First Cycle Games*

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First cycle games (FCG) are played on a finite graph by two players who push a token along the edges until a vertex is repeated, and a simple cycle is formed. The winner is determined by some fixed property Y of the sequence of labels of the edges (or nodes) forming this cycle. These games are traditionally of interest because of their connection with infinite-duration games such as parity and mean-payoff games.

We study the memory requirements for winning strategies of FCGs and certain associated infinite duration games. We exhibit a simple FCG that is not memoryless determined (this corrects a mistake in *Memoryless determinacy of parity and mean payoff games: a simple proof* by Björklund, Sandberg, Vorobyov (2004) that claims that FCGs for which Y is closed under cyclic permutations are memoryless determined). We show that $\Theta(n)!$ memory (where n is the number of nodes in the graph), which is always sufficient, may be necessary to win some FCGs. On the other hand, we identify easy to check conditions on Y (i.e., Y is closed under cyclic permutations, and both Y and its complement are closed under concatenation) that are sufficient to ensure that the corresponding FCGs and their associated infinite duration games are memoryless determined. We demonstrate that many games considered in the literature, such as mean-payoff, parity, energy, etc., satisfy these conditions. On the complexity side, we show (for efficiently computable Y) that while solving FCGs is in PSPACE, solving some families of FCGs is PSPACE-hard.

1 Introduction

First cycle games (FCGs) are played on a finite graph by two players who push a token along the edges of the graph until a simple cycle is formed. Player 0 wins the play if the sequence of labels of the edges (or nodes) of the cycle satisfies some fixed cycle property Y, and otherwise Player 1 wins. For instance, if every vertex has an integer priority, the cycle property $Y = \mathsf{cyc\text{-}Parity}$ states that the largest priority occurring on the cycle should be even. For a fixed cycle property Y, we write $\mathrm{FCG}(Y)$ for the family of games over all possible arenas with this winning condition. We are motivated by two questions: Under what conditions on Y is every game in $\mathrm{FCG}(Y)$ memoryless determined? What is the connection between FCGs and infinite-duration games?

First cycle games. First, we give a simple example showing that first cycle games (FCGs) are not necessarily memoryless determined, even if Y is closed under cyclic permutations (i.e., even if winning depends on the cycle but not on how it was traversed), contrary to the claim in [2][Page 370]. We then show that, for a graph with n nodes, whereas no winning strategy needs more than $(n-1)!$ memory (since this is enough to remember the whole history of the game), some FCGs require at least $\Omega(n!)$ memory. To complete the picture, we analyse the complexity of solving FCGs and show that it is PSPACE-complete. More specifically, we show that if one can decide in PSPACE whether a given cycle satisfies the property

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Y, then solving the games in $\mathrm{FCG}(Y)$ is in PSPACE; and that even for a trivially computable cycle property Y (namely, that the cycle ends with the label 0), solving the games in $\mathrm{FCG}(Y)$ is PSPACE-hard.

First Cycle Games and Infinite-Duration Games. The main object used to connect FCGs and infinite-duration games (such as parity games) is the *cycles-decomposition* of a path. Informally, a path is decomposed by pushing the edges of the path onto a stack; as soon as a cycle is detected in the stack it is popped and output, and the algorithm continues. We then say that a winning condition W (such as the parity or energy winning condition) is *Y-greedy on $\mathscr{A}$* if in the game on arena $\mathscr{A}$ and winning condition W, Player 0 is guaranteed to win if he ensures that every cycle in the cycles-decomposition of the play satisfies Y, and Player 1 is guaranteed to win if she ensures that every cycle in the cycles-decomposition does not satisfy Y. We prove a *Transfer Theorem:* if W is Y-greedy on $\mathscr{A}$, then the winning regions in the following two games on arena $\mathscr{A}$ coincide, and memoryless winning strategies transfer between them: the infinite duration game with winning condition W, and the FCG with winning condition Y.

To illustrate the usefulness of the concept of being Y-greedy, we instantiate the definition to well-studied infinite-duration games: i) the parity winning condition (the largest priority occurring infinitely often is even) is Y-greedy on every arena $\mathscr{A}$ where $Y = \textsf{cyc-Parity}$, ii) the mean-payoff condition (the mean payoff is at least ν) is $\textsf{cyc-MeanPayoff}_\nu$-greedy on every arena $\mathscr{A}$ (where $\textsf{cyc-MeanPayoff}_\nu =$ average payoff is at least ν), and iii) for every arena $\mathscr{A}$ with vertex set V, and largest weight W, the energy condition stating that the energy level is always non-negative starting with initial credit $W(|V|-1)$ is $\textsf{cyc-Energy}$-greedy on $\mathscr{A}$ (where $\textsf{cyc-Energy} =$ the energy level is non-negative).

In order to prove memoryless determinacy of certain FCGs (and related infinite-duration games) we generalise techniques used to prove that mean-payoff games are memoryless determined (Ehrenfeucht and Mycielski [4]). Given a cycle property Y, we first consider the infinite duration games $\mathrm{ACG}(Y)$ (all cycles), and $\mathrm{SCG}(Y)$ (suffix all-cycles). A game in the family $\mathrm{ACG}(Y)$ requires Player 0 to ensure that every cycle in the cycles-decomposition of the play (starting from the beginning) satisfies Y. A game in the family $\mathrm{SCG}(Y)$ requires Player 0 to ensure that every cycle in the cycles-decomposition of *some suffix* of the play satisfies Y. As was done in [4], reasoning about infinite and finite duration games is intertwined – in our case, we simultaneously reason about games in $\mathrm{FCG}(Y)$ and $\mathrm{SCG}(Y)$. We define a property of arenas, which we call *Y-unambiguous*, and prove a *Memoryless Determinacy Theorem*: a game from $\mathrm{FCG}(Y)$ whose arena $\mathscr{A}$ is Y-unambiguous is memoryless determined. Combining this with the Transfer Theorem above, we also get that if $\mathscr{A}$ is Y-unambiguous, then any game with a winning condition W that is Y-greedy on $\mathscr{A}$, is memoryless determined[1].

Although checking if an arena is Y-unambiguous may not be hard, it has two disadvantages: it involves reasoning about infinite paths and it involves reasoning about the arena whereas, in many cases, memoryless determinacy is guaranteed by the cycle property Y regardless of the arena (this is the case for example with $Y = \textsf{cyc-Parity}$). Therefore, we also provide easy to check 'finitary' sufficient conditions on Y (namely that Y is closed under cyclic permutations, and both Y and its complement are closed under concatenation) that ensure Y-unambiguity of every arena, and thus memoryless determinacy for all games in $\mathrm{FCG}(Y)$. We demonstrate the usefulness of these conditions by observing that typical cycle properties are easily seen to satisfy them, e.g., $\textsf{cyc-Parity}, \textsf{cyc-MeanPayoff}_\nu, \textsf{cyc-Energy}$.

We conclude by noting that, in particular, if Y is closed under cyclic permutations, and both Y and its complement are closed under concatenation, then games with winning condition W are memoryless determined on every arena $\mathscr{A}$ for which W is Y-greedy on $\mathscr{A}$. As noted above, for many winning

[1] Taking Y to be $\textsf{cyc-GoodForEnergy}$ (defined to be that either the energy level is positive, or it is zero and the largest priority occurring is even) and noting that for every arena $\mathscr{A}$ we have: i) $\mathscr{A}$ is Y-unambiguous and, ii) the game in $\mathrm{ACG}(Y)$ over $\mathscr{A}$ is Y-greedy on $\mathscr{A}$; we obtain a proof of [3][Lemma 4] that no longer relies on the incorrect result from [2].

conditions W (such as mean-payoff, parity, and energy winning conditions) it is easy to find a cycle property Y satisfying the mentioned closure conditions, and for which W is Y-greedy on the arena of interest. This provides an easy way to deduce memoryless determinacy of these classic games.

Related work. As just discussed, this work extends [4], finds a counter-example to a claim in [2], and supplies a proof of a lemma in [3]. Conditions that ensure (or characterise) which games have memoryless strategies appear for example in [1, 5, 6]. However, all of these deal with infinite duration games and do not exploit the connection to finite duration games.

Due to space limitations, proofs appear in the full version of the article.

2 Definitions

In this paper all games are two-player turn-based games of perfect information played on finite graphs. The players are called Player 0 and Player 1.

Arena An *arena* is a labeled directed graph $\mathscr{A} = (V_0, V_1, E, \mathbb{U}, \lambda)$ where

1. V_0 and V_1 are disjoint sets of vertices of Player 0 and Player 1, respectively; the set of vertices of the arena $V := V_0 \cup V_1$ is non-empty.
2. $E \subseteq V \times V$ is a set of edges with no dead-ends (i.e., for every $v \in V$ there is some edge $(v,w) \in E$);
3. $\mathbb{U}$ is a set of possible labels.
4. $\lambda : E \to \mathbb{U}$ is a *labeling* function, used by the winning condition.

Typical choices for $\mathbb{U}$ are $\mathbb{R}$ and $\mathbb{N}$. Games in which vertices are labeled instead of edges can be modeled by ensuring $\lambda(v,w) = \lambda(v,w')$ for all $v,w,w' \in V$. Similarly, games in which vertices are labeled by elements of $\mathbb{U}'$ and edges are labeled by elements of $\mathbb{U}''$ can be modeled by labeling edges by elements of $\mathbb{U}' \times \mathbb{U}''$. As usual, if $u = e_1 e_2 \cdots$ is a (finite or infinite) sequence of edges in the arena, we write $\lambda(u)$ for the string of labels $\lambda(e_1)\lambda(e_2)\cdots$.

Plays and strategies A *play* $\pi = \pi_0, \pi_1, \ldots$ in an arena is an infinite[2] sequence over V such that $(\pi_j, \pi_{j+1}) \in E$ for all $j \in \mathbb{N}$. The node π_0 is called the *starting* node of the play. We denote the set of all plays in the arena $\mathscr{A}$ by $plays(\mathscr{A})$. A *strategy* for Player i is a function $S : V^* V_i \to V$ such that if $u \in V^*$ and $v \in V_i$ then $(v, S(uv)) \in E$. A strategy S for Player i is *memoryless* if $S(uv) = S(u'v)$ for all $u,u' \in V^*, v \in V_i$. A play π is *consistent* with S, where S is a strategy for Player i, if for every $j \in \mathbb{N}$ such that $\pi_j \in V_i$, it is the case that $\pi_{j+1} = S(\pi_0 \cdots \pi_j)$. A strategy S for Player i is *generated by a Moore machine* if there exists a finite set M of *memory states*, an *initial state* $m_I \in M$, a *memory update* function $\delta : V \times M \to M$, and a *next-move function* $\rho : V \times M \to V$ such that if $u = u_0 u_1 \cdots u_l$ is a prefix of a play with $u_l \in V_i$ then $S(u) = \rho(u_l, m_l)$ where m_l is defined inductively by $m_0 = m_I$ and $m_{i+1} = \delta(u_i, m_i)$. A strategy S is *finite-memory* if it is generated by some Moore machine. A strategy *S uses memory at most k* if it is generated by some Moore machine with $|M| \leq k$. A strategy *S uses memory at least k* if every Moore machine generating S has $|M| \geq k$.

Games, Winning Conditions, and Memoryless Determinacy A *game* is a pair $(\mathscr{A}, O)$ where $\mathscr{A} = (V_0, V_1, E, \mathbb{U}, \lambda)$ is an arena and $O \subseteq plays(\mathscr{A})$ is an *objective* (usually induced by the labeling). If either V_0 or V_1 is empty, then the game (A, O) is called a *solitaire game*. A play π in a game $(\mathscr{A}, O)$ is *won by Player* 0 if $\pi \in O$, and *won by Player* 1 otherwise. A strategy S for Player i is *winning starting from a node* $v \in V$ if every play π that starts from v and is consistent with S is won by Player i.

[2] For simplicity, we consider plays of both finite and infinite duration games to be infinite. However, in a finite duration game (and thus in any FCG) the winner is determined by a finite prefix of the play, and the moves after this prefix are immaterial.

A *winning condition* is a set $W \subseteq \mathbb{U}^\omega$. If W is a winning condition and $\mathscr{A}$ is an arena, the objective $O_W(\mathscr{A})$ induced by W is defined as follows: $O_W(\mathscr{A}) = \{v_0v_1v_2\cdots \in plays(\mathscr{A}) \mid \lambda(v_0,v_1)\lambda(v_1,v_2)\cdots \in W\}$. Here are some standard winning conditions:

- The *parity condition* PARITY consists of those infinite sequences $c_1c_2\cdots \in \mathbb{N}^\omega$ such that the largest label occurring infinitely often is even.
- For $\nu \in \mathbb{R}$, the *ν-mean-payoff condition* consists of those infinite sequences $c_1c_2\cdots \in \mathbb{R}$ such that $\limsup_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} c_i$ is at least ν.
- The *energy condition* for a given *initial credit* $r \in \mathbb{N}$, written ENERGY(r), consists of those infinite sequences $c_1c_2\cdots \in \mathbb{Z}^\omega$ such that $r + c_1 + \cdots + c_k \geq 0$ for all $k \geq 1$.
- The *energy-parity condition* ENERGY-PARITY(r) is defined as consisting of $(c_1,d_1)(c_2,d_2)\cdots \in \mathbb{N}\times\mathbb{Z}$ such that $c_1c_2\cdots$ is in PARITY and $d_1d_2\cdots$ is in ENERGY(r).

The *(memoryless) winning region* of Player i is the set of vertices $v \in V$ such that Player i has a (memoryless) winning strategy starting from v. A game is *pointwise memoryless for Player i* if the memoryless winning region for Player i coincides with the winning region for Player i. A game is *uniform memoryless for Player i* if there is a memoryless strategy for Player i that is winning starting from every vertex in that player's winning region.

A game is *determined* if the winning regions partition V. A game is *pointwise memoryless determined* if it is determined and it is pointwise memoryless for both players. A game is *uniform memoryless determined* if it is determined and uniform memoryless for both players.

Cycles-decomposition A *cycle* in an arena $\mathscr{A}$ is a sequence of edges $(v_1,v_2)(v_2,v_3)\cdots(v_{k-1},v_k)(v_k,v_1)$.

Define an algorithm that processes a play $\pi \in plays(\mathscr{A})$ and outputs a sequence of cycles: at step 0 start with empty stack; at step j push the edge (π_j,π_{j+1}), and if for some k, the top k edges on the stack form a cycle, this cycle is popped and output, and the algorithm continues to step $j+1$. The sequence of cycles output by this algorithm is called the *cycles-decomposition of π*, and is denoted by $cycles(\pi)$. The *first cycle of π* is the first cycle in $cycles(\pi)$. For example, if $\pi = vwxwvs(xyz)^\omega$, then $cycles(\pi) = (w,x)(x,w),(v,w)(w,v),(x,y)(y,z)(z,x),(x,y)(y,z)(z,x),\ldots$, and the first cycle of π is $(w,x)(x,w)$.Note that $cycles(\pi)$ is such that at most $|V|-1$ edges of π do not appear in it (i.e, they are pushed but never popped – like the edge (v,s) in the example above). As we show in the full version, this allows one to reason, for instance, about the initial credit problem for energy games (cf. [3]).

Cycle properties A *cycle property* is a set $Y \subseteq \mathbb{U}^*$, used later on to define winning conditions for games. Here are some cycle properties that we refer to in the rest of the article:

1. Let cyc-EvenLen be those sequences $c_1c_2\cdots c_k \in \mathbb{U}^*$ such that k is even.
2. Let cyc-Parity be those sequences $c_1\cdots c_k \in \mathbb{N}^*$ such that $\max_{1\leq i\leq k} c_i$ is even.
3. Let cyc-Energy be those sequences $c_1\cdots c_k \in \mathbb{Z}^*$ such that $\sum_{i=1}^{k} c_i \geq 0$.
4. Let cyc-GoodForEnergy be those sequences $(c_1,d_1)\cdots(c_k,d_k) \in (\mathbb{N}\times\mathbb{Z})^*$ such that either $\sum_{i=1}^{k} d_i > 0$, or both $\sum_{i=1}^{k} d_i = 0$ and $c_1\cdots c_k \in$ cyc-Parity.
5. Let cyc-MeanPayoff$_\nu$ be those sequences $c_1\cdots c_k \in \mathbb{R}^*$ such that $\frac{1}{k}\sum_{i=1}^{k} c_i \leq \nu$, for some $\nu \in \mathbb{R}$.
6. Let cyc-MaxFirst be those sequences $c_1\cdots c_k \in \mathbb{N}^*$ such that $c_1 \geq c_i$ for all $1 \leq i \leq k$.
7. Let cyc-EndsZero be those sequences $c_1\cdots c_k \in \mathbb{N}^*$ such that $c_k = 0$.

If $Y \subseteq \mathbb{U}^*$ is a cycle property, write $\neg Y$ for the cycle property $\mathbb{U}^* \setminus Y$. We isolate two important classes of cycle properties (the first is inspired by [2]):

1. Say that Y is *closed under cyclic permutations* if $ab \in Y$ implies $ba \in Y$, for all $a \in \mathbb{U}, b \in \mathbb{U}^*$.
2. Say that Y is *closed under concatenation* if $a \in Y$ and $b \in Y$ imply that $ab \in Y$, for all $a, b \in \mathbb{U}^*$.

Note that the cycle properties 1-5 above are closed under cyclic permutations and concatenation; and that $\neg$cyc-EvenLen is closed under cyclic permutations but not under concatenation.

First Cycle Games (FCGs) Given a cycle property $Y \subseteq \mathbb{U}^*$, and an arena $\mathscr{A} = (V_0, V_1, E, \mathbb{U}, \lambda)$, let the objective $O_{\mathrm{FCG}(Y)}(\mathscr{A}) \subseteq plays(\mathscr{A})$ be such that $\pi \in O_{\mathrm{FCG}(Y)}(\mathscr{A})$ iff $\lambda(u) \in Y$ where u is the *first cycle* in the cycles-decomposition of π. The family $\mathrm{FCG}(Y)$ of *first cycle games of* Y consists of all games of the form $(A, O_{\mathrm{FCG}(Y)}(\mathscr{A}))$ where $\mathscr{A}$ is an arena with labels in $\mathbb{U}$. For instance, $\mathrm{FCG}(\text{cyc-Parity})$ consists of those games such that Player 0 wins iff the largest label occurring on the first cycle is even.[3]

3 Finite Duration Cycle Games (on being first)

In this section we analyse the memory required for winning strategies in first cycle games, and the complexity of solving these games. We begin by correcting a mistake in [2].

Proposition 1. *There exists a cycle property Y closed under cyclic permutations and a game in* $\mathrm{FCG}(Y)$ *that is not pointwise memoryless determined.*

To see this, consider a game where Player 1 chooses from $\{a, b\}$ and Player 0 must match the choice. This clearly requires Player 0 to have memory. The claim follows by simply encoding this game as a FCG. For example, let the cycle-property Y be cyc-EvenLen, let the vertex set be $\{v_1, v_2, v_3, v_4\}$, let $V_0 = \{v_1\}$, and let the edges be $\{(v_1, v_2), (v_2, v_1), (v_1, v_3), (v_3, v_2), (v_2, v_4), (v_4, v_1)\}$.

We now consider the difference between pointwise and uniform memoryless determinacy of FCGs.

Theorem 1. 1. *Solitaire FCGs are pointwise memoryless determined.*

2. *There is a solitaire FCG that is not uniform memoryless determined.*
3. *If cycle property Y is closed under cyclic permutations, and a game from* $\mathrm{FCG}(Y)$ *is pointwise memoryless for Player i, then that game is uniform memoryless for Player i.*

Proposition 2. 1. *For a FCG on an arena with n vertices, if Player i wins from v, then every winning strategy for Player i starting from v uses memory at most* $(n-1)!$.

2. *For every n there exists a FCG on an arena with* $3n+1$ *vertices, and a vertex v, such that every winning strategy for Player* 0 *starting from v uses memory at least $n!$.*

The first item is immediate since $(n-1)!$ is enough to remember the whole history of the game up to the point a cycle is formed. The proof of the second item is by showing a game where Player 1 can "weave" any possible permutation of n nodes, whereas in order to win Player 0 must remember this permutation. The construction is in the full version of the paper.

Finally, we analyse the complexity of solving FCGs with efficiently computable cycle properties.

Theorem 2. 1. *If Y is a cycle property for which solving membership is in PSPACE, then the problem of solving games in* $\mathrm{FCG}(Y)$ *is in PSPACE.*

2. *The problem of solving games in* $\mathrm{FCG}(\textit{cyc-EndsZero})$ *is PSPACE-complete.*

[3] Formally, then, first cycle games are of infinite duration, although the winner is determined after the first cycle appears on the play.

Sketch. For the first item, observe that solving the game amounts to evaluating the finite AND-OR tree obtained by unwinding the arena into all possible plays, up to the point on each play where a cycle is formed; nodes belonging to Player 0 are 'or' nodes, nodes belonging to Player 1 are 'and' nodes, and a leaf is marked by 'true' iff the cycle formed on the way to it is in Y. Since this tree has depth at most n (the size of the arena), and since we assumed membership in Y is in PSPACE, marking the leaves can be done in PSPACE. So evaluating the tree can be done in PSPACE.

For the second item, note that Generalised Geography can be thought of as a first cycle game in which Player i nodes are labeled by i, and $Y = \mathsf{cyc\text{-}EndsZero}$. Note that computing Y is computationally trivial, but solving Generalised Geography is PSPACE-hard (see for instance [7][Theorem 8.11]). □

4 Infinite Duration Cycle Games

4.1 On being greedy

We start by defining two types of infinite duration games called the *All-Cycles* and the *Suffix All-Cycles* games, whose winning condition is derived from Y. Informally, All-Cycles games are games in which Player 0 wins iff all cycles in the cycles-decomposition of the play are in Y, and Suffix All-Cycles Games are games in which Player 0 wins iff all cycles in the cycles-decomposition of *some suffix* of the play are in Y. Formally, for arena $\mathscr{A} = (V_0, V_1, E, \mathbb{U}, \lambda)$ and cycle property $Y \subseteq \mathbb{U}^*$, we define two objectives $O \subseteq plays(\mathscr{A})$ and corresponding families of games as follows:

1. $\pi \in O_{\mathrm{ACG}(Y)}(\mathscr{A})$:if $\lambda(u) \in Y$ for *all cycles* u in $cycles(\pi)$.
2. $\pi \in O_{\mathrm{SCG}(Y)}(\mathscr{A})$:if some *suffix* π' of π satisfies that $\lambda(u) \in Y$ for all cycles u in $cycles(\pi')$. [4]

Define the corresponding families of games:

1. The family $\mathrm{ACG}(Y)$ of *all-cycles games of* Y consists of all games of the form $(\mathscr{A}, O_{\mathrm{ACG}(Y)}(\mathscr{A}))$.
2. The family $\mathrm{SCG}(Y)$ of *suffix all-cycles games of* Y consists of all games of the form $(\mathscr{A}, O_{\mathrm{SCG}(Y)}(\mathscr{A}))$.

Definition 1. *Say that a* game $(\mathscr{A}, O)$ is Y-greedy *if* $O_{\mathrm{ACG}(Y)}(\mathscr{A}) \subseteq O$ *and* $O_{\mathrm{ACG}(\neg Y)}(\mathscr{A}) \subseteq V^\omega \setminus O$. *Say that a* winning condition W is Y-greedy on arena $\mathscr{A}$ *if the game* $(\mathscr{A}, O_W)$ *is* Y*-greedy.*

Intuitively, W being Y-greedy on $\mathscr{A}$ means that Player 0 can win the game on arena $\mathscr{A}$ with winning condition W if he ensures that every cycle in the cycles-decomposition of the play is in Y, and Player 1 can win if she ensures that every cycle in the cycles-decomposition of the play is not in Y.

For instance, the winning condition PARITY (the largest priority occurring infinitely often is even) is $\mathsf{cyc\text{-}Parity}$-greedy on every arena $\mathscr{A}$, the ν-mean-payoff condition (the $\limsup$ average is at least ν) is $\mathsf{cyc\text{-}MeanPayoff}_\nu$-greedy on every arena $\mathscr{A}$, and the energy condition (stating that the energy level is always non-negative starting with initial credit $W(|V|-1)$, where W is the largest weight and V are the vertices of the arena $\mathscr{A}$) is $\mathsf{cyc\text{-}Energy}$-greedy on $\mathscr{A}$.

Theorem 3 (Transfer)**.** *Let* $(\mathscr{A}, O)$ *be a* Y*-greedy game, and let* $i \in \{0,1\}$.

1. *The winning regions for Player i in the games* $(\mathscr{A}, O)$ *and* $(\mathscr{A}, O_{\mathrm{FCG}(Y)}(\mathscr{A}))$ *coincide.*
2. *For every memoryless strategy S for Player i starting from v in arena* $\mathscr{A}$*: S is winning in the game* $(\mathscr{A}, O)$ *if and only if S is winning in the game* $(\mathscr{A}, O_{\mathrm{FCG}(Y)}(\mathscr{A}))$.

[4]Note that this is *not* the same as saying that $\lambda(u) \in Y$ for all but finitely many cycles u in $cycles(\pi)$. For instance, let Y be the property that the cycle has odd length, and take $\pi := (v_1 v_2 v_1 v_3 v_2 v_4)^\omega$. Note that i) decomposing the suffix π' starting with the second vertex results in all cycles having odd length, and ii) it is not the case that almost all cycles in the cycles-decomposition of π have odd length (in fact, they all have even length).

Corollary 1. *Let W be Y-greedy on arena $\mathscr{A}$. Then the game $(\mathscr{A}, O_W)$ is determined, and is pointwise (uniform) memoryless determined if and only if the game $(\mathscr{A}, O_{\mathrm{FCG}(Y)}(\mathscr{A}))$ is pointwise (uniform) memoryless determined.*

4.2 On being unambiguous

Definition 2. *An arena $\mathscr{A}$ is Y-*unambiguous *if $O_{\mathrm{SCG}(Y)}(\mathscr{A}) \cap O_{\mathrm{SCG}(\neg Y)}(\mathscr{A}) = \emptyset$.*

Lemma 1. *If $\mathscr{A}$ is Y-unambiguous then the game $(\mathscr{A}, O_{\mathrm{SCG}(Y)}(\mathscr{A}))$ is Y-greedy.*

Theorem 4 (Memoryless Determinacy)**.** *If arena $\mathscr{A}$ is Y-unambiguous, then the game $(\mathscr{A}, O_{\mathrm{FCG}(Y)}(\mathscr{A}))$ is pointwise memoryless determined. If Y is also closed under cyclic permutations, then this game is uniform memoryless determined.*

It is of interest to note that the proof of this theorem is a generalisation of the proof used in [4] for showing memoryless determinacy of mean-payoff games. As in [4], our proof reasons about infinite plays. More specifically, we obtain from Theorem 3 and Lemma 1 that the winning regions of each player in the games $(\mathscr{A}, O_{\mathrm{SCG}(Y)}(\mathscr{A}))$ and $(\mathscr{A}, O_{\mathrm{FCG}(Y)}(\mathscr{A}))$ coincide, and then go on and use this fact to derive memoryless strategies for the game $(\mathscr{A}, O_{\mathrm{FCG}(Y)}(\mathscr{A}))$.

Corollary 2. *Suppose arena $\mathscr{A}$ is Y-unambiguous.*

1. *If $(\mathscr{A}, O)$ is Y-greedy, then the game $(\mathscr{A}, O)$ is pointwise memoryless determined.*
2. *The games $(\mathscr{A}, O_{\mathrm{SCG}(Y)}(\mathscr{A}))$ and $(\mathscr{A}, O_{\mathrm{ACG}(Y)}(\mathscr{A}))$ are pointwise memoryless determined.*

If in addition Y is closed under cyclic permutations, then these game are uniform memoryless determined.

Proof. For the first item combine Theorems 3 and 4. For the second, use Lemma 1 and the fact that $(\mathscr{A}, O_{\mathrm{ACG}(Y)}(\mathscr{A}))$ is always Y-greedy. For the final statement apply Theorem 1 item 3. ☐

We now provide a simple sufficient condition on Y — that does not involve reasoning about cycles-decompositions of infinite paths — that ensures that every arena $\mathscr{A}$ is Y-unambiguous:

Theorem 5. *Let $Y \subseteq \mathbb{U}^*$ be a cycle property. If Y is closed under cyclic permutations*[5]*, and both Y and $\neg Y$ are closed under concatenation, then every arena $\mathscr{A}$ is Y-unambiguous.*

It is easy to check that the following cycle properties satisfy the hypothesis of Theorem 5: cyc-Parity, cyc-Energy, cyc-MeanPayoff$_\nu$, and cyc-GoodForEnergy. On the other hand, $\neg$cyc-EvenLen is not closed under concatenation, whereas cyc-MaxFirst is not closed under cyclic permutations.

We conclude with the main result of this section:

Corollary 3. *Suppose Y is closed under cyclic permutations, and both Y and its complement are closed under concatenation. Then the following games are uniform memoryless determined for every arena $\mathscr{A}$: $(\mathscr{A}, O_W)$ if W is Y-greedy on $\mathscr{A}$, $(\mathscr{A}, O_{\mathrm{SCG}(Y)}(\mathscr{A}))$, and $(\mathscr{A}, O_{\mathrm{ACG}(Y)}(\mathscr{A}))$.*

We believe that Corollary 3 provides a practical and easy way of deducing that many infinite duration games are uniform memoryless determined, as follows: exhibit a cycle property Y that is closed under cyclic permutations and both Y and $\neg Y$ are closed under concatenation, such that the winning condition W is Y-greedy on the arena A of interest. Finding such a Y is usually easy since it is simply a ‘finitary’ version of the winning condition W. For example, uniform memoryless determinacy of parity games, mean-payoff games, and energy-games, can easily be deduced by considering the cycle properties cyc-Parity, cyc-MeanPayoff$_\nu$, and cyc-Energy.

[5] It may be worth noting that Y is closed under cyclic permutations iff so is $\neg Y$.

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