}(x) + \alpha_i'(x)}{\alpha_i(x) + \bar{\alpha}_{-i}(x)} \leq \frac{\alpha_i'(x)}{\alpha_i(x)}. \end{aligned}$$

Substituting into Equation (1) and recalling that $\alpha_i'(x) \leq \alpha_i(x)$ for all $x \notin T$, we obtain that

$$\gamma \leq \prod_{x \in C} \left(\frac{\alpha_i'(x)}{\alpha_i(x)} \right)^{\alpha_i(x)} \leq \prod_{x \in T} \left(\frac{\alpha_i'(x)}{\alpha_i(x)} \right)^{\alpha_i(x)}.$$

Taking logarithms gives $\log \gamma \leq \sum_{x \in T} \alpha_i(x) \log(\alpha_i'(x)/\alpha_i(x))$.

Let $A = \sum_{x \in T} \alpha_i(x)$. We note that if $A = 0$, then $\log \gamma \leq 0$, meaning the manipulation α_i' is not profitable. Suppose, instead, that $A > 0$. Then, using Jensen's inequality, we observe that

$$\log \gamma \leq A \sum_{x \in T} \frac{\alpha_i(x)}{A} \log \frac{\alpha_i'(x)}{\alpha_i(x)} \leq A \log \left(\sum_{x \in T} \frac{\alpha_i(x) \cdot \alpha_i'(x)}{A \cdot \alpha_i(x)} \right) \leq A \log \left(\sum_{x \in T} \frac{\alpha_i'(x)}{A} \right) \leq A \log(1/A).$$

The function $A \to A \log(1/A)$ is maximized on $(0, 1]$ at $A = 1/e$, where it takes value $1/e$. Hence $\log \gamma \leq 1/e$, and so $\gamma \leq e^{1/e}$. Since the voter, the profile, and the misreport were arbitrary, the incentive ratio of NASH on the Cobb–Douglas domain is at most $e^{1/e}$.

It remains to show the lower bound holds asymptotically. Fix $n \geq 2$ and consider two candidates, x and y. Let voter 1's true exponent vector be $\alpha_1 = (t, 1-t)$, where $t \in (0, 1)$, and let every other voter have exponent vector $\alpha_k = (0, 1)$. If voter 1 reports truthfully, NASH returns $p = (t/n, 1 - t/n)$. If voter 1 instead reports $\alpha_1' = (1, 0)$, then NASH returns $q = (1/n, (n-1)/n)$. Thus voter 1's utility ratio is

$$\frac{u_1(q)}{u_1(p)} = \frac{\left(\frac{1}{n}\right)^t \left(\frac{n-1}{n}\right)^{1-t}}{\left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{1-t}} = \left(\frac{1}{t}\right)^t \left(\frac{n-1}{n-t}\right)^{1-t}.$$

Setting $t = 1/e$ and letting $n \to \infty$, the second factor converges to 1, while the first factor equals $e^{1/e}$. Hence the upper bound $e^{1/e}$ is tight as a supremum over all profiles. □

C Proof of Lemma 1

In this section, we prove Lemma 1. We note that, in our proof, we will not use that we defined utility functions to be continuous. Instead, we only require that NASH is well-defined, i.e., that the maximum $\max_{q \in \Delta(C)} \prod_{i \in N} u_i(q)$ exists for each considered utility profile U. This attainment property is implied by continuity but may fail without it: for example, consider the following concave utility profile U with two voters and an arbitrary number of candidates $m \geq 2$. The first voter's utility is given by $u_1(p) = p(x_1)$ and the second one by $u_2(p) = 0$ if $p(x_1) = 1$ and $u_2(p) = 1$ otherwise. In this profile, NASH is not well-defined as the Nash welfare increases as we put more budget on x_1, but drops to 0 when $p(x_1) = 1$. As we show next, except for such paradoxical situations, Lemma 1 holds. Moreover, based on this insight, it follows that Theorem 2 does not need continuous utility functions but only that NASH is well-defined before and after the manipulation.

Lemma 1. *Fix an arbitrary concave utility profile U.*

(1) It holds that $u_i(\mathsf{NASH}(U)) > 0$ for all voters $i \in N$.

(2) For all distributions $q \in \Delta(C)$, it holds that $\sum_{i \in N} \frac{u_i(q) - u_i(\mathsf{NASH}(U))}{u_i(\mathsf{NASH}(U))} \leq 0$.

(3) For all voters $i \in N$, it holds that $e^{-1} \leq \frac{\prod_{j \in N \setminus \{i\}} u_j(\mathsf{NASH}(U))}{\prod_{j \in N \setminus \{i\}} u_j(\mathsf{NASH}(U_{-i}))} \leq 1$.

Proof. Fix a concave utility profile U and let p denote the distribution chosen by NASH for U. We will prove the three claims of this lemma separately.

Proof of Claim (1): As our first claim, we will show that $u_i(p) > 0$ for all voters $i \in N$. To this end, we observe that, since utility functions are non-negative and non-zero, there is for each voter i a distribution q_i such that $u_i(q_i) > 0$. We will show that this implies that $u_i(p_{uni}) > 0$ for all $i \in N$, where p_{uni} is the distribution that assigns a share of $\frac{1}{m}$ to each candidate. To see this, let p_x be the distribution assigning share 1 to x, for each candidate $x \in C$. Then, it holds that

$$p_{uni} = \frac{1}{m} q_i + \sum_{x \in C} \frac{1 - q_i(x)}{m} p_x.$$

By the concavity and non-negativity of u_i, we derive that $u_i(p_{uni}) \geq \frac{1}{m} u_i(q_i) + \sum_{x \in C} \frac{1 - q_i(x)}{m} u_i(p_x) \geq \frac{1}{m} u_i(q_i) > 0$. Since this holds for every voter $i \in N$, this means that $\prod_{i \in N} u_i(p_{uni}) > 0$. Finally, because p maximizes the product of the voters' utilities, $\prod_{i \in N} u_i(p) \geq \prod_{i \in N} u_i(p_{uni}) > 0$ and thus also $u_i(p) > 0$ for all $i \in N$.

Proof of Claim (2): For proving this claim, we let q denote an arbitrary distribution; our goal is to show that $\sum_{i \in N} \frac{u_i(q) - u_i(p)}{u_i(p)} \leq 0$. To this end, we first observe that, by Claim (1), $u_i(p) > 0$ for all voters $i \in N$, so this sum is well-defined. Next, we consider the line segment $x_t = (1 - t)p + tq$ for $t \in [0, 1]$, which connects p and q in the simplex. Further, for every voter $i \in N$, we define by $\phi_i(t) = u_i(x_t)$ her utility for x_t and by $s_i(t) = \frac{\phi_i(t) - \phi_i(0)}{t}$ for $t \in (0, 1]$ the slope in her utility. Now, we fix a voter $i \in N$ and aim to prove that the right limit of s_i at 0 exists and is finite, i.e., $\lim_{t \downarrow 0} s_i(t) \in \mathbb{R}$. To this end, we observe that ϕ_i is concave as it is merely a different representation of the voter's utility along the line segment between p and q. Consequently, s_i is non-increasing, because it holds for all $t', t \in (0, 1]$ with $t' < t$ that

$$\begin{aligned}
s_i(t') &= \frac{\phi_i(t') - \phi_i(0)}{t'} \\
&= \frac{\phi_i\left(\frac{t'}{t} t + (1 - \frac{t'}{t})0\right) - \phi_i(0)}{t'} \\
&\geq \frac{\frac{t'}{t}\phi_i(t) + (1 - \frac{t'}{t})\phi_i(0) - \phi_i(0)}{t'} \\
&= \frac{\phi_i(t) - \phi_i(0)}{t} \\
&= s_i(t).
\end{aligned}$$

Next, we show that s_i is also bounded. Assume for contradiction that this is not true. Since s_i is bounded from below by $s_i(1)$ (as s_i is non-increasing), there is for every value $r \in \mathbb{R}$ a point $t \in (0, 1]$ such that $s_i(t) > r$. Moreover, since s_i is non-increasing, we may assume that t is arbitrarily close to 0. Our goal is to show that this contradicts that p maximizes the Nash welfare. To this end, let M be a constant such that $\phi_i(t) \leq M$ for all $t \in [0, 1]$. Such a constant exists since ϕ_i is a concave, non-negative, and real-valued function. In particular, concavity shows for every $\delta \in [0, \frac{1}{2}]$ that $\phi_i(\frac{1}{2}) \geq \frac{1}{2}\phi_i(\frac{1}{2} - \delta) + \frac{1}{2}\phi_i(\frac{1}{2} + \delta)$. By non-negativity, this means that $2\phi_i(\frac{1}{2}) \geq \phi_i(\frac{1}{2} - \delta)$ and $2\phi_i(\frac{1}{2}) \geq \phi_i(\frac{1}{2} + \delta)$ for $\delta \in [0, \frac{1}{2}]$ and thus $2\phi_i(\frac{1}{2}) \geq \phi_i(x)$ for all $x \in [0, 1]$.

We choose a value $t^* \in (0, \frac{1}{2}]$ such that $s_i(t^*) > 2nM$ and show that the distribution x_{t^*} has a higher Nash welfare than $x_0 = p$. To this end, we observe that, by concavity and non-negativity of the individual utility functions, it holds that $u_j(x_{t^*}) \geq (1 - t^*)u_j(x_0) > 0$ for all $j \in N$. Hence, it suffices to prove that $\sum_{j \in N} \log u_j(x_{t^*}) > \sum_{j \in N} \log u_j(x_0)$,

or, equivalently, that $\frac{1}{t^*}\sum_{j\in N}(\log u_j(x_{t^*}) - \log u_j(x_0)) > 0$. We first consider the voters $j \in N \setminus \{i\}$. For each of these voters, it holds that $\log u_j(x_{t^*}) - \log u_j(x_0) \geq \log((1-t^*)u_j(x_0)) - \log u_j(x_0) = \log \frac{(1-t^*)u_j(x_0)}{u_j(x_0)} = \log(1-t^*)$. Further, it holds for every $t \in (0,1)$ that $\log 1 - t \geq -\frac{t}{1-t}$. Since $t^* \in (0, \frac{1}{2}]$, it follows that $\log 1 - t^* \geq -\frac{t^*}{1-t^*} \geq -1$. In summary, this means that

$$\sum_{j\in N\setminus\{i\}} \frac{\log u_j(x_{t^*}) - \log u_j(x_0)}{t^*} \geq \sum_{j\in N\setminus\{i\}} -\frac{t^*}{t^*(1-t^*)} \geq -2(n-1).$$

We next turn to voter i and note that $\phi_i(t^*) > \phi_i(0)$ since $s_i(t^*) = \frac{\phi_i(t^*)-\phi_i(0)}{t^*} > 0$. Further, by the mean value theorem applied to the logarithm, there is for all values $a, b \in \mathbb{R}_{>0}$ with $a < b$ a constant $c \in (a,b)$ such that $\log b - \log a = \frac{b-a}{c}$. Setting $b = \phi_i(t^*)$ and $a = \phi_i(0)$, this shows that there is a constant $c \in (\phi_i(0), \phi_i(t^*))$ such that $\log \phi_i(t^*) - \log \phi_i(0) = \frac{\phi_i(t^*)-\phi_i(0)}{c}$. Because $c \leq \phi_i(t^*) \leq M$, we have that $\log \phi_i(t^*) - \log \phi_i(0) \geq \frac{\phi_i(t^*)-\phi_i(0)}{M}$. By dividing by t^* and using the definition of s_i, we get that

$$\frac{\log \phi_i(t^*) - \log \phi_i(0)}{t^*} \geq \frac{\phi_i(t^*) - \phi_i(0)}{t^* M} = \frac{s_i(t^*)}{M} > 2n.$$

In combination, this proves that $\frac{1}{t^*}\sum_{j\in N}(\log u_j(x_{t^*}) - \log u_j(x_0)) > 0$, as desired. This contradicts that $x_0 = p$ maximizes the Nash welfare, so there is a constant M' such that $s_i(t) \leq M'$ for all $t \in (0,1]$. Finally, since s_i is non-increasing and bounded, $\lim_{t\downarrow 0} s_i(t)$ exists and is finite.

Based on this observation, we define $z_i = \lim_{t\downarrow 0} s_i(t)$ for all $i \in N$ and note that this value has multiple important implications. First, since all s_i are non-increasing, it holds that $s_i(t) \leq z_i$ for all $t \in (0,1]$ and thus $\phi_i(t) - \phi_i(0) \leq t z_i$. This shows that ϕ_i is (right) continuous at 0. Further, by setting $t = 1$, we get that $u_i(q) - u_i(p) = u_i(x_1) - u_i(x_0) \leq z_i$.

Now, to derive our separation inequality, we consider the Nash welfare along our line segment and let $\Phi(t) = \sum_{i\in N} \log \phi_i(t)$ for all $t \in [0,1)$. This function is well-defined since concavity and non-negativity imply that $\phi_i(t) \geq (1-t)u_i(x_0) > 0$. Further, since $p = x_0$ maximizes the Nash welfare, we have that $\Phi(0) \geq \Phi(t)$ for all $t \in (0,1)$, or, equivalently, that $\frac{\Phi(t)-\Phi(0)}{t} \leq 0$ for all $t \in (0,1)$. This means that $\limsup_{t\downarrow 0} \frac{\Phi(t)-\Phi(0)}{t} \leq 0$, too. Motivated by this observation, we will consider the term $\limsup_{t\downarrow 0} \frac{\log \phi_i(t) - \log \phi_i(0)}{t}$ for the individual voters $i \in N$. Using again the mean value theorem for the logarithm, there is for every $t \in (0,1)$ a constant $c(t)$ between $\phi_i(t)$ and $\phi_i(0)$ such that $\log \phi_i(t) - \log \phi_i(0) = \frac{\phi_i(t)-\phi_i(0)}{c(t)}$. Further, by right continuity of ϕ_i at 0, it holds that $\phi_i(t)$ converges to $\phi_i(0)$ as t approaches 0. This implies that $c(t)$ also converges to $\phi_i(0)$ because it lies always between $\phi_i(0)$ and $\phi_i(t)$. Using the definition of s_i, we conclude that

$$\limsup_{t\downarrow 0} \frac{\log \phi_i(t) - \log \phi_i(0)}{t} = \limsup_{t\downarrow 0} \frac{\phi_i(t) - \phi_i(0)}{t c(t)} = \limsup_{t\downarrow 0} \frac{\phi_i(t) - \phi_i(0)}{t \phi(0)} = \frac{z_i}{\phi(0)}.$$

Since this holds for all voters $i \in N$, it follows that

$$\limsup_{t\downarrow 0} \frac{\Phi(t) - \Phi(0)}{t} = \limsup_{t\downarrow 0} \sum_{i\in N} \frac{\log \phi_i(t) - \log \phi_i(0)}{t} = \sum_{i\in N} \frac{z_i}{\phi_i(0)}.$$

Finally, by combining our insights, we derive that

$$\sum_{i\in N} \frac{u_i(q) - u_i(p)}{u_i(p)} = \sum_{i\in N} \frac{\phi_i(1) - \phi_i(0)}{\phi_i(0)} \leq \sum_{i\in N} \frac{z_i}{\phi_i(0)} = \limsup_{t\downarrow 0} \frac{\Phi(t) - \Phi(0)}{t} \leq 0.$$

This concludes the proof of our claim.

Proof of Claim (3): For our last claim, we analyze the change in the Nash welfare when a voter i abstains from the election. To this end, let $q = \mathsf{NASH}(U_{-i})$ be the chosen distribution when voter i abstains. Our goal is to show that $e^{-1} \leq \frac{\prod_{j\in N\setminus\{i\}} u_j(p)}{\prod_{j\in N\setminus\{i\}} u_j(q)} \leq 1$. First, we note that $\prod_{j\in N\setminus\{i\}} u_j(q) > 0$ and $\prod_{j\in N\setminus\{i\}} u_j(p) > 0$ by Claim (1). For our

upper bound, it now suffices to observe that, by definition, q maximizes $\prod_{j \in N \setminus \{i\}} u_j(q')$ among all distributions q'. Thus, we immediately get that $\frac{\prod_{j \in N \setminus \{i\}} u_j(p)}{\prod_{j \in N \setminus \{i\}} u_j(q)} \leq 1$.

For the lower bound, we use the separation inequality in Claim (2). By applying this inequality for q, we derive that $\sum_{j \in N} \frac{u_j(q) - u_j(p)}{u_j(p)} \leq 0$. Next, we define for every voter $j \in N$ a constant d_j such that $1 + d_j = \frac{u_j(q)}{u_j(p)}$. By the definition of these constants, it holds that $\frac{u_j(q) - u_j(p)}{u_j(p)} = \frac{(1+d_j)u_j(p) - u_j(p)}{u_j(p)} = d_j$ for all $j \in N$. By substituting this into our separation inequality for all voters $j \in N \setminus \{i\}$ and rearranging it, we derive that $\sum_{j \in N \setminus \{i\}} d_j \leq \frac{u_i(p) - u_i(q)}{u_i(p)}$. Next, since our utilities are non-negative, the right hand side is bounded by $\frac{u_i(p) - u_i(q)}{u_i(p)} \leq \frac{u_i(p)}{u_i(p)} = 1$, which implies that $\sum_{j \in N \setminus \{i\}} d_j \leq 1$.

Furthermore, we observe that $1 + d_j \leq e^{d_j}$ for all $j \in N$. Thus, it holds that

$$\prod_{j \in N \setminus \{i\}} (1 + d_j) \leq \prod_{j \in N \setminus \{i\}} e^{d_j} = e^{\sum_{j \in N \setminus \{i\}} d_j} \leq e.$$

This proves our lower bound because

$$\frac{\prod_{j \in N \setminus \{i\}} u_j(p)}{\prod_{j \in N \setminus \{i\}} u_j(q)} = \frac{1}{\prod_{j \in N \setminus \{i\}} (1 + d_j)} \geq \frac{1}{e}.$$

This completes the proof of this lemma.[6] □

D Detailed Discussion of Experiments

In this appendix, we present additional statistics for our experiments (Section 4); see Figures 4 to 6.

	NASH					EGAL					FUT					MP				
n	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq
10	1.20	1.45	1.32	0.09	1.00	1.64	2.69	1.88	0.21	1.00	1.06	2.00	1.20	0.10	0.46	1.31	5.00	1.60	0.35	0.79
20	1.20	1.48	1.29	0.07	1.00	1.83	2.93	2.14	0.26	1.00	1.13	3.00	1.33	0.16	0.74	1.49	6.00	1.78	0.59	0.95
30	1.19	1.43	1.28	0.07	1.00	1.86	3.09	2.20	0.25	1.00	1.16	4.00	1.33	0.22	0.82	1.69	10.00	2.33	1.04	0.97
40	1.18	1.41	1.26	0.06	1.00	1.86	3.61	2.19	0.25	1.00	1.19	3.00	1.43	0.21	0.85	1.87	15.00	3.00	1.51	0.98
50	1.17	1.37	1.24	0.06	1.00	1.86	3.23	2.25	0.27	1.00	1.19	3.00	1.42	0.20	0.90	2.03	24.00	3.00	2.02	0.99
60	1.16	1.38	1.24	0.06	1.00	1.84	3.14	2.19	0.26	1.00	1.19	3.00	1.40	0.20	0.91	2.11	21.00	4.00	1.95	0.99
70	1.15	1.39	1.23	0.06	1.00	1.82	3.00	2.19	0.26	1.00	1.17	3.00	1.33	0.18	0.89	2.25	19.00	4.00	2.30	0.98
80	1.14	1.35	1.21	0.06	1.00	1.78	2.80	2.19	0.27	0.99	1.18	2.50	1.33	0.17	0.93	2.39	25.00	5.00	2.61	0.99
90	1.13	1.31	1.20	0.06	1.00	1.76	2.79	2.19	0.27	0.98	1.17	2.60	1.33	0.16	0.92	2.50	37.00	5.00	2.92	0.99
100	1.12	1.32	1.20	0.06	1.00	1.74	2.80	2.19	0.28	0.97	1.17	3.00	1.33	0.17	0.92	2.52	26.00	5.33	2.82	0.99

Figure 4: Full summary of our simulations with the impartial culture. Avg, Max, 0.9P, and Std respectively indicate the average, maximum, 90%-percentile, and standard deviation of the incentive ratio. Freq denotes the percentage of manipulable profiles.

[6] For fixed n, Lemma 3.3 of Cole *et al.* [27] implies a slightly improved bound on the ratio $\frac{\prod_{j \in N \setminus \{i\}} u_j(p)}{\prod_{j \in N \setminus \{i\}} u_j(q)}$. In particular, this result entails that $\prod_{j \in N \setminus \{i\}} (1 + d_j) \leq (1 + \frac{1}{n-1})^{n-1}$, so $\frac{\prod_{j \in N \setminus \{i\}} u_j(p)}{\prod_{j \in N \setminus \{i\}} u_j(q)} \geq (1 + \frac{1}{n-1})^{-(n-1)}$. On the other hand, as n grows, this bound converges to $1/e$, so our analysis is tight in the limit.

	NASH					EGAL					FUT					MP				
n	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq
10	1.15	1.57	1.30	0.11	0.89	1.50	2.25	1.60	0.15	0.99	1.05	2.00	1.17	0.09	0.34	1.22	3.00	1.50	0.27	0.64
20	1.20	1.56	1.35	0.11	1.00	1.62	2.40	1.78	0.15	1.00	1.11	3.00	1.25	0.16	0.63	1.36	7.00	1.62	0.57	0.87
30	1.20	1.57	1.34	0.10	1.00	1.66	2.67	1.80	0.15	1.00	1.12	2.00	1.30	0.15	0.68	1.51	10.00	2.00	1.01	0.90
40	1.19	1.59	1.34	0.10	1.00	1.69	2.67	1.83	0.15	1.00	1.14	3.00	1.33	0.19	0.71	1.53	11.00	2.00	0.96	0.91
50	1.20	1.57	1.33	0.10	1.00	1.71	2.78	1.83	0.15	1.00	1.15	3.00	1.35	0.20	0.76	1.67	18.00	2.50	1.40	0.91
60	1.19	1.58	1.33	0.10	1.00	1.72	2.79	1.88	0.16	1.00	1.15	5.00	1.33	0.23	0.77	1.73	18.00	2.50	1.75	0.93
70	1.18	1.63	1.30	0.10	1.00	1.72	2.50	1.88	0.15	1.00	1.14	3.00	1.36	0.19	0.77	1.78	23.00	2.50	2.00	0.91
80	1.18	1.59	1.31	0.10	1.00	1.73	2.57	1.93	0.17	1.00	1.15	3.50	1.36	0.20	0.73	1.71	22.00	2.67	1.71	0.90
90	1.18	1.64	1.31	0.10	1.00	1.74	2.57	1.89	0.16	1.00	1.14	7.00	1.33	0.28	0.73	1.79	26.00	2.73	2.03	0.88
100	1.18	1.60	1.31	0.10	1.00	1.74	2.57	1.86	0.16	1.00	1.14	3.00	1.35	0.21	0.75	1.79	30.00	2.50	2.31	0.89

Figure 5: Full summary of our simulations with the euclidean model. Avg, Max, 0.9P, and Std respectively indicate the average, maximum, 90%-percentile, and standard deviation of the incentive ratio. Freq denotes the percentage of manipulable profiles.

	NASH					EGAL					FUT					MP				
n	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq	Avg	Max	0.9P	Std	Freq
10	1.18	1.60	1.33	0.11	0.89	1.50	2.57	1.71	0.19	0.98	1.03	2.00	1.11	0.08	0.31	1.15	3.00	1.33	0.19	0.62
20	1.18	1.52	1.29	0.08	1.00	1.64	3.06	1.88	0.20	1.00	1.06	2.00	1.17	0.11	0.55	1.22	5.00	1.40	0.31	0.82
30	1.17	1.50	1.27	0.08	1.00	1.71	3.00	2.00	0.21	1.00	1.09	2.00	1.20	0.15	0.64	1.26	7.00	1.50	0.43	0.88
40	1.15	1.44	1.26	0.08	1.00	1.73	2.67	2.00	0.20	1.00	1.09	2.00	1.25	0.13	0.68	1.29	11.00	1.50	0.55	0.89
50	1.14	1.45	1.26	0.08	1.00	1.72	2.57	2.00	0.17	1.00	1.10	2.50	1.25	0.16	0.68	1.35	12.00	1.50	0.86	0.91
60	1.13	1.43	1.25	0.08	1.00	1.74	2.76	2.00	0.18	1.00	1.09	2.00	1.25	0.13	0.68	1.28	7.00	1.50	0.49	0.90
70	1.12	1.39	1.24	0.08	1.00	1.74	2.78	1.93	0.17	1.00	1.10	3.00	1.25	0.14	0.70	1.29	8.00	1.50	0.49	0.91
80	1.12	1.44	1.25	0.09	1.00	1.73	2.57	1.80	0.16	1.00	1.10	2.33	1.25	0.14	0.70	1.27	10.00	1.50	0.49	0.91
90	1.12	1.43	1.25	0.09	1.00	1.74	2.36	1.80	0.17	1.00	1.11	2.00	1.25	0.13	0.72	1.28	11.00	1.50	0.54	0.93
100	1.11	1.46	1.24	0.09	1.00	1.73	2.40	1.80	0.18	0.99	1.10	2.00	1.25	0.12	0.70	1.29	11.50	1.50	0.60	0.90

Figure 6: Full summary of our simulations with the truncated Mallows model. Avg, Max, 0.9P, and Std respectively indicate the average, maximum, 90%-percentile, and standard deviation of the incentive ratio. Freq denotes the percentage of manipulable profiles.