Best-of-Both-Worlds Fairness in Committee Voting with Ordinal Preferences

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Abstract

Choosing a committee by lottery can improve representation in expectation, but the committee ultimately selected must also be representative. We ask how far these two demands can be combined when voters rank candidates, possibly with ties. We show that every weak-ranking profile admits a lottery satisfying cutoff group fair share (cutoff-GFS) and fractional rank proportional justified representation plus (Rank-PJR+) ex ante, and inclusion proportionality for solid coalitions (IPSC) ex post. Such a lottery can be computed in polynomial time on unrestricted weak rankings. Under strict rankings, IPSC is proportionality for solid coalitions (PSC). In contrast, requiring rank justified representation (Rank-JR) ex post, and hence also Rank-PJR+, can be incompatible with rank individual fair share (Rank-IFS) ex ante, already under strict rankings. We also classify the complexity of exactly implementing prescribed marginals: it is NP-complete for PSC under strict rankings and for Rank-PJR+ under weak rankings, and Σ_2^{P}-complete for IPSC under weak rankings.

Keywords: Committee voting, Proportional representation, Randomization, Best-of-both-worlds fairness, Ordinal preferences, Computational social choice

1. Introduction

A lottery can distribute representation fairly in expectation and still elect a committee that leaves a cohesive group without a voice. Randomization allows groups to share indivisible seats through their chances of representation, but those chances must translate into meaningful guarantees for the committee actually elected. Ranked ballots make this challenge particularly rich. Voters may disagree about their favourite candidates while agreeing on a broader set of preferred representatives, and the same voter may belong to several such groups at different depths of her ranking. We seek lotteries that reconcile these overlapping claims, giving voters fair representation in expectation while ensuring that every committee drawn is proportionally representative.

Why use a lottery at all? Committee slots are indivisible, so a deterministic rule must sometimes favour one group over an equally large one. If the voters split into two equal groups and the number of seats is odd, each group deserves half the seats, but every committee gives one group more than half. Tossing a coin to decide which group gets the extra seat gives each group exactly half in expectation, and every possible committee still gives each group its share rounded down. Fairness in expectation alone, however, does not prevent an unrepresentative committee from being drawn:

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tossing the coin to decide which group receives *all* the seats is equally fair in expectation. The *best-of-both-worlds* approach asks for both: a lottery that is fair before the draw and whose every outcome is fair after it [1, 2]. Our running example (Example 1) casts this coin-toss situation as a ranked profile with ties, and we follow it through the definitions and both algorithms.

Prior work combines fairness before and after the draw for approval-based committees [2, 3]. For strict rankings, Peters [4] shows that a lottery can satisfy a fractional version of proportionality for solid coalitions (PSC) ex ante while selecting only PSC committees. We ask how far this guarantee extends when voters may be indifferent and when ex-ante representation is measured across their rank cutoffs.

> *Can we give voters their fair share across rank cutoffs in expectation while ensuring that every committee drawn from the lottery is proportionally representative?*

Under weak rankings, verifying whether a given committee satisfies inclusion PSC (IPSC) is coNP-complete [5, Proposition 1], and deciding whether prescribed rational marginals admit an IPSC implementation is Σ_2^{P}-complete (Theorem 7). Nevertheless, our main result shows that every weak-ranking profile admits a polynomial-time computable lottery satisfying cutoff-GFS and fractional Rank-PJR+ ex ante and IPSC ex post (Theorems 1 and 3). Its support contains at most $m + 1$ committees.

The two ex-ante guarantees express fairness across rank cutoffs. Cutoff-GFS lets each voter choose one of her rank cutoffs and gives every group, within the union of its members' chosen cutoffs, at least the sum of their individual fair shares. For example, when $k = 2$, each voter can claim $2/n$ expected winners from a cutoff containing at least two candidates. Fractional Rank-PJR+ demands more for a group whose members share a candidate within their cutoffs: unless that candidate is selected with certainty, the union of these cutoffs must receive the group's full Hare share $k|S|/n$.

Two ideas make the construction possible. We first fix how much each voter spends on each indifference class and preserve these totals throughout rounding. We then choose how this spending is split among tied candidates jointly with the lottery. This freedom allows the marginals to retain both fractional guarantees while every committee in the support satisfies IPSC.

For strict rankings, Peters [4] already establishes fractional PSC, a PSC-supported implementation of Phragmén's fractional rule (PFR), and candidate monotonicity. His conclusion asks whether these results extend to weak preferences. We establish the fairness and implementation extension with IPSC under the Hare quota, which strengthens generalized PSC and coincides with PSC on strict rankings [5]. We also show that PFR satisfies cutoff-GFS and fractional Rank-PJR+. Both of our constructions implement exactly PFR on strict profiles and therefore inherit its candidate monotonicity there. Candidate monotonicity of our weak-ranking construction is not established here.

Results for approval ballots do not simply carry over to rankings. An approval ballot is a two-tier weak ranking, so our positive results specialize to approval profiles (Lemma 2), but the converse transfer fails. A ranking induces a chain of nested cutoffs, and the same voter may belong to different cohesive groups at different depths while holding a single budget of k/n. Running an approval rule on every cutoff would either give each voter several budgets or dilute every entitlement $k|S|/n$. Our rules must therefore decide at which depth each voter's budget is spent, while the ex-post guarantee protects groups at all depths simultaneously.

Our first algorithmic tool is *Class Eating* (Definition 8, Algorithm 1), a polynomial-time process that extends PFR to weak rankings. Voters eat their indifference classes at equal speed, each class as a whole, and maximum-flow computations check that the reservations remain jointly feasible.

The process fixes how much each voter spends on each of her classes without choosing among tied candidates, and every realization of the final reservations has marginals satisfying cutoff-GFS and fractional Rank-PJR+ (Theorem 2). Our second tool, *payment-transfer rounding* (Algorithm 2), reads a realization as voter payments and first fills candidates whose residual top supporters can fund a seat. It then packs the remaining budgets into blocks that each receive one seat in a matching decomposition. Every committee it draws satisfies IPSC, giving a direct compatibility proof.

Our third tool, and a contribution of independent interest, is a polynomial-time rounding technique (Algorithm 3 and Theorem 3). Its key idea is to reverse the order of payment-transfer rounding: instead of fixing the spending and then searching for representative committees that implement it, it generates committees that are representative by construction and only afterwards chooses their probabilities. Prices on the candidates steer a variant of EAR towards different representative committees, and an LP-duality argument, made constructive by the ellipsoid method, shows that each committee need only be no cheaper than the cheapest admissible spending at those prices. The technique thus avoids every hard step: it never tests whether a committee satisfies IPSC, never searches for smallest solidly supported sets (Proposition 2), and never implements prescribed marginals.

We complement this with an impossibility result. For every $k \geq 2$, Theorem 4 gives a strict-ranking profile on which Rank-IFS ex ante conflicts with Rank-JR ex post. Both cutoff-GFS and fractional Rank-PJR+ imply Rank-IFS, and Rank-PJR+ implies Rank-JR, so neither fractional guarantee can be combined with Rank-JR, let alone Rank-PJR+, ex post. This reveals a sharp distinction between IPSC and Rank-PJR+ as ex-post guarantees. Rank-PJR+ implies IPSC, while Rank-JR and IPSC are incomparable. IPSC can always be combined with both fractional guarantees, even when preferences contain ties, whereas requiring Rank-PJR+, or even Rank-JR, in every realized committee can preclude basic individual fairness in expectation. The conflict is specifically ordinal: on approval profiles, polynomial-time lotteries combine GFS ex ante with extended justified representation (EJR) ex post [2]. It is the deeper rank cutoffs that create the conflict.

Both fractional fairness guarantees can also be combined, in polynomial time, with efficiency in the sense of stochastic dominance (SD-efficiency, Theorem 5). Every implementation of the resulting marginals has SD-efficient support, and under strict rankings a polynomial-time implementation can additionally ensure PSC ex post. Once ties are allowed, the approval impossibility of Jin et al. [6, Theorem 1] implies that SD-efficiency can conflict with ex-post IPSC (Remark 2). The efficiency guarantee thus complements our main fairness result, while this impossibility limits how far the two can be combined.

Finally, we study how to implement a *given* fractional committee, which differs from finding some fair lottery. Exact implementation by PSC committees under strict rankings and by Rank-PJR+ committees under weak rankings is NP-complete, and exact implementation by IPSC committees under weak rankings is Σ_2^{P}-complete (Theorems 6 and 7). On approval profiles, where IPSC is PJR+, exact IPSC implementation is only NP-complete; the jump to Σ_2^{P} comes from rankings with more than two tiers. Fractional IPSC verification is coNP-complete (it is polynomial on approval profiles), whereas cutoff-GFS admits polynomial-time separation and fractional Rank-PJR+ admits polynomial-time violation detection (Theorem 9, Proposition 3). These hardness results concern prescribed marginals, whereas Theorem 3 computes some compatible lottery by choosing its marginals.

Overall, fairness before and after the draw can be guaranteed for all ranked profiles, but the choice of ex-post axiom matters. Figure 1 and Table 2 summarize the boundaries. The next section places our results in the context of ordinal committee rules and earlier work on fair lotteries.

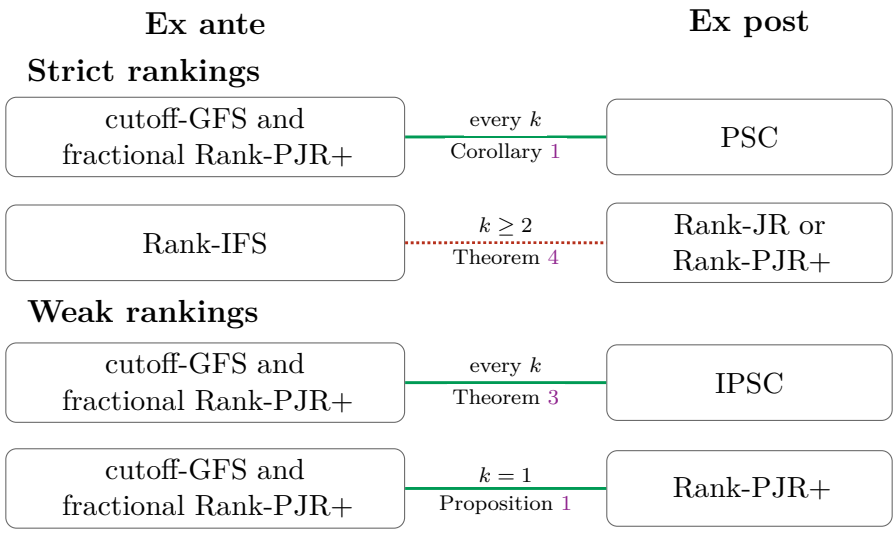


Figure 1: Ex-ante/ex-post compatibility. Green solid lines denote polynomial-time construction for every profile in the stated domain; red dotted lines denote a profile on which no compatible lottery exists. The strict Rank-IFS obstruction also applies to weak rankings and to cutoff-GFS, fractional Rank-PJR+, and fractional IPSC ex ante. The positive fairness results also give fractional IPSC ex ante.

2. Related Work

Representation from rankings. Classical ordinal committee rules select representatives by aggregating how voters rank the candidates: the Chamberlin–Courant rule rewards well-represented voters, and Monroe's rule balances the numbers of voters assigned to representatives [7, 8]. The axiomatic comparison of these and other ordinal rules by Elkind et al. [9] gives background for the distinction between selecting a representative committee and requiring a seat for each sufficiently large cohesive group. Dummett's PSC formalizes the latter requirement [10]. Aziz and Lee [11] develop the Expanding Approvals Rule for ranked ballots. Unlike these deterministic guarantees, we ask whether every draw of a lottery can satisfy proportionality while the lottery meets fairness requirements in expectation.

Weak rankings and robust proportionality. Ties call for care in deciding which candidates a group may claim. Aziz and Lee [12] introduce IPSC in their study of ordinal participatory budgeting. Brill and Peters [5] develop Rank-PJR+ by testing representation across rank cutoffs; the axiom is stronger than IPSC and remains polynomial-time verifiable for weak rankings. Aziz et al. [13] design a committee-monotone rule satisfying PSC and IPSC, while Bardal et al. [14] measure proportionality on ranked election data and find that large solid coalitions are scarce there. These results motivate the two sides of our lottery: cutoffs express broader agreements ex ante, and IPSC gives a solid-coalition guarantee in each realized committee. Our negative results identify the point at which stronger ex-post rank guarantees conflict with those ex-ante demands.

Fair lotteries. Individual and group fair shares for dichotomous fair mixing are studied by Aziz et al. [15]; justified representation for approval committees originates with Aziz et al. [16]. Aziz et al. [2] introduce best-of-both-worlds fairness for approval-based committees, and Suzuki and Vollen [3] strengthen its fractional guarantee by a flow construction. Approval ballots are the two-tier case of our model, so our results specialize to approval profiles. The approval results do not extend to rankings, however: for instance, the ranking analogue of their GFS–EJR compatibility fails (Theorem 4). Aziz et al. [17] extend the lottery perspective to participatory budgeting, where

different project costs create an additional implementation problem. Their unit-cost cardinal IFS–JR impossibility provides the proof pattern for our Rank-IFS–Rank-JR separation. Closest to our ordinal setting, Peters [4] constructs a lottery supported on PSC committees for strict preferences. The PFR rule of Peters [4] derives from Phragmén's Ordered Rule, surveyed by Janson [18]. Class Eating adapts the extension of probabilistic serial [19] to indifferences by Katta and Sethuraman [20], which tracks feasibility by parametric network flows. In our setting, voters reserve mass for whole indifference classes, the split among tied candidates is left open, and ex-post committee representation requires a further rounding step. The matching decomposition is a standard dependent-rounding operation [21]. Finally, Airiau et al. [22] study fair portioning from rankings, a related fractional public-decision model without our ex-post committee constraints.

3. Model and Core Axioms

Let $N = [n]$ be the set of voters, C a set of m candidates, and $k \le m$ the size of a committee $W \subseteq C$. A fractional committee $p = (p_c)_{c \in C}$ satisfies $0 \le p_c \le 1$ and $\sum_{c \in C} p_c = k$. A lottery implements p when $p_c = \Pr[c \in W]$ for every $c \in C$. Every fractional committee has an implementation by size-k committees, but the committees in its support need not satisfy a given fairness axiom.

Each voter i has a weak order (a complete and transitive preference) $\succsim_i$ over C; we write $c \succ_i d$ for strict preference and $c \sim_i d$ for indifference. Equivalently, we list her indifference classes, *classes* for short, from best to worst:

$$E_i^1 \succ_i E_i^2 \succ_i \cdots \succ_i E_i^{t_i},$$

where E_i^1 contains her most preferred candidates, E_i^2 her most preferred among the rest, and so on. Let $E_i^{\le j} = E_i^1 \cup \cdots \cup E_i^j$ be the union of her top j classes, with $E_i^{\le 0} = \emptyset$. A candidate's rank is determined by her class:

$$\mathrm{rank}_i(c) = 1 + |E_i^{\le j-1}| \qquad (c \in E_i^j),$$

one more than the number of candidates i strictly prefers to c. Tied candidates thus share a rank, and the next class starts after all of them: if $a \succ_i \{b, c\} \succ_i d$, the ranks of a, b, c, d are $1, 2, 2, 4$. Voter i's rank cutoffs are

$$A_i^r = \{c \in C : \mathrm{rank}_i(c) \le r\},$$

so every cutoff consists of whole classes: $A_i^r = E_i^{\le j}$ for the unique j with $|E_i^{\le j-1}| < r \le |E_i^{\le j}|$. In the example, $A_i^2 = A_i^3 = E_i^{\le 2} = \{a, b, c\}$. Equivalently, A_i^r consists of the candidates that voter i weakly prefers to her r-th choice, for any way of breaking her ties; this is the reading used by Aziz and Lee [11]. Thus $E_i^{\le 1} \subsetneq \cdots \subsetneq E_i^{\le t_i} = C$ are exactly voter i's distinct cutoffs, and $E_i^{\le j} = \{d : d \succsim_i c\}$ is the upper contour set of every $c \in E_i^j$. The fairness axioms are stated with rank cutoffs, because a rank means the same for every voter; our algorithms work with the classes E_i^j and their unions $E_i^{\le j}$, which list each distinct cutoff once. Under strict rankings, $t_i = m$ and $A_i^r = E_i^{\le r}$ has r members. For $B \subseteq C$, classes and cutoffs within B are defined in the same way from $\succsim_i$ restricted to B; we write the latter as $A_i^{r,B}$. For a lottery implementing p, $\sum_{c \in A_i^r} p_c$ is the expected number of selected candidates from this cutoff. We view an approval ballot as a two-tier weak ranking, with approved candidates tied above the rest. A coalition $S \subseteq N$ has Hare entitlement $k|S|/n$. Unless stated otherwise, coalitions and supported candidate sets are nonempty, and an integral entitlement ℓ lies in $[k]$.

For a vector v indexed by candidates or voters and a set X, we write $v(X) = \sum_{z \in X} v_z$; for instance, $p(D) = \sum_{c \in D} p_c$ for $D \subseteq C$. We distinguish two levels of evaluation. Ex-post axioms constrain every committee that may be realized, whereas ex-ante axioms constrain the marginal

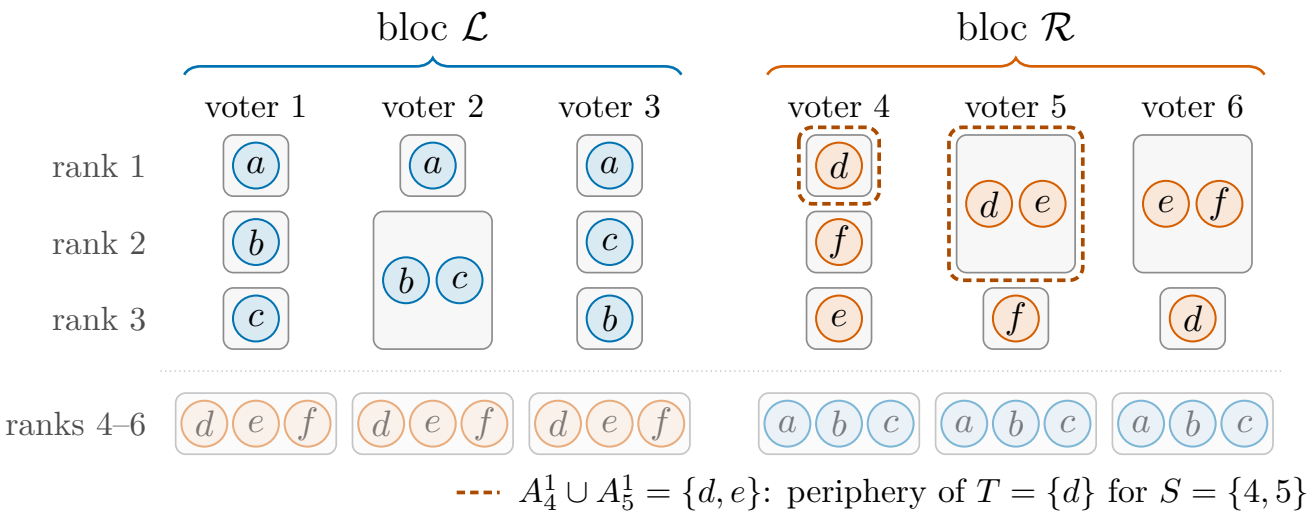


Figure 2: The running example: six voters, $k = 3$ seats, so each voter holds $k/n = 1/2$ and two voters form a quota. Each box is an indifference class; a class occupying several rank rows is a tie, and every cutoff A_i^r contains whole boxes. Candidates of bloc $\mathcal{L}$ are blue, those of bloc $\mathcal{R}$ orange.

inclusion probabilities generated by the lottery. We first introduce the relevant ex-post representation benchmarks and then their fractional and cutoff-based counterparts.

Throughout, PSC and IPSC use the Hare quota n/k: a coalition S meets the entitlement threshold for ℓ seats when $|S| \geq \ell n/k$. Our results concern these versions of the axioms and do not assert their Droop-quota counterparts.

Definition 1 (PSC [10, 11])**.** Under strict rankings, S solidly supports T if $c \succ_i d$ for all $i \in S$, $c \in T$, and $d \in C \setminus T$. A committee W satisfies PSC if every such S, T and every integer $\ell \in [k]$ with $k|S|/n \geq \ell$ satisfy $|W \cap T| \geq \min\{\ell, |T|\}$.

For weak rankings, S *generalized solidly supports* T if the same comparisons hold weakly: $c \succsim_i d$ for all $i \in S$, $c \in T$, and $d \in C \setminus T$. The *periphery* of (S, T) consists of the candidates that some member of S ranks at least as high as some candidate of T. For a generalized solid coalition S for T, this set is $D(S, T) = \bigcup_{i \in S} A_i^{|T|}$, the candidates that some supporter ranks at least as high as her $|T|$-th choice, and $T \subseteq A_i^{|T|}$ for every $i \in S$. Indeed, let E_i^s be the worst class of supporter i that meets T. Then $T \subseteq E_i^{\leq s}$, and fewer than $|T|$ candidates lie strictly above E_i^s, so $A_i^{|T|} = E_i^{\leq s}$. Under strict preferences, the periphery is exactly T.

Definition 2 (Inclusion PSC [12])**.** A committee W satisfies *inclusion PSC (IPSC)* if, whenever S generalized solidly supports T and $\frac{k|S|}{n} \geq \ell$ for an integer $\ell \in [k]$,

$$T \subseteq W \quad \text{or} \quad |W \cap D(S, T)| \geq \ell.$$

IPSC is our principal ex-post requirement. Under strict rankings, it reduces to PSC. We relate it to the rank-based axioms of Brill and Peters [5] after Definition 3 and Lemma 2.

Example 1 (Running example: ballots and IPSC)**.** Six voters elect $k = 3$ of the candidates $C = \{a, b, c, d, e, f\}$, so each voter holds $k/n = 1/2$ and every two voters are entitled to one seat. They form two blocs, $\mathcal{L} = \{1, 2, 3\}$ and $\mathcal{R} = \{4, 5, 6\}$ (Figure 2):

$$\begin{aligned} 1&: a \succ b \succ c \succ \{d, e, f\}, & 4&: d \succ f \succ e \succ \{a, b, c\}, \\ 2&: a \succ \{b, c\} \succ \{d, e, f\}, & 5&: \{d, e\} \succ f \succ \{a, b, c\}, \\ 3&: a \succ c \succ b \succ \{d, e, f\}, & 6&: \{e, f\} \succ d \succ \{a, b, c\}. \end{aligned}$$

Each bloc is entitled to 3/2 seats, so, as in the introduction, every committee favours one of them. Cutoffs contain whole indifference classes; for instance, $A_2^2 = A_2^3 = \{a, b, c\}$ and $A_5^1 = A_5^2 = \{d, e\}$.

Any two voters of $\mathcal{L}$ generalized solidly support $\{a\}$ with entitlement one, so IPSC places a in every committee. In $\mathcal{R}$, the pair $\{4, 5\}$ generalized solidly supports $T = \{d\}$, since both voters rank d in their top class. Because voter 5 ties d with e, the periphery $A_4^1 \cup A_5^1 = \{d, e\}$ is larger than T. Similarly, $\{5, 6\}$ supports $\{e\}$ with periphery $\{d, e, f\}$. The remaining demands are implied by these (Appendix C), so a committee satisfies IPSC exactly when it contains a and meets $\{d, e\}$. Thus $\{a, b, e\}$ satisfies IPSC although it omits d, the only candidate that voters 4 and 5 both rank first. In contrast, $\{a, b, f\}$ violates IPSC: f lies outside the periphery $\{d, e\}$, even though every voter of $\mathcal{R}$ ranks f among her top three.

Definition 3 (Rank-JR and Rank-PJR+ [5])**.** Here JR abbreviates justified representation, and PJR+ denotes the robust strengthening of proportional justified representation. A committee W satisfies Rank-JR if $W \cap (\bigcup_{i \in S} A_i^r) \neq \emptyset$ for every rank r and every S with $k|S|/n \geq 1$ and $\bigcap_{i \in S} A_i^r \neq \emptyset$. It satisfies Rank-PJR+ if for every rank r, integer $\ell \in [k]$, S with $k|S|/n \geq \ell$, and $c \in \bigcap_{i \in S} A_i^r$,

$$c \in W \quad \text{or} \quad |W \cap (\bigcup_{i \in S} A_i^r)| \geq \ell.$$

Rank-PJR+ implies Rank-JR (take $\ell = 1$). It also implies IPSC: if S generalized solidly supports $T \not\subseteq W$, pick $c \in T \setminus W$; since $T \subseteq A_i^{|T|}$ for every $i \in S$, Rank-PJR+ at rank $|T|$ yields the IPSC demand. The Expanding Approvals Rule (EAR) computes a Rank-PJR+ committee even for weak rankings [5, 11].

Definition 4 (Fractional Rank-PJR+)**.** A fractional committee p satisfies fractional Rank-PJR+ if for every $r \in [m]$, nonempty $S \subseteq N$, and $c \in \bigcap_{i \in S} A_i^r$,

$$p_c = 1 \quad \text{or} \quad \sum_{z \in \bigcup_{i \in S} A_i^r} p_z \geq \frac{k|S|}{n}.$$

Fractional Rank-PJR+ retains the coalition's full Hare entitlement $k|S|/n$ rather than rounding it down, unless the common candidate is selected with certainty. Randomization allows the fractional remainder to be supplied in expectation, even though an integral committee can guarantee only whole seats. Consequently, an integral committee satisfying Rank-PJR+ need not satisfy fractional Rank-PJR+ when viewed as a fractional committee.

Example 2 (Integral and fractional entitlements)**.** Let $n = 3$, $k = 2$, and $C = \{a, b, d\}$, with approval ballots $\{a, b\}, \{a, b\}, \{d\}$ interpreted as two-tier weak rankings. The committee $W = \{a, d\}$ satisfies Rank-PJR+: a coalition with cutoff union $\{a, b\}$ contains at most two voters, so its integer entitlement is at most one seat, supplied by a. All other cutoff unions are $\{d\}$ or C, and their demands are met. Nevertheless, voters 1 and 2 and their unselected common candidate b witness a violation of fractional Rank-PJR+: the incidence vector p of W satisfies

$$p_b = 0, \qquad p(\{a, b\}) = 1 < \frac{k \cdot 2}{n} = \frac{4}{3}.$$

A supported set T is *fully included* in a fractional committee p if $p_c = 1$ for each $c \in T$.

Definition 5 (Fractional IPSC)**.** A fractional committee p satisfies *fractional IPSC* if, for every pair (S, T) such that S generalized solidly supports T,

$$p(T) = |T| \quad \text{or} \quad p(D(S, T)) \geq \frac{k|S|}{n}.$$

Fractional Rank-PJR+ implies fractional IPSC: if T is not fully included, choose $c \in T$ with $p_c < 1$ and apply the former axiom at rank $|T|$. Under strict rankings, fractional IPSC coincides with *fractional PSC*, which requires $p(T) \geq \min\{k|S|/n, |T|\}$ for every solid coalition S supporting T. Fractional IPSC likewise retains the unrounded Hare entitlement, subject to its full-inclusion alternative. Cutoff-GFS, defined next, instead sums the individual shares capped by each chosen cutoff's size.

Definition 6 (Cutoff group fair share (cutoff-GFS))**.** A fractional committee p satisfies cutoff-GFS if, for every nonempty $S \subseteq N$ and every choice of ranks $(r_i)_{i \in S} \in [m]^S$,

$$\sum_{z \in \bigcup_{i \in S} A_i^{r_i}} p_z \geq \frac{1}{n} \sum_{i \in S} \min\{k, |A_i^{r_i}|\}. \tag{1}$$

Definition 7 (Rank individual fair share (Rank-IFS))**.** A fractional committee p satisfies Rank-IFS if $p(A_i^r) \geq \min\{k, |A_i^r|\}/n$ for every voter i and rank r. These are exactly the single-voter restrictions of cutoff-GFS.

Our main positive results impose cutoff-GFS and fractional Rank-PJR+ simultaneously.

The random-dictator benchmark. Let W_i consist of voter i's k most-preferred candidates, breaking a boundary tie arbitrarily. Random dictator selects each W_i with probability $1/n$. Since $|W_i \cap A_i^r| = \min\{k, |A_i^r|\}$, its marginals p^{RD} satisfy, for every coalition and chosen cutoffs,

$$p^{\mathrm{RD}}\left(\bigcup_{i \in S} A_i^{r_i}\right) \geq \frac{1}{n} \sum_{i \in S} |W_i \cap A_i^{r_i}| = \frac{1}{n} \sum_{i \in S} \min\{k, |A_i^{r_i}|\}.$$

Thus the random-dictator marginals satisfy cutoff-GFS, and hence Rank-IFS, extending Aziz et al. [2, Proposition 3.6] to rank cutoffs. Many other vectors satisfy these bounds as well; the incompatibility in Theorem 4 concerns every vector satisfying Rank-IFS. Random dictator need not satisfy fractional Rank-PJR+: on a strict profile with $n = 4$ and $k = 2$, if exactly two voters rank c first and the other two omit c from their top two, it gives $p_c = 1/2$, whereas fractional Rank-PJR+ requires $p_c = 1$ for the first pair at rank one.

Example 3 (Running example: ex-ante demands)**.** In Example 1, the three voters of $\mathcal{L}$ have the same top-three cutoff $\{a, b, c\}$. Cutoff-GFS for $\mathcal{L}$ with these cutoffs demands $p(\{a, b, c\}) \geq 3 \cdot 3/6 = 3/2$, and likewise $p(\{d, e, f\}) \geq 3/2$. Together the two demands exhaust the total mass $k = 3$, so cutoff-GFS gives each bloc exactly $3/2$ expected seats, and no single committee satisfies it. Shallower cutoffs give smaller demands, such as $p(\{d, e\}) \geq (1 + 2)/6 = 1/2$ from voter 4's cutoff $\{d\}$ and voter 5's cutoff $\{d, e\}$.

Fractional Rank-PJR+ demands more for groups that share a candidate. Voters 1 and 2 share a at rank one and have entitlement one, while the union of their first cutoffs is just $\{a\}$; both alternatives of the axiom therefore require $p_a = 1$. The pair $\{4, 5\}$ shares d at rank one, so the axiom requires $p(\{d, e\}) \geq 1$ (the alternative $p_d = 1$ implies this). Table C.4 lists all demands of both axioms on this profile.

Random dictator selects $\{a, b, c\}$ or $\{d, e, f\}$ with probability $1/2$ each: a coin decides which bloc takes all three seats. Its marginals, all equal to $1/2$, satisfy cutoff-GFS but violate fractional Rank-PJR+, and both committees it draws violate IPSC. The lotteries we construct below toss the coin only for the third seat.

Lemma 1 (The individual cutoff consequence)**.** *Fractional IPSC, and hence fractional Rank-PJR+, implies Rank-IFS.*

Proof. Voter i generalized solidly supports $T = A_i^r$, with periphery T. If T is fully included, $p(T) = |T|$; otherwise fractional IPSC gives $p(T) \geq k/n$. Either bound implies $p(T) \geq \min\{k, |T|\}/n$. □

Lemma 2 (The approval restriction)**.** *On two-tier approval profiles, cutoff-GFS is GFS, Rank-JR is JR, and Rank-PJR+ and IPSC are both PJR+. Fractional Rank-PJR+ is fractional PJR+.*

Proof. Write A_i for voter i's approved set. Each nonempty cutoff is A_i or C. A cutoff union containing C makes its inequality automatic; all other cutoff-GFS demands are exactly $p(\bigcup_{i\in S} A_i) \geq n^{-1} \sum_{i\in S} \min\{k, |A_i|\}$. The rank demands with union different from C are exactly the JR and PJR+ demands on approval unions, with the same common-candidate and entitlement tests. For IPSC, any periphery meeting a supporter's disapproved tier is C and imposes no restriction. Otherwise $T \subseteq \bigcap_{i\in S} A_i$; a missing $c \in T$ invokes PJR+. Conversely, every PJR+ demand is the IPSC demand for $T = \{c\}$. The fractional rank statement follows identically. □

The lemma transfers results only from rankings to approval ballots. Approval results do not transfer back, because a voter's cutoffs form a chain of approval-like sets that share one budget of k/n.

Rank-JR and IPSC are incomparable. Two voters approving $\{a, b\}$ with $k = 2$ admit the JR committee $\{a, c\}$, which violates PJR+ and hence IPSC. In the other direction, on the strict profiles of Theorem 4, the lottery of Corollary 1 satisfies Rank-IFS and is supported on PSC (hence IPSC) committees, so at least one of these committees fails Rank-JR.

Example 4 (Independent fractional requirements)**.** With $n = k = 2$, approval sets $\{a\}$ and $\{b\}$, and candidates a, b, c, the vector $(1/2, 1/2, 1)$ satisfies GFS but fails fractional PJR+: a singleton voter needs mass one on her approved candidate. Conversely, for $n = 3$, $k = 2$, and approval sets $\{a, b\}, \{b, c\}, \{c, a\}$, the vector $(p_a, p_b, p_c, p_d) = (4/9, 4/9, 4/9, 2/3)$ satisfies fractional PJR+. A common-candidate group has at most two voters; singletons receive $8/9 \geq 2/3$, and two-voter unions receive $4/3$. GFS fails for N, whose approval union has mass $4/3 < 2$. By Lemma 2, these examples prove independence of cutoff-GFS and fractional Rank-PJR+.

Figure 3 records the implications used below.



Figure 3: Implications among the axioms for weak rankings; an arrow denotes an implication within one panel. Cutoff-GFS and fractional Rank-PJR+ are incomparable (Example 4), as are IPSC and Rank-JR. Under strict rankings, IPSC is PSC and fractional IPSC is fractional PSC.

4. Fair Lotteries and Compatibility

Our main result is the following.

Theorem 1 (Both fractional guarantees with ex-post IPSC)**.** *Every weak-ranking profile and every committee size k admit a lottery satisfying cutoff-GFS and fractional Rank-PJR+ ex ante and IPSC ex post.*

The direct compatibility construction has two steps. First, voters eat their indifference classes at equal speed, without choosing among tied candidates (Class Eating, Section 4.1). Every realization of these reservations already satisfies both fractional guarantees (Theorem 2), but its marginals need not be implementable by IPSC committees (Example 6). Second, we read a realization as voter payments, transfer payments within voters' ties, and pack them into payment blocks of mass one, each funded by voters who generalized solidly support a common set; a matching decomposition then gives each block one seat (Algorithm 2, Section 4.2). The transfers keep the payments a realization of the same reservations, so both ex-ante guarantees survive, while the block construction enforces IPSC in every drawn committee. Section 4.3 then gives a polynomial-time construction on unrestricted weak rankings, using a different rounding method. Finally, Section 4.4 shows that the ex-post requirement cannot be replaced by Rank-JR (Theorem 4).

4.1. Class Eating and fractional fairness

We let voters reserve mass for entire classes and defer how it is spread within each class. Reservations remain jointly feasible, and a voter advances exactly when her current class is full in every realization. Under strict rankings this is ordinary simultaneous eating: each voter eats her favourite remaining candidate at unit speed and moves on when it is exhausted. Under weak rankings a voter may have several favourites, and committing her to one of them would resolve her ties prematurely. She therefore eats her whole top class at once: she reserves mass for the class as a set, and only at the end is it settled which candidates in the class receive it. The result is a reservation vector rather than a single fractional committee. The ex-ante guarantees hold for every way of settling the ties, and the rounding constructions of Sections 4.2 and 4.3 exploit exactly this freedom.

Definition 8 (Class Eating)**.** A *reservation vector* $x = (x_{ij})_{i\in N,\, j\in[t_i]}$ assigns a mass $x_{ij} \geq 0$ to each voter i and each of her classes E_i^j. Payments $y = (y_{ic}) \geq 0$, with $y_i = (y_{ic})_{c\in C}$ the payments of voter i, *realize* x if every candidate receives at most one unit, $\sum_i y_{ic} \leq 1$, and each voter spends her reservation for each class inside that class:

$$y_i(E_i^j) = x_{ij} \qquad (i \in N,\ j \in [t_i]). \tag{2}$$

The vector x is *feasible* if some payments realize it, and realizing payments give candidate marginals $p_c = \sum_i y_{ic}$. For a committee size $k \leq m$, let x^{CE} be the final reservation vector computed by Algorithm 1. The *Class Eating* rule is the correspondence

$$\mathrm{CE}(\succsim, k) = \Big\{ p \in [0,1]^C : p_c = \sum_i y_{ic}\ (c \in C) \text{ for some payments } y \text{ realizing } x^{\mathrm{CE}} \Big\}.$$

In words, x_{ij} is the mass voter i has eaten from her j-th class E_i^j. Realizing payments settle the ties: mass reserved for a class must be spent inside it, but how it is split among the tied candidates is left open, and y_{ic} is then what voter i pays for candidate c. Feasibility is a Hall condition: x is feasible exactly when no collection X of reservations exceeds the number of candidates in their

classes, $\sum_{(i,j)\in X} x_{ij} \leq |\bigcup_{(i,j)\in X} E_i^j|$. The rule returns a set of fractional committees because it fixes how much each voter spends on each of her classes, but not how tied candidates share it.

Algorithm 1 increases all voters' active reservations at unit speed. A voter's active reservation is *blocked* if it cannot be increased by any positive amount while all other reservations stay fixed. When no further common increase is feasible, every voter whose reservation is blocked advances to her next class. Feasibility of a reservation vector is a bipartite maximum-flow test, with source capacities x_{ij} and candidate capacities one.

Algorithm 1 (Class Eating)**.**
Input: weak rankings $(\succsim_i)_{i\in N}$ and a committee size $k \leq m$.
Output: the final reservation vector x^{CE}.
Initialization: put $\tau = 0$, $x_{ij} = 0$ for all i, j, and $j_i = 1$ for every voter i; voter i's *active* reservation is her reservation for her current class $E_i^{j_i}$. Voter i *leaves* E_i^j when j_i is increased beyond j.

1. **Reserve at equal speed.** Let δ be the largest $d \in [0, k/n - \tau]$ such that x stays feasible when every active reservation is increased by d. The increment δ may be zero. Increase every active reservation by δ and set $\tau \leftarrow \tau + \delta$. If $\tau = k/n$, go to step 3.
2. **Advance blocked voters.** Let N_{blk} be the set of voters whose active reservation cannot be increased by any positive amount while all other reservations stay fixed. Simultaneously set $j_i \leftarrow j_i + 1$ for every $i \in N_{\mathrm{blk}}$, keep x unchanged, and return to step 1.
3. **Return.** Set $x^{\mathrm{CE}} = x$ and return it.

Step 1 asks how long all voters can keep eating at unit speed from their current classes before the reservations stop being feasible. The answer δ is the time until some voter's current class becomes full in every realization, or until the end of the process, whichever comes first; it is zero when such a class is full already. Step 2 moves exactly these voters on: a reservation that cannot be increased even on its own is one whose class is full in every realization (Lemma 3). A class containing a candidate that some realization leaves underfull is not full, and its voter keeps eating there; in the running example, voter 5 is never moved out of $\{d, e\}$, since her mass can always be shifted to e. The process runs until time k/n, so each voter reserves exactly her budget and the reservations total k, one unit per seat.

Why voters wait until a class is forced full. Moving a voter on as soon as *some* realization fills her class would be premature: she would start paying for worse candidates while a candidate she likes at least as much might still finish with spare capacity. Waiting until the class is full in every realization yields the invariant on which both fractional guarantees and both rounding constructions rest: if c ends underfull, no voter who has c in her class E_i^s ever paid for anything below E_i^s (Lemma 3). Blocking happens to groups. When the Hall inequality of a collection of reservations becomes tight, these reservations use up exactly the candidates in their classes, so all of their classes are full in every realization and the voters holding them advance together; this is the bottleneck structure exploited by Katta and Sethuraman [20]. An increment $\delta = 0$ arises when a voter advances into a class that others have already filled in every realization. She is then blocked at once and advances again, so exhausted classes are simply skipped.

Only the reservations are fixed during the process; their realization may be rearranged at any stage. The final reservation vector is uniquely determined without choosing among tied candidates, but its realization need not be unique. A rational realization, and hence an output of Class Eating, can be computed by maximum flow.

Lemma 3 (Saturation and reservation guarantees)**.** *Algorithm 1 is well defined, its output is uniquely determined, and it can be implemented in polynomial time. If voter i leaves her class E_i^j, every candidate in $E_i^{\le j}$ is full in every realization from that point onwards. The final reservations satisfy*

$$\sum_{s \le j} x_{is} \ge \frac{\min\{k, |E_i^{\le j}|\}}{n}. \tag{3}$$

For every final realization, if $p_c < 1$ and $c \in E_i^s$, then $x_{ij} = 0$ for all $j > s$.

In words: a voter leaves a class only when it is full, and a full class stays full. All voters together eat at speed n, so the candidates of $E_i^{\le j}$ cannot all be full before time $|E_i^{\le j}|/n$; a voter therefore stays within her first j classes until time $\min\{k/n, |E_i^{\le j}|/n\}$, which is (3). Finally, if a candidate c ends underfull, no voter was ever moved past it: if $c \in E_i^s$, all of voter i's mass sits in her first s classes. This is the source of the payment-order property (5) on which both rounding constructions rely.

Proof. Well-definedness and the running time are proved in Appendix A.1. For the remaining claims, the key observation is that a reservation is blocked exactly when its entire class is full in every realization. If some realization y of x left $c \in E_i^j$ underfull, increasing both y_{ic} and x_{ij} slightly would give another realization, contradicting blocking. Conversely, suppose every candidate of E_i^j is full in every realization of x. Given payments realizing x with x_{ij} increased by ε, remove ε from voter i's payments inside E_i^j. This realizes x and leaves a candidate of E_i^j underfull, which is impossible, so the reservation is blocked. This fullness persists: from a realization of any later, componentwise larger reservation vector, reduce each voter's payments inside each class to her earlier reservation for it. No candidate receives more, so every previously forced candidate is still full. When voter i leaves E_i^j, the classes she left before are full as well, so all of $E_i^{\le j}$ is full.

At time τ, each voter's reservations total τ, and all reservations total $n\tau$. If voter i leaves E_i^j at time τ, fullness therefore gives $n\tau \ge |E_i^{\le j}|$. Her final reservation for her first j classes equals this departure time. If she never leaves E_i^j, all her k/n units remain in those classes. These two cases prove (3).

Finally, a positive reservation for a class after E_i^s would require voter i to leave E_i^s, which would force c to be full in every final realization. This proves the last claim. □

Under strict rankings every class is a single candidate, so Algorithm 1 is ordinary simultaneous eating, stopped at time k/n: each voter eats her current candidate until it is full and then moves to her next one. Its realization is unique, and its marginals are the PFR marginals [4].

Theorem 2 (Simultaneous fractional guarantees)**.** *Every weak-ranking profile admits a fractional committee that satisfies cutoff-GFS and fractional Rank-PJR+ simultaneously. Consequently, it also satisfies fractional IPSC. These guarantees hold for every output of Class Eating (Definition 8).*

Both guarantees are read off from where each voter's mass sits at the end. For cutoff-GFS, each voter's first r ranks hold at least $\min\{k, |A_i^r|\}/n$ of her mass by (3), and the union of a coalition's cutoffs receives at least the mass its members hold there. For fractional Rank-PJR+, an underfull

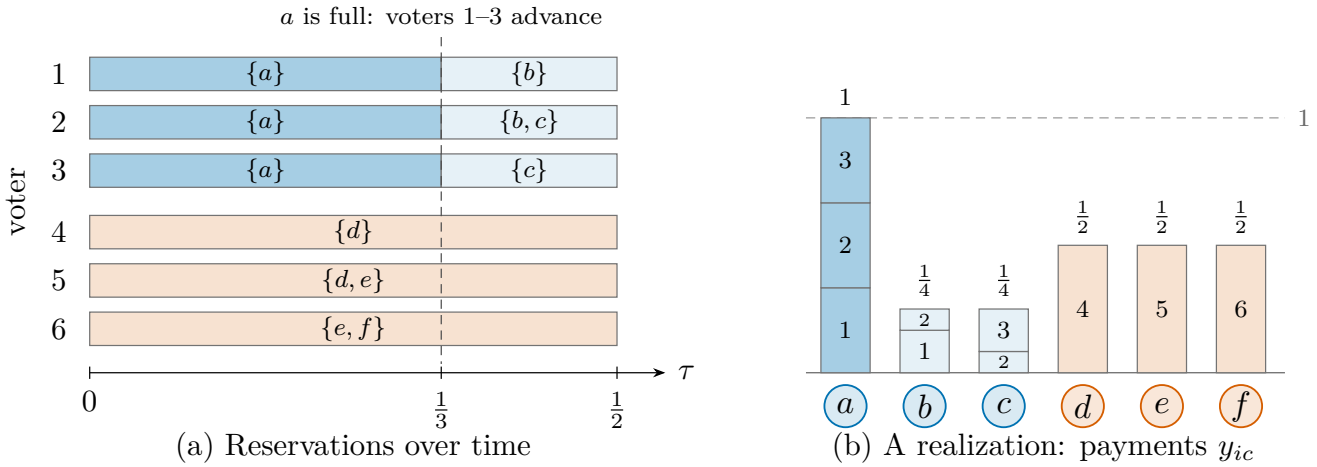


Figure 4: Class Eating on the running example. (a) Each bar shows the class a voter is eating; at $\tau = 1/3$ candidate a is full in every realization, so voters 1–3 move to their next classes. (b) One realization of the final reservations; numbers inside the bars identify the paying voter, and light segments are paid from second classes.

candidate c in a common cutoff shows that no member of S was ever moved past c, so all of their mass lies inside the union of their rank-r cutoffs. The alternative $p_c = 1$ in the axiom is exactly the case in which some members may have moved on. The first argument uses only equal speed, and the second only the forced-full rule. Neither looks inside a class: both use only the class totals $y_i(E_i^j) = x_{ij}$. This is why they hold for every realization, and why they survive every later transfer of payments between candidates that a voter is indifferent between.

Proof. Let x be the vector returned by Algorithm 1 and choose any payments y realizing it. Each voter pays k/n, every candidate receives at most one, and the marginals total k. Every cutoff A_i^r equals some $E_i^{\leq j}$ (Section 3), so (2) and Lemma 3 give

$$\sum_{c \in A_i^r} y_{ic} = \sum_{s \leq j} x_{is} \geq \frac{\min\{k, |A_i^r|\}}{n}. \tag{4}$$

For any coalition S and voter-specific cutoffs, its members' payments therefore yield

$$p\left(\bigcup_{i \in S} A_i^{r_i}\right) \geq \sum_{i \in S} \sum_{c \in A_i^{r_i}} y_{ic} \geq \frac{1}{n} \sum_{i \in S} \min\{k, |A_i^{r_i}|\}.$$

This proves cutoff-GFS.

Now let $c \in \bigcap_{i \in S} A_i^r$ have $p_c < 1$. For each $i \in S$, let $c \in E_i^s$. By Lemma 3, her entire budget is reserved for her first s classes. By (2), all her k/n units are therefore paid inside $E_i^{\leq s} \subseteq A_i^r$, hence into the cutoff union, proving fractional Rank-PJR+ and hence fractional IPSC. □

Example 5 (Running example: Class Eating)**.** Figure 4 runs Algorithm 1 on Example 1, where $k/n = 1/2$. Every voter starts with her top class. The voters of $\mathcal{L}$ eat $\{a\}$ at combined speed three, so at $\tau = 1/3$ candidate a is full in every realization. Their reservations are then blocked, and they advance to their second classes $\{b\}$, $\{b, c\}$ and $\{c\}$, respectively. No reservation of $\mathcal{R}$ is ever blocked: voter 5 can always pay for e instead of d, and voter 6 for f instead of e, so no candidate of $\mathcal{R}$ is full in every realization. Reservation stops at $\tau = 1/2$. Each voter of $\mathcal{L}$ has then reserved $1/3$ for $\{a\}$ and $1/6$ for her second class, and each voter of $\mathcal{R}$ has reserved $1/2$ for her top class.

In the realization shown in Figure 4(b), voter 2 splits her second reservation equally between b and c. Its marginals are

$$p^0 = (p_a, p_b, p_c, p_d, p_e, p_f) = \left(1, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

As Theorem 2 guarantees, p^0 satisfies both fractional axioms, and several of their demands are tight. The bloc demands hold with equality, $p^0(\{a, b, c\}) = p^0(\{d, e, f\}) = 3/2$. Since $p^0_d < 1$, Lemma 3 keeps voters 4 and 5 in their top classes, so their whole budgets are paid into $\{d\} \cup \{d, e\}$. This gives $p^0(\{d, e\}) = 1$, exactly the pair's fractional Rank-PJR+ demand.

What the rounding inherits. Class Eating hands the rounding stage one structural property. Payment order (Lemma 4 below) says that a voter never pays for a candidate she ranks below an underfull candidate. Consider a generalized solid coalition S for T whose set is not fully funded, and pick $c \in T$ with $p_c < 1$. Every member $i \in S$ pays only for candidates she weakly prefers to c, and all of these lie in $A_i^{|T|}$. Hence the whole budget $k|S|/n$ of S is paid inside the periphery $D(S, T)$, though not necessarily inside T: in Example 5, voter 5 may pay for e rather than d. IPSC asks for as many winners in this periphery as the money there can buy, rounded down. In this sense IPSC is the integral counterpart of payment order, and it is the natural ex-post target for lotteries built on Class Eating. The task of the rounding is to make this money buy seats in the periphery in every drawn committee, not merely in expectation.

4.2. A direct compatibility construction

This subsection proves Theorem 1. Starting from a realization of the reservations returned by Algorithm 1, Algorithm 2 transfers mass only within indifference classes and then decomposes the resulting payments into committees. More precisely, each voter's total payment to each of her indifference classes is the same at the end as in the initial realization. Saturation and block formation only change how she splits it among tied candidates, and the committees drawn implement the final payments exactly. This is why the ex-ante guarantees survive the rounding. Every output of Class Eating satisfies both fractional guarantees (Theorem 2), but, as the following example shows, not every output can be implemented by IPSC committees.

Example 6 (Reservation marginals may need transfers)**.** Let $n = 4$ and $k = 2$, with approval sets

$$\{a, b\}, \qquad \{a, d, e\}, \qquad \{a\}, \qquad \{a, c, d\}.$$

Every voter reserves $1/2$ for her top class, and no class is left before time $1/2$. Realizing these reservations at b, e, a, c, respectively, gives $(p_a, p_b, p_c, p_d, p_e) = (1/2, 1/2, 1/2, 0, 1/2)$, which satisfies both fractional guarantees by Theorem 2. The pairs $\{1, 3\}, \{2, 3\}, \{3, 4\}$ each support $\{a\}$ with entitlement one. A committee omitting a must therefore contain b, meet $\{d, e\}$, and meet $\{c, d\}$. Its only possibility is $\{b, d\}$. Hence every IPSC lottery satisfies $p_d \geq 1 - p_a$, which this realization violates. The saturation step of Algorithm 2 below changes this realization by transfers within ties. First, all four voters have a in their top class and together hold two units. Moving voter 1's payment from b to a fills a. After fixing a, voters 2 and 4 each retain $1/2$ and have d in their residual top class, their top class among the candidates not yet fully funded. Moving their payments from e and c to d fills d. No residual budget remains, so the algorithm returns the IPSC committee $\{a, d\}$ with probability one.

Even when a realization can be implemented by IPSC committees, an arbitrary lottery with the same marginals need not be supported on them. The draw must be correlated with who paid for what.

Example 7 (Good marginals, bad lottery)**.** Let $n = 3$ and $k = 2$, so each voter holds $2/3$, with

$$1:\ a \succ b \succ c \succ d, \qquad 2:\ b \succ a \succ c \succ d, \qquad 3:\ \{c,d\} \succ a \succ b.$$

No reservation is ever blocked: voters 1, 2 and 3 eat $\{a\}$, $\{b\}$ and $\{c,d\}$, respectively, without competition until time $2/3$. One realization has marginals $(p_a, p_b, p_c, p_d) = (2/3, 2/3, 1/3, 1/3)$. Voters 1 and 2 hold $4/3 \geq 1$ and solidly support $\{a,b\}$, whose periphery is $\{a,b\}$. A single voter holds less than one unit, and every set supported by voter 3 together with voter 1 or 2 is all of C. Hence a committee satisfies IPSC exactly when it meets $\{a,b\}$. The lottery $\frac{2}{3}\{a,b\} + \frac{1}{3}\{c,d\}$ has the marginals above but draws the violating committee $\{c,d\}$ with probability $1/3$. The pair's money pays for $4/3$ seats in expectation, yet this lottery gives the pair two seats in one draw and none in the other. The realization itself is not at fault; Example 9 implements it by IPSC committees.

Idea of the rounding. Examples 6 and 7 identify the two jobs of the rounding. It may have to change the realization, which is free ex ante as long as payments move only within ties, and it must correlate the draw with who paid for what. Read a reservation realization as payments: each voter pays k/n, and each candidate receives at most one. A seat thus costs one unit, and a coalition's Hare entitlement $k|S|/n$ is exactly the money it holds. Fully funded candidates form a set F and belong to every committee. Outside F, payment order puts each voter's payments in her top remaining indifference class. We call this class her *residual top class*, the part of her k/n that she pays outside F her *residual budget*, and the voters whose residual top class contains a candidate c the *residual top supporters* of c. First fill any candidate whose residual top supporters jointly hold at least one unit, moving their payments within ties. Repeat until no such candidate remains. A candidate outside F that lies in no residual top class has no residual top supporters and is never filled; by payment order it receives no payment at all, and no block will pay it either. This *saturation* step makes every later redistribution within residual top classes feasible: even all of a candidate's top supporters together can pay it less than one. Saturation plays the role that electing a candidate who reaches the quota plays in rules such as EAR: a candidate wanted at the top level by a full unit of budget is funded outright, before any money is committed elsewhere. Because the transfers stay inside ties, this costs nothing ex ante (Lemma 4). Without it, more than one unit of block money could later be placed on a single candidate, which the matching step cannot accommodate.

Now keep F fixed and pack the remaining budgets into *payment blocks* of mass one. Choose a smallest *eligible* set T, whose generalized solid supporters still hold at least one unit of available budget. Take one unit from these voters and place each contribution in its voter's residual top class within T. Solid support ensures that this intersection is nonempty. Record the block and continue until all budgets are assigned. A matching decomposition then gives each block one seat, filled by a candidate it paid for, with different blocks receiving different candidates. This is where the correlation comes from: every drawn committee gives every block a seat, so voters who jointly fund a block are represented in every draw, not only on average. A block is a joint purchase: voters who agree on a set pool exactly the price of one seat, and the seat must go to the set they agree on. The point of a block is the following. Its seat is bought with one unit of money contributed only by voters who all generalized solidly support T, and it goes to a member of T. Since $T \subseteq A_i^{|T|}$ for every supporter i, the seat lies in $A_i^{|T'|}$ for every contributor i and every $|T'| \geq |T|$, and hence in the periphery of every pair (S, T') in which S contains a contributor and $|T'| \geq |T|$. A smallest label T makes the seat count for as many coalitions as possible.

For IPSC, an underrepresented coalition cannot run out of money: each fixed winner or earlier block with a winner in its periphery accounts for at most one unit of its payments. While it retains

a full unit, smallest-set-first selection ensures that any block using its money pays only candidates inside that periphery. Put differently, as long as a coalition still holds a full unit, its own set T is on the menu of eligible sets, so every block that spends its money carries a label no larger than T and buys a seat inside its periphery. Its money can leak out of the periphery only after it has already bought the seats it is entitled to. Saturation is done before grouping, so blocks never need to be repaired or rebuilt. Under strict rankings, no payment moves.

The two guarantees rest on different parts of the construction. The ex-ante guarantees depend only on where money ends up: any placement of each residual budget within the voter's residual top class is admissible (Lemma 5), so the block payments in step 3(c) below may be chosen freely. The ex-post guarantee depends only on the labels T_t and on which voters fund each block: whatever the placement, the matching gives each block a candidate it paid for, hence a member of $T_t \setminus F$.

Bookkeeping. The algorithm uses the following data.

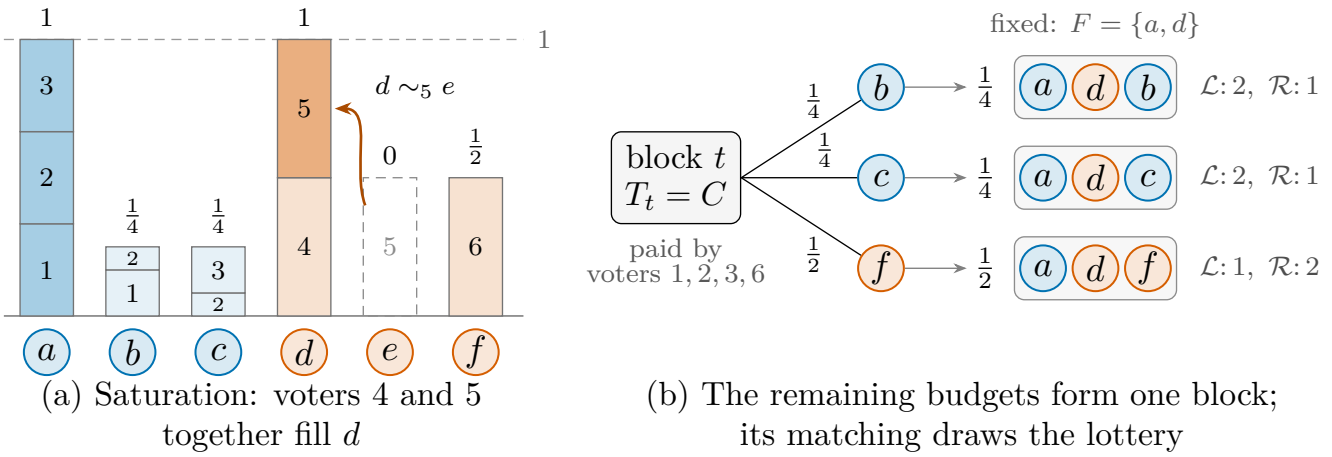
- *Payments* $y_{ic} \geq 0$, initially obtained from a reservation realization, with $\sum_c y_{ic} = k/n$ and $p_c = \sum_i y_{ic} \leq 1$. Let $F = \{c : p_c = 1\}$ and $B = C \setminus F$. Payments to F are fixed and never enter a block.
- *Residual budgets* $b_i = \sum_{c \in B} y_{ic}$, with $I = \{i : b_i > 0\}$. For $i \in I$, the *residual top class* L_i is her top class within B, that is, $L_i = A_i^{1,B} = E_i^s \cap B$ for the first class E_i^s that meets B. These data are recomputed during saturation and then kept fixed. The remaining number of seats is $k' = k - |F| = \sum_{i \in I} b_i$.
- *Available budgets* a_i, initially b_i, are the amounts not yet assigned to blocks. For $T \subseteq C$, let $N(T)$ be all voters who generalized solidly support T in the original rankings. The set T is eligible if $T \not\subseteq F$ and $\sum_{i \in N(T) \cap I} a_i \geq 1$, that is, if its supporters can still jointly buy a seat outside F.
- A *payment block* t consists of a set T_t and contributions $u_{ic}^t \geq 0$ satisfying

$$\sum_{i,c} u_{ic}^t = 1, \qquad u_{ic}^t > 0 \implies i \in N(T_t) \cap I, \quad c \in L_i \cap T_t.$$

  Voter i's contribution to it is $u_i^t = \sum_c u_{ic}^t$, and its payment to candidate c is $\sum_i u_{ic}^t$. A voter may contribute to several blocks, but her total contribution cannot exceed b_i.

Algorithm 2 (Payment-transfer rounding).
Input: weak rankings $(\succsim_i)_{i\in N}$ and a committee size $k \le m$.
Output: a rational lottery over size-k committees.

1. **Initialize.** Run Algorithm 1 and compute rational payments y realizing x^{CE} by maximum flow. Compute p, F, B, I, b, L as above.
2. **Saturate.** While some $c \in B$ satisfies $\sum_{i\in I: c\in L_i} b_i \ge 1$, move exactly $1 - p_c$ to c from these voters' payments to other members of their L_i, in a fixed order and splitting the last payment if needed. Update y, p, add c to F, and recompute B, I, b, L. Then set $k' = k - |F|$ and $a_i = b_i$. If $k' = 0$, return F with probability one.
3. **Form blocks.** For $t = 1, \dots, k'$:
   (a) Choose an eligible set T_t of minimum cardinality.
   (b) Take one unit of available budget from $N(T_t) \cap I$, splitting the last contribution if needed. Let $u_i^t \le a_i$ be voter i's contribution, so $\sum_i u_i^t = 1$.
   (c) For each contributor, choose rational block payments u_{ic}^t within $L_i \cap T_t$ with $\sum_c u_{ic}^t = u_i^t$. Set $a_i \leftarrow a_i - u_i^t$.
4. **Round.** Form the bipartite graph between the k' blocks and B in which block t and candidate c are joined with weight $\sum_i u_{ic}^t$, the block's payment to c. Decompose these weights into rationally weighted matchings that cover every block and use only positive edges. Return each matching's candidates together with F, with the probability assigned to that matching.



(a) Saturation: voters 4 and 5 together fill d

(b) The remaining budgets form one block; its matching draws the lottery

Figure 5: Algorithm 2 on the running example. (a) Voters 4 and 5 have d in their residual top classes and jointly hold one unit. Moving voter 5's payment from e to d fixes d before any block is formed. (b) The remaining budgets of voters 1, 2, 3 and 6 total one unit and form a single block with $T_t = C$. Matching it to b, c or f yields the three committees, with the seats each bloc wins.

Example 8 (Running example: payment-transfer rounding). We run Algorithm 2 on Example 1, starting from the realization p^0 of Example 5 (Figure 5).

Saturation. Initially $F = \{a\}$. Voters 1–3 have residual budget 1/6 each, and voters 4–6 have

1/2 each. Their residual top classes are

$$L_1 = \{b\}, \quad L_2 = \{b, c\}, \quad L_3 = \{c\},$$
$$L_4 = \{d\}, \quad L_5 = \{d, e\}, \quad L_6 = \{e, f\}.$$

Both d and e have one unit of residual top-supporting budget. Choose d in step 2 and move voter 5's 1/2 from e to d. This fills d and gives $F = \{a, d\}$. The remaining budgets are $b_1 = b_2 = b_3 = 1/6$ and $b_6 = 1/2$, with the same residual top classes. The top-support sums are now 1/3 for each of b, c and 1/2 for each of e, f, so saturation is complete. Choosing e instead gives an analogous run (Appendix C.3).

Block formation. The residual budgets total one, so an eligible set must be supported by all four remaining voters, in particular by voters 1 and 6. Such a set contains a top choice of each of them, which lies in the other voter's bottom class, so by generalized solidity it is all of C. Thus $T_t = C$. Retain the payments on b, c, f, including voter 2's split between b and c. The single block pays 1/4 to each of b and c and 1/2 to f: the halves left over by the two blocs are pooled into one seat.

Matching. Matching the block to b, c or f returns

$$\tfrac{1}{4}\{a, b, d\} + \tfrac{1}{4}\{a, c, d\} + \tfrac{1}{2}\{a, d, f\}.$$

Each committee contains a and d, so it satisfies IPSC by Example 1. Each bloc wins two seats with probability 1/2: the coin is tossed only for the third seat. Since the transfer stays inside voter 5's top tie, the payments still realize the same reservations (Lemma 4), so by Theorem 2 the final marginals $\left(1, \frac{1}{4}, \frac{1}{4}, 1, 0, \frac{1}{2}\right)$ still satisfy cutoff-GFS and fractional Rank-PJR+. Voter 6's singleton fractional Rank-PJR+ demand $p(\{e, f\}) \geq 1/2$ is now tight. Her Rank-IFS and cutoff-GFS bound on this cutoff is 1/3 (Table C.4). Saturation already settles the IPSC demands in this example, and matching allocates the remaining seat. This run changes the marginals, although p^0 itself also admits an IPSC implementation (Appendix C.3).

In the running example the only block has label C, so the labels play no role there. They do the work in the next example.

Example 9 (Block labels at work)**.** Run Algorithm 2 on Example 7, starting from the realization $(p_a, p_b, p_c, p_d) = (2/3, 2/3, 1/3, 1/3)$. Nothing is fully funded and every residual top-support sum is 2/3, so $F = \emptyset$ and saturation changes nothing. Every set of at most two candidates other than $\{a, b\}$ is generalized solidly supported by at most one voter, who holds only 2/3, whereas $\{a, b\}$ is supported by voters 1 and 2, who hold 4/3. Hence $T_1 = \{a, b\}$. Block 1 takes 2/3 from voter 1, placed on a, and 1/3 from voter 2, placed on b. The remaining 1/3 of voter 2 and 2/3 of voter 3 can only be pooled under a set both of them support, which is C, so $T_2 = C$; voter 2 places her 1/3 on b and voter 3 places 1/3 on each of c and d. Block 1 pays 2/3 to a and 1/3 to b, and block 2 pays 1/3 to each of b, c, d. One decomposition into matchings gives

$$\tfrac{1}{3}\{a, b\} + \tfrac{1}{3}\{a, c\} + \tfrac{1}{3}\{b, d\}.$$

This lottery has the same marginals as the one in Example 7, but it never draws $\{c, d\}$: block 1 is funded only by voters 1 and 2 and labelled $\{a, b\}$, so its seat always goes to a or b. The pair's leftover 1/3 is pooled with voter 3's budget under the label C. As in the running example, only this rounding remainder is left to chance.

The next two lemmas justify saturation and the freedom to allocate block contributions within residual top classes. The same preprocessing will also be used in Section 4.3.

Lemma 4 (Payment order and reservation preservation)**.** *Payments y realizing the final reservations satisfy*

$$p_c < 1, \quad y_{id} > 0 \quad \Longrightarrow \quad d \succsim_i c. \tag{5}$$

Transfers within indifference classes preserve the original reservations whenever candidate capacities are respected.

Proof. If $p_c < 1$ and $c \in E_i^s$, Lemma 3 puts all of voter i's reservations on her first s classes, so by (2) she spends her whole budget inside $E_i^{\leq s}$. Every positive payment therefore goes to a candidate weakly preferred to c, proving (5). A transfer from d to $c \sim_i d$ stays inside one class of voter i, so it changes none of her class totals $y_i(E_i^j)$. If candidate capacities are respected, the payments therefore still realize the original reservations. □

Lemma 5 (Residual flexibility)**.** *The saturation step runs in polynomial time and yields*

$$k' := \sum_{i \in I} b_i = k - |F|, \qquad \sum_{i \in I:\, c \in L_i} b_i < 1 \quad (c \in B). \tag{6}$$

Keeping the payments to F, every redistribution of each b_i within L_i realizes the original reservations and hence satisfies both fractional guarantees.

Indeed, saturation fills one more candidate at each step, and at termination all of a remaining candidate's top supporters together hold less than one unit. This bounds its total payment under any redistribution. The transfer details and proof are in Appendix B.1.

The steps are well defined and terminate. After saturation, F, B, I, b, L remain fixed. Before block t, the total available budget is $k' - t + 1 \geq 1$. Since every voter supports C, an eligible set exists. If i contributes to a chosen set T_t, then $L_i \cap T_t \neq \emptyset$: otherwise she would strictly prefer every member of L_i to every member of $T_t \setminus F$, contradicting solid support. Thus step 3(c) can allocate every contribution. After exactly k' blocks, all budgets are assigned, and Lemma 5 ensures that each remaining candidate's total payment is below one. These bounds do not make step 3(a) polynomial: finding a smallest eligible set is NP-hard (Proposition 2).

We can now prove the main result.

Proof of Theorem 1. Apply Algorithm 2. By Lemma 5, the final payments realize x^{CE} and satisfy both ex-ante guarantees. If $k' = 0$, these payments are the incidence vector of F. Otherwise, the $k' = k - |F|$ blocks hold all residual budgets, with row sums one and column sums

$$\sum_t \sum_i u_{ic}^t \leq \sum_{i \in I : c \in L_i} b_i < 1 \qquad (c \in B).$$

Thus their weights give a fractional bipartite matching saturating every block. By integrality of the bipartite matching polytope, these weights decompose into a rational convex combination of matchings that saturate every block, so the resulting lottery implements the final marginals and every committee has size k.

Every supported committee satisfies IPSC. We show that an underrepresented coalition would still have a full unit available when the first block using its payments receives a winner outside its periphery. The count behind this is simple: each winner in the periphery can absorb at most one unit of the coalition's $k|S|/n \geq \ell$ units, and an underrepresented coalition sees fewer than ℓ

winners there. Fix a committee W in the support of the lottery and suppose a generalized solid coalition S for T violates IPSC with entitlement ℓ. Put

$$D = D(S,T), \qquad q = \frac{k|S|}{n}, \qquad s = |W \cap D|.$$

Then $T \not\subseteq W$ and $s < \ell \leq q$. Pick $c \in T \setminus W$. Since $F \subseteq W$, we have $c \in B$ and $p_c < 1$. Payment order puts every payment of S in D.

Associate each block with its winner in the matching defining W. Each fixed candidate and each block contains at most one unit of S's payments. Some block using these payments must therefore have its winner outside D; otherwise $q \leq s$. Let t be the first such block. Before block t is formed, the fixed candidates and earlier blocks receiving S's payments have distinct associated winners, all in D. They account for at most s units, so $\sum_{i \in S \cap I} a_i \geq q - s \geq \ell - s \geq 1$. Hence T is eligible and minimum-cardinality selection gives $|T_t| \leq |T|$. Choose $i \in S$ with $u_i^t > 0$. Then $i \in N(T_t)$, so solidity gives

$$T_t \subseteq A_i^{|T_t|} \subseteq A_i^{|T|} \subseteq D.$$

The block's matched winner lies in $T_t \setminus F \subseteq D$, a contradiction. □

The construction selects an output of Class Eating that admits an IPSC implementation. It may change the initial realization and does not implement every vector satisfying the fractional axioms, or even every output of Class Eating (Example 6). Finding a smallest eligible generalized solid set is NP-hard, even at the first search for an eligible set (Proposition 2). The construction in Section 4.3 bypasses this search and runs in polynomial time on unrestricted weak rankings. Under strict rankings, Class Eating is PFR and each residual top class is a singleton, so no payment moves. The construction therefore implements the PFR marginals by PSC committees, as does the construction of Peters [4]. Theorem 1 also does not follow from the approval constructions [2, 3]. They round with respect to one approved set per voter, whereas the periphery $D(S,T)$ protected by IPSC changes with $|T|$, so one voter's payments must serve coalitions at several depths.

A fractional version of EAR does not suffice: it can violate cutoff-GFS and return marginals with no IPSC implementation (Appendix A.7).

4.3. Polynomial-time construction on weak rankings

We first presented a direct construction because its payment blocks and matching decomposition give a simpler proof of compatibility. Its smallest-eligible-set search is NP-hard, however, so a different approach is needed for polynomial-time computation. We now give such a construction. It shares the Class Eating payments, saturation step and underlying budget argument, but replaces payment blocks and matching by a weighted committee oracle and convex decomposition. The two constructions proceed in opposite orders. Algorithm 2 first decides where every unit of budget is spent, through the blocks, and then draws committees that implement this spending; its hard step is choosing the block labels so that every drawn committee is representative. Algorithm 3 first generates representative committees, one candidate at a time and without ever choosing a solid set, and only then chooses probabilities for them so that the resulting marginals correspond to some admissible spending. Presenting both constructions separates the simpler compatibility argument from the additional machinery used to establish efficient computation.

Theorem 3 (Polynomial-time fair lotteries)**.** *For every weak-ranking profile and every committee size k, one can compute in polynomial time a lottery satisfying cutoff-GFS and fractional Rank-PJR+ ex ante and IPSC ex post. The lottery lists at most $m+1$ committees with rational probabilities of polynomial encoding length.*

The construction has three layers. Preprocessing, shared with Algorithm 2, fixes the fully funded candidates and leaves each voter's residual budget on her residual top class. An inner layer, the weighted committee oracle (Algorithm 4), builds for any candidate prices one committee that is representative by construction and not too cheap at those prices. An outer layer (Lemma 9) chooses the prices, collects the oracle's committees and mixes them so that the average has admissible marginals. The inner layer is thus responsible for ex-post fairness and the outer layer for ex-ante fairness; prices are the only channel between them. The algorithm chooses the marginals together with the lottery; it does not implement a prescribed output of Class Eating. Its marginals are nevertheless an output of Class Eating. As in Algorithm 2, they arise from the saturated payments by moving each voter's residual budget only within her residual top class (Lemma 5), so no voter's probability moves between her indifference classes. Under strict rankings every residual top class is a single candidate, so the lottery implements exactly the PFR marginals.

Algorithm 3 (Polynomial-time rounding)**.**
Input: weak rankings and a committee size k.
Output: a rational lottery over size-k committees.

1. Perform steps 1–2 of Algorithm 2 to obtain F, B, I, b, L. Let P be the polytope in (7). If $k' = k - |F| = 0$, return F with probability one.
2. Use the weighted committee oracle (Algorithm 4) in the convex-intersection procedure of Lemma 9. Obtain residual committees $W_1, \dots, W_q$ and probabilities $\lambda_1, \dots, \lambda_q$ whose average incidence vector lies in P.
3. Return $F \cup W_j$ with probability λ_j for each j.

Residual marginals. Use the fixed residual data from Section 4.2. By Lemma 5, every vector in

$$P = \left\{ \left(\sum_{i \in I} z_{ic} \right)_{c \in B} : z \geq 0, \quad z_{ic} = 0 \ (c \notin L_i), \quad \sum_{c \in L_i} z_{ic} = b_i \ (i \in I) \right\} \tag{7}$$

extends, by setting $p_c = 1$ on F, to marginals satisfying both fractional guarantees. In words, P is the set of residual marginals obtainable by letting each voter spread her residual budget b_i over her residual top class L_i in any way; it is the freedom that Lemma 5 provides. A lottery is therefore ex-ante fair as soon as its residual marginals lie in P. The oracle below does not produce points of P: it produces committees, and it is the lottery over them that must land in P. If $k' = 0$, the output is F. Otherwise, all ranks and cutoffs used by the residual oracle refer to the rankings restricted to B. A *residual generalized solid coalition* is a nonempty set $S \subseteq I$ that generalized solidly supports a nonempty set $T \subseteq B$ in these restricted rankings. We write $A_i^{r,B}$ for these cutoffs and $D_B(S,T) = \bigcup_{i \in S} A_i^{|T|,B}$ for the corresponding periphery. These rankings and the classes L_i remain fixed during the oracle, even after candidates are selected.

The weighted committee oracle. The main new ingredient is the following lemma. It requires only a lower bound on the committee's weight, not exact maximization over all representative committees.

Lemma 6 (Weighted committee oracle)**.** *For residual data satisfying (6) with $k' > 0$ and any signed rational candidate weights w, one can compute in polynomial time a committee $W \subseteq B$ of size k' such that*

$$|W \cap D_B(S,T)| \geq \lfloor b(S) \rfloor \quad \text{for every residual generalized solid coalition } S \text{ for } T, \tag{8}$$

$$w(W) \geq \sum_{i \in I} b_i \min_{d \in L_i} w_d = \min_{p \in P} w \cdot p, \tag{9}$$

where $b(S) = \sum_{i \in S} b_i$ and $w(W) = \sum_{c \in W} w_c$.

Condition (8) says that W is representative on the residual instance: every residual coalition receives, within its periphery, as many winners as its residual budget pays for. Condition (9) says that, at prices w, W is worth at least the cheapest admissible spending. Representation does not depend on w; the prices only steer which representative committee is produced, so that the convex-intersection step can combine the outputs into a lottery with marginals in P. What makes this tractable is that P is described by the same budgets that the oracle spends. At prices w, the cheapest point of P values each unit of voter i's budget at the cheapest member of L_i, and the oracle below matches this unit by unit. Under strict rankings, P is a single point, the PFR marginals, and the comparison still applies. Under ties, the freedom within L_i is what allows the comparison with the cheapest member, rather than with wherever a fixed realization happens to place the budget. For an arbitrary prescribed vector no such comparison is available, and exact implementation by IPSC committees is Σ_2^{P}-complete (Theorem 7).

The oracle is a weighted, budget-based variant of EAR. It scans ranks in increasing order and selects a candidate once a full unit of budget supports it. At each rank, it repeatedly deletes candidates with insufficient support, counting only voters whose fixed top class meets the surviving set. If candidates survive, it selects one of maximum weight and charges one unit to its supporters, first exhausting every voter whose top class contains it. It then continues deleting at the same rank, advancing only when no candidate survives. We explain the role of each rule below; the pseudocode and proof then make the argument precise.

Oracle charges are only accounting: a voter may be charged for a candidate below her top class. Top-class feasibility is required of the lottery's marginals, not of each oracle committee.

The role of each rule. Each rule serves one of the two conditions of Lemma 6. The *activity* rule serves the weight bound. A voter is counted, and charged, only while her top class still meets the surviving set, so the selected candidate of maximum weight is worth at least some member of her top class, and hence at least its cheapest member. Every unit of money therefore buys a candidate worth at least what the same unit buys at the cheapest point of P. Activity also makes deletions cascade: deleting a candidate can deactivate a voter and lower the support of others, which is why the surviving set, which we call the *rank-r core*, is computed by repeated peeling. The term is borrowed from the k-core of a graph, the largest subgraph in which every vertex has degree at least k; it is unrelated to the core as a notion of coalitional stability. The *maximum-weight* choice lets the prices steer the oracle towards different representative committees. The *exhaustion* rule and the *smallest-rank* rule serve representation. Exhaustion is affordable because, after saturation, the voters whose top class contains a given candidate hold less than one unit together. It ensures that

a voter who still has money has lost no member of her top class. Since her top class meets every set she generalized solidly supports, each such set keeps an unselected member as long as she has money left.

From sets to ranks. The smallest-rank rule plays the role of the smallest-eligible-set rule of Algorithm 2. There, a coalition S for T that still holds a unit keeps T eligible, so every block that uses its money has a label of size at most $|T|$ and buys a seat inside $D(S,T)$. Here, the same coalition keeps the rank-$|T|$ core nonempty. Each of its members with remaining budget still has her whole top class unselected, this class meets T, and every unselected member of T lies in every member's rank-$|T|$ cutoff; so S alone gives each unselected member of T support at least one. Selection therefore happens at some rank $r \le |T|$, and a charged member of S receives a candidate in her rank-r cutoff, hence in $D(S,T)$. Because $T \subseteq A_i^{|T|}$ for every supporter, the rank $|T|$ can stand in for the size of T. There are exponentially many candidate sets but only m ranks to scan, and this is the source of the polynomial running time.

Throughout the oracle, B, I, b, L and the restricted cutoffs $A_i^{r,B}$ remain fixed. Let $U \subseteq B$ be the unselected candidates and a_i the available budgets, initially $U = B$ and $a_i = b_i$. For a candidate set $Q \subseteq U$, voter i is active if $L_i \cap Q \neq \emptyset$. Candidate c's support at rank r in Q is

$$\sum_{i \in I:\, L_i \cap Q \neq \emptyset,\ c \in A_i^{r,B}} a_i.$$

Starting from $Q = U$, repeatedly delete a candidate with support below one. Write Q_r for the surviving set, called the rank-r core. Deletion can only deactivate voters and reduce support, so this is the largest subset of U in which every candidate has support at least one. It is independent of deletion order and is computable in polynomial time.

As budgets decrease and U shrinks, every rank-r core can only shrink. A previously deleted candidate therefore cannot return at that rank, and an empty core stays empty. We can scan ranks once, retaining the surviving set after each selection and advancing only when it becomes empty. This always selects from the smallest rank with a nonempty core.

Algorithm 4 (Weighted committee oracle)**.**
Input: residual data satisfying (6), $k' > 0$, and signed rational weights w_c for $c \in B$.
Output: a committee $W \subseteq B$ of size k'.

1. Initialize $W = \emptyset$, $U = Q = B$, $r = 1$, and $a_i = b_i$ for $i \in I$.
2. Repeatedly delete from Q any candidate whose rank-r support in Q is below one. If $Q = \emptyset$, increase r by one, set $Q = U$, and repeat this step.
3. Choose $c \in Q$ of maximum weight w_c, and put $N_c = \{i \in I : L_i \cap Q \neq \emptyset,\ c \in A_i^{r,B}\}$. Charge exactly one unit to N_c, first taking the entire available budget of every voter with $c \in L_i$, then taking from the other members of N_c in a fixed order, splitting the last budget if needed. Write u_i for the charge to i, with $u_i = 0$ outside N_c.
4. Set $a_i \leftarrow a_i - u_i$, add c to W, and remove c from both U and Q. If $|W| = k'$, return W. Otherwise return to step 2 at the same rank r.

Proof of Lemma 6. We first establish feasible charges and the invariant that keeps unselected solid sets available. We then prove completion, the weight bound, and representation.

Feasible charges and the invariant. For a selected candidate $c \in Q_r$, the support condition gives $\sum_{i \in N_c} a_i \geq 1$. Every voter with $c \in L_i$ belongs to N_c, and their mandatory charges total at most

$$\sum_{i \in I : c \in L_i} a_i \leq \sum_{i \in I : c \in L_i} b_i < 1.$$

Thus the remaining charge can be taken from the other members of N_c. Each selection charges exactly one unit and exhausts every voter whose fixed top class contains the selected candidate. Consequently, the algorithm maintains

$$a_i > 0 \quad \Longrightarrow \quad L_i \subseteq U. \tag{10}$$

This holds initially because $U = B$. Deleting candidates from Q does not change U. When c is selected and removed from U, every voter with $c \in L_i$ is exhausted, while all other positive-budget voters retain their entire top class in U.

Completion. Before each round, the total available budget is $k' - |W| \geq 1$. At rank $|B|$, every positive-budget voter is active for U by (10) and approves all candidates, so $Q_{|B|} = U \neq \emptyset$. Indeed, $k' = \sum_{i \in I} b_i \leq \sum_{c \in B} \sum_{i \in I : c \in L_i} b_i < |B|$, so U cannot be empty before k' selections. Each round selects a new candidate and charges exactly one unit. After k' rounds every budget is exhausted.

Weight. Every charged voter i has some $d \in L_i \cap Q_r$. Since c has maximum weight in Q_r, $w_c \geq w_d \geq \min_{e \in L_i} w_e$. The charges of the round sum to one, so

$$w_c \geq \sum_{i \in I} u_i \min_{d \in L_i} w_d.$$

Summing over rounds gives the inequality in (9), since voter i is charged b_i in total. The equality follows from (7): each voter independently places her budget on a minimum-weight member of L_i. No sign restriction on w is used.

Representation. For every residual generalized solid coalition S for T, each supporter has $L_i \cap T \neq \emptyset$. Otherwise she would strictly prefer every member of L_i to every member of T, contradicting solid support. In particular,

$$b(S) \leq \sum_{c \in T} \sum_{i \in I : c \in L_i} b_i < |T|.$$

Thus including all of T would already imply (8); no inclusion alternative is needed.

Suppose instead that the final committee has $s = |W \cap D_B(S,T)| < \lfloor b(S) \rfloor$. The s selections in the periphery charge S at most s units. Since the algorithm exhausts $b(S)$, there is a first round that charges S for a selection outside its periphery. Just before it, $a(S) := \sum_{i \in S} a_i \geq b(S) - s \geq 1$. Let $T_0 = T \cap U$. By (10), every positive-budget member of S has her whole top class in U. That class meets T, so it meets T_0. In particular, $T_0 \neq \emptyset$.

Consider peeling at rank $|T|$. As long as T_0 survives, every positive-budget member of S is active and approves all of T_0. Every candidate in T_0 therefore has support at least $a(S) \geq 1$. No first deletion from T_0 is possible, so $Q_{|T|} \neq \emptyset$. Since the algorithm always selects at the smallest rank with a nonempty core, the selected rank satisfies $r \leq |T|$. A charged member i of S then has the selected candidate in $A_i^{r,B} \subseteq A_i^{|T|,B} \subseteq D_B(S,T)$, a contradiction.

Running time. There are $k' < |B|$ selections. The rank increases at most $|B| - 1$ times, and each candidate is deleted or selected at most once per rank. Thus there are at most $|B|^2$ removals.

Recomputing all supports after each removal or charge takes polynomial time. Supports are sums of rational budgets. Every charge is a current budget or one minus a sum of current budgets, so all available budgets remain multiples of the common denominator of the b_i. Thus all arithmetic and comparisons take polynomial time. □

The oracle on the running example. After the saturation of Example 8, $F = \{a, d\}$, $B = \{b, c, e, f\}$ and $k' = 1$, with residual budgets $b_1 = b_2 = b_3 = 1/6$ and $b_6 = 1/2$ and residual top classes $L_1 = \{b\}$, $L_2 = \{b, c\}$, $L_3 = \{c\}$ and $L_6 = \{e, f\}$. At rank 1, candidate b has support $b_1 + b_2 = 1/3$ and is deleted, which deactivates voter 1; then c, e and f are deleted in turn, so the rank-1 core is empty. At rank 2 the restricted cutoffs are $\{b, c\}$ for voters 1–3 and $\{e, f\}$ for voter 6, so every candidate has support $1/2$ and the core is again empty. At rank 3 every remaining voter approves every remaining candidate, each candidate has support one, and the core is all of B. The oracle selects the candidate of maximum weight and charges all four voters in full. With prices favouring b, c, e or f it thus returns $\{b\}$, $\{c\}$, $\{e\}$ or $\{f\}$. Here P consists of the vectors with $p_b + p_c = 1/2$, $p_b, p_c \in [1/6, 1/3]$ and $p_e + p_f = 1/2$. The convex-intersection step returns probabilities whose average lies in P, for instance $\frac{1}{4}\{b\} + \frac{1}{4}\{c\} + \frac{1}{2}\{f\}$, which after adding F is the lottery of Example 8.

The oracle on Example 7. Here $k' = 2$ and every rule of the oracle is exercised. Saturation changes nothing, so $F = \emptyset$, $B = C$, every residual budget is $2/3$, and $L_1 = \{a\}$, $L_2 = \{b\}$, $L_3 = \{c, d\}$. At rank 1 each candidate is supported by a single voter, so the core is empty. At rank 2 the cutoffs are $\{a, b\}$, $\{a, b\}$ and $\{c, d\}$; candidates c and d have support $2/3$ and are deleted, while a and b have support $4/3$, so the core is $\{a, b\}$.

Suppose the prices favour a. Voter 1 is exhausted, since $a \in L_1$, and voter 2 pays the remaining $1/3$. The rank-2 core is now empty. At rank 3, with cutoffs $\{a, b, c\}$, $\{a, b, c\}$ and $\{a, c, d\}$, candidate c initially has support $1/3 + 2/3 = 1$, but voter 2 is active only through b, which voter 2 alone supports. Once b is deleted, voter 2 becomes inactive and c loses voter 2's support: a voter's money may be counted only while some member of her top class is still in contention. As d is supported by voter 3 alone, the rank-3 core is empty. At rank 4, each of b, c, d has support one, and the oracle returns $\{a, b\}$, $\{a, c\}$ or $\{a, d\}$.

Suppose instead the prices favour b. Voter 2 is exhausted, and voter 1 pays $1/3$ and keeps $1/3$; her top class $\{a\}$ is still unselected, as (10) requires. At rank 3, candidates a and c are each supported by voters 1 and 3, with $1/3 + 2/3 = 1$, whereas d is supported by voter 3 alone. The core is $\{a, c\}$, and the oracle returns $\{a, b\}$ or $\{b, c\}$.

In either case the first seat goes to a or b, and the oracle never returns $\{c, d\}$. Here $P = \{(2/3, 2/3, s, 2/3 - s) : 0 \le s \le 2/3\}$, and, for instance, $\frac{1}{3}\{a, b\} + \frac{1}{3}\{a, d\} + \frac{1}{3}\{b, c\}$ is a lottery over oracle committees with marginals $(2/3, 2/3, 1/3, 1/3) \in P$. No single committee would do. The committee $\{a, b\}$ is representative, but its marginals $(1, 1, 0, 0)$ lie outside P, since voter 3's budget buys nothing. Prices detect this: at $w = (-1, -1, 0, 0)$, the committee $\{a, b\}$ costs $-2 < -4/3 = \min_{p \in P} w \cdot p$, whereas the oracle returns a committee such as $\{a, c\}$, which costs -1. Prices that penalize a and b steer the oracle towards voter 3's candidates, while representation keeps a or b in every output.

Remark 1 (Each rule of Algorithm 4 is needed)**.** Example 7 also shows what fails if a rule is dropped. *Activity:* at prices $w = (10, 9, 0, 0)$, where $\min_{p \in P} w \cdot p = 38/3$, the oracle selects a at rank 2, charging $2/3$ to voter 1 and $1/3$ to voter 2. Without activity, c would reach support $1/3 + 2/3 = 1$ at rank 3, and the output $\{a, c\}$ would cost $10 < 38/3$: voter 2's last third would buy c, worth 0, although the cheapest point of P spends it on b, worth 9. With activity, voter 2 stops counting once b is deleted, and the oracle returns $\{a, b\}$. *Exhaustion:* if the unit for a were

charged 2/3 to voter 2 and 1/3 to voter 1, voter 1 would keep 1/3 but could never be active again, and no candidate would ever reach support one. *Smallest rank:* selecting at rank 4 with prices favouring c and d would return $\{c, d\}$, which violates IPSC.

From weighted committees to a lottery. Read w as prices. A point lies in P exactly when it passes every *price test*: at every price vector w, it costs at least $\min_{p \in P} w \cdot p$, the cost of the cheapest admissible spending. If no lottery over representative committees had marginals in P, the convex hull of these committees and P would be disjoint polytopes, and a separating hyperplane would give prices at which every representative committee is strictly cheaper than every point of P. The weight bound (9) rules this out, so a suitable lottery exists. The condition is also necessary: if some lottery over representative committees has marginals in P, then at every price vector some committee in its support costs at least $\min_{p \in P} w \cdot p$. The weight bound thus asks for exactly what is needed. In particular, the oracle never has to find a best representative committee, only one that is no cheaper than the cheapest point of P, and it never has to test whether a given committee is representative, which for IPSC is coNP-complete: its committees are representative by construction.

The same argument is also the algorithm, following the "ellipsoid against hope" pattern of Papadimitriou and Roughgarden [23]. Call w a *bad price* if $w \cdot (p - v) \geq 1$ for all $p \in P$ and all representative committees v; no bad price exists. For a proposed w, the cheapest point p^w of P and the oracle's committee v^w give an inequality $w \cdot (p^w - v^w) \geq 1$ that every bad price satisfies but w violates. The oracle is thus a separation oracle, not for the convex hull of representative committees, which would require optimizing over them, but for the empty set of bad prices. The ellipsoid method certifies this emptiness after polynomially many queries because every such inequality has small encoding length whatever w was: v^w is a 0/1 vector, and p^w is a vertex of P built from the fixed budgets. This bounded size is what makes the method stop; it is the point that Jiang and Leyton-Brown [24] had to repair in the correlated-equilibrium application. Once the collected inequalities have no common solution, Farkas' lemma gives weights λ_j with $\sum_j \lambda_j v^j = \sum_j \lambda_j p^j$. Each refutation pairs a committee with an admissible spending, and mixing the pairs so that the two sides balance yields a lottery whose marginals lie in P. In game terms, the oracle answers every announced price vector, and the minimax theorem turns these answers into a single lottery. The approach is related to randomized metarounding [25, 26]; Appendix B.2 proves the precise lemma. We now combine it with the weighted committee guarantee and verify IPSC after adding F.

Proof of Theorem 3. Apply Algorithm 3. If $k' > 0$, let V consist of the incidence vectors of residual committees satisfying (8). The weighted oracle supplies the condition of Lemma 9. Minimizing over P is easy: each voter puts all of b_i on a minimum-weight member of L_i. The resulting coordinates are sums of the fixed budgets, so their encoding length is polynomial independently of w. We obtain a lottery over at most $|B| + 1 \leq m + 1$ residual committees with average incidence vector in P. By Lemma 5, adding F gives both fractional guarantees and committees of size k. The same fractional guarantees hold when $k' = 0$ and the output is the single committee F.

For IPSC, the fixed candidates and the residual committee share the work: each fixed candidate in a coalition's periphery absorbs at most one unit of its money, and the oracle covers what is left. Fix a generalized solid coalition S for T in the original rankings and an integer $1 \leq \ell \leq k|S|/n$. Consider an output $F \cup W$, with $W = \emptyset$ when $k' = 0$. If $T \subseteq F$, IPSC holds. Otherwise choose $c \in T \setminus F$, write $D = D(S, T)$, and let $f = |F \cap D|$. In the final preprocessing payments, $p_c < 1$. Payment order puts all payments of S inside D, so its residual budget satisfies

$$b(S \cap I) \geq k|S|/n - f \geq \ell - f.$$

	Class Eating (Algorithm 1)	Payment-transfer rounding (Algorithm 2)	Polynomial-time rounding (Algorithms 3 and 4)
Decides	how much each voter spends on each of her classes	who pays for which candidate, then which committees are drawn	which committees are drawn, then their probabilities
Ex-ante source	equal eating speed and the forced-full rule	payments move only within ties (Lemma 5)	residual marginals lie in P (Lemma 5)
Ex-post source	—	blocks with smallest eligible labels	selection at the smallest rank with a nonempty core
Correlation	—	matching of blocks to seats	mixture of oracle committees
Running time	polynomial (maximum flow)	NP-hard label search	polynomial (ellipsoid method)

Table 1: The three constructions at a glance. Both roundings start from the same saturated Class Eating payments and preserve every voter's spending on each of her indifference classes.

If $f \geq \ell$, the fixed candidates suffice. Otherwise this inequality gives $b(S \cap I) \geq 1$, and hence $k' > 0$. In the restricted rankings, $S \cap I$ generalized solidly supports $T' = T \setminus F$. Its periphery $D' = D_B(S \cap I, T')$ is contained in $D \setminus F$: every member of D' is weakly above a member of $T' \subseteq T$ for some voter in $S \cap I \subseteq S$. Therefore (8) gives

$$|W \cap D'| \geq \lfloor b(S \cap I) \rfloor \geq \ell - f.$$

Together with the f fixed candidates, this gives at least ℓ winners in D. Thus every output committee satisfies IPSC.

The preprocessing, weighted oracle, and convex-intersection procedure all run in polynomial time and use rationals of polynomial encoding length. The bound is in the ordinary rational bit model; no strongly polynomial running time is asserted. □

Table 1 summarizes how the three constructions divide the work.

Corollary 1 (The strict-ranking case)**.** *For every strict-ranking profile and every committee size, one can compute in polynomial time a lottery satisfying cutoff-GFS and fractional Rank-PJR+ ex ante, and PSC (equivalently, IPSC) ex post.*

Proof. Apply Theorem 3, since IPSC and PSC coincide under strict rankings. □

4.4. The boundary of compatibility

The preceding constructions combine both fractional guarantees with IPSC ex post, first through a direct compatibility proof and then in polynomial time. We now show that requiring Rank-JR ex post can already rule out Rank-IFS ex ante, even under strict rankings.

Theorem 4 (Individual cutoff fairness conflicts with Rank-JR)**.** *For every $k \geq 2$, there is a strict-ranking profile on which no lottery satisfies Rank-IFS ex ante and Rank-JR ex post. This remains true when Rank-IFS is replaced by cutoff-GFS, fractional Rank-PJR+, fractional IPSC, or (on strict rankings) fractional PSC, and when Rank-JR is replaced by Rank-PJR+.*

Proof. Set $n = k + 2$. Let b be a candidate, and give each voter $i \in [n]$ a private block X_i of k candidates; the blocks are pairwise disjoint and do not contain b. Voter i ranks the candidates of X_i in any strict order in her first k positions, ranks b next, and ranks all remaining candidates after b in any strict order.

Let W be a Rank-JR committee. If $b \notin W$, then the k seats of W meet at most k of the $k + 2$ disjoint blocks X_i. Choose two untouched blocks X_i, X_j. At rank $k + 1$ their voters' cutoffs are $A_i^{k+1} = X_i \cup \{b\}$ and $A_j^{k+1} = X_j \cup \{b\}$. These have common candidate b and a union disjoint from W, although the pair is entitled to representation because $2k/(k + 2) \geq 1$. This contradicts Rank-JR. Thus any lottery supported on Rank-JR committees has $p_b = 1$.

For each voter i, the rank-k individual cutoff inequality required by Rank-IFS, and hence by cutoff-GFS, gives $p(X_i) \geq \min\{k, |X_i|\}/n = k/n$. Since the $k+2$ blocks are disjoint, $\sum_{i=1}^{k+2} p(X_i) \geq k$. Together with $p_b = 1$, this exceeds the total mass k.

For the remaining assertions, cutoff-GFS implies Rank-IFS by definition, and fractional IPSC (hence fractional Rank-PJR+ and, on strict rankings, fractional PSC) implies Rank-IFS by Lemma 1. Finally, Rank-PJR+ implies Rank-JR. □

Peters [4, Example 5] already establishes incompatibility between fractional PSC and ex-post Rank-PJR for two seats. Theorem 4 strengthens the obstruction to the weaker individual requirement Rank-IFS and the weaker ex-post requirement Rank-JR, for every $k \geq 2$. It leaves the strict-ranking PSC implementation in Corollary 1 intact, since PSC does not imply Rank-JR. In the running example, every committee drawn in Example 8 even satisfies Rank-PJR+ (Appendix C); the theorem shows that no lottery meeting Rank-IFS ex ante can guarantee this in general.

The private-block argument adapts the unit-cost cardinal IFS–JR obstruction of Aziz et al. [17, Theorem 6.2] to rank cutoffs and every committee size $k \geq 2$. The obstruction has no approval counterpart: on approval profiles, GFS ex ante is compatible even with EJR ex post [2]. Here the ex-ante demand is made at rank k and the ex-post demand at rank $k + 1$, whereas an approval ballot imposes both on the same set.

The one-seat case completes the boundary.

Proposition 1 (One-seat compatibility and implementation)**.** *For $k = 1$, every fractional Rank-PJR+ vector on a weak-ranking profile is implementable by Rank-PJR+ singletons. Choosing a voter uniformly and then one of her top candidates gives cutoff-GFS and fractional Rank-PJR+ ex ante and Rank-PJR+ ex post.*

Proof. Only the grand coalition can make a one-seat Rank-PJR+ demand. Let r_* be its first rank with a common cutoff candidate and put $U = \bigcup_{i \in N} A_i^{r_*}$. Fractional Rank-PJR+ forces $p(U) = 1$, whether a common candidate is fully included or not. Every singleton in U meets all integral demands: smaller ranks have no common candidate, and larger cutoff unions contain U. Drawing d with probability p_d therefore implements any such vector. In the stated construction, the winner is a top candidate of some voter and hence lies in U, so every drawn singleton satisfies Rank-PJR+. For any coalition S and chosen cutoffs, the drawn voter belongs to S with probability $|S|/n$, in which case the winner lies in her cutoff. Every tested union therefore receives mass at least $|S|/n$, which proves both fractional guarantees. □

5. Efficiency

Both fractional fairness guarantees can be combined with SD-efficiency by choosing a suitable reservation realization. A fractional committee p is *stochastic-dominance (SD) efficient* if no other

fractional committee q has $q(A_i^r) \geq p(A_i^r)$ for all i, r, with strict inequality for at least one pair. An integral committee is SD-efficient if its incidence vector is; that is, fractional committees are allowed as potential SD-improvements. On approval profiles, the committee version is fractional Pareto optimality: the comparison may use fractional committees, so it is stronger than ordinary integral Pareto optimality [27].

Theorem 5 (Fairness and efficiency)**.** *For every weak-ranking profile and $k \leq m$, Efficient Class Eating (Definition 9) computes in polynomial time marginals satisfying cutoff-GFS, fractional Rank-PJR+, and SD-efficiency. Every implementation of these marginals has SD-efficient support, and an implementation with polynomial support can be computed in polynomial time. Under strict rankings, the same marginals also admit a polynomial-time implementation by PSC (equivalently IPSC) committees.*

To choose the realization, treat each reservation x_{ij} as an auxiliary approval agent who approves the cutoff $E_i^{\leq j}$ and whose budget is the reserved amount. The efficient full-spending lemma (Lemma 8) provides a spending flow that still spends each reservation inside its class (Lemma 7), so it realizes the reservations, and whose marginals are Pareto efficient with respect to all cutoff totals, including those of cutoffs with zero reservation. Since the comparison is with *all* fractional committees, these marginals are SD-efficient. Both fairness guarantees hold because the flow realizes the original reservations (Theorem 2). Under strict rankings, the realization is PFR, which is SD-efficient and admits a PSC implementation. The formal rule and complete proof are in Appendix A.2.

Efficient support follows from a general replacement argument. If a positive-probability committee had a fractional SD-improvement, replacing its probability mass by an implementation of that improvement would SD-improve the original marginals. Thus every implementation of any SD-efficient vector has SD-efficient support.

The choice of realization matters. With one seat and two voters approving $\{a, b\}$ and $\{b, c\}$, each voter reserves $1/2$ for her top class. Realizing these reservations at a and c gives $(1/2, 0, 1/2)$, which is SD-dominated by selecting b with certainty. Assignment efficiency of the payment matrix does not exclude this outcome: each voter already pays only for her top class. Efficiency here concerns the collective committee marginals.

Remark 2 (SD-efficiency and ex-post IPSC impossibility)**.** On the approval instance of Jin et al. [6, Theorem 1], every JR committee has a fractional improvement that strictly increases every voter's approved-candidate total. Averaging these improvements over any JR-supported lottery strictly improves every approval cutoff, so on the corresponding two-tier ranking profile, no lottery over Rank-JR committees has SD-efficient marginals. By Lemma 2, the same holds for Rank-PJR+ and IPSC, including lotteries concentrated on a single committee. This obstruction is inherited directly from approval profiles.

6. Exact Implementation

Finding *some* fair object is easy for every axiom we consider. The Expanding Approvals Rule runs in polynomial time and returns a committee satisfying PSC under strict rankings [11] and Rank-PJR+, hence IPSC, under weak rankings [5]. Class Eating computes, in polynomial time, a vector satisfying cutoff-GFS and fractional Rank-PJR+, hence fractional IPSC (Theorem 2). Moreover, Theorem 3 computes a lottery combining both fractional guarantees with IPSC ex post.

The harder tasks are verifying a given object and implementing *prescribed* marginals. The latter arises when an ex-ante rule or optimization procedure first fixes candidate inclusion probabilities and the planner must then realize them using only committees satisfying an ex-post requirement. For an ex-post axiom Φ, let 1_W denote the incidence vector of committee W. The fair-committee polytope of Φ is $P_\Phi = \mathrm{conv}\{1_W : |W| = k,\ W \text{ satisfies } \Phi\}$, and exact implementation asks whether a given rational vector p lies in P_Φ. Table 2 separates what was known from what we prove.

Property and domain	Verify a given object	Implement marginals
PSC, strict rankings	polynomial [11, Prop. 11]	NP-complete (Theorem 6)
Rank-PJR+, weak rankings	polynomial [5]	NP-complete (Theorem 6)
IPSC, weak rankings	coNP-complete [5, Prop. 1]	Σ_2^{P}-complete (Theorem 7)
Fractional Rank-PJR+, weak rankings	polynomial (Proposition 3)	—
Fractional IPSC, weak rankings	coNP-complete (Theorem 9)	—
Cutoff-GFS, weak rankings	polynomial (Proposition 3)	—
IPSC, weak rankings, fixed $k \geq 2$	coNP-complete (Theorem 8)	coNP-complete (Theorem 8)

Table 2: Verification and exact implementation. Entries citing a paper were previously known; entries citing a theorem, proposition or appendix are proved here. For integral rows an object is a committee; for fractional rows it is a marginal vector. The last column concerns lotteries over committees satisfying the integral axiom; a dash marks fractional properties, for which implementation by committees is not the relevant question. Completeness uses polynomial-time many-one reductions. In every row, a fair object can be found in polynomial time.

A yes certificate for exact implementation uses at most $m + 1$ rationally weighted committees: choose an affinely independent support by Carathéodory's theorem and solve its marginal equations; Cramer's rule bounds the weights' encoding lengths polynomially. Exact implementation is therefore in NP for PSC and Rank-PJR+, whose verification is polynomial, and in Σ_2^{P} for IPSC, checking the committees with a coNP oracle. Both upper bounds are tight, as the following theorems show.

Theorem 6 (NP-complete exact implementation)**.** *Exact implementation of rational marginals is NP-complete for PSC under strict rankings and for Rank-PJR+ under weak rankings. For Rank-PJR+, hardness already holds on approval profiles. Exact IPSC implementation restricted to approval profiles is also NP-complete. All reductions are polynomial-time many-one reductions and use only marginals in* $\{0, 2/3, 1\}$*; the approval restriction uses* $\{2/3, 1\}$*.*

Theorem 7 (Exact IPSC implementation)**.** *Under weak rankings, exact implementation of rational marginals by IPSC committees is* Σ_2^{P}*-complete under polynomial-time many-one reductions. Hardness holds with at most four indifference classes per voter and marginals in* $\{0, 1/2, 1\}$*.*

Theorem 8 (Fixed-size IPSC implementation)**.** *For every fixed integer* $k \geq 2$*, deciding whether prescribed rational marginals admit an IPSC implementation is coNP-complete under polynomial-time many-one reductions. Hardness already holds for integral marginals and at most four indifference classes per voter. Integral IPSC verification is coNP-complete under the same restrictions.*

The proofs are in Appendix A.4 and Appendix A.5. These classifications rule out polynomial-time implementation or complete separation for the three fair-committee polytopes unless P = NP. They concern arbitrary prescribed marginals and do not contradict the constructions in Section 4, which choose the marginals themselves. On approval profiles, where IPSC coincides with PJR+

(Lemma 2), integral and fractional verification are polynomial (5; Proposition 3) and exact IPSC implementation is NP-complete (Theorem 6). Unless P = NP (respectively NP = Σ_2^P), the coNP- and Σ_2^P-hardness results for IPSC therefore require ballots with more than two tiers; our reductions use four.

The IPSC implementation polytope also differs from fractional IPSC: a lottery supported on IPSC committees need not satisfy fractional IPSC.

Example 10 (Ex-post IPSC need not imply fractional IPSC)**.** Let $n = k = 3$, with approval sets $\{a, b\}$, $\{a, c\}$, and $\{d, e\}$. Both $\{a, d, e\}$ and $\{b, c, d\}$ satisfy IPSC. Their equal mixture has $p_a = 1/2$ and $p(\{a, b, c\}) = 3/2 < 2$. Coalition $\{1, 2\}$ generalized solidly supports $\{a\}$, whose periphery is $\{a, b, c\}$. Thus its fractional IPSC inequality fails. Both integral vectors also satisfy fractional Rank-PJR+: their PJR+ demands have integral entitlements since $n = k$. Their mixture violates that axiom for the same common candidate a and coalition $\{1, 2\}$. Thus fractional Rank-PJR+ is nonconvex too. Its violation detection in Proposition 3 is not convex-set separation.

Finally, the compatibility construction of Algorithm 2 has its own computational bottleneck, already at its first selection step.

Proposition 2 (Smallest eligible solid set)**.** *Given weak rankings, a fully funded set F, residual budgets b_i, and an integer bound, deciding whether some generalized solid coalition S for a set T with $T \not\subseteq F$ and $\sum_{i \in S} b_i \geq 1$ has $|T|$ at most this bound is NP-complete. Hardness holds at the first execution of step 3(a) of Algorithm 2.*

The proof is in Appendix A.6. The proposition concerns this particular selection step. Computing some lottery with both fractional guarantees and IPSC ex post is polynomial-time by Theorem 3.

7. Conclusions

We have given a polynomial-time construction of a lottery satisfying cutoff-GFS and fractional Rank-PJR+ ex ante and IPSC ex post on every weak-ranking profile. Under strict rankings, the ex-post guarantee is PSC. Stronger rank-based ex-post representation meets a sharp boundary: for every $k \geq 2$, Rank-JR ex post conflicts with even Rank-IFS ex ante, already under strict rankings. Exact implementation of prescribed marginals is NP-complete for PSC under strict rankings and for Rank-PJR+ under weak rankings, and Σ_2^P-complete for IPSC under weak rankings.

The comparison between IPSC and Rank-PJR+ gives a substantive reason to distinguish ex-ante and ex-post proportionality. Although the two axioms coincide on approval profiles, their compatibility with fairness in expectation differs sharply on ranked ballots: IPSC accommodates both cutoff-GFS and fractional Rank-PJR+ on every weak-ranking profile, whereas ex-post Rank-PJR+, and even the weaker Rank-JR, can conflict with Rank-IFS. This makes IPSC a natural ex-post benchmark for ordinal lotteries: it protects proportional representation for solid coalitions while allowing stronger fairness guarantees across rank cutoffs in expectation.

Beyond the specific axioms, the polynomial-time rounding is a general technique for best-of-both-worlds problems whose ex-post fair outcomes cannot be recognized or optimized over efficiently. It needs only a convex ex-ante target over which linear functions can be minimized, and a rule that, for any prices, builds one fair outcome that is no cheaper than the target's cheapest point; such outcomes exist for every price vector exactly when a suitable lottery exists. The weighted committee oracle is such a rule for IPSC. We expect the technique to apply to other combinations of ex-ante and ex-post axioms, for instance in participatory budgeting and fair division, wherever a price-guided version of an ex-post fair rule can be designed.

The polynomial-time construction uses an ellipsoid-based convex decomposition. A direct combinatorial construction with the same guarantees would give a simpler implementation and remains an interesting direction. The two constructions suggest where the difficulty lies. Payment-transfer rounding ties every seat to payments made inside voters' top classes, but needs smallest solid sets; the weighted oracle replaces sets by ranks, but its charges are only accounting and may fall outside a voter's top class. A combinatorial rule would have to combine rank-based selection with payments that stay within ties.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

The main contributions of the author were conceptualizing the project, proposing the relevant concepts, designing the Class Eating rule and its exponential-time rounding, and writing up the results. The author acknowledges the use of OpenAI's GPT Sol 6.1 and Anthropic's Fable 5.1 as interactive assistants for writing out the detailed proofs, deriving the complexity results, and, at a later stage of the project, devising a polynomial-time rounding rule that builds on ideas from the exponential-time rounding proposed by the author.

Appendix A. Technical Proofs

Appendix A.1. Well-definedness and computation of Class Eating

Remaining proof of Lemma 3. We show that every nonterminal event advances a voter and that the event times have polynomial encoding length. At every stage, x is feasible, and each voter's reservations total the elapsed time τ. The maximum common increment in step 1 of Algorithm 1 exists because its feasible region is nonempty, closed, and bounded.

If the process has not reached time k/n, maximality of the increment implies that some active reservation is blocked. Otherwise, for each voter i there would be an $\varepsilon_i > 0$ such that x stays feasible when her active reservation alone is increased by ε_i. Reducing the extra amounts to a common $\varepsilon > 0$ with $\varepsilon \le \min_i \varepsilon_i$ and $\varepsilon/n \le k/n - \tau$, and averaging the n realizing payments, shows that x stays feasible when every active reservation is increased by ε/n, contradicting maximality. No voter can be blocked at her last class $E_i^{t_i}$ before termination: that class would then be full in every realization, and so would her earlier classes, which she has left; all m candidates would be full, although the reservations total $n\tau < k \le m$. Thus advancing every blocked voter is well defined. There are at most $\sum_i (t_i - 1)$ advances, including those following a zero increment, and the process terminates. Each event's maximal increment and set of blocked voters are uniquely determined by the current reservations, so the final reservation vector is uniquely determined as well.

The largest common increment is computed by the polynomial-size LP consisting of (2), with every active reservation increased by δ and $0 \le \delta \le k/n - \tau$. For each active row, an analogous LP maximizing its individual increment detects whether it is blocked. To bound encoding lengths, let J be the set of reservation rows (i, j), and let $J_{\mathrm{act}} \subseteq J$ be the active rows. The fractional Hall conditions for feasible reservations are

$$\sum_{(i,j)\in X} x_{ij} \le \left| \bigcup_{(i,j)\in X} E_i^j \right| \qquad (X \subseteq J).$$

Consequently, every nonterminal positive maximal increment has the form

$$\delta = \frac{\left|\bigcup_{(i,j)\in X} E_i^j\right| - \sum_{(i,j)\in X} x_{ij}}{|X \cap J_{\mathrm{act}}|} \quad \text{for some } X \text{ with } 1 \le |X \cap J_{\mathrm{act}}| \le n.$$

Starting with a common denominator n, each event multiplies that denominator by at most n; a zero increment or the terminal increment $k/n - \tau$ needs no larger factor. The polynomially many events therefore have polynomial bit length. Scaling the final flow network by this denominator gives integral capacities, so realizing payments with the same denominator can be computed by an integral maximum-flow algorithm. This proves polynomial time for the whole construction. □

The efficient realizations in Appendix A.2 let each reservation be spent anywhere in its cutoff. The next lemma shows that this gives nothing new.

Lemma 7 (Classes versus cutoffs)**.** *Let $y \geq 0$ satisfy $\sum_i y_{ic} \leq 1$ for every candidate, and let each voter spend her final reservations in her cutoffs: $y_i(C) = \sum_j x_{ij}$ and $y_i(E_i^{\leq j}) \geq \sum_{s \leq j} x_{is}$ for all j, where $x = x^{\mathrm{CE}}$. Then y realizes x, that is, $y_i(E_i^j) = x_{ij}$ for all i and j.*

Proof. Because a voter's cutoffs are nested, Hall's theorem lets us split her payments among her reservations so that x_{ij} is paid inside $E_i^{\leq j}$. Consider any event of Algorithm 1, let F be the set of reservations (i, j) whose classes their voters have left by the end of this event, and let $K = \bigcup_{(i,j) \in F} E_i^j$. We first show $\sum_{(i,j) \in F} x_{ij} = |K|$. By Lemma 3, every candidate of K is full in every realization. By max-flow duality, the candidates that are full in every realization form a set $U = \bigcup_{(i,j) \in Y} E_i^j$ for a set Y of positive reservations with $\sum_{(i,j) \in Y} x_{ij} = |U|$. The reservations in Y pay only inside U and therefore fill it, so no other reservation pays in U. Each reservation in Y has its whole class inside U; it is therefore blocked, and its voter leaves that class at this event unless she has left it before. Hence $Y \subseteq F$, and in any realization $K \subseteq U$ is paid only by F. As K is full and the classes of F lie in K, the claimed equality follows, and it persists because reservations in F never change again.

Now consider the split of y. Every reservation in F is paid inside its cutoff, which consists of classes its voter has left, so inside K. By the equality just shown, the reservations in F fill K, and no other reservation pays in K. Take a reservation (i, j) with $j \geq 2$ and $x_{ij} > 0$, and the event at which voter i left E_i^{j-1}. At that event she leaves no other class, so $(i, j) \notin F$ while $E_i^{\leq j-1} \subseteq K$. Hence (i, j) pays nothing in $E_i^{\leq j-1}$ and is paid inside E_i^j. Reservations with $j = 1$ are paid inside E_i^1 anyway, so $y_i(E_i^j) = x_{ij}$. □

Appendix A.2. Efficient realizations

Regard each reservation row $(i, j) \in J$ as an auxiliary agent who approves $E_i^{\leq j}$ and has a rational budget $x_{ij} \geq 0$, where $\sum_{(i,j) \in J} x_{ij} = k$. A *full spending flow* has nonnegative f_{ijc}, zero outside $E_i^{\leq j}$, with $\sum_c f_{ijc} = x_{ij}$ for every row and $\sum_{(i,j) \in J} f_{ijc} \leq 1$ for every candidate. Its column sums form a fractional committee.

Lemma 8 (Efficient full spending)**.** *If a full spending flow exists, one can compute one in polynomial time whose column vector is Pareto efficient among all fractional committees of size k with respect to the utilities $p \mapsto p(E_i^{\leq j})$, $(i, j) \in J$, including those of zero-budget agents.*

Definition 9 (Efficient Class Eating)**.** Run Algorithm 1 to obtain the reservations x. To choose their realization, apply Lemma 8 to all rows (i, j), including those with $x_{ij} = 0$, and return the resulting column sums. By Lemma 7, they belong to $\mathrm{CE}(\succsim, k)$.

Proof of Lemma 8. We construct a full spending flow together with positive utility weights certifying its efficiency. Make a network with capacities x_{ij} on $s \to (i, j)$, $k + 1$ on $(i, j) \to c$ for $c \in E_i^{\leq j}$, and one on $c \to t$. A full spending flow exists by assumption, so the maximum flow value is k. For $Q \subseteq C$, the restricted network retains just the candidates in Q.

Initially put $w_{ij} = 1$ for every agent, including those with zero budget. Score candidate c by $\sigma_c = \sum_{(i,j):\, c \in E_i^{\preceq j}} w_{ij}$, fix $\theta = \max_c \sigma_c$ at these initial weights, and let C_θ contain all candidates scoring θ. Start with zero flow and repeat the following steps.

(1) Augment the current flow to a maximum flow in the network restricted to C_θ. Use simple augmenting paths, which never decrease any candidate-to-sink flow. If its value is k, return it.

(2) Let U be the set of auxiliary agents reachable from s in the residual graph of this restricted maximum flow. Put

$$d_U(c) = |\{(i,j) \in U : c \in E_i^{\preceq j}\}|, \qquad \alpha = \min_{\substack{c \notin C_\theta \\ d_U(c) > 0}} \frac{\theta - \sigma_c}{d_U(c)}.$$

Increase w_{ij} by α for every $(i,j) \in U$, update the scores, and add all candidates whose score has reached θ to C_θ.

The update is well defined whenever the flow value is below k. The full network then has an augmenting path. Its first candidate outside C_θ is preceded by an agent reachable in the restricted residual graph, so some agent in U approves a candidate outside C_θ. All such outside candidates have score below θ, and the update admits at least one of them without taking another above θ.

If an agent $(i,j) \in U$ approves an admitted candidate c, the residual capacity on $(i,j) \to c$ is positive, since its capacity is $k+1$. Thus c is also reachable. Its sink arc must be full, or the restricted flow would have an augmenting path. Every admitted candidate whose score increases therefore already has marginal one, and subsequent augmentations preserve that marginal. It follows that a final score above θ implies $p_c = 1$. A final score below θ belongs to an unadmitted candidate and implies $p_c = 0$.

Each update adds a candidate, so there are at most m updates. If all candidates have entered, the restricted maximum flow has value k and the algorithm stops. For any other size-k fractional committee q, the final scores satisfy

$$\sum_c \sigma_c(q_c - p_c) = \sum_c (\sigma_c - \theta)(q_c - p_c) \le 0.$$

Indeed, coordinates above the threshold have $p_c = 1$, and those below it have $p_c = 0$. Hence p maximizes $\sum_{(i,j)} w_{ij}\, p(E_i^{\preceq j})$ over all size-k fractional committees. Every w_{ij} is positive, which proves Pareto efficiency for all the stated utilities, including those of zero-budget agents.

Use a polynomial-time augmenting-path algorithm in each round. Flows retain a common denominator from the input capacities. Each score update divides by an integer at most $|J|$, so the polynomially many weight updates also have polynomial bit length. The whole procedure therefore runs in polynomial time. □

Proof of Theorem 5. Weak rankings. Algorithm 1 produces feasible rational reservations in polynomial time by Lemma 3. Apply Lemma 8 to all rows (i,j), including those with zero budgets. Its resulting column vector p is Pareto efficient among all size-k fractional committees for the utilities $p(E_i^{\preceq j})$. These are precisely the voters' distinct SD cutoff coordinates, so p is SD-efficient. Since $p \in \mathrm{CE}(\succsim, k)$ by Lemma 7, Theorem 2 supplies both fractional fairness guarantees. The replacement argument in Section 5 gives efficient support for every implementation.

Polynomial-support implementation. Fix any candidate order $c_1, \dots, c_m$, put $s_0 = 0$ and $s_j = \sum_{a=1}^{j} p_{c_a}$, and draw u uniformly from $[0,1)$. Include c_j exactly when

$$\lfloor s_j + u \rfloor - \lfloor s_{j-1} + u \rfloor = 1.$$

Each difference is zero or one because $0 \le p_{c_j} \le 1$. Their sum is k and their expectations are the marginals p_{c_j}. The committee changes only at the fractional parts of $-s_j$. There are at most $m-1$ distinct internal breakpoints, so sorting them gives at most m committees with rational probabilities in polynomial time.

Strict rankings. Class Eating reduces to PFR, so the realization has the PFR marginals. If $p_c > 0$, some voter i ate c, and every candidate in $B = \{d : d \succ_i c\}$ is fully funded. For any SD-dominating vector q, its cutoff inequality at c gives

$$q(B) + q_c \ge p(B) + p_c = |B| + p_c.$$

Since $q(B) \le |B|$, we have $q_c \ge p_c$. This is automatic when $p_c = 0$, and $p(C) = q(C) = k$ therefore forces $q = p$. Thus PFR is SD-efficient. Its polynomial-time PSC implementation is supplied by Corollary 1, and every supported committee is SD-efficient by the same replacement argument. □

Appendix A.3. Verification of the ex-ante axioms

Theorem 9 (Fractional IPSC verification)**.** *Given weak rankings and a rational fractional committee, deciding whether it satisfies fractional IPSC is coNP-complete. Hardness holds even when the vector is integral and every voter has the same selected unique first choice.*

Proof of Theorem 9. A certificate of failure consists of a coalition S and a supported candidate set T. The generalized-solid comparisons, the inclusion test $p(T) < |T|$, and the inequality $p(D(S,T)) < k|S|/n$ can all be checked in polynomial time. Thus verification is in coNP.

We reduce from Clique. Given a graph $G = (V, E)$ and target size $h \ge 3$, add isolated vertices until $n = |V|$ is divisible by $q = h - 1$ and $n \ge 2q$; this preserves h-cliques. Set $k = n/q$. Introduce a candidate b, graph candidates c_i for $i \in V$, and $k-1$ dummies $d_1, \dots, d_{k-1}$. For each vertex i, voter i has the following weak ranking:

$$b \ \succ_i \ c_i \ \succ_i \ \{c_j : ij \in E\} \ \succ_i \ \Big(\{c_j : j \ne i,\ ij \notin E\} \cup \{d_1, \dots, d_{k-1}\}\Big).$$

Candidates in braces are tied, and empty tiers are omitted. Let p be the incidence vector of $W = \{b, d_1, \dots, d_{k-1}\}$. Candidate b is the unique selected top choice of every voter.

If $Q \subseteq V$ is an h-clique, take $S = Q$ and $T = \{b\} \cup \{c_i : i \in Q\}$. Each supporter weakly ranks every member of T above every candidate outside it. For each $i \in Q$, the bottom tier starts after rank $h + 1 = |T|$, since i has at least $h - 1$ neighbors in Q. Hence the periphery $D = \bigcup_{i \in Q} A_i^{|T|}$ contains no dummy, so $p(D) = 1$. But $p(T) = 1 < |T|$ and $k|S|/n = h/(h-1) > 1$. Fractional IPSC fails.

Conversely, let (S, T) violate fractional IPSC. If a dummy belongs to the periphery D, then all dummies do: they are tied at the bottom for every voter. This yields $p(D) = k$, contrary to $p(D) < k|S|/n \le k$. Thus D has no dummy. It contains b, so $p(D) = 1$ and $|S| > h - 1$. Since T is not fully selected, it contains a graph candidate. For every $i \in S$, generalized solid support then forces $c_i \in T$, because voter i ranks c_i strictly above every other graph candidate. If $i, j \in S$ were nonadjacent, voter i would tie $c_j \in T$ with all dummies, putting those dummies in D. Consequently S is a clique with at least h vertices. This completes the reduction. □

Proposition 3 (Checking the ex-ante axioms)**.** *For rational vectors, fractional Rank-PJR+ admits polynomial-time verification with a violating-demand witness, and cutoff-GFS admits a polynomial-time separation oracle.*

Proof. Fix a rank r and a candidate c with $p_c < 1$; there are polynomially many such pairs. Minimize

$$p\left(\bigcup_{i\in S} A_i^r\right) - \frac{k|S|}{n} \quad \text{over } S \subseteq \{i : c \in A_i^r\}.$$

Weighted coverage minus a modular function is submodular, so this minimum can be computed in polynomial time, and a negative minimum is exactly a fractional Rank-PJR+ violation for this r, c.

For cutoff-GFS, put

$$g_i(D) = \max_{r:A_i^r\subseteq D} \min\{k, |A_i^r|\}, \qquad g_i(D) = 0 \text{ if no cutoff qualifies.}$$

The cutoff-GFS inequalities are equivalent to $p(D) \geq n^{-1}\sum_i g_i(D)$ for every $D \subseteq C$: choose for each voter a maximizing cutoff contained in D, and conversely take D to be the union in (1). To separate, list voter i's distinct cutoffs $\emptyset = E_i^{\leq 0} \subsetneq \cdots \subsetneq E_i^{\leq t_i}$. Then $g_i(D)$ is a nonnegative linear combination of indicators $\mathbf{1}\{E_i^{\leq j} \subseteq D\}$, with coefficients $\min\{k, |E_i^{\leq j}|\} - \min\{k, |E_i^{\leq j-1}|\}$. Each indicator is supermodular. Hence $p(D) - n^{-1}\sum_i g_i(D)$ is submodular and can be minimized in polynomial time. □

Appendix A.4. Exact-implementation reductions

Appendix A.4.1. NP-completeness of exact implementation

Proof of Theorem 6. We reduce from 3-Colourability [28]. Given a graph G on $r \geq 1$ vertices, form a graph H with vertices (v, a) for $v \in V(G)$ and $a \in \{0, 1, 2\}$. For each v, join its three copies into a triangle. For each edge uv of G, join (u, a) to (v, a) for every a. Write $V_H = V(H)$ and $M_E = |E(H)| = 3r + 3|E(G)|$. Every independent set of H has size at most r, and one of size r specifies a proper three-colouring of G. Equivalently, a vertex cover of size $2r$ exists exactly when G is three-colourable. Moreover, cyclically shifting a proper colouring gives three size-$2r$ vertex covers in which each vertex of H occurs exactly twice.

Approval profiles. Make one voter per edge of H, approving its two endpoints, and add a set Z of $M_E - 2r$ candidates approved by nobody. Set $n = k = M_E$ and prescribe $p_v = 2/3$ for $v \in V_H$ and $p_d = 1$ for $d \in Z$. These marginals sum to k. Every supported committee must be $W = Z \cup X$ with $|X| = 2r$. JR requires X to cover every edge: an unrepresented edge voter alone has entitlement one. Hence any implementation by JR committees, and in particular by PJR+ committees, yields a vertex cover of size $2r$ and thus a proper colouring.

Conversely, every vertex cover X gives a PJR+ committee $Z \cup X$. For a group S with a common approved candidate $c \notin W$, the distinct edges of S must be covered by their distinct other endpoints. Its approval union therefore contains at least $|S| = k|S|/n$ selected candidates. This is PJR+, which on approval profiles is equivalent to both Rank-PJR+ and IPSC (Lemma 2). A proper colouring therefore supplies three cyclic covers, whose equal lottery, with Z added, implements the prescribed vector. Polynomial verification of PJR+ also gives NP membership for the approval restriction of IPSC.

Strict rankings. Keep V_H, Z and $k = M_E$. For each edge $\eta = \{u, v\}$ of H, introduce private candidates a_η, b_η and two voters. Let P be the set of all private candidates. The two voters of η rank

$$a_\eta \succ u \succ v \succ b_\eta \succ [V_H \setminus \{u, v\}] \succ [Z] \succ [P \setminus \{a_\eta, b_\eta\}],$$
$$b_\eta \succ v \succ u \succ a_\eta \succ [V_H \setminus \{u, v\}] \succ [Z] \succ [P \setminus \{a_\eta, b_\eta\}].$$

Each bracket is ordered strictly in any fixed order. There are $n = 2M_E$ voters, so $n/k = 2$. Prescribe the same marginals on $V_H \cup Z$ and marginal zero on P. Again a supported committee must be $Z \cup X$, $|X| = 2r$. For each edge, its two voters solidly support their common top-four set $\{a_\eta, b_\eta, u, v\}$ and have entitlement one. PSC therefore requires X to cover that edge.

If X is a vertex cover, all PSC demands hold. A one-voter coalition has no positive integer entitlement. A coalition consisting of the two voters of one edge has entitlement one; every set they solidly support contains their top-four set and hence a selected endpoint. Finally, consider a solid coalition containing voters from distinct edges. Its common prefix contains each member's private first choice. In the other edge's ballot that candidate is below all of $V_H \cup Z$. The common prefix thus contains every member of W, meeting any possible PSC demand. This proves that $Z \cup X$ is PSC exactly when X is a vertex cover. The same equal lottery over three cyclic covers proves the reduction. □

Appendix A.4.2. Σ_2^{P}*-completeness of exact IPSC implementation*

Proof of Theorem 7. The upper bound was established in Section 6. For hardness, start with the standard Σ_2^{P}-complete problem

$$\exists x\ \forall y\ \neg\varphi(x, y),$$

where φ is a 3CNF formula [29]. We may assume that x has $r \geq 1$ variables.

Making the target marginals independent of the assignment. Replace each existential variable x_j by the exclusive-or of two bits u_j, v_j. Introduce a witness variable z_j and let

$$\psi(u, v, z, y) = \varphi(z, y) \wedge \bigwedge_{j=1}^{r} [z_j \leftrightarrow (u_j \oplus v_j)].$$

Encode each displayed equivalence by its four clauses of size three. For fixed u, v, the formula ψ is satisfiable in z, y exactly when $\varphi(u \oplus v, y)$ is satisfiable in y. Pad with clauses $(w \vee \neg w \vee w')$ on fresh witness variables until the clause count is even and at least four; this leaves satisfiability unchanged. Write ν for the resulting clause count. Put $t = \nu/2$, and enumerate the $R = 2r$ bits of (u, v) as the assignment variables.

Form the literal-occurrence graph H: it has one vertex per literal occurrence of ψ, and two vertices are adjacent exactly when they come from different clauses and their literals are not complementary. A clique has at most ν vertices; a ν-clique chooses a consistent literal from every clause. Add isolated vertices until the number g of vertices is divisible by t. Since every clause is nonempty, $g \geq \nu = 2t$.

Candidates, voters, and marginals. Use a candidate b, a candidate c_w for every vertex of H, a pair $a_{j,0}, a_{j,1}$ for each of the R assignment variables, and a set Z of $g/t - 1$ fixed dummy candidates. For a literal occurrence w on assignment variable j, let M_w contain the one marker $a_{j,\varepsilon}$ whose value ε makes that literal false. Set $M_w = \emptyset$ for witness-variable occurrences and the added isolated vertices. Vertex w supplies one voter with ranking

$$b \ \succ\ c_w \ \succ\ \Big(\{c_z : z \in N_H(w)\} \cup M_w\Big) \ \succ\ \text{all remaining candidates.}$$

The third and last tiers are ties, and empty tiers are omitted. For every j, add t voters who tie $\{a_{j,0}, a_{j,1}\}$ first and tie all other candidates last. Set

$$n = g + Rt, \qquad k = g/t + R, \qquad n/k = t,$$

and prescribe

$$p_b = p_d = 1 \ (d \in Z), \qquad p_{a_{j,0}} = p_{a_{j,1}} = \tfrac{1}{2}, \qquad p_{c_w} = 0.$$

The total marginal mass is $1 + |Z| + R = k$. Every supported committee contains b and Z and excludes all c_w. The t voters for marker pair j have entitlement one and generalized solidly support either singleton with periphery exactly that pair. IPSC forces at least one of its markers to be selected. There are exactly R remaining positions, so every IPSC committee in the support has the form

$$W_\alpha = \{b\} \cup Z \cup \{a_{j,\alpha_j} : j \in [R]\}$$

for an assignment α to (u, v).

A representation violation is a satisfying completion. We prove

$$\begin{aligned} &W_\alpha \text{ satisfies IPSC} \\ &\quad\iff\quad \psi(\alpha, z, y) \text{ is unsatisfiable in its witness variables.} \end{aligned} \tag{A.1}$$

Suppose first that ψ has a satisfying completion. Choose one true occurrence from each clause, forming a ν-clique Q. The graph voters of Q generalized solidly support $T = \{b\} \cup \{c_w : w \in Q\}$. For each such voter, the bottom tier starts after position $\nu + 1 = |T|$, since it has at least $\nu - 1$ neighbours. The periphery contains b, graph candidates, and possibly markers that falsify selected literals. None of those markers is selected in W_α. It contains no member of Z. Hence the periphery has only one selected candidate, although $|Q|/t = 2$ and $T \not\subseteq W_\alpha$. This violates IPSC.

Conversely, let (S, T, ℓ) be an IPSC violation by W_α. The periphery cannot be all of C. If S contains a marker-pair voter, T must be contained in that voter's top pair: including a last-tier candidate would put all of C in the periphery. No graph voter generalized solidly supports a nonempty subset of that pair, because it omits her unique first choice b. No voter from another marker pair supports it either. Thus S lies within one group of t pair voters; its entitlement is at most one and its periphery contains the pair's selected marker. Such a group cannot violate IPSC.

Therefore S consists only of graph voters. Since b is selected and $T \not\subseteq W_\alpha$, we have $T \neq \{b\}$, so generalized solidity forces $b \in T$ and $c_w \in T$ for every $w \in S$. No member of T can lie in any supporter's bottom tier. Consequently any two vertices of S are adjacent, and $|S| \leq \nu$. Since the periphery contains selected b, a violation requires $\ell \geq 2$. The quota condition $\ell \leq |S|/t$ now forces $\ell = 2$ and $|S| = \nu$. Each supporter's third tier is in the periphery, because $|T| \geq \nu + 1$. None of its markers M_w can be selected, or the periphery would contain at least two winners. Thus S chooses one mutually consistent literal per clause, and no chosen assignment-variable literal is false under α. These literals extend to a satisfying completion of ψ, proving (A.1).

Completing the reduction. If an assignment x makes $\varphi(x, y)$ unsatisfiable, take $(u, v) = (0^r, x)$ and $(u, v) = (1^r, \neg x)$. Both have $u \oplus v = x$, so both corresponding committees are IPSC by (A.1). The assignments complement every one of the R bits. Their equal lottery therefore has exactly the prescribed half marginals on all markers, as well as the prescribed zero and one marginals. Conversely, any IPSC implementation has a supported W_α. By (A.1), its $x = u \oplus v$ makes $\varphi(x, y)$ unsatisfiable. This is a polynomial-time many-one reduction. □

Appendix A.5. Fixed-seat IPSC implementation

Proof of Theorem 8. Hardness. We first note that, for each fixed $k \geq 2$, it is NP-complete to decide whether a graph on n vertices, with $k \mid n$, contains a clique of size n/k. Indeed, given a graph G

with r vertices and target $h \geq 2$, put

$$a = \max\left\{0, \left\lceil \frac{r - kh}{k-1} \right\rceil\right\}, \qquad b = kh - r + (k-1)a.$$

Join G to a new a-clique, and then add b isolated vertices. The new graph has $n = k(h+a)$ vertices and has an n/k-clique exactly when G has an h-clique. The construction is polynomial for fixed k.

We now adapt the clique construction of Brill and Peters [5, Proposition 1]. Take a graph H on n vertices with $k \mid n$. Introduce a candidate c_i and a voter for each vertex i, together with k guard candidates $d_1, \ldots, d_k$. Voter i ranks

$$c_i \succ \{c_j : ij \in E(H)\} \succ \{d_1, \ldots, d_k\} \succ \{c_j : j \neq i,\ ij \notin E(H)\}, \tag{A.2}$$

where braces are ties and empty tiers are omitted. Prescribe the integral vector of $W = \{d_1, \ldots, d_k\}$. We show

$$W \text{ violates IPSC} \iff H \text{ has a clique of size at least } n/k. \tag{A.3}$$

If S is such a clique, take $T = \{c_i : i \in S\}$. Every voter in S generalized solidly supports T. Each has at least $|S| - 1$ neighbours, so the guard tier starts after rank $|T|$. Its periphery contains no guard, although $k|S|/n \geq 1$ and $T \not\subseteq W$. Thus IPSC fails.

Conversely, consider an IPSC violation (S, T). Since the guards are tied in every ballot, a periphery meeting the guard tier contains all k selected candidates and cannot violate IPSC. The violating periphery therefore avoids every guard. All candidates in T must lie above the guard tier for each supporter. Generalized solidity and the unique first choice force $c_i \in T$ for every $i \in S$. For distinct $i, j \in S$, candidate c_j must then be above the guards for voter i, so $ij \in E(H)$. Thus S is a clique. A violation needs a positive integer entitlement, hence $|S| \geq n/k$. This proves (A.3).

The reduction proves coNP-hardness of integral IPSC verification. An integral marginal vector has only its own committee as a possible implementation, so it proves the implementation hardness as well.

Membership in coNP. Failure of integral IPSC has the polynomial witness (S, T, ℓ). For implementation, enumerate all $\binom{m}{k} = O(m^k)$ committees; this is polynomial because k is fixed. A certificate of nonimplementability lists some committees to remove, with an IPSC-violation witness for each. Verify all witnesses, then solve the polynomial-size linear feasibility system asking whether p is a convex combination of the remaining committees. Accept the certificate exactly when that system is infeasible.

This verifier is sound because it removes only unfair committees. It is complete because, for a nonimplementable vector, a certificate can list every unfair committee and one witness for each. There are only polynomially many committees. Therefore nonimplementation belongs to NP, proving the claimed coNP upper bound. □

Appendix A.6. The smallest eligible solid set

Proof of Proposition 2. A pair (S, T) is a polynomial certificate. We reduce from Clique with target size $h \geq 3$. Add isolated vertices until the graph has n vertices, with $n \geq h$ and h dividing n. Use a candidate b, vertex candidates c_i, and h guard candidates. The padded vertex count n is also the number of voters. Vertex i supplies a voter with ranking

$$b \succ c_i \succ \{c_j : ij \in E\} \succ \{\text{guards}\} \succ \{c_j : j \neq i,\ ij \notin E\},$$

with each brace a tie and empty tiers omitted. Set $k = 1 + n/h$. At time $1/n$, the top-class reservations jointly fill b. Each voter then reserves another $1/h$ for her second class $\{c_i\}$ before

termination, so every realization pays these additional amounts to the private second choices. Thus $F = \{b\}$ and every residual budget is $1/h$. Each residual top class is the singleton $\{c_i\}$, so every residual top-support sum is at most $1/h < 1$. Saturation makes no change before the first block selection.

An eligible set therefore needs at least h supporters. If $T \not\subseteq \{b\}$ is generalized solidly supported by S, it contains b and each c_i with $i \in S$. Hence $|T| \leq h+1$ forces $|S| = h$ and $T = \{b\} \cup \{c_i : i \in S\}$. If two supporters are nonadjacent, their solidity comparison would also require all guards in T, a contradiction. Thus S is an h-clique. Conversely, an h-clique supports exactly such an eligible set. Use the bound $h + 1$. □

Appendix A.7. Fractional EAR

Remark 3 (Fractional EAR)**.** A natural divisible analogue of EAR gives every voter a budget of k/n and every candidate capacity one. Starting at the first rank cutoff, repeatedly choose an underfull candidate with positive remaining support and purchase as much of her remaining capacity as her supporters' budgets permit, charging these supporters proportionally. Expand the common cutoff only when no further purchase is possible. This procedure satisfies fractional Rank-PJR+, but need not satisfy cutoff-GFS or admit an IPSC-supported implementation.

For cutoff-GFS, let $n = 4$, $k = 3$, and consider

$$1, 2 : \ a \sim b \sim c \succ d, \qquad 3 : \ b \succ d \succ a \sim c, \qquad 4 : \ b \succ a \succ d \succ c.$$

Funding b first leaves every voter with budget $1/2$. Voters 1 and 2 then fully fund a, while voters 3 and 4 fund d as their lower cutoffs open. The resulting marginals are $(p_a, p_b, p_c, p_d) = (1, 1, 0, 1)$. Choosing cutoffs $\{a, b, c\}$ for voters 1 and 2, $\{b\}$ for voter 3, and $\{a, b\}$ for voter 4, cutoff-GFS requires

$$p(\{a, b, c\}) \geq \frac{3 + 3 + 1 + 2}{4} = \frac{9}{4},$$

whereas this union receives only two units.

For IPSC implementation, the procedure can also return the marginals in Example 6: purchasing b, e, c, a in that order at the first cutoff exhausts all four voters' budgets. As shown there, these marginals admit no lottery supported on IPSC committees. Thus fractional EAR alone does not establish Theorem 1.

Appendix B. Polynomial-Time Construction

This appendix justifies the saturation step shared by the two constructions in Sections 4.2 and 4.3 and proves the constructive convex-intersection lemma. The weighted committee algorithm and the proof of Theorem 3 appear in the main text. We use the payment and cutoff notation introduced there.

Appendix B.1. Preprocessing and residual flexibility

Proof of Lemma 5. Maintain $F = \{c : p_c = 1\}$ and $B = C \setminus F$, and recompute b_i, $I = \{i : b_i > 0\}$, and L_i after each transfer step. Payment order puts every residual payment of voter i in L_i: a candidate she pays must be weakly above every underfull candidate.

Suppose a candidate $c \in B$ has $\sum_{i \in I : c \in L_i} b_i \geq 1$. All payments to c come from these voters. Their residual payments to other candidates therefore total at least $1 - p_c$. Transfer exactly this amount to c, taking payments in a fixed order and splitting the last one if needed. Each transfer is within the donor's top class. Candidate c becomes full, donor candidates remain underfull, and

payments to F do not change. By Lemma 4, all reservations and hence payment order are preserved. Add c to F and repeat.

Each step fixes one more candidate, so there are at most k steps. At termination every residual top-support sum is below one, and $\sum_{i\in I} b_i = k - |F|$. Each step uses sums, differences, and minima of rational payments, preserving their common denominator. Thus the procedure runs in polynomial time with polynomial encoding length.

Now redistribute each b_i arbitrarily within L_i. Candidate c receives at most $\sum_{i\in I: c\in L_i} b_i < 1$, so all capacities are respected. Members of L_i lie in one original class, so, as in the proof of Lemma 4, the redistribution changes none of voter i's class totals. Payments to F remain fixed, so the redistributed payments still realize x^{CE}, and Theorem 2 gives both fractional guarantees. □

Appendix B.2. Constructive convex intersection

The following form of separation and convex decomposition turns the signed-weight oracle into an explicit lottery. It is related to randomized metarounding [25, 26]; we give the polytope version needed here.

Lemma 9 (Constructive convex intersection)**.** *Let $P \subseteq \mathbb{R}^B$ be a nonempty polytope. Suppose that, for every rational w, a minimizing point p^w of $w \cdot p$ over P can be computed in polynomial time, with encoding length bounded by a polynomial in the instance size independently of w. Let $V \subseteq \{0,1\}^B$. Suppose a polynomial-time oracle returns $v^w \in V$ with*

$$w \cdot v^w \geq \min_{p\in P} w \cdot p.$$

Then, using polynomially many oracle calls, one can compute in polynomial time a convex combination of at most $|B| + 1$ oracle outputs that belongs to P, with rational coefficients of polynomial encoding length.

Proof. If $B = \emptyset$, every oracle output lies in P. Otherwise put $d = |B| \geq 1$ and consider the inequalities $w \cdot (p - v) \geq 1$ for all $p \in P$ and $v \in V$. No rational w satisfies them: compute p^w and v^w, for which $w \cdot (p^w - v^w) \leq 0$. Use the violated inequality $w \cdot (p^w - v^w) \geq 1$ as a cut in the ellipsoid method.

Let $\varphi \geq d$ be a uniform polynomial bound on the encoding length of each cut, including its right-hand side. Every finite collection of cuts defines a polyhedron of facet complexity at most φ. By Schrijver [30, Theorem 10.2], if this polyhedron is nonempty, it contains a rational point w^0 of encoding length at most $4d^2\varphi$. This bound depends on d and φ, not on the number of collected cuts.

In particular, $\|w^0\|_\infty \leq 2^{4d^2\varphi}$, and doubling w^0 leaves slack at least one in every cut. Each coefficient vector has Euclidean norm at most $d2^\varphi$, so the ball of radius $\rho = (2d2^\varphi)^{-1}$ centred at $2w^0$ satisfies every cut. It lies in the box $[-H, H]^d$ with $H = 2^{4d^2\varphi+2}$. Thus the same box and ball radius work for every feasible finite collection of cuts, independently of how many cuts it contains.

Run the ellipsoid method on that box for the polynomial number of iterations prescribed by these radii. Every query returns a cut. The current ellipsoid contains every point in the box satisfying the cuts collected so far. At termination its volume is too small to contain a ball of the guaranteed radius, so the collected cuts are infeasible.

Index these cuts by j, writing them as $w \cdot (p^j - v^j) \geq 1$. Their infeasibility implies that the origin lies in the convex hull of the differences $p^j - v^j$. Otherwise a strictly separating hyperplane, scaled appropriately, would satisfy every cut. Solve the polynomial-size linear program

$$\lambda_j \geq 0, \qquad \sum_j \lambda_j = 1, \qquad \sum_j \lambda_j (p^j - v^j) = 0.$$

A basic feasible solution has at most $|B|+1$ positive coefficients, all of polynomial encoding length. Finally, $\sum_j \lambda_j v^j = \sum_j \lambda_j p^j \in P$ by convexity. Neither membership testing for V nor exact weight maximization over V is required. □

Appendix C. Details of the Running Example

Throughout, $n = 6$, $k = 3$, $\mathcal{L} = \{1, 2, 3\}$ and $\mathcal{R} = \{4, 5, 6\}$, with the ballots of Example 1. Every voter holds $k/n = 1/2$, so a coalition S is entitled to $|S|/2$ in expectation and to $\lfloor |S|/2 \rfloor$ seats ex post. Table C.3 lists the voters' cutoffs.

voter	distinct cutoffs	voter	distinct cutoffs
1	$\{a\} \subset \{a,b\} \subset \{a,b,c\} \subset C$	4	$\{d\} \subset \{d,f\} \subset \{d,e,f\} \subset C$
2	$\{a\} \subset \{a,b,c\} \subset C$	5	$\{d,e\} \subset \{d,e,f\} \subset C$
3	$\{a\} \subset \{a,c\} \subset \{a,b,c\} \subset C$	6	$\{e,f\} \subset \{d,e,f\} \subset C$

Table C.3: The nested cutoffs of the running example.

Appendix C.1. Ex-post demands

Every voter's bottom class has three candidates and starts at rank four. If a set T supported by S contains a bottom-class candidate of some $i \in S$, generalized solidity puts the three candidates that i ranks above it into T. Then $|T| \geq 4$ and $A_i^{|T|} = C$, so the periphery is C and the IPSC demand holds for every committee. A coalition meeting both blocs supports only such sets: a supported set contains a top choice of each supporter, and the top choices of one bloc lie in the bottom class of the other. The remaining generalized solid coalitions with positive entitlement lie within one bloc and have $|S| \in \{2, 3\}$, hence entitlement one. They are:

coalition S	supported set T	periphery
any $S \subseteq \mathcal{L}$ with $\|S\| \geq 2$	$\{a\}$ and $\{a,b,c\}$	$\{a\}$ and $\{a,b,c\}$
$\{1,2\}$	$\{a,b\}$	$\{a,b,c\}$
$\{2,3\}$	$\{a,c\}$	$\{a,b,c\}$
$\{4,5\}$	$\{d\}$	$\{d,e\}$
$\{5,6\}$	$\{e\}$	$\{d,e,f\}$
any $S \subseteq \mathcal{R}$ with $\|S\| \geq 2$	$\{d,e,f\}$	$\{d,e,f\}$

The pairs with periphery $\{a\}$ force $a \in W$, which meets every demand of $\mathcal{L}$. In $\mathcal{R}$, the pair $\{4,5\}$ with $T = \{d\}$ requires $W \cap \{d,e\} \neq \emptyset$, which meets the others. Hence the IPSC committees are exactly the seven committees

$$\{a,b,d\},\ \{a,b,e\},\ \{a,c,d\},\ \{a,c,e\},\ \{a,d,e\},\ \{a,d,f\},\ \{a,e,f\}.$$

They are also exactly the Rank-PJR+ committees: Rank-PJR+ demands with a positive entitlement and a union other than C again come from two or three voters of one bloc. In $\mathcal{L}$, every such union contains a, while in $\mathcal{R}$ the union is $\{d,e\}$ for the pair $\{4,5\}$ at rank one and $\{d,e,f\}$ otherwise. In particular, these committees satisfy Rank-JR, and every committee drawn in Example 8 satisfies Rank-PJR+, although Theorem 4 shows that a lottery meeting Rank-IFS ex ante cannot guarantee even Rank-JR ex post in general.

Appendix C.2. Ex-ante demands

Every cutoff other than C lies within one bloc. For $D \neq C$, the largest cutoff-GFS demand on D, namely $n^{-1}\sum_i g_i(D)$ in the notation of the proof of Proposition 3, is therefore the sum of the demands on $D \cap \{a, b, c\}$ and $D \cap \{d, e, f\}$. So cutoff-GFS amounts to the inequalities for the sets D in Table C.4, omitting sets with demand zero. Each fractional Rank-PJR+ demand with union $U \neq C$ also comes from voters of one bloc. Since U contains the common candidate c, the alternative $p_c = 1$ implies $p(U) \geq 1$, so a demand with entitlement $1/2$ or 1 is simply $p(U) \geq k|S|/n$. The remaining demands, of $\mathcal{L}$ or $\mathcal{R}$ as a whole, hold whenever $p_a = 1$, $p(\{a, b, c\}) \geq 3/2$ and $p(\{d, e, f\}) \geq 3/2$. The fractional Rank-PJR+ column of the table lists the resulting bounds, each of which is itself required by the axiom.

set D	cutoff-GFS	fractional Rank-PJR+	$p^{\mathrm{RD}}(D)$	$p^0(D)$	$p^1(D)$
$\{a\}$	$\geq 1/2$	≥ 1	$\mathbf{1/2}$	1	1
$\{a, b\}$, $\{a, c\}$	$\geq 2/3$	$\geq 1/2$	1	5/4	5/4
$\{a, b, c\}$	$\geq 3/2$	$\geq 3/2$	3/2	3/2	3/2
$\{d\}$	$\geq 1/6$	$\geq 1/2$	1/2	1/2	1
$\{d, e\}$	$\geq 1/2$	≥ 1	1	1	1
$\{d, f\}$	$\geq 1/3$	$\geq 1/2$	1	1	3/2
$\{e, f\}$	$\geq 1/3$	$\geq 1/2$	1	1	1/2
$\{d, e, f\}$	$\geq 3/2$	$\geq 3/2$	3/2	3/2	3/2

Table C.4: The ex-ante demands of the running example, and the mass given by random dictator, by the reservation realization p^0 of Example 5, and by the final marginals p^1 of Example 8. Every other demand of either axiom is implied by these rows. The only violation, in bold, is random dictator's $p_a = 1/2$.

In particular, the two rows with bound $3/2$ show that neither axiom is met by a single committee, which would need two seats in each bloc. Random dictator satisfies every cutoff-GFS demand but fails $p_a = 1$.

Appendix C.3. The rounding

The other tie-break. If saturation chooses e first, voter 6's payment moves from f to e, which becomes fully funded. No further candidate can be saturated. The remaining budgets belong to voters 1–4 and total one. They form a single block with $T_t = C$, because a set supported by voters 1 and 4 is all of C. Keeping their payments in place yields $\frac{1}{4}\{a, b, e\} + \frac{1}{4}\{a, c, e\} + \frac{1}{2}\{a, d, e\}$, again a lottery over IPSC committees that tosses a fair coin for the third seat.

Transfers and implementability. Saturation changes p^0 in this run, although the initial vector also has an IPSC implementation: $\frac{1}{2}\{a, d, f\} + \frac{1}{4}\{a, b, e\} + \frac{1}{4}\{a, c, e\}$ implements p^0 by IPSC committees. Example 6 gives a profile on which a reservation realization admits no IPSC implementation.

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