

Automated Discovery of Geometrical Theorems in GeoGebra

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We describe a prototype of a new experimental GeoGebra command and tool, *Discover*, that analyzes geometric figures for salient patterns, properties, and theorems. This tool is a basic implementation of automated discovery in elementary planar geometry. The paper focuses on the mathematical background of the implementation, as well as methods to avoid combinatorial explosion when storing the interesting properties of a geometric figure.

1 Introduction

In this paper we introduce a new GeoGebra command and tool *Discover*, which is available in the development GitHub repository¹. This research is closely related to the former project *Automated Geometer*² (see [1, 2, 3] for further details).

Given a Euclidean geometry construction drawn in GeoGebra, suppose a user wants to know if a given object O has some “interesting features,” such as relevant theorems or properties. This object can be a point, a line, a circle, or something else, although in the current implementation O will always be a point contained in a given construction. Without any further user input, the *Discover* command will then analyze O for its interesting and relevant features, and present them to the user as a list of formulas, as well as graphical illustrations.

For example, let ABC an arbitrary triangle, and let D and E be the midpoints of BC and AC , respectively (Fig. 1). Has D any noteworthy features? Yes: DE is parallel to AB , independent of the position of A , B and C . Indeed, the command $\text{Discover}(D)$ confirms this observation with the output shown in

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¹<https://github.com/kovzol/geogebra-discovery>

²<https://github.com/kovzol/ag>

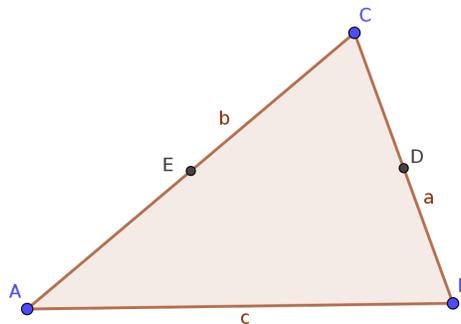


Figure 1: Initial setup for a discovery

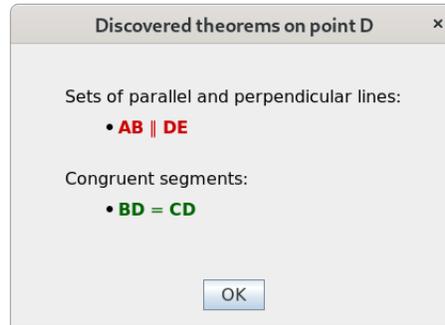


Figure 2: Output window of the Discover command that reports the *Midline theorem*

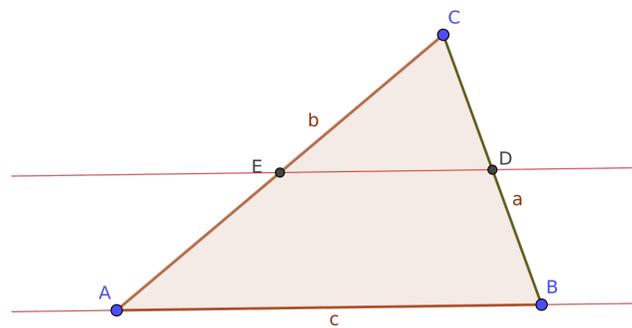


Figure 3: Further output of the Discover command

Fig. 2; GeoGebra adds lines DE and AB in the same color (Fig. 3). (Note, however, that the current implementation of GeoGebra does not report that $2 \cdot |DE| = |AB|$.) Also, the software reports the somewhat trivial finding that the segments BD and CD are congruent, with BD and CD highlighted in the same color. (The fact that “ AE and CE are congruent” is not reported because these segments do not have a direct relationship with the point D .)

This output can be obtained by selecting the Discover tool  in GeoGebra’s toolbox, and then clicking on the point D . This functionality is currently implemented in both GeoGebra versions Classic 5 and 6 and will soon be incorporated into the official GeoGebra versions.

The Discover command calculates its outputs with the following algorithm: First, all points are analyzed to determine whether they are the same as another point. Second, all possible point triplets are examined for collinearity. Third, all possible subsets containing four points on the figure are checked for concyclicity. With knowledge of collinear points, separate lines can be uniquely defined, in order to determine whether they are parallel. Next, congruent segments can be identified by considering the pairs of all possible point pairs. Finally, perpendicular lines are identified. This combination of numerical and symbolic processes provides the results of the Discover command.

Our second example illustrates a more complicated problem. A regular hexagon $ABCDEF$ is given in Fig. 4. Point G is defined as the intersection of AD and BE . In addition, $H = BE \cap CF$, $I = AD \cap CF$. The points G , H and I may have trivial differences in their numerical representations, but in the geometrical

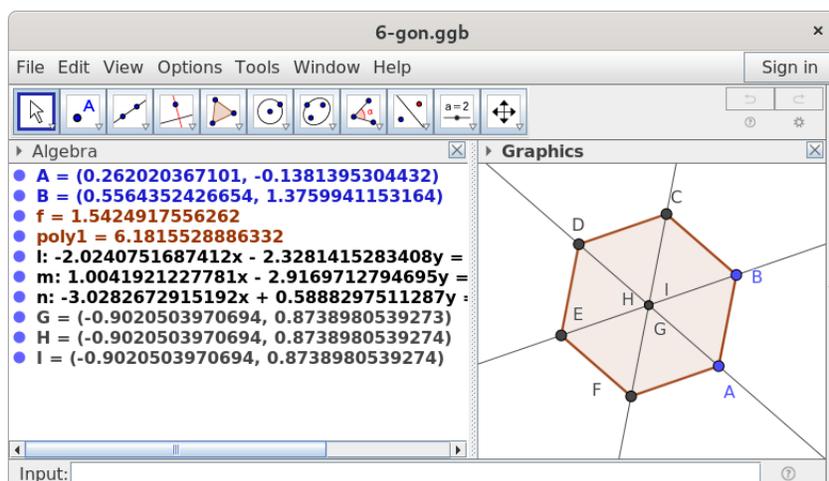


Figure 4: Initial setup for another discovery

sense they should be equal. For example, the y -coordinates of G and H , as computed numerically by GeoGebra, differ only in the least significant decimal place shown. However, the final calculations to prove that they are identical will be symbolic and exact.

To determine any interesting features of point F , we enter the command `Discover(F)`. GeoGebra reports a set of properties in a message box and adds some additional outputs to the initial setup (Fig. 5).

Here we learn that the points G, H and I are all equal, as stated above. We also see that concyclic points are reported as a single item, not as separate data. In addition, parallel line sets and their perpendicular supplementors are classified into three different sets, each colored with the same color. This allows the user to distinguish among the rectangular grids related to the objects of the figure. Finally, there are three sets of congruent segments. This approach in computation and reporting helps avoid combinatorial explosion.

2 Mathematical background

The above mentioned strategies have some similarities to the ones introduced in [4], but here we focus on minimizing the number of objects that have to be compared in the process that practically compares all objects with all other objects.

Our current implementation deals with *points*, *lines*, *circles* and *parallel lines* (or *directions*) and their perpendicular supplementors, and *congruent segments*.

A *geometric point* P is a GeoGebra object, described by the `GeoPoint` class³. While we will not provide a detailed definition of a geometric point, generally speaking, it is an object with a very complex structure containing two real coordinates, several style settings (including size and color, for example) and other technical details that are used in the application. Some geometric points are dependent of other geometric points or other geometric objects—this hierarchy is stored in the set of `GeoPoints`, too.

Independent of the detailed definition of a geometric point, we can still define the notion of *point* in our context.

³See GeoGebra's source code at github.com/geogebra/geogebra for more details.

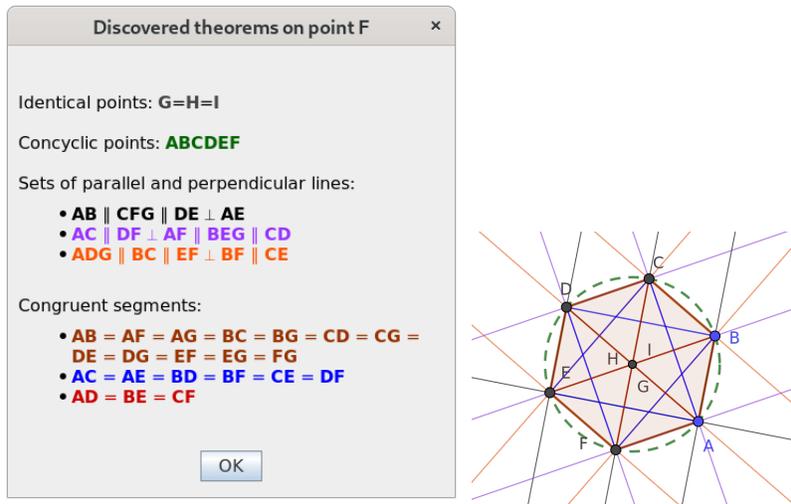


Figure 5: Output window that reports several theorems related to point F , and the geometric output of discovery

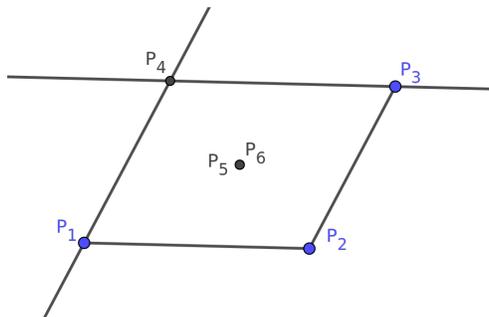


Figure 6: Points P_5 and P_6 are defined as midpoints of opposite vertices of parallelogram $P_1P_2P_3P_4$

Definition 1. A set of geometric points $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ is called a point if for all different $P, Q \in \mathcal{P}$ the points P and Q are identical in general.

Henceforth, unless otherwise mentioned, we will consider points according to the definition above, not as geometric points.

Here, we do not precisely define when two points are identical *in general*. Instead, we will illustrate the concept of point identity with the following example. Consider geometric points P_1, P_2, P_3 and P_4 that form a parallelogram. Now define P_5 and P_6 as the midpoint of P_1 and P_3 , and P_2 and P_4 , respectively. With these definitions, P_5 and P_6 are identical, because the diagonals of a parallelogram always bisect each other. In a dynamic geometry setting like GeoGebra, this simply means that by changing some points of the set $\{P_1, P_2, P_3, P_4\}$, the points P_5 and P_6 will still share the same position in the plane. (See Fig. 6. Here the construction is determined only by the points P_1, P_2 and P_3 : after they are freely chosen, the point P_4 must be dependent and uniquely defined as the intersection of the two parallel lines to P_1P_2 and P_2P_3 , respectively, through P_3 and P_1 .)

Statements considered “generally true” are true in most typical cases, but not 100% of the time, especially in cases of degenerate objects. For example, it is generally true that altitudes of a triangle meet at a point—but not always, since a degenerate triangle “usually” has three parallel “altitudes,”

unless two (or even three!) vertices of the triangle coincide. (See [5] for more details on the concept of general truth and degeneracies.)

Definition 2. A set of points $\ell = \{P_1, P_2, \dots, P_n\}$ is called a line if for all different $P, Q, R \in \ell$ the points P, Q and R are collinear in general.

For example, the set $\ell = \{C, F, G\}$ in Fig. 5 forms a line.

Definition 3. A set of points $\mathcal{C} = \{P_1, P_2, P_3, \dots, P_n\}$ is called a circle if for all different $P, Q, R, S \in \mathcal{C}$ the points P, Q, R and S are concyclic in general.

Definition 4. A set of lines $\vec{D} = \{\ell_1, \dots, \ell_n\}$ is called parallel lines (or a direction) if for all different $\ell, m \in \vec{D}$ the lines ℓ and m are parallel in general.

Definition 5. A pair of parallel lines \vec{D}_1, \vec{D}_2 is called perpendicular or parallel lines if for all different $\ell \in \vec{D}_1, m \in \vec{D}_2$ the lines ℓ and m are parallel or perpendicular in general. We will use the notation $\vec{D}_1 \perp \vec{D}_2$ in this case.

Definition 6. A set $\bar{s} = \{P, Q\}$ of two points is called a segment.

Definition 7. A set of segments $s = \{\bar{s}_1, \dots, \bar{s}_n\}$ is called equal length segments (or congruent segments) if for all different $\bar{s}_1, \bar{s}_2 \in s$ the segments \bar{s}_1 and \bar{s}_2 are equally long in general.

In fact, GeoGebra Discovery uses a more general concept of being identical: it allows two points (or two objects) to have a kind of relationship also if it is true just *on parts* (see [11] for more details).

The main idea of storing the objects is that points, lines, circles, directions (and their perpendicular supplementors) and equally long segments designate equivalence classes, that is:

Theorem 1. Let ℓ and m be lines. Then, for all different points $P, Q, R \in m$, if $\{P, Q\} \subset \ell$, then $R \in \ell$; that is, $\ell = m$.

Proof. In Euclidean geometry⁴ two points always designate a unique line. □

Theorem 2. Let \mathcal{C} and \mathcal{D} be circles. Then, for all different points $P, Q, R, S \in \mathcal{D}$, if $\{P, Q, R\} \subset \mathcal{C}$, then $S \in \mathcal{C}$; that is, $\mathcal{C} = \mathcal{D}$.

Proof. In Euclidean geometry three non-collinear points always designate a unique circle. □

Theorem 3. Let \vec{D} and \vec{E} be directions. Let $\ell \in \vec{D}$ and $m \in \vec{E}$. If $\ell \parallel m$ in general, then $\vec{D} = \vec{E}$.

Proof. This follows immediately from the transitive property of parallelism. □

Theorem 4. The relation \perp is an equivalence relation on the parallel lines.

Proof. Reflexivity and symmetry are obvious. To check transitivity we assume that $\vec{D}_1 \perp \vec{D}_2$ and $\vec{D}_2 \perp \vec{D}_3$ hold and verify these four possible cases:

	$\vec{D}_2 \parallel \vec{D}_3$	$\vec{D}_2 \perp \vec{D}_3$
$\vec{D}_1 \parallel \vec{D}_2$	$\vec{D}_1 \parallel \vec{D}_3$	$\vec{D}_1 \perp \vec{D}_3$
$\vec{D}_1 \perp \vec{D}_2$	$\vec{D}_1 \perp \vec{D}_3$	$\vec{D}_1 \parallel \vec{D}_3$

We note that it is a requirement that all the lines are defined in the *plane*. □

⁴The statement also holds in absolute geometry.

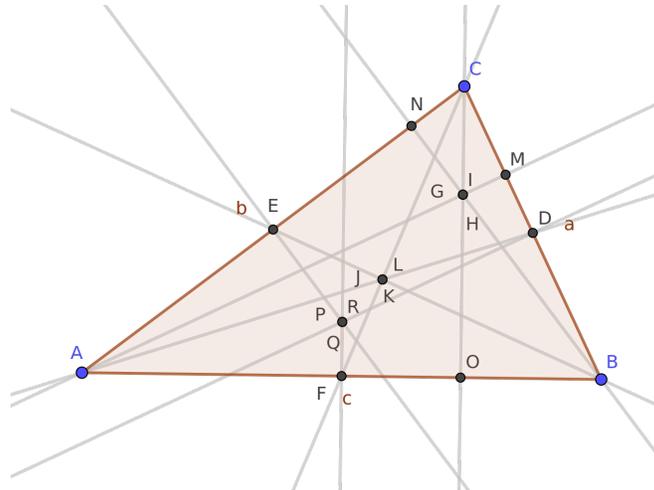


Figure 7: Euler line

Theorem 5. Let s and t be segments. Let $\bar{u} \in s$ and $\bar{v} \in t$. If $|\bar{u}| = |\bar{v}|$ in general, then $s_1 = s_2$.

Proof. This is an immediate consequence of the transitive property of equality of lengths. \square

By using these theorems we can maintain a minimal set of objects during discovery. See [12] for a more detailed description of the technical internals.

3 Examples of Discover with selected theorems

GeoGebra is a well-known and widely used software tool in education, with meaningful potential for using geometric discovery and exploration to teach elementary geometry. Even so, the range of mathematical knowledge is broad, including secondary school topics, international math competitions, and higher level mathematics. Below we examine selected theorems confirmed in the current implementation of Discover.

3.1 Euler line

The Euler line is a line determined from any triangle that is not regular. It passes through the orthocenter, the circumcenter and the centroid. The problem is shown in Fig. 7. With discovery on point P , the relevant theorems are listed in Fig. 8. The Euler line theorem implicitly includes several simple theorems, including concurrency of the altitudes ($G = H = I$, these points being the pairwise intersections of the altitudes), concurrency of the medians of a triangle ($J = K = L$, the generated points being the pairwise intersections of the medians), and concurrency of the perpendicular bisectors of the altitudes ($P = Q = R$, pairwise intersections as above).

3.2 Nine-point circle

The nine-point circle passes through nine significant points of an arbitrary triangle, namely:

- the midpoint of each side of the triangle,

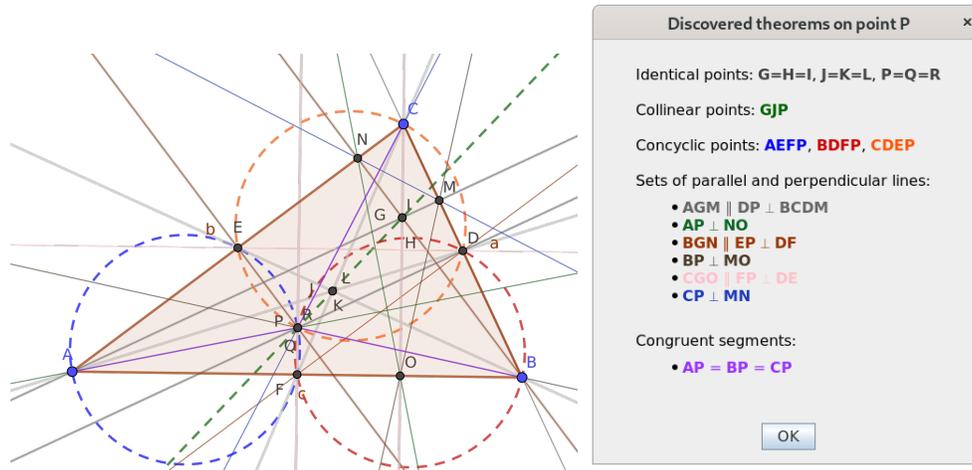


Figure 8: Output of the command Discover (P)

- the foot point of each altitude,
- the midpoint of the line segment from each vertex of the triangle to the orthocenter.

With discovery on the midpoint D of side BC , the appropriate theorems are reported in Fig. 9. The nine-point circle theorem implicitly includes several other simple theorems. In addition, the graphical result suggests further theorems: segments DJ , EK , and FL are congruent and concurrent; these three segments are also diameters of the nine-point circle; and their intersection designates the center of the nine-point circle. This can be easily confirmed by performing another discovery.

3.3 A contest problem

In 2010, at the 51st International Mathematics Olympiad in Astana, Kazakhstan, the following shortlisted problem was proposed by United Kingdom:

Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P . The lines BP and DF meet at point Q . Prove that $AP = AQ$.

After constructing an acute triangle with GeoGebra Discovery (noting that no matching figure can be drawn for a non-acute input angle), we start discovery on point Q .

The discovered theorems appear in Fig. 10. We learn a few unexpected properties: $DP \parallel EQ$, and the points C, D, P, Q , and A, F, P, Q are concyclic.

4 Discussion

4.1 Trivial statements and theorems

After the Discover command determines the salient properties for a given input, it displays relevant *theorems* but does not report *trivial statements*. One example of a trivial statement occurs in Fig. 2, in which the collinearity of points B, C and D and of points A, C and E are not reported. By defining D as the midpoint of BC , this collinearity is implicitly assumed, so it does not make any sense to reiterate this.

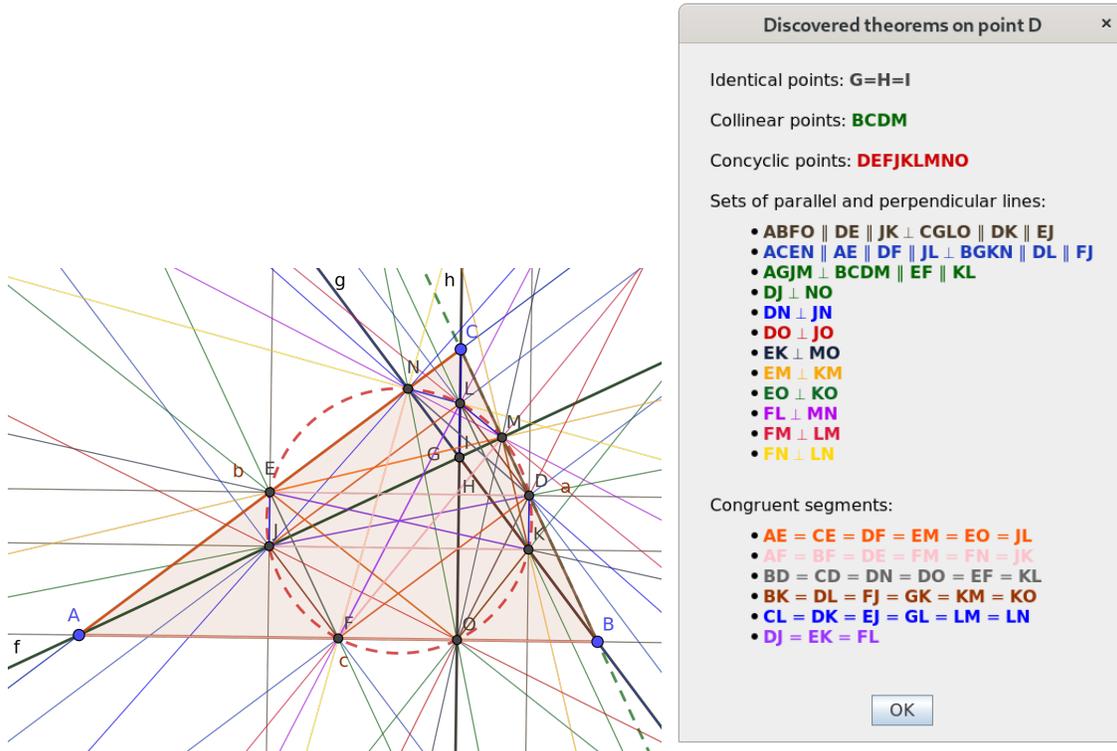


Figure 9: Output of the command Discover (D)

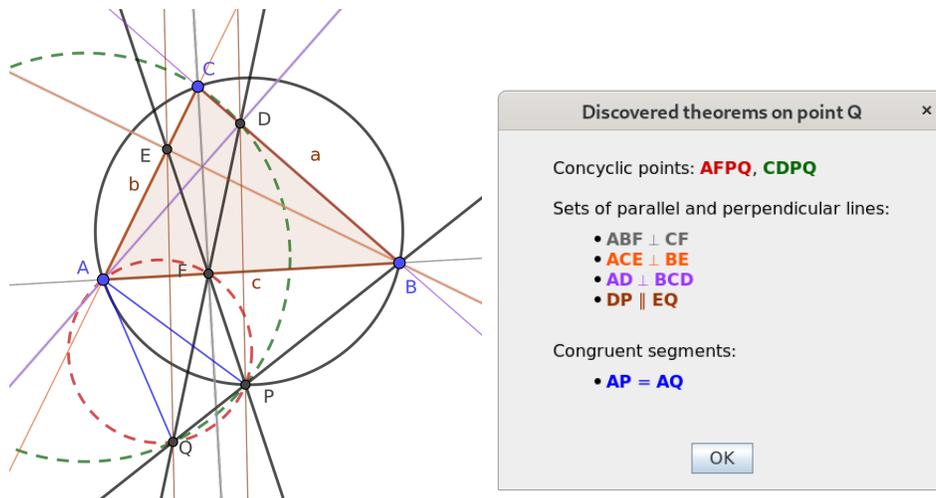


Figure 10: Output of the command Discover (Q)

The question of which properties are considered trivial or not has been a topic of long term discussion. Over 30 years ago, in his famous book, Larry Wos asked, “What properties can be identified to permit an automated reasoning program to find new and interesting theorems, as opposed to proving conjectured theorems?” [17]. This decision, at some level, does become a judgment call that may vary depending on the expertise of the user. Most users would regard the above statement $BD = CD$ as trivial, given that D is the midpoint of BC . On the other hand, explicitly stating this property might be helpful for some beginners. Currently, GeoGebra Discovery regards as trivial all statements that directly follow from the parameters of the original problem, as demonstrated in the example above. However, more detailed criteria to determine which theorems to display could be clarified in the future. Recent research may help in crafting effective algorithms that prioritize identifying the most interesting theorems while filtering out less important statements (see [8, 15]).

4.2 Combinatorial explosion and computational complexity

By using the classes of the equivalence relations, the number of statements to be checked can be decreased significantly.

For each possible statement, a numerical check is first performed. Unfortunately, for some exotic coordinates of the points in the figure, the numerical check can be completely misleading. For example, very large numbers can sometimes result in numerically unstable computations. Regardless, if a numerical check is positive, then the statement is added to the list of conjectures, but if it is negative, no conjecture is registered. As a consequence, while our implementation may miss some true statements (due to numerical errors), it will not output false statements.

For each conjecture, a symbolic check will be performed. If the symbolic check is positive, then the statement will be saved as a theorem. If the symbolic check is negative, then the statement will be removed from the list of conjectures. If the symbolic check cannot determine whether a conjecture is true or false, the conjecture is removed from the list.

A special case of a conjecture is $P_1 = P_2$ for each two geometric points. If this conjecture cannot be proven or disproven symbolically, then the discovery process will be halted. The user will then be notified that the construction must be redrawn in a different way, or else no output can be produced. This is required to maintain the consistency of the internal data.

Symbolic checks usually require more time than numerical verifications. The underlying computation uses Gröbner bases that require at most double exponential time of the number of variables [14] according to the algebraic translation of the given figure. Usually, the number of variables are double the number of geometric points in the figure (since there are two coordinates for each).

GeoGebra internally sets 5 seconds for the maximal execution time of each symbolic check. After timeout the result of the symbolic check will be undecided.

Complexity can be decreased by hiding some points in the figure. This allows for faster processing and avoids an overly cluttered output containing too much information.

4.3 User interface enhancements

Currently only points can be investigated. In a future version, a set of points, segments, lines, circles or a set of these could be allowed as input.

At the moment the computation process cannot be interrupted by the user. Given a large number of points in the figure, the calculation can be time consuming. For example, investigating the relationships

of a regular 20-gon may require about 4 minutes on a modern personal computer (in our test a Lenovo ThinkPad E480 with an i7 processor, 16 GB RAM, Ubuntu Linux 18.04, was used).

The current version, which is based on GeoGebra Classic 5, performs better than the one on Classic 6. The latter is a web implementation of the GeoGebra application and uses a WebAssembly compilation of the computer algebra system Giac. Even if the code is reasonably fast as embedded code in a web page, Classic 6 underperforms the native technology: the same hardware is unable to handle the input of the regular 20-gon, with Google Chrome 83 crashing after 12 minutes of computation.

4.4 Angles

In a complex algebraic geometry setting, the study of angles is not as straightforward as investigating other objects. For a future version, however, this feature would be an important improvement.

By combining algebraic and pure geometric observations, however, simple theorems on angle equality could be easily detected. For example, Fig. 10 states that points A , F , P , Q are concyclic. The inscribed angle theorem automatically implies $\angle QAP = \angle QFP$, among others.

4.5 Stepwise suggestions

Prior research (see [10, p. 46]) proposed that collecting the interesting new objects in a figure could be done stepwise, similarly to GeoGebra’s former feature “special objects.” For our midline theorem example (Fig. 1), this meant that after constructing the triangle ABC and then midpoint D , the system automatically displayed the segments BD and CD . The user could then accept these newly generated segments or remove them from the system. Then, by creating midpoint E , the system could show lines AB and DE to visualize parallelism.

However, this “special objects” feature was recently removed from GeoGebra after feedback that many users found this feature confusing. How stepwise suggestions might be implemented in GeoGebra Discovery remains a topic of future investigation.

5 Related work

We now discuss several projects that share some similarity to GeoGebra Discover but differ in significant ways.

First, GeoGebra Discovery is not the first tool that systematically displays confirmed theorems in a geometric figure. We refer the reader to

- Zlatan Magajna’s *OK Geometry*⁵ [13],
- Jacques Gressier’s *Géométrie*⁶ and
- *Java Geometry Expert*⁷ (JGEX) [18].

These systems are available free of charge, but the source code is available only for JGEX. On the other hand, GeoGebra Discovery focuses on an intuitive user interface and proofs in the most mathematical sense.

⁵www.ok-geometry.com

⁶geometrix.free.fr

⁷<https://github.com/yezheng1981/Java-Geometry-Expert>

Second, we note that there is a growing interest in creating algorithms related to success completion of secondary school or undergraduate mathematics entrance exams. (See [6, 16, 7], among others.) Sometimes these projects rely significantly on techniques used in the underlying computational methods. Also, these projects are often related to artificial intelligence and Big Data rather than to computational mathematics.

Third, we mention a theoretical issue. The idea to store a geometric point only once if it is identical to another one was previously described in Kortenkamp's work [9, 9.3.1]. This concept is a main design element in the dynamic geometry software Cinderella, which never stores a geometric point twice if the two variants are identical in general.

GeoGebra has a different design concept by allowing the user an arbitrary number of identical points to be defined. From the theorem prover's point of view, GeoGebra's concept is more difficult to handle, and a kind of translation is required to have a different data structure by using the concepts from Section 2.

Finally, we note that GeoGebra Discovery proves the truth in a different manner from Cinderella, with Cinderella using a probabilistic method, and GeoGebra Discovery literally *proving* all the deduced facts.

6 Conclusion

We described a prototype of the `Discover` command that is available in an experimental version of GeoGebra, called GeoGebra Discovery. This tool facilitates geometric analysis to determine important properties and theorems based on the user's input. Our current implementation can be directly downloaded from <https://github.com/kovzol/geogebra/releases/tag/v5.0.641.0-2021Sep03>.

Active testing and sample outputs of the `Discover` command can be seen at https://prover-test.geogebra.org/job/GeoGebra_Discovery-discovertest/. Currently, 19 test problems are successfully demonstrated, including Brahmagupta's theorem, Napoleon's theorem, Thebault's first two theorems, as well as some simpler statements. Areas for future development are listed in Section 4.

7 Acknowledgments

The `Discover` command is a result of a long collaboration of several researchers. The project was initiated by Tomás Recio in 2010, and several other researchers joined, including Francisco Botana and M. Pilar Vélez, to name just the most prominent collaborators. The development and research work was continuously monitored and supported by the GeoGebra Team. Special thanks to Markus Hohenwarter, project director of GeoGebra.

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