# Confluence of Conditional Term Rewrite Systems via Transformations

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Conditional term rewriting is an intuitive yet complex extension of term rewriting. In order to benefit from the simpler framework of unconditional rewriting, transformations have been defined to eliminate the conditions of conditional term rewrite systems.

Recent results provide confluence criteria for conditional term rewrite systems via transformations, yet they are restricted to CTRSs with certain syntactic properties like weak left-linearity. These syntactic properties imply that the transformations are sound for the given CTRS.

This paper shows how to use transformations to prove confluence of operationally terminating, right-stable deterministic conditional term rewrite systems without the necessity of soundness restrictions. For this purpose, it is shown that certain rewrite strategies, in particular almost U-eagerness and innermost rewriting, always imply soundness.

## 1 Introduction

### 1.1 Background and Motivation

Conditional term rewrite systems (CTRSs) are term rewrite systems in which rewrite rules may be bound to certain conditions. Such systems are a widely accepted extension of unconditional term rewrite systems (TRSs) that has been investigated for decades but they are more complex than unconditional TRSs. Several properties of unconditional rewriting are not satisfied anymore or change their intuitive meaning and many criteria for TRSs cannot be applied. Thus, there have been efforts to develop transformations that map CTRSs into unconditional TRSs, for instance in [4, 6, 12, 22, 1].

Transformations are supposed to simplify the original CTRS by eliminating the conditions. This way, properties of the CTRS can be proved by using the simpler, unconditional TRS. Yet, for this purpose one must ensure that the rewrite relation of the transformed TRS does not give rise to rewrite sequences that are not possible in the original CTRS, a property called soundness.

The aim of this paper is to prove that if the transformed TRS is confluent, then the CTRS is also confluent, without the necessity to also prove soundness. This main result is applicable to right-stable deterministic CTRSs that are transformed into terminating TRSs and it significantly improves other, similar confluence results like the ones in [11] and [17] because there are no syntactic restrictions required that imply soundness (in particular weak left-linearity). In fact, it also holds for CTRSs for which the used transformation is unsound. This result leads to a new method to prove confluence of CTRSs that can be easily automated and it leaves space for further improvements.

### 1.2 Overview and Outline

In order to prove properties of CTRSs using transformations, one must prove that the transformation is suitable for the given purpose. [13] introduces the notions of *soundness* and *completeness* of a certain

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class of transformations, so-called unravelings. Informally, soundness means that if the transformed TRS gives rise to a rewrite sequence in the transformed TRS then this rewrite sequence is also possible in the original CTRS. Completeness is the opposite of soundness, i.e. that a rewrite sequence in the CTRS also exists in the transformed TRS.

Completeness is usually implied by the structure of transformations but soundness is more difficult to prove and not satisfied in general. Yet, soundness is needed to prove properties like non-termination or confluence. In many papers it is proved that certain syntactic properties like (weak) left-linearity imply soundness for a certain transformation (see e.g. [13]).

Soundness and confluence of the transformed system implies confluence of the original CTRS (see e.g. [11]), yet there is not yet a positive or a negative result whether soundness is essential (although confluence of the transformed CTRS does not imply soundness which was shown in [9]). This paper will answer this question by first showing that innermost derivations are always sound and then show that this in fact implies confluence if the transformed TRS is terminating.

The following section recalls some basics and notions of (conditional) term rewriting. Section 3 introduces the most common unravelings of CTRSs. In Section 4 a rewrite strategy called *almost U-eager derivations* is introduced and it is proved that it implies soundness. Based on this, further results are shown, in particular soundness of innermost rewrite sequences. These results are used in Section 5 to prove confluence of CTRSs. Finally, the results are summarized and similar results in the literature and possible perspectives are discussed.

## 2 Preliminaries

This paper follows basic notions and notations as they are defined in [3] and [19]. Basic knowledge of (conditional) term rewriting is assumed. Some less common notions are recalled in the following.

The set of all terms over a signature $\mathscr{F}$ and an infinite but countable set of variables $\mathscr{V}$ is denoted as $\mathscr{T}(\mathscr{F},\mathscr{V})$. In the following $\mathscr{T}$ is used if $\mathscr{F}$ and $\mathscr{V}$ are clear from context. The set of variables in a term $s$ is $\mathscr{V}ar(s)$. For a set of variables $X$, $\vec{X}$ denotes the sequence of variables in $X$ in some arbitrary but fixed order. The set of positions of a term $s$ is denoted as $\mathscr{P}os(s)$, $s|_p$ is the subterm of $s$ at position $p$ and $s[t]_p$ represents the term $s$ after inserting the term $t$ at position $p$. If $p \leq q$ ($p < q$), then $q$ is below (strictly below) $p$. Otherwise $q$ is above $p$ ($p \geq q$) or parallel to $p$ ($p \parallel q$).

A substitution $\sigma$ is a mapping from variables to terms that is implicitly extended to terms. In the following, the common postfix notation $s\sigma$ is used for the term $s$ with the substitution $\sigma$ applied. This notation is extended to substitutions, i.e. $\sigma\tau$ corresponds to $\sigma\tau(x) = \tau(\sigma(x))$.

A rewrite rule $\alpha$ is a pair of two terms $(l,r)$, denoted as $l \to r$, where $\mathscr{V}ar(r) \subseteq \mathscr{V}ar(l)$. A term rewrite system (TRS) is a pair $\mathscr{R} = (\mathscr{F},R)$ of a signature and a set of rules. In the following, the signature will often be left implicit and slightly abusing notation $\mathscr{R}$ will be used instead of $R$.

A rewrite step from a term $s$ to a term $t$ at a position $p$ using a rule $\alpha$ is denoted as $s \to_{p,\alpha,\mathscr{R}} t$. Some labels are skipped if they are clear from context or irrelevant. A single rewrite step is written as $\to$, the transitive closure is $\to^+$, the reflexive and transitive closure is $\to^*$. $\leftarrow$ ($\leftarrow^*$) is the inverse of $\to$ ($\to^*$) and $\leftrightarrow$ ($\leftrightarrow^*$) is $\leftarrow \cup \to$ ($(\leftarrow \cup \to)^*$). A rewrite sequence $u \to^*_{\mathscr{R}} v$ in some TRS $\mathscr{R}$ is normalizing if $v$ is a normal form in $\mathscr{R}$.

The set of *one-step descendants* $q\backslash A$ of a position $q$ in a term $s$ w.r.t. the rewrite step $A : s \to_{p,l\to r} t$ is the set of positions

$$q\backslash A = \begin{cases} \{q\} & \text{if } q \leq p \text{ or } p \parallel q \\ \{p.q'.q'' \mid r|_{q'} = l|_{p'}\} & \text{if } l|_{p'} \text{ is a variable and } q = p.p'.q'' \\ \emptyset & \text{otherwise} \end{cases}$$

The one-step descendant relation is defined as $\{(p,q) \mid p \in q\backslash A\}$. The descendant relation is the reflexive, transitive closure of the one-step descendant relation, extended to rewrite sequences. The ancestor relation is the inverse of the descendant relation. By slight abuse of terminology a term $t|_{q'}$ will be referred to as the (one-step) descendant of a term $s|_q$ if $q'$ is a (one-step) descendant of $q$.[1]

A conditional rule is a triple $(l,r,c)$, usually denoted as $l \to r \Leftarrow c$ where $l,r$ are terms and $c$ is a conjunction of equations $s_1 = t_1,\dots,s_k = t_k$. In this paper we only consider oriented conditional rules in which equality is defined as reducibility $\to^*$. A conditional term rewrite system (CTRS) $\mathscr{R}$ over some signature $\mathscr{F}$ consists of conditional rules. The underlying TRS $\mathscr{R}_u$ contains the unconditional part of the conditional rules $\mathscr{R}_u = \{l \to r \mid l \to r \Leftarrow c \in \mathscr{R}\}$.

Let $\mathscr{R}_n$ be the following TRSs:

$$\mathscr{R}_0 = \emptyset$$
$$\mathscr{R}_{n+1} = \{l\sigma \to r\sigma \mid l \to r \Leftarrow c \in \mathscr{R} \text{ and } s\sigma \to^*_{\mathscr{R}_n} t\sigma \text{ for all } s \to^* t \in c\}$$

A CTRS $\mathscr{R}$ gives rise to the rewrite step $u \to_{\mathscr{R}} v$ if there is an $n$ such that $u \to_{\mathscr{R}_n} v$. The minimal such $n$ is the *depth* of the rewrite step.

A conditional rule is of type 1 if there are no extra variables ($\mathscr{V}ar(r) \cup \mathscr{V}ar(c) \subseteq \mathscr{V}ar(l)$). It is of type 3 if all extra variables occur in the conditions ($\mathscr{V}ar(r) \subseteq \mathscr{V}ar(c) \cup \mathscr{V}ar(l)$). A *normal* conditional rule is an oriented 1-rule in which for every condition $s_i \to^* t_i$ ($i \in \{1,\dots,k\}$), $t_i$ is a ground normal form w.r.t. $\mathscr{R}_u$. A *deterministic* conditional rule is an oriented 3-rule $l \to r \Leftarrow s_1 \to^* t_1,\dots,s_k \to^* t_k$ such that $\mathscr{V}ar(s_i) \subseteq \mathscr{V}ar(l,t_1,\dots,t_{i-1})$ for all $i \in \{1,\dots,k\}$. A CTRS is a deterministic CTRS (DCTRS) if all rules are deterministic conditional rules.

A CTRS is *right-stable* if for all conditional rules $l \to r \Leftarrow s_1 \to^* t_1,\dots,s_k \to^* t_k$, $t_i$ is either a linear constructor term or a ground irreducible term (w.r.t. $\mathscr{R}_u$), and $\mathscr{V}ar(t_i) \cap \mathscr{V}ar(l,s_1,t_1,\dots,s_{i-1},t_{i-1},s_i) = \emptyset$ for all $i \in \{1,\dots,k\}$. In the following only right-stable DCTRSs are considered.

## 3 Unravelings

Unravelings are a simple class of transformations from CTRSs to TRSs that was introduced in [13]. In the same paper Marchiori also introduces multiple specific unravelings, in particular the simultaneous unraveling $\mathbb{U}_{sim}$ for normal 1-CTRSs. This unraveling splits a conditional rule $\alpha : l \to r \Leftarrow s_1 \to^* t_1,\dots,s_k \to^* t_k$ into two unconditional rules:

$$l \to U^\alpha(s_1,\dots,s_k,\overrightarrow{\mathscr{V}ar(l)})$$
$$U^\alpha(t_1,\dots,t_k,\overrightarrow{\mathscr{V}ar(l)}) \to r$$

The sequential unraveling that was introduced in [18] (a similar unraveling was already defined in [14]) extends this approach to DCTRSs.

[1] From this definition it follows that $t|_p$ is a one-step descendant of $s|_p$ in a rewrite step $s \to_p t$. This case is sometimes excluded from the descendant relation.

**Definition 1** (sequential unraveling $\mathbb{U}_{seq}$ [18])**.** *Given a deterministic conditional rule* $\alpha : l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k$*,* $\mathbb{U}_{seq}$ *translates the rule into a set of unconditional rules:*

$$\mathbb{U}_{seq}(\alpha) = \left\{ \begin{array}{rl} l \to U_1^\alpha(s_1, \vec{X_1}) & \textit{(introduction rule)} \\ U_1^\alpha(t_1, \vec{X_1}) \to U_2^\alpha(s_2, \vec{X_2}) & \textit{(switch rule)} \\ \vdots \qquad \vdots & \qquad \vdots \\ U_{k-1}^\alpha(t_{k-1}, \vec{X_{k-1}}) \to U_k^\alpha(s_k, \vec{X_k}) & \textit{(switch rule)} \\ U_k^\alpha(t_k, \vec{X_k}) \to r \qquad & \textit{(elimination rule)} \end{array} \right\}$$

*where* $X_i = \mathscr{V}ar(l, t_1, \ldots, t_{i-1})$*. For an unconditional rule* $\alpha$*,* $\mathbb{U}_{seq}(\alpha) = \{\alpha\}$*. The unraveled CTRS* $\mathbb{U}_{seq}(\mathscr{R})$ *then is defined as* $\bigcup_{\alpha \in \mathscr{R}} \mathbb{U}_{seq}(\alpha)$*.*

In the following, $\mathbb{U}_{seq}(\mathscr{F})$ denotes the signature of the unraveled TRS $\mathbb{U}_{seq}(\mathscr{R})$. The new function symbols $\mathbb{U}_{seq}(\mathscr{F}) \setminus \mathscr{F}$ are *U-symbols*. Terms rooted by a U-symbol are *U-terms*. A terms $s$ is a *mixed term* if it contains U-terms ($s \in \mathscr{T}(\mathbb{U}_{seq}(\mathscr{F}), \mathscr{V})$, short $\mathbb{U}_{seq}(\mathscr{T})$), otherwise it is an *original term* ($s \in \mathscr{T}$). In U-terms of some $\mathbb{U}_{seq}(\mathscr{R})$, the first argument encodes the *conditional argument* while the other *variable arguments* contain the variable bindings.

A rewrite step in the transformed TRS in which an introduction (switch/elimination) rule is applied is an *introduction step* (*switch step*/*elimination step*).

According to the original definition an unraveling $\mathbb{U}$ is complete, i.e. $u \to_{\mathscr{R}}^* v$ implies $u \to_{\mathbb{U}(\mathscr{R})}^* v$. It is sound if $u \to_{\mathbb{U}(\mathscr{R})}^* v$ implies $u \to_{\mathscr{R}}^* v$ for all $u, v \in \mathscr{T}$.

The unraveling $\mathbb{U}_{seq}$ encodes all variable bindings in its U-terms even if they are not used anymore. In [5] the variable bindings are optimized, leading to the optimized sequential unraveling $\mathbb{U}_{opt}$ ([15]). In this unraveling variables are not encoded if they are not required in a later condition or the right-hand side of the conditional rule:

$$\mathbb{U}_{opt}(\alpha) = \left\{ l \to U_1^\alpha(s_1, \vec{X_1}), U_1^\alpha(t_1, \vec{X_1}) \to U_2^\alpha(s_2, \vec{X_2}), \ldots, U_k^\alpha(t_k, \vec{X_k}) \to r \right\}$$

where $X_i = \mathscr{V}ar(l, t_1, \ldots, t_{i-1}) \cap \mathscr{V}ar(t_{i+1}, s_{i+2} \ldots, s_k, t_k, r)$.

Optimizing the variable bindings in unravelings has advantages in some cases because less terms have to be considered in proofs. In [7] several soundness results for $\mathbb{U}_{opt}$ are shown, in particular soundness for U-eager rewrite sequences. Formally, a derivation $u_0 \to_{p_0} u_1 \to_{p_1} \cdots \to_{p_{n-1}} u_n$ in some $\mathbb{U}(\mathscr{R})$ is *U-eager* if U-terms are immediately rewritten, i.e., $p \leq p_i$ for all U-terms $u_i|_p$.

Yet, this optimization has some drawbacks. For instance, two terms that are not joinable in the original CTRS rewrite to the same mixed term because a variable binding is erased. Because of this phenomenon, the main result of this paper does not hold for $\mathbb{U}_{opt}$.

**Example 2** (unsoundness for confluence of the optimized unraveling)**.** *Consider the following DCTRS and its transformed terminating TRS using the optimized unraveling:*

$$\mathscr{R} = \left\{ \begin{array}{c} a \to s(b) \\ \searrow \\ s(c) \\ s(x) \to A \Leftarrow B \to^* C \end{array} \right\} \qquad \mathbb{U}_{opt}(\mathscr{R}) = \left\{ \begin{array}{r} a \to s(b) \\ \searrow \\ s(c) \\ s(x) \to U_1^\alpha(B) \\ U_1^\alpha(C) \to A \end{array} \right\}$$

$\mathbb{U}_{opt}(\mathscr{R})$ *is confluent since the only critical pair* $\langle s(b), s(c) \rangle$ *gives rise to the common reduct* $U_1^\alpha(B)$ *and the transformed TRS is terminating. Yet,* $\mathscr{R}$ *is not confluent because* $a$ *rewrites to* $s(b)$ *and* $s(c)$ *but the condition of the conditional rule is never satisfied so that* $s(b)$ *and* $s(c)$ *are irreducible.*

Since $\mathbb{U}_{seq}$ preserves all variable bindings of the left-hand side of a conditional rule it is possible to extract these bindings and insert them into the corresponding left-hand side, thus obtaining the back-translation tb (defined in [8], similar mappings are used in the proofs in [13] and [19]).

**Definition 3** (back-translation tb)**.** *Let $\mathscr{R} = (\mathscr{F}, R)$ be a CTRS, then $\mathtt{tb} : \mathbb{U}_{seq}(\mathscr{T}) \mapsto \mathscr{T}$ is defined as follows:*

$$\mathtt{tb}(s) = \begin{cases} s & \text{if } s \text{ is a variable} \\ f(\mathtt{tb}(s_1), \ldots, \mathtt{tb}(s_k)) & \text{if } s = f(s_1, \ldots, s_k) \text{ and } f \in \mathscr{F} \\ l\sigma & \text{if } s = U_i^{\alpha}(w, v_1, \ldots, v_m) \text{ and} \\ & \alpha = l \to r \Leftarrow s_1 \to^* t_1, \ldots s_k \to^* t_k \end{cases}$$

*where $\sigma$ is defined as $x_i\sigma = \mathtt{tb}(v_i)$ where $\overrightarrow{\mathscr{V}ar(l, t_1, \ldots, t_{i-1})} = x_1, \ldots, x_m$.*

In the following tb will sometimes be extended to substitutions ($x\mathtt{tb}(\sigma) = \mathtt{tb}(x\sigma)$ for $x \in \mathscr{D}om(\sigma)$). The back translation allows us to define soundness such that it also extends to mixed terms: A rewrite sequence $u \to^*_{\mathbb{U}_{seq}} (R) v$ ($u \in \mathscr{T}$) is sound if $u \to^*_{\mathscr{R}} \mathtt{tb}(v)$.

## 4 Soundness and Completeness of Transformations

The transformation $\mathbb{U}_{seq}$ is not sound for DCTRSs in general. This was first shown by Marchiori in [13] using a normal 1-CTRS that consists of multiple non-linear rules. For DCTRSs we presented another example in [9].

**Example 4** (unsoundness [9])**.** *Consider the following DCTRS and its unraveling*

$$\mathscr{R} = \left\{ \begin{array}{c} \begin{array}{ccc} a & \to & c \\ & \searrow\!\!\!\!\!\nearrow & \\ b & \to & d \end{array} \\ \begin{array}{ccc} s(c) & \to & t(k) \\ & \searrow & \\ & & t(l) \end{array} \\ g(x,x) \to h(x,x) \\ f(x) \to \langle x, y \rangle \Leftarrow s(x) \to^* t(y) \end{array} \right\} \qquad \mathbb{U}_{seq}(\mathscr{R}) = \left\{ \begin{array}{c} \begin{array}{ccc} a & \to & c \\ & \searrow\!\!\!\!\!\nearrow & \\ b & \to & d \end{array} \\ \begin{array}{ccc} s(c) & \to & t(k) \\ & \searrow & \\ & & t(l) \end{array} \\ g(x,x) \to h(x,x) \\ f(x) \to U_1^{\alpha}(s(x), x) \\ U_1^{\alpha}(t(y), x) \to \langle x, y \rangle \end{array} \right\}$$

*In $\mathbb{U}_{seq}(\mathscr{R})$, there is the following reduction sequence:*

$$g(f(a), f(b)) \to^* g(U_1^{\alpha}(s(c), d), U_1^{\alpha}(s(c), d)) \to h(U_1^{\alpha}(s(c), d), U_1^{\alpha}(s(c), d)) \to^* h(\langle d, k \rangle, \langle d, l \rangle)$$

*Yet, this derivation is not possible in $\mathscr{R}$ because there is no common reduct of $f(a)$ and $f(b)$ that rewrites to both, $\langle d, k \rangle$ and $\langle d, l \rangle$.*

The CTRSs of Example 4 and the counterexample of [13] are syntactically very complex. Based on this observation it was shown that many syntactic properties imply soundness: Left-linearity (normal 1-CTRSs [13]/[19], DCTRSs [16]), weak left-linearity, right-linearity (normal 1-CTRSs [8], DCTRSs [9]), non-erasingness (normal 1-CTRSs [8], 2-DCTRSs [9], counterexample for 3-DCTRSs [9]) and weak right-linearity (DCTRSs [7]).

The CTRS of Example 4 is not confluent and in [8] it is shown that the simultaneous unraveling is sound for confluent normal 1-CTRSs. Yet, this result does not hold for DCTRSs:

**Example 5** (unsoundness for confluence [9]). *Let $\mathscr{R}$ be the CTRS of Example 4 and $\mathscr{R}'$ be the CTRS consisting of the unconditional rules*

$$\mathscr{R}' = \{c \to e \leftarrow d, k \to e \leftarrow l, s(e) \to t(e)\}$$

*Then, $\mathscr{R} \cup \mathscr{R}'$ and $\mathbb{U}_{seq}(\mathscr{R} \cup \mathscr{R}')$ are confluent, yet, the argument of Example 4 still holds so that the reduction sequence $g(f(a), f(b)) \to^* h(\langle d,k\rangle, \langle d,l\rangle)$ in the transformed TRS is still unsound. Nonetheless, the last term of the unsound derivation can be further reduced to the irreducible term $h(\langle e,e\rangle, \langle e,e\rangle)$. The derivation $g(f(a), f(b)) \to^* h(\langle e,e\rangle, \langle e,e\rangle)$ is sound.*

Although the previous example shows that confluence of the transformed TRS is not sufficient for soundness, it also shows that (in contrast to Example 4) the last term of the unsound derivation can be further reduced. In fact, all normalizing derivations in confluent DCTRSs are sound [9].

The original definition of unravelings in [13] states that an unraveling must be complete and preserve the original signature. Based on the definition of the unravelings it is not surprising that completeness is satisfied in all cases. In the following the proof of [13] is adapted to $\mathbb{U}_{seq}$. The proof will be useful to motivate a rewrite strategy that implies soundness.

**Lemma 6** (completeness of $\mathbb{U}_{seq}$). *Let $\mathscr{R}$ be an oriented CTRS and $s,t$ be two terms such that $s \to_{\mathscr{R}} t$, then $s \to^+_{\mathbb{U}_{seq}(\mathscr{R})} t$.*

*Proof.* By induction on the depth $n$ of the rewrite step $s \to_{n,\mathscr{R}} t$. If $n = 0$, then the applied rule $\alpha$ is an unconditional rule and $\alpha \in \mathbb{U}_{seq}(\mathscr{R})$.

Otherwise, let $\alpha : l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k$ be the rule applied in $s \to_{n,\mathscr{R}} t$ so that $s = C[l\sigma]$ and $t = C[r\sigma]$. By the definition of the depth, $s_i\sigma \to^*_{\mathscr{R}_{n-1}} t_i\sigma$ for all $i \in \{1,\ldots,k\}$. By the induction hypothesis, there are derivations $s_i\sigma \to^*_{\mathbb{U}_{seq}(\mathscr{R})} t_i\sigma$. Thus, there is the following derivation in $\mathbb{U}_{seq}(\mathscr{R})$:

$$\begin{aligned} l\sigma &\to U_1^\alpha(s_1\sigma, \vec{X}_1\sigma) \to^* U_1^\alpha(t_1\sigma, \vec{X}_1\sigma) \to U_2^\alpha(s_2\sigma, \vec{X}_2\sigma) \to^* U_2^\alpha(t_2\sigma, \vec{X}_2\sigma) \to \cdots \\ &\to U_1^\alpha(s_k\sigma, \vec{X}_k\sigma) \to^* U_k^\alpha(t_k\sigma, \vec{X}_k\sigma) \to r\sigma \end{aligned}$$

□

The previous completeness result constructs a derivation in $\mathbb{U}_{seq}(\mathscr{R})$ in which first the U-term is introduced, then the conditional argument is rewritten and finally the U-term is eliminated. The definition of the U-eager rewrite strategy is based on such derivations but it also allows rewrite steps inside variable bindings.

In U-eager derivations, after a U-term is introduced only rewrite steps inside this U-term are allowed until it is eliminated. Rewrite steps outside of U-terms are forbidden. The reason for this limitation is that in a derivation in some $\mathbb{U}_{opt}(\mathscr{R})$ one obtains mixed terms that have no meaning in the original CTRS. For instance, in Example 2 the mixed term $U_1^\alpha(B)$ is a common reduct of $s(b)$ and $s(c)$. Yet, there is no such term in the original CTRS.

For $\mathbb{U}_{seq}$, mixed terms can be back-translated to the left-hand side of the conditional rule because all variable bindings are preserved. Therefore, U-eagerness for some $\mathbb{U}_{seq}(\mathscr{R})$ can be generalized to also allow rewrite steps outside of U-terms even if they are not eliminated. In such *almost U-eager rewrite sequences* if a U-term is not rewritten it is considered to represent a failed conditional evaluation and thus the arguments of such a U-term and the U-term itself must not be rewritten anymore. Rewrite steps above such U-terms, including erasing rewrite steps, are allowed.

**Definition 7** (almost U-eager derivations)**.** *Let $\mathscr{R}$ be a DCTRS. A derivation $D : u_0 \to_{p_0} u_1 \to_{p_1} \cdots \to_{p_{n-1}} u_n$ in $\mathbb{U}_{seq}(\mathscr{R})$ is* almost U-eager*, if for every rewrite step $u_i \to_{p_i} u_{i+1}$, if there is a U-term $u_i|_q$ such that $q \leq p_i$, then also $q \leq p_{i-1}$ ($i \in \{1,\ldots,n-1\}$).*

This way, rewrite steps in U-terms are always grouped in such derivations which makes tracking terms easier. Furthermore, U-terms that represent intermediate evaluation steps of conditions are isolated from other rewrite steps. Rewrite steps above U-terms that are rewritten in a later rewrite step are vitally important for unsoundness. Observe that the unsound derivation of Example 4 is not almost U-eager and that in the unsound derivation a non-linear rewrite step is applied above a U-term.

The proof for soundness of almost U-eager derivations will use the same proof structure that was already used in [9]. First, we recall the following lemma that states that rewrite steps in variable and conditional arguments can be extracted from derivations.

**Lemma 8** (extraction lemma of [9])**.** *Let $\mathscr{R}$ be a DCTRS and $D : u_0 \to_{p_0} u_1 \to_{p_1} \cdots \to_{p_{n-1}} u_n$ be a derivation in $\mathbb{U}_{seq}(\mathscr{R})$ ($u_0 \in \mathscr{T}$). If $u_n|_p = U_i^{\alpha}(w, \vec{X}_i\sigma_{i+1})$ where $\alpha$ is the conditional rule $l \to r \Leftarrow s_1 \to^* t_1,\ldots,s_k \to^* t_k$, then there is an index $m$ and a position $q$ such that $u_m|_q$ is an ancestor of $u_n|_p$ and there are substitutions $\sigma_1,\ldots,\sigma_i$ such that $u_m|_q = l\sigma_1$ and the following derivations can be extracted from D:*

- *$s_j\sigma_j \to^*_{\mathbb{U}_{seq}(\mathscr{R})} t_j\sigma_{j+1}$ ($j \in \{1,\ldots,i-1\}$),*
- *$x\sigma_j \to^*_{\mathbb{U}_{seq}(\mathscr{R})} x\sigma_{j+1}$ ($j \in \{1,\ldots,i\}$, $x \in X_j$), and*
- *$s_i\sigma_i \to^*_{\mathbb{U}_{seq}(\mathscr{R})} w$.*

*Furthermore, in the reductions above for every single rewrite step $u \to v$ there is an index $m' \in \{m+1,\ldots,n-1\}$ and a position $q'$ such that $u_{m'}|_{q'} = u$ and $u_{m'+1}|_{q'} = v$.*

In the following this extraction lemma will be used implicitly.

Next, a monotony result on $\mathtt{tb}$ is shown.

**Lemma 9** (monotony of $\mathtt{tb}$)**.** *Let $\mathscr{R}$ be a DCTRS. If $u \to_{p,\mathbb{U}_{seq}(\mathscr{R})} v$ for $u,v \in \mathbb{U}_{seq}(\mathscr{T})$ and $\mathtt{tb}(u|_p) \to^*_{\mathscr{R}} \mathtt{tb}(v|_p)$ then $\mathtt{tb}(u|_q) \to^*_{\mathscr{R}} \mathtt{tb}(v|_{q'})$ for all $q \in \mathscr{P}os(u)$ and descendants $q'$ of $q$.*

*Proof.* By case distinction on $p$ and $q$: If $p < q$ or $p \parallel q$, then $u|_q = v|_{q'}$, hence also $\mathtt{tb}(u|_q) = \mathtt{tb}(v|_{q'})$.

Otherwise, if $q \leq p$, then there is only one descendant of $u|_q$ which is $v|_q$. Let $q.q' = p$. Then by induction on $|q'|$, if $q = p$ then $\mathtt{tb}(u|_q) \to^*_{\mathscr{R}} \mathtt{tb}(v|_{q'})$ is equivalent to the assumption $\mathtt{tb}(u|_p) \to^*_{\mathscr{R}} \mathtt{tb}(v|_p)$.

For the induction step, let $q' = i.q''$. There are the following cases based on the term $u|_q$: If $u|_q = f(u_1,\ldots,u_n)$ where $f \in \mathscr{F}$ is an original symbol, then $\mathtt{tb}(u|_q) = f(\mathtt{tb}(u_1),\ldots,\mathtt{tb}(u_n))$ and $\mathtt{tb}(v|_q) = f(\mathtt{tb}(u_1),\mathtt{tb}(u_{i-1}),\mathtt{tb}(v|_{q.i}),\mathtt{tb}(u_{i+1}),\ldots,\mathtt{tb}(u_n))$. By the induction hypothesis $\mathtt{tb}(u_i) \to^* \mathtt{tb}(v|_{q.i})$, thus also $\mathtt{tb}(u|_q) \to^* \mathtt{tb}(v|_q)$.

The remaining case is that $u|_q$ is a U-term $U_j^{\alpha}(w,x_1,\ldots x_n)\sigma$. If $i = 1$, then the rewrite step is inside the conditional argument so that the variable bindings are unmodified and $\mathtt{tb}(u|_q) = \mathtt{tb}(v|_q)$. Otherwise, $v|_q = U_j^{\alpha}(w,x_1,\ldots x_n)\sigma'$ where $x_j\sigma = x_j\sigma'$ for all $j \in \{1,\ldots,i-2,i,\ldots,n\}$. By the induction hypothesis, $\mathtt{tb}(x_{i-1}\sigma) \to^* \mathtt{tb}(x_{x_{i-1}}\sigma')$. Hence, $\mathtt{tb}(u|_q) = l\,\mathtt{tb}(\sigma)$ where $\mathtt{tb}(v|_q) = l\,\mathtt{tb}(\sigma')$ and thus $\mathtt{tb}(u|_q) \to^* \mathtt{tb}(v|_q)$. □

In the next lemma, single rewrite steps of a derivation are translated using $\mathtt{tb}$.

**Lemma 10** (technical key lemma)**.** *Let $\mathscr{R}$ be a right-stable DCTRS and let $u_0 \to_{p_0} u_1 \to_{p_1} \cdots \to_{p_{n-1}} u_n$ be an almost U-eager derivation in $\mathbb{U}_{seq}(\mathscr{R})$ where $u_0 \in \mathscr{T}$. Then, $\mathtt{tb}(u_i|_{p_i}) \to^*_{\mathscr{R}} \mathtt{tb}(u_{i+1}|_{p_i})$ ($i \in \{0,\ldots,n-1\}$).*

*Proof.* In the following, assume w.l.o.g. that for all substitutions, mapped terms do not share variables with the domain, i.e. , $\mathscr{D}om(\sigma) \cap \mathscr{V}ar(x\sigma) = \emptyset$ for all $x \in \mathscr{D}om(\sigma)$.

By induction on the length of the derivation $n$: If $n = 0$, the result holds vacuously.

Otherwise $\mathtt{tb}(u_i|_q) \to^*_{\mathscr{R}} \mathtt{tb}(u_{i+1}|_{q'})$ for all one-step descendants $u_{i+1}|_{q'}$ of $u_i|_q$ by the induction hypothesis and Lemma 9. Consequently also $\mathtt{tb}(u_i|_q) \to^*_{\mathscr{R}} \mathtt{tb}(u_j|_{q''})$ for all descendants $u_j|_{q''}$ of $u_i|_q$ ($1 \leq i < j < n$).

By case distinction on the rule applied in the last rewrite step $u_{n-1} \to_{\alpha,p_{n-1}} u_n$: If the applied rule is an unconditional original rule $l \to r \in \mathscr{R}$, then $u_{n-1}|_{p_{n-1}} = l\sigma$, $u_n|_{p_{n-1}} = r\sigma$, $\mathtt{tb}(u_{n-1}|_{p_{n-1}}) = l\,\mathtt{tb}(\sigma)$ and $\mathtt{tb}(u_n|_{p_{n-1}}) = r\,\mathtt{tb}(\sigma)$.

If the applied rule is an introduction rule or a switch rule, $\mathtt{tb}(u_{n-1}|_{p_{n-1}}) = \mathtt{tb}(u_n|_{p_{n-1}})$.

Finally, if the applied rule is an elimination rule, then by the definition of almost U-eagerness, all preceding rewrite steps are below $p_{n-1}$ up to the introduction step of the U-term, i.e., if the conditional rule is $\alpha : l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k$, then there is an $m$ such that $u_m|_{p_m} = l\sigma_1$, $p_m = p_{n-1}$, $p_m \leq p_i$ for all $i \in \{m, \ldots, n-1\}$ and the derivation $u_m|_{p_m} \to^*_{\mathbb{U}_{seq}(\mathscr{R})} u_n|_{p_{n-1}}$ is

$$l\sigma_1 \to U_1^\alpha(s_1\sigma_1, \vec{X}_1\sigma_1) \to^* U_1^\alpha(t_1\sigma_2, \vec{X}_1\sigma_2) \to U_2^\alpha(s_2\sigma_2, \vec{X}_2\sigma_2) \to^* \cdots$$
$$\to^* U_k^\alpha(t_k\sigma_{k+1}, \vec{X}_k\sigma_{k+1}) \to r\sigma_{k+1}$$

By the induction hypothesis, $\mathtt{tb}(x\sigma_i) \to^*_{\mathscr{R}} \mathtt{tb}(x\sigma_{i+1})$ and $\mathtt{tb}(s_i\sigma_i) \to^*_{\mathscr{R}} \mathtt{tb}(t_i\sigma_{i+1})$ for all $x \in X_i$.

Let $\sigma$ be the combined substitution $\mathtt{tb}(\sigma_1)\mathtt{tb}(\sigma_2)\cdots\mathtt{tb}(\sigma_{k+1})$, then $s_i\sigma \to^*_{\mathscr{R}} s_i\,\mathtt{tb}(\sigma_i)$ and $t_i\,\mathtt{tb}(\sigma_{i+1}) = t_i\sigma$ by right-stability. Hence, the conditions are satisfied for $\sigma$ and $l\sigma \to_{\mathscr{R}} r\sigma$. Furthermore, $l\sigma = l\sigma_1$. Thus, $l\,\mathtt{tb}(\sigma_1) \to_{\mathscr{R}} r\sigma \to^*_{\mathscr{R}} r\,\mathtt{tb}(\sigma_{k+1})$. □

Finally we prove soundness of almost U-eager rewrite sequences.

**Lemma 11** (soundness of almost U-eager derivations)**.** *Let $\mathscr{R}$ be a right-stable DCTRS. If $u_0 \to_{p_0} u_1 \cdots \to_{p_{n-1}} u_n$ is an almost U-eager derivation in $\mathbb{U}_{seq}(\mathscr{R})$ ($u_0 \in \mathscr{T}$) then $u_0 \to^*_{\mathscr{R}} \mathtt{tb}(u_n)$.*

*Proof.* By induction on the length of the derivation, if $n = 0$ the result holds vacuously. Otherwise, by Lemma 10, $\mathtt{tb}(u_{n-1}|_{p_{n-1}}) \to^*_{\mathscr{R}} \mathtt{tb}(u_n|_{p_{n-1}})$. By Lemma 9, $\mathtt{tb}(u_{n-1}) \to^*_{\mathscr{R}} \mathtt{tb}(u_n)$. Since by the inductive hypothesis $u_0 \to^*_{\mathscr{R}} \mathtt{tb}(u_{n-1})$ finally $u_0 \to^*_{\mathscr{R}} \mathtt{tb}(u_n)$. □

This result can be used to prove soundness for other rewrite strategies. Next, it is shown that innermost derivations can be converted into almost U-eager derivations, thus proving soundness of innermost rewriting. For this reason, innermost derivations are translated into almost U-eager derivations.

**Lemma 12** (innermost to almost-U-eager)**.** *Let $\mathscr{R}$ be a DCTRS and let $u_0 \to_{p_0} u_1 \to_{p_1} \cdots \to_{p_{n-1}} u_n$ be an innermost derivation ($u_0 \in \mathscr{T}$). Then there is an innermost, almost U-eager derivation $u_0 \to^*_{\mathbb{U}_{seq}(\mathscr{R})} u_n$.*

*Proof.* By induction on the length $n$ of the derivation. If $n = 0$ the result holds vacuously. Otherwise, by the induction hypothesis, the derivation $u_0 \to^*_{\mathbb{U}_{seq}(\mathscr{R})} u_{n-2} \to_{p_{n-2},\mathbb{U}_{seq}(\mathscr{R})} u_{n-1}$ is innermost and almost U-eager.

By case distinction on the last rewrite step $u_{n-1} \to_{p_{n-1}} u_n$: If $p_{n-1}$ is not below a U-term, then $u_0 \to^*_{\mathbb{U}_{seq}(\mathscr{R})} u_n$ is already almost U-eager. Otherwise, there is a U-term $u_{n-1}|_q$ and $q \leq p_{n-1}$. If there are multiple nested U-terms, let $u_{n-1}|_q$ be the innermost such U-term. By case distinction on $p_{n-2}$ and $q$: The case $p_{n-2} < q$ is not possible because of the assumption that the derivation is innermost.

If $q \leq p_{n-2}$, then $u_0 \to^*_{\mathbb{U}_{seq}(\mathscr{R})} u_n$ is almost U-eager.

If $q \parallel p_{n-2}$, then let $m$ be the largest value such that $p_{n-m}, p_{n-m+1}, \ldots, p_{n-2}$ are parallel to $q$. Since $u_0 \in \mathscr{T}$, $m < n$. Then, $u_{n-m}|_q = u_{n-1}|_q$ and $q \leq p_{n-m-1}$. Therefore, the following rewrite sequence in $\mathbb{U}_{seq}(\mathscr{R})$ is in fact U-eager:

$$\begin{aligned} u_0 \to^* u_{n-m-1} &\to_{p_{n-m-1}} u_{n-m} \to_{p_{n-1}} u_{n-m}[u_n|_{p_{n-1}}]_{p_{n-1}} \to_{p_{n-m}} \\ &\to_{p_{n-m}} u_{n-m+1}[u_n|_{p_{n-1}}]_{p_{n-1}} \to_{p_{n-m+1}} u_{n-m+2}[u_n|_{p_{n-1}}]_{p_{n-1}} \to_{p_{n-m+2}} \cdots \\ &\to_{p_{n-3}} u_{n-2}[u_n|_{p_{n-1}}]_{p_{n-1}} \to_{p_{n-2}} u_n \end{aligned}$$

□

Since almost U-eager rewrite sequences are sound this implies soundness.

**Lemma 13** (soundness of innermost derivations)**.** *Let $\mathscr{R}$ be a right-stable DCTRS. Let $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$ be an innermost derivation such that $u \in \mathscr{T}$. Then, $u \to^*_{\mathscr{R}} \mathtt{tb}(v)$.*

*Proof.* By Lemma 12, there is an almost U-eager derivation $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$. By Lemma 11, $u \to^*_{\mathscr{R}} \mathtt{tb}(v)$. □

**Theorem 14** (soundness of innermost derivations)**.** *$\mathbb{U}_{seq}$ is sound for innermost derivations for right-stable DCTRSs.*

*Proof.* By Lemma 13, if $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$ is an innermost derivation, then there is a derivation $u \to^*_{\mathscr{R}} \mathtt{tb}(v)$. □

Innermost derivations are therefore sound. Nonetheless, innermost rewriting is not suitable to simulate conditional rewriting in general because they are not complete. This can be easily seen in CTRSs in which the conditions are satisfiable but not innermost-satisfiable.

**Example 15** (incompleteness of innermost rewriting)**.** *Consider the following CTRS and its unraveled TRS:*

$$\mathscr{R} = \left\{ \begin{array}{c} a \to b \\ f(a) \to b \\ A \to B \Leftarrow f(a) \to^* b \end{array} \right\} \qquad \mathbb{U}_{seq}(\mathscr{R}) = \left\{ \begin{array}{c} a \to b \\ f(a) \to b \\ A \to U_1^{\alpha}(f(a)) \\ U_1^{\alpha}(b) \to B \end{array} \right\}$$

*In $\mathscr{R}$, the condition $f(a) \to^* b$ is satisfied (although there is no innermost derivation $f(a) \to^*_{\mathscr{R}} b$), therefore, $A$ rewrites to $B$. This derivation is innermost (yet, notice that the conditional evaluation is not). Nonetheless, in $\mathbb{U}_{seq}(\mathscr{R})$ the only innermost derivation starting from $A$ is $A \to U_1^{\alpha}(f(a)) \to U_1^{\alpha}(f(b))$ where the last term is irreducible. In particular, there is no innermost derivation for $A \to^*_{\mathbb{U}_{seq}(\mathscr{R})} B$.*

Nonetheless, we obtain completeness if the transformed TRS is confluent and terminating:

**Proposition 16** (completeness for innermost rewriting)**.** *Let $\mathscr{R}$ be a right-stable DCTRS such that $\mathbb{U}_{seq}(\mathscr{R})$ is confluent and terminating. Then, if $u \to^*_{\mathscr{R}} v$ ($u, v \in \mathscr{T}$) such that $v$ is irreducible (w.r.t. $\mathscr{R}$), then there is an innermost derivation $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v'$ such that $\mathtt{tb}(v') = v$.*

*Proof.* Because of completeness of $\mathbb{U}_{seq}$, there is a derivation $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$. By confluence and termination there is a unique normal form $w \in \mathbb{U}_{seq}(\mathscr{T})$ of $u$ and $v$ in $\mathbb{U}_{seq}(\mathscr{R})$ and there is an innermost derivation $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} w$.

Finally, the assumption that $v$ is a normal form in $\mathscr{R}$ and Lemma 13 imply that $\mathtt{tb}(v) = w'$ for all $w' \in \mathbb{U}_{seq}(\mathscr{T})$ such that $v \to^*_{\mathbb{U}_{seq}(\mathscr{R})} w'$. □

Next, we prove soundness for DCTRSs that are transformed into confluent and terminating TRSs. For this purpose, observe that if a TRS is confluent and terminating, then for every derivation $u \to^* v$ such that $v$ is a normal form there is an innermost derivation $u \to^* v$. This observation can be combined with Theorem 14 that states that innermost, normalizing rewrite sequences in some right-stable DCTRS are sound:

**Lemma 17** (soundness for confluent and terminating TRSs)**.** *Let $\mathscr{R}$ be a right-stable DCTRS such that $\mathbb{U}_{seq}(\mathscr{R})$ is terminating and confluent and let $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$ be a normalizing rewrite sequence ($u \in \mathscr{T}$). Then, $u \to^*_{\mathscr{R}} \mathtt{tb}(v)$.*

*Proof.* $\mathbb{U}_{seq}(\mathscr{R})$ is terminating and confluent, and $v$ is a normal form in the derivation $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$. Therefore, there is an innermost derivation $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$. By Lemma 13 this implies $u \to^*_{\mathscr{R}} \mathtt{tb}(v)$. □

**Theorem 18** (soundness for normalizing rewrite sequences)**.** *Let $\mathscr{R}$ be a right-stable DCTRS such that $\mathbb{U}_{seq}(\mathscr{R})$ is confluent and terminating. Then $\mathbb{U}_{seq}$ is sound for reductions to normal forms.*

*Proof.* Straightforward from Lemma 17. □

The previous theorem is interesting because it shows that [9, Theorem 9] (soundness for reductions to normal forms of confluent DCTRSs), also holds if only the transformed TRS is known to be confluent.

## 5 Confluence of Conditional Term Rewrite Systems

Our goal is to prove that if $\mathbb{U}_{seq}(\mathscr{R})$ is confluent, then also $\mathscr{R}$ is confluent. For this purpose we introduce another soundness property, soundness for joinability.

**Definition 19** (soundness for joinability)**.** *An unraveling $\mathbb{U}$ is sound for joinability for a CTRS $\mathscr{R}$ if for all terms $u, v \in \mathscr{T}$ such that $u \downarrow_{\mathbb{U}(\mathscr{R})} v$ also $u \downarrow_{\mathscr{R}} v$.*

Soundness for joinability is important in connection with confluence because it allows us to prove confluence of a DCTRS via confluence of the transformed TRS.

There is an important connection between soundness for joinability and confluence.

**Lemma 20** (soundness for joinability and confluence)**.** *Let $\mathscr{R}$ be a CTRS such that $\mathbb{U}_{seq}(\mathscr{R})$ is confluent and $\mathbb{U}_{seq}$ is sound for joinability, then $\mathscr{R}$ is confluent.*

*Proof.* Consider two terms $u, v \in \mathscr{T}$ such that $u \leftrightarrow^*_{\mathscr{R}} v$. Since $\mathbb{U}_{seq}$ is complete by Lemma 6, $u \leftrightarrow^*_{\mathbb{U}_{seq}(\mathscr{R})} v$. $\mathbb{U}_{seq}(\mathscr{R})$ is confluent so that $u \downarrow_{\mathbb{U}_{seq}(\mathscr{R})} v$. By soundness for joinability this implies $u \downarrow_{\mathscr{R}} v$. □

It remains to prove soundness for joinability of right-stable DCTRSs for which the transformed TRS is confluent. Theorem 18 shows that confluence and termination of the transformed TRS imples soundness for normalizing derivations. Since every term is terminating this implies soundness for joinability:

**Lemma 21** (soundness for joinability)**.** *Let $\mathscr{R}$ be a right-stable DCTRS such that $\mathbb{U}_{seq}(\mathscr{R})$ is confluent and terminating, and let $u \downarrow_{\mathbb{U}_{seq}(\mathscr{R})} v$ ($u, v \in \mathscr{T}$), then $u \downarrow_{\mathscr{R}} v$.*

*Proof.* Since $\mathbb{U}_{seq}(\mathscr{R})$ is confluent and terminating, $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} v$ implies that there is an irreducible term $w \in \mathbb{U}_{seq}(\mathscr{T})$ such that $u \to^*_{\mathbb{U}_{seq}(\mathscr{R})} w \leftarrow^*_{\mathbb{U}_{seq}(\mathscr{R})} v$. Since $\mathscr{R}$ is right-stable, Lemma 17 implies $u \to^*_{\mathscr{R}} \mathtt{tb}(w) \leftarrow^*_{\mathscr{R}} v$. □

Thus we obtain our main result:

**Theorem 22** (soundness for confluence)**.** *Let $\mathscr{R}$ be a right-stable DCTRS such that $\mathbb{U}_{seq}(\mathscr{R})$ is confluent and terminating, then $\mathscr{R}$ is confluent.*

*Proof.* By Lemma 21, $\mathbb{U}_{seq}$ is sound for joinability for $\mathscr{R}$. Since $\mathbb{U}_{seq}(\mathscr{R})$ is confluent, Lemma 20 implies that $\mathscr{R}$ is confluent. $\square$

This confluence result is remarkable because it also holds for CTRSs for which $\mathbb{U}_{seq}$ is unsound like the CTRS of Example 5.

**Example 23** (unsound confluent CTRS)**.** *Let us recall the right-stable DCTRS of Example 5 and its transformed TRS.*

$$
\mathscr{R} = \left\{ \begin{array}{l}
\quad a \to c \to e \\
\quad\;\; \searrow\!\!\!\!\!\!\nearrow \;\; \nearrow \\
\quad b \to d \\
\quad k \to e \\
\quad\;\; \nearrow \\
\quad l \\
s(c) \to t(k) \\
\quad\quad \searrow \\
\quad\quad\quad t(l) \\
s(e) \to t(e) \\
g(x,x) \to h(x,x) \\
f(x) \to \langle x,y \rangle \Leftarrow s(x) \to^{*} t(y)
\end{array} \right\}
\qquad
\mathbb{U}_{seq}(\mathscr{R}) = \left\{ \begin{array}{l}
\quad a \to c \to e \\
\quad\;\; \searrow\!\!\!\!\!\!\nearrow \;\; \nearrow \\
\quad b \to d \\
\quad k \to e \\
\quad\;\; \nearrow \\
\quad l \\
s(c) \to t(k) \\
\quad\quad \searrow \\
\quad\quad\quad t(l) \\
s(e) \to t(e) \\
g(x,x) \to h(x,x) \\
f(x) \to U_1^{\alpha}(s(x),x) \\
U_1^{\alpha}(t(y),x) \to \langle x,y \rangle
\end{array} \right\}
$$

*The transformed TRS $\mathbb{U}_{seq}(\mathscr{R})$ is confluent because it is terminating and all critical pairs are joinable. Therefore, by Theorem 22, $\mathscr{R}$ is also confluent.*

Although termination of the transformed TRS seems to be a major limitation, [11] proves that for an unraveling similar to $\mathbb{U}_{seq}$, (weakly-)left-linearity (which implies soundness) and confluence of the transformed TRS implies confluence of the original CTRS. Currently it is not known whether Theorem 22 also holds for DCTRSs that are transformed into non-terminating and non-left-linear TRSs.

# 6 Conclusion

## 6.1 Summary

Transformations have been used as a tool to prove termination and confluence of conditional term rewrite systems for a long time. For confluence the problem is that the rewrite relation of the transformed system may give rise to rewrite sequences that are not possible in the original system, i.e. the transformation may not be sound.

We use the so-called sequential unraveling, a simple transformation for deterministic CTRSs that was introduced in [18] based on [14].

Recent results (e.g. in [11]) show that confluence of the transformed system (using the sequential unraveling) implies confluence of the original system if the transformation is sound. There are many syntactic restrictions like (weak) left-linearity that imply soundness, yet, for non-left-linear CTRSs for which the transformation is not sound there are no such results yet. Lemma 17 shows that if the transformed system is terminating and confluent, normalizing derivations are always sound. This result is interesting because a similar result was shown in [9] for confluent CTRSs.

This lemma holds because innermost rewrite sequences in the transformed system are always sound (Theorem 14). Since soundness for normalizing derivations implies soundness for joinability (which implies soundness for confluence) we finally can show that a right-stable, deterministic CTRS is confluent if the transformed TRS is confluent and terminating (Theorem 22).

It should be pointed out that it is not yet known whether termination is really needed in this result. If there is a counterexample for this we know that it must be non-left-linear, non-terminating and confluent.

### 6.2 Related Work and Perspectives

In [11], we presented a confluence criterion for CTRSs based on soundness and confluence of the transformed system for an unraveling similar to $\mathbb{U}_{seq}$. [17] contains a similar result for the structure preserving transformation of [21].

Yet all these results have in common that they require some syntactic criterion like (weakly) left-linearity of the CTRS that implies soundness. Theorem 22 is a significant improvement to these results because it is also applicable to non-linear CTRSs for which the transformation is unsound.

There are many confluence results for CTRSs in the literature and one similar result is [2, Theorem 4.1], stating that every strongly deterministic TRS that is quasi-reductive and has joinable critical pairs is confluent. This result does not use transformations but it can be seen that critical pairs in the CTRS correspond to one or more critical pairs in the transformed system while termination of the transformed TRS implies quasi-reductiveness [20]. Hence, it subsumes Theorem 22.

Yet, Theorem 22 has some advantages over [2, Theorem 4.1]. In particular, it does not use the framework of conditional rewriting. Checking for joinability of terms in CTRSs is easier in the transformed unconditional TRS which is important for automated confluence proofs.

The main result does not extend any previous results but rather is a novel approach to prove confluence. It uses a simple transformation and a very general proof structure. Hence, the result might be improved in the future e.g. by relaxing the requirements for confluence or termination. Termination is only needed for two purposes: To show that for every normalizing rewrite sequence there is also an innermost rewrite sequence, and to prove that soundness for normalizing rewrite sequences implies soundness for joinability.

Finally, adapting the result to more complex transformations that have better properties towards preserving confluence (in particular *structure-preserving transformations*, most notably the transformations of [1] and its extension to DCTRSs in [10]) might improve this result further.

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