

Dual Density Operators and Natural Language Meaning

Daniela Ashoush and Bob Coecke

University of Oxford

Department of Computer Science

dan.ashoush@gmail.com - bob.coecke@cs.ox.ac.uk

Density operators allow for representing ambiguity about a vector representation, both in quantum theory and in distributional natural language meaning. Formally equivalently, they allow for discarding part of the description of a composite system, where we consider the discarded part to be the context. We introduce dual density operators, which allow for two independent notions of context. We demonstrate the use of dual density operators within a grammatical-compositional distributional framework for natural language meaning. We show that dual density operators can be used to simultaneously represent: (i) ambiguity about word meanings (e.g. queen as a person vs. queen as a band), and (ii) lexical entailment (e.g. tiger \Rightarrow mammal). We provide a proof-of-concept example.

1 Introduction

In [20] von Neumann introduced *density operators* in order to give a description of quantum systems for which we don't have perfect knowledge about their state, but rather, there is a probability distribution describing the likeliness to be in a each state. The result is not a standard probability distribution, but one that also accounts for the 'probabilistic distance' between vectors as given by the Born-rule, i.e. the square-modulus of the inner-product [15].

However, vectors are not only used to represent the states of quantum systems. In natural language processing (NLP) they are also used to represent the meanings of words [19, 24], and (some function of) the inner-product is typically taken to be a similarity measure. However, the meanings of many words are ambiguous, that is, the same word is used to describe very different things: "queen" can be a monarch, a rock band, a bee, or a chess piece. Also in this case it is very natural to use density operators in order to allow for a lack of knowledge on which meaning (i.e. which vector) is intended [17, 21, 22]. Since density operators admit 'mixing', we can now mix the vectors representing the distinct meanings of an ambiguous word into a density operator representing that ambiguous word:

$$\text{queen} := \frac{1}{4} (\text{queen-monarch} + \text{queen-bee} + \text{queen-band} + \text{queen-chess})$$

Besides accounting for similarity of words, a vector representation of word meanings also allows for compositional reasoning: given the grammatical structure of a phrase or a sentence and the meanings of the words therein, one can compute the meaning of that phrase or sentence [13, 16, 18]. The crux to doing so is the fact that vectors inhabit a category of a structure that matches the structure of grammar [9], resulting in meanings of words being 'teleported' through a sentence [7].

Moreover, this algorithm for computing phrase and sentence meanings from word meanings carries over to density operators, since via Selinger's CPM-construction [25] these also inhabit a category of the appropriate structure. It is indeed an important feature of the framework of [13] that it is not attached to a particular representation of word-meanings. The passage to density operators also allows for retaining standard empirical methods, hence resulting in data-driven and grammar-driven compositional reasoning about ambiguous words [22]. This allows one, for example, to observe how the ambiguity (measured by e.g. von Neumann entropy) vanishes thanks to the disambiguating role other words play in the sentence.

Now, ambiguity is not the only feature of natural language that is not captured by a plain vector representation. Many pairs of words have a clear entailment-relationship, for example:

$$\text{tiger} \Rightarrow \text{big cat} \Rightarrow \text{mammal} \Rightarrow \text{vertebrate} \Rightarrow \text{animal}$$

While plain vectors living in a vector space do not come with any kind of structure that can capture these entailment-relationships, density operators can be partially ordered [10, 4, 27], and this partial order can then be interpreted as lexical entailment [3, 2, 4].¹ Since the space of density operators embeds in a vector space, we can rely on sums in order to construct general meanings from more specific ones e.g.:

$$\text{big cat} := \frac{1}{N} (\text{lion} + \text{tiger} + \text{cheetah} + \text{leopard} + \dots)$$

This brings up a dilemma: should we either use density operators to express ambiguity, or, lexical entailment? We resolve this dilemma by introducing *dual density operators*. These are mathematical entities which admit ‘two independent dimensions’ of being a density operator. Moreover, just like ordinary density operators, these dual density operators inhabit a category of the appropriate structure for composing meanings, which is obtained by twice applying the CPM-construction. Hence they allow for data-driven and grammar-driven compositional reasoning about meanings of sentences, accounting for ambiguity as well as lexical entailment.

In the following section, we provide a direct construction of dual density operators, using standard Dirac notation. In Section 3, we provide the corresponding categorical construction. Then, we provide an example encoding of meanings both involving ambiguity and lexical entailment, and in the following section we compose these meanings. Finally, in Section 6, we axiomatise categories resulting from twice applying the CPM-construction, exposing two contexts and two corresponding discarding operations.

2 Direct construction

Given a set of normalised vectors $\{|\varphi_i\rangle\}$ in a finite-dimensional inner-product space H and a probability distribution $\{p_i\}$ we form a *density operator for H* as follows:

$$(\{|\varphi_i\rangle\}, \{p_i\}) \mapsto \rho_{\text{operator}} := \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$$

That is, first we replace each vector by the pair consisting of the vector (a.k.a ‘ket’) and its adjoint (a.k.a ‘bra’), which together form a rank 1 operator. Then, we make a weighted sum. Alternatively, instead of taking the adjoint of the vector, we could take its conjugate, and instead of an operator obtain a two-system vector:

$$(\{|\varphi_i\rangle\}, \{p_i\}) \mapsto |\rho\rangle := \sum_i p_i |\varphi_i\rangle\overline{|\varphi_i\rangle} \quad (1)$$

One big advantage of *density vectors for H* as compared to density operators, is that density vectors still live in a vector space $H \otimes \overline{H}$, where \overline{H} is the conjugate space,² so that we can simply repeat construction (1). Doing so we obtain:

$$(\{|\rho_k\rangle\}, \{p'_k\}) \mapsto \sum_k p'_k |\rho_k\rangle\overline{|\rho_k\rangle} \quad (2)$$

¹In the first two of these papers, the ordering is taken to be a preorder for the sake of simplicity, with the induced equivalence classes corresponding to the lattice of closed subspaces, i.e quantum logic [5]. In the case of the partial orders of [10, 4], quantum logic embeds within the ordering of density operators.

²For simplicity, one could take H to be self-dual so that $\overline{H} = H$. However, some of the categorical constructions are directly guided by distinguishing between these two spaces.

We will follow the convention that conjugation of a state in $H \otimes H$ also swaps the states:

$$\overline{|\rho_1\rangle|\rho_2\rangle} = \overline{|\rho_2\rangle|\rho_1\rangle}$$

and hence chaining (1) and (2) together we obtain:

$$(\{|\varphi_{ik}\rangle\}, \{p_{ik}\}, \{p'_k\}) \mapsto \sum_k p'_k \left(\sum_i p_{ik} |\varphi_{ik}\rangle \overline{|\varphi_{ik}\rangle} \right) \overline{\left(\sum_j p_{jk} |\varphi_{jk}\rangle \overline{|\varphi_{jk}\rangle} \right)} \quad (3)$$

So, we obtain a vector in $H \otimes \overline{H} \otimes H \otimes \overline{H}$:

$$\Phi := \sum_{ijk} p_{ik} p_{jk} p'_k |\varphi_{ik}\rangle \overline{|\varphi_{ik}\rangle} |\varphi_{jk}\rangle \overline{|\varphi_{jk}\rangle} \quad (4)$$

As is obvious from the form in the RHS of (3), the vector Φ can be seen as a density vector for $H \otimes \overline{H}$. However, if we swap the 2nd and 4th vectors in (4), we obtain another density vector for $H \otimes \overline{H}$:

$$\sum_{ijk} p_{ik} p_{jk} p'_k |\varphi_{ik}\rangle \overline{|\varphi_{jk}\rangle} |\varphi_{jk}\rangle \overline{|\varphi_{ik}\rangle} = \sum_{ijk} p_{ik} p_{jk} p'_k \left(|\varphi_{ik}\rangle \overline{|\varphi_{jk}\rangle} \right) \overline{\left(|\varphi_{ik}\rangle \overline{|\varphi_{jk}\rangle} \right)} \quad (5)$$

Hence, the vector Φ can be thought of in two manners as a density vector for $H \otimes \overline{H}$, and hence, can be thought of in two manners as a density operator for $H \otimes \overline{H}$.

We will refer to vectors in $H \otimes \overline{H} \otimes H \otimes \overline{H}$ of the form (4) as *Dual density operators for H* . Since to any dual density operator Φ correspond two density vectors for $H \otimes \overline{H}$:

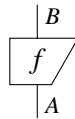
$$\Phi_1 := \sum_{ijk} p_{ik} p_{jk} p'_k \left(|\varphi_{ik}\rangle \overline{|\varphi_{ik}\rangle} \right) \overline{\left(|\varphi_{jk}\rangle \overline{|\varphi_{jk}\rangle} \right)} \quad \Phi_2 := \sum_{ijk} p_{ik} p_{jk} p'_k \left(|\varphi_{ik}\rangle \overline{|\varphi_{jk}\rangle} \right) \overline{\left(|\varphi_{ik}\rangle \overline{|\varphi_{jk}\rangle} \right)}$$

and hence two density operators for $H \otimes \overline{H}$, all features of density operators apply in two-fold to dual density operators. For example, there are two notions of eigenvectors, two notions of spectrum, two notions of entropy, two notions of (im)purity, and so on.

3 Categorical construction

The direct construction of density vectors from vectors is an instance of a general category-theoretic construction, called the *CPM-construction*, which not only applies to inner-product spaces, but to any structure that can be organised in a so-called dagger compact closed category [25]. Moreover, in the case of inner-product spaces, it doesn't just generate density vectors in that case, but also completely positive maps. In general, we again obtain a dagger compact closed category, so we can apply the CPM-construction as many times as we wish.

What this construction does is most easily seen in terms of the diagrammatic language of dagger compact closed categories [26].³ In this language, inner-product spaces are represented by wires, and linear maps by boxes:

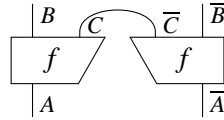


³Please see [11] for a tutorial.

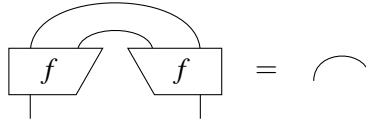
Vectors in H , when represented as linear maps from the vector space field \mathbb{K} (seen as a one-dimensional inner-product space) into H , correspond to boxes without inputs, which in general we represent by triangles. Conjugation is represented by horizontal reflection of these boxes, and we will make use of one special linear map with two inputs, and no outputs, i.e. an *effect*, which we represent by a cap:

$$\cap : H \otimes \bar{H} \rightarrow \mathbb{K} :: |\varphi\rangle|\bar{\varphi}'\rangle \mapsto \langle\varphi'|\varphi\rangle$$

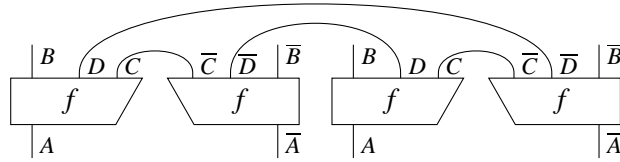
The CPM-construction boils down to passing from general boxes to those of the form:



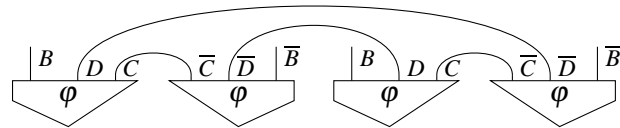
When comparing this diagrams to the form (2), the cup corresponds to the summation, the type C to the set of indices, and the probabilities are absorbed within the boxes. In fact, the vectors that we obtain in this manner are not normalised, and if we want to restrict to normalised ones, we require ‘trace preservation’:



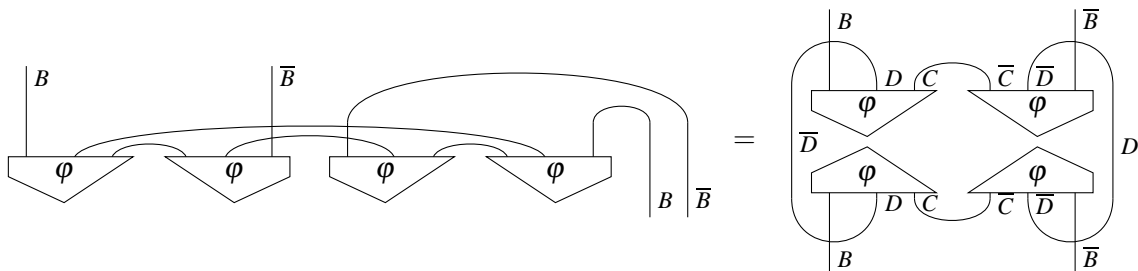
The CPM^2 -construction means applying the CPM-construction twice, yielding boxes of the form:



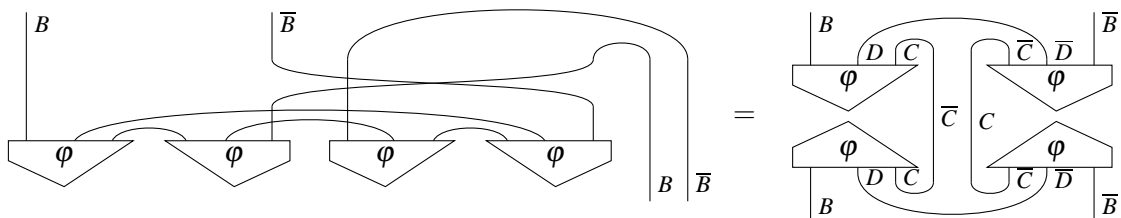
and the dual density operators Φ are then of the form:



The density operator Φ_1 is obtained by bending two wires down:



and the density operator Φ_2 by doing the same after swapping the 1st and 3th wire:



Note also that from the above it is obvious that the two density operators Φ_1 and Φ_2 exist on ‘equal footing’. More specifically, there is an isomorphism which takes the density operators of the form Φ_1 to those of the form Φ_2 , which is realised by swapping the SW wire and the NE wire.

Moreover, it also becomes clear that the two notions of (im)purity are independent, in the case of Φ_1 depending on the ‘size’ of D , while in the case of Φ_2 it depends on the size of C , since it are wires of these respective types that connect the inputs and the outputs of the respective density operators.

4 Ambiguity and lexical entailment

Dual density operators now provide a natural setting to accommodate both ambiguity and lexical entailment in natural language. Given a dual density operator Φ , the first density operator Φ_1 accounts for entailment, while the dual structure, in addition, allows one to express ambiguity. Theoretically, all meanings and their entailment relationships are encoded as density operators on H and their partial ordering. Here, all meanings are to be conceived as unambiguous, cf. “queen” as monarch and “queen” as rock band each have their own dedicated density operator. Then, by construction (2), we can introduce ambiguity. For example, let “Beirut” be the ambiguous word with unambiguous meanings “Beirut city” and “Beirut band”. The city of Beirut has neighbourhoods “Ashrafieh”, that we will denote by “A”, and “Monot”, that we will denote by “M”, while the band has members “Zach”, denoted by “Z”, and “Paul”, denoted by “P”. We can use density operators:

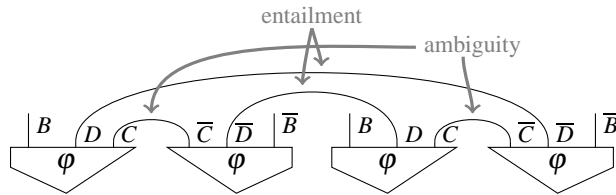
$$\text{“Beirut city”} := A\bar{A} + M\bar{M} \qquad \text{“Beirut band”} := Z\bar{Z} + P\bar{P}$$

in order to express that A and M entail “Beirut city” and Z and P entail “Beirut band”, and we obtain the unambiguous meaning by first turning these in dual density operators and then adding them:

$$\begin{aligned} \text{“Beirut”} &:= (A\bar{A} + M\bar{M})(A\bar{A} + M\bar{M}) + (Z\bar{Z} + P\bar{P})(Z\bar{Z} + P\bar{P}) \\ &:= A\bar{A}A\bar{A} + A\bar{A}M\bar{M} + M\bar{M}A\bar{A} + M\bar{M}M\bar{M} + Z\bar{Z}Z\bar{Z} + Z\bar{Z}P\bar{P} + P\bar{P}Z\bar{Z} + P\bar{P}P\bar{P} \end{aligned}$$

Note that we did not add weights in order to keep the notation simple.

Remark 4.1. The procedure outlined above is not the only one for building meaning involving both ambiguity and lexical entailment. An alternative one is presented in the first author’s MSc thesis [1], which relates lexical entailment and ambiguity directly to Φ_1 and Φ_2 respectively:



The relationship between the alternative encodings is subject to currently ongoing research.

5 Interacting meanings

In [13] a mathematical framework is proposed which allows for the computation of the meaning of sentences in terms of their constituents. This framework unifies two orthogonal but complementary models of meaning.

The first one formalises the grammar of natural language, for example, in terms of *pregroups* $(P, \leq, \cdot, 1, (-)^l, (-)^r)$ where $(P, \leq, \cdot, 1)$ is a partially ordered monoid, $(-)^l$ and $(-)^r$ are unary operations on P , called the left and right adjoints, satisfying the following inequalities for all $a \in P$:

$$a^l \cdot a \leq 1 \leq a \cdot a^l \qquad a \cdot a^r \leq 1 \leq a^r \cdot a$$

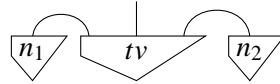
In what follows, we omit the “.” and replace “ \leq ” by “ \rightarrow ”. To see how pregroups model grammar, we fix two basic grammatical types $\{n, s\}$, where n is the grammatical type for *noun*, and s is the grammatical type for *sentence*. Compound types are formed by adjoining and juxtaposing basic types: a transitive verb interacts with a subject to its left and an object to its right, to produce a sentence that is grammatically valid. Transitive verbs are therefore assigned the type $n^r s n^l$, and a transitive sentence reduces to a valid grammatical sentence as follows:

$$n(n^r s n^l)n = (n n^r)s(n^l n) \rightarrow s$$

The second approach concerns the distributional model of meaning, in which words are represented by vectors in finite-dimensional inner-product spaces. While this model does not account for grammar, it does provide a reliable meaning for words. The algorithm of [13] exploits the fact that pregroups on the one hand, when viewed as thin monoidal categories, and inner-product spaces and linear maps on the other hand, are both examples of compact-closed categories. Then, via a strong monoidal functor between these two categories, grammatical reductions are mapped on a linear map:

$$[n(n^r s n^l)n \rightarrow s] \mapsto \begin{array}{c} \frown \quad | \quad \smile \end{array}$$

which then when applied to meaning vectors, gives the meaning of a sentence:



Clearly, the use of a category of inner-product spaces and linear maps is not at all essential; it suffices to have any compact-closed category, or even more general, a category that matches the structure of the chosen categorial grammar [9]. Since the CPM-construction maps a dagger compact closed category on a dagger compact closed category [25], rather than using vectors, we can use density operators to represent meanings, or, of course, dual density operators.

To illustrate this, let us go back to our example involving Beirut. We seek to show that the meaning of ambiguous words ‘collapses’ when enough context is provided. For this, we will compute the meanings of two noun phrases: “Beirut that plays at Beirut”, and “Beirut that Beirut plays at”. We expect the former to be “Beirut band”, and the latter to be “Beirut city”. We already gave the meaning of “Beirut”, so it suffices to give the meaning of “play-at”. It is a transitive verb which we take to be non-ambiguous, and atomic. Hence, in essence it is described by a vector in $N \otimes S \otimes N$ where N is the space in which we describe nouns, namely the one we used to construct “Beirut”, and S is the sentence space, which for the sake of simplicity we choose to be $\{\perp, \top\}$, where \perp stands for “false” and \top for “true”. A natural way for constructing the meaning of a verb, is to simply take pairs of objects and subjects which ‘obey’ that verb, with a “true”-symbol in the middle. Therefore, for “play-at” as a vector in $N \otimes S \otimes N$ we set:

$$\text{play-at}_{N \otimes S \otimes N} := Z \top A + P \top A$$

meaning that Zach and Paul play in neighbourhood Ashrafieh. As a dual density operator this gives:

$$\text{“play-at”} := (Z \top A + P \top A)(\bar{Z} \top \bar{A} + \bar{P} \top \bar{A})(Z \top A + P \top A)(\bar{Z} \top \bar{A} + \bar{P} \top \bar{A})$$

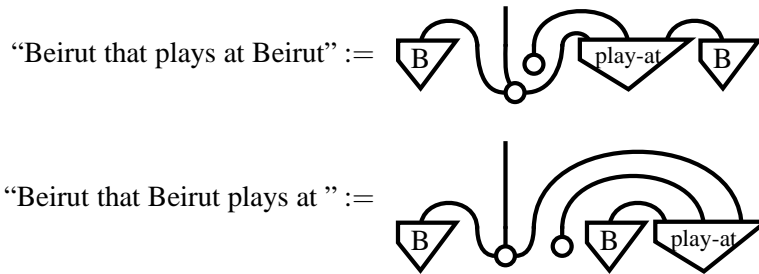
We follow [23] in order to assign meaning to the relative pronoun “that”. Diagrammatically, this boils down to the use of ‘spiders’, and category-theoretically, the use of special commutative Frobenius algebras. Given an ONB we will make use of:

$$\begin{array}{c} \cup \\ \circ \end{array} : \mathbb{K} \rightarrow H \otimes H \otimes H :: 1 \mapsto \sum_i |iii\rangle \qquad \begin{array}{c} | \\ \circ \end{array} : \mathbb{K} \rightarrow H :: 1 \mapsto \sum_i |i\rangle$$

The grammatical type of “that” used as a subject relative pronoun is $N \otimes N \otimes S \otimes N$, while as an object relative pronoun it is $N \otimes N \otimes N \otimes S$, and we set:

$$\text{“that”}_{subj} := \begin{array}{c} N \quad N \quad S \quad N \\ \cup \\ \circ \end{array} \qquad \text{“that”}_{obj} := \begin{array}{c} N \quad N \quad N \quad S \\ \cup \\ \circ \end{array}$$

So:



where the use of bold-wires indicates that all meanings are dual density operators. A somewhat tedious direct computation of these diagrams then indeed yields:

$$\text{“Beirut that plays at Beirut”} := \text{“Beirut-band”} \qquad \text{“Beirut that Beirut plays at”} := \text{“Beirut-city”}$$

Both results are consistent with our expectations and accurately model the case where enough context is provided to disambiguate the meaning of a word. Further examples are provided in [1].

6 Axiomatic characterisation

Density operators allow for discarding part of the description of a composite system, where the discarded part corresponds to the environment or context. As shown in [8, 12], the CPM-construction can be recast in terms of an *environment structure* on a dagger compact closed category \mathbf{C} , which consists of a designated effect $\top_A : A \rightarrow I$ for each object A in \mathbf{C} , called *discarding*, obeying $\top_I = 1_I$, $\top_{A \otimes B} = \top_A \otimes \top_B$, and $(\top_A)_* = \top_{A^*}$, together with an all-objects-including sub-dagger compact closed category \mathbf{C}_Σ of *pure morphisms*, which is such that for all pure morphisms f, g we have:

$$\begin{array}{c} \begin{array}{c} \top \\ \square \\ f \\ \square \\ \top \end{array} = \begin{array}{c} \top \\ \square \\ g \\ \square \\ \top \end{array} \iff \begin{array}{c} \top \\ \triangle \\ f \\ \triangle \\ \top \end{array} = \begin{array}{c} \top \\ \triangle \\ g \\ \triangle \\ \top \end{array} \end{array} \tag{6}$$

Applying (6) to the specific case of vectors yields:

$$|\psi\rangle\langle\psi| = |\varphi\rangle\langle\varphi| \iff |\psi\rangle = |\varphi\rangle$$

which has been called *preparation-state agreement* [8]. It can then be shown that a dagger compact closed category \mathbf{C} carrying an environment structure is isomorphic to $\mathbf{CPM}(\mathbf{C}_\Sigma)$, and applying the CPM-construction to a dagger compact closed category \mathbf{C} which satisfies preparation-state agreement induces an environment structure on \mathbf{C} [8, 12].

Similarly, a dual-environment structure on a dagger compact closed category \mathbf{C} consists of two discarding effects $\top_{1,A}, \top_{2,A} : A \rightarrow I$ for each object A of \mathbf{C} , together with an all-objects-including sub-dagger compact closed category \mathbf{C}_{Σ^2} of *pure morphisms*, which is such that for all pure morphisms f, g we have:

Now, a dagger compact closed category \mathbf{C} carrying a dual-environment structure is isomorphic to $\mathbf{CPM}^2(\mathbf{C}_{\Sigma^2})$, and applying the \mathbf{CPM}^2 -construction to a dagger compact closed category \mathbf{C} which satisfies the preparation-state agreement axiom induces a dual-environment structure on \mathbf{C} .

The proof of this fact can be found in [1], as well as a generalization to multiple applications of the CPM-construction, resulting in multiple discarding operations.

7 Discussion and outlook

Firstly, we applied the CPM-construction twice, in order to accommodate two linguistic features, but there is no reason to stop there: more applications would enable one to accommodate more natural language features.

Secondly, the same ‘trick’ does not only apply to vectors in inner-product spaces, but any candidate model of meaning that can be structured in a dagger compact closed category. One example of other models currently being studied in [6] are based on Gärdenfors’ conceptual spaces. [14].

Thirdly, density operators were borrowed from physics in order to represent ambiguity, perfectly matching their quantum-theoretical interpretation in terms of lack of knowledge. When providing them with a partial ordering in order to represent lexical entailment, one actually went beyond the standard practice in physics, although a subset of the ordering is Birkhoff-von Neumann quantum logic. However, dual density operators are an entirely new kind of mathematical entity that (to our knowledge) have never been used in physics. This of course does not exclude that there is a natural application for them.

Fourthly, of course, we only provided one very simple proof-of-concept example in support of our claims. More involved examples as well as empirical evidence are needed to firmly establish dual density operators as a useful tool for representing natural language meaning.

Finally, many books have been written on density operators. Several things that don’t make sense for vectors, emerge for density operators, like diagonalisability, spectrum, entropy and so on. Dual density operators are yet again a new entity, and hence new basic mathematics needs to be developed.

For example, we know that construction (1) and application of the CPM-construction to inner-product spaces yields the same result. However, this isn’t entirely true anymore for construction (3) and twice applying the CPM-construction to inner-product spaces. Indeed, in ongoing work in collaboration with Maaïke Zwart we have characterised the dual density operators obtained via (3) as a proper subset of those that arise from twice applying the CPM-construction. This is only the beginning, and much more remains to be discovered, for which we refer to a future publication.

References

- [1] D. Ashoush (2015): *Categorical Models of Meaning: Accommodating for Lexical Ambiguity and Entailment*. Master's thesis, University of Oxford.
- [2] E. Balkır, D. Kartsaklis & M. Sadrzadeh (2015): *Sentence Entailment in Compositional Distributional Semantics*. International Symposium of Artificial Intelligence and Mathematics, 2016.
- [3] E. Balkır, M. Sadrzadeh & B. Coecke (2016): *Distributional sentence entailment using density matrices*. In: *Topics in Theoretical Computer Science*, Springer, pp. 1–22, doi:10.1007/978-3-319-28678-5.
- [4] D. Bankova, B. Coecke, M. Lewis & D. Marsden (2016): *Graded Entailment for Compositional Distributional Semantics*. arXiv:1601.04908.
- [5] G. Birkhoff & J. von Neumann (1936): *The logic of quantum mechanics*. *Annals of Mathematics* 37, pp. 823–843, doi:10.2307/1968621.
- [6] J. Bolt, B. Coecke, F. Genovese, M. Lewis, D. Marsden & R. Piedeleu (2016): *Interacting Conceptual Spaces*. In: *Semantic Spaces at the Intersection of NLP, Physics and Cognitive Science*.
- [7] S. Clark, B. Coecke, E. Grefenstette, S. Pulman & M. Sadrzadeh (2014): *A quantum teleportation inspired algorithm produces sentence meaning from word meaning and grammatical structure*. *Malaysian Journal of Mathematical Sciences* 8, pp. 15–25. arXiv:1305.0556.
- [8] B. Coecke (2008): *Axiomatic description of mixed states from Selinger's CPM-construction*. *Electronic Notes in Theoretical Computer Science* 210, pp. 3–13, doi:10.1016/j.entcs.2008.04.014.
- [9] B. Coecke, E. Grefenstette & M. Sadrzadeh (2013): *Lambek vs. Lambek: Functorial vector space semantics and string diagrams for Lambek calculus*. *Annals of Pure and Applied Logic* 164, pp. 1079–1100, doi:10.1016/j.apal.2013.05.009.
- [10] B. Coecke & K. Martin (2011): *A partial order on classical and quantum states*. In B. Coecke, editor: *New Structures for Physics*, Lecture Notes in Physics, Springer, pp. 593–683, doi:10.1007/978-3-642-12821-9.
- [11] B. Coecke & É. O. Paquette: In B. Coecke, editor: *New Structures for Physics*, doi:10.1007/978-3-642-12821-9.
- [12] B. Coecke & S. Perdrix (2010): *Environment and classical channels in categorical quantum mechanics*. In: *Proceedings of the 19th EACSL Annual Conference on Computer Science Logic (CSL)*, Lecture Notes in Computer Science 6247, pp. 230–244, doi:10.1007/978-3-642-15205-4.
- [13] B. Coecke, M. Sadrzadeh & S. Clark (2010): *Mathematical foundations for a compositional distributional model of meaning*. In J. van Benthem, M. Moortgat & W. Buszkowski, editors: *A Festschrift for Jim Lambek*, *Linguistic Analysis* 36, pp. 345–384. arXiv:1003.4394.
- [14] Peter Gärdenfors (2014): *The Geometry of Meaning: Semantics Based on Conceptual Spaces*. MIT Press.
- [15] A. M. Gleason (1957): *Measures on the closed subspaces of a Hilbert space*. *Journal of Mathematics and Mechanics* 6, pp. 885–893, doi:10.1512/iumj.1957.6.56050.
- [16] E. Grefenstette & M. Sadrzadeh (2011): *Experimental Support for a Categorical Compositional Distributional Model of Meaning*. In: *The 2014 Conference on Empirical Methods on Natural Language Processing*, pp. 1394–1404. arXiv:1106.4058.
- [17] D. Kartsaklis (2014): *Compositional Distributional Semantics with Compact Closed Categories and Frobenius Algebras*. Ph.D. thesis, University of Oxford.
- [18] D. Kartsaklis & M. Sadrzadeh (2013): *Prior disambiguation of word tensors for constructing Sentence vectors*. In: *The 2013 Conference on Empirical Methods on Natural Language Processing*, ACL, pp. 1590–1601.
- [19] K. Lund & C. Burgess (1996): *Producing high-dimensional semantic spaces from lexical co-occurrence*. *Behavior Research Methods, Instruments & Computers* 28, pp. 203–208, doi:10.3758/BF03204766.
- [20] J. von Neumann (1932): *Mathematische Grundlagen der Quantenmechanik*. Springer-Verlag. Translation, *Mathematical foundations of quantum mechanics*, Princeton University Press, 1955.
- [21] R. Piedeleu (2014): *Ambiguity in Categorical Models of Meaning*. Master's thesis, University of Oxford.

- [22] Robin Piedeleu, Dimitri Kartsaklis, Bob Coecke & Mehrnoosh Sadrzadeh (2015): *Open System Categorical Quantum Semantics in Natural Language Processing*. In: *Proceedings of the 6th Conference on Algebra and Coalgebra in Computer Science (CALCO)*, Nijmegen, Netherlands, doi:10.4230/LIPIcs.CALCO.2015.270.
- [23] M. Sadrzadeh, S. Clark & B. Coecke (2013): *The Frobenius anatomy of word meanings I: subject and object relative pronouns*. *Journal of Logic and Computation* Advance Access, doi:10.1093/logcom/ext044.
- [24] H. Schütze (1998): *Automatic word sense discrimination*. *Computational linguistics* 24(1), pp. 97–123.
- [25] P. Selinger (2007): *Dagger compact closed categories and completely positive maps*. *Electronic Notes in Theoretical Computer Science* 170, pp. 139–163, doi:10.1016/j.entcs.2006.12.018.
- [26] P. Selinger (2011): *A survey of graphical languages for monoidal categories*. In B. Coecke, editor: *New Structures for Physics*, Lecture Notes in Physics, Springer-Verlag, pp. 275–337, doi:10.1007/978-3-642-12821-9.
- [27] J. M. M. van de Weteringen: *A Classification of Entropic Partial Orders*. Preprint.