

Computer Assisted Proofs and Automated Methods in Mathematics Education

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This survey paper is an expanded version of an invited keynote at the ThEdu'22 workshop, August 2022, in Haifa (Israel). After a short introduction on the developments of CAS, DGS and other useful technologies, we show implications in Mathematics Education, and in the broader frame of STEAM Education. In particular, we discuss the transformation of Mathematics Education into exploration-discovery-conjecture-proof scheme, avoiding usage as a black box. This scheme fits well into the so-called 4 C's of 21st Century Education. Communication and Collaboration are emphasized not only between humans, but also between machines, and between man and machine. Specific characteristics of the outputs enhance the need of Critical Thinking. The usage of automated commands for exploration and discovery is discussed, with mention of limitations where they exist. We illustrate the topic with examples from parametric integrals (describing a “cognitive neighborhood” of a mathematical notion), plane geometry, and the study of plane curves (envelopes, isoptic curves). Some of the examples are fully worked out, others are explained and references are given.

1 Introduction

1.1 A long history made very short, with a personal touch.

During the 1980's, a computer-assisted proof was not always easily accepted by the community. At the beginning of the era of machine-assisted computations, only numerical algorithms existed, allowing approximate results. These algorithms could not always provide all the solutions to a given problem.

Later, algorithms for symbolic computations began to be developed and implemented. Sometimes the researcher had to write in a general language a specific program for his/her problem. M. Schaps did in the 80's for classification of generic associative algebras [38, 39] and wrote programs for computing Hochschild cohomology groups and groups of automorphisms of given algebras, each one allowing checking the validity of the results of the other one. Vanishing of the homology group ensures that the given algebra is rigid, whence defines a component in the variety \mathbf{Alg}_n which parameterizes the n -dimensional associative algebras. For local algebras, i.e. algebras of the form $\mathbb{F}[X]/\mathcal{I}$, where \mathcal{I} is an ideal in the polynomial ring $\mathbb{F}[X]$, we began using computations of Gröbner bases. The project yielded the classification of generic algebras in dimensions 6, 7 and 8; see [38]. Smaller dimensions were studied previously.

Other researchers in the world were already working using computer computations, but at that time a disclaimer was sometimes added about the non-responsibility of the journal regarding the computations. The computations were time-consuming (for one local algebra, the computation could take 20 minutes, with a 4.77Hz PCs. In order to read the output we had to print it (on wide continuous sheets) and we stored the printouts in case somebody would request to check them.

Development and usage of software did not remain the exclusive property of researchers. Of course, as long as computers were too big and too expensive, they could not enter the regular classroom. With

the first hand-held devices and the personal computers, things changed. Hand-held calculators worked numerically, but quite quickly algorithms for symbolic computations were developed, and implemented in a variety of devices. The Derive software was contained on a small floppy disk, a consequence of the choice of algorithms. For example, the computation by Derive of definite integrals is based on a theorem, which is both easy to prove and rarely presented in textbooks [24]. In [69], we proposed to Derive a parametric improper integral. Derive computed it immediately with the general parameter; at that time, no other system could do it, but for small integer values of the parameter. Later a “clone” of the software was implemented in the TI92. Today, GeoGebra has versions for PCs, iPads and smartphones. In the recent period, mobile versions of the different mathematical software allowed the development of outdoor mathematical activities [6, 5].

Classical methods of teaching mathematics led sometimes teachers to think that mathematics are a fixed domain of knowledge. Even teacher trainers eventually claim this. But the world has changed profoundly, and the various available technologies transformed mathematics into an experimental domain, and discovering novelties at an early stage of education. P. Quaresma says ([61]): “Scientific research and education at all levels are concerned primarily with the discovery, verification, communication, and application of scientific knowledge. Learning, reusing, inventing, and archiving are the four essential aspects of knowledge accumulation in mankind’s civilization process. In this cycle of knowledge accumulation, which has been supported for thousands of years by written books and other physical means, rigorous reasoning has always played an essential role. Nowadays this process is becoming more and more effective due to the availability of new paradigms based on computer applications. Geometric reasoning with such computer applications is one of the most attractive challenges for future accumulation and dissemination of knowledge.”

In this paper, we relate mostly to the two kinds of mathematical software, namely Computer Algebra Systems (CAS) and Dynamic Geometry Systems (DGS)¹. Actually, the distinction between them fades more and more. For example, GeoGebra began as a DGS, but developed in other directions, and a CAS called Giac [52] is embedded in it. New features include also tools for Augmented Reality, allowing outdoor activities, among others; see [12, 53].

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1.2 Computer Algebra Systems.

Decades ago, advanced specialized programs have been written, such as GAP - an acronym for Groups, Algorithms, Programming for Group Theory (its scope is much broader now), FeliX for Number Theory, Macaulay2 for Algebraic Geometry, CoCoA for Computations in Commutative Algebra, etc. All these are examples of Computer Algebra Systems (CAS). General purpose CAS began also to integrate various mathematical fields, both symbolic and numerical algorithms; they are now powerful multi-domains assistants. The web offers also an interactive usage of platforms, where different CAS work in the background; this is the case with the education oriented platform WIMS.

A core feature of a CAS is the ability to manipulate symbolic mathematical expressions, in a way similar to the traditional manual computations. Of course, the syntax of the commands can vary and some CAS may be thought as more user-friendly than others. This can be an issue when working with students.

¹There exist other kinds of software, proof assistants and theorem provers, whose importance cannot be overemphasized. We refer the interested reader to recent contributions to ThEdu, also in this volume

Among the needed features are a programming language (enabling the user to enter his/her own algorithms), an interpreter and a simplifier. A **simplify** command, with several options is important (pattern recognition may not be enough to lead to use an efficient algorithm). Note that for a given computation, the output may be quite different from what the user would have obtained by hand (if possible), and that two different packages may give different outputs. Simplifying helps the user to understand that different output formats may determine the same mathematical object. Some examples in Section 4 illustrate this fact.

Algorithms for Calculus, Linear Algebra, and other domains have to be used by students and teachers. Therefore, the CAS must include a large library of algorithms and special functions. We refer to a Wikipedia page² for a more exhaustive list of requested features, including some abilities of interest to the developers. The first list on that page does not mention the plotting features: visualization, plotting, animations. They appear in a second list, not less important. We illustrate this in Section 4.

The Wikipedia page adds that a CAS should also include a programming language, allowing users to implement their own algorithms, arbitrary-precision numeric operations, exact integer arithmetic and number theory functionality, editing of mathematical expressions in two-dimensional form, plotting graphs and parametric plots of functions in two and three dimensions, and animating them, drawing charts and diagrams, APIs for linking it on an external program such as a database (we discuss later the need for a dialog between different systems), or using in a programming language to use the computer algebra system, string manipulation such as matching and searching, add-ons for use in applied mathematics such as physics, bioinformatics, computational chemistry and packages for physical computation, etc. Some CAS include graphic production and editing such as computer-generated imagery and signal processing as image processing and sound synthesis. DGS may fulfill a certain number of these requirements.

As a visualization of the large variety of existing packages, Figure 1 shows the Google answer to a search for Computer Algebra System.



Figure 1: Google visual answer to a search for a CAS

The danger is to use a CAS as a black box. We mean that the user writes a command, without any idea how it works. He/she enters the data, receives an output and relies on it. It is common sense that an educator will not go this way.

1.3 Dynamic Geometry Systems.

Later, other packages called Dynamic Geometry Systems (DGS) appeared: Geometric SketchPad, Cinderella, Cabri Geometer, GeoGebra, etc.. The needs for programming did not disappear, but a core feature is interactivity. The man-and-machine interaction is different from what is generally offered by a CAS. A Dynamic Geometry System (DGS) provides ways of representing and manipulating geometric objects that are not possible with traditional paper-pencil work, using a compass and straightedge. These various environments provide opportunities for students to explore geometric objects, to measure

²https://en.wikipedia.org/wiki/Computer_algebra_system

objects on the screen and outdoor. They can help students to develop different understandings of many properties and theorems. With the apparition of dragging and measuring offered by a DGS, teachers were sometimes reluctant to use them, as they feared that the role of the proof will fade. Mariotti [55] reflected on these fears, but proposed a more positive view. Her reflection was based on work with the Cabri software.

A DGS offers both button-driven commands and written commands, using a syntax similar to what exists in a CAS. A command has eventually a written version and a button, sometimes with slightly different affordances. For example, in GeoGebra's button for attaching a point to an object has a version as a written command working in cases the button does not. As a whole, this double feature buttons-written commands contribute to the software's user-friendliness.

The main dynamical features are the dragging points and slider bars. The basic objects are free points, and other constructions depend on these free points. Dragging a free point with the mouse induces automatically changes in every object depending on it. For example, the midpoint I of a segment AB depends on the endpoints A and B . Dragging A (or B , one at a time) with the mouse changes I . A slider bar corresponds to a real parameter; it allows to define and dynamically change objects depending on this parameter. For multi-parameter constructs, a slider is defined for each parameter (generally the software requests this automatically). Using sliders, animated graphs and constructs are available. Dragging and sliders transformed Geometry into an experimental field. The main words in teaching Geometry are now exploration and experimentation. Changes using dragging and sliders, provide infinitely many examples of the situation under study. This is not a proof, but it provides conviction. Moreover, a conjecture can be enounced, and then has to be proven. The sequence exploration-conjecture-proof is ubiquitous when working with a DGS.

Animations are also offered by CAS, with an important difference. With a CAS, an animation is defined by a specific command with some options (size, number of frames per second, etc.). Then the animation is run without an human intervention. The human can look at what happens, sometimes can intervene, but this is minor. An animation with a DGS can be almost automatic: the Animation On option available with a slider is an important feature, sometimes accompanied by the Trace On option for the object under study. It can be mouse-driven, i.e. driven by the human. Examples are given in Section 4. The interested reader can also refer to a dedicated Wikipedia page³.

Mastering the moves with the hand makes the software some kind of a prothesis for the human; Debray says that being able to build a prothesis is what makes the man a human [45]. Maybe do we deal now with an augmented human? We prefer to speak about a human, totally human, with a strong digital literacy.

1.4 Core issues with man-and-machine and machine-and-machine interaction

The availability of both CAS and DGS opened large new fields of study, in research and in education. Their sets of respective affordances are distinct, with a non empty intersection. Important results can be derived by networking with the two kinds of technology [64, 32], one of them being more powerful geometry and dynamics and the other having stronger algebraic capabilities. After all, the study of a mathematical object is based on different registers of representation (algebraic, numerical, geometric, etc.) [48]. For example, some packages are devoted to 2D only or to 3D only. GeoGebra has both possibilities, the 2D window being fully synchronized with the xy -plane in the 3D window. Of course, they are also synchronized with the algebraic window. Considering different representations of the same

³http://en.wikipedia.org/wiki/Interactive_geometry_software

object, i.e., multiple points of view, enable on the one hand to build bridges between mathematical fields (v.i. Section 2), and on the other hand, to derive new theorems or new proofs of classical theorems (such as in [40]). Until now, in order to have benefit of both environments, CAS and DGS, we generally need to copy-paste from one system to the other. The (lack of) communication between CAS and DGS has been addressed for a long time [64, 65, 32], we wish that solutions will be found in the next future. Actually, one of them was the embedding of a CAS into GeoGebra [52], but as of today DGS and CAS are still different. In the recent years, features of dynamic geometry entered CAS and core CAS components have been implemented into DGS [52]. Work is still needed to have full integration of all the possibilities.

Besides this machine-machine communication and collaboration, new behaviours appeared in man-and-machine communication. This had also a great influence on communication and collaboration between humans. Shall we mention that the communication between humans and between humans and machines developed new aspects during the Covid-19 crisis [29]?

The joint usage of the different technologies, including websurfing and data mining, enable huge advances in the various components of the so-called 4 C's of 21st Century Education: Communication, Collaboration, Critical Thinking and Creativity [20]. The two first C's are generally considered between humans, but man-and-machine and machine-and-machine communication and collaboration are crucial. In [33], we propose even a 5th C, namely Curiosity, a must for human's proceeding further. New didactic situations can be considered and new ways to deal with them. Mathematics Education has been transformed from the traditional Definition-Theorem-Proof scheme into an exploratory domain. The sequence can be now Exploration-Discovery-Conjecture-Proof (or disproof).

The 4C's are the conceptual basis of the STEAM Education approach. It can be illustrated by examples and activities in Abstract Algebra, Combinatorics together with Integrals, the study of Plane Algebraic Curves and Algebraic Surfaces and also Mathematics and Arts. We refer to the papers in [63].

1.5 Numerical vs symbolic representations

Between the symbolic data and the plot, a lot of numerical data is computed, eventually displayed on demand. Numerical data induces also the need to master the approximations [30]. This numerical data is obtained, after choosing a mesh (in 3D) or a partition of the interval of definition (in 2D), by interpolation. This may lead to strange plots [78, 79]. This problem can be overcome either with options for discontinuity or using a non standard mesh (Maple proposes about 20 of non standard meshes for 3D plots).

For other needs, the differences between numerical and symbolic algorithms led GeoGebra's developers to propose two kinds of output, for example, to the **Relation** command⁴. Figure 2 shows screenshots of the output. The regular Relation command provides an answer based on numerical data only. The "More" button runs other algorithms, that time symbolic, the output is more precise. It is an efficient tool in Education, providing assistance to the teacher in developing new activities.

2 Cognitive neighborhood of a mathematical notion

Either with a single student or with a class, the teacher faces some endeavours

⁴We used here a beta-version of *GeoGebra Discovery*, a package developed by Z. Kovács, working on the basis of the regular GeoGebra version. It contains a more advanced version of the **Relation** command. The package is freely downloadable from <https://github.com/kovzol/geogebra-discovery> (check there for the last updated version).

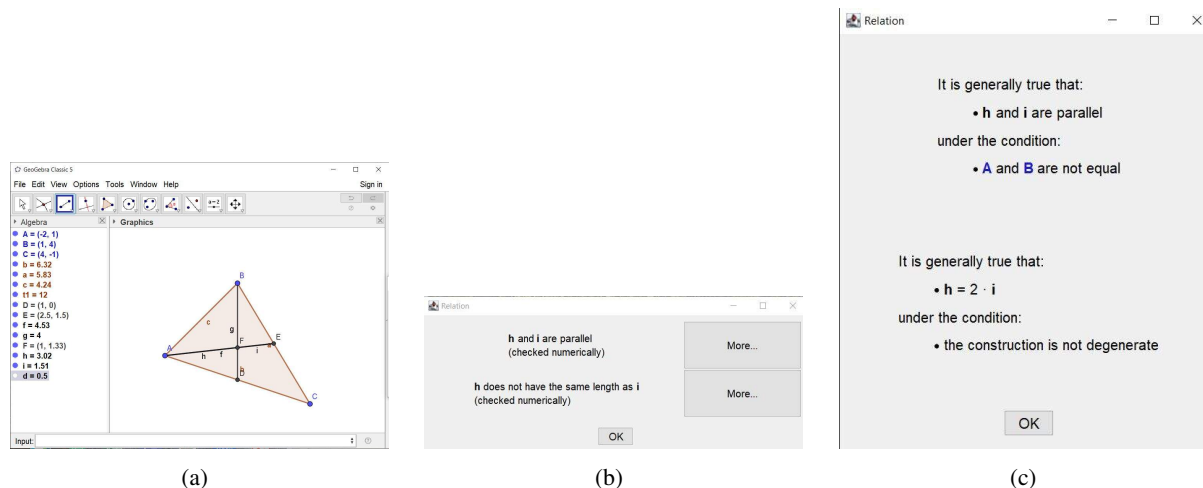


Figure 2: Pop-ups offered by the Relation command

1. Stimulate students' curiosity for interlaced techniques, using more than one of the newly available technologies.
2. Make Mathematics more attractive, and show it as a living field of knowledge by discovering new tracks.
3. Discover links between apparently different fields. In a traditional curriculum, courses are generally taught as separate topics, often without bridges between them. For example, some teachers are reluctant to introduce example for Calculus into a course in Linear Algebra, despite the natural structure of a vector space of the set of continuous (resp. differentiable) functions over an interval in \mathbb{R} .
4. Explore other mathematical objects "looking like" (Consolidation?) the mathematical object of study. Such a task is generally built by the teacher, i.e. in this context, the teacher is active and creative.

An advanced step in teaching integrals is provided by definite integrals depending on one (ore more) parameter. This is a more abstract situation than ordinary definite integrals, important in applied situations [42]. The study of parametric definite integrals provides intra-mathematics connections: integration, combinatorics, applications to physical problems, and they may be strengthened by topics in history of mathematics. The mathematical notions, the bridges between them and the instrumented techniques, accompanied by an appropriate technological discourse (a term coined by Artigue [3]) build what we call a cognitive neighborhood. In the exploration of such a cognitive neighborhood, teachers and students consider mathematical objects "looking like" the original mathematical object of study. Such a task is generally built by the teacher, i.e. in this context, the teacher is active and creative; the student reproduces the teacher's working steps. Such activities incite the students to search for related material. The exploration of the student's Zone of Proximal Development (ZPD [73]) builds step by step an always larger neighborhood.

Links to neighboring mathematical topics, to "real-world" situations can be discovered. Here the student is more autonomous and can develop more initiative. Actually, both the educator and the student are creative.

Figure 3 shows such a construct for parametric integrals, as explored in [43, 44].

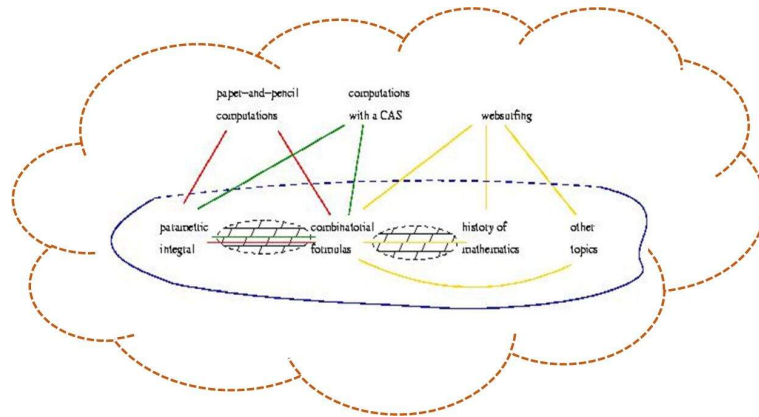


Figure 3: Visualization of a cognitive neighborhood

3 Next step: Automated Deduction in Geometry (ADG)

This field of R & D has seen tremendous developments during the last decades, tens of papers and of conference presentations have been devoted to the advances. Therefore we will mention only a small sample. The interested reader can use the vast bibliography in the few papers that we mention, and also have a look in the proceedings of conferences such as ACA (Applications of Computer Algebra, especially the special sessions on Education, but not only) and ADG conference. In the “far” past, a short survey appeared in [26].

In [10] Botana and Abánades write the following definition of ADG, “as the study and development of computer programs designed to prove geometry theorems”. Then they traced the domain back to 1963 when Gelernter [49] connected it to Artificial Intelligence. They wrote: “However, the real flourishing of the field came in the early 1980’s with the development by Wu of an algebraic method based on Ritt’s characteristic set for proving a restricted set of geometry theorems [76]. Impressive results by several authors using Wu’s method [77, 75, 76] encouraged researchers to consider other algebraic methods, among which those based on Gröbner bases [19] proved to be the most relevant.”

Along the years, other researchers and developers joined the domain. The definitions became more precise, in parallel with the continuous software developments. In 2016, we could find the following description [1]: “By automatic proving of elementary geometry theorems, we refer to the theorem proving approach via computational algebraic geometry methods, as initiated by Wu forty years ago, and popularized by the book of Chou [21]. Roughly speaking, the idea is to provide algorithms, using computer algebra methods, for confirming (or refuting) the truth of some given geometric statement. More precisely, the goal is to decide whether a given statement is generally true or not, i.e. true except for some degenerate cases, to be described by the algorithm.”

The developments of different kinds of software, in the present case DGS and Automatic Provers enabled automated reasoning and dynamic geometry together. The collaboration of the different software revealed very successful. In 2020, Kovács and Recio write [53]: “Along the last half century, automated deduction in elementary geometry has increasingly become one of the most successful achievements in the field of automated reasoning. Along these decades various methods and techniques have been studied and developed for automated proving and discovering of elementary geometry statements. On the other hand, dynamic geometry software systems have emerged, such as Cabri Geometry, C.a.R., Cinderella, DrGeo, GeoGebra, The Geometer’s Sketchpad, Geometry Expert, Geometry Expressions or Kig, with an

ever-increasing presence in mathematics education. Some of them possess a large number of users (over thirty million) all around the world. The merging of these two tools (automatic proving and dynamic geometry) is, thus, a very natural, challenging and promising issue, currently involving logic, symbolic computation, software development, algebraic geometry and mathematics education experts from all over the world.”

A central topic in Geometry is the determination of geometric loci. Numerous results have been obtained, some of them are presented in [9, 8]. We explore some loci in subsection 4.2. Commands for automated exploration and discovery of geometric loci have been developed and implemented in GeoGebra. We illustrate them in subsection 4.2.

In subsection 4.3, we address another topic, namely envelopes of parametric families of plane curves. Despite Thom’s complained [71] (back in 1962) that it disappeared from the syllabus, research did not stop and numerous papers have been devoted to envelopes, software developments and Mathematics Education. A small sample is [15, 14, 39, 13, 27, 33]. For practical situations, approximate methods had also to be developed; for example, see [67, 60]. An automated command has been implemented in GeoGebra for the determination of envelopes of families of plane curves. Examples are given in subsection 4.3. The first computations consist in solving a non-linear system of equations, which is feasible using a CAS. The output consists in a list of parametric presentations of curves, and may need to be simplified, whence the importance of the `simplify` command. Examining the equations together with plotting them reveal that the components are complementary. This is to be proven, generally with algebraic tools. First the parametric presentations have to be translated into polynomials, then Gröbner packages can be applied. These packages may not be implemented in the DGS, and work has to be transferred to another CAS, generally using copy-paste. The obtained polynomials generate an ideal in a polynomial ring. By elimination of the parameter, an implicit equation can be derived for the envelope. This is performed with a package for Gröbner bases computations; see [23, 57] for the theory of Gröbner bases and [68] for rational curves. Using once again the CAS, the obtained polynomial can be factored, revealing whether the envelope is reducible or not. Of course, the CAS offers plotting features, and their output will give a confirmation to what has already been obtained with the DGS.

Different definitions of an envelope are given in [18]; we illustrate them with automated methods in subsection 4.3. Offsets are worked out as loci, the difference between them and envelopes are made clear by using a DGS, as in [28].

The third topic that we will address later is the determination of isoptics curves of plane curves. It has been studied for years [22, 56, 70]. Working in a technology-rich environment enabled new results [35, 37]. More recently inner isoptics have been studied [36]. Some dynamics have been introduced into the study of isoptic in [34]. As of today, no specific command has been implemented for the discovery of isoptics; nevertheless the existing features allow exploration, computations and confirmation of the results. This is the topic of subsection 4.4.

To close this section, we wish to mention the book by Pavel Pech [59]. A chapter is devoted to automatic theorem proving and automatic discovery. The subsequent chapters deal with classical theorems in Geometry by means of polynomials with CoCoA, a freely downloadable software for computing in polynomial rings; see [2].

4 Other examples

We present now a couple for mathematical activities for which technology may have a crucial role. For exploration, discovery and proofs, we use here GeoGebra and Maple.

4.1 Experiment the continuity of a function

Mastering the $\varepsilon - \delta$ - definition is often difficult for a student. Exploration with technology makes things easier to understand. In order to make things easier, more intuitive, we developed an applet with GeoGebra⁵. Figure 4 shows two screenshots. The sliders enable to change independently the value of ε and of δ , which is useful for a first exploration. The student notes that ε determines a horizontal band and δ a vertical one, whose intersection is a rectangle centered at the point $(x_0, f(x_0))$, the question being whether the graph of the function passes fully in the rectangle. Discontinuity appears when ε is quite small and no value of δ enables this. Later, the teacher may propose another applet⁶, where the

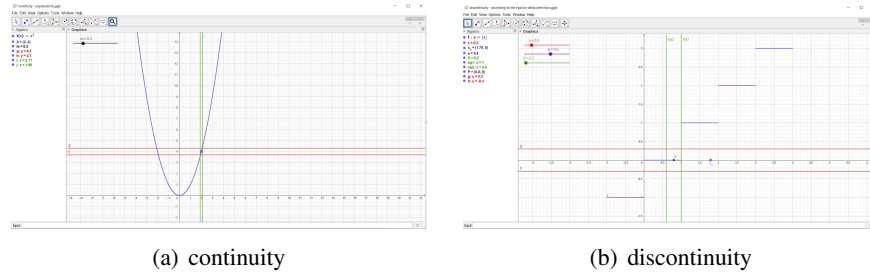


Figure 4: Exploration of the continuity of a function according to the $\varepsilon - \delta$ - definition

dependence of δ on ε is utilized, as shown in Figure 5 and only one slider appears. This applet requires preliminary human work to discover a formula for the dependence of δ on ε . Therefore it is not suitable for exploration, more suitable for illustration, and even be useful for consolidation of the new knowledge. Of course, the best situation is when the student is able to develop such an applet, and does not rely on the teacher for this. A student may develop suitable skills for this. If not the teacher may offer some scaffolding ([58], pages 105-107).

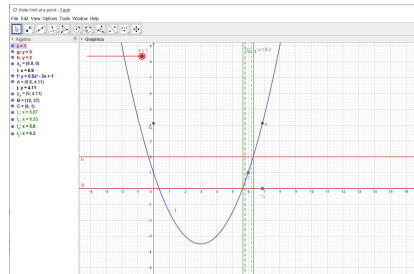


Figure 5: A more advanced step for illustration of continuity

4.2 Exploration of a geometric locus

GeoGebra has several versions of an automated command for the determination of a geometric locus. According to the situation, whether the tracer is geometrically related to the mover or the tracer has to

⁵Available at <https://www.geogebra.org/m/bkj8872v>

⁶<https://www.geogebra.org/m/zsb6p2vd>

fulfill a given condition (a Boolean expression), the output has different forms: it can be a plot of a curve or a plot together with an implicit equation. Recent developments provide plots of regions in the plane⁷.

4.2.1 Classical and less classical constructions.

Take two points A and B in the plane. The geometric locus of points M such that the ratio AM/BM is equal to k is the perpendicular bisector of the segment AB if $k = 1$, and a circle otherwise. This can be explored using the Locus command and a slider for letting the ratio vary (see <https://www.geogebra.org/m/vqpkd57t>). For each value of the parameter an implicit equation is provided.

The applet <https://www.geogebra.org/m/v2kdhrus> shows the construction of the loci of the incenter and excenters of a triangle, where two vertices are fixed and the third one moves along a line; see Figure 6(b). Here the version of the command is `Locus(<Point Creating Locus line >, <Point>)` and the output is a plot of the locus, but no equation is found.

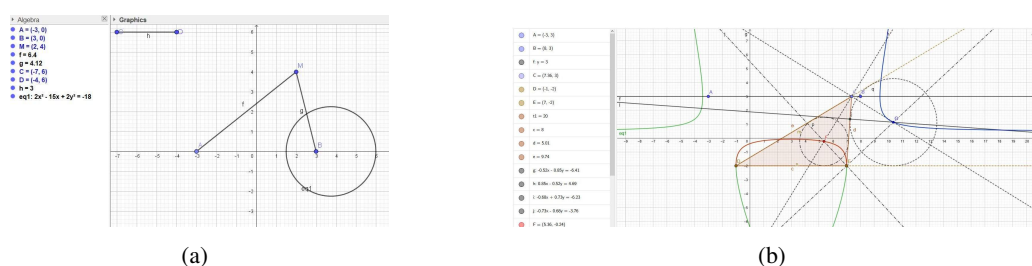


Figure 6: Geometric loci

The same kind of construction can be applied looking for the geometric locus of points M such that the angle $\angle AMB = \theta$ fro given θ . The output is a circle. In this case, the teacher has to draw the students' attention that, actually, the geometric locus is the circle without the two points A and B , as if $M \in \{A, B\}$, there is no angle at all. A good opportunity to educate to Critical Thinking [20].

4.2.2 Beyond the directrix of a parabola.

Let \mathcal{P} be the parabola whose equation is $y = x^2$. Denote by B a point on \mathcal{P} and by C a point on the y -axis of \mathcal{P} . Let C' be the image of C by the reflection about the tangent to \mathcal{P} at B . Two examples are on display in Figure 7. The `Locus(<Point Creating the Line>, <Point>)` creates the dotted plot without providing an equation. A database of plane curves may be searched in order to determine which curve has been obtained, and to try to find a suitable equation for it. Students observed that the limiting case between smooth curves and curves with a double point is when the point C is the focus of the parabola \mathcal{P} , in which case the geometric locus of C' is the directrix of the parabola.

4.2.3 A crosscurve.

Let be given a circle centered at the origin. We explore the geometric locus of the midpoint of the intercepts with the axes of the tangents to the circles. An automated command is available in GeoGebra, the Locus command. Several version are proposed, and the student has to choose among them. A snapshot of a session⁸ is displayed in Figure 8: a purely geometric construct is performed, enabling the

⁷Derive did it, representing inequalities in 2 variables graphically.

⁸<https://www.geogebra.org/m/bmdfd3hm>

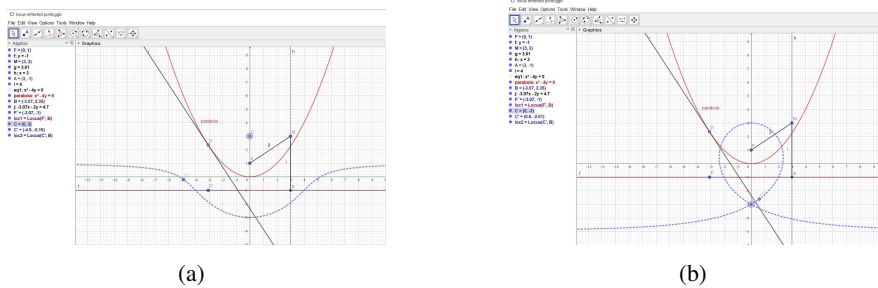


Figure 7: The locus of reflected parabola y -axis point about tangents

usage of the Locus (<Point>, <Point>) command. Here the tracer is the midpoint and the mover is the point on the circle (for details on the terminology see [54]). Identification of the obtained curve may not

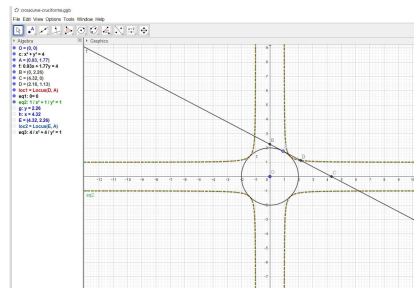


Figure 8: Automated exploration of a geometric locus

be easy. In our example, the circle has radius 2. The geometric locus has equation

$$\frac{1}{x^2} + \frac{1}{y^2} = 1 \quad (1)$$

This equation has been derived by either by hand or with a CAS. In reverse direction, it could have been discovered in a catalogue of classical plane curves⁹, but this an unilluminating search. Maybe a websearch based on a picture will help, we did not try it.

Remark 1 Equation (2) can be transformed into a polynomial equation, namely

$$x^2 + y^2 = x^2 y^2. \quad (2)$$

This may be an indication that packages for polynomial computations may be useful for the current exploration. Here too, Critical Thinking has to be applied as the domains of validity are different.

4.2.4 A sextic.

A sextic is a curve of degree 6. A *sextic of Maclaurin* is the geometric locus of the point of intersection of two lines, which are each revolving at constant rates about different points called poles. Figure 9

⁹Such as the Mathcurve website. The curve appears there at <https://mathcurve.com/courbes2d.gb/cruciforme/cruciforme.shtml>

shows three examples, according to the ratio r of angular velocities being equal to $1/3$, -1 or 2 (or, equivalently, to $1/2$). In the 1st case, the obtained curve is a strophoid. Note the different numbers of components. These are screenshots from a GeoGebra applet¹⁰; exploration is made possible by the definition of several sliders.

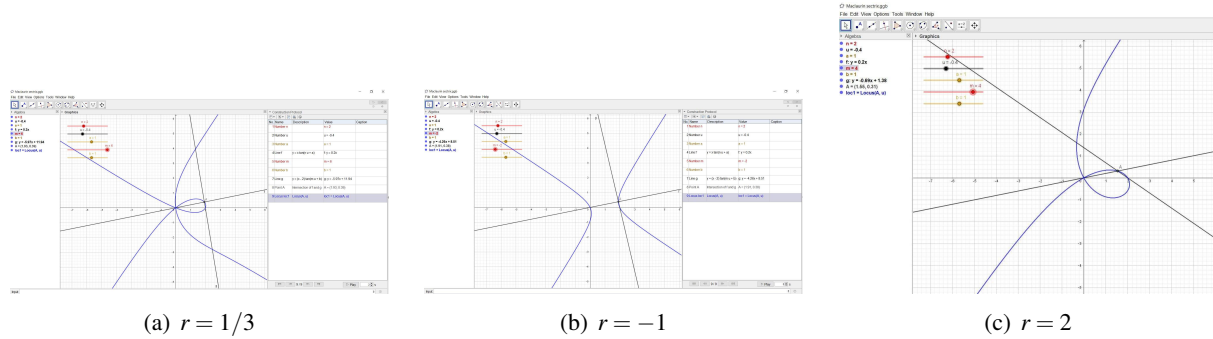


Figure 9: sextrices of MacLaurin

Catalogues of plane curves show them as separate objects. The slider provides continuous changes in the values of the parameter, leading to discover that several seemingly different curves belong actually to a single family. The apparition of a straight component requires more theoretical developments.

4.3 Envelopes

There exist 4 different definitions of an envelope; see [18] (Chap. 5). Kock [50] gives three of them, calling them respectively synthetic, impredicative and analytic. We will explore the examples, according to the different definitions, and see how automated methods are applicable. More advanced examples may be found in [40, 27, 28]; dedicated databases provide also numerous examples.

Definition 2 [Synthetic] Let \mathcal{C}_u be a family of real plane curves dependent on a real parameter u . The envelope \mathcal{E} is the union of the characteristic points M_u , where the characteristic point M_u is the limit point of intersections $\mathcal{C}_u \cap \mathcal{C}_{u+\varepsilon}$ as $h \rightarrow 0$. In other words, the envelope is the set of limit points of intersections of nearby curves \mathcal{C}_u .

Definition 3 [Impredicative] The envelope \mathcal{E} is a curve such that at each of its points, it is tangent to a unique curve from the given family. The locus of points where \mathcal{E} touches \mathcal{C}_u is called the \mathcal{E} -characteristic point M_u .

Definition 4 [Analytic] Suppose that the family of curves \mathcal{C}_u is given by an equation $F(x, y, u) = 0$ (where u is a real parameter and F is differentiable with respect to u), then an envelope \mathcal{E} is determined by the solution of the system of equations:

$$\begin{cases} F(x, y, u) = 0 \\ \frac{\partial F}{\partial u} F(x, y, u) = 0 \end{cases}, \quad (3)$$

In Definition 4, the envelope is described as the projection onto the (x, y) -plane of the points, in the 3-dimensional (x, y, u) -space, belonging to the surface with equation $F(x, y, u) = 0$ and having tangent plane

¹⁰<https://www.geogebra.org/f/kwzvga3rpn>

parallel to the u -axis (or being singular points and, thus, not having tangent plane, properly speaking). See [18], p.102. Note that the analytic definition 4 is the only one given by Berger [7](sections 9.6.7 and 14.6.1) and by Rovenski [66]. This last book gives details on how to work out envelopes using Maple.

Remark 5 We chose to denote the parameter by u , as in GeoGebra t has a special role.

4.3.1 An envelope of a family of lines

We consider a family of lines given by the equation $F(x, y, u) = 0$, where $F(x, y, u) = x + uy + u^2$. Such a family has been studied with Derive in [35]; we study it here in order to show the usage of automated commands, if possible. Figure 10(a) shows a first exploration, using the slider and Trace On¹¹. The speed of the mouse on the slider conditions the density of the lines in the output. Not plotting too many lines retains the visual impression of a family of lines, and not a fully colored area.

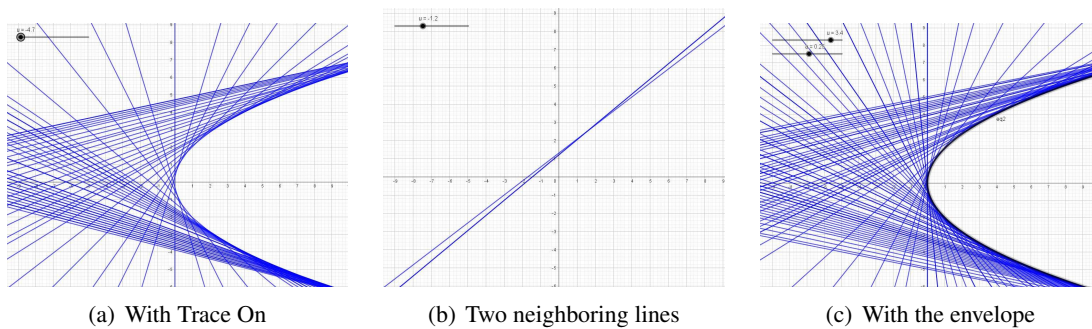


Figure 10: Exploration of a family of lines

Figure 10 shows a GeoGebra session using its embedded CAS to determine characteristic points, according to Definition 2. These points have coordinates $M_u = (u^2 + u\varepsilon, -2u - \varepsilon)$. We have: $\lim_{\varepsilon \rightarrow 0} M_{u+\varepsilon} = (u^2, -2u)$. This is a parametric presentation of the parabola whose equation is $y = -4x^2$. Figure 10(b) models two neighboring lines L_u and $L_{u+\varepsilon}$. Their intersection is given by:

$$\begin{cases} x + uy + u^2 = 0 \\ x + (u + \varepsilon)y + (u + \varepsilon)^2 = 0 \end{cases} \quad (4)$$

The obtained parabola is plotted in Figure 10(c).

Applying this method in the general case shows that Definition 2 implies Definition 4; see [18, 40]. GeoGebra has a command for computing envelopes, but its syntax does not fit the above problem. The command has syntax `Envelope(<Path>, <Point>)`; this means that the command has been constructed to determine the envelope of a family of curves (the Path) depending geometrically on a tracer (the Point). It does not fit a purely algebraic setting, as in the example above.

4.3.2 A nephroid as an envelope.

Let \mathcal{U} be the unit circle centered at the origin. Consider the point A on \mathcal{U} and a circle \mathcal{C}_A centered at A and tangent to the y -axis. If it exists, denote by \mathcal{E} the envelope of the family of circles. Figure 11(a)

¹¹<https://www.geogebra.org/f/qt5pdbah42>. Note that the increment for the slider a has to be put to 0.05 in order to have an accurate plot.

shows a screen shot of a first experimentation with GeoGebra¹², using Trace On for the circles.

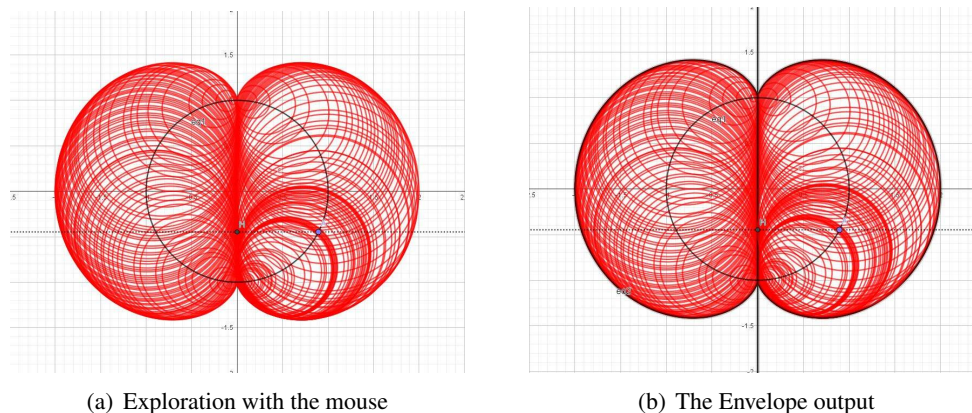


Figure 11: Exploration of a nephroid as envelope of circles

Figure 11(b) shows a screenshot of the applet after usage of the Envelope command.

- Draw a line through A perpendicular to the y -axis;
- Determine the point of intersection of this line with the y -axis, denoted by H (this may be different with every new experimentation);
- Plot a circle whose center is A and which passes through H .

The Envelope command is effective. The output has two components:

- A curve plotted in the geometric window (the dotted curve of the figure);
- An implicit equation in the algebraic window.

Note that the equation is of degree 7. Actually the polynomial is reducible and can be written as follows:

$$x(4x^6 + 12x^4y^2 - 12x^4 + 12x^2y^4 - 24x^2y^2 - 15x^2 + 4y^6 - 12y^4 + 12y^2 - 4) = 0 \quad (5)$$

This means that the result is the union of the y -axis (given by the vanishing of the first term) and a sextic. A quick websearch yields that this sextic is a nephroid. Note that, if the Mathcurve site (<https://mathcurve.com/courbes2d.gb/nephroid/nephroid.shtml>) is read, the equation given there has to be modified, because of the respective roles of the coordinate axes. It should be observed that the y -axis is “too big”. It corresponds to the factor x in the polynomial of degree 7 (Equation (5) obtained in the algebraic window, but the geometric data indicates that only a segment of the axis is relevant. The superfluous parts are a consequence of the algebraic computations which determine a closure in the Zariski topology.

The question is now: can we improve the automated work in order to obtain the “true” answer? The answer is yes, as we show now.

Consider the following parametric presentation for the unit circle $(x, y) = (\cos u, \sin u)$, $u \in [0, 2\pi]$. Then a generic equation for the circles is

$$(x - \cos(u))^2 + (y - \sin(u))^2 - \cos^2 u = 0. \quad (6)$$

¹²<https://www.geogebra.org/m/fxjgmkqu>

Denote by $F(x, y, u)$ the left hand-side in Equation (6). In this case, the system of equations of Definition 4 reads as follows:

$$\begin{cases} (x - \cos(u))^2 + (y - \sin(u))^2 - \cos^2 u = 0 \\ x \sin u - y \cos u + \sin 2u = 0 \end{cases} \quad (7)$$

Solving the system with Maple, we obtain the following output:

$\{x = 0, y = \sin(u)\}, \{x = 2*\cos(u)^3, y = -2*\sin(u)^3 + 3*\sin(u)\}$

The first component describes a segment on the y -axis, which is really accurate. The second component is a parametric presentation of the nephroid. Figure 12 shows two screenshots of an animation performed by Maple: For the reader's sake, we include here the Maple code. Note that the axes have

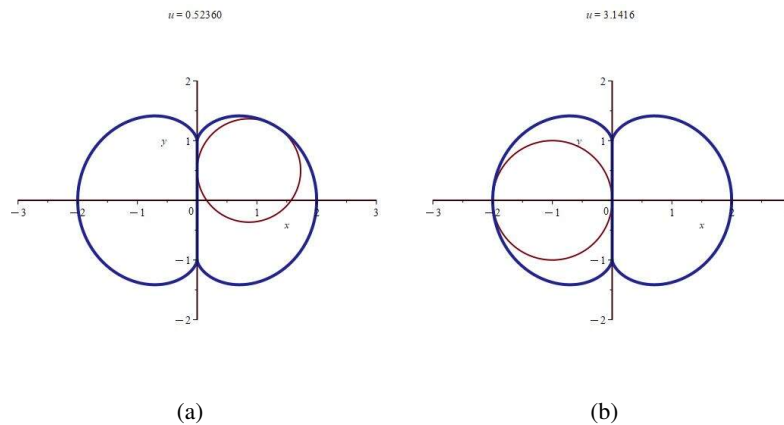


Figure 12: Plots of circles and together with the envelope with Maple

been defined explicitly in order to have a more readable plot.

```
F := (x - cos(u))^2 + (y - sin(u))^2 - cos(u)^2;
derF := diff(F, u);
expand(%);
solve({F = 0, derF = 0}, {x, y});
neph := plot({[0, sin(u), u = 0 .. 2*Pi],
  [2*cos(u)^3, -2*sin(u)^3 + 3*sin(u), u = 0 .. 2*Pi]},
  scaling = constrained, thickness = 4, color = navy);
axes := implicitplot({x = 0, y = 0}, x = -3 .. 3, y = -2 .. 2);
circles := animate(implicitplot, [F = 0], u = 0 .. 2*Pi, thickness = 2);
display(axes, neph, circles);
```

From this point, the derivation of an implicit equation requires applying the methods described in [28, 32, 37, 36], transforming trigonometric functions into rational expressions, then transforming the data into polynomials which generate ideals in a polynomial ring, and finally utilising elimination algorithms.

4.4 Isoptics of plane curves.

Let \mathcal{C} be a plane curve and θ a given angle. The θ -isoptic curve of \mathcal{C} is the geometric locus of points M in the plane through which passes a pair of tangents making an angle equal to θ . For conics and $\theta = 90^\circ$, the isoptic curves are called orthoptics and have been known for a long time. These are the directrix of a parabola, the director circle of an ellipse, and the director circle of a hyperbola (when it exists,

depending on the angle between the asymptotes). On Figure 13(a) the isoptics an ellipse are displayed for two complementary angles. Figure 13 shows the orthoptic curve (i.e. the isoptic for right angles) of a Fermat curve of degree 6.

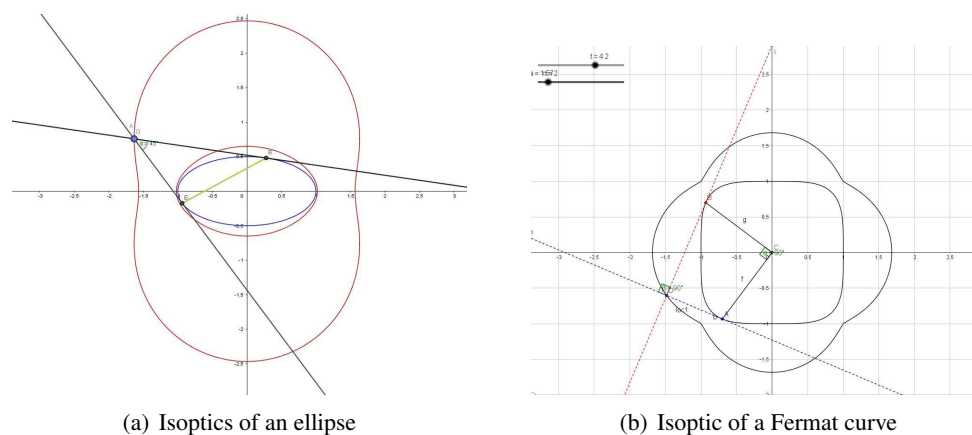


Figure 13: Isoptic curves

For general θ , isoptic curves of conics and of Fermat curves have been studied, for example, in [22, 56, 35, 41, 37]. The study relies strongly on the usage of software, but not on isoptics-dedicated commands. The main tools are solvers for non-linear equations, and Gröbner packages.

In these works, plots of isoptics are provided, but as a consequence of algebraic computations and on a one by one basis. A first step towards automated exploration has been made, by developing tools for automated coloring [31, 34]. These tools provide visualization together of several isoptics. The variations of the parameter are translated into coloring differences. Confirmation is obtained a posteriori, using GeoGebra's dragging of a point on the given curve¹³.

In every case, the availability of the automated command Locus enabled to plot immediately the curve. Moreover, as in other situations, a slider (or more than one) offer the possibility to explore not only one curve at a time, but a family depending continuously on one (or more) parameter. Nevertheless, this command did not provide an implicit equation of the curve. The derivation of an implicit equation requires algebraic machinery. Here a command to solve equations is needed; as mentioned previously, such a command is a standard in any CAS. Of course, it has to contain pattern recognition, as different problems lead to different algebraic settings (polynomial equations, trigonometric equations, etc...).

The reward of such activities is multiple:

- Work is interactive, i.e. it is performed as a dialog between man and machine. This is a nice opportunity to experience the mutual influence between the user and the software, and the teacher can observe a different processes of instrumental genesis among the students; see [72, 3, 4].
- Generally, the catalogues of curves present them as a discrete set of objects. The automated exploration reveals often that different curves belong actually to one larger family, the various curves corresponding to different values of the parameters. The author described such a situation in [28], where the machine-and-machine communication is crucial.

¹³The interested reader can try the applets <https://www.geogebra.org/m/a2zpetsc> for isoptics of an ellipse, <https://www.geogebra.org/m/kvbnpzt3> for inner isoptics of an ellipse, and <https://www.geogebra.org/m/yjgsbbpk> for isoptics of a Fermat curve. Other applets are also available on the site.

- These activities can stand by themselves, but can also be a promo aimed at broadening horizons. When working in class, a couple of students reacted with a great “waoool!” when the curves appeared.

A central issue for the determination of isoptics and envelopes is solving systems of non linear equations. The result consists in parametric representation of plane curves. these parametric equations are rarely rational. Computational work may sometimes been performed in order to transform the expression into rational expressions and afterwards to obtain polynomials. If possible, Gröbner methods (such as Elimination) can be applied to implicitize the parametric presentation. This is not always possible. Even when polynomials can be obtained, Elimination may be too time-consuming or too memory-consuming for the CAS to answer; we experienced that when preparing [51]: the computations have been launched on two different computers with different characteristics, none gave an answer.

5 Conclusions and some thoughts for next steps

We presented activities around automated exploration, discovery and proof using GeoGebra and Maple. Actually a few drops in a vast ocean. First, note that we used the current version of GeoGebra, and the companion package called GeoGebra-Discovery, developed by Z. Kovács. It contains several automated commands, sometimes specific to the package, sometimes an extension of commands existing in GeoGebra. For example, the Relation command exists in GeoGebra and is numerical. The GeoGebra-Discovery version extends it with symbolic algorithms; we showed an example in subsection 1.4. Other systems are also available for the same kind of tasks. For example see [11] on the usage of Sage for automated work. An important decision to be made by the educator is to choose the software which will be used. This choice has multiple faces. First of all, it depends on which systems are available at the teacher’s institution. This is a consequence of pedagogical choices, but not less on financial decision of the administrators. It depends also on the teacher’s literacy with regards to the different available packages. Of course, not every CAS or DGS is suitable for every level of students. Button-driven software may be easier to use than a software which requires mastering the syntax of the commands. Moreover, button-driven commands are often accompanied by a description of the command when right-clicking on the button, a very helpful feature. The existence of an interactive website for examples and tutorials is a strong help for the student.

The level of the students’ background is crucial. Automated methods are not intended to be used as a blackbox, but are intended to incite the user to develop new approaches, and to achieve a more profound insight and understanding of the questions under study. The *black box - white box* issue has been analyzed for example in [46, 47]). Sometimes, the CAS helps to bypass a lack of theoretical knowledge, but then it is important to “go back” and have the students fill the gap and understand what was hidden in the activities and the software usage [25]. Critical Thinking and Creativity have to be at work together. Before presenting the new methods, the teacher has to be informed of the actual level of the students, and then will be able to construct a curriculum adapted to the exploration of their Zone of Proximal Development (ZPD); see [73]. We recall that “the ZPD refers to the learner’s ability to successfully complete tasks with the assistance of more capable other people, and for this reason it is often discussed in relation to assisted or scaffolded learning. The creation of ZPDs involves assistance with the cognitive structuring of learning tasks and sensitivity to the learner’s current capabilities” (Walker, [74]). The author of this survey insists that his students work by pairs. The communication between them and their subsequent collaboration contribute to a reinforcement of the benefits that receive from the scaffolding by the teacher. The student’s “waoow!” mentioned previously was such a consequence, and the mathematical

discovery made on that occasion was an incitement to create something new. That has been done.

The versatility of exploratory tools offered by a DGS and a CAS enable different students in the same class to experience different ways to solve the same problem. For STEAM oriented students, this opens numerous opportunities to create models, animations, etc. We illustrated that in subsection 4.3. The scaffolding offered by the teacher has therefore to be more personally adapted, and the exploration of the student's ZPD becomes more and more personal. Some kind of joint ZPD (for 2 students) is also created. Smartphones, and before that walkmen (who remember them? and other electronic devices, created some disconnection between humans. Here, the fact that students are more free to experience and explore with their personal device, and then share their discoveries with their classmates, makes the class experience richer, more meaningful and more interesting. Outdoor activities contribute also to attract students to learn more mathematics, as these are based on their everyday environment and to their cultural background. The author teaches practical courses on "technology in Mathematics education" aimed either at pre-service or in-service teachers. The technological discourse has always to be adapted, according to the mathematical level, the cultural background and other characteristics of the students. Artigue [3] emphasizes also that the new technological knowledge is an integral part of the new mathematical knowledge acquired by the students. New topics or renewed topics can be proposed and explored, at an earlier stage of the student's cursus than in the past, whence a need to analyse new didactic situations and didactic transposition [16, 17]. All this is part of the new paradigms evoked by P. Quaresma [61] (v.s. Section 1). It is not surprising that the instrumental genesis (see [72, 3, 4], etc.) of every single student and of the class as a whole is totally different every year. Of course, this is relevant also for pre-service engineers.

Finally, we should once again emphasize that automated methods for exploration, discovery and proof, are a new approach aimed at developing new skills and to emphasize a new kind of understanding. New developments are always made. Quoting once again P. Quaresma (op.cit.): "Geometric reasoning with such computer applications is one of the most attractive challenges for future accumulation and dissemination of knowledge." For example, recent works are aimed at the study of inequalities [62] and their plots in the plane. Something was already available with Derive, but modern developments are richer. This kind of developments, together with the personal teaching-learning processes deserve study of a new kind of instrumental genesis. From another point of view, with new achievements for an efficient and fruitful automated dialog between different kinds of software, new pairs CAS-DGS (either distinct or embedded one on the other) provide a new artifact which has to be transformed into an instrument [3, 72]. A new loop in a cognitive-educative spiral.

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