Pure Maps between Euclidean Jordan Algebras

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We propose a definition of purity for positive linear maps between Euclidean Jordan Algebras (EJA) that generalizes the notion of purity for quantum systems. We show that this definition of purity is closed under composition and taking adjoints and thus that the pure maps form a dagger category (which sets it apart from other possible definitions.) In fact, from the results presented in this paper, it follows that the category of EJAs with positive contractive linear maps is a †-*effectus*, a type of structure originally defined to study von Neumann algebras in an abstract categorical setting. In combination with previous work this characterizes EJAs as the most general systems allowed in a generalized probabilistic theory that is simultaneously a †-effectus. Using the dagger structure we get a notion of †-positive maps of the form $f = g^\dagger \circ g$. We give a complete characterization of the pure †-positive maps and show that these correspond precisely to the Jordan algebraic version of the sequential product $(a,b) \mapsto \sqrt{a}b\sqrt{a}$. The notion of †-positivity therefore characterizes the sequential product.

1 Introduction

A commonly used technique when studying the foundations of quantum theory, is to consider generalized theories that only exhibit some part of the features of conventional quantum theory. In this way, it becomes more clear what specific properties of quantum theory lead to certain structure. One of the first generalized quantum theories to be studied were the Euclidean Jordan algebras (EJAs) [17]. Besides the matrix algebras of complex self-adjoint matrices of conventional quantum theory, other examples of EJAs are the set of real symmetric matrices of real-valued quantum theory, or the set of self-adjoint matrices over the quaternions. Quite soon after the introduction of EJAs, a full characterization of EJAs was given [18] that showed that these examples almost completely exhaust the possibilities. The Koecher–Vinberg theorem [21] is a major result stating that any ordered vector space with a homogeneous self-dual positive cone is a Euclidean Jordan algebra. It is this theorem that explains the ubiquity of EJAs in reconstructions of quantum theory [3, 30, 22, 33, 23, 31]. Understanding the differences and similarities between regular quantum theory as described by complex matrix algebras, and the more general EJAs is an active topic of research. In this paper we will study the notion of *pure maps* in EJAs and show that they have many of the same properties as those found in quantum theory.

The concept of purity has proven very useful in the field of quantum information. In the context of states, it can be considered a resource in various protocols and computations [13, 5] and the possibility of purification of states is considered to be one of the characteristic features differentiating quantum theory from its classical counterpart [6, 7]. While there is a generally accepted definition of purity for states, when it comes to quantum channels there are several proposed definitions of purity in play, each of which has its drawbacks. There is for instance the definition of *atomicity* used in reconstructions of quantum theory [7]. This definition is very general in that it can be defined for any generalized probabilistic theory [4], but it has the drawback that even a canonical map like the identity will not always be pure. Purity can also be defined in terms of leaks [23], or using orthogonal factorization [12]. These definitions

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work well when considering finite dimensional spaces with a well-behaved sequential product, but when considering more general quantum systems like von Neumann algebras, they fail to reproduce many desirable properties. More fundamentally, both definitions require a notion of tensor product, the existence of which for EJAs is an open question [2]. Instead we will use a definition originally introduced for studying pure maps between von Neumann algebras [27, 26, 25], and that also defines the main concept of purity in *effectus theory* [8, 28]. While the other definitions of purity are related to the existence of *purifications* of states, this definition is related to the existence of *Paschke dilations* [26]. Following this definition, a map is pure when it is a composition of a *filter* and a *corner*. Filters represent a certain restricted class of measurements (named after polarization filters). Corners correspond to the act of restricting a system to a subsystem. Filters and corners are defined by universal properties, and thus this definition of purity makes sense even in the very general setting of effectus theory. In effectus theory, filters and corners are known as *quotients* and *comprehensions* respectively, since they correspond to those logical operations (with the reversed direction of maps). Quotients and comprehension appear in a multitude of categories (outside effectus theory) as a chain of adjunctions [9].

In this paper we show that Euclidean Jordan algebras indeed have pure maps as described above, and furthermore that the pure maps can be organised in a dagger category. This is noteworthy for several reasons. First, the other definitions of purity on quantum systems are in general not closed under composition, or taking the adjoint. Hence this shows that this definition of purity seems to be the right one for EJAs. Second, that the pure maps form a dagger category, which is the main result needed to show that the category $\mathbf{EJA}_{psu}$ of EJAs with positive subunital maps between them is a †-*effectus* [28]. The notion of †-effectus was introduced to give an abstract categorical framework for studying von Neumann algebras, and in fact the only known non-trivial example of a †-effectus was the category of von Neumann algebras. This paper therefore establishes that $\mathbf{EJA}_{psu}$ is indeed quite similar to the category of von Neumann algebras. Finally, it was shown in previous work by the third author [30] that a generalised probabilistic theory which is also a †-effectus, must be a subcategory of $\mathbf{EJA}_{psu}$. Hence, this paper proves the converse result, establishing a novel characterization of the category of Euclidean Jordan algebras.

To prove that the composition of pure maps is pure, we have found a new generalization of the *polar decomposition theorem* to EJAs, which might be of independent interest.

Motivated by effectus theory, we also undertake a study of the *possibilistic* structure of the maps between EJAs. This means that we do not look at the actual probabilities, but instead only consider whether probabilities are nonzero. In this way we define a duality between maps that we call $\diamond$-*adjointness*, that generalises the regular adjoints present by the Hilbert space structure of EJAs. We can similarly also define $\diamond$-self-adjointness and $\diamond$-positivity. We completely characterize the pure $\diamond$-positive maps and show that they exactly correspond to a generalization of the sequential product maps $b \mapsto \sqrt{a}b\sqrt{a}$ in von Neumann algebras, (and we'll deduce from this that pure $\diamond$-positivity and †-positivity coincide.) This result can be seen as a characterization of the sequential product like the ones given in [15, 32, 25].

The paper is structured as follows. In the next section we review some of the basic structure present in Euclidean Jordan algebras. We also refer the reader to the appendix for a concise largely self-contained introduction to EJAs. In section 3 we introduce the notion of *filters* and *corners* and study our proposed notion of purity. Finally in section 4 we use the possibilistic $\diamond$ structure to characterize the sequential product maps as the unique pure $\diamond$-positive maps.

2 Preliminaries

We begin by giving a definition of Euclidean Jordan algebras and some motivating examples. Afterwards we will review some of the basic theory necessary to develop our results.

Definition 1. A *Jordan algebra* $(E,*,1)$ is a real unital commutative (possibly non-associative) algebra satisfying the *Jordan identity*: $(a*b)*(a*a) = a*(b*(a*a))$. We call a Jordan Algebra E with an inner product $\langle\cdot,\cdot\rangle$ a *Euclidean Jordan algebra* (EJA) when the inner product turns E into a real Hilbert space with $\langle a*b,c\rangle = \langle b,a*c\rangle$ for all $a,b,c \in E$.

Note 2. In the original (and frequently used) definition, one additionally requires a Euclidean Jordan algebra to be finite dimensional. The possibly infinite-dimensional version we use is also called a JH-algebra [11], although we additionally require unitality.

Example 3. Let F be the field of real numbers, the field of complex numbers, or the division algebra of the quaternions. Let $A \in M_n(F)$ be an $n \times n$ matrix over F. We call A *self-adjoint* when $A_{ij} = \overline{A_{ji}}$ where $\overline{\lambda}$ denotes the standard involution on F. We let the set of self-adjoint matrices be denoted by $M_n(F)^{\mathrm{sa}}$. This set is an Euclidean Jordan Algebra with the Jordan product $A*B := \frac{1}{2}(AB+BA)$, inner product $\langle A,B\rangle := \mathrm{tr}(AB)$ and identity matrix as unit.

If $n = 3$ and F is the algebra of octonions, then the algebra $M_3(F)^{\mathrm{sa}}$ is also a Euclidean Jordan algebra, which is called *exceptional*.

Example 4. For any real Hilbert space H (possibly infinite-dimensional), the set $E := H \oplus \mathbb{R}$ is a Euclidean Jordan Algebra with $(a,t)*(b,s) := (sa+tb, \langle a,b\rangle + ts)$ and $\langle (a,t),(b,s)\rangle = \langle a,b\rangle + ts$. Such EJAs are called *spin-factors*.

The Jordan–von Neumann–Wigner classification theorem [18] asserts that any finite-dimensional EJA is the direct sum of the finite-dimensional examples given above. This statement is still true for our possibly infinite-dimensional class of EJAs, but then the spin-factors must also be allowed to be infinite-dimensional (a proof of this fact can be found in the appendix as corollary 54).

Definition 5. Let E be an EJA. We call $a \in E$ *positive* (and write $a \geq 0$) when there exists a $b \in E$ such that $a = b*b$. We write $a \geq c$ when $a - c \geq 0$. We call a linear map between EJAs $f\colon E \to F$ *positive* when $f(a) \geq 0$ for all $a \geq 0$. A map f is called *unital* whenever $1_F = f(1_E)$ and *subunital* provided $f(1_E) \leq 1_F$. We let $\mathbf{EJA}_{psu}$ denote the category of Euclidean Jordan algebras with positive linear subunital maps between them.

For the matrix algebras the definition of positivity coincides with the regular definition of a positive matrix. For a spin factor $H \oplus \mathbb{R}$ we have $(a,t) \geq 0$ iff $t \geq \|a\|_2$, so that the set of positive elements forms the positive light-cone in a Lorentzian space-time.

Definition 6. A positive unital linear map $\omega\colon E \to \mathbb{R}$ is called a *state* on E. An *effect* is a positive subunital linear map $a\colon \mathbb{R} \to E$ and corresponds to an $a \in E$ with $0 \leq a \leq 1$. We will also call such an element $0 \leq a \leq 1$, an effect.

Because E is a Hilbert space, any map $f : E \to \mathbb{R}$ is of the form $f(a) = \langle a,b\rangle$ for some specific b. As we will see that the inner product of positive elements is positive (proposition 47), any state on an EJA will be given by a positive $a \in E$ with $\langle a,1\rangle = 1$. For an effect a we write $a^\perp := 1 - a$ to denote its *complement*.

We wish to define a notion of purity for the category $\mathbf{EJA}_{psu}$. The canonical example of a pure map in a quantum system is the Lüders map $B \mapsto ABA$ for some fixed matrix A, and hence we would desire the generalization of these maps to EJAs to also be pure. It also seems reasonable that any isomorphism should be pure. As we are working on a Hilbert space, every map has an adjoint, and it makes sense to let the adjoint of a pure map be pure again. Finally, as we want our pure maps to be closed under composition, any composition of the examples here above should also be pure. We will see how to make these ideas exact, and in particular how to generalize the idea of a Lüders map to arbitrary EJAs. But first we need to know more about the structure of EJAs.

Theorem 7. *Let $(E,*,1)$ be an EJA. Sums of positive elements are again positive, so that the set of positive elements forms a* cone*. Furthermore:*

1. *The unit* 1 *is a* strong Archimedean unit*: for all $a \in E$ there exists $n \in \mathbb{N}$ such that $-n1 \leq a \leq n1$, and if $a \leq \frac{1}{n}1$ for all $n \in \mathbb{N}, n \neq 0$ then $a \leq 0$.*
2. *As a consequence the algebra is an* order unit space *so that $\|a\| := \inf\{r \in \mathbb{R}\ ;\ -r1 \leq a \leq r1\}$ is a norm, called the* order unit norm*, for which $a \leq \|a\|1$.*
3. *The order unit norm defined above, and the norm induced by the inner product are equivalent so that the topologies they induce are the same. The space is therefore also complete in the order unit norm topology.*
4. *The space is* self-dual*: $a \geq 0$ if and only if $\langle a,b\rangle \geq 0$ for all $b \geq 0$.*
5. *The algebra E is bounded directed complete and every state is normal so that E is in fact a* JBW-algebra *[16].*

Proof. Since to our knowledge there isn't a textbook solely dedicated to infinite-dimensional EJAs we supply a relatively self-contained proof of these claims in the appendix. □

Definition 8. For each $a \in E$ let $L_a : E \to E$ be the left-multiplication operator of a, that is: $L(a)b := a * b$. The seemingly oddly named but important *quadratic representation* of the Jordan algebra is a linear map $Q_a \colon E \to E$ for each $a \in E$, defined by $Q_a := 2L(a)^2 - L(a^2)$.

The definition of the quadratic representation Q_a might look arbitrary, but in the case of a matrix algebra with the standard Jordan multiplication they are exactly the Lüders maps $Q_A B = ABA$. The quadratic representation maps will therefore act as the Jordan equivalent of these maps in quantum theory. The following proposition establishes a few basic properties of the maps Q_a, that can be easily checked to hold for a matrix algebra.

Proposition 9. *For any EJA E and $a,b,c \in E$, the following hold.*

1. $Q_1 = \mathrm{id}$.
2. $Q_a 1 = a^2$.
3. $\langle Q_a b, c\rangle = \langle b, Q_a c\rangle$.
4. $Q_a b = 0 \iff Q_b a = 0 \iff a * b = 0$.
5. $Q_{Q_a b} = Q_a Q_b Q_a$ *(this is known in the literature as the* fundamental equality*).*
6. *Q_a is invertible if and only a is invertible. In that case, we have $Q_a^{-1} = Q_{a^{-1}}$.*
7. *Q_a is a positive operator(even when a is not positive). If it is invertible, it is an order automorphism.*

Proof. Points 1,2 and 3 are trivial. Point 4 can be found in [1, Lemma 1.26]. Point 5 is usually proven using MacDonalds theorem (see [16, Theorem 2.4.13]), but see also [29]. Point 6 is proven in [1, Lemma 1.23] and finally 7 is given by [1, Theorem 1.25]. □

Note that while the Jordan multiplication maps L_a are not positive, and therefore are not maps in the category $\mathbf{EJA}_{psu}$, the quadratic representation maps Q_a are positive and when $\|a\| \leq 1$ the maps are subunital so that they do happen to lie in $\mathbf{EJA}_{psu}$.

As the fundamental equality $Q_{Q_a b} = Q_a Q_b Q_a$ will be important in the proofs below, let's unfold it for matrix algebras. The expression $Q_{Q_A B}C$ is equal to $(ABA)C(ABA)$, while the right-hand side of the equation $Q_A Q_B Q_A C$ is equal to $A(B(ACA)B)A$. Thus the equality follows (in this special case) from associativity of matrix multiplication.

Definition 10. We call an element p of an EJA an *idempotent* iff $p^2 := p * p = p$. An idempotent is automatically positive and below the identity. We call an idempotent *atomic* if there is no non-zero idempotent strictly smaller than it. Two idempotents p and q are *orthogonal* when $p * q = 0$ or equivalently $\langle p, q \rangle = 0$.

In a matrix algebra the idempotents are precisely the projections. The quadratic representation map Q_p is then the projection map $Q_p(a) = pap$. The following proposition again contains claims that are easily verified to hold true in any matrix algebra, but which for general EJAs must be proven with a bit of care.

Proposition 11. *Let p be an idempotent. Then Q_p is idempotent, and furthermore for $0 \leq a \leq 1$ we have*

$$Q_p a = 0 \iff \langle a, p \rangle = 0 \quad \textit{and} \quad Q_p a = a \iff a \leq p \iff p * a = a.$$

Proof. The fundamental equality (number 5 of the previous proposition) implies that when p is an idempotent: $Q_{Q_p 1} = Q_{p^2} = Q_p = Q_p Q_1 Q_p = Q_p Q_p$, so that Q_p is a positive idempotent operator, symmetric with respect to the inner product. When $Q_p a = 0$ we have $0 = \langle Q_p a, 1 \rangle = \langle a, Q_p 1 \rangle = \langle a, p \rangle$. Conversely, if $\langle a, p \rangle = 0$, then $\langle Q_p a, 1 \rangle = 0$. As $Q_p a$ is positive, we must have $c^2 = Q_p a$ for some c and so $0 = \langle c, c \rangle$, hence $c = 0$ and $Q_p a = 0$. Now suppose $Q_p a = a$, then because we have $a \leq 1$ we also have by positivity of Q_p, $a = Q_p a \leq Q_p 1 = p$. When $a \leq p$ then by definition there is a $r \geq 0$ such that $a + r = p$. Now $0 = \langle 0, 1 \rangle = \langle p * p^\perp, 1 \rangle = \langle p, p^\perp \rangle = \langle a + r, p^\perp \rangle = \langle a, p^\perp \rangle + \langle r, p^\perp \rangle$. By self-duality (see theorem 7) each of these terms is positive so that we must have $0 = \langle a, p^\perp \rangle = \langle a, Q_{p^\perp} 1 \rangle = \langle Q_{p^\perp} a, 1 \rangle$. Since $Q_{p^\perp} a \geq 0$ this can only be the case when $Q_{p^\perp} a = 0$ so that $p^\perp * a = 0$ from which we get $p * a = a$. To complete the proof, assume $p * a = a$. We will show $Q_p a = a$. This follow readily from the definition and assumption: $Q_p a = 2p * (p * a) - (p * p) * a = 2a - a = a$. □

Like in quantum theory, we have a spectral theorem for elements of a Euclidean Jordan algebra.

Proposition 12. *Let a be an element of an EJA. Then there exists a number n, real numbers $\lambda_1, \ldots, \lambda_n$ and orthogonal atomic idempotents $p_1, \ldots, p_n$ such that $a = \sum_{i=1}^n \lambda_i p_i$.*

Proof. Proven in the appendix under Corollary 46. □

Again like in quantum theory, we can for each element consider its 'range' where it acts non-trivially. We will denote this by a 'ceiling': $\lceil a \rceil$. The ceiling will play an important role later when we want to restrict an EJA to certain subspaces.

Proposition 13. *For an effect $a \in E$ of an EJA E (i.e. $0 \leq a \leq 1$), we can find $\lambda_i > 0$ and orthogonal atomic idempotents p_i with $a = \sum_i \lambda_i p_i$. With such a decomposition, we define $\lceil a \rceil = \sum_i p_i$. This is the least idempotent above a (and thus independent of choice of decomposition). We denote the de Morgan dual by $\lfloor a \rfloor = \lceil a^\perp \rceil^\perp$. This is the greatest idempotent below a.*

Proof. Simply leaving out those terms where $\lambda_i = 0$, we can find $\lambda_i \neq 0$ and orthogonal atomic idempotent p_i with $a = \sum_i \lambda_i p_i$. As the inner product of positive elements is positive, we find $0 \leq \langle a, p_i\rangle = \lambda_i \langle p_i, p_i\rangle$. As $\langle p_i, p_i\rangle > 0$ we must have $\lambda_i > 0$, as promised.

Next, we prove that $\lceil a\rceil$ is the least idempotent above a. Let q be idempotent such that $a \leq q$. Then also $\lambda_i p_i \leq q$. By proposition 11 we then have $q * (\lambda_i p_i) = \lambda_i p_i$ so that also $q * p_i = p_i$. Hence $q * \lceil a\rceil = \sum_i q * p_i = \sum_i p_i = \lceil a\rceil$. Again by proposition 11 we conclude that $\lceil a\rceil \leq q$. Thus $\lceil a\rceil$ is indeed the least idempotent above a. □

3 Filters and Corners

With the preliminaries out of the way we will start to look at additional structure that is present in the category $\mathbf{EJA}_{psu}$. The proofs in this section are heavily inspired by [25, 27] where the existence of this structure was shown for the category of von Neumann algebras. As stated in the introduction, our notion of purity is based on *filters* and *corners*. In this section we will give their formal definition and establish their existence.

Definition 14. Let $q \in E$ be an effect. A *corner* for q is a positive subunital linear map $\pi\colon E \to \{E|q\}$ such that $\pi(1) = \pi(q)$ and which is *initial* with this property: if $g\colon E \to F$ is another positive subunital linear map such that $g(1) = g(q)$ then there must exist a *unique* $\overline{g}\colon \{E|q\} \to F$ such that $\overline{g} \circ \pi = g$. In the form of a diagram:

$$\begin{tikzcd} \{E|q\} \arrow[d, dotted, "\overline{g}"'] & E \arrow[l, "\pi"'] \arrow[dl, "g"] \\ F & \end{tikzcd}$$

Note 15. The name of ‘corner’ is inspired by the appearance of these maps when considering matrix algebras, in which case they can be arranged to project onto a corner of the matrix. When $g(1) = g(q)$ we of course have $g(q^\perp) = 0$, and hence everything orthogonal to q is ‘thrown away’ by this map. Thus these maps project onto a subspace *where q holds*. The universal property tells us that $\{E|q\}$ is the largest such subspace.

Definition 16. Let $q \in E$ be an effect. A *filter* for q is a positive subunital linear map $\xi : E_q \to E$ such that $\xi(1) \leq q$ and that is *final* with this property: if $f\colon F \to E$ is another positive subunital linear map such that $f(1) \leq q$ then there must exist a unique $\overline{f}\colon F \to E_q$ such that $\xi \circ \overline{f} = f$. In the form of a diagram:

$$\begin{tikzcd} E_q \arrow[r, "\xi"] & E \\ F \arrow[u, dotted, "\overline{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

Note 17. The name ‘filter’ comes from the function of these maps in quantum theory as describing the act of updating the action of effects based on previous measurement outcomes, i.e. filtering them. The universal property can be interpreted as stating that E_q is the smallest subsystem of E that can faithfully represent all effects below q.

Since both these types of maps satisfy a universal property, they are (for a given effect) unique up to isomorphism. In particular, given a corner $\pi\colon E \to \{E|q\}$ and an isomorphism $\Theta\colon \{E|q\} \to F$ the map $\Theta \circ \pi$ is again a corner (for q), and furthermore any corner for q is of this form. Similarly when $\xi : E_q \to E$ is a filter, and we have an isomorphism $\Theta\colon F \to E_q$, the map $\xi \circ \Theta$ is also a filter, and any filter for q is of this form. The objects $\{E|q\}$ and E_q are therefore also unique up to isomorphism. In this section we will see that there is a canonical choice of corner and filter for every effect.

As promised, a pure map is defined to be a composition of a corner and a filter:

Definition 18. We call a positive subunital linear map between EJAs $f\colon E \to F$ *pure* when there exists some corner π and some filter ξ (not necessarily for the same effect) such that $f = \xi \circ \pi$.

Note that at the moment it is not yet clear whether pure maps are closed under composition or whether there are any pure maps at all. First, we will study corners a bit more, for which we need some preparation.

Proposition 19. *[1, Proposition 1.43] (Peirce-decomposition) Let E be an EJA with an idempotent $p \in E$. Then $E_1(p) := Q_p(E) := \{Q_p(a) \ ; \ a \in E\}$ is a sub-EJA of E consisting precisely of those elements of E for which $Q_p(a) = a$.*

Definition 20. Let E be an EJA with an effect $q \in E$. Then $\{E|q\} := E_1(\lfloor q \rfloor) = \{E|\lfloor q \rfloor\}$ and $E_q := E_1(\lfloor q^\perp \rfloor^\perp) = E_1(\lceil q \rceil)$.

For an idempotent p, we have $\{E|p\} = E_p$. After a few brief lemmas, we will show that $\{E|q\}$ and E_q are (the objects for) a corner and a filter respectively.

Lemma 21. *Let $\omega\colon E \to \mathbb{R}$ be any positive linear map such that $\omega(p) = \omega(1)$ for some idempotent p. Then $\omega(Q_p a) = \omega(a)$ for all a.*

Proof. Given such a map ω we can define $\langle a,b\rangle_\omega := \omega(a * b)$ which is a bilinear positive semi-definite form. It then satisfies the Cauchy–Schwarz inequality: $|\langle a,b\rangle_\omega|^2 \le \langle a,a\rangle_\omega \langle b,b\rangle_\omega$. Since $\omega(p) = \omega(1)$ we also have $\omega(p^\perp) = 0$. But then $|\omega(p^\perp * a)|^2 \le \omega(p^\perp * p^\perp)\omega(a * a) = 0$ so that $\omega(p^\perp * a) = 0$. Then obviously $\omega(p * a) = \omega(a)$ from which we also get $\omega(p * (p * a)) = \omega(a)$. Unfolding the definition of Q_p we then get $\omega(Q_p a) = \omega(a)$. □

Corollary 22. *Let $g\colon E \to W$ be a positive linear map between EJAs such that $g(p) = g(1)$ for some idempotent p. Then $g(Q_p a) = g(a)$ for all a.*

Proof. Follows by the previous lemma because the states separate the maps. □

Lemma 23. *Let $g\colon E \to F$ be any positive linear map between EJAs such that $g(q) = g(1)$ for some effect q. Then $g(\lfloor q \rfloor) = g(1)$.*

Proof. $g(q) = g(1)$ means that $g(q^\perp) = 0$. Write $q^\perp = \sum_i \lambda_i p_i$ where $\lambda_i > 0$, then $0 = g(q^\perp) = \sum_i \lambda_i g(p_i)$. Since g is a positive map and $\lambda_i > 0$ and $p_i \ge 0$ this implies that $g(p_i) = 0$. But since $\lceil q^\perp \rceil = \sum_i p_i$ by proposition 13, we see $g(\lceil q^\perp \rceil) = 0$, so that $g(\lfloor q \rfloor) = g(\lceil q^\perp \rceil^\perp) = g(1)$. □

Proposition 24. *Let q be an effect of an EJA E. Define $\pi_q\colon E \to \{E|q\} = E_1(\lfloor q \rfloor)$ to be $\pi_q = r \circ Q_{\lfloor q \rfloor}$ where $r : E \to E_1(\lfloor q \rfloor)$ is the orthogonal projection map with respect to the Hilbert space structure. Then π_q is a corner for q. We will refer to this map as the* standard corner *for q.*

Proof. First of all we have $\pi_q(1) = (r \circ Q_{\lfloor q \rfloor})(1) = r(\lfloor q \rfloor) = r \circ Q_{\lfloor q \rfloor}(q) = \pi_q(q)$. Now suppose $g\colon E \to F$ is a positive subunital linear map such that $g(q) = g(1)$. We must show that there is a unique $\overline{g} : \{E|q\} \to F$ such that $\overline{g} \circ \pi_q = g$.

By the previous lemma $g(\lfloor q \rfloor) = g(1)$. Define $\overline{g} : E_1(\lfloor q \rfloor) \to F$ as the restriction of g. To prove that $\overline{g} \circ \pi_q = g$, we need to show that $g(a) = g(Q_{\lfloor q \rfloor} a)$ for all a. This follows from corollary 22. For uniqueness suppose we have a $h\colon E_1(\lfloor q \rfloor) \to F$ such that $h \circ \pi_q = g = \overline{g} \circ \pi_q$. Let $a \in E_1(\lfloor q \rfloor)$, then we can see a as an element of E with $\pi_q(a) = a$, so that $h(a) = h(\pi_q(a)) = \overline{g}(\pi_q(a)) = \overline{g}(a)$, as desired. □

For a positive $q = \sum_i \lambda_i p_i$ we can define a positive square root $\sqrt{q} := \sum_i \sqrt{\lambda_i} p_i$. This is the unique positive element such that $\sqrt{q} * \sqrt{q} = q$.

Proposition 25. *Let q be an effect of an EJA E. Define $\xi_q\colon E_q \to E$ to be the map $\xi_q := Q_{\sqrt{q}} \circ \iota$ where ι is the inclusion $\iota\colon E_q = E_1(\lceil q \rceil) \to E$, then ξ_q is a filter for q. We will refer to this map as the* standard filter *for q.*

Proof. Clearly $\xi_q(1) = q$. We need to show that this map is final with respect to this property. To this end, assume $f\colon F \to E$ is any positive subunital linear map with $f(1) \leq q$. We have to show that there is a unique $\overline{f} : F \to E_q$ such that $\xi_q \circ \overline{f} = f$.

Clearly $f(1) \leq q \leq \lceil q \rceil$. Thus for all $0 \leq p \leq 1$ we have $f(p) \leq \lceil q \rceil$ so that $f(p) \in E_1(\lceil q \rceil)$ which means that we can restrict the codomain of f to $E_1(\lceil q \rceil) = E_q$. Writing q as $q = \sum_i \lambda_i p_i$ for some $\lambda_i > 0$ and orthogonal atomic projections p_i, we see it has a pseudo-inverse $q^{-1} := \sum_i \lambda_i^{-1} p_i$ such that $Q_{\sqrt{q^{-1}}} q = q * q^{-1} = \lceil q \rceil$. In particular $Q_{\sqrt{q^{-1}}} f(p) \leq Q_{\sqrt{q^{-1}}} q = \lceil q \rceil$. It follows that the map $\overline{f} : W \to E_q$ given by $\overline{f}(a) = Q_{\sqrt{q^{-1}}} f(a)$ is subunital and obviously $(\xi_q \circ \overline{f})(a) = Q_{\sqrt{q}} Q_{\sqrt{q^{-1}}} f(a) = Q_{\lceil q \rceil} f(a) = f(a)$ by proposition 11.

Now for uniqueness, suppose that we have a $g : F \to E_q$ such that $\xi_q \circ g = f$. Then $Q_{\lceil q \rceil} \circ \iota \circ g = Q_{\sqrt{q^{-1}}} \circ Q_{\sqrt{q}} \circ \iota \circ g = Q_{\sqrt{q^{-1}}} \circ f$. As $Q_{\lceil q \rceil}$ acts as the identity on all elements coming from $E_1(\lceil q \rceil)$ it can be removed from the expression. By taking the corestriction of both sides to $E_1(\lceil q \rceil)$ we see that $g = Q_{\sqrt{q^{-1}}} \circ f = \overline{f}$, as desired. □

Let q be an effect in some EJA E. Note that the EJA associated to the standard filter of q is $E_q = E_1(\lceil q \rceil)$ while the EJA of the standard corner is $\{E|q\} = E_1(\lfloor q \rfloor)$. Therefore, when q is not an idempotent, we have $E_q \neq \{E|q\}$and hence we cannot compose the standard filter and corner of q. However, if one takes the the standard corner of $\lceil q \rceil$ instead of q, then $E_q = \{E|\lceil q \rceil\}$ and the filter and corner can indeed be composed. It is easy to see that this composition $\xi_q \circ \pi_{\lceil q \rceil}$ equals $Q_{\sqrt{q}}$. This shows that the Q_a maps are indeed pure (for positive a). Also note that the standard filter and the standard corner for the unit $1 \in E$ are simply the identity and since a filter composed with an isomorphism is still a filter we see that indeed all isomorphisms are pure. Next we will show that pure maps are closed under composition.

3.1 The polar decomposition theorem

The composition of two filters is again a filter, which can be shown in the general setting of an effectus [28, 197IX]. In our setting, it is easy to see that the composition of two corners is again a corner. Suppose we know that a composition of a filter with a corner 'in the wrong order' can be written 'in the correct order', i.e. that we can always write $\pi \circ \xi$ as $\xi' \circ \pi'$ for some different filter ξ' and corner π'. Then when we have pure maps $f = \xi_1 \circ \pi_1$ and $g = \xi_2 \circ \pi_2$ their composition is $f \circ g = \xi_1 \circ \pi_1 \circ \xi_2 \circ \pi_2$ and we can interchange π_1 and ξ_2 to get a composition of two corners with two filters, which is indeed pure. So what we need to show to establish that our definition of purity is closed under composition is that filters and corners can be interchanged as assumed before. It is sufficient to prove this for the standard corner and filter. To summarize, we must show that for a given effect q and idempotent p there exist effects a and b such that $\pi_p \circ \xi_q = \xi_a \circ \Phi \circ \pi_b$ where Φ is some isomorphism.

The same problem of establishing that pure maps are closed under composition in von Neumann algebras is related to the existence of *polar decompositions* of elements. By applying the polar decomposition to $\sqrt{p}\sqrt{q}$ for positive p and q we have a partial isometry u such that $\sqrt{p}q\sqrt{p} = u(\sqrt{q}p\sqrt{q})u^*$. The isomorphism Φ above is then the conjugation map $a \mapsto uau^*$ restricted to the appropriate domains. Partial isometries can also be defined for EJAs, and an analogous polar decomposition theorem can be stated:

Definition 26. Let $\Phi\colon E \to E$ be a positive linear map on an EJA E. We denote its adjoint with respect to the inner product by Φ^*, that is the unique linear map with $\langle \Phi^*(a), b\rangle = \langle a, \Phi(b)\rangle$. We call Φ a *partial isometry* when $\Phi\Phi^*$ and $\Phi^*\Phi$ are projections.

Theorem 27. *Polar Decomposition: Let p and q be positive elements of a Euclidean Jordan algebra E. There exists a partial isometry $\Phi : E \to E$ such that $Q_q Q_p = \Phi Q_{\sqrt{Q_p q^2}}$, $\Phi(1) = \lceil Q_q p\rceil$, $\Phi^*(1) = \lceil Q_p q\rceil$, $\Phi^*\Phi = Q_{\lceil Q_p q\rceil}$ and $\Phi\Phi^* = Q_{\lceil Q_q p\rceil}$.*

To see how this is related to polar decomposition note that if we plug in the unit in $Q_q Q_p$ that we will get $Q_q p^2 = Q_q Q_p 1 = \Phi Q_{\sqrt{Q_p q^2}} 1 = \Phi(Q_p q^2)$. This polar decomposition theorem should not be confused with the already established notion of polar decomposition in Jordan algebras (see for instance [14, Ch. VI]) that asserts the existence of a Jordan isomorphism between any two maximal collections of orthogonal atomic idempotents in a simple EJA. In the theory of generalized probabilistic theory, this property is also known as *strong symmetry* [3].

The rest of this section is dedicated to proving theorem 27 and showing how it proves that pure maps are closed under composition.

First need a new notion:

Definition 28. Let $f : E \to F$ be a positive linear map between EJAs. The *image* of f (if it exists) is the smallest effect q such that $f(q) = f(1)$. We will denote the image of f by $\operatorname{im} f$.

Proposition 29. *Any positive linear map $f\colon E \to F$ between Euclidean Jordan algebras has an image. This image is always an idempotent.*

Proof. Because of lemma 23 a positive linear map f satisfies $f(q) = f(1)$ if and only if $f(\lfloor q\rfloor) = f(1)$ so we can restrict to effects which satisfy $q = \lfloor q\rfloor$; viz. the idempotents.

By theorem 7 EJAs are JBW-algebras (see [1]) so that the idempotents form a complete lattice. Furthermore, all states are normal, meaning they preserve infima. Because the states separate the maps, all maps are also normal. We conclude that $\operatorname{im} f = \inf\{p\ ;\ p^2 = p, f(p) = f(1)\}$ exists and that $f(\operatorname{im} f) = f(\inf\{p\ ;\ f(p) = f(1)\}) = \inf_p f(p) = f(1)$. □

Proof of Theorem 27. Let $\Phi = Q_q Q_p Q_{(Q_p q^2)^{-1/2}}$ so that $\Phi^* = Q_{(Q_p q^2)^{-1/2}} Q_p Q_q$because $Q_a^* = Q_a$ for all a. Then $Q_{(Q_p q^2)^{1/2}} \Phi^* = Q_{\lceil Q_p q^2\rceil} Q_p Q_q = Q_p Q_q$. By taking adjoints we then get $Q_q Q_p = \Phi Q_{\sqrt{Q_p q^2}}$ as desired. Note that

$$\Phi^*\Phi \;=\; Q_{(Q_p q^2)^{-1/2}} Q_p Q_q Q_q Q_p Q_{(Q_p q^2)^{-1/2}} \;=\; Q_{(Q_p q^2)^{-1/2}} Q_{Q_p q^2} Q_{(Q_p q^2)^{-1/2}} \;=\; Q_{\lceil Q_p q^2\rceil}$$

by application of the fundamental equality. Since $\lceil Q_p q^2\rceil = \lceil Q_p \lceil q^2\rceil\rceil = \lceil Q_p\lceil q\rceil\rceil = \lceil Q_p q\rceil$ this can be simplified to $\Phi^*\Phi = Q_{\lceil Q_p q\rceil}$. Because $\Phi^*\Phi$ is a projection we can use [20, Proposition 6.1.1] to conclude that $\Phi\Phi^*$ must be projection as well. By a simple calculation $\Phi^*(1) = \lceil Q_p q \rceil$ so that it remains to show that $\Phi(1) = \lceil Q_q p \rceil$ and that $\Phi\Phi^*(1) = \lceil Q_q p \rceil$ since this latter condition (in combination with the knowledge that $\Phi\Phi^*$ is a projection) is sufficient to conclude that $\Phi\Phi^* = Q_{\lceil Q_q p \rceil}$.

Suppose $\Phi^*(s) = 0$ then $\langle 1, \Phi^*(s) \rangle = 0 = \langle Q_p Q_q s, (Q_p q^2)^{-1/2} \rangle$. Since $\lceil (Q_p q^2)^{-1/2} \rceil = \lceil Q_p q^2 \rceil$ this gives $0 = \langle Q_p Q_q s, Q_p q^2 \rangle = \langle s, Q_q Q_{p^2} Q_q 1 \rangle = \langle s, Q_{Q_q p^2} 1 \rangle = \langle s, Q_q p^2 \rangle$. We conclude that $\Phi^*(s) = 0$ if and only if $s \perp \lceil Q_q p \rceil$ so that $\operatorname{im} \Phi^* = \lceil Q_q p \rceil$. We of course also have $0 = \langle 1, \Phi^*(s) \rangle = \langle \Phi(1), s \rangle$ so that $\lceil \Phi(1) \rceil = \operatorname{im} \Phi^* = \lceil Q_q p \rceil$. Because $\langle 1, (\Phi(1))^2 \rangle = \langle \Phi(1), \Phi(1) \rangle = \langle \Phi^* \Phi(1), 1 \rangle = \langle \lceil Q_p q \rceil, 1 \rangle = \langle \Phi^*(1), 1 \rangle = \langle 1, \Phi(1) \rangle$ we conclude that $\Phi(1) = (\Phi(1))^2$ so that $\Phi(1) = \lceil \Phi(1) \rceil = \lceil Q_q p \rceil$.

By a similar argument as above we can show that $\operatorname{im} \Phi\Phi^* = \lceil (\Phi\Phi^*)(1) \rceil$ which gives $(\Phi\Phi^*)(1) \leq \operatorname{im} \Phi\Phi^* \leq \operatorname{im} \Phi^* = \lceil Q_q p \rceil$. For the other direction we recall that we had $Q_q Q_p = \Phi Q_{\sqrt{Q_p q^2}}$ so that $Q_{Q_q p^2} = Q_q Q_p Q_p Q_q = \Phi Q_{\sqrt{Q_p q^2}} Q_{\sqrt{Q_p q^2}} \Phi^* = \Phi Q_{Q_p q^2} \Phi^* \leq \| Q_{Q_p q^2} \| \Phi\Phi^*$. By inserting the unit into the expression and taking the ceiling we are left with $\lceil Q_q p \rceil = \lceil Q_{Q_q p^2} 1 \rceil \leq \lceil (\Phi\Phi^*)(1) \rceil$. □

Proposition 30. *Let $\xi_q : E_1(\lceil q \rceil) \to E$, $\xi_q = Q_{\sqrt{q}} \colon \iota$ be the standard filter of an effect q and $\pi_p \colon E \to E_1(p)$, $\pi_p = r \circ Q_p$ be the standard corner of an idempotent effect p. Then $\pi_p \circ \xi_q = \xi_a \circ \Phi \circ \pi_b$ where a and b are some effects and Φ is an isomorphism. In other words: $\pi_p \circ \xi_q$ is pure.*

Proof. Define the shorthand $q\&p := Q_{\sqrt{q}}(p)$. Let $f = \pi_p \circ \xi_q : E_{\lceil q \rceil} \to E_p$. Because $f(1) = \pi_p(\xi_q(1)) = \pi_p(q) = p\&q$ we see that there must exist $\overline{f} : E_{\lceil q \rceil} \to E_{\lceil p\&q \rceil}$ such that $\xi_{p\&q} \circ \overline{f} = f$ where $\xi_{p\&q} : E_{\lceil p\&q \rceil} \to E_p$ by the universal property of the filter. This $\overline{f}$ is given by $\overline{f} = Q_{(p\&q)^{-1/2}} \circ f$ so that $\overline{f}(1) = Q_{(p\&q)^{-1/2}}(p\&q) = \lceil p\&q \rceil = 1$ since the codomain is $E_{\lceil p\&q \rceil}$. We will ignore the restriction and inclusion maps present in the filter and corner so that we can write $f = Q_p Q_{\sqrt{q}}$ and similarly $\overline{f} = Q_{(p\&q)^{-1/2}} Q_p Q_{\sqrt{q}}$.

Similar to the argument used in the proof of theorem 27 we can show that $\operatorname{im} \overline{f} = \lceil q\&p \rceil$. Then we can use the universal property of the corner to find a map $\Phi : E_{\lceil q\&p \rceil} \to E_{\lceil p\&q \rceil}$ such that $\Theta \circ \pi_{\lceil q\&p \rceil} = \overline{f}$. Because $\overline{f}$ and $\pi_{\lceil q\&p \rceil}$ are unital, Φ has to be unital as well. Note that Φ is just a restriction of $\overline{f}$ to the appropriate domain and that $\overline{f} = Q_{(p\&q)^{-1/2}} Q_p Q_{\sqrt{q}}$ is exactly the same as Φ^* in the proof of theorem 27. We can conclude as a consequence that $\Phi\Phi^* = Q_{\lceil p\&q \rceil}$ while $\Phi^*\Phi = Q_{\lceil q\&p \rceil}$. These are of course the identity maps on $E_{\lceil p\&q \rceil}$ respectively $E_{\lceil q\&p \rceil}$ so that $\Phi^* = \Phi^{-1}$. We conclude that $f = \xi_{p\&q} \circ \overline{f} = \xi_{p\&q} \circ \Phi \circ \pi_{\lceil q\&p \rceil}$ where Φ is an isomorphism. □

Corollary 31. *The composition of pure maps is pure.*

Proof. Let f_1 and f_2 be pure, then $f_i = \xi_i \circ \Theta_i \circ \pi_i$, so that $f_1 \circ f_2 = \xi_1 \circ \Theta_1 \circ \pi_1 \circ \xi_2 \circ \Theta_2 \circ \pi_2 = \xi_1' \circ \xi' \circ \Theta' \circ \pi' \circ \pi_2'$ by the previous proposition and writing $\xi_1 \circ \Theta_1 = \xi_1'$ and $\Theta_2 \circ \pi_2 = \pi_2'$ where ξ_1' and π_2' are again a filter respectively a corner. But now since a composition of filters is again a filter and a composition of corners is again a corner we see that $f_1 \circ f_2$ is indeed pure. □

4 Diamond adjointness and positivity

Since Euclidean Jordan algebras are also Hilbert spaces, we can find for any positive map an adjoint with respect to the inner product. This means that the category of all EJAs with positive (not necessarily subunital) maps is a dagger category. The adjoint of a subunital map is not necessarily subunital again however, so that $\mathbf{EJA}_{psu}$ is *not* a dagger category. However, the set of pure maps is closed under taking adjoints (which can be shown by a simple case analysis), so that this restricted category *is* a dagger category. This is not an accident: a consequence of the results in this section will be that $\mathbf{EJA}_{psu}$ is a †-*effectus*, a type of structure already introduced as an abstract version of the category of von Neumann algebras where the pure maps also form a dagger category. For more information regarding †-effectuses we refer to [28].

An important notion in an effectus is that of $\diamond$-adjointness. This is a possibilistic alternative to adjointness that can be defined even when there is no obvious choice of dagger. In this section we will study $\diamond$-adjointness in $\mathbf{EJA}_{psu}$ and show that it behaves similarly to $\diamond$-adjointness in von Neumann algebras. In particular we will give a characterization of pure $\diamond$-self-adjoint maps and show that a pure $\diamond$-*positive* map $f : E \to E$ is completely determined by its image at the unit: $f = Q_{\sqrt{f(1)}}$ and thus that the only pure $\diamond$-positive maps are the quadratic representation maps Q_a for some positive a. As these quadratic representation maps are the Jordan equivalent of the *sequential product* map $b \mapsto aba$ [32], this can be seen as a new characterization of the sequential product.

Definition 32. Let $f : E \to F$ be a positive subunital map and write $\mathrm{Idem}(E)$ for the set of idempotents of E. Define the maps $f^\diamond\colon \mathrm{Idem}(F) \to \mathrm{Idem}(E)$ and $f_\diamond\colon \mathrm{Idem}(E) \to \mathrm{Idem}(F)$ by

$$f^\diamond(p) \;=\; \lceil p \circ f \rceil \qquad \text{and} \qquad f_\diamond(q) \;=\; \mathrm{im}(Q_q \circ f).$$

We say that $f : E \to F$ is $\diamond$-*adjoint* to $g : F \to E$ when $f^\diamond = g_\diamond$ or equivalently $f_\diamond = g^\diamond$ [28]. We call $f : E \to E$ $\diamond$-*self-adjoint* when f is $\diamond$-adjoint to itself, and we call f $\diamond$-*positive* when there exists a $\diamond$-self-adjoint g such that $f = g \circ g$.

It can be shown that $f^\diamond(p) \le q^\perp$ iff $f_\diamond(q) \le p^\perp$ so that the diamond defines a Galois connection between the orthomodular lattices of idempotents. As a result we get a functor $\diamond : \mathbf{EJA}_{psu} \to \mathbf{OMLatGal}$ [28]. Note that $\diamond$-self-adjointness is weaker than regular self-adjointness:

Proposition 33. *Any self-adjoint operator $f : E \to E$ on an EJA E is $\diamond$-self-adjoint. In particular Q_a is $\diamond$-self-adjoint for any $a \in E$. Consequently, Q_a is $\diamond$-positive for positive a.*

Proof. Let f be any self-adjoint operator. It suffices to show $f^\diamond(s) \le t^\perp \iff f^\diamond(t) \le s^\perp$ for all idempotents $s,t \in E$ (see [28, §207III]). This is equivalent to

$$\langle f^\diamond(s), t\rangle \;=\; 0 \quad \iff \quad \langle s, f^\diamond(t)\rangle \;=\; 0 \qquad (s,t \in E \text{ idempotents}). \tag{1}$$

By the spectral theorem $\langle \lceil q \rceil, s\rangle = 0 \iff \langle q, s\rangle = 0$ for any positive q and idempotent s, so (1) is equivalent to $\langle f(s), t\rangle = 0 \iff \langle s, f(t)\rangle = 0$, which clearly holds as f is self-adjoint.

Pick any positive $a \in E$. By the fundamental identity, we have $Q_a = Q_{\sqrt{a}^2} = Q^2_{\sqrt{a}}$, so Q_a is the square of a $\diamond$-self-adjoint map, hence $\diamond$-positive. $\square$

The rest of this section contains the necessary work to prove the following theorem characterizing the pure $\diamond$-positive maps:

Theorem 34. *Let $g : E \to E$ be a pure $\diamond$-positive map and write $p := g(1)$, then $g = Q_{\sqrt{p}}$.*

First, we will need a well-known fact about Jordan algebras for which we need a short lemma.

Lemma 35. *An effect p is called* order-sharp *when $q = 0$ whenever both $q \le p$ and $q \le p^\perp$. An effect p is order-sharp if and only if it is an idempotent.*

Proof. Let a be an order-sharp effect and write $a = \sum_i \lambda_i p_i$. Let $r_i = \min\{\lambda_i, 1-\lambda_i\}$, then $r_i p_i \le a$ and $r_i p_i \le a^\perp = 1 - a$ which implies that $r = 0$, so either $\lambda_i = 1$ or $\lambda_i = 0$ for all i. But then as a is a sum of orthogonal idempotents it is also an idempotent. For the other direction suppose a is idempotent. Let $q \le a$ and $q \le a^\perp$. By $q \le a$ we know that $Q_a q = q$, but we also have $Q_a q \le Q_a a^\perp = 0$ so that $q = 0$. $\square$

Proposition 36. *A unital order isomorphism between EJAs is a Jordan isomorphism: that is, it preserves the Jordan multiplication.*

Proof. Let $\Theta\colon E \to F$ be any unital order-isomorphism between EJAs. As $2(a*b) = (a+b)^2 - a^2 - b^2$ it suffices to show $\Theta(a)^2 = \Theta(a^2)$ for any $a \in E$. Write $\sum_i \lambda_i p_i = a$ for the spectral decomposition of a. As idempotents are exactly the order-sharp elements by the previous lemma and idempotents p, q are orthogonal iff $p \leq 1 - q$, we see that $\Theta(p_i)$ are also pairwise orthogonal idempotents. Thus $\Theta(a)^2 = (\sum_i \lambda_i \Theta(p_i))^2 = \Theta(\sum_i \lambda_i^2 p_i) = \Theta(a^2)$, as desired. □

Corollary 37. *Let $\Theta : E \to F$ be any unital order-isomorphism between Euclidean Jordan Algebras. Then, for any $a, b \in E$, we have $\Theta(Q_a b) = Q_{\Theta(a)}\Theta(b)$. That is: $\Theta \circ Q_a = Q_{\Theta(a)} \circ \Theta$. Equivalently: $Q_a \circ \Theta = \Theta \circ Q_{\Theta^{-1}(a)}$.*

The next few results involve the notion of *faithfulness*. A map $f : E \to F$ is called faithful when for any positive a the equation $f(a) = 0$ implies $a = 0$. A map f is faithful if and only if $\operatorname{im} f = 1$.

Lemma 38. *Let $f : E \to E$ be a faithful pure $\diamond$-self-adjoint map between EJAs. Then $f = Q_{\sqrt{f(1)}} \circ \Theta$ for some unital Jordan isomorphism Θ with $\Theta(\sqrt{f(1)}) = \sqrt{f(1)}$ and $\Theta = \Theta^{-1}$.*

Proof. First, we collect some basic facts. As f is pure, we have $f = \xi \circ \pi$ for some filter ξ and corner π. Note that $\operatorname{im} f = \operatorname{im} \pi$ as ξ is faithful and $f(1) = \xi(1)$ as π is unital. Hence, by $\diamond$-self-adjointness of f, we have $\lceil \xi(1) \rceil = \lceil f(1) \rceil = f^\diamond(1) = f_\diamond(1) = \operatorname{im} f = \operatorname{im} \pi$. Next, by the universal properties of filters and corners, there exist order isomorphisms Θ_1, Θ_2 such that $\xi = \xi_{\xi(1)} \circ \Theta_1$ and $\pi = \Theta_2 \circ \pi_{\operatorname{im} \pi}$, so that $f = \xi_{\xi(1)} \circ \Theta_1 \circ \Theta_2 \circ \pi_{\operatorname{im}\pi} = \xi_{f(1)} \circ \Theta \circ \pi_{\operatorname{im} f}$ defining $\Theta := \Theta_1 \circ \Theta_2$.

We assumed f is faithful, i.e. $\operatorname{im} f = 1$. So $\pi_{\operatorname{im} f} = \pi_1 = \mathrm{id}$. For brevity, write $q := \sqrt{f(1)}$. As $\lceil q \rceil = \lceil f(1) \rceil = \lceil \xi(1) \rceil = \operatorname{im} \pi = 1$, we have $\xi_q = Q_q$ and so $f = Q_q \circ \Theta$.

As seen in the proof of proposition 33 if f is $\diamond$-self-adjoint we have $\langle f(a), b \rangle = 0 \iff \langle a, f(b) \rangle = 0$. In this case this translates to $0 = \langle Q_q \Theta(a), b \rangle = 0 \iff 0 = \langle a, Q_q \Theta(b) \rangle = \langle Q_q a, \Theta(b) \rangle = \langle \Theta^{-1} Q_q a, b \rangle$. This implies that $\lceil Q_q \Theta(a) \rceil = \lceil \Theta^{-1} Q_q a \rceil = \Theta^{-1}(\lceil Q_q a \rceil)$ for all a. Write $q = \sum_i \lambda_i q_i$ where the q_i are atomic. Then we have $Q_q q_i = \lambda_i^2 q_i$. Filling in $a = q_i$ we then get $\lceil Q_q \Theta(q_i) \rceil = \Theta^{-1}(\lceil Q_q q_i \rceil) = \Theta^{-1}(\lceil \lambda_i^2 q_i \rceil) = \Theta^{-1}(q_i)$. The right-hand side is atomic as Jordan isomorphisms preserve atomicity, so the left-hand side must also be atomic. Since $\lceil b \rceil$ is atomic if and only if b is proportional to an atomic predicate we then get $Q_q \Theta(q_i) = \mu_i \Theta^{-1}(q_i)$ for some $0 < \mu_i < 1$. By composing with Θ this becomes $(\Theta Q_q \Theta)(q_i) = \mu_i q_i$. Now we note that:

$$\sum_i \lambda_i^2 \Theta(q_i) \;=\; \Theta(q^2) \;=\; \Theta Q_q \Theta(1) \;=\; \sum_i \Theta Q_q \Theta(q_i) \;=\; \sum_i \mu_i q_i$$

Now let $p_j = \sum_{i, \lambda_i = \lambda_j} q_i$ and $r_j = \sum_{i, \mu_i = \mu_j} q_j$. Then we can write $\sum_i \lambda_i^2 \Theta(q_i) = \sum_j \lambda_j^2 \Theta(p_j)$ and $\sum_i \mu_i q_i = \sum_j \mu_j r_j$ where in the sums on the right-hand side each of the λ_j and each of the μ_j is distinct. Since Θ preserves orthogonality this means we get two orthogonal decompositions that are equal: $\sum_j \lambda_j^2 \Theta(p_j) = \sum_j \mu_j r_j$. By uniqueness of such decompositions we then have $\lambda_j^2 = \mu_j$ and $\Theta(p_j) = r_j$ (where we assume for now that we have ordered the eigenvalues from high to low). But of course since the λ_j^2 and μ_j agree, the p_j and the r_j will also agree by their definition, so that $\Theta(p_j) = p_j$. Finally, we get $\Theta(q) = \sum_j \lambda_j \Theta(p_j) = \sum_j \lambda_j p_j = q$.

Now $\Theta Q_q = Q_{\Theta(q)} \Theta = Q_q \Theta$ so the Θ commutes with Q_q. Note that since $\lceil q \rceil = 1$, q will be invertible. Let $g = Q_q$. Then $g^\diamond \Theta^\diamond = (\Theta g)^\diamond = (g\Theta)^\diamond = f^\diamond = f_\diamond = (g\Theta)_\diamond = g_\diamond \Theta_\diamond = g^\diamond (\Theta^{-1})^\diamond$. Now g^{-1} is not a subunital map, but it can be scaled downwards until it is, in which case $gg^{-1} = \lambda \mathrm{id}$ for some $\lambda > 0$, in which case $g^\diamond(g^{-1})^\diamond = \mathrm{id}$. Since $g^\diamond$ has an inverse, we see that $g^\diamond\Theta^\diamond = g^\diamond(\Theta^{-1})^\diamond$ can only hold when $\Theta^\diamond = (\Theta^{-1})^\diamond$. From this and $\Theta(\lceil a\rceil) = \lceil\Theta(a)\rceil$ it follows that $\Theta = \Theta^{-1}$. □

Proposition 39. *Let $g\colon E\to E$ be a faithful pure $\diamond$-positive map and let $p := g(1)$, then $g = Q_{\sqrt{p}}$.*

Proof. Since g is $\diamond$-positive there must exist some pure $\diamond$-self-adjoint $f : E \to E$ such that $g = ff$. Since g is faithful, the f must be faithful as well since $1 = \mathrm{im}\, g = \mathrm{im}\, ff \leq \mathrm{im}\, f$. By the previous lemma $f = Q_q\Theta$ for some q and $\Theta(q) = q$ and $\Theta = \Theta^{-1}$. But then $g = ff = Q_q\Theta Q_q\Theta = Q_qQ_{\Theta(q)}\Theta\Theta = Q_qQ_q\Theta^{-1}\Theta = Q_{q^2}$. Now $g(1) = Q_{q^2}1 = q^4$, so that $g = Q_{\sqrt{p}}$ where $p = q^4$. □

Proof of theorem 34. Let f be $\diamond$-self-adjoint so that in particular $\mathrm{im}\, f = \lceil f(1)\rceil$. We can then corestrict f to $E \to E_1(\mathrm{im}\, f)$. Using corollary 22 we also get $f = fQ_{\mathrm{im}\, f}$ so that we can factor f as $f = \xi_{\mathrm{im}\, f}\circ\overline{f}\circ\pi_{\mathrm{im}\, f}$ where $\overline{f} : E_1(\mathrm{im}\, f) \to E_1(\mathrm{im}\, f)$. Note that $\xi_{\mathrm{im}\, f}$ is nothing but the inclusion map into E. It is also easy to see that $\overline{f} = \pi_{\mathrm{im}\, f}\circ f\circ\xi_{\mathrm{im}\, f}$. Then $\overline{f}(1) = \pi_{\mathrm{im}\, f}(f(\mathrm{im}\, f)) = f(\mathrm{im}\, f) = f(1)$. Because f is pure, $\overline{f}$ is also pure as it is a composition of pure maps. When f is $\diamond$-self-adjoint we have $\overline{f}^\diamond = (\pi_{\mathrm{im}\, f}\circ f\circ\xi_{\mathrm{im}\, f})^\diamond = \xi^\diamond_{\mathrm{im}\, f}\circ f^\diamond\circ\pi^\diamond_{\mathrm{im}\, f} = (\pi_{\mathrm{im}\, f})_\diamond\circ f_\diamond\circ(\xi_{\mathrm{im}\, f})_\diamond = \overline{f}_\diamond$. Here we have used that $\xi^\diamond_s(t) = \lceil\xi_s(t)\rceil = \lceil t\rceil = t$ since $t\in E_1(s)$ and $(\pi_s)_\diamond(t) = \mathrm{im}\,\pi_t\circ\pi_s = \mathrm{im}\,\pi_t = t$ which also follows because $t\leq s$, so that $\xi^\diamond_s = (\pi_s)_\diamond$.

When f is $\diamond$-self-adjoint we get $\mathrm{im}\, f^2 = f^\diamond(f_\diamond(1)) = f^\diamond(\mathrm{im}\, f) = f^\diamond(1) = f_\diamond(1) = \mathrm{im}\, f$. For a $\diamond$-self-adjoint f we then get $\overline{f}^2 = \pi_{\mathrm{im}\, f}\circ f\circ\xi_{\mathrm{im}\, f}\circ\pi_{\mathrm{im}\, f}\circ f\circ\xi_{\mathrm{im}\, f} = \pi_{\mathrm{im}\, f}\circ f^2\circ\xi_{\mathrm{im}\, f} = \overline{f^2}$. We conclude that when $g = f\circ f$ is $\diamond$-positive, $\overline{g} = \overline{f^2} = \overline{f}^2$ is also $\diamond$-positive. Since $\overline{g}$ is faithful we have already established that $\overline{g} = Q_{\sqrt{p}}$ where $p = \overline{g}(1) = g(1)$ and $\mathrm{im}\, g = \lceil g(1)\rceil = \lceil p\rceil$. Now $g = \xi_{\mathrm{im}\, g}Q_{\sqrt{p}}\pi_{\mathrm{im}\, g} = Q_{\sqrt{p}}Q_{\lceil p\rceil} = Q_{\sqrt{p}}$. □

We're now ready to conclude that $\mathbf{EJA}^{\mathrm{op}}_{psu}$ is a †-effectus. The details of †-effectuses are beyond the scope we chose for this paper; for those we refer the reader to [28, §173–215].

Theorem 40. *$\mathbf{EJA}^{\mathrm{op}}_{psu}$ is a †-effectus (as defined in [28, §215]).*

Proof. It is straightforward to show that $\mathbf{EJA}^{\mathrm{op}}_{psu}$ is an effectus, see eg. [28, §191]. The existence of corners (prop. 24), filters (prop. 25), images (prop. 29) and the fact that the complement of an idempotent is again idempotent, shows that $\mathbf{EJA}^{\mathrm{op}}_{psu}$ is a $\diamond$-effectus. [28, §206] This, combined with the characterization of $\diamond$-positive maps (thm. 34) and the fact that pure maps are closed under composition (cor. 31), gives us that $\mathbf{EJA}^{\mathrm{op}}_{psu}$ is a &-effectus. [28, §211] To see, finally, that we have a †-effectus at hand, it is sufficient to show the three conditions from [28, §215III].

1. Every predicate $0\leq p\leq 1$ must have a unique square root: there must exist a unique $0\leq q\leq 1$ such that $q\&q = p$. This is true by the spectral theorem (corollary 46).
2. We must have $Q(\sqrt{Q_{\sqrt{p}}q})^2 = Q_{\sqrt{p}}Q(\sqrt{q})^2Q_{\sqrt{p}}$. Rewritten to $Q(Q_{\sqrt{p}}q) = Q_{\sqrt{p}}Q_qQ_{\sqrt{p}}$, this becomes the familiar fundamental equality for the quadratic representation.
3. The filter of an idempotent must map idempotents to idempotents. This is clearly true for the standard filter of an idempotent (being an inclusion) and hence for any as isomorphisms preserve idempotents as well. □

5 Conclusion

We have shown that the definition of purity for von Neumann algebras [27, 26, 25] and effectus theory [8, 9] also works well in the category of positive contractive linear maps between Euclidean Jordan algebras. In particular, this definition of purity is closed under composition and the dagger, and includes the equivalents of the Lüders maps for Jordan algebras. We have also shown that the possibilistic notion of $\diamond$-adjointness from effectus theory translates to these algebras, and we have used it to give a new characterization of the 'sequential product' maps $b \mapsto Q_a b$ for positive a.

With the results in this paper we know that the filter-corner definition of purity works well in von Neumann algebras and in Euclidean Jordan algebras. This begs the question whether our results concerning pure maps generalize to the class of *JBW-algebras* [16], which include all von Neumann algebras and EJAs. We consider this to be a challenging topic for future research.

The precise relationship between the different notions of purity found in the literature and the one studied in this paper is yet to be determined. When restricted to complex matrix algebras all the definitions coincide, but when direct sums of simple algebras are considered the definitions sometimes diverge. For example, the identity map on a direct sum is not atomic and therefore not pure in the sense of [7]. Also, the adjoint of a pure map in the sense of [12, 23] on a direct sum need not be pure.

Then finally there is the matter of the different notions of positivity for maps between Euclidean Jordan algebras: they can be *superoperator positive* (mapping positive elements to positive elements), *operator positive* as linear maps between Hilbert spaces, and, of course, $\diamond$-positive. Any relation? While preliminary investigations reveal none between superoperator positivity and operator positivity, $\diamond$-positivity does seem to be connected with operator positivity.

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A Basic structure of EJAs

EJAs are commonly defined to be finite-dimensional. The infinite-dimensional algebras we study are also known as JH-algebras [11, 10] (where we additionally require the existence of a unit). In this appendix we will give a relatively self-contained proof that the EJAs we use are JBW-algebras and that every element has a finite spectral decomposition.

Proposition 41. *For every EJA E there is a constant $r > 0$ such that, for all $a, b \in E$,*

$$\| a * b \|_2 \;\leq\; r \|a\|_2 \|b\|_2, \tag{2}$$

where $\|c\|_2 \equiv \sqrt{\langle c, c\rangle}$ denotes the Hilbert norm. In particular, $*$ is uniformly continuous with respect to the Hilbert norm.

Proof. The trick is to apply the *uniform boundedness principle* (see e.g. [24]) twice, which states that any collection $\mathscr{T}$ of bounded operators from a Banach space X to a normed vector space Y that is bounded pointwise, i.e. $\sup_{T\in\mathscr{T}} \|Tx\| < \infty$ for all $x \in X$, is uniformly bounded in the sense that $\sup_{T\in\mathscr{T}} \|T\| < \infty$.

For the moment fix $a \in E$. Our first step is to show that the operator $a * (\,\cdot\,)\colon E \to E$ is bounded. To this end consider the collection of linear functionals $\langle b, a * (\,\cdot\,)\rangle : E \to \mathbb{R}$, where $b \in E$ with $\|b\|_2 \leq 1$. These are bounded operators, since $\langle b, a*(\,\cdot\,)\rangle = \langle a * b, (\,\cdot\,)\rangle$, and as a collection they are bounded pointwise, since $|\langle b, a * c\rangle| \leq \|b\|_2 \|a * c\|_2 \leq \|a * c\|_2 < \infty$. Hence

$$r_a \;:=\; \sup_{\|b\|_2 \leq 1} \| \langle a * b, (\,\cdot\,)\rangle \| \;<\; \infty$$

by the uniform boundedness principle. Since in particular $(\|a * b\|_2)^2 = \langle a * b, a * b\rangle \leq r_b \|a * b\|_2$ for all $b \in E$ with $\|b\|_2 \leq 1$, we get $\|a * b\|_2 \leq r_a$ for all $b \in E$ with $\|b\|_2 \leq 1$, and thus $\|a * b\|_2 \leq r_a \|b\|_2$ for any $b \in E$. In other words, the linear operator $a * (\,\cdot\,)\colon E \to E$ is bounded.

Now, to prove equation (2) it suffices to show that $\sup_{\|a\|_2 \leq 1} \|a * (\,\cdot\,)\|$ is finite. For this, in turn, it suffices, by the uniform boundedness principle, to show given $b \in E$ that $\sup_{\|a\|_2 \leq 1} \|a * b\|_2 < \infty$. Since $\|a * b\|_2 = \|b * a\|_2 \leq \|b * (\,\cdot\,)\| \, \|a\|_2 \leq \|b * (\,\cdot\,)\| < \infty$ for all $a \in E$ with $\|a\|_2 \leq 1$, this is indeed the case. □

To proceed we need some basic algebraic properties of Jordan algebras, which are most conveniently expressed with some additional notation.

Notation 42. Let E be a Jordan algebra.

1. We write $a^0 := 1, \quad a^1 := a, \quad a^2 := a * a, \quad a^3 := a * a^2, \quad a^4 := a * a^3, \ldots$. Note that since $*$ is not associative it's not a priori clear whether equations like $a^4 = a^2 * a^2$ hold.

2. Given $a \in E$ we denote the linear operator $E \to E\colon b \mapsto a * b$ by L_a.
   Given two linear operators $S, T\colon E \to E$ we write $[S, T] := ST - TS$ for the commutator of S and T.

Proposition 43. *Given a Jordan algebra E, and $a,b,c \in E$, we have*

1. $[L_a, L_{a^2}] = 0$; $[L_b, L_{a^2}] = 2[L_{a*b}, L_a]$; *and* $[L_a, L_{b*c}] + [L_b, L_{c*a}] + [L_c, L_{a*b}] = 0$;
2. $L_{a*(b*c)} = L_a L_{b*c} + L_b L_{c*a} + L_c L_{a*b} - L_b L_a L_c - L_c L_a L_b$;
3. $a^n * (b * a^m) = (a^n * b) * a^m$ *and* $a^n * a^m = a^{n+m}$ *for all* $n, m \in \mathbb{N}$.

Proof. 1. The first equation, $[L_a, L_{a^2}] = 0$, is just a reformulation of the Jordan identity:

$$L_a L_{a^2} b \equiv a * (b * a^2) = (a * b) * a^2 \equiv L_{a^2} L_a b.$$

Note that $[L_{a+b}, L_{(a+b)^2}] - [L_{a-b}, L_{(a-b)^2}] = 4[L_a, L_{a*b}] + 2[L_b, L_{a^2}] + 2[L_b, L_{b^2}]$—just expand both sides. Applying $[L_d, L_{d^2}] = 0$ with $d = b$, $a+b$, $a-b$, we get $[L_b, L_{a^2}] = -2[L_a, L_{a*b}] = 2[L_{a*b}, L_a]$. Similarly, one gets $[L_a, L_{b*c}] + [L_b, L_{c*a}] + [L_c, L_{a*b}] = 0$ by expanding $2[L_{(a+c)*b}, L_{a+c}] - [L_b, L_{(a+c)^2}] \equiv 0$.

2. Since $[L_a, L_{b*c}] + [L_b, L_{c*a}] + [L_c, L_{a*b}] = 0$, we have, for all $d \in E$,

$$(L_a L_{b*c} + L_b L_{c*a} + L_c L_{a*b})d = (b*c)*(a*d) + (c*a)*(b*d) + (a*b)*(c*d).$$

Since the right-hand side of this equation is invariant under a switch of the roles of a and d, so must be the left-hand side, which gives us the professed equality after some rewriting:

$$\begin{aligned}(L_a L_{b*c} + L_b L_{c*a} + L_c L_{a*b})d &= (L_d L_{b*c} + L_b L_{c*d} + L_c L_{d*b})a && a \leftrightarrow d \\ &\equiv (L_{a*(b*c)} + L_b L_a L_c + L_c L_a L_b)d && \text{rewriting.}\end{aligned}$$

3. By repeatedly applying the equation for $L_{a*(b*c)}$ from 2 it is clear that L_{a^n} and L_{a^m} may both be written as polynomial expressions in L_a and L_{a^2}. Since L_a and L_{a^2} commute by the Jordan identity, so will L_{a^n} and L_{a^m} commute. Whence $a^n * (b * a^m) = (a^n * b) * a^m$.

Finally, seeing that $a^n * a^m = a^{n+m}$ is only a matter of induction over m. Indeed, $a^n * a^0 = a^n * 1 = a^n$, and if $a^n * a^m = a^{n+m}$ for all n for some fixed m, we get $a^n * a^{m+1} = a^n * (a * a^m) = (a^n * a) * a^m = a^{n+1} * a^m = a^{n+m+1}$. □

Corollary 44. *Let $a \in E$ be an element of an EJA. Let $C(a)$ denote the closure of the algebra generated by a, then $C(a)$ is a commutative associative algebra.*

Proof. Point 3 of proposition 43 allows us to see that the smallest Jordan subalgebra of E that contains a consists of all real polynomials $\sum_{n=0}^{N} \lambda_n a^n$ over a, and is therefore associative. Since the Jordan multiplication is continuous (by proposition 2) the closure $C(a)$ of this associative subalgebra will again be an associative subalgebra. □

Proposition 45. *An associative EJA is isomorphic as an algebra to $\mathbb{R}^n$ with pointwise multiplication for some $n \in \mathbb{N}$.*

Proof. Let E be an associative EJA and let $L_a : E \to E$ denote the Jordan multiplication operator of $a \in E$. This gives rise to a map $L : E \to B(E)$ that is linear (since $L_{a+b} = L_a + L_b$ and $L_{\lambda a} = \lambda L_a$), multiplicative (by associativity $L_{a*b} = L_a L_b$), unital ($L_1 = \mathrm{id}$), injective (since $L_a 1 = a$) and positive (L_a is self-adjoint and $L_{a^2} = L_a^2$ is therefore a positive operator). The map is also order-reflecting. To see this we first note that the algebra $C(L_a)$ generated by L_a in $B(E)$ is equal to the set $L(C(a)) := \{L_b \,;\, b \in C(a)\}$. Now if $L_a \geq 0$ in $B(E)$, then it has a square root which lies in $C(L_a) = L(C(a))$, so that we can find a $b \in C(a)$ with $L_{b^2} = L_b^2 = \sqrt{L_a}^2 = L_a$ so that a is indeed positive in E. We conclude that E is order-isomorphic to some closed subspace of $B(E)$ and thus that E is a complete Archimedean order unit space.

The product of positive elements is positive, since indeed: $a^2 * b^2 = (a*a)*(b*b) = (a*b)*(a*b) = (a*b)^2$. By Kadison's representation theorem [19] any complete Archimedean order unit space with unital multiplication that preserves positivity (like E), is isomorphic as an algebra to $C(X)$, the real-valued continuous functions on some compact Hausdorff space X.

Thus without loss of generality, we may assume $E = C(X)$. It is sufficient to show X is discrete (for then X must be finite by compactness). For $x \in X$, write $\delta_x \colon C(X) \to \mathbb{R}$ for the bounded linear map $\delta_x(f) = f(x)$. As $E = C(X)$ is assumed to be a Hilbert space, there must be an $\hat{x} \in C(X)$ with $\delta_x(f) = \langle \hat{x}, f \rangle = f(x)$ for all $f \in C(X)$. As $\langle \hat{x}g, f \rangle = \langle \hat{x}, gf \rangle = (gf)(x) = g(x)f(x) = \langle \hat{x}, g \rangle \langle \hat{x}, f \rangle = \langle \langle \hat{x}, g \rangle \hat{x}, f \rangle$ for all $f \in C(X)$, we must have $\hat{x}g = \langle \hat{x}, g \rangle \hat{x}$. In particular $\hat{x}\hat{y} = \langle \hat{x}, \hat{y} \rangle \hat{x}$ and with similar reasoning $\hat{x}\hat{y} = \langle \hat{x}, \hat{y} \rangle \hat{y}$. Assume $x \neq y$. Then $\hat{x} \neq \hat{y}$, but by the previous $\langle \hat{x}, \hat{y} \rangle \hat{x} = \langle \hat{x}, \hat{y} \rangle \hat{y}$. So that necessarily $0 = \langle \hat{x}, \hat{y} \rangle = \hat{x}(y)$ for all $y \neq x$. As $\hat{x} \neq 0$ and $\hat{x}$ is continuous, we see $\{x\}$ is open and so X is discrete. □

Corollary 46. *Let a be an element of an EJA. Then there exist real numbers λ_i and orthogonal idempotents p_i such that $a = \sum_{i=1}^n \lambda_i p_i$ for some n.*

Proof. Let $C(a)$ denote the EJA generated by a. This is an associative algebra by corollary 44 so that by proposition 45 we have $C(a) \cong \mathbb{R}^n$ for some n. Since $\mathbb{R}^n$ is obviously spanned by orthogonal idempotents we see that indeed $a = \sum_{i=1}^n \lambda_i p_i$. □

Proposition 47. *An element $a \in E$ is positive (i.e. a square) if and only if $\langle a, b \rangle \geq 0$ for all positive b.*

Proof. If p is an idempotent then $\langle p, a \rangle \geq 0$ if a is positive [10, p. 107]. As a result if $b = \sum_i \lambda_i p_i$ with $\lambda_i \geq 0$ and with the p_i idempotents we have $\langle a, b \rangle \geq 0$. Now for the other direction suppose $\langle a, b \rangle \geq 0$ for all positive b. Write $a = \sum_i \lambda_i p_i$ with the λ_i not necessary positive, and where the p_i are orthogonal. We then have $\langle p_i, p_j \rangle = \langle 1, p_i * p_j \rangle = \langle 1, 0 \rangle = 0$ so that $0 \leq \langle a, p_j \rangle = \lambda_j \langle p_j, p_j \rangle$. Since $p_j \neq 0$ this is only possible when $\lambda_j \geq 0$. This holds for all j so that we conclude that $a \geq 0$. □

Corollary 48. *Let E be an EJA. The set of positive elements is closed under addition. More specifically E is an Archimedean order unit space.*

Proof. By the previous proposition $a \geq 0$ if and only if $\langle a, b \rangle \geq 0$ for all $b \geq 0$. But then if $c \geq 0$ we obviously have $\langle a + c, b \rangle = \langle a, b \rangle + \langle c, b \rangle \geq 0$ for all positive b so that indeed $a + c \geq 0$. Suppose now that $a \leq \frac{1}{n}1$ for all $n \in \mathbb{N}$. By proposition 45 the associative algebra generated by a (which contains $\frac{1}{n}1$) is isomorphic to $\mathbb{R}^n$. Since this space is Archimedean we conclude that $a \leq 0$ in $\mathbb{R}^n$ so that also $a \leq 0$ in E. In the same way we can find for any $a \in E$ a number $n \in \mathbb{N}$ so that $-n1 \leq a \leq n1$ so that E is indeed an Archimedean order unit space. □

Proposition 49. *Let E be an EJA. The topologies induced by the Hilbert norm and by the order unit norm are equivalent.*

Proof. In order to show that the topologies are the same we need to show that the norms are equivalent. Let $\|a\|$ denote the order unit norm and $\|a\|_2$ the Hilbert norm. We need to find constants $c, d \in \mathbb{R}_{>0}$ such that $c\|a\|_2 \leq \|a\| \leq d\|a\|_2$ for all $a \in E$.

Note that $\|a\|_2^2 = \langle a, a \rangle \leq \|a\|^2 \langle 1, 1 \rangle = \|a\|^2 \|1\|_2^2$ by self-duality, so that we already have one side of the inequality.

Any $a \in E$ can be written as $a = \sum_i \lambda_i p_i$ where the p_i are nonzero by the previous corollary so that $\|a\| = \max\{|\lambda_i|\}$. Now $\|a\|_2^2 = \langle a, a \rangle = \sum_i \lambda_i^2 \|p_i\|_2^2 \geq \sum_i \lambda_i^2 \inf\{\|p_j\|_2^2\} \geq \max\{|\lambda_i^2|\} \inf\{\|p_j\|_2^2\} =$

$\|a\|^2 \inf\{\|p_j\|_2^2\}$ so if we can find some constant $R > 0$ such that $\|p\|_2 \geq R$ for all nonzero idempotents p we are done.

Let $R = \inf_{p\neq 0, p^2=p} \|p\|_2$. If $R \neq 0$ we are done, so suppose $R = 0$. In this case there exists a sequence of idempotents (p_i) such that $\|p_i\|_2 \to 0$. We can then pick a subsequence such that $\|p_k\|_2 \leq 2^{-k}/k$. Now let $q_n = \sum_{i=1}^n i p_i$. Let $n \geq m$. We have $\|q_n - q_m\|_2 = \|\sum_{k=m}^n k p_k\|_2 \leq \sum_{k=m}^n k \|p_k\|_2 \leq \sum_{k=m}^n k 2^{-k}/k$ so that the (q_n) form a Cauchy sequence in the Hilbert norm. Since E is a Hilbert space it must converge to some $q \in E$ and since it is also an increasing sequence and the set of positive elements is closed in the Hilbert norm by proposition 47 we must have $q \geq q_n$ so that $\|q\| \geq \|q_n\| \geq n$ for all n which is a contradiction. □

Proposition 50. *Let E be an EJA. Then E is a JB-algebra.*

Proof. By corollary 48 E is an Archimedean order unit space. By definition E is complete in the Hilbert norm topology and by the previous proposition this topology is equivalent to the order unit topology. We conclude that E is a complete Archimedean order unit space. By [1, Theorem 1.11], E will then be a JB-algebra when the implication $-1 \leq a \leq 1 \implies 0 \leq a^2 \leq 1$ holds. So suppose $-1 \leq a \leq 1$. By the spectral theorem $a = \sum_i \lambda_i p_i$ and we must have $-1 \leq \lambda_i \leq 1$. But then $a^2 = \sum_i \lambda_i^2 p_i$ so that indeed $0 \leq a^2 \leq 1$. □

Proposition 51. *Let E be an EJA. Then E is bounded directed complete and furthermore every state is normal.*

Proof. Let $(a_i)_{i\in I}$ be a bounded upwards directed set. By translation we can take all a_i to be positive. Define for $b \geq 0$ the state $\omega(b) := \sup_{i\in I}\langle a_i, b\rangle$. This supremum exists since the a_i are bounded and the inner product between positive elements is again positive. This map can obviously be extended by linearity to the entirety of E. Since E is a Hilbert space we conclude that there must exist an $a \in E$ such that $\omega(b) = \langle a, b\rangle$ for all $b \in E$. We claim that this a is the lowest upper bound. That it is an upper bound follows by the self-duality of the order. Suppose $a_i \leq c$ for some c. Then $c - a_i \geq 0$ so that $\langle c - a_i, b\rangle \geq 0$ for all $b \geq 0$ so that $\langle c, b\rangle \geq \langle a_i, b\rangle$. By taking the supremum over the a_i's we see then that $\langle c, b\rangle \geq \langle a, b\rangle$. Again by self-duality we conclude that $c \geq a$.

For any state $\omega' : E \to \mathbb{R}$ we can find a $b \in E$ such that $\omega' = \langle \cdot, b\rangle$. As the previous argument shows, suprema of elements are defined in terms of these states so that they must preserve those suprema. □

Proposition 52. *Let p be an idempotent of an EJA. Then there exist orthogonal atomic idempotents p_i such that $p = \sum_i p_i$.*

Proof. If p is atomic we are done, so suppose it is not. Then by definition we can find $0 \leq a \leq p$ such that $a \neq \lambda p$ for some λ. Using corollary 46 write $a = \sum_i \lambda_i q_i$. If all the $q_i = p$ then $a = \lambda p$ so there must be a $q_i \neq p$. Pick this one. We have $\lambda_i q_i \leq p$. By proposition 11 we then have $Q_p(\lambda_i q_i) = \lambda_i q_i$. This of course implies $Q_p q_i = q_i$ so that again by proposition 11 we have $q_i \leq p$. We can now repeat this procedure with p replaced by q_i and $p - q_i$ to get a family of orthogonal idempotents that sum up to p. We claim that this process stops after a finite amount of iterations. By assumption the resulting idempotents are then atomic.

Suppose the process does not halt after a finite amount of iterations. Then we are left with a countable collection of orthogonal idempotents $(q_i)_i$. By equation (1) we have in any Jordan algebra for any a and b: $[L_a, L_{b^2}] + 2[L_b, L_{a*b}] = 0$ Let $a = q_i$ and $b = q_j$ with $i \neq j$ so that $a * b = 0$, then we conclude that $[L_{q_i}, L_{q_j}] = 0$ and thus that $q_i * (a * q_j) = (q_i * a) * q_j$. The algebra spanned by the $(q_i)_i$ is therefore associative. As this algebra is necessarily infinite-dimensional this is in contradiction to proposition 45. □

Proposition 53. *Let E be an EJA. Then E is a type I JBW-algebra of finite rank.*

Proof. By proposition 50 E is a JB-algebra. By definition, a JBW-algebra is a JB-algebra that is bounded directed complete and that is separated by normal states. Proposition 51 therefore has established that E is indeed a JBW-algebra. A JBW-algebra is of type I if below every idempotent we can find an atomic idempotent. This is true by proposition 52. By this same proposition we can write the identity as a finite sum of atomic idempotents, so that the space is indeed of finite rank. □

Corollary 54. *Let E be an EJA. Then there exists a finite-dimensional EJA E_{fin} and an EJA E_{inf} that is a direct sum of infinite-dimensional spin-factors such that E is isomorphic as an EJA to $E_{fin} \oplus E_{inf}$.*

Proof. By the previous corollary E is a type I JBW-algebra of finite rank. It is therefore isomorphic to a finite direct sum of type I JBW-factors of finite rank. These factors have been classified in [16]. They are either finite-dimensional or they are infinite-dimensional spin-factors. □